Derivation of the Wave Equation

The objective is to provide an explanation of how the wave equation applies to the vibration of an elastic string. This will be done using the mass-spring material developed in Section 3.8, and a little trigonometry.

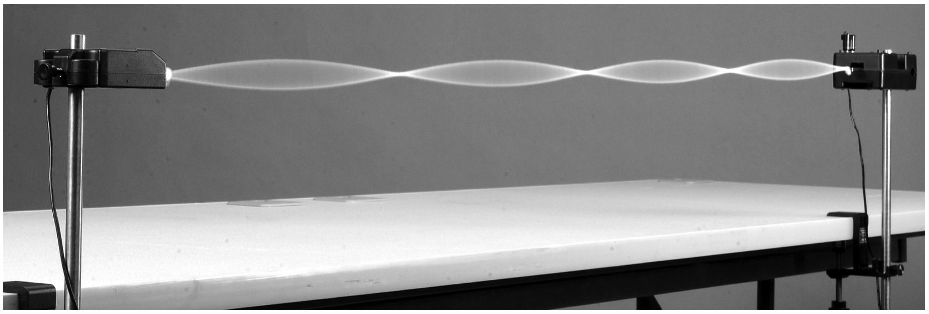

Figure 1: Time-lapsed photograph of a vibrating string. It is vibrating at approximately the fourth harmonic, or natural frequency, of the string. This solution is given in equation (7.68) of the textbook, with $n = 4$.

There are numerous online videos of vibrating strings. Here are a few examples: Example 1, Example 2, and Example 3.

(I) Modeling a String using Atoms and Springs

An unstretched elastic string can be thought of as made up of a row of atoms, each a distance ℓ_0 apart. Adjacent atoms in this case are connected by a spring as illustrated in the figure below.

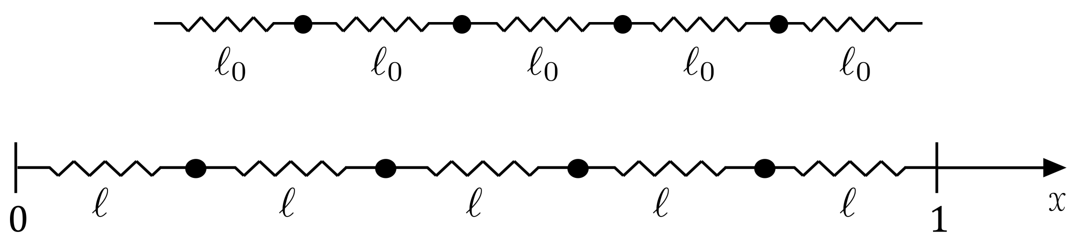

Figure 2: Upper: Unstretched row of atoms and springs making up the string. Lower: Spacing of atoms and springs after stretching the string so it occupies the interval $0 \le x \le 1$.

Assume that the string is stretched, with one end held at $x = 0$ and the other held at, say, $x = 1$ (see above figure). The atoms in the string are now a distance ℓ apart, where $\ell > \ell_0$.

As usual, we assume that the restoring force in each spring is proportional to how far it is stretched. Specifically, when stretching the distance between the atoms from ℓ_0 to ℓ the resulting restoring force is assumed to be

$$F = E\frac{\ell - \ell_0}{\ell_0}\,. \tag{1}$$

This formula differs slightly from Hooke's Law used in Section 3.8 in that we are now normalizing the stretch distance by the original length. A material for which this holds is said to be linearly elastic, and the positive constant E is called the Young's modulus for the material.

(II) Vertical Motion

As the string deflects, it's assumed that the atoms in the string move vertically as illustrated in Figure 3. According to (1), the force F_b in the spring on the right, and the force F_a in the spring on the left, are

$$F_b = E\frac{\ell_b - \ell_0}{\ell_0}, \qquad F_a = E\frac{\ell_a - \ell_0}{\ell_0}.$$

The respective force vectors are $\mathbf{F}_b = F_b(\cos\theta_b, \sin\theta_b)$ and $\mathbf{F}_a = -F_a(\cos\theta_a, \sin\theta_a)$.

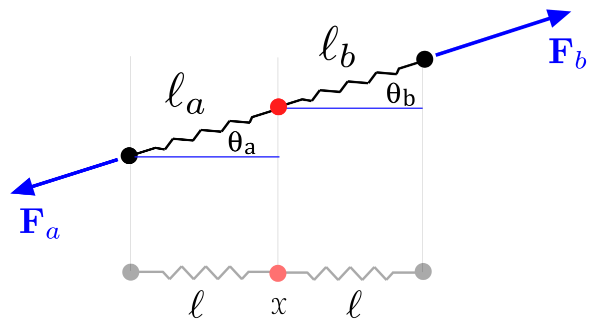

Figure 3: Stretched springs between atoms as the atoms move vertically. The rest position of each atom is also shown (using faded colors).

(III) Some Trigonometry

The vertical displacement of the red atom in Figure 3 is $u(x,t)$. The displacement of the atom on the left is $u(x-\ell,t)$, and the displacement of the one on the right is $u(x+\ell,t)$. From this we get:

$$\cos\theta_b = \frac{\ell}{\ell_b}, \quad \sin\theta_b = \frac{u(x+\ell,t) - u(x,t)}{\ell_b},$$
$$\cos\theta_a = \frac{\ell}{\ell_a}, \quad \text{and} \quad \sin\theta_a = \frac{u(x,t) - u(x-\ell,t)}{\ell_a}.$$

Also,

$$\ell_b = \sqrt{\ell^2 + \big(u(x+\ell,t) - u(x,t)\big)^2}, \quad \text{and} \quad \ell_a = \sqrt{\ell^2 + \big(u(x,t) - u(x-\ell,t)\big)^2}.$$

(IV) Newton's Second Law: $\mathbf{F} = m\mathbf{a}$

In this problem $\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b$ and $\mathbf{a} = (0, u_{tt})$. So, from the first component of $\mathbf{F} = m\mathbf{a}$ we conclude that $F_b\cos\theta_b - F_a\cos\theta_a = 0$. From this, and the formulas in (III) and (IV), it follows that $\ell_a = \ell_b$.

This means that the second component of $\mathbf{F}$ is

$$
\begin{aligned}
F_b \sin\theta_b - F_a \sin\theta_a &= E\frac{\ell_b-\ell_0}{\ell_0}\frac{u(x+\ell,t)-u(x,t)}{\ell_b} - E\frac{\ell_a-\ell_0}{\ell_0}\frac{u(x,t)-u(x-\ell,t)}{\ell_a} \\
&= E\frac{\ell_b-\ell_0}{\ell_0}\frac{u(x+\ell,t)-2u(x,t)+u(x-\ell,t)}{\ell_b}\,. \qquad (2)
\end{aligned}
$$

We now use the fact that the atomic spacing is much smaller that the overall length of the string, which means that $\ell \ll 1$. So, using Taylor's theorem,

$$
\begin{aligned}
u(x+\ell,t) &= u(x,t) + \ell u_x(x,t) + \frac{1}{2}\ell^2 u_{xx}(x,t) + \frac{1}{6}\ell^3 u_{xxx}(x,t) + \cdots \\
&\approx u(x,t) + \ell u_x(x,t) + \frac{1}{2}\ell^2 u_{xx}(x,t).
\end{aligned}
$$

Similarly, $u(x-\ell,t) \approx u(x,t) - \ell u_x(x,t) + \frac{1}{2}\ell^2 u_{xx}(x,t)$, and

$$
\ell_b = \sqrt{\ell^2 + (u(x+\ell,t)-u(x,t))^2} \approx \sqrt{\ell^2 + \ell^2 u_x(x,t)^2} = \ell\sqrt{1+u_x^2}\,.
$$

Substituting these into (2) yields

$$
\begin{aligned}
F_b \sin\theta_b - F_a \sin\theta_a &\approx E\frac{\ell_b-\ell_0}{\ell_0}\frac{\ell u_{xx}}{\sqrt{1+u_x^2}} \\
&\approx E\ell(\lambda\sqrt{1+u_x^2}-1)\frac{u_{xx}}{\sqrt{1+u_x^2}}\,,
\end{aligned}
$$

where $\lambda = \ell/\ell_0$ is a constant satisfying $\lambda > 1$.

The remaining term to determine is the mass m. This is the mass of the string segment associated with the red atom in Figure 3. If ρ is the mass density of the material that the string is made of, and if the original unstretched string segment is cylindrical with a cross-section area of A, then $m = \ell_0 A\rho$.

Combining the above results, we obtain the nonlinear partial differential equation

$$
\frac{E\lambda}{A\rho}\left(\lambda\sqrt{1+u_x^2}-1\right)\frac{u_{xx}}{\sqrt{1+u_x^2}} = u_{tt}\,. \qquad (3)
$$

(V) Final Assumption

The last assumption is that the string's amplitude is small when compared to distances along the x-axis over which its amplitude changes. More specifically, it's assumed that $u_x^2 \ll 1$ so that $\sqrt{1+u_x^2} \approx 1$. The PDE (3) then reduces to the wave equation

$$
c^2 u_{xx} = u_{tt}\,,
$$

where $c^2 = E\lambda(\lambda-1)/(\rho A)$ is a positive constant.

(VI) Summary

The principal assumptions made in the above derivation are:

(a) The string is linearly elastic.

(b) The string, before starting to move, is under tension.

(c) The points in the string move vertically.

(d) The string's motion has a relatively small amplitude.