Numerical Linear Algebra with Applications S'23
Exam 1 Name: ______________________________

Make sure to show your work (answers without supporting work will receive no credit).

1. Assume that $\mathbf{A}$ is a symmetric 10×10 matrix with 10 different eigenvalues (some positive and some negative). Explain how to use the power method (twice) to determine λ_a and λ_b so that they are eigenvalues for $\mathbf{A}$ and all other eigenvalues satisfy $\lambda_a \leq \lambda \leq \lambda_b$. You can assume that $\lambda_a \neq -\lambda_b$, and the order in which you compute them is not important.

2. This problem concerns the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & & & \\ 2 & 5 & 2 & & \\ & & \ddots & \ddots & \ddots \\ & & & & 2 \\ & & & 2 & 5 \end{pmatrix}$$

a) Show that the matrix is positive definite.

b) The results using the power method are shown below. What is, approximately, the second largest eigenvalue? Make sure to explain why.

k	v_k	$\dfrac{v_k - v_{k-1}}{v_{k-1} - v_{k-2}}$
1	4.719622127223	
2	5.591880438394	8.72e−01
3	5.888784613194	3.40e−01
4	5.971538275911	2.79e−01
5	5.992845180809	2.57e−01
6	5.998209277724	2.52e−01
7	5.999552248068	2.50e−01
8	5.999888063756	2.50e−01
9	5.999972016739	2.50e−01
10	5.999993004312	2.50e−01

c) Explain how to use the result from part (b) to compute the second largest eigenvalue.

3. a) Why is a random vector $\mathbf{y}$ used to start the power method?

b) Why is the modified Gram-Schmidt method generally a better choice that the standard (classic) Gram-Schmidt method?

c) Why is the QR approach for solving the linear least squares problem generally a better choice that using the Moore-Penrose pseudo-inverse?

4. Fit the model function $g(x) = v_1(x - 1) + v_2 x^3$ to the data: $(0, 0)$, $(1, -9)$, $(2, -6)$.

5. Use the Householder method to find a QR factorization of

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

6. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \end{pmatrix}.$$

a) For what values of a, if any, is $\mathbf{A}$ irreducible?

b) For what values of a, if any, is $\mathbf{A}$ a probability matrix?

7. Find the least squares solution of the following:

$$\begin{aligned} x + y &= 1 \\ 2x &= 1 \\ -y &= 1 \end{aligned}$$

Extra Credit: Answer Problem 1 but do not assume that $\lambda_a \neq -\lambda_b$ (i.e., your method works whether or not $\lambda_a \neq -\lambda_b$).

Workspace