Summary: IVP Solvers (Chapter 7)

Methods for solving the differential equation

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

Method	**Difference Equation**	τ_j	**Properties**
Euler	$\mathbf{y}_{j+1} = \mathbf{y}_j + k\mathbf{f}_j$	$O(k)$	E; C. A-S
Backward Euler	$\mathbf{y}_{j+1} = \mathbf{y}_j + k\mathbf{f}_{j+1}$	$O(k)$	I; A-S
Trapezoidal	$\mathbf{y}_{j+1} = \mathbf{y}_j + \dfrac{k}{2}(\mathbf{f}_j + \mathbf{f}_{j+1})$	$O(k^2)$	I; A-S
Heun **RK2**	$\mathbf{y}_{j+1} = \mathbf{y}_j + \dfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$ where $\mathbf{k}_1 = k\mathbf{f}_j, \quad \mathbf{k}_2 = k\mathbf{f}(t_{j+1}, \mathbf{y}_j + \mathbf{k}_1)$	$O(k^2)$	E; C. A-S
Classic **RK4**	$\mathbf{y}_{j+1} = \mathbf{y}_j + \dfrac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$ where $\mathbf{k}_1 = k\mathbf{f}_j,\ \mathbf{k}_2 = k\mathbf{f}(t_j + \frac{k}{2}, \mathbf{y}_j + \frac{1}{2}\mathbf{k}_1),$ $\mathbf{k}_3 = k\mathbf{f}(t_j + \dfrac{k}{2}, \mathbf{y}_j + \dfrac{1}{2}\mathbf{k}_2),$ $\mathbf{k}_4 = k\mathbf{f}(t_{j+1}, \mathbf{y}_j + \mathbf{k}_3)$	$O(k^4)$	E; C. A-S
Lobatto **RK4**	$\mathbf{y}_{j+1} = \mathbf{y}_j + \dfrac{1}{12}(\mathbf{k}_1 + 5\mathbf{k}_2 + 5\mathbf{k}_3 + \mathbf{k}_4)$ where $\mathbf{k}_1 = k\mathbf{f}_j,\ \mathbf{k}_2 = k\mathbf{f}\Big(t_j + \alpha k, \mathbf{y}_j + \alpha\mathbf{k}_1\Big)$ $\mathbf{k}_3 = k\mathbf{f}(t_j - \alpha k, \mathbf{y}_j - r\mathbf{k}_2),$ $\mathbf{k}_4 = k\mathbf{f}(t_{j+1}, \mathbf{y}_j + \mathbf{k}_1 + 5\beta\mathbf{k}_{21} + 5\alpha\mathbf{k}_{31}$	$O(k^4)$	E; C. A-S

Table 1: The step size is $k = t_{j+1} - t_j$, $\mathbf{f}_j = \mathbf{f}(t_j, \mathbf{y}_j)$, $\mathbf{f}_{j+1} = \mathbf{f}(t_{j+1}, \mathbf{y}_{j+1})$, and τ_j is the truncation error. Also, E=explicit, I=implicit, A-S=A-stable, and C. A-S=conditionally A-stable. In addition, $\alpha = (1 - 1/\sqrt{5})/2$, $\beta = -(1 + 3/\sqrt{5})/4$, $\mathbf{k}_{21} = \mathbf{k}_2 - \mathbf{k}_1$, and $\mathbf{k}_{31} = \mathbf{k}_3 - \mathbf{k}_1$.

Type	Difference Approximation	Truncation Term
Forward	$y'(t_j) \approx \frac{y(t_{j+1})-y(t_j)}{k}$	$\tau_j = -\frac{1}{2}ky''(\eta_j)$
Backward	$y'(t_j) \approx \frac{y(t_j)-y(t_{j-1})}{k}$	$\tau_j = \frac{1}{2}ky''(\eta_j)$
Centered	$y'(t_j) \approx \frac{y(t_{j+1})-y(t_{j-1})}{2k}$	$\tau_j = -\frac{1}{6}k^2y'''(\eta_j)$
One-sided	$y'(t_j) \approx \frac{-y(t_{j+2})+4y(t_{j+1})-3y(t_j)}{2k}$	$\tau_j = \frac{1}{3}k^2y'''(\eta_j)$
One-sided	$y'(t_j) \approx \frac{3y(t_j)-4y(t_{j-1})+y(t_{j-2})}{2k}$	$\tau_j = \frac{1}{3}k^2y'''(\eta_j)$
Centered	$y''(t_j) \approx \frac{y(t_{j+1})-2y(t_j)+y(t_{j-1})}{k^2}$	$\tau_j = -\frac{1}{12}k^2y''''(\eta_j)$

Table 2: Numerical differentiation formulas when using equally spaced points with $k = t_{j+1} - t_j$. The point η_j is located between the left- and rightmost points used in the formula.

Rule	Integration Formula
Right Box	$\int_{t_j}^{t_{j+1}} f(x)dx = kf(t_{j+1}) + O(k^2)$
Left Box	$\int_{t_j}^{t_{i+1}} f(x)dx = kf(t_j) + O(k^2)$
Midpoint	$\int_{t_{j-1}}^{t_{j+1}} f(x)dx = 2kf(t_j) + \frac{k^3}{3}f''(\eta_i)$
Trapezoidal	$\int_{t_j}^{t_{j+1}} f(x)dx = \frac{k}{2}\left[f(t_j) + f(t_{j+1})\right] - \frac{k^3}{12}f''(\eta_j)$
Simpson	$\int_{t_{j-1}}^{t_{j+1}} f(x)dx = \frac{k}{3}\left[f(t_{j+1}) + 4f(t_j) + f(t_{j-1})\right] - \frac{k^5}{90}f''''(\eta_j)$

Table 3: Numerical integration formulas. The points $t_1, t_2, t_3, \ldots$ are equally spaced with step size $k = t_{j+1} - t_j$. The point η_j is located within the interval of integration.