Numerical Computing F'22
Exam 2 Name: ______________________________

There are 7+ questions. You can use a crib sheet but no calculators or other notes are to be used. Make sure to show your work, any answer without supporting work will receive no credit.

1. Interpolation is going to be used to approximate the function $f(x) = x^3$ on the interval $-1 \leq x \leq 2$, using $x_1 = -1$, $x_2 = 0$, and $x_3 = 2$.
a) Find the piecewise linear interpolation function.

b) Find the quadratic interpolation function.

2. Find $s_2(x)$ so $s(x)$ is a natural cubic spline for $-1 \leq x \leq 1$.

$$s(x) = \begin{cases} -2 + x + 6x^2 + 2x^3 & \text{if} \quad -1 \leq x \leq 0, \\ s_2(x) & \text{if} \quad 0 \leq x \leq 1. \end{cases}$$

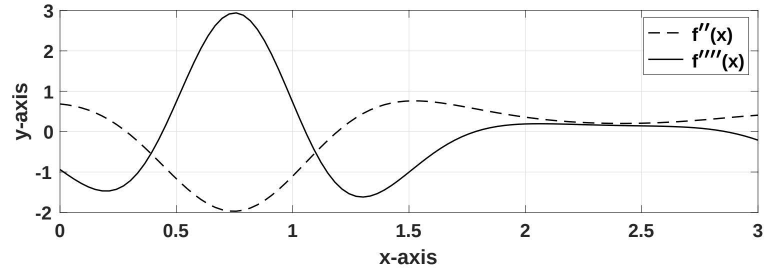
3. Suppose that $x_1 = -1$, x_2=0, $x_3 = 1$, and

$$s(x) = B_1(x) - B_3(x) + B_4(x), \quad \text{for } -1 \leq x \leq 1.$$

a) What data points (x_i, y_i) were used to produce this cubic spline?

b) Is this a natural cubic spline?

4. The second, $f''(x)$, and fourth, $f''''(x)$, derivative of a function $f(x)$ are plotted in the figure below for $0 \leq x \leq 3$.



a) For piecewise linear interpolation of $f(x)$, how many data points are needed to guarantee that the error is less than 10^{-8}?

b) To compute $\int_0^3 f(x)dx$ using the composite Simpson's rule, how small does the step size h have to be to guarantee that the error is less than 10^{-9}?

5. What is the principal reason that Lagrange interpolation should be used instead of the direct approach for polynomial interpolation?

6. The midpoint rule states that $\displaystyle\int_{x_i}^{x_{i+1}} f(x)dx \approx hf(c_i)$, where $c_i = x_i + \frac{1}{2}h$. Show that the error is $O(h^3)$. Do not use the theorem we derived for the error for the midpoint rule (you are being asked to derive that result in this problem).

7. The data for a function $f(x)$ are given below. You are to use as many of the data points as possible to evaluate $\int_0^6 f(x)dx$ using the given method.

x	0	1	2	4	6
$f(x)$	-1	1	0	-2	1

a) Use the trapezoidal rule to evaluate the integral.

b) Use Simpson's rule to evaluate the integral.

8. Suppose an integration rule has the form

$$\int_a^b f(x)dx \approx w_1 f\big(a + \frac{1}{4}\ell\big) + w_2 f\big(b\big),$$

where $\ell = b - a$. Find the values of w_1 and w_2 that maximize the precision.

Extra Credit: With the derived values for w_1 and w_2 in Problem 8, it is found that

$$\int_a^b f(x)dx = w_1 f\big(a + \frac{1}{4}\ell\big) + w_2 f\big(b\big) + K\ell^3 f''(\eta),$$

for some η with $a < \eta < b$. Find K.

Worksheet