11
Filters: Analysis

11.1 INTRODUCTION

In the previous chapter, we experimentally determined filter properties in passive circuits with a capacitor (C) and a resistor (R) — RC circuits. We observed that we could distinguish a pass band, a transition band, and a stop band in the frequency response of such a filter (e.g., Fig. 10.1). These frequency characteristics can be used to define four basic types of filters: low-pass, high-pass, band-pass, and band-reject. In this chapter, we analyze the same RC filter with time domain and frequency domain techniques we introduced in previous chapters.

The gradually changing, frequency-dependent output observed from the RC circuit (in the previous chapter) demonstrates that this analog RC-filter response is far from that of an ideal filter, which would completely remove undesirable frequency components while leaving the components of interest unaltered (Fig. 11.1). Because ***analog*** filters are electronic circuits obeying the laws of physics, they behave in a ***causal*** fashion (i.e., the output cannot be determined by the input in the future but must be determined by present or past input). Unfortunately, this makes it impossible to construct an analog filter with ideal characteristics because the inverse transform of the ideal profile (***a finite block of frequencies***, such as the one depicted in blue in Fig. 11.1) creates an impulse response (≡ the inverse transform of the frequency response) that violates causality because the response to an impulse at $t = 0$ includes values $\neq 0$ for $t < 0$ (Appendix 11.1).

Filter types that do not behave as causal linear time invariant (LTI) systems do exist, but we will not consider these (more unusual) filter types here.

11.2 THE RC CIRCUIT

Figure 11.2 shows a diagram and the associated ordinary differential equation (ODE) for the simple low-pass RC filter we explored in Chapter

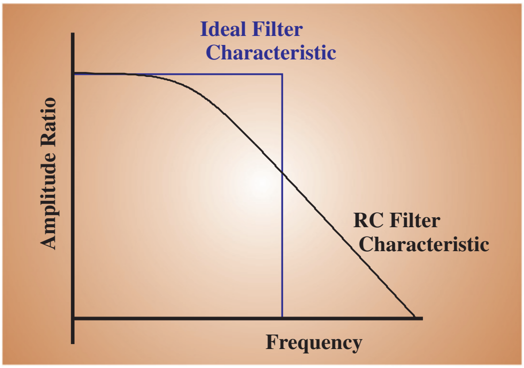

Figure 11.1 Low-pass filter frequency characteristic. An ideal characteristic (blue) would completely remove high frequencies while passing lowfrequency components unaltered. In real filters, such as the RC circuit (black), this ideal characteristic is compromised.

10. An overview of the time and frequency domain properties associated with passive electronic components can be found in Appendix 11.2.

We can analyze this filter in several ways; all approaches generate an equivalent end result.

11.2.1 Continuous Time

In ***continuous time*** analysis, we can solve the ODE (Fig. 11.2) in several ways:

1. *Directly in the time domain.* Denoting y as the output and x as the input (Fig. 11.2), we can describe the RC circuit with the differential equation:

$$x = RC\frac{dy}{dt} + y \tag{11.1}$$

Setting the forcing term x to 0 to find the unforced solution, we get

$$\frac{dy}{dt} = -\frac{1}{RC}y \rightarrow \frac{dy}{y} = -\frac{dt}{RC} \rightarrow \ln(y) = -\frac{t}{RC} \rightarrow$$
$$y = e^{\frac{1}{RC}} \tag{11.2}$$

This solution is not the only one; any solution in the form $y = Ae^{-t/RC} + B$, with A and B as constants, will work. In this case, one usually solves for A and B by using the output values at $t = 0$ and large t ($t \rightarrow \infty$). For $t \rightarrow \infty$, the first term $Ae^{-t/RC} \rightarrow 0$, hence B is the output at $t \rightarrow \infty$. For $t = 0$, the output is $A + B$. In most cases, the output at ∞ is zero and the solution becomes $y = y_0e^{-t/RC}$ with y_0 as output at $t = 0$.

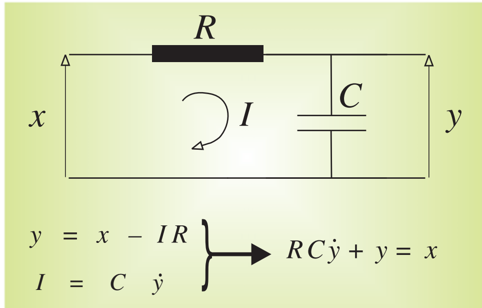

Figure 11.2 RC low-pass filter diagram and the associated ODE. Current (I) passes through the resistor (R) and capacitor (C). High-frequency components of input x are attenuated in output y.

Given an input x, a ***particular solution*** may be added to the ***unforced solution*** in order to obtain the ***general solution***. Often the choice for evaluating a particular solution depends on the forcing term (i.e., the input) x. For instance, if the input x is a sine wave with frequency f, we may find a particular solution of the form $A\sin(2\pi ft) + B\cos(2\pi ft)$; if x is an exponential function (e.g., $3e^{-2t}$) the particular solution of the same form (Ae^{-2t}) is expected.

2. *Directly in the frequency domain.* Using the formula for the impedance Z for a capacitor C as $Z = \dfrac{1}{j\omega C}$ together with Ohm's law we get

$$x = i\left[R + \frac{1}{j\omega C}\right] \quad \text{and} \quad y = i\frac{1}{j\omega C} \rightarrow x = j\omega Cy\left[R + \frac{1}{j\omega C}\right] \tag{11.3}$$

This results in an input-output relationship in the frequency domain — that is, the frequency response:

$$\frac{y}{x} = \frac{1}{1 + j\omega RC} \tag{11.4}$$

3. *Indirectly by using the Laplace or Fourier transform.* Using the unit impulse as the input to our ODE/filter (i.e., $x = \delta$), we get the following transforms:

$\delta \Leftrightarrow 1$	for both the Laplace and Fourier transforms
$y \Leftrightarrow Y(s) \quad \text{or} \quad Y(j\omega)$	for the Laplace and Fourier transforms, respectively
$\dfrac{dy}{dt} \Leftrightarrow sY(s) \quad \text{or} \quad j\omega Y(j\omega)$	for the Laplace and Fourier transforms, respectively

The Laplace-transformed ODE is therefore

$$1 = RCsY(s) + Y(s) \rightarrow Y(s) = H(s) = \frac{1}{1 + RCs} \tag{11.5}$$

In this case, Y(s) is the transfer function H(s) because the input is the unit impulse δ. Using the Fourier transform instead of the Laplace transform, we can determine the filter's frequency response:

$$Y(j\omega) = H(j\omega) = \frac{1}{1 + RCj\omega} \tag{11.6}$$

Using a table for Laplace transform pairs (Appendix 9.1), we can find the inverse transform for the transfer function (in the Laplace domain), generating the filter's impulse response function $h(t)$:

$$h(t) = (1/RC)e^{-t/RC} \quad \text{for} \quad t \geq 0 \tag{11.7}$$

The inverse of the Fourier transform in Equation (11.6) generates the same result. Note that we obtain an exponential function for $t \geq 0$ only where all output for $t < 0$ is supposed to be zero; this results in a single-sided Fourier transform pair that is equivalent to the single-sided Laplace transform used earlier.

Since we are dealing with a linear system and we know the RC-circuit's transfer function (Equation (11.5)), we can in principle determine the filter's output $y(t)$ to an arbitrary input function $x(t)$. In the time domain, this can be done using convolution:

$$y(t) = h(t) \otimes x(t) = x(t) \otimes h(t)$$

In the s-domain we can obtain the Laplace transform $Y(s)$ of time domain output $y(t)$ by multiplication of the transfer function (Equation (11.5)) with the Laplace transform of the input. For instance, if we want to determine the output caused by a step $U(t)$ at the input, we have the following transform pairs:

$$x(t) = U(t) \Leftrightarrow \frac{1}{s} \quad \text{(see Appendix 9.1)}$$

$$h(t) \Leftrightarrow \frac{1}{1 + RCs} \quad \text{(the transfer function)}$$

$$y(t) = h(t) \otimes x(t) \Leftrightarrow Y(s) = H(s)X(s) = \frac{1}{1 + RCs}\frac{1}{s} = \frac{1}{RC}\left[\frac{1}{s + 1/RC}\frac{1}{s}\right]$$

Using partial fraction expansion (see Appendix 9.3) and the table for Laplace transform pairs (Appendix 9.1), we find that the solution in the time domain is

$$y(t) = 1 - e^{-t/RC} \quad \text{for} \quad t \geq 0. \tag{11.8}$$

Here we determined the time domain response by finding the inverse transform of the solution in the s-domain. A graphical representation of the convolution procedure applied to the unit step and the exponential impulse response in the time domain is shown in Figure A8.1-1. Not surprisingly, the outcomes of the direct convolution procedure and the Laplace transform method are the same.

11.2.2 Discrete Time

In **discrete time**, the ODE for the RC circuit can be approximated with a difference equation. One technique to obtain an equivalent difference equation is by using a numerical approximation (such as the Euler technique) for the differential Equation (11.1). Alternatively, if the sample interval is very small relative to the time constant (RC), one can approximate Equation (11.1) in discrete time by (Appendix 11.3):

$$x(n) = RC\frac{y(n) - y(n-1)}{\Delta t} + y(n)$$

Simplifying notation by substituting $A = \frac{RC}{\Delta t}$ we obtain

$$x(n) = y(n)[A+1] - Ay(n-1) \rightarrow y(n) = \frac{x(n) + Ay(n-1)}{A+1} \tag{11.9}$$

The difference equation can be solved in the following ways:

1. *Numerically by direct calculation.* A difference equation such as Equation (11.9) where $y(n)$ is expressed as a function of a given input time series can easily be implemented in a MATLAB script. Graphically, a block diagram can be used as the basis for such an implementation (e.g., Fig. 11.3).

   To mimic our experimentally obtained data in Chapter 10, type in the following parameters for the filter in the MATLAB command window:

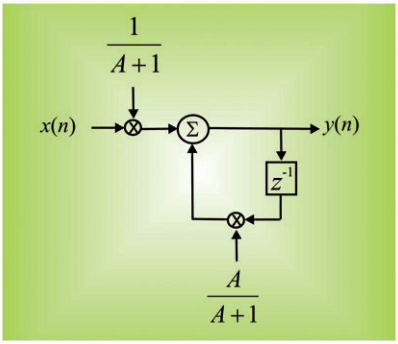

Figure 11.3 Discrete version of a continuous time analog low-pass RC filter. This block diagram depicts the algorithm for the difference Equation (11.9). The $y(n\text{-}1)$ term in Equation (11.9) is indicated by the delay operator z^{-1}.

```
sr=400;
dt=1/sr;
R=10^4;
C=3.3e-6;
tau=R*C;
A=tau/dt;
t=0:dt:1;
x=ones(length(t),1);
y(1)=0;
```

Now, type in the following line representing the recursive algorithm of Equation (11.9):

```
for n=2:length(t); y(n)=(A/(A+1))*y(n-1)+x(n)/(A+1);end;
```

You can study the outcome by plotting the results for the output and for the input in the same figure:

```
figure; hold;
plot(t,y,'r')
plot(t,x,'k')
axis([-0.1 1 0 1.1])
```

If you want, you can add axis labels and a title to the graph:

```
xlabel ('Time (s)');
ylabel ('Amplitude (V)')
title('Low pass filter response (red) to unit step input (black)');
```

The result of the plot is identical to the sketch of the filter response (shown in red in this example) to a unit step function (shown in black in this example). You can compare your finding with the example in Figure 10.5.

2. *Indirectly by using the z-transform.* The difference Equation (11.9) can be transformed into the z-domain:

$$x(n) \Leftrightarrow X(z)$$
$$y(n) \Leftrightarrow Y(z)$$
$$y(n-1) \Leftrightarrow z^{-1}Y(z)$$

Substituted in the difference equation we get

$$X(z) = (A+1)Y(z) - Az^{-1}Y(z)$$

As in the s- or Fourier domains, the transfer function $H(z)$ is a ratio of the output to the input:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(A+1) - Az^{-1}} = \frac{1}{A+1}\frac{1}{1 - \dfrac{A}{A+1}z^{-1}} \tag{11.10}$$

Using the table for z-transform pairs in Appendix 9.1 we can determine that the inverse transform is

$$h(n) = \frac{1}{A+1}\left[\frac{A}{A+1}\right]^n = \frac{A^n}{(A+1)^{n+1}} \tag{11.11}$$

This result can be used directly to simulate the impulse response for the discrete version of the low-pass filter.

11.3 THE EXPERIMENTAL DATA

The experiment described in Chapter 10 resulted in the measured response of the filter to step and sine wave inputs. In the analysis in this chapter, we found that the unit step response of the filter can be represented by Equation (11.8) and can be numerically calculated by using the MATLAB commands described in Section 11.2, part 1. If you create the graphs using these commands, the fit between the theoretical and the measured step response in Figure 10.5 (top-right plot) will be obvious.

The plot of the output/input amplitude ratio versus frequency in Figure 10.5 (bottom-right plot) is the filter frequency response characteristic, which corresponds to Equation (11.4) or (11.6). In these equations, the output/input relationship is a complex-valued function (including a real and imaginary part) of frequency and the details of how to relate this complex function to measured data will be further discussed in the following chapter; for now it is obvious that Equations (11.4) and (11.6) both represent a function that decreases with frequency, which is consistent with a low-pass characteristic.

The conclusion that the frequency response of the filter is complex is directly related to the presence of the capacitance, which necessarily implies an imaginary impedance (Fig. A11.2). In circuits where only resistors are involved, the impedance is real (i.e., equal to R, Fig. A11.2) and consequently the frequency response is also real. A real-valued frequency response (i.e., the imaginary component is zero; see also Chapter 12, Fig. 12.3) indicates that there is no change of phase (i.e., $\phi = 0$ in Fig. 12.3) between a sine wave at the output relative to the input. This principle can also be extended to a wider context — for instance, in modeling experiments in slices of brain tissue — where the extracellular medium can be considered mainly resistive (capacitance can be neglected); in such a medium, the frequency response is real and no phase changes occur. On the other hand, in cases with transition layers between media such as membranes, membrane capacitance plays a critical role and phase changes in membrane current across such barriers may be significant.

APPENDIX 11.1

An ideal filter characteristic passes a finite block of frequencies unaltered (let's say, up to a certain frequency ω_c) while completely removing frequencies outside the pass band from the signal (blue, Fig. 11.1). Since the filter characteristic $H(j\omega)$ is an even function, it is typically only shown for $\omega > 0$.

If one calculates the inverse Fourier transform of the product of the filter characteristic $H(j\omega)$ (which is already in the frequency domain) and the Fourier transform of the unit impulse function $\delta(t)$ (i.e., 1; see Equation (6.9)), one obtains the unit impulse response $h(t)$:

To summarize,

$$1 \Leftrightarrow \delta(t)$$
$$H(j\omega)\begin{cases} 1 & if \quad |\omega| \le \omega_c \\ 0 & if \quad |\omega| > \omega_c \end{cases} \tag{A11.1-1}$$

Because $H(j\omega)$ is 0 outside the $\pm\ \omega_c$ range, we may change the integration limits in the inverse Fourier transform:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{j\omega t}d\omega = \frac{1}{2\pi jt}\left[e^{j\omega t}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jt}\left[e^{j\omega_c t} - e^{-j\omega_c t}\right] \tag{A11.1-2}$$

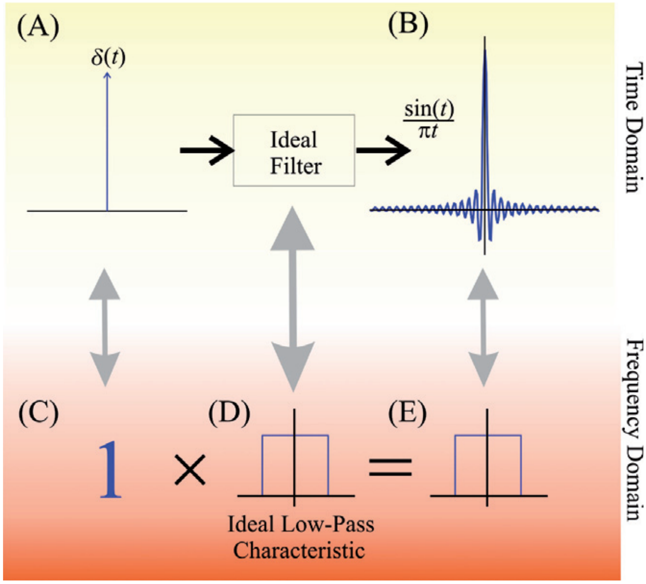

Figure A11.1 The ideal low-pass filter would completely remove high-frequency components and leave the low-frequency components unaltered. In the frequency domain, this would correspond to a rectangular frequency response (D); *note that here the negative frequencies are also depicted*. In the frequency domain, the output (E) is the product of input (C) and the frequency response (D). The time domain response of this filter (B) to a unit impulse (A) precludes the existence of such an ideal device because a nonzero component is present in the response at $t < 0$ (i.e., there is a response before the input is given at $t = 0$, therefore the filter cannot exist because it violates causality).

Using Euler's relation, this evaluates to the so-called sinc function:

$$h(t) = \frac{\sin(\omega_c t)}{\pi t} \tag{A11.1-3}$$

Figure A11.1 shows that $h(t)$ exists for $t < 0$ whereas the input $\delta(t)$ occurs only at $t = 0$; this indicates that such an ideal filter is a ***noncausal*** system. In the analog world where systems must behave causally, such a filter cannot be made, but only approximated.

In the digital world, other problems are associated with implementing an ideal filter. In Figure A11.1, it can be seen that there are oscillations in the impulse response $h(t)$ from $-\infty$ to ∞, and its frequency domain equivalent has an infinitely steep slope. In the first place, neither of these properties can be represented in a real digital system with finite memory. Further, when an ideal filter is convolved with transients at the input (e.g., a square wave), this causes a ripple in its output. An example of a square wave approximated with a finite number (five) of sine waves (i.e., a truncated spectrum) was shown in Chapter 5, Figure 5.2; in this example, a ripple effect in the square wave approximation is clearly visible. This example mimics the effect of a simple truncation of the higher frequency components of a square wave just as an ideal low-pass filter would do. Interestingly, while the ripple frequency increases with an increased number of component sine waves in the approximation (strangely), the individual oscillations in the ripple have fixed amplitude ratios (first described by Gibbs in the 19th century). For an ideal filter with a square wave input (with zero-mean and an equal duty cycle), the first oscillation is an overshoot (see also Fig. 12.1, Chapter 12) with an amplitude that is always 18% of the expected step amplitude. The MATLAB script pr11_1.m (included on the CD) simulates the effect of truncating the spectral content of a square wave.

APPENDIX 11.2

The resistor, inductor, and capacitor are the passive components used in electronic circuits for filtering. The symbols used for them in circuit diagrams and their properties in the time and frequency domains are summarized in Figure A11.2.

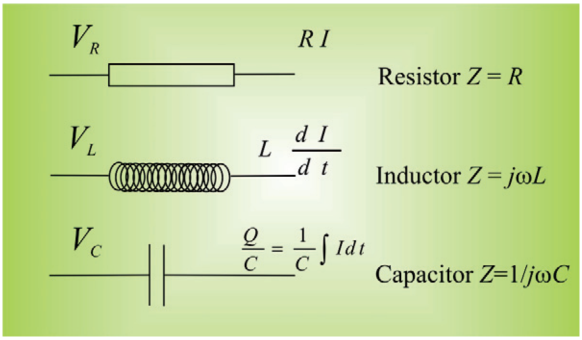

Figure A11.2 Electronic components, their relationships between current and potential in the time domain, and their representations of impedance in the frequency domain.

APPENDIX 11.3

The solutions to differential equations can be approximated with the Euler method. This algorithm integrates such equations with an iterative approach. If $\dot{y} = f(y)$, and one wants to estimate a point y_n from the previous value y_{n-1} with an interval distance of Δt, one can use a linear approximation of the function at hand and estimate the difference between y_n and y_{n-1} by the derivative at y_{n-1} multiplied by the distance:

$$y_n = y_{n-1} + \dot{y}_{n-1}dt = y_{n-1} + f(y_{n-1})\Delta t \quad \text{(A11.3-1)}$$

Equation (A11.3-1) is a difference equation that approximates the differential equation $\dot{y} = f(y)$.

Alternatively, one can use knowledge about the solution of the differential equation to describe a difference equation. For instance, a difference equation that is equivalent to Equation (11.1) is

$$y_n = e^{-\Delta t/RC} y_{n-1} + (1 - e^{-\Delta t/RC})x_n \quad \text{(A11.3-2)}$$

Here we use the solution for Equation (11.1) to relate the output at n with previous output and input. Using the unforced solution of Equation (11.1) $Ae^{-t/RC} + B$, we can solve for A and B. We assume zero output for $t \to \infty$, we set the initial value to y_{n-1}, and we set the time difference between y_n and y_{n-1} to Δt; this results in $y_n = y_{n-1}e^{-\Delta t/RC}$, which is the first term in Equation (A11.3-2). This term indicates that, for subsequent values of n, the output signal decays following an exponential with a time constant $\Delta t/RC$. If there is no input x, the second term in Equation (A11.3-2) is 0, and this is the whole story. However, in the presence of input we must add a particular solution to obtain the general one. Let's assume that there is a constant input with amplitude x_n, and we know from our experiments that the low-pass filter will respond with a constant output (there will be no decay). This behavior leads to the second term in Equation (A11.3-2) in which the correction factor $(1 - e^{-\Delta t/RC})$ for x_n is required to compensate for the leakage factor $e^{-\Delta t/RC}$ in the first term, thus maintaining the output y constant for constant input x.

Now we can show that (A11.3-2) can be approximated by Equation (11.9) when $\Delta t \ll RC$ (i.e., for small values of the exponent). Here we repeat Equation (11.1) and the approximation used in section 11.2.2:

$$x = RC\frac{dy}{dt} + y \quad \text{and approximation } x(n) = RC\frac{y(n) - y(n-1)}{\Delta t} + y(n) \quad \text{(A11.3-3)}$$

We can rewrite the approximation as

$$RCy(n) + \Delta t y(n) = RCy(n-1) + \Delta t x(n)$$

$$\rightarrow y(n) = \frac{RC}{RC + \Delta t} y(n-1) + \frac{\Delta t}{RC + \Delta t} x(n) \qquad \text{(A11.3-4)}$$

The correction factor for $y(n-1)$ can be written as

$$\frac{1}{1 + \dfrac{\Delta t}{RC}} \approx \frac{1}{e^{\Delta t/RC}} = e^{-\Delta t/RC} \quad \textit{if} \quad \Delta t \ll RC \qquad \text{(A11.3-5)}$$

Here we used the power expansion of the exponential $\left[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$, where we set all higher-order terms to zero (i.e., $e^x = 1 + x$, with $x = \Delta t/RC$). Usually this is a reasonable thing to do since we sample relatively frequently so that Δt is small relative to the time constant. Similarly, we can write the factor for $x(n)$ in Equation (A11.3-4) as

$$1 - \frac{RC}{RC + \Delta t} = 1 - \frac{1}{1 + \dfrac{\Delta t}{RC}} \approx 1 - \frac{1}{e^{\Delta t/RC}} = 1 - e^{-\Delta t/RC} \quad \textit{if} \quad \Delta t \ll RC \qquad \text{(A11.3-6)}$$

Combining Equations (A11.3-4) through (A11.3-6), we get the expression in Equation (A11.3-2) again. It should be noted that the approximations in the Euler approach and the approximation in Equation (11.9) are only suitable for smaller time intervals because the error in each step will be compounded in the following steps. When $\Delta t \ll RC$ is not a valid assumption, or when higher precision is required, either the approach in (A11.3-2) or a more accurate integration algorithm (such as a higher order Runge-Kutta algorithm) is preferable.