The Frequency Domain 3

3.1 THE DISCRETE-TIME FOURIER TRANSFORM

The concept of the frequency domain is as important for understanding discrete-time signals and systems as it is for understanding the continuous-time case. A careful examination of the spectrum of a signal provides important clues about how that signal should be analyzed. The frequency response of a system tells us how the system will respond to unknown, and possibly difficult to characterize, inputs. This chapter focuses on the discrete-time Fourier transform (DTFT) and the frequency response of discrete-time systems. These are the most important of the frequency-domain analysis tools. It also discusses a number of properties of the DTFT. The chapter concludes with a brief discussion of filtering. A more complete discussion of the latter topic can be found in the companion text *Digital Filters: A Computer Laboratory Text*.

The discrete-time Fourier transform, which serves as the Fourier representation of a discrete-time signal, is defined by the summation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{3.1}$$

It is a periodic function in the real variable ω with a period of 2π. The DTFT is equivalent to the representation of a signal as a sequence of samples, since the samples can be computed from the DTFT and vice versa. The integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega \tag{3.2}$$

is used to evaluate the inverse DTFT. Note that the range of integration is limited to one period of $X(e^{j\omega})$. These two formulas resemble those that define the Fourier series representation of a periodic function. In fact, the sample values $x[n]$ can be identified with the Fourier series coefficients of $X(e^{j\omega})$. Table 3.1 contains an abbreviated list of discrete-time Fourier transform pairs.

Table 3.1. A Short List of DTFT Pairs[a]

$x[n]$	$\Longleftrightarrow$	$X(e^{j\omega})$
1	$\Longleftrightarrow$	$2\pi\delta((\omega))_{2\pi}$
$e^{j\omega_0 n}$	$\Longleftrightarrow$	$2\pi\delta((\omega-\omega_0))_{2\pi}$
$\cos\omega_0 n$	$\Longleftrightarrow$	$\pi\delta((\omega-\omega_0))_{2\pi}+\pi\delta((\omega+\omega_0))_{2\pi}$
$\sin\omega_0 n$	$\Longleftrightarrow$	$\frac{\pi}{j}\delta((\omega-\omega_0))_{2\pi}-\frac{\pi}{j}\delta((\omega+\omega_0))_{2\pi}$
$\sum_{k=-\infty}^{\infty}\delta[n-kN]$	$\Longleftrightarrow$	$\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\left(\omega-\frac{2\pi k}{N}\right)\right)_{2\pi}$
$a^n u[n],\ \lvert a\rvert<1$	$\Longleftrightarrow$	$\frac{1}{1-ae^{-j\omega}}$
$x[n]=\begin{cases}1, & \lvert n\rvert\le N\\ 0, & \lvert n\rvert>N\end{cases}$	$\Longleftrightarrow$	$\frac{\sin[\omega(N+\frac{1}{2})]}{\sin[\frac{\omega}{2}]}$
$\frac{\sin\alpha n}{\pi n}\quad 0<\alpha<\pi$	$\Longleftrightarrow$	$X(\omega)=\begin{cases}1, & 0<\lvert\omega\rvert<\alpha\\ 0, & \alpha<\lvert\omega\rvert\le\pi\end{cases}$
$\delta[n]$	$\Longleftrightarrow$	1
$u[n]$	$\Longleftrightarrow$	$\frac{1}{1-e^{-j\omega}}+\pi\delta((\omega))_{2\pi}$
$\delta[n-n_0]$	$\Longleftrightarrow$	$e^{-j\omega n_0}$
$(n+1)a^n u[n],\quad \lvert a\rvert<1$	$\Longleftrightarrow$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+k-1)!}{n!(k-1)!}a^n u[n],\quad \lvert a\rvert<1$	$\Longleftrightarrow$	$\frac{1}{(1-ae^{-j\omega})^k}$

[a]The DTFT is always periodic in ω with period 2π. Thus impulses in the frequency domain are actually impulse trains with a period of 2π. This is indicated by the notation $((\cdot))_{2\pi}$. $\delta(\omega)$ is a Dirac delta function.

The Fourier transform of a continuous-time signal is defined by the integral

$$X_a(\Omega)=\int_{-\infty}^{\infty}x_a(t)e^{-j\Omega t}dt. \tag{3.3}$$

A number of similarities and differences between equations (3.1) and (3.3) become evident when they are compared. For example, both are complex functions, and thus are displayed in terms of real and imaginary parts or in terms of magnitude and phase. Unlike the Fourier transform, however, the DTFT is periodic in frequency with period 2π. When $x[n]$ is of finite length, the DTFT can be evaluated numerically for selected frequencies, but the Fourier transform cannot, in general. The units of the frequency variables are also different: ω has units of radians while Ω is measured in radians per second.

The DTFT is used frequently because many signals, systems, and signal processing operations can be described more conveniently in the frequency domain than in the time domain. A number of properties of the DTFT are listed in Table 3.2; these can be very helpful in making difficult problems easy to solve.

Table 3.2. Properties of the Discrete-Time Fourier Transform[a]

$x[n]$	$\Longleftrightarrow$	$X(e^{j\omega})$
$x[n-n_0]$	$\Longleftrightarrow$	$e^{-j\omega n_0}X(e^{j\omega})$
$e^{j\omega_0 n}x[n]$	$\Longleftrightarrow$	$X(e^{j(\omega-\omega_0)})$
$x^*[n]$	$\Longleftrightarrow$	$X^*(e^{-j\omega})$
$x[-n]$	$\Longleftrightarrow$	$X(e^{-j\omega})$
$x[n]*y[n]$	$\Longleftrightarrow$	$X(e^{j\omega})Y(e^{j\omega})$
$x[n]y[n]$	$\Longleftrightarrow$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\,d\theta$
$x[n]-x[n-1]$	$\Longleftrightarrow$	$(1-e^{-j\omega})X(e^{j\omega})$
$\sum_{k=-\infty}^{n} x[k]$	$\Longleftrightarrow$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\delta((\omega))_{2\pi}$
$nx[n]$	$\Longleftrightarrow$	$j\frac{dX(e^{j\omega})}{d\omega}$
$\Re e\{x_e[n]\}$	$\Longleftrightarrow$	$ev\{\Re e(X(e^{j\omega}))\}$
$j\,\Im m\{x_e[n]\}$	$\Longleftrightarrow$	$jev\{\Im m(X(e^{j\omega}))\}$
$\Re e\{x_O[n]\}$	$\Longleftrightarrow$	$jod\{\Im m(X(e^{j\omega}))\}$
$j\,\Im m\{x_O[n]\}$	$\Longleftrightarrow$	$od\{\Re e(X(e^{j\omega}))\}$

[a]The symbol $\delta((\omega))_{2\pi}$ is the Dirac delta function. The notation $((\cdot))_{2\pi}$ reflects the fact that the DTFT is always periodic in ω with period 2π.

Two additional properties relate measurements in the time domain to measurements in the frequency domain. *Parseval's relation* relates signal energy in the time domain to measurements made from the DTFT.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega.$$

The *initial value relations*

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$$

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

are frequently useful for checking computations.

The exercises in this section examine the frequency domain and the ways in which the discrete-time Fourier transform may be used as an analysis tool.

– *EXERCISE 3.1.1.* **Response of a System to a Complex Exponential**

An LTI system is completely characterized by its impulse response. Here the output of an LTI system is examined when the input is a complex exponential. Consider the system with impulse response $h[n]$ given by

$$h[n] = 0.03\delta[n] + 0.4\delta[n-1] + 0.54\delta[n-2] + 0.2\delta[n-3] - 0.2\delta[n-4].$$

Create a signal file containing this impulse response using the *create file* option in x siggen. Generate a set of six 128-point complex exponentials of the form $e^{j\omega_k n}$ with the following discrete-time frequencies: $\omega_1 = \pi/6$, $\omega_2 = \pi/3$, $\omega_3 = \pi/2$, $\omega_4 = 2\pi/3$, $\omega_5 = 5\pi/6$, and $\omega_6 = 8\pi/9$. This may be done easily by using the *exponential* $Ke^{j\alpha n}$ option in x siggen. The parameters for α should be complex in this case, e.g., 0 $\pi/2$, 0 $\pi/3$, etc. Display the real and imaginary parts of the complex exponential and observe that they are sinusoids with unity amplitude. Convolve each complex exponential with the system impulse response, $h[n]$, to form $y_1[n], y_2[n], \ldots, y_6[n]$.

(a) Examine and record the magnitude of each of the output sequences in the steady state region, i.e., $4 \le n \le 128$. The function x view2 can be used to examine the magnitudes of these outputs one above the other. You will observe that the outputs are essentially constant. The only differences are that the magnitudes of the output sequences may differ from those of the inputs and that the output sequences may have been shifted slightly in time.

(b) Construct a graph with amplitude along the vertical axis and frequency in the range $(0 \le \omega \le \pi)$ along the horizontal axis. Plot the peak magnitude values for $y_1[n], y_2[n], \ldots, y_6[n]$ on the graph—one point for each of the six frequencies.

Connect the points to form a continuous plot. Your result should approximate the discrete-time Fourier transform magnitude, $|H(e^{j\omega})|$..

(c) The discrete-time Fourier transform, or DTFT, is a frequency-domain representation that is most often displayed in the frequency range $-\pi \leq \omega < \pi$. Plots of the DTFT are continuous curves. The computer is actually displaying discrete sample values of the DTFT and connecting these points with straight lines. Use the `x dtft` function to display the DTFT magnitude, $|H(e^{j\omega})|$. Sketch the transform magnitude in the range $0 \leq \omega \leq \pi$. How does it compare to your plot in part (b)?

The discrete-time Fourier transform of a signal or system can be interpreted as its response to an infinite set of complex sinusoids that collectively span the frequency range.

—*EXERCISE 3.1.2.* **Symmetry in the Fourier Transform**

In Chapter 2 the concept of even and odd symmetry was introduced. It was shown that any real sequence, $x[n]$, could be expressed as a sum of its even and odd parts where the even part was defined as

$$ev\{x[n]\} = (x[n] + x[-n])/2$$

and the odd part of $x[n]$ was defined as

$$od\{x[n]\} = (x[n] - x[-n])/2.$$

This concept was also extended so that any arbitrary complex sequence, $\tilde{x}[n]$, could be expressed in terms of the even and odd parts of its real and imaginary components, i.e.,

$$\tilde{x}[n] = \underbrace{a_e[n]}_{\text{real-even}} + \underbrace{a_o[n]}_{\text{real-odd}} + j\underbrace{b_e[n]}_{\text{imag-even}} + j\underbrace{b_o[n]}_{\text{imag-odd}}.$$

The DTFT can be similarly expressed in terms of these components,

$$\tilde{X}(e^{j\omega}) = \underbrace{C_e(e^{j\omega})}_{\text{real-even}} + \underbrace{C_o(e^{j\omega})}_{\text{real-odd}} + j\underbrace{D_e(e^{j\omega})}_{\text{imag-even}} + j\underbrace{D_o(e^{j\omega})}_{\text{imag-odd}}.$$

In this exercise the even and odd symmetry properties with respect to the discrete-time Fourier transform are investigated.

(a) Generate an arbitrary 5-point complex sequence, $\tilde{x}[n]$. Do this by using `x rgen` to create two random 5-point real sequences. Use `x gain` with the *gain* parameter equal to "0 1" (which represents the complex number j) to transform one of the real sequences into a purely imaginary signal, then add the sequences together using `x add`. Sketch the real and imaginary parts of $\tilde{x}[n]$.

(b) Using the computer, decompose $\tilde{x}[n]$ into its four components, $a_e[n]$, $a_o[n]$, $b_e[n]$, $b_o[n]$, and provide a sketch for each. The functions **x realpart** and **x imagpart** can be used to extract the real and imaginary parts, respectively. The even and odd components can be obtained by using the **x reverse**, **x add**, **x subtract**, and **x gain** functions. Next compute the DTFT of $\tilde{x}[n]$. Decompose $\tilde{X}(e^{j\omega})$ into its components $C_e(e^{j\omega})$, $C_o(e^{j\omega})$, $D_e(e^{j\omega})$, and $D_o(e^{j\omega})$ in the same way. Recall from Chapter 1 that the function **x dtft** generates a 513 point file "_dtft_" that contains samples of the transform. (Note that this file name begins and ends with underbars.) You may reverse, add, and process this file appropriately to generate $C_e(e^{j\omega})$, $C_o(e^{j\omega})$, $D_e(e^{j\omega})$, and $D_o(e^{j\omega})$.

(c) Now individually take the discrete-time Fourier transform of $a_e[n]$, $a_o[n]$, $jb_e[n]$, and $jb_o[n]$. Summarize your findings by matching the time-domain and frequency-domain components listed below.

	Time Domain	*Frequency Domain*	
real-even	$a_e[n]$	$C_e(e^{j\omega})$	real-even
real-odd	$a_o[n]$	$C_o(e^{j\omega})$	real-odd
imaginary-even	$jb_e[n]$	$jD_e(e^{j\omega})$	imaginary-even
imaginary-odd	$jb_o[n]$	$jD_o(e^{j\omega})$	imaginary-odd

These symmetry properties that you have observed apply for any arbitrary complex sequence.

– *EXERCISE 3.1.3.* **Proving the Symmetry Properties**

In Chapter 2 and in the previous exercise the concept of a decomposition into even and odd signal components was discussed. You saw that any arbitrary signal (real or complex) could be expressed in terms of real-even, real-odd, imaginary-even, and imaginary-odd components. In the previous exercise you were shown experimentally that each of these components transforms to one of four related frequency-domain components.

Consider the definitions for the even and odd components in the time domain and the frequency domain:

$$a_e[n] = \tfrac{1}{2}(a[n] + a[-n])$$

$$a_o[n] = \tfrac{1}{2}(a[n] - a[-n])$$

$$C_e(e^{j\omega}) = \tfrac{1}{2}(C(e^{j\omega}) + C(e^{-j\omega}))$$

$$C_o(e^{j\omega}) = \tfrac{1}{2}(C(e^{j\omega}) - C(e^{-j\omega}))$$

where $a[n]$ and $C(e^{j\omega})$ are assumed to be purely real signals. Using the definition of the DTFT, prove analytically that the relationships you found in the previous exercise are always true. (*Hint:* You may wish to use the fact that

$$\sum_{n=-\infty}^{\infty} f[n] = 0$$

if $f[n]$ is odd. You may also wish to determine whether each of the components is either even or odd. Then determine if each is real or imaginary.)

EXERCISE 3.1.4. **Hermitian Symmetry**

The two previous exercises established the relationships between the real, imaginary, even, and odd parts of signals in the time and frequency domains. A special case of this general result occurs when the time signal, $x[n]$, is purely real. In this case, the real part of the DTFT is an even function and the imaginary part is an odd function. This condition is known as *Hermitian symmetry*.

(a) Using the function `x dtft` sketch the real and imaginary parts of the DTFT of the signal $cc[n]$ contained in the file `cc`. Observe the symmetry in the real and imaginary parts. What type of symmetry is present in the magnitude and phase of the DTFT of this signal, i.e., which is an even function and which is odd?

(b) Now use `x dtft` to examine the DTFT of the *complex* signal, $dd[n]$ contained in the file `dd`. Sketch the real and imaginary parts of its DTFT. Also sketch the magnitude and phase.

(c) It is very common to display the DTFT only over the range from $0 \leq \omega \leq \pi$ due to Hermitian symmetry. Based on your observations in part (b), when would it be inappropriate to display the transform in this limited range?

EXERCISE 3.1.5. **Periodicity of the Discrete-Time Fourier Transform**

The discrete-time Fourier transform shares many characteristics with the continuous-time Fourier transform. A major difference, however, is that the DTFT is periodic in ω with period 2π.

To examine the issue of periodicity, consider the definition

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \cdots + x[-1]e^{j\omega} + x[0] + x[1]e^{-j\omega} + x[2]e^{-j2\omega} + \cdots . \end{aligned}$$

Carefully examine each term in the DTFT and observe that it repeats in frequency every 2π radians. This is because $e^{-j2\pi n} = 1$ for $n = 0, \pm 1, \pm 2, \ldots$.

Use `x dtft` to compute the DTFT of the signal $aa[n]$ contained in the file `aa`. By examining the `x dtft` display, estimate the numerical values for each of the following using the periodicity property:

(a) $|AA(e^{j\pi/4})|$.

(b) $|AA(e^{j3\pi/4})|$.

(c) $|AA(e^{j9\pi/4})|$.

(d) $|AA(e^{j17\pi/4})|$.

Note that `x dtft` only displays the DTFT in the range between $-\pi$ and π.

—*EXERCISE 3.1.6.* **The Shift Property**

The shift property of the DTFT says that shifting a signal in time by n_0 causes its DTFT to be multiplied or modulated by the complex exponential $e^{-j\omega n_0}$. In other words, if $x[n]$ has a DTFT $X(e^{j\omega})$, then $x[n-n_0]$ has DTFT $X(e^{j\omega})e^{-j\omega n_0}$.

(a) Display and sketch the 15-point signal $ee[n]$ contained in the file ee using x view. Using x dtft, display and sketch the imaginary part. Observe that since $ee[n]$ is an odd function, the DTFT is purely imaginary (i.e., its real part is zero), and consequently only one sketch is necessary for its complete representation. Practically speaking, the real part of the DTFT shown in the plot is only approximately zero due to limited numerical precision in the computation.

(b) Consider the signals

$$y_0[n] = ee[n-1]$$

and

$$y_1[n] = ee[n+1].$$

Using the x lshift and x dtft functions, sketch their frequency responses. You will observe that the resulting frequency response is no longer purely imaginary.

Euler's formula states that

$$e^{\pm j\omega n_0} = \cos\omega n_0 \pm j\sin\omega n_0.$$

Therefore, the shifted signals will have the form

$$Y_i(e^{j\omega}) = EE(e^{j\omega})\cos\omega n_0 \pm jEE(e^{j\omega})\sin\omega n_0. \tag{3.4}$$

Display and sketch the real and imaginary parts of $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$ and observe the frequency-domain effects of shifting in time.

(c) By changing the summation variable, n in the DTFT definition (i.e., letting $n \rightarrow n-n_0$), derive the shift property. (NCR)

— *EXERCISE 3.1.7.* **The Modulation Property**

The modulation property is the dual of the shift property. It states that multiplying a sequence, $x[n]$, by a complex exponential $e^{j\alpha n}$ shifts its DTFT in frequency by α. If $x[n]$ has the DTFT $X(e^{j\omega})$, then $x[n]e^{j\alpha n}$ has the DTFT $X(e^{j(\omega-\alpha)})$. Consider the signal $ee[n]$ contained in the file ee. Modulate this signal by $e^{j\pi n/4}$ using x cexp.

(a) Display and sketch $|EE(e^{j\omega})|$ the DTFT magnitude of $ee[n]$, and also the DTFT magnitude of $e^{j\pi n/4}ee[n]$.

(b) Analytically derive the modulation property by explicitly taking the DTFT of $x[n]e^{j\alpha n}$.

— *EXERCISE 3.1.8.* **The Convolution Property**

The convolution property is one of the most important and useful properties of the DTFT. It states that when two signals are convolved, their respective DTFTs are multiplied. In other words, the DTFT of $x[n] * h[n]$ is $X(e^{j\omega})H(e^{j\omega})$. Consider the sequences $ee[n]$ and $ff[n]$ contained in the files ee and ff.

(a) Display and sketch the magnitude responses of $EE(e^{j\omega})$ and $FF(e^{j\omega})$. Based on these plots, sketch the magnitude of $EE(e^{j\omega})\ FF(e^{j\omega})$.

(b) Using the x convolve function, compute

$$y[n] = ee[n] * ff[n].$$

Display and sketch the DTFT magnitude of $y[n]$.

(c) Consider deriving the convolution property. Recall that the convolution of two signals $x[n]$ and $h[n]$ is defined by the convolution sum

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m].$$

Apply the DTFT definition to the convolution sum. Observe that this double summation may be rewritten as

$$\sum_{m=-\infty}^{\infty} x[m] \underbrace{\sum_{n=-\infty}^{\infty} h[n-m]e^{-j\omega n}}_{\text{DTFT of } h[n-m]}$$

since $x[m]$ has no dependence on the variable n. Using the definition of the DTFT and the shift property, show that this expression reduces to $X(e^{j\omega})H(e^{j\omega})$.

— *EXERCISE 3.1.9.* **The Multiplication Property**

The multiplication and convolution properties are duals in the sense that multiplication in the time domain implies convolution in the frequency domain and convolution in the time domain implies multiplication in the frequency domain. The DTFT of a product of two sequences is the convolution of their individual DTFTs, i.e.,

$$\begin{aligned} \text{DTFT}\{x[n]h[n]\} &= \frac{1}{2\pi}X(e^{j\omega}) \circledast H(e^{j\omega}) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})\, d\theta \end{aligned} \tag{3.5}$$

where the circled star is used to denote a form of convolution called *circular* or *periodic* convolution.

(a) Taking the DTFT of $x[n]h[n]$ results in

$$\mathrm{DTFT}\{x[n]h[n]\} = \sum_{n=-\infty}^{\infty} x[n]h[n]e^{-j\omega n}.$$

By expressing $x[n]$ as the inverse DTFT of $X(e^{j\omega})$ and substituting, this equation becomes

$$\mathrm{DTFT}\{x[n]h[n]\} = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\theta n}\, d\theta}_{x[n]}\, h[n]e^{-j\omega n}.$$

By interchanging the order of the sum and integration operations, show analytically that the multiplication property is valid.

(b) Circular convolution is a topic that will be considered in more detail later in this text. For the present we will consider only a simple example. Display and sketch the DTFT magnitudes of the sequences $ff[n]$ and $gg[n]$ contained in files ff and gg. The circular convolution of $FF(e^{j\omega})$ and $GG(e^{j\omega})$ is very much like their linear convolution except that the two signals are periodic and the integration is confined to one period. Display and sketch the magnitude of

$$FF(e^{j\omega}) \circledast GG(e^{j\omega})$$

using the multiplication property defined in equation (3.5), i.e., use the functions `x multiply` and `x dtft`.

— *EXERCISE 3.1.10.* **The Initial Value Theorem**

The initial value theorem states that

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega,$$

which is proportional to the area under a graph of the DTFT. The analogous relationship with the domains reversed is

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n].$$

These properties can be very useful in solving certain problems.

(a) Write the definition of the inverse DTFT. Let n equal zero and show that the result reduces to the initial value theorem.

(b) Now use the definition of the forward DTFT to derive the other initial value relationship.

(c) Using the `x lshift` function, create the signal $vv[n] = gg[n+5]$, where $gg[n]$ is the signal contained in the file gg. Now use `x dtft` to display the real and imaginary parts. Sketch the real part of $VV(e^{j\omega})$ and estimate the area under the curve. Since x dtft produces an output file _dtft_ containing the 513 samples of the DTFT used in the graphics, the function x summer can be used to sum the frequency domain samples. To obtain the area under the curve, you must multiply this sum by $2\pi/513$. Now examine $vv[n]$ using x view and record the value of $vv[0]$. Verify that

$$vv[0] = \frac{1}{2\pi}(\text{area under } VV(e^{j\omega})).$$

— *EXERCISE 3.1.11.* **Some Fourier Transform Pairs**

A number of signals arise frequently in digital signal processing. Remembering their discrete-time Fourier transforms can be very useful. This exercise will add to the list of familiar discrete-time Fourier transform pairs.

(a) Compute and sketch the DTFT magnitudes of the following finite length sequences using x siggen, x multiply, and x dtft:

(i) $\sin(\frac{\pi n}{8})(u[n] - u[n-64])$.

(ii) $\cos(\frac{\pi n}{8})(u[n] - u[n-64])$.

(iii) $u[n] - u[n-32]$.

(iv) $\frac{\sin(\pi n/2)}{\pi n}(u[n+20] - u[n-21])$. You may use x fdesign with the rectangular window option to create $\sin[(\pi n/2)/\pi n]$.

(v) A 64-point square wave with period 8.

(vi) A 64-point impulse train with period 8.

Notice that the discrete-time Fourier transforms of these signals are slightly different from the Fourier transforms of the related continuous-time signals. Explain the differences.

(b) It can be shown analytically that the discrete-time Fourier transform of a sine wave is a pair of impulses; however, the results that you observed on the computer were not impulses. Explain this apparent contradiction.

3.2 FILTERS

The term "filter" describes a very large variety of different systems. However, it is commonly used to refer to LTI systems with frequency-selective frequency responses. These filters allow certain regions of the spectrum to remain undisturbed while other spectral regions are attenuated. Familiar examples include lowpass, highpass, and bandpass filters.

Filters are specified by their frequency-domain magnitude characteristics. Ideally, the frequency response of a filter should display only two types of behavior: it should have one or more spectral regions of constant nonzero amplitude (called the *passband(s)*) and one or more regions where the frequency response is zero (called the *stopband(s)*). These ideal filters are not realizable in practice. Realizable filters will also contain intermediate regions called *transition bands*. Because the frequency response must be continuous, a transition band must lie between each passband and stopband.

Realizable filters are designed to approximate ideal filters to within certain prescribed tolerances. These tolerances are defined by the widths of the transition regions, the maximum passband deviation, and the maximum stopband deviation. These parameters are illustrated in Fig. 3.1 for a lowpass filter. The effect is to define a region on the graph in which the magnitude response of the filter must lie. The ideal frequency response is shown by the solid black line. The following parameters define the match between the digital filter and the ideal:

δ_p passband ripple,
δ_s stopband ripple,
ω_ℓ lower cutoff frequency,
ω_u upper cutoff frequency,
ω_c nominal cutoff frequency.

For high-quality filters that closely approximate the ideal response, the ripples in the stopband may not be visible in a plot of the magnitude response. In these cases, the log magnitude response, defined as $20\log_{10}|H(e^{j\omega})|$, may be more useful. The vertical axis of the log magnitude plot is measured in decibels and clearly shows the variation in the stopband region. Thus, an alternate way of specifying the maximum stopband deviation is through the *attenuation* where

$$\text{attenuation} \triangleq -20\log_{10}\delta_s \quad \text{(dB)}.$$

The *phase response* of the filter is defined as

$$\angle H(e^{j\omega}) = \tan^{-1}\frac{\Im m\{H(e^{j\omega})\}}{\Re e\{H(e^{j\omega})\}} + \frac{\pi}{2}(1 + \operatorname{sgn}(\Re e\{H(e^{j\omega})\})).$$

Since the phase function is a multivalued function, the principal value is generally displayed. This results in a function with an amplitude that ranges between $-\pi$ and $+\pi$. The phase response of a filter can be important in certain applications, although historically greater attention has been given to the magnitude response.

Digital filters can be divided into two general categories based on their lengths: finite impulse response (FIR) filters and infinite impulse response (IIR) filters. FIR filters have impulse responses that are finite in duration; they can be implemented conveniently using convolution. IIR filters, on the other hand, are infinite in duration and are implemented using difference equations. Much can be said about digital filters, their properties, and methods for their design, but this is the topic of the companion text, *Digital Filters: A Computer Laboratory Text*. The few exercises that follow try to provide some limited familiarity with this topic.

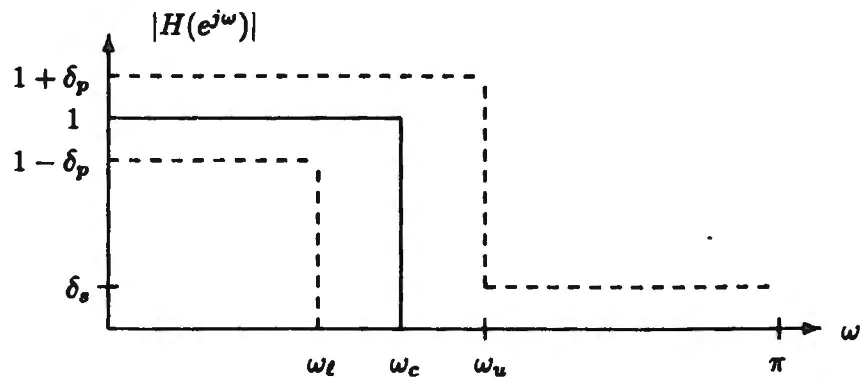


Figure 3.1. Tolerances for specifying the acceptable performance of a lowpass filter. The ideal filter $H(e^{j\omega})$ is shown by the solid line.

– *EXERCISE 3.2.1.* **Digital Filters**

When the frequency response of a linear time-invariant system allows a certain region of the spectrum to be passed undisturbed and rejects or nullifies other frequency regions, the system is often called a filter. Consider the two FIR filters $h_1[n]$ and $h_2[n]$ with impulse responses

$$\begin{aligned} h_1[n] &= 0.036\delta[n] - 0.036\delta[n-1] - 0.29\delta[n-2] + 0.56\delta[n-3] \\ &\quad -0.29\delta[n-4] - 0.036\delta[n-5] + 0.036\delta[n-6] \\ h_2[n] &= -0.09\delta[n] + 0.12\delta[n-1] + 0.5\delta[n-2] \\ &\quad +0.5\delta[n-3] + 0.12\delta[n-4] - 0.09\delta[n-5]. \end{aligned}$$

In addition, consider the IIR filter that is described by the difference equation

$$\begin{aligned} &y[n] - 0.57y[n-1] + 0.88y[n-2] - 0.26y[n-3] + 0.09y[n-4] \\ &\quad = 0.11x[n] + 0.27x[n-1] + 0.37x[n-2] + 0.27x[n-3] + 0.11x[n-4]. \end{aligned}$$

Use the *create file* option of `x siggen` to create files for each of these filters. Display and sketch the magnitude responses for each filter using `x dtft`. Classify each as a lowpass or highpass. What are the approximate cutoff frequencies and transition widths in radians for each of these filters? What is the stopband attenuation (in dB) for each? Estimate these values by inspecting the DTFT plots.

– *EXERCISE 3.2.2.* **Time-Domain Filter Characteristics**

In the preceding exercise, you examined the frequency characteristics of some LTI filters. In this exercise, you will investigate time-domain characteristics of ideal lowpass filters.

The impulse response, $h[n]$, of an ideal lowpass filter can be expressed analytically as

$$h[n] = \frac{\sin \omega_0 n}{\pi n}.$$

Give a rough sketch of this signal for $\omega_0 = \pi/2$, $\pi/4$, $\pi/6$, and $\pi/8$. The function x fdesign (using the rectangular window option) can be used to construct lowpass filters that have this time domain characteristic.

(a) Design four 32-point lowpass filters with cutoff frequencies as specified above. Examine them carefully in both the time domain and the frequency domain. In the time domain each impulse response has a peaked cluster of samples centered in the middle of the sequence. This cluster is called the *main lobe.* How is the width of the main lobe in the impulse response related to the width of the passband of the filter?

(b) The step response of a filter is the output when the input is an ideal unit step, $u[n]$. In this exercise, approximate the step function by a block of length 100 created using x siggen. Sketch the step responses for the set of lowpass filters used in part (a). Truncate your result to 50 samples using x truncate to avoid displaying the erroneous points due to the block approximation of the unit step. Determine the relationship between the slope of the step response in the middle of the transition and the main lobe width of the impulse response. Observe that the ripple heights (overshoot and undershoot) in the step response are not significantly affected by the nominal cutoff frequency.

— *EXERCISE 3.2.3.* **Filter Design**

There are many programs available for designing FIR filters. This text does not address filter design procedures. However, the use of filters and the ability to design them are important if complex systems are to be constructed. The program x fdesign is a filter design program based on the *window design procedure.* Three choices for the window can be chosen: a rectangular window, a Hanning (Von Hann) window, or a Hamming window. Design a 32-point and a 64-point lowpass filter with cutoff $\pi/3$ using each of these window options. Measure the attenuation and transition band in each case using x dtft.

(a) Which window gives the greatest attenuation? What is the value of that attenuation? Does it seem to depend on the filter length?

(b) Which option gives the narrowest transition band for a given filter length? How does the transition band depend on the filter length?

— *EXERCISE 3.2.4.* **Filtering a Sinusoid and a Random Sequence**

The frequency domain provides useful insight in understanding filtering. Create a file containing the filter $h_1[n]$ defined in Exercise 3.2.1 using the *create file* option of x siggen.

(a) Display and sketch the DTFT magnitude of this filter.

(b) Create the following two signals using x siggen, x add, and x rgen.

(i) The first 256 points of

$$x_1[n] = \sum_{k=1}^{4} \sin \frac{\pi k}{5} n.$$

(ii) A 256-point random signal $x_2[n]$.

Display and sketch the DTFT magnitudes of $x_1[n]$ and $x_2[n]$.

(c) Consider $x_1[n]$ and $x_2[n]$ as inputs to the filter with impulse response $h_1[n]$. Without the aid of the computer, sketch the DTFT magnitude of the outputs. Verify your results using the computer.

An LTI system can be used to alter the spectral shape of the input. For the random signals generated by x rgen, the DTFT has energy that is uniformly distributed in the frequency domain. Thus the frequency response of the LTI system shapes the spectrum of the random signal.

3.3 LINEAR PHASE

The DTFT is a complex function that can be expressed in terms of either its real and imaginary parts or its magnitude and phase—the latter being more common. An important subclass of signals is the class for which the DTFT phase is linear. Real, linear phase signals have the property that their discrete-time Fourier transforms can be written as $A(e^{j\omega})e^{-j\omega n_0}$ where $A(e^{j\omega})$ is either purely real or purely imaginary. In this context, $A(e^{j\omega})$ plays the role of an amplitude function and $-\omega n_0$ plays the role of the phase. Since this phase component is a linear function of ω, signals of this form are termed *linear-phase sequences.*

The linear-phase condition imposes a well-defined symmetric structure on the signal in the time domain. For odd length real sequences, the samples are either symmetric or antisymmetric about a midpoint as shown in Fig. 3.2. For even length sequences, the symmetry characteristics are similar, except that the point of symmetry lies midway between two sample values. For finite length sequences that begin at N_1 and end at N_2 $(N_1 < N_2)$, the midpoint, n_0 is at

ength: $L = N_2 - N_1 + 1$

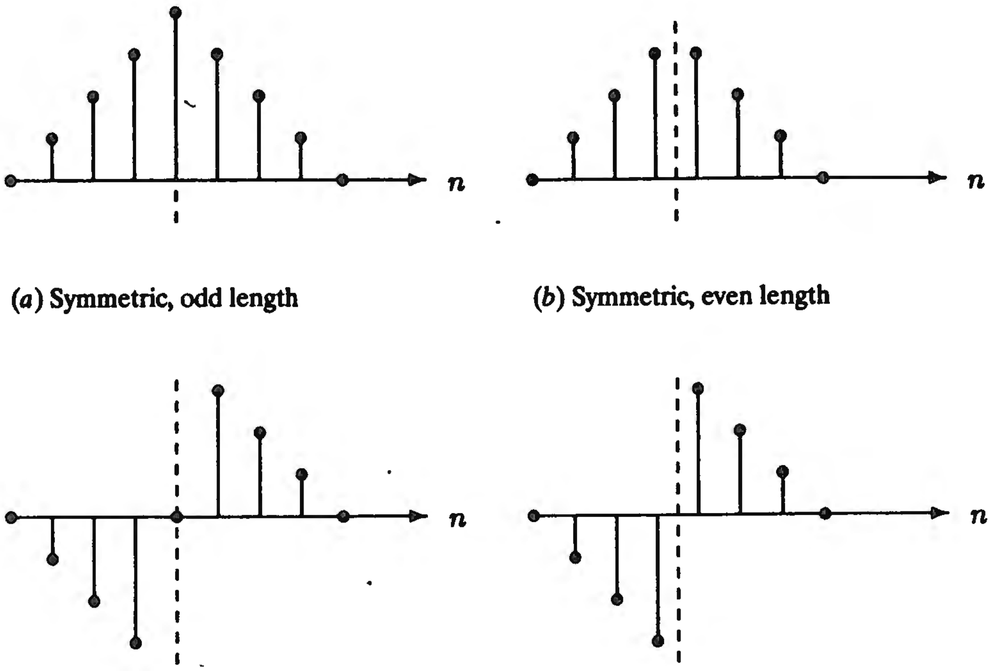
$$n_0 = \frac{N_2 + N_1}{2} = N_1 + \frac{L-1}{2}$$

When the sequence length is odd, n_0 is an integer, but when the sequence length is even, n_0 is a half integer.

An interesting subset of linear-phase sequences is the class of odd length sequences with a midpoint at $n = 0$. These are called *zero-phase* sequences because n_0 is zero and the phase term is also identically zero.

— *EXERCISE 3.3.1.* **Zero Phase**

Zero-phase sequences are odd length linear-phase sequences that are centered at the origin. Due to the sequence symmetry, the discrete-time Fourier transform of a



(*a*) Symmetric, odd length

(*b*) Symmetric, even length

(*c*) Antisymmetric, odd length

(*d*) Antisymmetric, even length

Figure 3.2. The four types of linear-phase sequences.

real odd length, symmetric (even function) zero-phase sequence $h[n]$ can always be expressed as

$$H(e^{j\omega}) = \sum_{n=-L}^{L} h[n]e^{-j\omega n} = h[0] + 2\sum_{n=1}^{L} h[n]\cos\omega n. \tag{3.6}$$

Create an arbitrary 13-point zero-phase sequence, $e[n]$. There are many ways to do this, one of which is to use x **fdesign**. Sketch the sequence and its DTFT. Now create

$$f[n] = e[n] + \delta[n]$$

and sketch $F(e^{j\omega})$. Compare $E(e^{j\omega})$ and $F(e^{j\omega})$ and explain why they have similar spectral shapes.

—*EXERCISE 3.3.2.* **A Cascaded System**

The frequency response of an LTI system provides useful information about its behavior. Consider the LTI systems with impulse responses

$$\begin{aligned} h_1[n] = &-0.04\delta[n] + 0.04\delta[n-1] + 0.3\delta[n-2] - 0.6\delta[n-3] \\ &+0.3\delta[n-4] + 0.04\delta[n-5] - 0.04\delta[n-6] \end{aligned}$$

$$h_2[n] = 0.09\delta[n] - 0.12\delta[n-1] - 0.5\delta[n-2]$$
$$-0.5\delta[n-3] + 0.12\delta[n-4] + 0.09\delta[n-5].$$

Use the *create file* option of `x siggen` to create files for each filter.

(a) Using `x dtft` display and sketch the magnitude responses of $h_1[n]$ and $h_2[n]$.

(b) Now consider a new system composed of the cascade of $h_1[n]$ and $h_2[n]$. The impulse response of the new system, which we will call $h_3[n]$, can be shown to equal

$$h_3[n] = h_1[n] * h_2[n].$$

Without the aid of the computer, give a rough sketch of the magnitude response of $h_3[n]$. Verify your answer by explicitly computing $h_3[n]$ and taking its DTFT.

—*EXERCISE 3.3.3.* **A Parallel System**

Consider two LTI systems defined by the impulse responses

$$h_1[n] = 0.04\delta[n] - 0.04\delta[n-1] - 0.3\delta[n-2] + 0.6\delta[n-3]$$
$$-0.3\delta[n-4] - 0.04\delta[n-5] + 0.04\delta[n-6]$$
$$h_2[n] = 0.09\delta[n] - 0.12\delta[n-1] - 0.5\delta[n-2]$$
$$-0.5\delta[n-3] + 0.12\delta[n-4] + 0.09\delta[n-5].$$

Use the *create file* option of `x siggen` to create files for these filters.

(a) Using `x dtft` display and sketch the magnitude responses for $h_1[n]$ and $h_2[n]$.

(b) Now consider a new system $h_3[n]$ composed of the sum of $h_1[n]$ and $h_2[n]$ where

$$h_3[n] = h_1[n] + h_2[n].$$

Without the aid of the computer, give a rough sketch of the magnitude response of $h_3[n]$. Verify your answer by explicitly computing $h_3[n]$ using `x add` and taking its DTFT magnitude.