9

Laplace and *z*-Transform

9.1 INTRODUCTION

This chapter briefly summarizes the use of the ***Laplace transform*** and the closely related ***z-transform***. The former is used in the analysis of continuous time systems, while the latter is the equivalent for discrete time (sampled) data sets. Both transforms are related to the Fourier transform. Therefore, those who are not familiar with spectral analysis should review Chapters 5 through 7 before proceeding with this chapter. The goal is to use the Laplace and z-transforms to analyze the input-output relationship of linear systems, which we will need specifically for the subsequent chapters that cover the application of analog and digital filters. The starting point for the mathematical description of these linear time invariant (LTI) systems is their associated differential and difference equations (Equations (8.1a) and (8.1.b)).

9.2 THE USE OF TRANSFORMS TO SOLVE ODEs

Solving ordinary differential equations (ODEs) and using convolution to analyze LTI systems can be mathematically complicated. In many cases, this task can be simplified considerably by transforming the problem into another domain (Fig. 9.1) where many operations can be performed algebraically. In the previous chapter, we showed, for example, that (complicated) convolution and correlation integrals in the time domain are equivalent to (simpler) multiplications in the frequency domain. Because the fundamental difference between the Fourier transform on one hand and the Laplace and z-transforms on the other is merely a change from the complex variable $j\omega$ to another complex variable s or z, ***we can extend the frequency domain results for convolution and correlation into the s- and z-domains***. The idea of using a ***transformation*** is to make use of

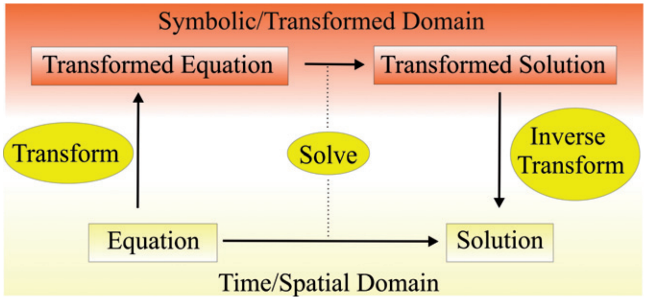

Figure 9.1 Transforms and problem solving.

properties that make a problem easier to solve in the transformed domain. Solving a multiplication problem by a transformation to logarithms is an example of such a procedure. The transformation allows substitution of addition for multiplication. For example, $3.56 \times 4.18 = 14.8808$ can be calculated directly with multiplication. On the other hand, if one could use a table for $\log_{10}$ values, we could find $\log_{10}(3.56) = 0.5514$ and $\log_{10}(4.18) = 0.6212$, and calculate $\log_{10}(3.56) + \log_{10}(4.18) = 1.1726$; the answer can then be obtained by looking up the inverse transformation of the resulting value in the table (i.e., $10^{1.1726} = 14.8808$). This example illustrates that we replace a multiplication by a (simpler) addition under the assumption that we can efficiently make use of a table of log transforms and their inverses.

In a similar fashion as the log transform, a solution of an ODE can be found by using the Laplace transform or the Fourier transform of the equation, while the z transform can be used for the solution of difference equations. As in the log transform presented earlier, the ***rationale*** for the discussed approach (summarized in Fig. 9.1) is that for some types of problems the solution in the transformed domain (plus the steps involved in finding both the transformation and inverse transformation from a table) is more easily calculated than with a direct approach of finding a solution in the time or spatial domain. Since deriving the transforms of arbitrary functions analytically is not often straightforward, a critical element in the relative ease of finding solutions in alternate domains is the existence of tables of Fourier, Laplace, and z-transform pairs. A few examples are summarized in the tables in Appendix 9.1. Extended versions of these tables can be readily found in general textbooks (e.g., Hsu, 1995; Northrop, 2003), while even more extended versions of such tables are available in specialized publications (e.g., Abramowitz and Stegun, 1975) or from specialized websites. Alternately, software packages such as Mathematica and the Symbolic Math Toolbox in MATLAB can calculate transforms and their inverses.

9.3 THE LAPLACE TRANSFORM

In Chapter 6, the Fourier transform was defined as

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{9.1}$$

The Laplace transform is similar:

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \tag{9.2}$$

Here the complex variable $j\omega$ is replaced by complex variable s with both real (σ) and imaginary ($j\omega$) parts: $s = \sigma + j\omega$. Here we show the one-sided Laplace transform with the integration limits from $0 \to \infty$. We focus here on the one-sided Laplace transform because we commonly deal with causal systems, which we begin to study while the system is in rest at some convenient point in time defined as $t = 0$. At this point in time, we also may start to perturb the system with an input signal, but we do not need to worry about the system for $t < 0$, hence the integration limits $0 \to \infty$ in Equation (9.2). A two-sided Laplace transform (with integration $-\infty \to \infty$) does exist, but because of its limited application in our context, it will not be discussed in this text. There are several reasons why transformed ODEs are simpler to solve than their untransformed counterparts. Primarily the evaluation of convolution and cross-correlation integrals is replaced by multiplication. In addition, dealing with differentiation (and integration) is also fairly straightforward in the transformed domain. For example, the Laplace transform $L[\ldots]$ of the derivative of $f(t)$ is

$$L\left[\frac{f(t)}{dt}\right] = \int_{0}^{\infty} \frac{f(t)}{dt} e^{-st}dt = \left[f(t)e^{-st}\right]_0^{\infty} + \int_{0}^{\infty} sf(t)e^{-st}dt = -f(0) + sF(s) \tag{9.3}$$

Notes:

1. The preceding integral was solved using integration by parts (Appendix 3.2; $\int u dv = uv - \int v du$, with $u = e^{-st}$ and $dv = f'(t)$).
2. Note that in Equation (9.3), the Laplace transform is symbolized by ***operator*** $L[\ldots]$.

Using the result for the derivative in Equation (9.3), we can apply the Laplace transform to an ODE describing an LTI system's input-output relationship (Chapter 8, Equation (8.1a)). Using the typical notation, we define $Y(s)$ as the transform of the system output $y(t)$ and $X(s)$ as the transform of input $x(t)$, and we further conveniently assume that all initial conditions $x(0)$, $x'(0)$, . . . , $y(0)$, $y'(0)$, . . . , and so on are zero. This allows us to transform the terms with $y(t)$ and $x(t)$ and their derivatives from Equation (8.1a) to the following:

$$\begin{aligned} x(t) &\Leftrightarrow X(s), \\ x'(t) &\Leftrightarrow sX(s), \\ x''(t) &\Leftrightarrow s^2X(s),\ etc \end{aligned}$$

and

$$\begin{aligned} y(t) &\Leftrightarrow Y(s), \\ y'(t) &\Leftrightarrow sY(s), \\ y''(t) &\Leftrightarrow s^2Y(s),\ etc \end{aligned}$$

resulting in

$$\left[A_n s^n + A_{n-1}s^{n-1} + \ldots + A_0\right]Y(s) = \left[B_m s^m + B_{m-1}s^{m-1} + \ldots + B_0\right]X(s) \tag{9.4}$$

where it should be noted that, with zero initial conditions, higher-order derivatives are simplified to the complex variable s raised to higher powers. Thus, the ratio between the output and input of the system can be represented in the Laplace domain by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{B_m s^m + B_{m-1}s^{m-1} + \ldots + B_0}{A_n s^n + A_{n-1}s^{n-1} + \ldots + A_0} \tag{9.5}$$

The function $H(s)$ is the Laplace transform of $h(t)$, the ***impulse response*** of the LTI system. $H(s)$ is also called the ***transfer function*** of the LTI system.

9.4 EXAMPLES OF THE LAPLACE TRANSFORM

9.4.1 The Transform of a Few Commonly Used Functions

The Laplace transform of the unit impulse function can be obtained by using the sifting property. Here it is important to assume that the domain of the impulse function includes zero as part of the integration limits of the one-sided Laplace transform. In some texts, this is specifically stressed by indicating the integration as $\int_{0^-}^{\infty}$; in the following, we will not use this

0^- notation explicitly. The Laplace transform of the unit impulse evaluates to

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt = e^{-0} = 1 \tag{9.6}$$

The Laplace transform of the unit step function $U(t)$ is:

$$L[U(t)] = \int_0^\infty 1e^{-st}dt = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s} \tag{9.7}$$

This result should not be too surprising considering the relationship we found between the Laplace transform of a function and its derivative in Equation (9.3). The unit impulse can be considered the derivative of the unit step (Chapter 2, Fig. 2.4A), and in the Laplace domain they differ by a factor s.

Some particularly important functions for analysis of linear systems are exponentials, sine, and cosine waves. Let's consider the exponential function e^{at} in which a can be a positive, negative, real, or complex number. Further, we will only consider the exponential for $t \geq 0$ (formally this can be thought of as multiplying by the unit step function: $U(t)e^{at}$). The Laplace transform is

$$L[e^{at}] = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^\infty = -\frac{1}{s-a}[0-1] = \frac{1}{s-a} \tag{9.8}$$

As usual we did not worry about the convergence of the integral here and implicitly assumed that the exponential at infinity is zero. More on the issue of convergence of integrals and existence of Laplace and z-transforms can be found in Appendix 9.2.

Sine and cosine waves can be expressed as exponential expressions using the Euler relationship $[e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)]$ and easily solved using the result found in Equation (9.8). For instance, $\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$ results in the following expression for the Laplace transform:

$$\begin{aligned} L[\sin(\omega t)] &= \frac{1}{2j}\int_0^\infty \left[\left(e^{j\omega t} - e^{-j\omega t}\right)\right]e^{-st}dt = \frac{1}{2j}\left[\int_0^\infty e^{j\omega t}e^{-st}dt - \int_0^\infty e^{-j\omega t}e^{-st}dt\right] \\ &= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\left[\frac{2j\omega}{s^2-(j\omega)^2}\right] = \frac{\omega}{s^2+\omega^2} \end{aligned} \tag{9.9}$$

where $j^2 = -1$. Using the same approach for the Laplace transform of a cosine wave, we obtain

$$L[cos(\omega t)] = \frac{1}{2}\int_0^\infty [(e^{j\omega t} + e^{-j\omega t})]e^{-st}dt = \frac{1}{2}\left[\int_0^\infty e^{j\omega t}e^{-st}dt + \int_0^\infty e^{-j\omega t}e^{-st}dt\right]$$
$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{1}{2}\left[\frac{2s}{s^2-(j\omega)^2}\right] = \frac{s}{s^2+\omega^2} \tag{9.10}$$

9.4.2 The Inverse Laplace Transform

The inverse $f(t)$ of the Laplace transform $F(s)$ can be obtained from the evaluation of a complex integral:

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \tag{9.11}$$

Unlike the inverse transform for the Fourier time domain pair, the inverse Laplace transform in Equation (9.11) is rarely used explicitly. Instead, the most common procedure to find the inverse Laplace transform of an expression is a two-step approach (Appendix 9.3):

1. Apply partial fraction expansion to separate the expression into a sum of basic components.
2. Use a lookup table to find the inverse transforms for each basic component.

Examples of this two-stage approach can be found in Section 9.4.3. A direct application of Equation (9.11) to obtain inverse Laplace transforms is not further covered in this text.

9.4.3 Application to Solving ODEs

In the following example, we will apply the Laplace transform technique to the simplified ***ion channel model*** we introduced in the previous chapter (Fig. 8.2). The ODE describing this system (see the legend Fig. 8.2) is

$$y + RC\frac{dy}{dt} = x \tag{9.12}$$

where R and C are constants corresponding to the membrane resistance and capacitance, respectively. If we probe this system using a unit impulse δ as input x, the output y is the system's impulse response h. Transforming each term of the equation into the Laplace domain gives

$$\begin{aligned} x &= \delta(t) \Leftrightarrow 1 \\ y &= h(t) \Leftrightarrow H(s) \\ \frac{dy}{dt} &= \frac{dh}{dt} \Leftrightarrow sH(s) \end{aligned} \tag{9.13}$$

The $\Leftrightarrow$ symbol indicates the Laplace transform pair.

Note: To represent the preceding differential, we applied Equation (9.3) and assumed that $h(0) = 0$. Remember that in this case we are really taking the value at 0^-, so we may assume that at the onset of time t the system's output is zero.

Substitution of (9.13) into Equation (9.12) and solving for $H(s)$ gives us the transformed ODE:

$$H(s) + RCsH(s) = 1 \rightarrow H(s) = \frac{1}{RC}\frac{1}{s + \frac{1}{RC}} \tag{9.14}$$

Notice that the right-hand side is equivalent to Equation (9.8) with $a = -\frac{1}{RC}$. This allows us to obtain the inverse transform easily without having to deal with evaluating the inverse transform integral (9.11). Thus, the output in the time domain, the impulse response for $t \geq 0$, is

$$y(t) = h(t) = \frac{1}{RC}e^{-\frac{t}{RC}} \tag{9.15}$$

Notice that arriving at expression (9.15), we simply moved the constant $1/RC$ over from the Laplace domain into the time domain, just as we would treat a constant when evaluating an integral equation. In this example, with the unit impulse at the input, finding the inverse was really simple; had we instead chosen the step function $U(t)$ as the input, we would have obtained

$$\begin{aligned} x = U(t) &\Leftrightarrow \frac{1}{s} \\ y &\Leftrightarrow F(s) \\ \frac{dy}{dt} &\Leftrightarrow sF(s) \end{aligned} \tag{9.16}$$

Substitution of these terms in the ODE (Equation (9.12)) gives

$$F(s) + RCsF(s) = \frac{1}{s} \rightarrow F(s) = \frac{1}{RC}\frac{1}{\left(s + \frac{1}{RC}\right)s} \tag{9.17}$$

Here, finding the inverse is slightly more difficult because the denominator of the second factor is a polynomial in s: $(s + 1/RC)s$. This form, where the denominator is a polynomial, is very common because the general form of the ODE describing an LTI system is a quotient of two polynomials (the form shown in Equation (9.5)). As is often the case, partial fraction expansion must be used to decompose Equation (9.17) in simpler terms that can be inverse transformed more readily. Please consult Appendix 9.3 if you need to refresh your mathematical skills in partial fraction expansion. Following partial fraction expansion, we find that the Laplace transform pair associated with Equation (9.17) is

$$F(s) = \frac{1}{RC}\frac{1}{\left(s + \frac{1}{RC}\right)s} = \frac{1}{s} - \frac{1}{s + \frac{1}{RC}} \Leftrightarrow y(t) = 1 - e^{-\frac{t}{RC}} \text{ for } t \geq 0 \tag{9.18}$$

In the following chapters on linear filters (LTI systems), the Laplace transform technique is used to solve input-output relationships. Because the filters we consider can be characterized by the general equation for an LTI system, the Laplace transform associated with Equation (8.1a) can be used to analyze these systems. If we define $X(s)$ and $Y(s)$ as the transforms of the input $x(t)$ and output $y(t)$, respectively, we can transform Equation (8.1a) into the Laplace domain:

$$\begin{aligned} &A_n s^n Y(s) + A_{n-1}s^{n-1}Y(s) + \ldots + A_0 Y(s) \\ &\quad = B_m s^m X(s) + B_{m-1}s^{m-1}X(s) + \ldots + B_0 X(s) \end{aligned} \tag{9.19}$$

As in Equation (9.4), here we have also assumed for convenience that all initial values for $x(t)$, $y(t)$, and their derivatives are zero. Further we assume the input $x(t)$ and its Laplace transform $X(s)$ are known; thus, the expression for output $Y(s)$ results in the quotient of two polynomials in which the order n of the denominator is typically greater than the order of the numerator m:

$$Y(s) = \frac{B_m s^m X(s) + B_{m-1}s^{m-1}X(s) + \ldots + B_0 X(s)}{A_n s^n + A_{n-1}s^{n-1} + \ldots + A_0} \tag{9.20}$$

As mentioned earlier, the common approach to finding the inverse of the transformed output $Y(s)$ is to use two steps: partial fraction expansion, followed by looking up the inverse transforms of the individual terms. As demonstrated in Appendix 9.3, we then find $y(t)$ as the combined result of the inverse transforms of the individual terms. The application of this technique will become clear from the examples in the following chapters.

9.5 THE Z-TRANSFORM

In the following text, we introduce the z-transform as the equivalent of the Laplace transform for discrete time. Subsequently we will show how this procedure can be useful for analyzing difference equations.

9.5.1 The Effect of Delay on the Laplace Transform

In Figure 9.2, we consider a translated function $f(t)$. The Laplace transform of the function in Figure 9.2 is $F(s)=\int_0^\infty f(t)e^{-st}dt$, and the Laplace transform of the right shifted version can be formulated as

$$L[f(t-\tau)]=\int_\tau^\infty f(t-\tau)e^{-st}dt \tag{9.21}$$

In the preceding, operator $L[\ldots]$ indicates the Laplace transform. Substituting $T = t - \tau$ (and consequently, $dt = dT$), we get

$$L[f(t-\tau)]=\int_0^\infty f(T)e^{-s(T+\tau)}dT=e^{-s\tau}\int_0^\infty f(T)e^{-sT}dT=e^{-s\tau}F(s) \tag{9.22}$$

that is, in general ***a delay τ in the time domain is transformed into the s-domain as a multiplication factor exp(−sτ)***. This result is critical for understanding the z-transform introduced in the following section.

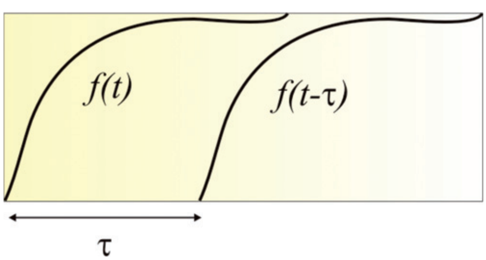

Figure 9.2 Function $f(t)$ and a delayed version $f(t-\tau)$.

9.5.2 Complex Variable z

The z-transform can be considered the equivalent of the Laplace transform for discrete time (sampled) signals. Whereas continuous systems are described by differential equations and approached with Laplace (or Fourier) techniques, the equations that relate to discrete signals are difference equations, such as Equation (8.1b). This type of equation includes terms such as $x(n)$, $x(n-1)$, $h(n-p)$, where n is integer time index. The complex variable z can be considered as the delay operator $exp(s\tau)$.

Consider a discrete time/sampled time series:

$$x(n) = x(0)\delta t + x(1)\delta(t-\tau) + x(2)\delta(t-2\tau) + \ldots + x(n)\delta(t-n\tau) + \ldots$$

and the Laplace transform of this series:

$$L[x(n)] = x(0) + x(1)e^{-s\tau} + x(2)e^{-2s\tau} + \ldots + x(n)e^{-ns\tau} + \ldots \tag{9.23}$$

By using the following definition:

$$e^{s\tau} \equiv z \quad \text{or} \quad e^{-s\tau} \equiv z^{-1} \tag{9.24}$$

we can rewrite Equation (9.23) as

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots + x(n)z^{-n} + \ldots \tag{9.25}$$

Note that in this equation, the constant time difference τ is not explicitly included anymore.

Difference equations such as Equation (8.1b) usually contain operations such as

$$\ldots \quad x(n) - x(n-1) \quad \ldots \tag{9.26}$$

In this example, the notation $(n-1)$ is shorthand for a shift of the ***whole*** time series. To perform this shift on time series x, we can shift $x(n)$ one position to the right in order to obtain $x(n-1)$ (Fig. 9.3). In the z-domain, we can now define the expressions in terms of $X(z)$ and z:

$$\begin{array}{lccc} \text{Original time series:} & x(n) & \Leftrightarrow & X(z) \\ \text{Time series shifted by 1 sample:} & x(n-1) & \Leftrightarrow & z^{-1}X(z) \\ & \ldots & & \ldots \\ & \ldots & & \ldots \\ \text{Time series shifted by } a \text{ samples:} & x(n-a) & \Leftrightarrow & z^{-a}X(z) \end{array} \tag{9.27}$$

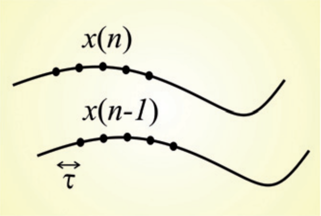

Figure 9.3 A right shift of a sampled time series is equivalent to a delay by one sample interval τ.

Using Equation (9.27), the subtraction in Equation (9.26) transformed into the z-domain becomes

$$\ldots \quad X(z) - z^{-1}X(z) \quad \ldots \tag{9.28}$$

This procedure is of ***general importance*** because any difference equation describing discrete time LTI systems such as that shown in Equation (8.1b) can be transformed into the z-domain following the same principle as that illustrated in the previous example.

Note: Concerning the lag operator, if you have consulted time series analysis in economics texts, you have probably encountered the lag operator (e.g., Hamilton, 1994). This operator, symbolized by **L**, not to be confused with the Laplace transform operator $L[\ldots]$ as shown in Equation (9.3), is similar to the z-transform. The difference is that in most signal processing and engineering texts, z denotes a variable (Equation (9.24)), while **L** is always considered an operator.

9.6 THE Z-TRANSFORM AND ITS INVERSE

In Equation (9.27), we introduced the z-transform of $x(n)$ as $X(z)$ without an explicit definition of how to derive $X(z)$ analytically (such as the definitions of the Fourier and Laplace transforms in Equations (9.1) and (9.2)). An alternative approach to introducing the z-transform is to create a discrete version of the Laplace transform in Equation (9.2). Here the integral is replaced by a summation, similar to the step made in going from the continuous Fourier transform to the discrete Fourier transform. Using the same definition as that used for z in Equation (9.24), with t substituted for

τ, we can transform a discrete time series $x(n)$ into the z-domain transform $X(z)$:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \tag{9.29}$$

The inverse transform is a (counterclockwise) contour integration:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{9.30}$$

which we present here for completeness; in the remainder of this text, we will not use this contour integral further. Rather, to obtain the inverse transform from the z-domain, we will follow the same approach as for the inverse Laplace transform (Section 9.4.2): first we perform partial fraction expansion (if needed), followed by looking up the inverse transform of each term in a table of z-transforms (see the example in Appendix 9.3).

9.7 EXAMPLE OF THE z-TRANSFORM

As an example for the z-transform, let us consider the algorithm for discrete differentiation of a time series $x(n)$ sampled with an interval Δ. Assume a signal differentiator system that outputs time series y, with y being the single time step differential of the input x. This differential can be ***approximated*** by taking the difference between subsequent samples:

$$y(n) = \frac{x(n) - x(n-1)}{\Delta} \tag{9.31}$$

Using $X(z)$ and $Y(z)$ as the z-transforms of $x(n)$ and $y(n)$, respectively, the z-transform of this difference equation becomes

$$Y(z) = \frac{X(z) - X(z) z^{-1}}{\Delta} = \frac{X(z)\left(1 - z^{-1}\right)}{\Delta} \tag{9.32}$$

The transfer function of our differentiator can now be determined:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{\Delta} = \frac{z - 1}{z\Delta} \tag{9.33}$$

If we set the interval Δ to one, the differentiator's transfer function becomes

$$H(z) = \frac{z-1}{z} = 1 - z^{-1} \tag{9.34}$$

Because we know the transfer function $H(z)$ of the differentiator, we can multiply the z-transform of its input with $H(z)$ to obtain the z-transformed output $Y(z)$: $Y(z) = X(z) \times H(z)$. Assuming an input a^n for $t \geq 0$ (i.e., $U(n)$ a^n) with its z-transform: $z/(z - a)$ (see Appendix 9.1, Table A9.2), we get the z-transform of the output as $(z - 1)/(z - a)$. To find the inverse of this z-domain function, a similar approach as with the Laplace transform is used: separate the expression into basic terms (usually by partial fraction expansion), then look up the solution for each component term in a table. An illustration of this procedure for the inverse transform of $(z - 1)/(z - a)$ with $a = \frac{1}{4}$ is given in Appendix 9.3. Further use and examples of the z-transform can be found in Chapters 11 through 13, in which digital filters are introduced.

APPENDIX 9.1

Laplace and *z*-Transforms

The following tables summarize a few Laplace and z-transform pairs, similar to the Fourier transform pairs in Table 6.1 (Chapter 6). In the tables that follow, we multiply the time domain functions with the unit step function U to stress that we are dealing with one-sided transforms in which it is assumed that $x = 0$ for t or $n < 0$.

Table A9.1 Laplace Transform Pairs

$x(t)$	$X(s)$
$\delta(t)$	1
$U(t)$	$\frac{1}{s}$
$U(t)e^{at}$	$\frac{1}{s-a}$
$U(t)\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$U(t)\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Table A9.2 z-Transform Pairs

$x(n)$	$X(z)$
$\delta(n)$	1
$U(n)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$
$U(n)a^n$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$
$U(n)\sin(\omega n)$	$\frac{[\sin(\omega)]z}{z^2-2[\cos(\omega)]z+1}$
$U(n)\cos(\omega n)$	$\frac{z^2-[\cos(\omega)]z}{z^2-2[\cos(\omega)]z+1}$

APPENDIX 9.2

Region of Convergence (ROC)

Throughout the text, we have generally used an optimistic approach with respect to the existence of transforms and convergence of integrals. Because we apply both Laplace and z-transforms as a tool for solving equations and we use tables to find the transforms and their inverses, we usually do not worry about the domain of existence. For the interested reader, we summarize a few comments about the existence of the Laplace and z-transform expressions in this appendix. In order for the transforms to exist, the associated integral/summation must be finite, similar to the Dirichlet conditions for the Fourier transform (Chapter 5, Section 5.3). Especially for functions representing a power such as e^{at} and a^n, the risk of the expressions exploding toward large values for t or n is clearly present. For example, in the integral in Equation (9.8), evaluating the Laplace transform of an exponential function, the term $e^{-(s-a)\infty}$ is zero only if the real part of $(s-a) > 0$. In this case, the integral exists (i.e., it evaluates to a finite value). In the case where a in e^{at} is a real number, the condition for the existence of the Laplace transform for e^{at} can be formulated as $\text{Re}(s) > a$ (Re symbolizing the real part σ of s); the graphical representation of this area in the s-plane is shown in Figure A9.1. The area satisfying the existence condition for the Laplace transform is called the ***region of convergence (ROC)***. The example of the ROC in Figure A9.1A is clearly only relevant for the exponential function e^{at}; other functions will have a different ROC.

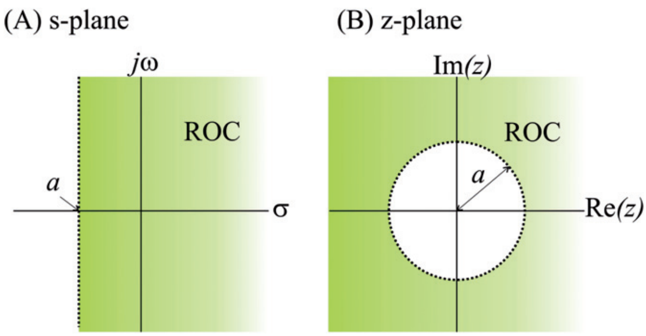

Figure A9.1 Examples of the region of convergence (A) of the Laplace transform of e^{at}, and (B) of the z-transform of a^n.

Just as in the Laplace transform, one can determine a region where the result of the summation in Equation (9.29) is finite. This area is the ROC for the z-transform (Fig. A9.1B) and as in the Laplace transform, it depends on the function at hand — that is, $x(n)$. Considering an example where $x(n) = a^n$, we can define the z-transform using Equation (9.29) as

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \quad \text{(A9.2-1)}$$

The summation in Equation (A9.2-1) is a series that converges only if $|az^{-1}| < 1 \rightarrow |z| > |a|$.

> *Note:* The convergence statement $|az^{-1}| < 1$ is provided without proof, but it is not completely counterintuitive since the power of a fraction smaller than 1 becomes very small for large powers n, whereas the power of a number larger than 1 grows increasingly large for increasing n.

The z-plane consists (as the s-plane) of real and imaginary components, and z (being a complex exponential, Equation (9.24)) is usually defined in polar coordinates; an example of the ROC for a^n is shown in Figure A9.1*B*.

APPENDIX 9.3

Partial Fraction Expansion

In finding the ***inverses of both Laplace and z-transforms***, it is often necessary to apply partial fraction expansion. This procedure is based on the fact that a rational function (the quotient of two polynomials) where the order of the numerator is lower than the denominator can be decomposed into a summation of lower-order terms. The partial fraction expansion will be reviewed in this appendix without further proof.

To illustrate the principle, we show the example in Equation (9.17):

$$F(s)=\frac{1}{RC}\frac{1}{\left(s+\frac{1}{RC}\right)s} \tag{A9.3-1}$$

To focus on the expansion, initially we ignore the constant factor $1/RC$ and focus on a function in the form $\frac{1}{\left(s+\frac{1}{RC}\right)s}=\frac{1}{(s-a)s}$, defining $a=-1/RC$ (note the minus sign) for a simpler notation. Since the order of the numerator is lower than the denominator, according to the algebra underlying the partial fraction expansion technique, we may state

$$\frac{1}{(s-a)s}=\frac{A}{s-a}+\frac{B}{s} \tag{A9.3-2}$$

Step 1 is to solve for A by multiplying through by the denominator of the first term on the right-hand side ($s-a$) and then setting $s=a$ in order to nullify the B coefficient:

$$\frac{1}{s}=A+\frac{B(s-a)}{s} \quad \text{and} \quad s=a\rightarrow A=\frac{1}{s}=\frac{1}{a} \tag{A9.3-3a}$$

Step 2 is to solve for B by multiplying through by the denominator of the second term s and then setting $s=0$ to eliminate the A term:

$$\frac{1}{s-a}=\frac{As}{s-a}+B \quad \text{and} \quad s=0\rightarrow B=\frac{1}{s-a}=-\frac{1}{a} \tag{A9.3-3b}$$

You probably noticed that in steps 1 and 2, we first multiply entire equations with expressions from the denominator of the separate terms and then conveniently choose a value that makes that expression zero; a strange trick, because it "feels" like division by zero, but it works!

Combining our results with the original expression in Equation (A9.3-2), we get

$$\frac{1}{(s-a)s}=\frac{1}{a}\frac{1}{(s-a)}-\frac{1}{a}\frac{1}{s}$$

Substituting $a = -1/RC$ we get

$$-\frac{RC}{\left(s+\frac{1}{RC}\right)}+\frac{RC}{s}$$

Finally we substitute our result into Equation (A9.3-1):

$$F(s)=\frac{1}{RC}\frac{1}{\left(s+\frac{1}{RC}\right)s}=\frac{1}{RC}\left[-\frac{RC}{\left(s+\frac{1}{RC}\right)}+\frac{RC}{s}\right]=\frac{1}{s}-\frac{1}{s+\frac{1}{RC}} \qquad \text{(A9.3-4)}$$

The inverse transform of both terms can be obtained easily by inverting the results we obtained in Equations (9.7) and (9.8):

$$y(t)=U(t)-U(t)e^{-\frac{1}{RC}} \quad \text{or} \quad y(t)=1-e^{-\frac{t}{RC}} \text{ for } t\geq 0 \qquad \text{(A9.3-5)}$$

Similar procedures are also commonly applied to find the inverses of the z-transform and the Fourier transform. There is one important condition that must be satisfied for this trick to work. The order of the numerator must be smaller than the order of the denominator! If this is not the case there are two procedures that can be followed:

1. Use polynomial division, or
2. Divide the entire expression temporarily by s and correct for this change later.

For example, the inverse of $\frac{s-a}{s+a}$ cannot be determined directly because the order of both numerator and denominator are the same. Using ***the division approach***:

$$\begin{array}{l} \qquad\;\; 1 \\ s+a\overline{)\,s-a} \\ \qquad \dfrac{s+a}{-2a} - \rightarrow \dfrac{s-a}{s+a}=1-\dfrac{2a}{s+a} \end{array} \qquad \text{(A9.3-6)}$$

These two terms can be easily transformed using the results obtained earlier (note that a = a constant!) in Equations (9.6) and (9.8):

$$\frac{s-a}{s+a} = 1 - \frac{2a}{s+a} \Leftrightarrow \delta(t) - 2ae^{-at} \text{ for } t \geq 0 \qquad \text{(A9.3-7)}$$

In this example, it was fairly easy to divide and find the inverse transform. The alternative is to follow the other approach and ***divide by s or z*** (in the case of the z-transform) and correct for this sleight of hand later. For example, the inverse transform of $\frac{z-1}{z-1/4}$ can be found by dividing through by z:

$$\frac{z-1}{z-1/4} \xrightarrow[\times\frac{1}{z}]{} \frac{z-1}{z(z-1/4)} = \frac{A}{z} + \frac{B}{z-1/4} \qquad \text{(A9.3-8)}$$

producing an expression that meets the order condition, and thus allows us to use the same approach for the partial fraction expansion followed in Equation (A9.3-3).
Step 1 is to multiply by the denominator of the first term z and then set $z = 0$:

$$\frac{z-1}{z-1/4} = A + \frac{Bz}{z-1/4} \quad \text{and} \quad z = 0 \quad \rightarrow \quad A = \frac{-1}{-1/4} = 4 \qquad \text{(A9.3-9a)}$$

Step 2 is to similarly multiply by the denominator of the second term $(z - \frac{1}{4})$ and then set $z = \frac{1}{4}$:

$$\frac{z-1}{z} = \frac{A(z-1/4)}{z} + B \quad \text{and} \quad z = \frac{1}{4} \quad \rightarrow \quad B = \frac{1/4-1}{1/4} = -3 \qquad \text{(A9.3-9b)}$$

Substituting out findings in (A9.3-9) back into Equation (A9.3-8) and correcting the result by multiplying it by z,

$$\frac{z-1}{z-1/4} \xrightarrow[\times 1/2]{} \frac{z-1}{z(z-1/4)} = 4\frac{1}{z} - 3\frac{1}{z-1/4} \xrightarrow[\times z]{} 4 - 3\frac{z}{z-1/4} \qquad \text{(A9.3-10)}$$

The form of the expression in Equation (A9.3-10) can be readily found in standard tables of the z-transform:

$$4 - 3\frac{z}{z-1/4} = 4 - 3\frac{1}{1-1/4z^{-1}} \Leftrightarrow 4\delta(n) - 3(1/4)^n\, U(n) \qquad \text{(A9.3-11)}$$

Here $\delta(n)$ and $U(n)$ are the discrete time versions of the unit impulse and unit step.