5

Real and Complex Fourier Series

5.1 INTRODUCTION

This chapter introduces the Fourier series in the real and the complex form. First we develop the Fourier series as a technique to represent arbitrary functions as a summation of sine and cosine waves. Subsequently we show that the complex version of the Fourier series is simply an alternative notation. At the end of this chapter, we apply the Fourier series technique to decompose periodic functions into their cosine and sine components.

Because the underlying principle is to represent waveforms as a summation of periodic cosine and sine waves with different frequencies, one can interpret Fourier analysis as a technique for examining signals in the ***frequency domain***. At first sight, the term *frequency domain* may appear to be a novel or unusual concept. However, in daily language we do use frequency domain descriptions; for instance, we use a frequency domain specification to describe the power line source as a 120-V, 60-Hz signal. Also, the decomposition of signals into underlying frequency components is familiar to most; examples are the color spectrum obtained from decomposing white light with a prism (Fig. 5.1), or decomposing sound into pure tone components.

An example showing an approximation of a square wave created from the sum of five sine waves is shown in Figure 5.2. This example can be reproduced with MATLAB script pr5_1.m. This example illustrates the basis of spectral analysis: a time domain signal (i.e., the (almost) square wave) can be decomposed into five sine waves, each with a different frequency and amplitude. The graph depicting these frequency and amplitude values in Figure 5.2 is a frequency domain representation of the (almost) square wave in the time domain. This task of deriving a frequency domain equivalent of a signal originally in the time or spatial domain is the topic of this chapter and Chapters 6 and 7. Here we introduce the Fourier series, and on the basis of this concept we introduce the continuous transform and its discrete version in Chapter 6. On the basis

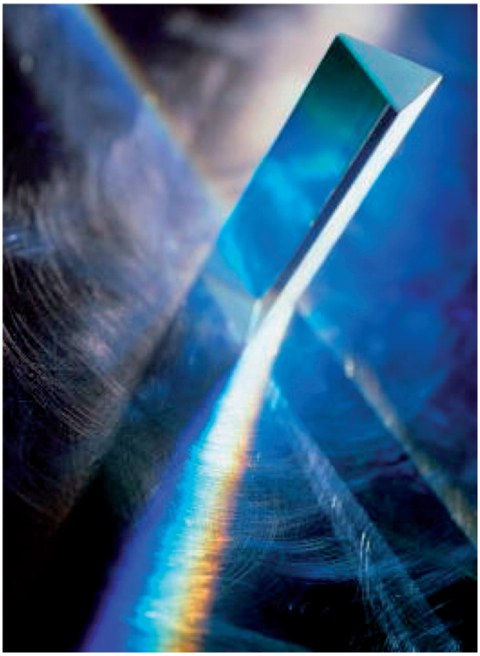

Figure 5.1 A prism performs spectral decomposition of white light in bands with different wavelengths that are perceived by us as different colors.

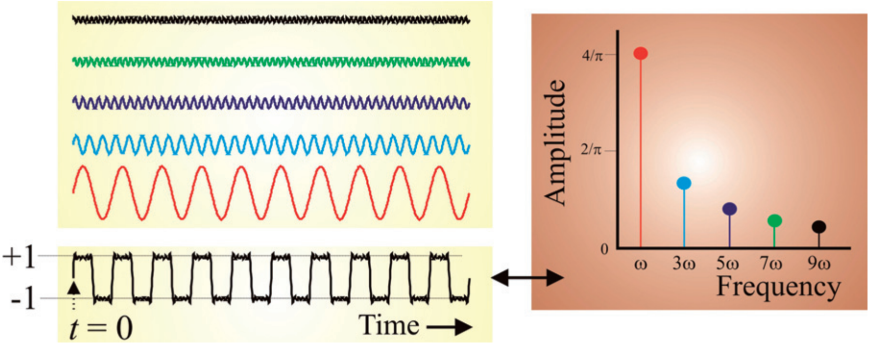

Figure 5.2 The sum of five sine waves approximates a square wave with amplitude ±1 (bottom trace). The amplitude of the sine waves decreases with frequency. The spectral content of the square wave is shown in a graph of amplitude versus frequency (right). The data can be obtained by running script pr5_1.m; the spectrum of a square wave is computed analytically in the second example in Section 5.4.

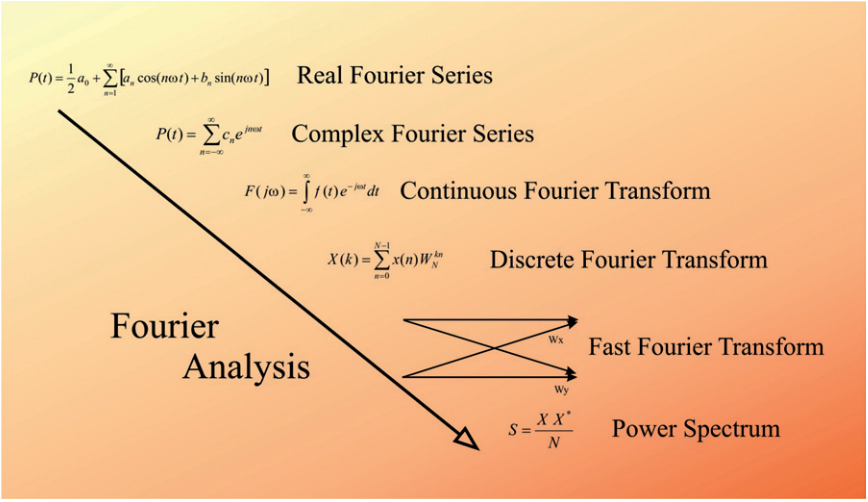

Figure 5.3 The relationship between different flavors of Fourier analysis. The real and complex Fourier series can represent a function as the sum of waves as shown in the example in Figure 5.2. The continuous and discrete versions of the Fourier transform provide the basis for examining real-world signals in the frequency domain. The computational effort to obtain a Fourier transform is significantly reduced by using the fast Fourier transform (FFT) algorithm. The FFT result can subsequently be applied to compute spectral properties such as a power spectrum describing the power of the signal's different frequency components.

of the discrete Fourier transform, we describe the development of algorithms to calculate the spectrum of a time series in Chapter 7. The order in which we proceed from the Fourier series to spectral analysis and specific algorithms is depicted in Figure 5.3. The end result is that you will understand the underlying math of the Fourier transform technique, you will have an idea of when to apply this powerful analytical tool, and you will understand what happens under the hood when you type the command fft or fft2 in MATLAB.

5.2 THE FOURIER SERIES

The Fourier series provides a basis for analysis of signals in the frequency domain. In this section, we show that a function $f(t)$ (such as the almost square wave in Fig. 5.2) with period T [i.e., $f(t) = f(t + T)$], frequency $f = 1/T$, and angular frequency ω defined as $\omega = 2\pi f$ can be represented by a series $P(t)$:

$$\boxed{\begin{aligned} P(t) &= \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t) + \ldots + b_1\sin(\omega t) + b_2\sin(2\omega t) + \ldots \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)] \end{aligned}} \tag{5.1}$$

with the first term $\frac{1}{2}a_0$ representing the ***direct current*** (DC) component; the remaining sine and cosine waves weighted by the a_n and b_n coefficients represent the ***alternating current*** (AC) components of the signal. The seemingly arbitrary choice of $\frac{1}{2}a_0$ allows us to obtain $\frac{2}{T}$ scaling for all coefficients (e.g., see Equations (5.6), (5.12), and (5.18)). However, in some definitions of the Fourier series, the first term is defined as a_0, leading to a difference of a factor of 2 in the end result shown in Equation (5.6).

5.2.2 Minimization of the Difference between $P(t)$ and $f(t)$

In the following sections, we derive the expressions for the coefficients a_n and b_n in Equation (5.1). Because it is easy to lose the big picture in the mathematical detail, we summarize the strategy in Figure 5.4 and relegate some of the mathematical detail to Appendices 5.1 and 5.2. Examples of how to apply Fourier series analysis to time series are given in Section 5.4; the reader who is not yet interested in the derivations described in the following paragraphs can simply proceed to these examples and apply Equations (5.6), (5.12), and (5.18) to calculate the Fourier series coefficients.

Two strategies are commonly used to derive the equations for coefficients a_n and b_n. One method begins by multiplying the terms of the series in Equation (5.1) with $\cos(N\omega t)$ or $\sin(N\omega t)$ with $N = 0, 1, 2, \ldots$ and integrating over a full period T associated with the lowest frequency ω. While it inevitably leads to the correct results, this approach is less intuitive because it starts from Equations (5.8) and (5.14) directly without particular justification. The other method, which in any case leads to the same result, starts with an evaluation of the difference between the Fourier series approximation $P(t)$ and function $f(t)$ itself. The difference is considered the error of the approximation — that is, the error E that is made by the approximation is $[P(t) - f(t)]$, which can be minimized by reducing E^2 over a full period T of the time series:

$$E^2 = \int_{t}^{t+T} [P(t) - f(t)]^2 dt \tag{5.2}$$

Assuming that the integral in Equation (5.2) exists, we can find the minimum of the error function by

$$\frac{\partial E^2}{\partial a_n} = 0 \quad \text{and} \quad \frac{\partial E^2}{\partial b_n} = 0$$

Substitution of the expression in Equation (5.2) into $\frac{\partial E^2}{\partial a_n}$ gives $\frac{\partial\left[\int\limits_t^{t+T} [P(t) - f(t)]^2 dt\right]}{\partial a_n}$. By reversing the order of differentiation and integration, we obtain $\frac{\int\limits_t^{t+T} \partial\{[P(t) - f(t)]^2 dt\}}{\partial a_n}$, which can be written as

$$2\int_T \left[(P(t) - f(t))\frac{\partial(P(t) - f(t))}{\partial a_n}\right]dt = 0 \tag{5.3}$$

The outcomes of the partial derivative expression $\frac{\partial(P(t) - f(t))}{\partial a_n} = \frac{\partial P(t)}{\partial a_n}$ for different a_n are summarized in Table 5.1.

Table 5.1 Evaluation of $\frac{\partial P(t)}{\partial a_n}$ for Different Values of n

Because for each partial derivative to a_n there is only one term in $P(t)$ containing a_n, the outcome is a single term for each value of n.

Index	Derivative
$n = 0$	$\frac{\partial P(t)}{\partial a_0} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial a_0} = \frac{1}{2}$
$n = 1$	$\frac{\partial P(t)}{\partial a_1} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial a_1} = \cos(\omega t)$
$n = 2$	$\frac{\partial P(t)}{\partial a_2} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial a_2} = \cos(2\omega t)$
n	$\frac{\partial P(t)}{\partial a_n} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial a_n} = \cos(n\omega t)$

Notes:

1. The area enclosed by a periodic function is independent of the starting point, so integration over a full period is insensitive to the value of t. For example, we may integrate from $0 \to T$ or from $-T/2 \to T/2$ and obtain the same result. Therefore we change the notation from $\int_0^T$ to $\int_T$ in Equation (5.3) to indicate that the integration is taken over the entire period without respect to the particular choice of integration limits.
2. In some of the textbook derivations of the Fourier series, ωt is substituted by a variable x; the integration limits over a full period then become: $0 \to 2\pi$ rad or $-\pi \to \pi$ rad.
3. The partial derivatives in the preceding equations are *not* with respect to t but to a_n and b_n. Because we evaluate $P(t)$ also as a function of a_n as well as of b_n, we should, strictly speaking, reflect that in the notation by using $\frac{\partial P(t, a_n, b_n)}{\partial a_n}$ and $\frac{\partial P(t, a_n, b_n)}{\partial b_n}$. In the text, we simplify this cumbersome notation to $\frac{\partial P(t)}{\partial a_n}$ and $\frac{\partial P(t)}{\partial b_n}$.

Minimization of equation (5.2) for b_n:

$$2\int_T \left[(P(t) - f(t)) \frac{\partial (P(t) - f(t))}{\partial b_n} \right] dt = 0 \tag{5.3b}$$

Again, the outcome of the partial derivative expression $\frac{\partial (P(t) - f(t))}{\partial b_n} = \frac{\partial P(t)}{\partial b_n}$ varies with the value of n (Table 5.2).

In the following sections, we use the obtained results to derive expressions for the coefficients a_n and b_n. To simplify matters, we will frequently rely on two helpful properties: the fact that (1) the integral of a cosine or sine wave over one or more periods evaluates to zero and (2) the orthogonal characteristics of the integrals at hand. The mathematical details of this approach can be found in Appendix 5.1.

5.2.2.1 Coefficient a_0

Returning to the a_n coefficients: for $n = 0$, we found that the derivative associated with minimization evaluates to $\frac{1}{2}$ (Table 5.1). Substitution of this result into Equation (5.3) gives us an expression for a_0:

$$2\int_T (P(t) - f(t)) \frac{1}{2} dt = \int_T P(t) dt - \int_T f(t) dt = 0 \to \int_T f(t) dt = \int_T P(t) dt \tag{5.4}$$

$$
\begin{aligned}
\int_T f(t)dt &= \int_T \left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]dt \\
&= \int_T \frac{1}{2}a_0 dt + \int_T a_1\cos(\omega t)dt + \int_T a_2\cos(2\omega t)dt \ldots \\
&\quad + \int_T b_1\sin(\omega t)dt + \ldots
\end{aligned} \tag{5.5}
$$

In Equation (5.5), the integrals of the cosine and sine terms (evaluated over T) equal zero (if this result is not obvious, review Equation (A5.1-3) in Appendix 5.1), leaving only the nonzero a_0 term:

$$
\int_T f(t)dt = \frac{1}{2}a_0\int_T dt = \frac{1}{2}a_0 T \rightarrow a_0 = \frac{2}{T}\int_T f(t)dt \tag{5.6}
$$

Note: The factor of 2 in Equation (5.6) originates from our choice to represent the first term in Equation (5.1) as $\frac{1}{2}a_0$. The first term in the Fourier series is therefore $\frac{1}{2}a_0 = \frac{1}{T}\int_T f(t)dt$, which is the mean of the function $f(t)$ in the interval T (see also Chapter 3, Section 3.2). In terms of electrical signals, this can also be thought of as the direct current (DC) component of $f(t)$.

5.2.2.2 Coefficients a_1 and a_n

For $n = 1$, we obtained $\cos(\omega t)$ for the partial derivative (Table 5.1); substituting this result into Equation (5.3a),

$$
\begin{aligned}
&2\int_T [(P(t) - f(t))\cos(\omega t)]dt = 2\int_T P(t)\cos(\omega t)dt - 2\int_T f(t)\cos(\omega t)dt = 0 \\
&\rightarrow \int_T f(t)\cos(\omega t)dt = \int_T P(t)\cos(\omega t)dt
\end{aligned} \tag{5.7}
$$

Filling in the terms for the Fourier series $P(t)$,

$$
\begin{aligned}
\int_T f(t)\cos(\omega t)dt &= \int_T \begin{bmatrix} \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + \\ b_1\sin(\omega t) + \ldots \end{bmatrix}\cos(\omega t)dt \\
&= \int_T \frac{1}{2}a_0\cos(\omega t)dt + \int_T a_1(\cos(\omega t))^2 dt + \\
&\quad \int_T a_2\cos(2\omega t)\cos(\omega t)dt \ldots + \int_T b_1\sin(\omega t)\cos(\omega t)dt + \ldots
\end{aligned} \tag{5.8}
$$

From Equation (5.8) we can solve for a_1 using the following logic:

1. The first term in the previous expression, the integral of $\cos(\omega t)$ over a full period $\int_T \frac{1}{2} a_0 \cos(\omega t)\, dt$, evaluates to zero. As mentioned earlier, this result is obtained because the area enclosed by a cosine function over a full period equals zero (Appendix 5.1, Equation (A5.1-3)).
2. The second term does not evaluate to zero, and we will address this term in Equation (5.9).
3. All remaining terms in Equation (5.8) are integrals over T that contain the following:
   a. The product of two cosine functions, which evaluate to zero because of orthogonal behavior (Appendix 5.1, Equation (A5.1-9))
   b. Or sine × cosine products, which evaluate to zero (Appendix 5.1, Equation (A5.1-9))

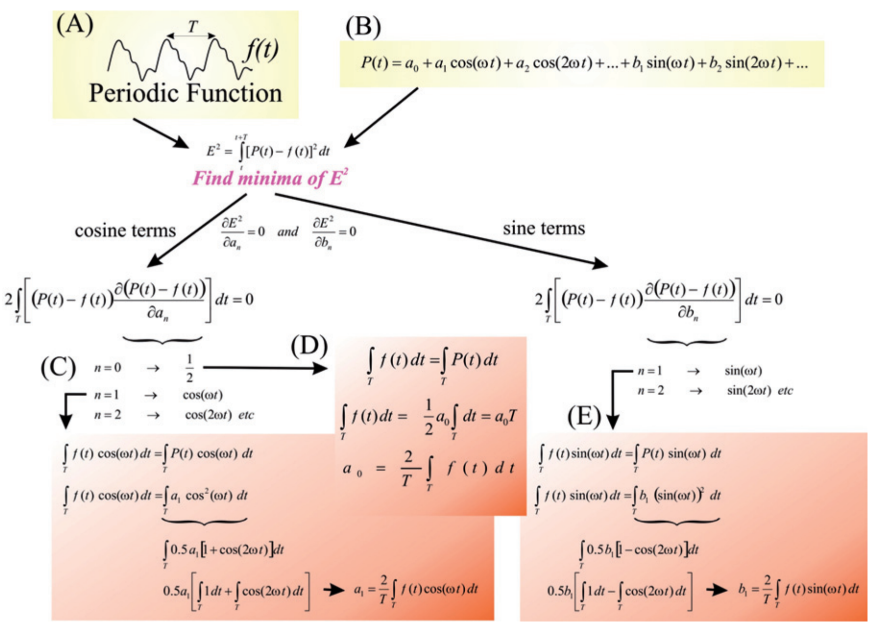

Figure 5.4 Overview of the real Fourier series representation of $f(t)$, a periodic function (A). (B) The real Fourier series $P(t)$. (C) and (D) Determination of coefficients a_0 and a_1 in $P(t)$. (E) The same as (C) for the b_1 coefficient (note that there is no b_0). Determination of a_n and b_n coefficients is similar to the procedure for a_1 and b_1.

Therefore, all the terms in Equation (5.8) evaluate to zero, except $\int_T a_1(\cos(\omega t))^2 dt$, allowing us to simplify Equation (5.7) to

$$\int_T f(t)\cos(\omega t)dt = \int_T a_1(\cos(\omega t))^2 dt \tag{5.9}$$

We use the special trigonometric relationships summarized in Equation (A5.1-6), with $A = \omega t$, and substitute the result in Equation (5.9):

$$\int_T f(t)\cos(\omega t)dt = \int_T a_1(\cos(\omega t))^2 dt = \int_T \frac{1}{2}a_1[1+\cos(2\omega t)]dt$$

The part after the equal sign can be further simplified:

$$\frac{1}{2}a_1\left[\int_T dt + \int_T \cos(2\omega t)dt\right] = \frac{1}{2}a_1\,[t]_o^T + 0 = \frac{T}{2}a_1 \tag{5.10}$$

The second integral in Equation (5.10) is a cosine evaluated over two periods and therefore evaluates to zero (Appendix 5.1). Solving Equation (5.10) for the coefficient:

$$a_1 = \frac{2}{T}\int_T f(t)\cos(\omega t)dt \tag{5.11}$$

This technique can be applied to find the other coefficients a_n. The integrals of the products $\cos(n\omega t) \times \cos(m\omega t)$ in the series all evaluate to zero with the exception of those in which $m = n$ (Appendix 5.1). The property that products of functions are zero unless they have the same coefficient is characteristic of ***orthogonal functions*** (Appendix 5.1). This leads to the general formula for a_n:

$$\boxed{a_n = \frac{2}{T}\int_T f(t)\cos(n\omega t)dt} \tag{5.12}$$

5.2.2.3 Coefficients b_1 and b_n

For $n = 1$, we obtained $\sin(\omega t)$ for the partial derivative (Table 5.2); substituting this result into Equation (5.3b):

$$\begin{aligned} &2\int_T [(P(t)-f(t))\sin(\omega t)]dt = 2\int_T P(t)\sin(\omega t)dt - 2\int_T f(t)\sin(\omega t)dt = 0 \\ &\quad\rightarrow \int_T f(t)\sin(\omega t)dt = \int_T P(t)\sin(\omega t)dt \end{aligned} \tag{5.13}$$

Substituting the terms for the Fourier series $P(t)$:

Table 5.2 Evaluation of $\frac{\partial P(t)}{\partial b_n}$ for Different Values of n

Index	Derivative
$n = 0$	Index does not exist for b coefficient.
$n = 1$	$\frac{\partial P(t)}{\partial b_1} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial b_1} = \sin(\omega t)$
$n = 2$	$\frac{\partial P(t)}{\partial b_2} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial b_2} = \sin(2\omega t)$
n	$\frac{\partial P(t)}{\partial b_n} = \frac{\partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]}{\partial b_n} = \sin(n\omega t)$

$$\begin{aligned}\int_T f(t)\sin(\omega t)dt &= \int_T \begin{bmatrix}\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots \\ + b_1\sin(\omega t) + \ldots\end{bmatrix}\sin(\omega t)dt \\ &= \int_T \frac{1}{2}a_0\sin(\omega t)dt + \int_T a_1\cos(\omega t)\sin(\omega t)dt + \\ &\quad \int_T a_2\cos(2\omega t)\sin(\omega t)dt \ldots + \int_T b_1(\sin(\omega t))^2 dt + \ldots\end{aligned} \tag{5.14}$$

For the same reasons used in deriving the expression for a_1 (***orthogonal function property***, Appendix 5.1), all terms except the one with the $(\sin(\omega t))^2$ evaluate to zero. Therefore, Equation (5.13) simplifies to

$$\int_T f(t)\sin(\omega t)dt = \int_T b_1(\sin(\omega t))^2 dt \tag{5.15}$$

Again using a trigonometric identity (A5.1-6), with $A = \omega t$ and substituting the result in Equation (5.15), we get

$$\begin{aligned}\int_T b_1(\sin(\omega t))^2 dt &= \int_T \frac{1}{2}b_1[1 - \cos(2\omega t)]dt \\ &= \frac{1}{2}b_1\left[\int_T dt - \int_T \cos(2\omega t)dt\right] = \frac{T}{2}b_1\end{aligned} \tag{5.16}$$

Using the property from Equation (A5.1-3), it can be seen that the second integral in Equation (5.16) evaluates to zero. Solving Equation (5.16) for the coefficient b_1:

$$b_1 = \frac{2}{T}\int_T f(t)\sin(\omega t)dt \tag{5.17}$$

Finally, applying the same procedure to solve for b_n:

$$\boxed{b_n = \frac{2}{T}\int_T f(t)\sin(n\omega t)dt} \tag{5.18}$$

This completes our task of finding the real valued Fourier series for function $f(t)$; starting from any function in the time (or spatial) domain, Equations (5.6), (5.12), and (5.18) allow us to determine all coefficients for the associated Fourier series.

5.3 THE COMPLEX FOURIER SERIES

The Fourier series of a periodic function is frequently presented in the complex form. In this section, we first introduce the complex version of the Fourier series and we then show its equivalence to the real Fourier series derived earlier. The notation for the complex Fourier series is

$$\boxed{P(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}} \tag{5.19}$$

The coefficients c_n in Equation (5.19) are defined as

$$\boxed{c_n = \frac{1}{T}\int_T f(t)e^{-jn\omega t}dt} \tag{5.20}$$

Just as in the real Fourier series formalism (Equations (5.6), and (5.12)), the $\int_T \ldots$ in Equation (5.18) indicates that the integral must be evaluated over a full period T, where it is not important what the starting point is (e.g., $-T/2 \to T/2$ or $0 \to T$). Note that as compared with the formulas for the coefficients in the real series (Equations (5.6), (5.12), and (5.18)), the sine and cosine terms are replaced by a complex exponential. In addition, comparing Equations (5.19) and (5.20), we see that the summation is performed from $-\infty$ to ∞ instead of from 0 to ∞ as in the real series.

In the following, we show that ***the real and complex Fourier series notations are equivalent***. The complex form of $P(t)$ in Equation (5.19) can be rewritten as

$$P(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega t} \tag{5.21}$$

Note: The integral in equation (5.20) is finite if

$$|c_n| = \left| \frac{1}{T} \int_T f(t) e^{-jn\omega t} dt \right| \le \frac{1}{T} \int_T |f(t)| \left| e^{-jn\omega t} \right| dt < \infty$$

Because $|e^{-jn\omega t}| = 1$, we may conclude that the integral must be finite if

$$\int_t^{t+T} |f(t)| dt < \infty$$

This is the so-called weak ***Dirichlet*** condition that guarantees the existence of the Fourier series. A function such as $f(t) = 1/t$ over an interval from 0 to T would fail this criterion. The other (strong) Dirichlet conditions are that the frequencies included in $f(t)$ are finite (a finite number minima and maxima) and that the number of discontinuities (abrupt changes) is also finite.

Although these criteria play a role in analysis of functions, for measured time series, they are irrelevant because these signals are always bounded within the measurement range and limited in frequency content by the bandwidth of the recording equipment.

We can change the polarity of the summation in the third term of Equation (5.21) from $-\infty \rightarrow -1$ to $1 \rightarrow \infty$. We correct this by changing the sign of n: in $c_n \rightarrow c_{-n}$ and in the exponent $e^{jn\omega t} \rightarrow e^{-jn\omega t}$. The result of this change is

$$P(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t} \tag{5.22}$$

Subsequently, we use Euler's relation $[e^{jx} = \cos(x) + j\sin(x)]$ to rewrite the coefficient c_n in Equation (5.20):

$$c_n = \frac{1}{T} \int_T f(t)[\cos(n\omega t) - j\sin(n\omega t)]dt \tag{5.23}$$

The expression in Equation (5.23) is a complex number. Because we can represent any complex number by its real (a_n) and imaginary (jb_n) parts, we may simplify:

$$\frac{1}{T} \int_T f(t)[\cos(n\omega t) - j\sin(n\omega t)]dt = \frac{1}{2}(a_n - jb_n) \tag{5.24}$$

with the factor $\frac{1}{2}$ in Equation (5.24), chosen for convenience, as will be shown in the text that follows. Further we can conclude from the expression in (5.24) that the real part of the equation is a cosine, which is an even symmetric function (Appendix 5.2). Therefore, we conclude that $a_n = a_{-n}$. The imaginary part is sine, an odd symmetric function (Appendix 5.2), therefore $b_n = -b_{-n}$ (Note the minus sign for the odd function). Using these properties of a_n and b_n, we have

$$c_n = \frac{1}{2}(a_n - jb_n), \quad \text{and} \quad c_{-n} = \frac{1}{2}(a_{-n} - jb_{-n}) = \frac{1}{2}(a_n + jb_n) \tag{5.25}$$

We substitute the results for c_n and c_{-n} in $P(t)$ in Equation (5.22):

$$P(t) = \frac{1}{2}(a_0 - jb_0) + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - jb_n)e^{jn\omega t} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n + jb_n)e^{-jn\omega t} \tag{5.26}$$

Here we can set the coefficient for the imaginary DC component to zero ($b_0 = 0$) because we want $P(t)$ to represent a real function $f(t)$. Using Euler's relation for the exponentials ($e^{jn\omega t}$ and $e^{-jn\omega t}$) in the preceding expression,

$$\begin{aligned} P(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\frac{1}{2}\underbrace{(a_n - jb_n)(\cos(n\omega t) + j\sin(n\omega t))}_{I} + \\ \sum_{n=1}^{\infty}\frac{1}{2}\underbrace{(a_n + jb_n)(\cos(n\omega t) - j\sin(n\omega t))}_{II} \end{aligned} \tag{5.27}$$

Evaluating the results for parts I and II in Equation (5.27),

$$\begin{aligned} I &\rightarrow \ a_n\cos(n\omega t) + ja_n\sin(n\omega t) - jb_n\cos(n\omega t) - j^2b_n\sin(n\omega t) \\ II &\rightarrow \ a_n\cos(n\omega t) - ja_n\sin(n\omega t) + jb_n\cos(n\omega t) - j^2b_n\sin(n\omega t) \\ I + II &\rightarrow \ 2a_n\cos(n\omega t) + 0 + 0 - 2j^2b_n\sin(n\omega t) \end{aligned}$$

As can be seen in the addition of $I + II$, the complex terms in the products under both the Σ operations cancel (also using $j^2 = -1$), and the final result becomes

$$\boxed{P(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]} \tag{5.28}$$

which is the same as the formula for a real Fourier series in Equation (5.1) (i.e., the complex series is equal to the real series described earlier).

5.4 EXAMPLES

In this section, we apply Fourier series to analyze time domain signals. In the first example, we apply the real series from Equation (5.1) to describe a ***triangular wave*** with period T and amplitude A shown in Figure 5.5. From inspection of the waveform, we can see directly that

1. a DC component is absent, therefore $a_0 = 0$, and
2. the time series is even (Appendix 5.2), that is, the sinusoidal odd components in the Fourier series are absent, therefore $b_n = 0$.

From these observations, we conclude that we only have to calculate the a_n coefficients of Equation (5.1) to obtain the representation of the real Fourier series. We can calculate these coefficients by integration over a full period from $-T/2$ to $T/2$. To avoid trying to integrate over the discontinuity at $t = 0$, we can break the function up into two components where we observe (Fig. 5.5) that

3. $A(t) = A + 4A\dfrac{t}{T}$ for $-T/2 \le t \le 0$, and
4. $A(t) = A - 4A\dfrac{t}{T}$ for $0 \le t \le T/2$

Applying Equation (5.12) for a_n and integrating over the interval $-T/2$ to $T/2$ in these two domains produces

$$a_n = \frac{2}{T}\int_{-T/2}^{0}\left(A + 4A\frac{t}{T}\right)\cos(n\omega t)dt + \frac{2}{T}\int_{0}^{T/2}\left(A - 4A\frac{t}{T}\right)\cos(n\omega t)dt \quad (5.29)$$

In this equation, we can separate the terms for A and $4At/T$ and pull the constants A, 4, and T out of the integration:

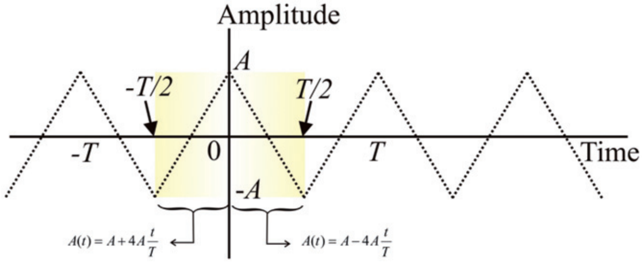

Figure 5.5 An even triangular waveform can be decomposed into a Fourier series that consists solely of cosine terms.

$$a_n = \underbrace{\frac{2A}{T}\int_{-T/2}^{0}\cos(n\omega t)dt + \frac{2A}{T}\int_{0}^{T/2}\cos(n\omega t)dt}_{I} + \underbrace{\frac{8A}{T^2}\int_{-T/2}^{0}t\cos(n\omega t)dt - \frac{8A}{T^2}\int_{0}^{T/2}t\cos(n\omega t)dt}_{II} \tag{5.30}$$

Evaluating part I of Equation (5.30),

$$\underbrace{\frac{2A}{T}\frac{1}{n\omega}[\sin(n\omega t)]_{-T/2}^{0} + \frac{2A}{T}\frac{1}{n\omega}[\sin(n\omega t)]_{0}^{T/2}}_{I} = 0 \tag{5.31}$$

This result is zero because all terms with sin(0) are zero. Further, notice that $\omega = 2\pi f = 2\pi/T$; therefore the term with $\sin(n\omega T/2)$ is equal to $\sin(n\pi) = 0$, and the term with $\sin(n\omega(-T/2))$ is equal to $\sin(-n\pi) = 0$.

The integrals in part II in Equation (5.30) can be evaluated by changing the variable from $t \to n\omega t$ and using integration by parts (Appendix 3.2):

$$\int t\cos(n\omega t)dt = \frac{1}{n^2\omega^2}\int \underbrace{(n\omega t)}_{u}\underbrace{\cos(n\omega t)d(n\omega t)}_{dv}$$
$$\text{With:}\begin{cases} u = n\omega t \to du = d(n\omega t) \\ dv = \cos(n\omega t)d(n\omega t) \to v = \sin(n\omega t) \end{cases} \tag{5.32}$$

Integrating by parts (Appendix 3.2),

$$\int \underbrace{(n\omega t)}_{u}\underbrace{\cos(n\omega t)d(n\omega t)}_{dv} = \underbrace{(n\omega t)\sin(n\omega t)}_{uv} - \int \underbrace{\sin(n\omega t)}_{v}\underbrace{d(n\omega t)}_{du}$$
$$= (n\omega t)\sin(n\omega t) + \cos(n\omega t) + C$$

When applying the integration limits over a full period, the sine terms are again zero (Appendix 5.1). When substituting our results in part II of Equation (5.30) and taking into account that part I is zero, we obtain

$$a_n = \frac{8A}{T^2}\frac{1}{n^2\omega^2}\left([\cos(n\omega t)]_{-T/2}^{0} - [\cos(n\omega t)]_{0}^{T/2}\right) \tag{5.33}$$

Using $\omega = 2\pi f = 2\pi/T$ to simplify this expression:

$$a_n = \frac{2A}{n^2\pi^2}\left(\left[\underbrace{\cos(0)}_{1} - \cos\underbrace{(-n\pi)}_{\substack{-1 \text{ for } n=odd \\ 1 \text{ for } n=even}}\right] - \left[\underbrace{\cos(n\pi)}_{\substack{-1 \text{ for } n=odd \\ 1 \text{ for } n=even}} - \underbrace{\cos(0)}_{1}\right]\right) \tag{5.34}$$

Depending on whether n is odd or even, the coefficients a_n are therefore $8A/(n\pi)^2$ or zero, respectively.

A second example is the ***square waveform*** we approximated with five sine waves in Figure 5.2. From knowledge of this waveform, we can conclude directly that

1. a DC component is absent, therefore $a_0 = 0$, and
2. the time series is odd (Appendix 5.2) — that is, the cosine (even) components in the Fourier series are absent, therefore $a_n = 0$.

We thus conclude that we only have to calculate the b_n coefficients of Equation (5.1) to obtain the expression for the real Fourier series. We can calculate these coefficients by integration over a full period from 0 to T. Again, splitting this period at the discontinuity:

3. $A(t) = A$ for $0 \le t \le T/2$, and
4. $A(t) = -A$ for $T/2 \le t \le T$

We use Equation (5.18) for the calculation of coefficient b_n, integrating over period T in two steps:

$$
\begin{aligned}
b_n &= \frac{2A}{T}\int_0^{T/2} A\sin(n\omega t)dt + \frac{2}{T}\int_{T/2}^{T} -A\sin(n\omega t)dt \\
&= \frac{2}{T}\left[\underbrace{\int_0^{T/2} \sin(n\omega t)dt}_{-\frac{1}{n\omega}[\cos(n\omega t)]_0^{T/2}} - \underbrace{\int_{T/2}^{T} \sin(n\omega t)dt}_{-\frac{1}{n\omega}[\cos(n\omega t)]_{T/2}^{T}}\right]
\end{aligned}
\tag{5.35}
$$

Using the results of this integration with $\omega = 2\pi f = 2\pi/T$, we simplify to

$$
\begin{aligned}
b_n &= \frac{2A}{T}\frac{1}{n\omega}\left([\cos(n\omega t)]_{T/2}^{T} - \left[\cos(n\omega t)_0^{T/2}\right]\right) \\
&= \frac{A}{n\pi}\left(\left[\underbrace{\cos(2\pi n)}_{1} - \underbrace{\cos(\pi n)}_{\substack{-1 \text{ for } n=odd \\ 1 \text{ for } n=even}}\right] - \left[\underbrace{\cos(\pi n)}_{\substack{-1 \text{ for } n=odd \\ 1 \text{ for } n=even}}\right] - \underbrace{\cos(0)}_{1}\right)
\end{aligned}
\tag{5.36}
$$

from which we conclude that the coefficients b_n are $4A/n\pi$ or zero, respectively, for odd or even n. This result is in agreement with the following MATLAB script that creates an approximation of a square wave with amplitude $A = 1$ (Fig. 5.2). In the example, we only use n = 1, 3, 5, 7, 9, resulting in amplitudes of $4/\pi$, $4/(3\ \pi)$, $4/(5\ \pi)$, $4/(7\ \pi)$, and $4/(9\ \pi)$.

The following is a part of the MATLAB program (pr5_1.m) that creates the harmonics; the amplitude coefficients are indicated in bold.

```
s1 = (4/pi)*sin(2*pi*f*t);
% the (4/pi) factor is to get a total amplitude of 1
% define harmonics with odd frequency ratio
s3 = (4/pi)*(1/3) * sin(2*pi*3*f*t);
s5 = (4/pi)*(1/5) * sin(2*pi*5*f*t);
s7 = (4/pi)*(1/7) * sin(2*pi*7*f*t);
s9 = (4/pi)*(1/9) * sin(2*pi*9*f*t);
```

$$b_n = \frac{2}{T}\int_0^{T/2} A\sin(n\omega t)\,dt + \frac{2}{T}\int_{T/2}^{T} -A\sin(n\omega t)\,dt = \frac{2A}{T}\left[\underbrace{\int_0^{T/2}\sin(n\omega t)\,dt}_{-\frac{1}{n\omega}[\cos(n\omega t)]_0^{T/2}} - \underbrace{\int_{T/2}^{T}\sin(n\omega t)\,dt}_{-\frac{1}{n\omega}[\cos(n\omega t)]_{T/2}^{T}}\right]$$

Sine waves s1–s9 correspond to the waves in the left panel of Fig. 5.2; amplitudes 4/π–4/(9π) are depicted in the spectral plot in Fig. 5.2.
If we extend the findings from these two examples to the complex Fourier series, we can conclude the following:

- In even functions, only the a_n coefficients are required. In the complex series approach, this translates into a series with real values only.
- With odd functions, the b_n coefficients are required. If there is no DC component ($a_0 = 0$ in Equation (5.1)), this translates into a series with solely imaginary numbers.
- A function that is neither even nor odd can be composed of even and odd components (Appendix 5.2); this results in a Fourier series that includes both real and imaginary values.

Because of the close relationship between the Fourier transform, introduced in the next Chapter 6, and the complex Fourier series, these conclusions remain relevant for the transform as well.

APPENDIX 5.1

In general, functions f_{mn} that produce a nonzero value only if $m = n$ are called orthogonal functions and they play an important role in signal processing. In this chapter, we derive the Fourier series using this property. More precisely, in this appendix we show that the coefficients a_n and b_n in Equations (5.6), (5.12), and (5.18) can be found because the following functions are orthogonal:

$$\int_T \sin(n\omega t)\sin(m\omega t)dt$$
$$\int_T \cos(n\omega t)\cos(m\omega t)dt \tag{A5.1-1}$$

Another important integral that always (also if $m = n$) evaluates to zero is

$$\int_T \sin(n\omega t)\cos(m\omega t)dt \tag{A5.1-2}$$

In the following, we show that the underlying reason for these properties in Equations (A5.1-1) and (A5.1-2) is that the integral of a sine or cosine wave over a number of periods (NT) with N = 1, 2, 3, . . . evaluates to zero:

$$\int_T \cos(N\omega t) = 0 \quad \text{and} \quad \int_T \sin(N\omega t) = 0 \tag{A5.1-3}$$

From the graphical presentation of a sine or cosine, this is clear without even performing the integration (Fig. A5.1); the areas enclosed by the waveform over one or more period(s) cancel since these functions are symmetric across the $y = 0$ axis.

An additional prerequisite to continue the derivation is the trigonometric functions that equate sine/cosine products to sums:

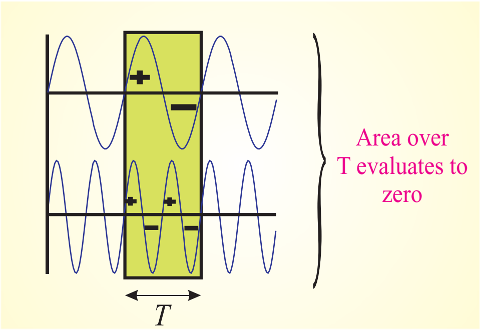

Figure A5.1 The areas enclosed by the positive and negative regions of a cosine or sine wave over at least one full period cancel; therefore, the integral of these trigonometric functions over one or more periods evaluates to zero.

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \qquad \text{(A5.1-4)}$$
$$\cos(A)\sin(B) = \frac{1}{2}[\sin(A-B) - \sin(A+B)]$$

For $A \neq B$, all of the preceding products generate two cosine or sine terms with a frequency $(A - B)$ or $(A + B)$. ***In Equation (A5.1-1), these become functions of ω or harmonics of the base frequency ω.*** For instance, if $A = 5\omega t$ and $B = 3\omega t$, the terms that are generated are the harmonics $2\omega t$ and $8\omega t$. As pointed out in Equation (A5.1-3), integrals of these harmonics over a period of base frequency ω all evaluate to zero — that is,

$$\int_T \frac{1}{2}[\cos(3\omega t) + \cos(8\omega t)]dt = \frac{1}{2}\int_T \cos(3\omega t)dt + \frac{1}{2}\int_T \cos(8\omega t)dt$$
$$= 0 + 0 \qquad \text{(A5.1-5)}$$

Special cases for products of trigonometric identities with $A = B$ can easily be derived from Equation (A5.1-4):

$$\cos(A)\cos(A) = \frac{1}{2}[\cos(0) + \cos(2A)]$$
$$\sin(A)\sin(A) = \frac{1}{2}[\cos(0) - \cos(2A)] \qquad \text{(A5.1-6)}$$
$$\cos(A)\sin(A) = \frac{1}{2}[\sin(0) - \sin(2A)]$$

Using $\cos(0) = 1$, we can conclude that integrals of the first two equations over a period T evaluate to $T/2$, for instance, for the top equation in (A5.1-6):

$$\int_T \frac{1}{2}[1 + \cos(2A)]dt = \frac{1}{2}\int_T 1dt + \frac{1}{2}\int_T \cos(2A)dt = \frac{1}{2}[t]_0^T + 0 = \frac{1}{2}T \qquad \text{(A5.1-7)}$$

Using $\sin(0) = 0$, the last equation in (A5.1-6) evaluates to zero in all cases:

$$\int_T \frac{1}{2}[0 + \sin(2A)]dt = \frac{1}{2}\int_T \sin(2A)dt = 0 \qquad \text{(A5.1-8)}$$

In summary, we conclude from the preceding that the integral (over one or more periods) of sine × sine and cosine × cosine products is an

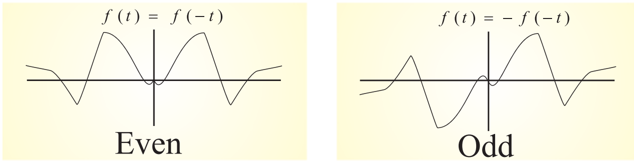

Figure A5.2 Example of an even and an odd function.

orthogonal function, whereas the integral of a sine × cosine product always evaluates to zero:

$$\boxed{\begin{aligned} &\int_T \cos(n\omega t)\cos(m\omega t)dt = \begin{cases} T/2 & \text{for} \quad m = n \\ 0 & otherwise \end{cases} \\ &\int_T \sin(n\omega t)\sin(m\omega t)dt = \begin{cases} T/2 & \text{for} \quad m = n \\ 0 & otherwise \end{cases} \\ &\int_T \sin(n\omega t)\cos(m\omega t)dt = 0 \quad \text{for all } m \text{ and } n \end{aligned}} \tag{A5.1-9}$$

The properties in Equation (A5.1-9) are used in the text to derive the expressions for the coefficients in the real Fourier series.

APPENDIX 5.2

In the development of the Fourier analysis, the distinction between odd and even symmetric functions plays an important role. Here we show that odd and even functions can be used to describe any function $f(t)$.

As Figure A5.2 shows, even functions are symmetrical around the vertical axis ($t = 0$): $f(t) = f(-t)$. An odd function is symmetric by reflection across both axes: $f(t) = -f(-t)$.

$$f_{even} = (f(t) + f(-t))/2 \tag{A5.2-1}$$

It is easily confirmed this is an even function by substituting $-t$, from which follows that $f_{even}(t) = f_{even}(-t)$. The odd component can be created as

$$f_{odd} = (f(t) - f(-t))/2 \tag{A5.2-2}$$

where substituting $-t$ reveals that $f_{odd}(t) = -f_{odd}(-t)$. The original function $f(t)$ can now be considered the sum of these even and odd parts:

$$f(t) = f_{even} + f_{odd} = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2} = \frac{2f(t)}{2} = f(t) \tag{A5.2-3}$$