8

LTI Systems, Convolution, Correlation, and Coherence

8.1 INTRODUCTION

In this chapter we present three important signal processing techniques that are based on linear time invariant (LTI) systems:

- Convolution
- Cross-correlation
- Coherence

The convolution operation allows us to relate an LTI system's input and output in the time domain. A related technique is calculation of cross-correlation between two different signals or between a signal and itself (called autocorrelation). Coherence is a related type of analysis used to correlate components in the frequency domain. The latter has the advantage that frequency-specific correlations can be determined, whereas cross-correlation in the time domain mainly reflects large amplitude components. We will show that techniques in the time domain have equivalents in the frequency domain and vice versa. For instance, convolution in the time domain is equivalent to multiplication in the frequency domain, and multiplication in the frequency domain corresponds to complex convolution in the time domain. The techniques to describe linear systems can be applied to characterize (the linear aspects of) physiological signals and will be applied in later chapters to develop analog and digital filters (Chapters 10 to 13). It is important to realize that the techniques described in this chapter reflect linear relationships and therefore they generally fail when strong nonlinear interactions are involved in a systems dynamics (Chapter 17).

8.2 LINEAR TIME INVARIANT (LTI) SYSTEM

The basic idea of a linear system is that it can be fully characterized by knowledge of its response r to a basic, simple input s (stimulus). If a suitable function is chosen for s, any arbitrary stimulus S can be decomposed into a set of these simple inputs (S = sum of several s). The defining feature of a linear system is that the compound response R associated with stimulus S is simply the sum of all responses to the set of simple ones — that is, the system's total reaction R is equal to the sum of the parts r (where R = sum of several r). From the engineering perspective, the most basic element of any signal is the unit impulse and therefore the most basic response of LTI systems is the unit impulse response. From this unit impulse response, all behavior of the linear system can be derived. The procedure for deriving this behavior is called convolution.

The systems considered in the remainder of this chapter are called linear time invariant (LTI). Following the logic of the preceding paragraph somewhat more rigorously, a system is linear if its output y is linearly related to its input x Fig. 8.1. Linearity implies that the output to a ***scaled*** version of the input $A \times x$ is equal to $A \times y$. Similarly, if input x_1 generates output y_1 and input x_2 generates y_2, the system's response to the combined input $x_1 + x_2$ is simply $y_1 + y_2$. This property (related to scaling) is called ***superposition***. The ***time invariant*** part of the LTI system indicates that the system's response does not depend on time — at different points in time (given the same initial state of the system) such a system's response y to input x is identical: if $x(t) \to y(t)$ then $x(t - \tau) \to y(t - \tau)$. Details of how these scaling and time invariant properties lead to the fourth property of an LTI system, ***convolution*** is further explained in Section 8.3. A system is considered nonlinear if it violates any one of the properties described.

In addition to the LTI constraint, we usually deal with ***causal*** systems (i.e., the output is related to previous or current input only).

Note: If a system only reacts to current input it is ***memoryless*** or ***static***. If a system's response is (also) determined on a previous or future input, it is ***dynamic***. In reality, we usually deal with causal systems whose output depends on previous, but not future, input.

In Figure 8.1, we represent a causal LTI system and show the response of this system to an arbitrary input function and to a unit impulse. The relationship between input $x(t)$ and output $y(t)$ can be described by (a set of) ordinary differential equations (ODEs). A special case of an input-output relationship shown in Figure 8.1 is the system's response h to a unit impulse δ; as we will demonstrate in Section 8.3, the LTI system's ***weighting function*** or ***impulse response function h*** can be used to link any input to its associated output.

Two simple examples of LTI systems are shown in Figure 8.2. The first example is a simple resistor network, which attenuates the input x. The other example is a simplified electrical equivalent circuit for a membrane

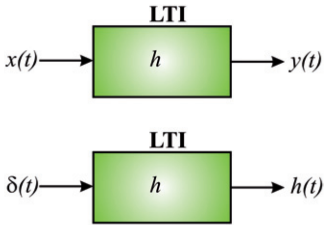

Figure 8.1 LTI system. Input-output relationship. The system's weighting function h and the unit impulse response.

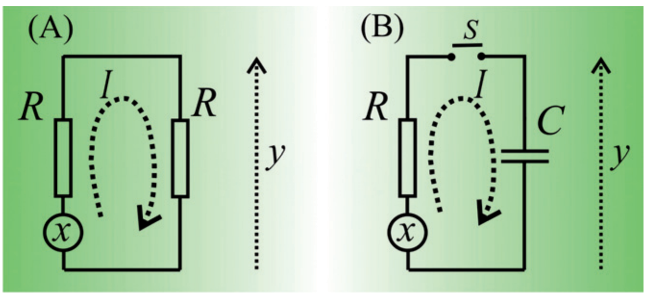

Figure 8.2 Two examples of input-output relationships. (A) A voltage divider consisting of two equal resistors. The potential at x is considered the input and the potential y across the second resistor is defined as the output. According to Kirchhoff's second law, the potentials in the loop must equal zero; in other words, the potential of x equals the potential drop over both resistors. The drop over the right resistor is equal to the output y. Because there are no branches, the current (I) is equal throughout the loop (Kirchhoff's first law). (B) A similar situation where the resistor is replaced by a capacitor can be considered as a simplified passive membrane model. Upon closing switch S, the ion channel with equilibrium potential x and conductivity $g = 1/R$ discharges over the membrane capacitance C causing a change in the membrane potential y. The following table summarizes the circuit's analysis and the input-output relationship.

	Circuit (A)		Circuit (B)
Kirchhoff's second	$x = IR + y$	Kirchhoff's second	$x = IR + y$
Kirchhoff's first and Ohm's law	$I = \frac{y}{R}$	Kirchhoff's first and capacitor	$I = C\frac{dy}{dt}$
Input-Output	$y = \frac{1}{2}x$	**Input-Output**	$y + RC\frac{dy}{dt} = x$

ion channel. The channel is modeled as a battery representing the equilibrium potential of the ion x in series with the channel's conductance $g = 1/R$. The ionic current charges the membrane capacitor C and therefore affects membrane potential y. In the analysis of these systems the relationship between input and output is described mathematically (see input-output in the Table with Fig. 8.2). These types of input-output relationships can all be generalized as

$$\begin{aligned} &A_n \frac{d^n y(t)}{dt^n} + A_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + A_0 y(t) \\ &= B_m \frac{d^m x(t)}{dt^m} + B_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \ldots + B_0 x(t) \end{aligned} \tag{8.1a}$$

for continuous time systems and

$$\begin{aligned} &A_n y(k-n) + A_{n-1} y(k-n+1) + \ldots + A_0 y(k) \\ &= B_m x(k-m) + B_{m-1} x(k-m+1) + \ldots + B_0 x(k) \end{aligned} \tag{8.1b}$$

for a discrete time system.

These equations link output with input in a generic fashion. In both Equations (8.1a) and (8.1b), usually $n > m$.

8.3 CONVOLUTION

8.3.1 Time Domain

8.3.1.1 *Continuous Time*

A key component in the analysis of linear systems is to relate input and output. The ***unit impulse response h(t)*** formalizes this relationship and can be considered the system's ***weighting function***. Furthermore, a mathematical operation defined as ***convolution*** determines the output of the LTI with a known impulse response for any given input. The general idea is that any input function can be decomposed in a sequence of weighted impulses. Here we follow the procedure developed in Chapter 2 and present an arbitrary input function as a series of unit impulses. More specifically, we use the sifting property from Equation (2.8) to represent input $x(t)$ as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{8.2}$$

Note that in Equation (8.2), we have changed the names of variables relative to those used in Equation (2.8). The variable t is substituted for Δ from

Equation (2.8), and τ is substituted for t. Further, we used the fact that δ is an even symmetric function: $\delta(t-\tau) = \delta(\tau - t)$ (in both cases we get $\delta(0)$ for $t = \tau$).

> *Note:* This change in notation from $(\tau - t)$ in Equation (2.8) to $(t - \tau)$ in Equation (8.2) is presented here to allow us in later steps to consider the response of a causal system with responses only for $t \geq \tau$.

Now by writing the LTI system's input as a set of weighted unit impulses, we can determine the output of the system. The system's response to a weighted impulse $x(\tau) \times \delta(t)$ is equal to $x(\tau) \times h(t)$ (***scaling*** by $x(\tau)$); the systems response to $\delta(t-\tau)$ is equal to $h(t-\tau)$ (***time invariance***). Combining both the scaling and time invariance, the response to a single-weighted impulse shifted in time can be characterized as

$$\underbrace{x(\tau)\delta(t-\tau)}_{Input} \rightarrow \underbrace{x(\tau)h(t-\tau)}_{Output} \tag{8.3}$$

Finally we can relate the system's response $y(t)$ to the input $x(t)$ as the sum, taken to the continuous integral limit, of all responses to the weighted impulses (***superposition***):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{8.4}$$

In a graphical representation of Equation (8.4) for each value of $y(t)$, one can consider the product $x(\tau)\ h(t-\tau)$ as the overlap of the two functions $x(\tau)$ and $h(-\tau)$ shifted by an amount equal to t (Appendix 8.1). The convolution integral in (8.4) is easier to interpret if one realizes that the time scale of the input x is represented by τ and that of the output y (or h if we consider the impulse response) by t. In reality, the output y at time t does not depend on the whole input signal x with τ ranging from $-\infty$ to ∞.

- First, we do not know the systems input at $\tau = -\infty$ (since we are not old enough). Therefore we usually bring a system into a rest state and we begin to perturb it with some input at a convenient point in time, which we define as $\tau = 0$. All input that occurs at $-\infty < \tau < 0$ can therefore be considered zero.
- Second, real systems are usually causal and do not respond to future input at $\tau \rightarrow \infty$ — that is, the impulse response h at time t depends only on current and previous input ($\tau \leq t$), meaning that the entire input signal for $\tau > t$ is irrelevant for the response at time t.

Combining these two considerations, we can change the integration limits in Equation (8.4) from $-\infty \to \infty$ to $0 \to t$:

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = x(t) \otimes h(t) \tag{8.5}$$

The $\otimes$ symbol, which we will use throughout this text, is often used to denote convolution. Convolution is commutative (Appendix 8.1), so we can also write $y(t)$ as the convolution of the system's impulse response $h(t)$ with the input $x(t)$:

$$y(t) = h(t) \otimes x(t) = \int_0^t h(\tau)x(t-\tau)d\tau \tag{8.6}$$

An example of the convolution principle is shown in Figure 8.3. The left column in Figure 8.3 shows different combinations of weighted unit

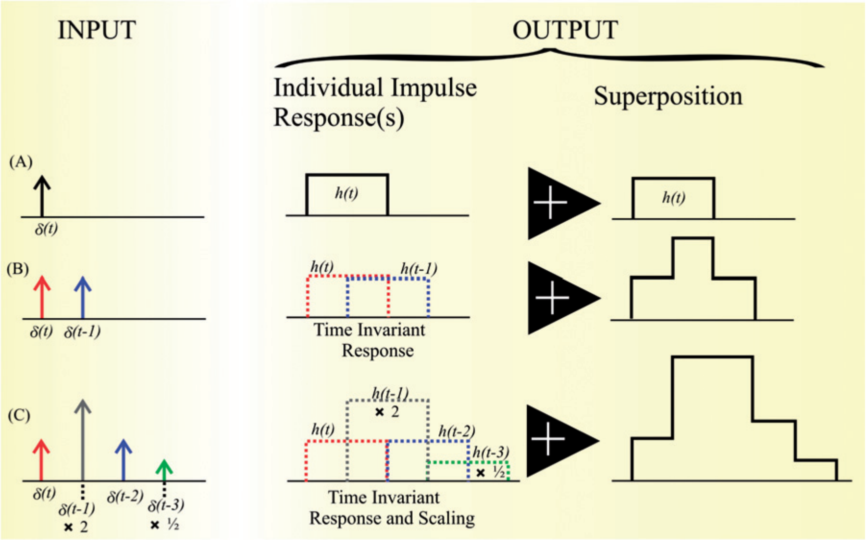

Figure 8.3 An example of the response of an LTI system to impulse functions. The example in (A) shows the system's response $h(t)$ to a single δ function. To keep this example simple, we have (arbitrarily) chosen a simple square pulse as the impulse response. (B) The sequence of two δ functions creates a compound response. This example shows (1) that the response to each δ function is identical and only shifted in time (time invariance) and (2) that the sum of these two responses $h(t)$ and $h(t-1)$ is the system's total response (superposition) to the combined input. (C) A sequence of four δ functions with different weights shows the same time invariance but also the scaling property — that is, $\delta(t-1) \times 2$ generates a response $h(t-1) \times 2$ and $\delta(t-3) \times \frac{1}{2}$ generates a response $h(t-3) \times \frac{1}{2}$. The system's response to the whole sequence is the superposition of all individual reactions. Note that scaling is not a separate property; it can be derived directly from superposition.

impulse functions. The second column depicts the individual unit impulse responses resulting from each of these input impulses. For instance, the $\delta(t-1)$ input generates $h(t-1)$ as output, the $\delta(t-1)\times 2$ input generates $h(t-1)\times 2$ as output, and so on. Finally, the last column in Figure 8.3 shows the superposition of the individual responses, corresponding to the convolution of the input with the impulse response function.

8.3.1.2 Discrete Time

The example in Figure 8.3 shows how one could interpret convolution for discrete events in time (δ functions). Applying the same logic explicitly in discrete time, a system's response can be interpreted in the same manner. Let's consider an example of an LTI system in discrete time using n to index time:

$$y(n) = 0.25x(n) + 0.5x(n-1) + 0.25x(n-2) \tag{8.7}$$

The system's response to a discrete unit impulse (at $n \geq 0$) would be

$$\begin{aligned} n=0 \quad & y(0)=0.25 \\ n=1 \quad & y(1)=0.5 \\ n=2 \quad & y(2)=0.25 \\ n>2 \quad & y(n)=0 \end{aligned} \tag{8.8}$$

Note that the impulse response in Equation (8.8) reproduces the weighting coefficients for $x(n)$, $x(n-1)$, and $x(n-2)$ in Equation (8.7). Given an input series more complex than a unit impulse, we would weight the impulse response with each of the terms from n to $n-2$.

An example of a discrete convolution can be examined with the following MATLAB script:

```
% pr8_1.m
% Discrete Convolution

d=1;                        % unit impulse
h=[.25 .5 .25];             % impulse-response
i=[20 20 20 12 40 20 20];          % input
ii=[20 20 40 12 20 20 20];         % reversed input
x=0:10;                            % x -axis

% Plot Routines
%————————————
figure
subplot(6,1,1),stem(x(1:length(d)),d)
axis([0 7 0 1.5]);
```

```
title(' Unit Impulse')
axis('off')

subplot(6,1,2),stem(x(1:length(h)),h)
axis([0 7 0 1.5]);
title('Impulse-response y=.25x(n)+.5x(n-1)+.25x(n-2)')

subplot(6,1,3),stem(x(1:length(i)),i)
axis([0 7 0 50]);
title(' LTI Input')

subplot(6,1,4),stem(x(1:length(ii)-1),ii(2:length(ii)))
axis([0 7 0 50]);
title(' Reversed LTI Input @ n=5')

subplot(6,1,5),stem(x(1:length(h)),h)
axis([0 7 0 1.5]);
title(' Impulse-response Again')
axis('off');

r5=.25*20+.5*40+.25*12;
subplot(6,1,6),stem(x(1:length(r5)),r5)
axis([0 7 0 50]);
title('Response @n=5 = .25*20+.5*40+.25*12 = 28')
xlabel('Sample #')
```

8.3.2 Frequency Domain

Convolution in the time domain can be a difficult operation, requiring evaluation of the integral in Equation (8.5) or (8.6); fortunately, in the s- or ω-domain convolution of functions can be simplified to a ***multiplication*** of their transformed versions:

$$x_1(t) \otimes x_2(t) \ \leftrightarrow X_1(\omega)X_2(\omega)$$
$$\text{with: } x_1(t) \leftrightarrow X_1(\omega) \quad \text{and} \quad x_2(t) \leftrightarrow X_2(\omega)$$

that is,

$$F\{x_1(t) \otimes x_2(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-j\omega t}dt \tag{8.9}$$

Changing the order of integration,

$$F\{x_1(t) \otimes x_2(t)\} = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dt\right]d\tau \tag{8.10}$$

The expression within the brackets is the Fourier transform of function x_2 shifted by an interval τ. Using $T = t - \tau$ ($\rightarrow t = T + \tau$ and $dt = dT$), this expression can be rewritten as

$$\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x_2(T)e^{-j\omega(T+\tau)}dT = e^{-j\omega\tau}\underbrace{\int_{-\infty}^{\infty} x_2(T)e^{-j\omega T}dT}_{X_2(\omega)} = X_2(\omega)e^{-j\omega\tau}$$

Substituting this result in Equation (8.10) gives

$$F\{x_1(t) \otimes x_2(t)\} = X_2(\omega)\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau = X_1(\omega)X_2(\omega) \qquad (8.11)$$

Expressing Equation (8.11) in English: the Fourier transform of the convolution of x_1 and x_2 (left-hand side) equals the product of the transforms X_1 and X_2 (right-hand side).

8.3.3 Complex Convolution

The Fourier transform of a product of two functions x and y in the time domain is $\int_{-\infty}^{\infty} x(t)y(t)e^{-j\omega t}dt$. Defining the Fourier transforms for x and y as X and Y, we can substitute the inverse of the Fourier transform $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda$ for x and obtain

$$\int_{-\infty}^{\infty} x(t)y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda\right)y(t)e^{-j\omega t}dt$$

Changing the order of integration we can write

$$\begin{aligned}
&= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda\right)y(t)e^{-j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\left(\int_{-\infty}^{\infty} y(t)e^{-j\omega t}e^{j\lambda t}dt\right)d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\underbrace{\left(\int_{-\infty}^{\infty} y(t)e^{-j(\omega-\lambda)t}dt\right)}_{Y(\omega-\lambda)}d\lambda \qquad (8.12) \\
&= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)Y(\omega-\lambda)d\lambda}_{X(\omega)\otimes Y(\omega)} = \frac{1}{2\pi}X(\omega) \otimes Y(\omega)
\end{aligned}$$

The expression $\frac{1}{2\pi}X(\omega) \otimes Y(\omega)$ is called the complex convolution, which is the frequency domain equivalent of the product of two functions in the time domain. We have seen this principle applied in Chapters 2 and 7 when evaluating the effects of sampling and truncation of continuous functions (Sections 2.3 and 7.1.1, and Figs. 2.6 and 7.5).

8.4 AUTOCORRELATION AND CROSS-CORRELATION

8.4.1 Time Domain

8.4.1.1 Continuous Time

Correlation between two time series or between a single time series and itself is used to find dependency between samples and neighboring samples. One could correlate, for instance, a time series with itself by plotting x_n versus x_n; it will be no surprise that this would result in a normalized correlation equal to 1. Formally the autocorrelation R_{xx} of a process x is defined as

$$R_{xx}(t_1, t_2) = E\{x(t_1)x(t_2)\} \tag{8.13}$$

Here the times t_1 and t_2 are arbitrary moments in time, and the autocorrelation demonstrates how a process is correlated with itself at these two different times. If the process is stationary, the underlying distribution is invariant over time and the autocorrelation therefore only depends on the offset $\tau = t_2 - t_1$:

$$R_{xx}(\tau) = E\{x(t)x(t+\tau)\} \tag{8.14}$$

Further, if we have an ergodic process, we may use a time average to define an autocorrelation function over the domain τ indicating a range of temporal offsets:

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)x(t+\tau)dt \quad \text{or} \quad R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \tag{8.15}$$

In some cases where the process at hand is not ergodic or if ergodicity is in doubt, one may use the term *time autocorrelation functions* for the expression in (8.15). These functions can be normalized to a range between –1 and 1 by dividing the end result by the variance of the process.

Applying a similar approach as in the preceding autocorrelation, the cross-correlation $\boldsymbol{R}_{xy}$ between two time series x and y can be defined as

$$R_{xy}(t_1, t_2) = E\{x(t_1)y(t_2)\} \tag{8.16}$$

If the processes are stationary, the underlying distributions are invariant over time and only the difference $\tau = t_2 - t_1$ is relevant:

$$R_{xy}(\tau) = E\{x(t)y(t+\tau)\} \tag{8.17}$$

Assuming ergodicity we can use a time average such that

$$R_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)y(t+\tau)dt \quad \text{or} \quad R_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t+\tau)dt \tag{8.18}$$

Note that the correlation functions as defined earlier may include DC components, if this component is removed we obtain the ***covariance function*** — that is,

$$C_{xx}(t_1, t_2) = E\{[x(t_1) - m(t_1)][x(t_2) - m(t_2)]\} \tag{8.19}$$

As with the Fourier transform (Chapter 6) in Equation (6.4) where we defined the transform in the limit of c_n with the period $T \to \infty$, we can define the correlation integral using Equations (8.15) and (8.18) as a starting point. In this definition (just as in Equation (6.4)), we remove the 1/T factor and obtain

$$z(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt \tag{8.20}$$

In the case where $y = x$ in the preceding integral, $z(\tau)$ represents the autocorrelation function. If the signals are demeaned, the integral in (8.20) is the covariance function.

8.4.1.2 Discrete Time

For a sampled time series x of a stationary and ergodic process, we can define the autocorrelation function R_{xx} in a similar fashion as in continuous time:

$$R_{xx}(n_1, n_2) = E\{x(n_1)x(n_2)\} \to R_{xx}(m) = E\{x(n)x(n+m)\} \tag{8.21}$$

Here the indices n_1, n_2, n, and m indicate samples in the time series. If we replace this expression with a time average,

$$R_{xx}(m) = E\{x(n)x(n+m)\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n)x(n+m) \tag{8.22}$$

Similarly for the cross-correlation function in discrete time, we obtain

$$R_{xy}(m) = E\{x(n)y(n+m)\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n)\,y(n+m) \tag{8.23}$$

In real signals we can maximize the epoch length from $-N$ to N in order to increase the accuracy of our correlation estimate, but it will, of course, always be a finite interval.

8.4.1.3 *Example*

We use Equation (8.7) to generate a time series and estimate the autocorrelation function of y given that x is a random variable with zero mean and a variance equal to one. For convenience here we reiterate the equation with the original numerical values replaced by coefficients a, b, and c:

$$y(n) = ax(n) + bx(n-1) + cx(n-2) \tag{8.24}$$

Because we know the underlying generator (Equation (8.24) and the probability function that characterize the nature of the input x, we can use $E\{y(n)\ y(n+m)\}$ (Equation (8.21)) to analytically determine the autocorrelation function of the time series for different temporal lags.
For lag $m = 0$:

$$E\{y(n)y(n)\} = E\left\{(ax(n) + bx(n-1) + cx(n-2))^2\right\} \tag{8.25}$$

In the evaluation of the preceding expression, the expectation $E\{x(n)\ x(m)\} = 0$ (because input x is a zero mean random variable) for all $n \neq m$, though for equal indices the expectation evaluates to the variance of the random input $E\{x(n)\ x(n)\} = \sigma^2$. Therefore, Equation (8.25) evaluates to

$$E\left\{a^2x(n)^2 + b^2x(n-1)^2 + c^2x(n-2)^2\right\} = \left(a^2 + b^2 + c^2\right)\sigma^2 \tag{8.26}$$

For $m = 1$:

$$\begin{aligned} &E\{y(n)y(n+1)\} \\ &\quad = E\{(ax(n) + bx(n-1) + cx(n-2))(ax(n+1) + bx(n) + cx(n-1))\} \end{aligned} \tag{8.27}$$

Table 8.1 Autocorrelation of $y(n) = 0.25x(n) + 0.5x(n-1) + 0.25x(n-2)$ for Different Lags m

Lag	$E\{y(n)\ y(n+m)\}$	Normalized: divide by $E\{y(n)^2\}$
$m = 0$	6/16 Equation (8.26)	1.00
$m = 1$	4/16 Equation (8.28)	0.67
$m = 2$	1/16 Equation (8.30)	0.17
$m > 2$	0 Equation (8.31)	0.00

Using the same properties as presented earlier (where $E\{x(n)\ x(m)\} = 0$ and $E\{x(n)\ x(n)\} = \sigma^2$), we can simplify Equation (8.27) to

$$(ab + bc)\sigma^2 \tag{8.28}$$

For $m = 2$:

$$E\{y(n)y(n+2)\} \\ = E\{(ax(n) + bx(n-1) + cx(n-2))(ax(n+2) + bx(n+1) + cx(n))\} \tag{8.29}$$

this simplifies to

$$ac\sigma^2 \tag{8.30}$$

For all $m > 2$:

$E\{y(n)\ y(n+m)\}$ evaluates to zero. For instance at $m = 3$, one obtains

$$\begin{aligned} &E\{y(n)y(n+3)\} \\ &= E\{(a\,x(n) + b\,x(n-1) + c\,x(n-2))(a\,x(n+3) + b\,x(n+2) + c\,x(n+1))\} \\ &= 0 \end{aligned} \tag{8.31}$$

For the particular values we used in Equation (8.7) ($a = 0.25$, $b = 0.5$, and $c = 0.25$) and the random process x with zero mean and unit variance, we obtain the autocorrelation values in Table 8.1 (second column). It is common to normalize the autocorrelation to reflect a value of one at zero lag, thereby preserving the mathematical relationship with nontime series statistics where the correlation of a data set with itself is necessarily unitary (third column in Table 8.1).

The outcome of the expectations summarized in Table 8.1 can be validated numerically against a time series produced using Equation (8.7) with a random input. It must be taken into account that this approach will give only estimates of the expected values in Table 8.1 based on the particular output of the random number generator.

The following script is a MATLAB routine to validate these analytically obtained values using a random input to generate time series y:

```
% pr8_2.m
% autocorrelation
clear;
le=10000;
x=randn(le,1);                % input

y(1)=0.25*x(1);
y(2)=0.25*x(2)+0.5*x(1);

for i=3:le;
  y(i)=0.25*x(i)+0.5*x(i-1)+0.25*x(i-2);
end;

tau=-(le-1):(le-1);
c=xcov(y,'coef');             % normalized
                              % autocorrelation

figure;
stem(tau,c);
title('Autocorrelation ');
xlabel('Lag');
ylabel('Correlation (0-1)');
axis([-10 10 -.1 1.1]);
```

8.4.2 Frequency Domain

A similar approach as that discussed for convolution in Section 8.3.2 can be used to relate ***auto- and cross-correlation*** to the power spectrum. In this approach, the correlation function can be denoted as follows:

$$z(t) = \int_{-\infty}^{\infty} x(\tau)h(t+\tau)d\tau \tag{8.32}$$

Note that the names of the delay and time variables are interchanged with respect to Equation (8.20). By doing this, we can express the Fourier transform of $z(t)$ in a manner similar to the procedure used for convolution in the explanation in Section 8.3.2:

$$\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t+\tau)d\tau\right]e^{-j\omega t}dt \tag{8.33}$$

Assuming again that we can change the order of integration,

$$Z(\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t+\tau)e^{-j\omega t}dt\right]d\tau \tag{8.34}$$

The term in between the brackets is the Fourier transform of function h with a time shift τ. It can be shown that such a shift in the time domain corresponds to multiplication by a complex exponential in the frequency domain (similar to the procedure followed in Section 8.3.2) — that is,

$$\int_{-\infty}^{\infty} h(t+\tau)e^{-j\omega t}dt = H(\omega)e^{j\omega\tau} \tag{8.35}$$

Substitution into the equation for $Z(\omega)$ gives

$$Z(\omega) = \int_{-\infty}^{\infty} x(\tau)H(\omega)e^{j\omega t}d\tau = H(\omega)\int_{-\infty}^{\infty} x(\tau)e^{j\omega t}d\tau \tag{8.36}$$

The integral can be decomposed using the Euler identity as

$$\int_{-\infty}^{\infty} x(\tau)e^{j\omega t}d\tau = \int_{-\infty}^{\infty} x(\tau)\cos(\omega\tau)d\tau + j\int_{-\infty}^{\infty} x(\tau)\sin(\omega\tau)d\tau \tag{8.37}$$

while the Fourier transform of $x(\tau)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega t}d\tau = \int_{-\infty}^{\infty} x(\tau)\cos(\omega\tau)d\tau - j\int_{-\infty}^{\infty} x(\tau)\sin(\omega\tau)d\tau \tag{8.38}$$

Comparing these two equations, one can see that the two expressions are complex conjugates, therefore $\int_{-\infty}^{\infty} x(\tau)e^{j\omega r}d\tau = X^*(\omega)$. Using this in the equation for $Z(\omega)$, one obtains

$$Z(\omega) = H(\omega)X^*(\omega) \tag{8.39}$$

The preceding equation finally shows that cross-correlation in the time domain equates to a multiplication of the transform of one function with the complex conjugate of the transform of the other function. If H and X are the same function, Z is the frequency transform of autocorrelation function. Also note that the product of the Fourier transform with its complex conjugate is also the definition of the ***power spectrum*** (the unscaled version, see Equation (7.1)). Therefore, the power spectrum of a function represents the same information as the function's autocorrelation function. The power spectrum of x is by definition a real valued function (i.e., XX^*). The autocorrelation function of x, being the inverse transform of the power spectrum, is therefore an even function (to review these relationships, see examples and concluding remarks in Chapter 5, Section 5.4).

> *Note:* Because the Fourier transform X of a real even signal is also real (without an imaginary component) and even, its complex conjugate X^* equals X. Therefore, convolution (Equation (8.11)) and correlation (Equation (8.39)) are identical for time series that are real and even.

The equivalence of cross-correlation in the frequency domain is an important property that will be used in the evaluation of LTI systems such as linear filters. As an application of this technique, we will show in Section 12.4 that the ratio of the power spectra of the output and input can be used to determine a filter's weighting function. In Sections 14.4 and 14.5 we apply the correlation techniques to spike trains.

8.5 COHERENCE

The ***coherence*** C between two signals x and y is defined as the ***cross-spectrum*** S_{xy} normalized by the power spectra S_{xx} and S_{yy}. To make the coherence, a dimensionless number between 0 and 1, S_{xy} is squared — that is,

$$C(\omega) = \frac{|S_{xy}(\omega)|^2}{S_{xx}(\omega)S_{yy}(\omega)} \tag{8.40}$$

In many applications, the square root of the previous expression is used as the amplitude coherence. Note that S_{xy} in the numerator of this equation will usually be a complex function, whereas S_{xx} and S_{yy} are both real functions. Because we want a real-valued function to express correlation at specific frequencies, we take the magnitude $|S_{xy}|$ of the complex series. If we calculate the normalized cross-spectrum as a complex number for a single frequency and a single trial, the outcome always has magnitude 1 and phase angle ϕ. For instance if we define

$$X(\omega) = a + bj$$

and

$$Y(\omega) = c + dj$$

we can write the expressions in Equation (8.40) as

$$
\begin{aligned}
S_{xx}(\omega) &= X(\omega)X^*(\omega) = (a+bj)(a-bj) = a^2+b^2,\\
S_{yy}(\omega) &= Y(\omega)Y^*(\omega) = (c+dj)(c-dj) = c^2+d^2, \quad \text{and}\\
|S_{xy}(\omega)|^2 &= |X(\omega)Y^*(\omega)|^2 = |(a+bj)(c-dj)|^2 = |(ac+bd)-j(ad-bc)|^2
\end{aligned}
\tag{8.41}
$$

Because S_{xy} is a complex number, the magnitude squared is the sum of the squares of the real and imaginary parts; in this case,

$$
= (ac+bd)^2 + (ad-bc)^2 = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \tag{8.42}
$$

Substituting the results in Equations (8.41) and (8.42) into Equation (8.40) shows that computation of the coherence of an individual epoch always results in one:

$$
C(\omega) = \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{(a^2+b^2)(c^2+d^2)} = 1 \tag{8.43}
$$

In practice, the coherence is typically estimated by averaging over several epochs or frequency bands — that is, the quantity S_{xy} is determined by averaging over n epochs, indicated by $\langle \ldots \rangle_n$ in the following equation:

$$
C(\omega) = \frac{\left|\langle S_{xy}(\omega)\rangle_n\right|^2}{\langle S_{xx}(\omega)\rangle_n \langle S_{yy}(\omega)\rangle_n} \tag{8.44}
$$

Note: The averaging of cross-spectrum S_{xy} occurs before the absolute value is taken. A common beginner's mistake is to average the absolute value in Equation (8.43); in this case, the outcome is always one!

When we determine $C(\omega)$ for a single frequency ω over different samples out of an ensemble, we obtain several vectors on the unit circle, typically with different phase angles for each sample (Fig. 8.4). The magnitude of sum of the individual vectors indicates the degree of coherence, and the resulting phase angle is the ***phase coherence***. It must be noted here that phase coherence must always be judged in conjunction with the magnitude of the vector; if, for example, the sum of the individual vectors is close to zero, indicating a low level of coherence, the associated phase angle has no real meaning.

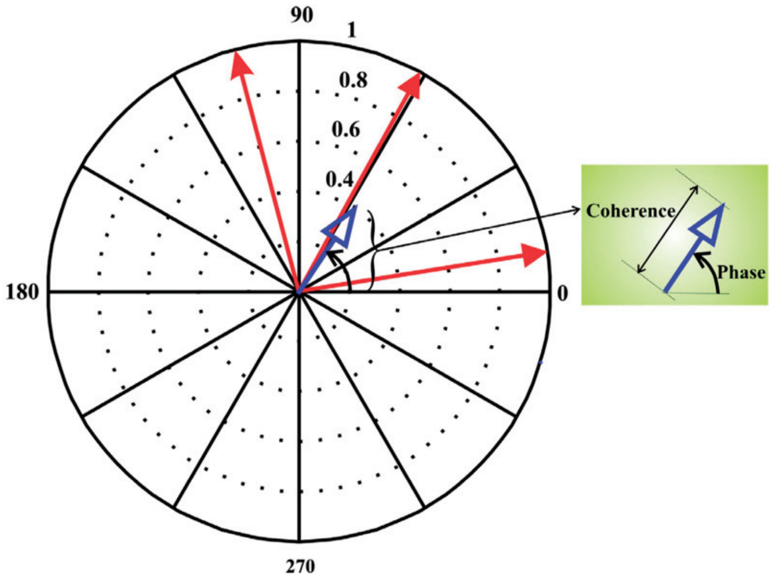

Figure 8.4 Coherence. The complex numbers indicated by red vectors in the complex plane represent different values for the normalized cross-spectrum obtained from different samples out of an ensemble. The blue arrow represents the average of these three numbers. The magnitude of this average is the amplitude coherence (often referred to as simply coherence), and the phase is the phase coherence. From this diagram it can be appreciated that phase coherence only has a meaning if the amplitude has a significant value.

An example of how to determine the coherence can be found in MATLAB file pr8_3.m. Here we calculate the coherence both explicitly from the spectral components and with the standard MATLAB routine cohere***:***

```
% pr8_3
% Coherence Study

clear;
N=8;
SampleRate=10;
t=[0 .1 .2 .3 .4 .5 .6 .7];

% Three Replications of Two Signals x and y
x1=[3 5 -6 2 4 -1 -4 1];
x2=[1 1 -4 5 1 -5 -1 4];
x3=[-1 7 -3 0 2 1 -1 -2];
y1=[-1 4 -2 2 0 0 2 -1];
```

```
y2=[4 3 -9 2 7 0 -5 1];
y3=[-1 9 -4 -1 2 4 -1 -5];
f=SampleRate*(0:N/2)/N;        % Frequency Axis

% Signals Combined
X=[x1 x2 x3];
Y=[y1 y2 y3];
T=[0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2 2.1 2.2 2.3];

cxy=cohere(X, Y, N, SampleRate, boxcar(8));    % direct MATLAB
                                               % command for coherence
                                               % boxcar(8) is a
                                               % Rectangular Window

% FFTs
fx1=fft(x1);
fx2=fft(x2);
fx3=fft(x3);

fy1=fft(y1);
fy2=fft(y2);
fy3=fft(y3);

%Power and Cross Spectra individual trials
Px1x1=fx1.*conj(fx1)/N;
Px2x2=fx2.*conj(fx2)/N;
Px3x3=fx3.*conj(fx3)/N;
MeanPx=mean([Px1x1', Px2x2', Px3x3']');         % Average the Trials

Py1y1=fy1.*conj(fy1)/N;
Py2y2=fy2.*conj(fy2)/N;
Py3y3=fy3.*conj(fy3)/N;
MeanPy=mean([Py1y1', Py2y2', Py3y3']');         % Average the Trials

Px1y1=fx1.*conj(fy1)/N;
Px2y2=fx2.*conj(fy2)/N;
Px3y3=fx3.*conj(fy3)/N;
MeanPxy=mean([Px1y1', Px2y2', Px3y3']');        % Average the Trials

% Calculate the Coherence, the abs command is to get
% the Magnitude of the Complex values in MeanPxy
C=(abs(MeanPxy).^2)./(MeanPx.*MeanPy);
```

```
 % Plot the Results
figure
plot(f,C(1:5),'k');
hold;
plot(f,cxy,'r*');
title(' Coherence Study red* MATLAB routine')
xlabel(' Frequency (Hz)')
ylabel(' Coherence')

figure;
for phi=0:2*pi;polar(phi,1);end;                    % Unit Circle
hold;

% Plot the individual points for the second frequency 2.5 Hz
plot(Px1y1(3)/sqrt((Px1x1(3)*Py1y1(3))),'r*')
plot(Px2y2(3)/sqrt((Px2x2(3)*Py2y2(3))),'r*')
plot(Px3y3(3)/sqrt((Px3x3(3)*Py3y3(3))),'r*')

% Plot the average
plot(MeanPxy(2)/sqrt((MeanPx(2)*MeanPy(2))),'k*');
title(' For individual Frequencies (e.g. here 2.5 Hz) all Three Points
(red) are on the Unit Circle, The Average black =<1')
```

8.5.1 Interpretation of the Coherence Values

The previous examples show that the magnitude r of a single coherence estimate is always 1. The use of the coherence metric therefore only makes sense if the value is determined repeatedly and subsequently averaged. Usually the coherence values are (1) averaged over different frequencies in a frequency band, (2) averaged for a given frequency band for different epochs, or (3) averaged over both frequencies and epochs of the signal.

There are different ways to evaluate statistical significance for coherence figures; in this paragraph, we discuss the simplest version. If we deal with an average of a set of vectors with length 1 and random phases, we can state that $E\{r^2\} = 1$ and $E\{r\} = 0$. As we saw in Chapter 4, the average estimate we obtain will improve as the SEM $= \frac{1}{\sqrt{N}}$, where N is the number of trials in the average. For example, if we average a coherence value over five frequencies in 20 epochs, we have $N = 100$ and a likely error in the estimate of expected value (0) of $\frac{1}{\sqrt{100}} = 0.1$. If we translate this into a 5% significance criterion of two standard deviations (i.e., $2 \times 0.1 = 0.2$), it means that all coherence values greater than 0.2 can be

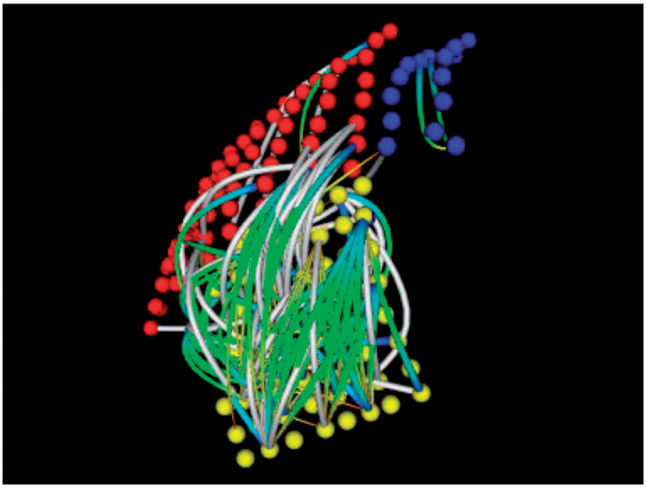

Figure 8.5 An example of coherence calculations associated with subdural electrode arrays implanted over the frontal cortex (red and blue, 1-cm spacing) and temporal cortex (green-yellow, 5-mm spacing) of a patient with medically intractable epilepsy. The colored pipes indicate pairs of electrodes with unusually high coherence between them. White pipes are not associated with a phase shift. Green pipes indicate a phase delay at the blue end of the pipe. These data were obtained as part of the surgical evaluation of the patient, who received a temporal lobectomy for treatment of seizures. (From V.L. Towle with permission.)

considered to deviate significantly from the null hypothesis of a random distribution of values.

8.5.2 Application of Coherence to EEG

An important hypothesis in neuroscience is that connectivity in the brain can be analyzed by determining the temporal relationships between activity patterns in different brain regions (e.g., Shaw, 1981). In studies where propagation of a well-defined temporal feature plays a significant role (such as propagating epileptic spikes), the preferred method is cross-correlation. However, if the relationship is based on similarity between background activity at different locations, the coherence metric is frequently applied (e.g., Towle et al., 1999). An example of a pattern of coherence across brain regions for a frequency band of 0.5 to 4.0 Hz is shown in Figure 8.5. Each dot represents the position of a cortical electrode, and the width of the interconnecting pipes denotes the level of coherence between the signals generated at those electrodes.

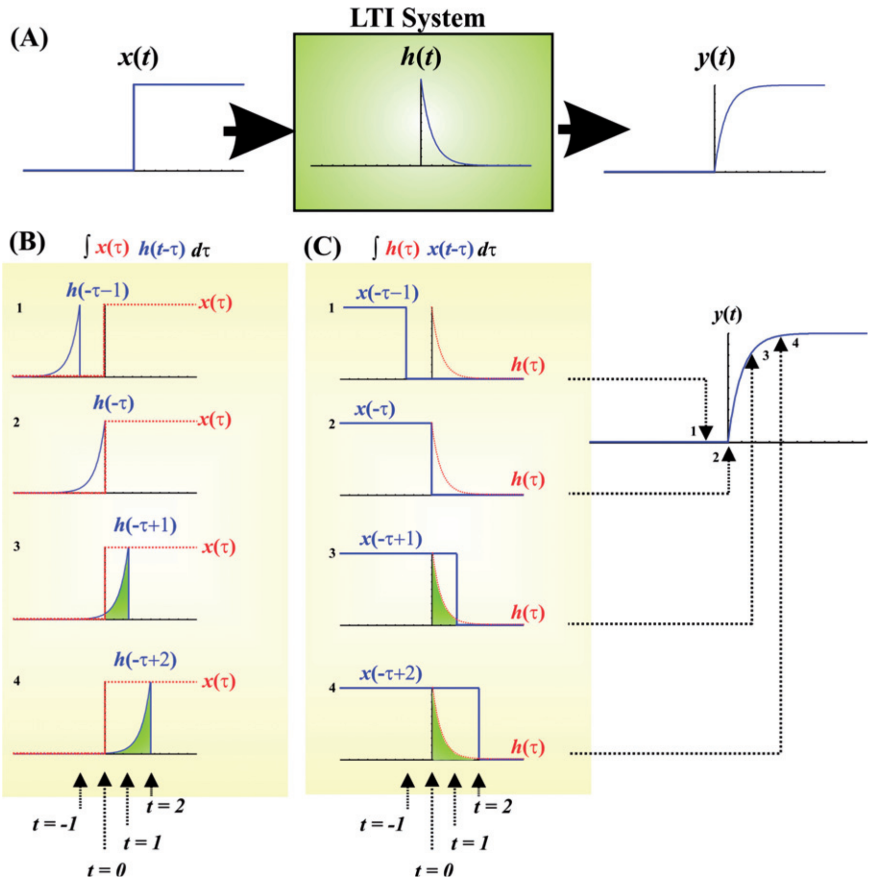

Figure A8.1 Graphical representation of the convolution process used to relate input and output functions of an LTI system. In this example, the input $x(t)$ is the unit step $U(t)$, the impulse response function $h(t)$ is e^{-t} for $t \geq 0$, and the output $y(t)$ is $1 - e^{-t}$ for $t \geq 0$. The convolution operation shifts the inverted impulse function along the input, and the area under the combined functions at time t is the output $y(t)$. It can be seen in (B) that for $t < 0$, there is no overlap and the output y is therefore zero. For $t = 1$ and $t = 2$, there is overlap and the area is indicated in green. This example also shows that for $t = 1$ integration limits can be established between $\tau = 0$ and $\tau = 1$, and for $t = 2$ the limits move from $0 \rightarrow 2$; more generally, the integration limits of the convolution integral required to determine y at time t are $0 \rightarrow t$. Comparing (B) and (C) shows that convolution is commutative.

APPENDIX 8.1

Here we consider an example of the application of convolution to relate the input and output of an LTI system by using its impulse response. In the example shown in Figure A8.1, we use input $x(t) = U(t)$, the unit step function, and impulse response function $h(t) = e^{-t}$. The output $y(t) = 1 - e^{-t}$ can be obtained by ***convolution***: $x(t) \otimes h(t)$ or $h(t) \otimes x(t)$.

The convolution depicted in Figure A8.1B can be obtained by using Equation (8.5):

$$y(t) = \int_0^t U(\tau)h(t-\tau)d\tau = \int_0^t e^{-(t-\tau)}d\tau = e^{-t}\int_0^t e^{\tau}d\tau = e^{-t}\left[e^{\tau}\right]_0^t = e^{-t}\left(e^t - 1\right) = 1 - e^{-t} \tag{A8-1.1}$$

Using Equation (8.6) for the convolution depicted in Figure A8.1C and applying the commutative property, we obtain the same result:

$$y(t) = \int_0^t h(\tau)U(t-\tau)d\tau = \int_0^t e^{-\tau}d\tau = \left[-e^{-\tau}\right]_0^t = 1 - e^{-t} \tag{A8-1.2}$$