16

Wavelet Analysis: Frequency Domain Properties

16.1 INTRODUCTION

In the discussion of digital filters (Chapter 13), we found that smoothing a signal by applying a window such as $y(n) = [x(n) + x(n-1)]/2$ in the time domain has its equivalent in the frequency domain; in this example, the smoothing window has a low-pass filter characteristic. This is precisely what wavelet and scaling signals do: they provide a time domain window of a particular shape, which is then translated over the signal and multiplied with the signal values (e.g., in Chapter 15 see Equations (15.4) and (15.5)). The spectral composition of scaling and wavelet signals is complementary. This becomes clear when comparing the level-1 Haar scaling signal and the level-1 Haar wavelet:

1. The Haar scaling signal produces the weighted average of the time domain signal, which contains the lower frequency components.
2. The Haar wavelet produces a weighted difference signal containing the higher frequency fluctuations.

In this example, the scaling signal acts as a low-pass filter and the wavelet acts as a band-pass filter which allows the higher-frequency components to persist in the fluctuation signal. As would be expected from our understanding of Fourier analysis a comparison of different level wavelets demonstrates that smaller scale (less dilated) wavelets have higher-frequency components than the large scale ones. The scaling signal and wavelet correlations with an input signal are therefore complementary, emphasizing the low- and high-frequency components, respectively.

16.2 THE CONTINUOUS WAVELET TRANSFORM (CWT)

The frequency domain properties of wavelets can be used to design a filter bank (see also Chapter 13, Section 13.7) where each wavelet at progressively greater scales (more dilated) passes a narrow, lower band

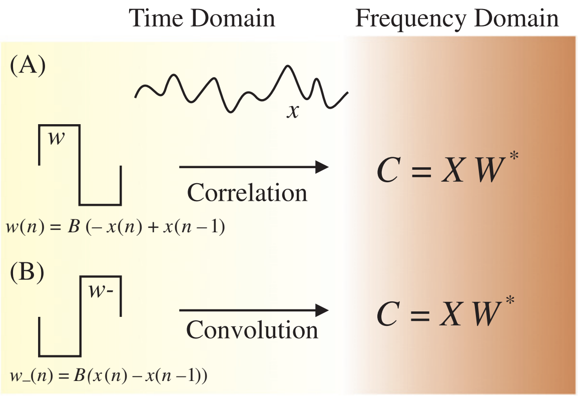

Figure 16.1 (A) Correlation of a signal x with a wavelet w is equivalent to convolution of x with the wavelet's reversed version w-, shown in (B). In both cases the frequency domain operation is the product of the Fourier transform of the input X with the complex conjugate of the wavelet W^*.

of frequencies from input signal x. The time domain procedure of the wavelet transform is to use wavelets w of different scales σ, move (translate) them over an interval τ along an input signal, and correlate the wavelet with the input at each of these scales and translations (Fig. 16.1A); for each τ and σ, the correlation c is given by

$$c(\sigma,\tau)=\int_{-\infty}^{\infty} x(t)\frac{1}{\sqrt{\sigma}}w^*\left(\frac{t-\tau}{\sigma}\right)dt \tag{16.1}$$

The * superscript indicates the complex conjugate of the wavelet; in the ***case of a real wavelet (such as the Haar and Daubechies wavelets), the * can be omitted***. Equation (16.1) represents the so-called continuous wavelet transform (CWT). Contrary to the usual $t + \tau$ (Equation (8.20)), we use $t - \tau$ in Equation (16.1) to emphasize that we translate the wavelet from left to right. The scaling term $\frac{1}{\sqrt{\sigma}}$ is used to preserve signal energy across the transform (compare with Equation (15.22) where the scale is 2^k). In this section, we use the relationship between correlation and convolution in the time and frequency domains; please review Chapter 8 if you need to refresh your memory about these topics. To evaluate the frequency domain properties associated with the time domain procedure in Equation (16.1), we define the following Fourier transform pairs:

$$c(\sigma,\tau)\Leftrightarrow C;\quad x(t)\Leftrightarrow X;\quad \frac{1}{\sqrt{\sigma}}w\left(\frac{t-\tau}{\sigma}\right)\Leftrightarrow W$$

This allows us to write the equivalent of the correlation in the frequency domain as

$$C = XW^* \tag{16.2}$$

where superscript * indicates the complex conjugate (see also Equation (8.39)).

For reasons that will become clear later, it is important to note what happens if we reverse the wavelet signal in the time domain w- in Fig. 16.1B; its frequency representation will change accordingly:

1. for the (cosine) even terms, nothing will change.
2. the (sine) odd terms will change sign (Chapter 5, Appendix 5.2).

If you have difficulty following this reasoning, please review the examples and conclusion in Chapter 5, Section 5.4. Because the sine terms are the complex terms in the Fourier transform, this means that the Fourier transform of the reversed wavelet is the complex conjugate of the Fourier transform of the wavelet:

$$\begin{aligned} &\text{if} && w^{\sigma} = \frac{1}{\sqrt{\sigma}} w\left(\frac{t-\tau}{\sigma}\right) \Leftrightarrow W \\ &\text{then} && w_{-}^{\sigma} = \frac{1}{\sqrt{\sigma}} w\left(-\frac{t-\tau}{\sigma}\right) \Leftrightarrow W^* \end{aligned} \tag{16.3}$$

Note the ***minus sign in the second equation in (16.3)***. From this we may conclude that the correlation of x with a wavelet at a given scale w^{σ} is the same as the convolution with the reversed wavelet w_{-}^{σ}. We may conclude this because the Fourier transform pair for convolution with the reversed wavelet at scale σ is

$$x \otimes w_{-}^{\sigma} \Leftrightarrow XW^* \tag{16.4}$$

From Equations (16.2) and (16.4) we can conclude that ***convolution of the input x with a reversed wavelet*** w_{-}^{σ} results in the identical frequency domain expression as ***correlation c of the signal with the (nonreversed) wavelet*** w^{σ}:

$$\begin{array}{l} x \otimes w_{-}^{\sigma} \Leftrightarrow XW^* = C \\ \quad ||| \\ c(\sigma, \tau) \end{array} \tag{16.5}$$

For even symmetric wavelets (such as the Mexican Hat wavelet, Fig. 16.2), the Fourier transform of the wavelet and its reversed version are necessarily identical; therefore, the correlation and convolution of an even wavelet with an input signal yield identical results.

Compared with the procedure for the wavelet transform discussed in Chapter 15, we change the approach slightly by translating the scaling signal (s) and wavelet (w) over the input (x) signal with one-step increments, instead of jumping in steps of 2 points. We can formalize the Haar scaling signal and wavelet-related operations as follows:

$$s(n) = \frac{1}{\sqrt{2}}(x(n) + x(n-1)) \quad \text{and} \tag{16.6a}$$

$$w(n) = \frac{1}{\sqrt{2}}(-x(n) + x(n-1)), \text{ respectively} \tag{16.6b}$$

Here $s(n)$ is the output of the scaling signal procedure and $w(n)$ is the output of the wavelet operation, both of which can be considered FIR/MA filters. Reversing the wavelet (so that we may use convolution) produces

$$w\_(n) = \frac{1}{\sqrt{2}}(x(n) - x(n-1)) \tag{16.6c}$$

Since the scaling signal is an even symmetric function, $s_-(n) = s(n)$ and no reversal is necessary. The scaling signal and reversed wavelet transfer functions in the z-domain (Chapter 9) $H_s(z)$ and $H_{w-}(z)$ are

$$\begin{aligned} H_s(z) &= \frac{S(z)}{X(z)} = \frac{1}{\sqrt{2}}(1 + z^{-1}) \quad \text{and} \\ H_{w-}(z) &= \frac{W_-(z)}{X(z)} = \frac{1}{\sqrt{2}}(1 - z^{-1}), \text{ respectively.} \end{aligned} \tag{16.7}$$

The frequency response can be obtained by substituting $z = \exp(j\omega T)$ with T being the sample interval (section 13.4). Alternatively, we can make this task very simple by using the MATLAB freqz command to obtain the Bode plots for both the scaling signal and wavelets. We can find the parameters required for this command by writing the signals in the form of Equation (13.2); for the scaling signal, we have the transfer function $\dfrac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}z^{-1}}{1}$. Type in the MATLAB command window:

```
b=[1/sqrt(2) 1/sqrt(2)];
a=[1];
freqz(b,a)
```

This will show the Bode plot for the Haar level-1 scaling signal, a low-pass filter. To evaluate the associated (reversed) wavelet coefficients, we type

```
bb=[1/sqrt(2) -1/sqrt(2)];
aa=[1];
freqz(bb,aa)
```

Note the "–" sign in bb. This will produce a graph representative of a high-pass filter characteristic. This specific finding can be generalized; ***scaling signals have a low-pass filter characteristic, while wavelets pass the higher-frequency components***.

An example for two scales of the Mexican Hat wavelet (MHW) is shown in Figure 16.2. Here we decompose a signal (Fig. 16.2B) that contains a low- and high-frequency component using the MHW at a large (Fig. 16.2A) and a small scale (Fig. 16.2C). In this example, we show the effect of two wavelets only; the procedure where one uses a set of wavelets to filter the signal at different frequency bands is the continuous wavelet transform (CWT) introduced in Equation (16.1); examples of CWT are given in Sections 16.3 and 16.4.

16.3 TIME FREQUENCY RESOLUTION

When performing spectral analysis on a sampled time series, the spectrum reveals frequency components in the input signal. Because the spectrum represents the whole time domain epoch, it is uncertain where exactly any particular frequency component is located in time. To increase resolution, one could reduce the size of the epoch of the input signal. This reduction, however, necessarily changes the resolution of the spectral analysis (Chapter 6, Fig. 6.3 and Chapter 7, Fig. 7.1). To illustrate the time frequency resolution of spectral analysis, let's consider a 10 s epoch sampled at 1000 Hz. This choice of parameters results in a spectrum with a resolution of 1/10 Hz up to the Nyquist frequency of 500 Hz. In this example, a spectral peak of a sinusoidal signal with a frequency of 30.06 Hz would be indicated by energy in the transform mainly between 30 Hz and 30.1 Hz. We cannot determine where this frequency component occurs in time because the 30- to 30.1-Hz component might be present throughout

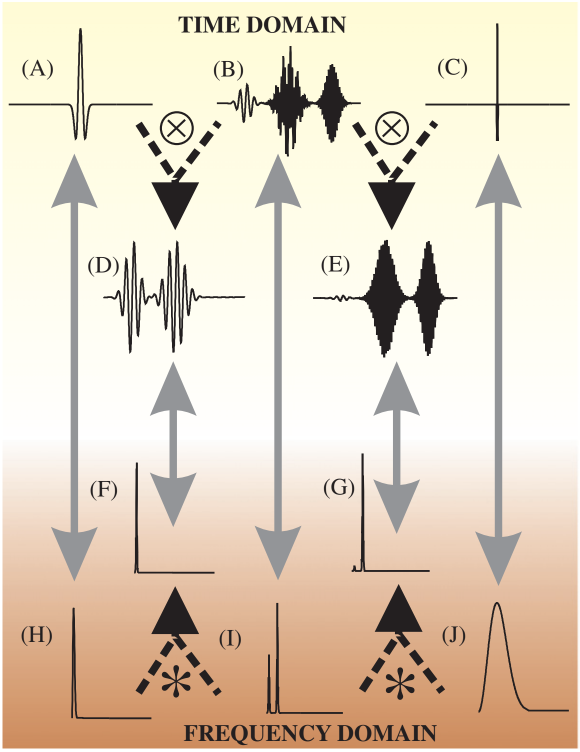

Figure 16.2 Example of using wavelet analysis as a filter bank. In this figure we show two scales of the Mexican Hat wavelet (MHW), a higher scale in (A) and a lower one in (C). (The wavelet transform with the example signal (B) shows the transform with higher-scale wavelet and a lower frequency component (D), whereas the lower scale shows the higher frequencies (E)). The preceding operation in the time domain can also be understood from the equivalent operations in the frequency domain. The transform of the original signal in (B) is shown in (I). It can be seen in (I) that there are two frequency components in the original signal. The higher-scale MHW in (A) has a transform containing low frequencies (H), the transform of the other lower scale wavelet in (C) is shifted to the higher frequencies (J). When using the MHW transforms as the filter characteristic, it can be seen that in one case the lower frequencies are predominant (F) and in the other case the higher frequencies predominate (G). The spectra in (F) and (G) correspond with time domain signals (D) and (E), respectively. The gray arrows indicate the Fourier transform pairs. The ⊗ and * symbols represent convolution and multiplication, respectively. Note the following: (1) The vertical scaling is optimized in each panel (i.e., not the same between panels). In the frequency domain, the amplitude spectra (not the raw Fourier transforms) are shown. (2) Because the MHW is an even symmetric function, convolution and correlation with the input are equivalent. (3) There is an inverse relationship between scale and frequency, as compared with the lower-scale wavelet in (C) and frequency plot (J), the higher-scaled wavelet in (A) is associated with the lower-frequency components (H).

the 10 s epoch, or could be localized in a burst somewhere within the 10 s epoch. We may conclude that the uncertainty of where this particular signal component occurs in time is 10 s and the uncertainty about its frequency value is between 30.0 and 30.1 Hz (i.e., 0.1 Hz). A reduction of the 10 s epoch to a 2 s window would give a 5× more precise (less uncertain) localization of the spectral components in time, because now the spectral components can be located somewhere within a 2 s window. However, this choice of a 2 s epoch results in a $\frac{1}{2}$ Hz frequency domain resolution up to the 500-Hz Nyquist limit. In this case, the energy of the 30.06-Hz component would appear in the spectrum mainly between 30 Hz and 30.5 Hz, increasing the uncertainty about the frequency to 0.5 Hz. The previous examples indicate that for the Fourier-based spectral analysis:

1. The size of the time domain epoch is proportional to the precision with which spectral components can be located in the time domain (i.e., time domain resolution of any of the spectral components equals the size of the selected epoch).
2. The size of the time domain epoch is inversely proportional to the resolution in the frequency domain (i.e., the spectral resolution = 1/epoch).

Therefore, any choice of the epoch length is always associated with a compromise between time and frequency resolution; it is impossible to choose an epoch length that will accommodate both a high temporal and a high spectral resolution. A very high temporal resolution (small epoch) is always associated with a low spectral resolution and vice versa. A low frequency must be detected by the choice of a long epoch, which is okay because low-frequency components are spread over longer epochs. However, having made the choice for a longer epoch of perhaps several seconds, the higher-frequency components (such as the 30.06 Hz-example shown earlier) can now be determined with high precision in the frequency domain but they cannot be precisely localized in time (e.g., the 30.05-Hz component with an intrinsic period only ~33 ms can only be localized within a 10 s or 2 s window in the preceding examples).

Compared to a single Fourier transform-based spectral analysis, continuous wavelet transforms have improved resolution of high-frequency components in the time domain. A low-scale wavelet correlates well with relatively high-frequency components, and the more the scale is stretched (a higher scale), the better the wavelet correlates with the lower-frequency components (Fig. 16.3). As in Equation (15.22), the wavelet scale σ is often expressed as 2^k, and the frequency associated with a particular wavelet scale is proportional to the inverse of the scale. For every subsequent integer value of k, there is a factor 2 difference in the frequency (i.e., in

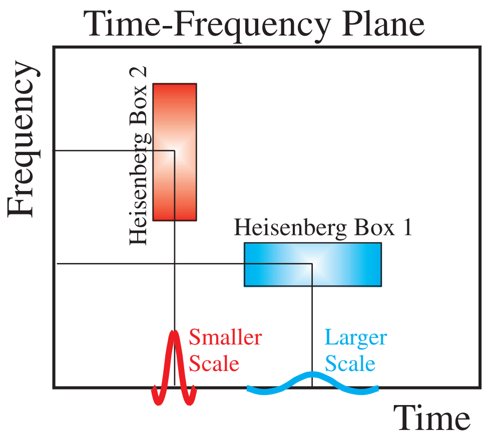

Figure 16.3 The time frequency plane and Heisenberg uncertainty boxes for a low-frequency signal component poorly localized in time but with reasonable spectral resolution (box 1) and for a higher-frequency tone with better time resolution but lower spectral resolution (box 2). This figure also demonstrates the features of the scalogram (Section 16.4) in which low-frequency components (longer period) are detected by larger scale wavelets (blue) and higher-frequency components are detected by smaller scale wavelets (red). This procedure creates a time resolution, which is appropriate for each frequency — that is, long epochs for slow oscillations and shorter ones for faster oscillations. In contrast, the classical Fourier transform-based spectrum has a fixed Heisenberg box for all frequencies determined by its epoch length (see the examples in Section 16.3).

musical terms there is an octave difference). All noninteger values between k and $k+1$, are steps (voices) within the octave. In some applications, the scale is therefore indicated as $2^{n/v}$ with $n = 1, 2, 3, \ldots, n \times v$ and v – number of voices.

An illustration of the uncertainty principle can also be seen in Figure 16.2, where low- and high-frequency components are distributed in different periods of the input signal epoch (Fig. 16.2B). The power spectrum of the entire epoch (Fig. 16.2I) acknowledges the presence of these spectral components without indicating where in the epoch these occur. In contrast, the MHW transform indicates more precisely (with less uncertainty) where each component is located in time (Figs. 16.2D and E). In empirically measured time series, the spectral components are not known a priori, and the simple approach illustrated in Figure 16.2 is not possible because one does not know in advance what scale(s) of wavelet to select. The solution to this problem is to explore a range of scales, similar to the Fourier transform or a filter bank where sets of frequencies are considered.

In the following MATLAB example (pr16_1), we use this approach and calculate a continuous wavelet transform on a signal including three bursts: the first one is a low-frequency burst, the last one is a high-frequency burst, and the middle is a combination of both low and high frequencies (the same signal as in Fig. 16.2B). The CWT with the MHW shows where in time the energy of these frequencies is located with much greater precision than would be possible using a single Fourier transform.

```
% pr16_1.m
% cwt analysis (continuous wavelet transform) using
% CONVOLUTION and CORRELATION
% using a Mexican hat wavelet (MHW)

clear;
msg=('Pls. wait and MAXIMIZE COLOR PLOTS!')
 N=2048;                   % # of points
maxlag=N/2;                % here maxlag is used to zoom in on the
                              correct part of C
C=zeros(128,2*N-1);              % initialize convolution array
CC=zeros(128,2*maxlag+1);        % initialize correlation array

figure
 % Input signal with m from 0 - 1
for n=1:N;
  m=(n-1)/(N-1);
  tg(n)=m;
  g(n)=sin(40*pi*m)*exp(-100*pi*(m-0.2)^2)+(sin(40*pi*m)+2*cos(160*pi*
m))*exp(-50*pi*(m-0.5)^2)+2*sin(160*pi*m)*exp(-100*pi*(m-0.8)^2);
end;

% Mexican Hat, a symmetrical real function
w=1/8;                     % NOTE: standard deviation parm w=1/8
index=1;

for k=0:128;               % Use 8 octaves and 16 voices 8 x 16 = 128
  s=2^(-k/16);             % 16 voices per octave
                     % Note that the scale decreases with k
  for n=1:N;
    % Mexican hat from -½ to ½
    m=(n-1)/(N-1)-1/2;           % time parameter
    if (k == 0)
      tmh(n)=m;            % time axis for the plot
    end;
    mh(n)=2*pi*(1/sqrt(s*w))*(1-2*pi*(m/(s*w))^2)*exp(-pi*(m/
```

```
(s*w))^2);
  end;

  if (k == 0)                          % plot wavelet example

    subplot(2,1,1), plot(tmh,mh,'k'); axis('tight')
    ylabel ('Amplitude');
    title(' Mexican Hat');
  end;

  % save the inverted scales
  scale(index)=1/s;

  % convolution of the wavelet and the signal
  C(index,:)=conv(g,mh);
  % Correlate the wavelet and the signal
  CC(index,:)=xcorr(g,mh,maxlag);

  index=index+1;

end;

% Plot the results
subplot(2,1,2), plot(tg,g,'r'); axis('tight')
xlabel ('Time ');
ylabel ('Amplitude');
title(' Original Signal containing 20 Hz and 80 Hz components(red)');

figure
pcolor(C(:,maxlag:5:2*N-maxlag).^2);
xlabel ('Time (Sample#/5)');
ylabel ('1/Scale#');
ttl=[' Convolution based Scalogram NOTE: Maxima of the CWT are
around the 1/scale #
                    (70) and (38). Ratio = ' num2str(scale(70)/scale(38))];
title(ttl);

figure
pcolor(CC(:,1:5:2*maxlag+1).^2);
xlabel ('Time (Sample#/5)');
ylabel ('1/Scale#');
ttl=[' Correlation based Scalogram NOTE: Maxima of the CWT are
around the scale #
                    (70) and (38). Ratio = ' num2str(scale(70)/scale(38))];
title(ttl);
```

Notes:
1. Because the MHW has even symmetry, the same result is obtained when using a cross-correlation between the wavelet and input signal instead of a convolution
2. To obtain the correct color display, ***maximize*** the figures generated by ***pr16_1.m***.

16.4 MATLAB WAVELET EXAMPLES

In MATLAB, several wavelet toolboxes are available. Here is a short description of the wavemenu command, which launches the graphical user interface for the Wavelet Toolbox. Type: help wavemenu, and the following description is displayed:

```
WAVEMENU Start the Wavelet Toolbox graphical user interface tools.
  WAVEMENU launches a menu for accessing the various
  graphical tools provided in the Wavelet Toolbox.
```


The wavemenu can be used for one-dimensional and two-dimensional wavelet transforms (Fig. 16.4). For two-dimensional analysis, the Lena image from the CD (lena_double.mat) can be loaded. A level-2 Haar transform of Lena generates the example shown in the previous chapter, Figure 15.6. The eeg.mat file included on the disk can be loaded to be used for one-dimensional analysis. The continuous wavelet one-dimensional generates a so-called scalogram, plotting the frequency components of the signal versus time. You can perform this analysis by selecting Continuous Wavelet 1-D in the menu; this will open a second window that will allow you to Load Signal in the File menu. This will open a dialog box that allows you to load input data such as the eeg.mat file. After loading the data, you can analyze the signal and select the wavelet, the range of scales, and the colormap.

Note: In the discussion of this topic, we will frequently use the term *scale*. Please keep in mind that ***scale*** (in the ***scalogram***) is associated with the width (dilation) of the wavelet signal and not with the ***scaling signal*** (*S*) introduced in Equation (15.2) in Chapter 15.

So-called joint time frequency analysis (JTFA, Northrop, 2003) is a broad class of techniques that generates representations of the spectral components from a time series. The most commonly applied procedure depicts the power of a set of frequency bands against time. In principle, these plots can be generated by displaying the graphs of spectral energy

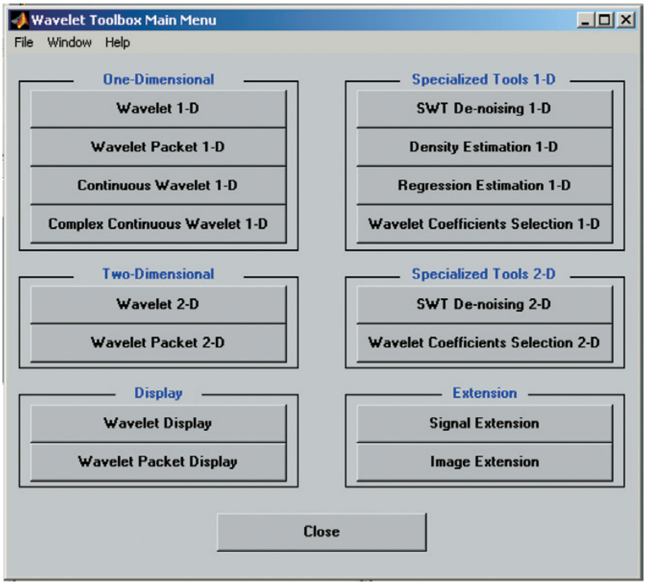

Figure 16.4 The MATLAB menu for wavelet analysis. Figures 16.5B, 15.6, and MRA such as the one shown in Figure 15.4 are a few examples of wavelet analyses that can be accomplished with the menu of this powerful toolbox. Reprinted with permission of The MathWorks, Inc.

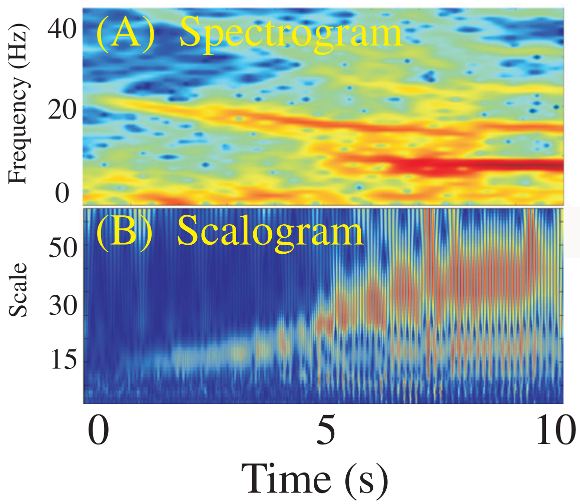

Figure 16.5 A spectrogram (A) and scalogram (B) of an EEG signal during the onset of an epileptic seizure.

in a waterfall format or by displaying spectral energy in a color-coded fashion (Fig. 16.5). Depending on whether the spectral parameters are determined with standard Fourier analysis or with a wavelet approach, the plot is called a ***spectrogram*** or ***scalogram***, respectively; an example of both techniques applied to an EEG epoch recorded during the onset of an epileptic seizure is shown in Figure 16.5. The input signal is clearly non-stationary: The seizure onset presents a drastic transition in the character of the EEG that becomes clearly visible both in the spectrogram (Fig. 16.5A) and the scalogram (Fig. 16.5B). In this example, the scalogram in Figure 16.5B can be compared with the spectrogram in Figure 16.5A. The spectrogram was created by applying a series of windowed Fourier transforms. Each transform is used to generate a power spectrum (or an amplitude spectrum) that then can be color coded. Using this procedure, each spectrum is represented as a colored vertical bar. These bars are then concatenated along a horizontal time scale. The scalogram in Figure 16.5B is produced in the same way as in pr16_1.m. For each scale of the wavelet, a convolution between signal and wavelet is determined. This generates a filtered trace of the input signal. For subsequent scales, the filtered traces are stacked and the amplitude values in the matrix of stacked traces are mapped onto a color code (see MATLAB command pcolor).

By comparing the spectrogram and scalogram in Figure 16.5, it can be seen that the time resolution of the scalogram is superior. Especially at the lower scales (corresponding to higher frequency), the contours are well defined in time; at higher scales, spectral components are less well defined because they correspond with lower frequencies. In the spectrogram, all frequency components are equally blurred in time due to the uncertainty created by the epoch length (Section 16.3).