Project work: A simple isentropic model

In isentropic models, the vertical coordinate is given by the potential temperature,

$$\theta = T\left(\frac{p_{ref}}{p}\right)^{R/c_p}$$

where p_{ref} = 1000 hPa is an (arbitrary) reference pressure, R = 287 J/(K kg) the gas constant for dry air, and c_p = 1004 J/(K kg) the specific heat of dry air at constant pressure. The vertical wind velocity is then defined as

$$\dot{\theta} = \frac{D\theta}{Dt}$$

In adiabatic flow regimes, the potential temperature is conserved,

$$\dot{\theta} = \frac{D\theta}{Dt} = 0.$$

This implies a vanishing vertical wind in isentropic coordinates which reduces the 3-dimensional system to a stack of two-dimensional θ-layers. For this reason, isentropic models have become very popular tools for idealized adiabatic flow problems.

In this project work, an isentropic model for an adiabatic, two-dimensional flow over a mountain ridge

$$ae^{-(\frac{x_i}{b})^2}$$

is programmed (Figure 1). An advective model formulation using centered finite differences will be used. As mentioned in the lecture notes, the equations for each isentropic layer are then formally identical with the shallow water equations.

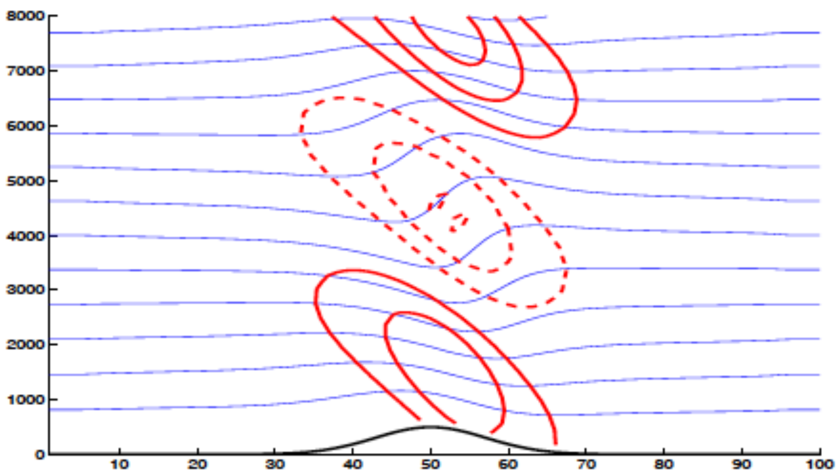


Figure 1: Gravity wave caused by adiabatic flow past a hill. Flow is from left to right. Potential temperature (thin blue lines) and horizontal wind (thick red lines).

Overview of the main equations in the adiabatic case

Simplifying assumptions in the adiabatic case:

Neglect earth's rotation:	f = 0
Adiabatic flow:	$\dot{\theta} = \frac{D\theta}{Dt} = 0$
Two-dimensional flow in (x, z) plane:	$\frac{\partial}{\partial y} = 0, v = 0$
Lower boundary is an isentropic surface:	$\theta(z = z_s) = \theta_s = \text{const}$

It follows a set of equations:

Horizontal momentum equation in *x*-direction:

$$\frac{Du}{Dt} = -\left(\frac{\partial M}{\partial x}\right)_\theta$$

with the simplified advection operator:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_\theta$$

and the Montgomery potential:

$$M = \phi + c_p T = gz + c_p T$$

Two-dimensional equation of continuity:

$$\frac{\partial \sigma}{\partial t} + \left(\frac{\partial \sigma u}{\partial x}\right)_\theta = 0$$

with the isentropic density:

$$\sigma = -\frac{1}{g}\frac{\partial p}{\partial \theta}$$

Hydrostatic relation:

$$\pi = \frac{\partial M}{\partial \theta} \quad \text{with} \quad \pi = c_p\left(\frac{p}{p_{ref}}\right)^{R/c_p}$$

Exercise 1

Derivation of the discretized equations

In this first exercise, the basic equations of the system have to be discretized using centered finite differences in both space and time dimensions. Let us assume a two-dimensional, adiabatic flow and neglect the earth's rotation. For the definition of the variables refer to the lecture notes. The staggering scheme is shown in Figure 2. Later, in Exercise 2, the discretized equations will be implemented into the Python source code.

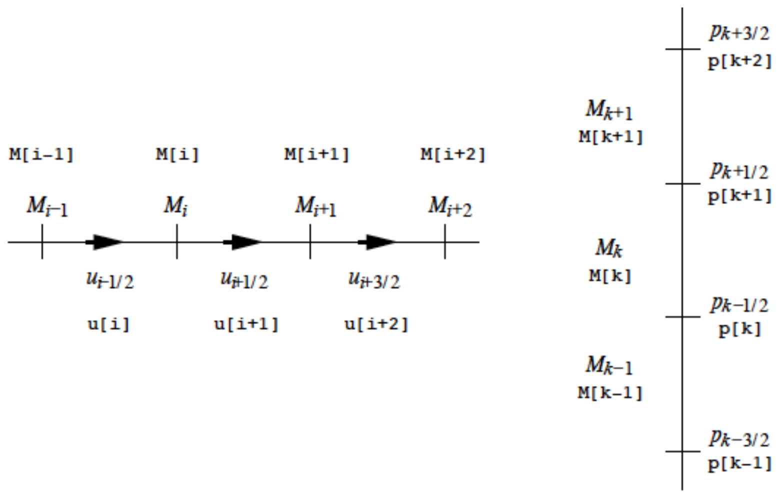


Figure 2: Staggering of the variables in horizontal and vertical dimensions

Main discrete model variables

$u_{i+1/2,k}$	Horizontal wind velocity
$\sigma_{i,k}$	Isentropic density
$M_{i,k}$	Montgomery potential
$p_{i,k+1/2}$	Pressure
$\pi_{i,k+1/2}$	Exner function

1 Prognostic computation of the isentropic density

Discretize the continuity equation using centered finite differences (time and space) and solve for σ^{n+1}. n is the index of the previously computed time step. Note that the velocity is given at staggered grid points and has to be interpolated to the unstaggered grid points.

Write the continuity equation in 4 ways:

As a formal equation

$$\frac{\partial \sigma}{\partial t} + \left(\frac{\partial \sigma u}{\partial x}\right)_\theta = 0,$$

in discretized form

$$\frac{\sigma_{i,k}^{n+1} - \sigma_{i,k}^{n-1}}{2\Delta t} + \cdots$$

solved for $\sigma_{i,k}^{n+1}$:

$$\sigma_{i,k}^{n+1} = \cdots$$

and finally, in Code-like notation, where the staggered variables have to get integer indices according to Figure 2:

$$\sigma^{n+1}(i,k) = \sigma^{n-1}(i,k) - \cdots$$

2 Prognostic computation of the velocity

Discretize the momentum equation and solve it for $u_{i-1/2,k}^{n+1}$. Also write this equation in the four different ways that were described in the last section. In contrast to the velocity, the Montgomery potential is given at (unstaggered) mass points.

3 Boundary conditions

Sections 1 and 2 yield $\sigma_{i,k}^{n+1}$ and $u_{i-1/2,k}^{n+1}$. An instantanous coupling between the vertical layers is given by the hydrostatic equation. The latter, in combination with the isentropic density, facilitates the computation of the next time step's Montgomery potential M^{n+1} out of σ^{n+1} and u^{n+1}.
Let the upper and lower model boundaries be constant isentropic planes at $\theta_{s(urface)}$ and $\theta_{t(op)}$, respectively. Furthermore, we assume a constant pressure

$$p(\theta_t) = p_0(\theta_t),$$

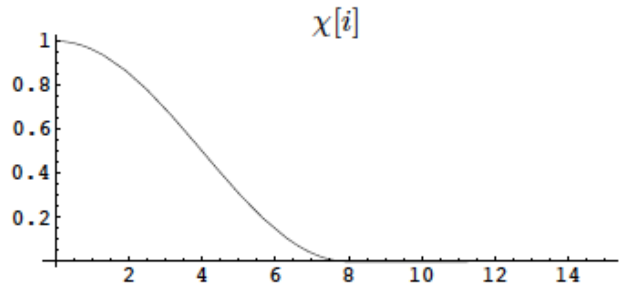
at the upper (horizontal) boundary. p_0 stands for the initial pressure distribution. For the lower boundary, the topography and the geopotential are known,

$$\phi(\theta_s) = g\, z_s.$$

We will start with periodic lateral boundaries and later on switch to relaxed boundary conditions. In the latter case the prognostic fields are relaxed towards their initial state boundary values. That is for any field $\phi(x, z)$:

$$\phi(x,z) = [1 - \chi(x)]\phi(x,z) + \chi(x)\phi_b(z)$$

where $\phi_b = \phi_1$ (left boundary) and $\phi_b = \phi_2$ (right boundary). The weighting function *χ(x)* is zero everywhere except near the boundaries. An example of such a weighting function is shown below for the left boundary. Its shape determines whether an arriving wave is reflected at the boundary or not.



4 Diagnostic computation of pressure

Discretize the isentropic density equation using finite differences, centered at *k*. It is recommended to solve the equation for $p_{i,k-1/2}^{n+1}$ because the equation should be computed from the upper boundary downwards ($p_0(\theta_t) = p_{i,k=nz+\frac{1}{2}}^{n+1}$, where *nz* is a number of vertical layers).

5 Diagnostic Computation of the Montgomery potential

For the computation of the Montgomery potential, let us define the Exner function

$$\pi_{i,k-1/2}^{n+1} = c_p \left(\frac{p_{i,k-1/2}^{n+1}}{p_{ref}} \right)^{R/c_p}$$

Discretize the hydrostatic equation, using centered differences at *k − 1/2*. In contrast to pressure diagnostics, this equation is computed upwards. Therefore, solving the equation for $M_{i,k}^{n+1}$ is recommended.

The lower boundary condition is given by

$$M_{i,k=1/2}^{n+1} = gz_s + \theta_s \pi_{i,k=1/2}^{n+1}$$

This equation has to be modified, because the Montgomery potential is given on the center of a layer (integer index), whereas θ, the lower boundary and the Exner function are given at the layer boundaries. We calculate $M_{i,k=1}^{n+1}$ using the hydrostatic relation ($\pi = \partial M / \partial \theta$)

$$M_{i,k=1}^{n+1} = M_{i,k=1/2}^{n+1} + \frac{\Delta\theta}{2} \pi_{i,k=1/2}^{n+1}$$

Note that $M_{i,k=1/2}^{n+1}$ is calculated above.

6 Initial conditions

Starting with a one-dimensional profile, we choose:

$$u(x, z, t = 0) = u_0$$

The topography is prescribed as

$$ae^{-(\frac{x_i}{b})^2}$$

where *a* is the maximum height of the mountain, *b* its full width at half maximum, *xi = i−nx/2+1* and finally *nx* the number of grid points in *x*-direction. The mountain height is increased from 0 (initialization) to its final height over a short, prescribed time span.

For the definition of the initial vertical stratification, the so-called Brunt-Väisälä frequency is used:

$$N^2 = \frac{g}{\theta} \frac{d\theta}{dz}$$

Its initial profile is uniform, $N(x,z,t = 0) = N_0$. Knowing the Brunt-Väisälä frequency, we can determine the Exner function, the pressure, the Montgomery potential, the isentropic density and the geometric height at time $t = 0$.