Complement Like Operator

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1 Introduction

Given a set X, we can uniquely identify the complement operator $^c : \mathcal{P}(X) \to \mathcal{P}(X)$ using 3 properties:

1. for all $a \subseteq X$ we have $a^c \cap a = \emptyset$
2. for all $a \subseteq X$ we have $a^c \cup a = X$
3. for all $a \subseteq X$ we have $(a^c)^c = a$

We wish to explore "complement like operators", an operator $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 2 out of those 3 properties:

Definition 1.1. $\blacksquare$-complement is an operator $\mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 1 and 2

Definition 1.2. $\bullet$-complement is an operator $\mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 1 and 3

Definition 1.3. $\star$-complement is an operator $\mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 2 and 3

As it turns out, there are no $\blacksquare$ operators, so in fact property 1 and 2 alone are enough to identify the complement operator.

Lemma 1.4. *There are no $\blacksquare$ operator*

Proof. Let * be $\blacksquare$ operator, By property 1 we have $x \in a^* \subseteq X$ implies $x \notin a$, so $a^* \subseteq a^c$.
Similarly by property 2 we have $x \notin a$ implies $x \in a^*$ so $a^c \subseteq a^*$, hence $a^c = a^*$. $\square$

So now we only need to consider $\star$ and $\bullet$ operators. The following theorems will justify only considering one of those 2 operators:

Theorem 1.5. *There is a canonical bijection between the set of all $\bullet$ operators and the set of all $\star$ operators on X.*

Proof. Let * be a $\star$ operator, then ^{c*c} is a $\bullet$ operator.

Indeed $(A^{c*c})^{c*c} = A^{c*cc*c} = A^{c**c} = A^{cc} = A$ and $x \in A^{c*c} \implies x \notin A^{c*} \implies x \notin A^{cc} = A \implies A \cap A^{c*c} = \emptyset$.

Moreover, this transformation is the inverse of itself: $^{c(c*c)c} = ^*$ $\square$

From here on we will only consider $\star$-complement operators.

2 Properties of $\star$-complement operator

One can ask, is there exists a $\star$-complement operator? i.e. does there exists a set X with $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $a^* \cup a = X$ and $a^{**} = a$ for all $a \subseteq X$ and * is not the complement operator?

For a Dedekind infinite X we can construct such operator pretty explicitly.

Definition 2.1. A set X is Dedekind finite set if whenever $Y \subsetneq X$, $|Y| < |X|$.
If X is not Dedekind finite, we call it Dedekind-infinite.

X is Dedekind-infinite if and only if it contains countably infinite subset, the $\Longleftarrow$ direction is trivial, for the other direction, take a bijection $f : X \to X \setminus \{*\}$, and look at the subset $\{f^n(*)\}_{n\in\omega}$ to find an infinite countable subset of X.

Lemma 2.2. *If X is Dedekind finite, then there is a $\star$-complement operator on X*

Proof. Because X contains a countable infinite subset, we can assume $\mathbb{Z} \subseteq X$, for $n \in \mathbb{Z}$ define $A_n = \{x \in \mathbb{Z} \mid x > n\}$ and let $A = \{A_n\}_{n\in\mathbb{Z}}$.

The operator $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by: $B^* = B^c$ for $B, B^c \notin A$, otherwise $A_n^* = A_{n-1}^c$ and $A_n^{c*} = A_{n+1}$.

Clearly $C^{**} = C$ for all $C \subseteq X$, and because $A_{n-1} \subseteq A_n$ we have $A_n^c \subseteq A_{n-1}^c$ so $C^* \cup C = X$. $\square$

From the construction we can notice how $(A, \subsetneq) \cong (\mathbb{Z}, <)$, as we will see soon, this isomorphism is appears in all $\star$-complement operators.

Definition 2.3. $\mathrm{cl}_f(A)$ is the closure of A under f, $\bigcup_{k\in\omega}\{f^k[A]\}$

Definition 2.4. For f a bijection, $\mathrm{clf}_f(A) = \mathrm{cl}_f(A) \cup \mathrm{cl}_{f^{-1}}(A)$

Theorem 2.5. *If * is $\star$-complement operator on X, then for each $a \subseteq X$ we have $clf_{*c}(a) = \{a\}$ or $(clf_{*c}(a), \subsetneq) \cong (\mathbb{Z}, <)$ and there exists at least one $a \subseteq X$ such that the latter holds. In addition, if a is finite or co-finite then $a^* = a^c$.*

Proof. Let ^{*c} be such operator, and assume $\mathrm{clf}_{*c}(a) \neq \{a\}$, because $a^* \supsetneq a^c$ such a exists, if $(a^{*c})^{*c} = a^{*c}$ then $(a^{*c})^* = a^*$ so $a^{*c} = a$, $a^{*c*c} \neq a$ as well because $a^{*c*c} \subsetneq a^{*c} \subsetneq a$, continuing it for both direction will finish the proof of the first part.

Assume that a is finite(resp. co-finite) and $a^* \neq a^c$ then $\mathrm{clf}_{*c}(a)$ has a $\subseteq$-minimum(resp. $\subseteq$-maximum), and hence is not isomorphic to $\mathbb{Z}$, contradiction. $\square$

In fact, $\mathrm{clf}_{*c}(a)$ can also be seen as the equivalence class $[a]_{\sim_*^\star}$ where $a \sim_*^\star b \iff \exists n \in \mathbb{Z}\ (A^{(*c)^n} = B)$, so $\{C \mid \exists a \subseteq X\ (\mathrm{clf}_{*c}(a) = C)\}$ is a partition of $\mathcal{P}(X)$.

Lemma 2.6. *If P is a partition of $\mathcal{P}(X)$ such that if $p \in P$ then either $|p| = 1$ or $(p, \subsetneq) \cong (\mathbb{Z}, <)$, and at least one $p \in P$ is the latter, then there exists $\star$-complement operator, *, such that $\{C \mid \exists a \subseteq X\ (clf_{*c}(a) = C)\} = P$.*

Proof. If $p \in P$ is such that $|p| = 1$ then $a^* = a^c$ for the $a \in p$.

If not, then $p = \{p_k\}_{k\in\mathbb{Z}} = \{\cdots, p_{-1} \subsetneq p_0 \subsetneq p_1 \cdots\}$, and let $p_k^* = p_{k-1}^c$ as we did in the proof of Lemma 2.2. □

We classify all X with a $\star$-complement operator.

Theorem 2.7. *The following are equivalent:*

1. *There exists a $\star$-complement operator on X*
2. *$\mathcal{P}(X)$ has a $\mathbb{Z}$-chain(ordered by $\subsetneq$)*
3. *X is countable union of infinite disjoint sets*
4. *$\mathcal{P}(X)$ is Dedekind infinite*

Proof. (1) $\iff$ (2) is clear by Lemma 2.2 and Theorem 2.5.

(2) $\implies$ (3) let $\{P_j\}_{j\in\mathbb{Z}}$ be a chain of subsets of X, define for $j \in \omega$, $C_j = P_j \setminus \bigcup_{0\le k<j} P_k$, then $\{\bigcup_{j\in\omega} C_{\langle k,j\rangle}\}_{k\in\omega} \cup \{X \setminus \bigcup_{i\in\omega} P_i\}$ is a countable family of infinite disjoint sets whose union is X, where $\langle\cdot,\cdot\rangle : \omega^2 \to \omega$ is a pairing function.

(3) $\implies$ (2), let $\{C_i\}_{i\in\omega}$ be countable family of infinite disjoint sets whose union is X, , reorder it to $\{D_i\}_{i\in\mathbb{Z}}$ and let $P_i = \bigcup_{k<i} D_k$ for each $i \in \mathbb{Z}$.

(3) $\implies$ (4) is trivial and (4) $\implies$ (3) is due to Tarski[1]: Let X be a set such that $\mathcal{P}(X)$ is Dedekind-infinite, then let $(X_i)_{i\in\omega}$ be a sequence of subsets of X, and define the function $F : X \to \mathcal{P}(\omega)$:

Let $a \in X$, then define $F(a)_n$ for $n \in \omega$ recursively: let $F(a)_n$ be the minimal $k \in \omega$ such that $\bigcup_{i\le n} X_{F(a)_i} \subsetneq \bigcup_{i<n} X_{F(a)_i}$ and $a \in X_k$, if not such k exists, let $F(a)_n = F(a)_{n-1}$, let $F(a) = \{F(a)_n\}_{n\in\omega}$.

If $F(a)$ is infinite we can use similar method as in the proof of (2) $\implies$ (3), if $F(a)$ is finite for all a we will note that $a \sim b \iff F(a) = F(b)$ is a equivalence relation, hence the underline equivalence classes are partition which is infinite and with injection to the set of finite subsets of ω, $\mathcal{P}_{<\omega}(\omega)$.

And because $|\mathcal{P}_{<\omega}(\omega)| = \aleph_0$, so does the partition of X, X is a countable union of infinite disjoint sets. □

Definition 2.8. Δ_1-finite set is a set that is not disjoint union of 2 infinite sets

Definition 2.9. Amorphous set is an infinite Δ_1-finite set.

Remark 2.10. If X is amorphous set, then there is no $\star$-complement operator on X, as countably many disjoint subset induce partition of X to infinite sets.

Theorem 2.11. *It is not provable in ZF that there are no amorphous sets.*

Proof. The proof can be found at Lévy[2] theorem 11. □

Corollary 2.12. *It is not provable in ZF that there exists a $\star$-complement operator on every infinite set.*

3 $\star$-strong complement operator

Now that we have shown some properties of $\star$-complement, we can ask "how far" can $\star$-complement be from the complement?

Given an infinite set X with a $\star$-complement operator * and $a \subseteq X$, we know by Theorem 2.5 that if a is finite or co-finite then $a^* = a^c$, can there exists a $\star$-complement where the other direction is also true? i.e. $a^* = a^c$ if and only if a is finite or co-finite?

Definition 3.1. An operator $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is called strong $\star$'-complement if

1. for all $a \subseteq X$ we have $a^c \cup a = X$
2. for all $a \subseteq X$ we have $(a^c)^c = a$
3. for all $a \subseteq X$ we have $a^* = a^c$ if and only if a is finite or co-finite

a strong $\star$'-complement operator is called strong $\star$-complement if it is also a $\star$-complement

Remark 3.2. A strong $\star$'-complement operators on X is a strong $\star$-complement operators if and only if X is not an amorphous set.

By Theorem 2.11, it is consistent with ZF that there exists strong $\star$'-complement operators that are not strong $\star$-complement operators, but interestingly the existence of such operators implies that there exists infinite sets without strong $\star$'-complement operators at all.

Theorem 3.3. *If every infinite set can be equipped with a strong $\star$'-complement operator then every strong $\star$'-complement operator is a strong $\star$-complement operator*

Proof. by Remark 3.2 all we need to prove is that there are no amorphous sets.

Assume the contrary and let X is amorphous, note that $2 \times X$ is an infinite set that is not amorphous (indeed $\{0\} \times X, \{1\} \times X$ is a partition of $2 \times X$ into 2 disjoint infinite sets), let * be strong $\star$'-complement operator on $2 \times X$, by Remark 3.2 this operator is strong $\star$-complement operator, in particular there exists a $\star$-complement operator on $2 \times X$.

by Theorem 2.7 there exists A_0, A_1, A_2 partition of $2 \times X$ into 3 infinite sets, clearly for each $j \in \{0, 1, 2\}$ there exists $i \in \{0, 1\}$ such that $\{i\} \times X \cap A_j$ is infinite, in particular, for some $i \in \{0, 1\}$ we have $j \neq k$ such that $\{i\} \times X \cap A_j$ and $\{i\} \times X \cap A_k$ are disjoint infinite subsets of $\{i\} \times X$, but $|\{i\} \times X| = |X|$, so $\{i\} \times X$ is amorphous, contradiction. $\square$

Similarly to $\star$-complement operator, we can classify the strong $\star$-complement operators using $\mathbb{Z}$-chains.

Theorem 3.4. *The following are equivalent:*

1. *There exists a strong $\star$-complement operator on X*

2. *There exists a partition of the infinite co-infinite subsets of X into $\mathbb{Z}$-chains (ordered by inclusion)*

Proof. (1) $\implies$ (2): let * be the strong $\star$-complement operator. For $a \subseteq X$ infinite co-infinite we have that $\mathrm{clf}_{*^c}(a) \neq \{a\}$ so by Theorem 2.5 $\{[a]_{\sim_*^\star} \mid a \text{ is infinite co-infinite}\}$ is the desired partition.

(2) $\implies$ (1): Let P be partition of $\mathcal{P}(X)$ be the partition of the infinite co-infinite subsets of X.

If $a \subseteq X$ is finite or co-finite, define $a^* = a^c$, otherwise, there exists $\{A_i\}_{i\in\mathbb{Z}} \in P$ such that $a = A_j$, so define $a^* = A$ □

4 Relation to Axiom of Choice

There are several questions we can ask about the $\star$-complement operator and AC:

1. Does the existence of a $\star$-strong complement is provable in ZF?
2. How strong exactly is the axiom "There exists a $\star$-complement operator on the power set of every infinite set"?
3. How strong exactly is the axiom "There exists a $\star$-strong complement operator on the power set of every infinite set"?

Theorem 2.3 is answers (2), that axiom is equivalent to:

$$\forall X \; (X \text{ is infinite} \implies \mathcal{P}(X) \text{ is Dedekind infinite})$$

If we to borrow definitions from Truss[3], then we have the following finiteness definition:

Definition 4.1. A cardinality κ is Δ_4-finite there is no surjective function from it to ω.

And, if we look at $\mathcal{P}(X)$ has a $\star$-complement operator as a finiteness definition, we have that it is equivalent to

$$\omega = \Delta_4$$

easily by theorem 2.3 form (3)

References

[1] Alfred Tarski. Sur les ensembles finis. *Fundamenta Mathematicae*, 6:45–95, 1924.

[2] John Truss. The independence of various definitions of finiteness. *Fundamenta Mathematicae*, 46:1—-13, 1958.

[3] John Truss. Classes of Dedekind Finite Cardinals. *Fundamenta Mathematicae*, 84:187–208, 1974.