Exercise 3

Holo

Saturday 13th May, 2023

Exercise 1. *Nice Names*

Part 1.1.

Let $f : \omega \to 2^{\mathbb{P}}$ be a function defined as $f(n) = \{p \in \mathbb{P} \mid p \Vdash \check{n} \in \sigma\}$ and let A_n be some maximal anti-chain of $f(n)$.

We will see that the nice name $\sigma^* = \bigcup_{n<\omega} \{\check{n}\} \times A_n$ is a nice name such that $0_{\mathbb{P}} \Vdash \sigma = \sigma^*$, or equivalently that $M[❀] \models \sigma_{❀} = \sigma^*_{❀}$ for all generic ideals ❀.

Fix some ❀ and $n \in \omega$ a natural, assume $M[❀] \models n \in \sigma_{❀}$, I claim that $❀ \cap A_n \neq \emptyset$, first remember that $g(n) = \{p \in \mathbb{P} \mid p \Vdash \check{n} \in \sigma \vee p \Vdash \check{n} \notin \sigma\} \supseteq f(n)$ is dense in $\mathbb{P}$, so extend A_n into B_n a maximal anti-chain in $g(n)$, because B_n is maximal anti-chain in a dense set, it is also maximal anti-chain in $\mathbb{P}$.

Let $p \in ❀ \cap B_n$, if $p \notin A_n$ it means that $p \Vdash \check{n} \notin \sigma$ which is false, hence $p \in ❀ \cap A_n$ which means by definition that $M[❀] \models n \in \sigma^*_{❀}$.

The direction of $M[❀] \models n \notin \sigma^*_{❀} \implies M[❀] \models n \notin \sigma_{❀}$ is just the contrapositive of the previous case.

The directions of $M[❀] \models n \in \sigma^*_{❀} \implies M[❀] \models n \in \sigma_{❀}$ and the contrapositive $M[❀] \models n \notin \sigma_{❀} \implies M[❀] \models n \notin \sigma^*_{❀}$ are directly from the definition of σ^*.

Part 1.2.

Let $\mathbb{P} = \mathrm{Add}(\omega, \omega_2)$, and note that $|\aleph_2| \leq |\mathbb{P}| \leq |[\aleph_0 \times \aleph_2 \times 2]^{<\omega}| = |[\aleph_2]^{<\omega}| = \aleph_2$.

Let $\mathcal{A}$ be the set of anti-chains of $\mathbb{P}$, because $\mathbb{P}$ is c.c.c. we have that $\aleph_2 = |\mathbb{P}| \leq |\mathcal{A}| \leq |[\aleph_2]^{\leq\omega}| = |[\aleph_2]^{<\omega} \cup [\aleph_2]^{\omega}| = \aleph_2 + |[\aleph_2]^{\omega}| = |[\aleph_2]^{\omega}| \leq \aleph_2^{\aleph_0} = \left(2^{\aleph_1}\right)^{\aleph_0} = 2^{\aleph_1 \times \aleph_0} = 2^{\aleph_1} = \aleph_2$

Notice that a function that sends $f : \omega \to \mathcal{A}$ to $\bigcup_{n<\omega} \{\check{n}\} \times f(n)$ is a bijection from the nice names to ${}^{\aleph_0}\mathcal{A}$ so the cardinality of the set of nice names is exactly $|\mathcal{A}|^{\aleph_0} = \aleph_2^{\aleph_0} = \aleph_2$

Part 1.3.

Let F be a bijection from the nice $\mathbb{P}$-names of M to $\aleph_2$, because $F, ❀ \in M[❀]$ and $M[❀] \models AC$ we can define inside of $M[❀]$ a function that for each $f \in 2^{\aleph_0}$ chooses some $\sigma \in \mathrm{dom}\,(F)$ such that $\sigma_{❀} = f$ and sends it to $F(\sigma)$, this is an injective function because F is injective and given $a \neq b \in G[❀]$ they are not evaluated from the same $\mathbb{P}$ name.

We have shown in class that $M[❀] \models 2^{\aleph_0} \geq \aleph_2$ and so because $M[❀]$ satisfy Cantor–Bernstein we have $M[❀] \models 2^{\aleph_0} = \aleph_2$.

Exercise 7. *Homogeneous Posets*

Part 7.1.

For 2 partial function p, q let $K(p,q) = \operatorname{dom}(p) \cap \operatorname{dom}(q)$ and $p^{(q)} = p \cup (q \restriction \operatorname{dom}(q) \setminus K(p,q))$.

Notice that $p^{(q)} \geq p$ and that if p, q are comparable then $p^{(q)} = \max(p,q)$.

Now let $\mathbb{P}$ be a poset of partial functions ordered by inclusion and $\pi : \mathbb{P} \to \mathbb{P}$ be a bijection that is bit-wise, that is $\operatorname{dom}(p) = \operatorname{dom}(\pi(p))$ and $\pi(p)(n)$ depends only on $p(n)$ and n for $n \in \operatorname{dom}(p)$, then we have that π is an automorphism.

Indeed let $\tau(n, p(n))$ be $(n, \pi(p)(n))$, because π is a bijection so is τ, and let $p \subseteq q$, if $(a,b) \in p$ then $(a,b) \in q$ so $\tau(a,b) \in \pi(p)$ and $\tau(a,b) \in \pi(q)$, and if $(a,b) \in \pi(p)$ then $\tau^{-1}(a,b) \in p$ so $\tau(\tau^{-1}(a,b)) = (a,b) \in \pi(q)$ so π is order-preserving.

Now fix some $p, q \in \mathbb{P}$ and lets define the automorphism $\pi_p^q : \mathbb{P} \to \mathbb{P}$ that swaps $p^{(q)}$ and $q^{(p)}$ by swapping $(n, p(n))$ with $(n, q(n))$ for all $n \in K(p,q)$ and let it not change any other pair.

Indeed π_p^q is bit-wise, $\pi_p^q(t)(n) = t(n)$ if $n \notin K(p,q)$ or $t(n) \notin \{p(n), q(n)\}$, otherwise if $t(n) = p(n)$ let $p(t)(n) = q(n)$ and vice versa.

To see it swaps $p^{(q)}$ with $q^{(p)}$ notice that $p^{(q)}, q^{(p)}$ have the domain of of $\operatorname{dom}(p) \cup \operatorname{dom}(q)$ and they agree on their domain apart from (maybe) $K(p,q)$, so let $n \in K(p,q)$ and we get that $p^{(q)}(n) = p(n)$, so $\pi_p^q(p^{(q)})(n) = q(n)$ by definition, and vice versa.

Because $\operatorname{Add}(\kappa, 1)$ and $Col(\omega, \lambda)$ are posets of partial functions ordered by inclusion we are done.

Part 7.2.

Let G be any generic we know that $\varphi(\overline{\check{x}_G})$ either holds in $M[G]$ or its negation holds, WLOG assume it holds, and take $p \in G$ such that $p \Vdash \varphi(\overline{\check{x}})$.

Let $q \in \mathbb{P}$ be any element, and let π be an automorphism that sends $r \geq p$ to $\underline{t \geq q}$, because $r \geq p$, it also forces $\varphi(\overline{\check{x}})$ and so from problem (6.2) we have that $t \Vdash \varphi(\overline{\pi(\check{x})})$ and from (6.1) we can conclude that $t \Vdash \varphi(\overline{\check{x}})$, so $\{p \in \mathbb{P} \mid p \Vdash \varphi(\overline{\check{x}})\}$ is dense above $0_{\mathbb{P}}$, hence $0_{\mathbb{P}}$ also forces that.