

# Exercise 1

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## Exercise 1.

### Part 1.1.

Let  $A \subseteq \Omega$  with probability 0, and  $B \subseteq \Omega$  any event with some probability  $\alpha$ .

Let  $B' = B \setminus A$ , then  $\mathbb{P}(A \cup B) = \mathbb{P}(A \cup B') = \mathbb{P}(A) + \mathbb{P}(B') = \mathbb{P}(B')$ .

We also have  $\mathbb{P}(B) = \mathbb{P}((B \cap A) \cup B') = \mathbb{P}(B \cap A) + \mathbb{P}(B')$

From 1.2,  $\mathbb{P}(B \cap A) \leq \mathbb{P}(A) \implies \mathbb{P}(B \cap A) = 0 \implies \mathbb{P}(B) = \mathbb{P}(B')$

### Part 1.2.

Let  $A \subseteq B$ , and let  $B' = B \setminus A$ , then  $\mathbb{P}(B) = \mathbb{P}(A \cup B') = \mathbb{P}(A) + \mathbb{P}(B') \geq \mathbb{P}(A)$  as  $\mathbb{P}(B') \geq 0$

### Part 1.3.

Let  $(\Omega, \mathbb{P})$  be any discrete probability space, let  $\omega \notin \Omega$ , and define  $(\Omega \cup \{\omega\}, \mathbb{P}^*)$  be discrete probability space defined as:  $\mathbb{P}^*(A) = \mathbb{P}(A \setminus \{\omega\})$ .

Clearly this is a probability space ( $\mathbb{P}^*(\Omega \cup \{\omega\}) = \mathbb{P}(\Omega) = 1$ , and given any countable set of disjoint subsets of  $\Omega \cup \{\omega\}$ , at most one of them contains  $\omega$ , removing the flower from this specific set and looking at the  $\sigma$ -additivity of  $\mathbb{P}$  gives the result)

It is also discrete, as if  $p$  is a discrete probability function inducing  $\mathbb{P}$ , then  $p^*$  defined as  $p$  on  $\Omega$  and  $p^*(\omega) = 0$  will induce  $\mathbb{P}^*$ .

In this probability space, let  $A \subseteq \Omega$ , then  $A \subsetneq A \cup \{\omega\}$  but  $\mathbb{P}^*(A) = \mathbb{P}^*(A \cup \{\omega\})$

### Part 1.4.

If  $\mathbb{P}(A \cap B) = \alpha \in [0, 1]$ , the only way for the inequality to fail is for  $\mathbb{P}(A) + \mathbb{P}(B) > 1 + \alpha$

Now let  $A', B'$  defined as in 1.1 and 1.2, then we have  $\mathbb{P}(A) = \mathbb{P}(A') + \alpha \leq 1 \implies \mathbb{P}(A) \leq 1 - \alpha$ , and similarly for  $B$  so  $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A') + \mathbb{P}(B') + 2\alpha > 1 + \alpha \implies \mathbb{P}(A') + \mathbb{P}(B') + \alpha > 1$ , but by the definition  $\alpha = \mathbb{P}(A \cap B)$ , and  $A', B', A \cap B$  are all disjoint, so we get that  $\mathbb{P}(A' \cup B' \cup (A \cap B)) > 1$ , contradiction.

### Part 1.5.

Let  $A', B'$  be as defined in 1.1 and 1.2.

We have  $\mathbb{P}(A) = \mathbb{P}(A') + \mathbb{P}(A \cap B)$  and  $\mathbb{P}(B) = \mathbb{P}(B') + \mathbb{P}(A \cap B)$

Notice that  $B' \cap A' = \emptyset$ , so adding the 2 equations we get  $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A' \cup B') + 2\mathbb{P}(A \cap B) \implies \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B) = \mathbb{P}(A' \cup B')$

But  $A' \cup B'$  is exactly  $A \Delta B$ , so we are done.

### Exercise 2.

Let  $\mathbb{P}$  be a probability function satisfying the conditions in the question.

Because  $\mathbb{N}$  is countable, so every subset of  $\mathbb{N}$ , so  $A = \cup_{n \in A} \{n\}$  is countable union of disjoint sets, hence  $\mathbb{P}(A) = \sum_{n \in A} \mathbb{P}(\{n\})$ , hence it is enough to show that there is a unique discrete probability function  $p$  on  $\mathbb{N}$  satisfying  $p(n) = 3p(n+1)$ .

Notice that given 2 such discrete probability functions that agree on a single number must be equal.

Let  $p(0) = \alpha$ , by definition of discrete probability function we have  $\sum_{n \in \mathbb{N}} \alpha/3^n = \alpha \cdot \sum_{n \in \mathbb{N}} 1/3^n = 1 \implies \alpha = \frac{1}{\sum_{n \in \mathbb{N}} 1/3^n}$ , hence any 2 discrete probability functions satisfying  $p(n) = 3p(n+1)$  must have the same value at 0, but this implies that they are equal.

$\mathbb{P}(\mathbb{N})$  must be 1, as  $\mathbb{P}$  is a probability function, and (assuming  $3\mathbb{N}$  means  $\{3n \mid n \in \mathbb{N}\}$ )  $\mathbb{P}(3\mathbb{N}) = \sum_{n \in \mathbb{N}} \alpha/3^{3n}$

### Exercise 3.

Define  $I_n = \{i \in I \mid a(i) \in [\frac{1}{n+1}, \frac{1}{n}]\}$  (where we treat  $\frac{1}{0}$  as  $+\infty$ ) and  $I = \bigcup I_n = \{i \in I \mid a(i) > 0\}$ .

$I$  is a countable union of sets, so if  $|I| > \aleph_0$ , there must be some  $n \in \mathbb{N}$  such that  $|I_n| \geq \aleph_0$  (I include 0 in  $\mathbb{N}$ )

But if  $J \subseteq I_n$  is finite, then  $\sum_J a(i) \geq \frac{|J|}{n+1}$ , and because  $I_n$  is infinite, we can take  $J$  to be as big as we want.

### Exercise 4.

All of the questions are about countable spaces, so when defining the probability function it is enough to only specify it on the singletons, and from  $\sigma$ -additivity the rest follows.

### Part 4.1.

$\Omega$  will be  $[6]^{10}$  modulo  $\sim$  where  $a \sim b$  if there exists a permutation on  $[6]$  such that  $\tau(a) = b$  (alternatively multisets of size 10 of elements from  $[6]$ ) and define  $\mathbb{P}(\{a\}) = \frac{|a|}{6^{10}}$  (remember that  $a$  is a set of elements in  $[6]^{10}$  that is an equivalent class, so it's size is the amount of ways to get a specific set of results).

Let  $\Omega' = [6]^{10}$  with uniform distribution  $\mathbb{P}'$ .

Now let  $A = \{x \in \Omega \mid 1 \in x\}$  (here we view  $x$  as a multiset), by construction,  $\mathbb{P}(A) = \mathbb{P}'(\bigcup A) = \mathbb{P}'(A')$  where  $A'$  is the set of elements from  $[6]^{10}$  that contains 1 at some point. Looking at  $\mathbb{P}([6]^{10} \setminus A') = (\frac{5}{6})^{10}$  and so  $\mathbb{P}(A) = 1 - \mathbb{P}'([6]^{10} \setminus A') = 1 - (\frac{5}{6})^{10}$

Let  $B = \{x \in \Omega \mid |\{z \in x \mid z = 1\}| = 1\}$  (here we view  $x$  as a multiset), as before this transform into  $B'$  the set of sequences with exactly one 1, and  $\mathbb{P}(B) = \mathbb{P}'(B')$ .

$\mathbb{P}'(B')$  is  $\frac{1}{6} \cdot (\frac{5}{6})^9 \cdot 10$  (the probability of rolling 1 first and then 9 other numbers, times 10 (getting 1 in any position instead of only the first place)).

#### Part 4.2.

$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  (the set of possible sums), and  $\mathbb{P}(\{a\}) = \frac{|\{x \in [6]^2 \mid \pi_0(x) + \pi_1(x) = a\}|}{6^2}$  where  $\pi_i$  is the projection to the  $i^{th}$  index.

We can count  $\mathbb{P}(\{2\}) = \frac{1}{36}, \mathbb{P}(\{3\}) = \frac{2}{36}, \mathbb{P}(\{4\}) = \frac{3}{36}, \mathbb{P}(\{5\}) = \frac{4}{36}, \mathbb{P}(\{6\}) = \frac{5}{36}$  and so  $\mathbb{P}(\{7, 8, 9, 10, 11, 12\}) = 1 - \mathbb{P}(\{2, 3, 4, 5, 6\}) = 1 - (\sum_{i=1}^5 \frac{i}{36})$

#### Part 4.3.

Assuming people's birth month is completely uniform, this probability space is the same as 4.1 but instead of dice with 6 sides, it is dice with 12 sides.

Similarly to 4.1 let  $A = \{x \in \Omega \mid \exists n(x(n) > 1)\}$  (where  $x$  is viewed as a multiset, and  $x(n)$  is the count of appearance of  $n$  in  $x$ ) and let  $A'$  the corresponding event in  $\Omega'$ .

Note that  $\mathbb{P}'([12]^{10} \setminus A')$  is exactly  $\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdots 3}{12^{10}}$ , so  $\mathbb{P}(A) = 1 - \frac{12!}{2 \cdot 12^{10}}$

#### Part 4.4.

Similarly to 4.3, we can view each basket as a label in a die, so the question can be modelled as 4.1 but with dice having 8 dice and rolling 12 dice instead of 10.

Let  $A = \{x \in \Omega \mid \forall n \in [8](n \in x)\}$  where  $x$  viewed as a multiset.

As usual let  $A'$  be the corresponding event in  $\Omega'$ .

Using Inclusion-Exclusion we have that

$$\mathbb{P}'([8]^{12} \setminus A') = \frac{\binom{8}{1}7^{12} - \binom{8}{2}6^{12} + \binom{8}{3}5^{12} - \binom{8}{4}4^{12} + \binom{8}{5}3^{12} - \binom{8}{6}2^{12} + \binom{8}{7}1^{12} - \binom{8}{8}0^{12}}{8^{12}}$$

And  $\mathbb{P}(A) = 1 - \mathbb{P}'([8]^{12} \setminus A')$

#### Part 4.5.

Let  $\Omega = [[13] \times \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}]^4$  (where  $[x]^n$  is the set of  $n$ -sized subset of  $x$ ) with uniform probability ( $\mathbb{P}(\{a\}) = \frac{1}{\binom{52}{4}}$ )

There are  $\binom{13}{4}$  many elements in  $\Omega$  that are all of Diamond suit, so from uniformity  $\mathbb{P}(\{x \in \Omega \mid \forall z \in x(\pi_1(z) = \diamondsuit)\}) = \frac{\binom{13}{4}}{\binom{52}{4}}$  (where  $\pi_i$  is the projection to the  $i^{th}$  index)

#### Part 4.6.

Let  $\Omega = \{M, F\} \times \{x \in \{n \in \mathbb{N} \mid n \neq 0\}^{<\omega} \mid \sum x = 16\}$  with uniform distribution. ( $X^{<\omega}$  denotes the finite sequences whose range is in  $X$ )

The right element each element of  $\Omega$  is indicate how many people of the same gender sit in a row, and the left index indicate if a Male or a Female sits first.

We want to calculate  $\mathbb{P}(\{x \in \Omega \mid \text{range}(\pi_1(x)) = \{1\}\})$  (where  $\pi_i$  is the projection to the  $i^{\text{th}}$  index), clearly  $|\{x \in \Omega \mid \text{range}(\pi_1(x)) = \{1\}\}| = 2$ .

$|\{x \in \{n \in \mathbb{N} \mid n \neq 0\}^{<\omega} \mid \sum x = 16\}|$  is well known to be  $2^{15}$ , hence  $|\Omega| = 2^{16}$  and so  $\mathbb{P}(\{x \in \Omega \mid \text{range}(\pi_1(x)) = \{1\}\}) = \frac{1}{2^{15}}$

#### Part 4.7.

Let  $\Omega = [[n]]^k \times [[n]]^m$  (where  $[x]^p$  is the  $p$ -sized subset of  $x$ ) with uniform probability.

The left index indicate the winning tickets, and the right index indicate the tickets a person got.

We want to calculate the probability of  $A = \{x \in \Omega \mid \pi_0(x) \cap \pi_1(x) \neq \emptyset\}$  where  $\pi_i$  is the projection to the  $i^{\text{th}}$  index.

Now fixing  $w \in [[n]]^k$ , we have  $k \cdot \binom{n-1}{m-1}$  many elements  $b \in [[n]]^m$  such that  $(w, b) \in A$ .

So the size of  $A$  is  $\binom{n}{k} \cdot k \cdot \binom{n-1}{m-1}$ , and  $|\Omega| = \binom{n}{k} \cdot \binom{n}{m}$  so  $\mathbb{P}(A) = k \cdot \frac{\binom{n-1}{m-1}}{\binom{n}{m}}$

#### Exercise 5. Bonus

##### Part 5.1.

If  $X \in \mathcal{F}$  then either  $X$  is finite, hence  $X^c$  is co-finite, or  $X$  is co-finite, hence  $X^c$  is finite.

For finite unions note that either all elements of the unions are finite, and hence the union is finite, or there exists at least 1 cofinite set in there, but any superset of a cofinite set is cofinite, hence the union is in  $\mathcal{F}$  in this case as well.

##### Part 5.2.

$\mathbb{P}(\emptyset) = 0$  because  $|\emptyset| = 0$  and  $\mathbb{P}(\Omega) = 1$  because  $|X| = \aleph_0$  by definition.

Given  $(A_n)_{n < k}$  a finite sequence of disjoint elements from  $\mathcal{F}$ , either all of  $A_n$  are finite, and then  $\bigcup A_n$  is finite and hence additivity present, or exactly one of  $A_n$  is cofinite (as there are no 2 disjoint cofinite subsets of  $\Omega$ ), and then  $\bigcup A_n$  is cofinite, hence has probability 1, and the additivity presented in this case as well.

##### Part 5.3.

WLOG we may assume  $\Omega = \mathbb{N}$  and then we can let  $A_n = \{n\}$ , each  $A_n$  has probability 0, but  $1 = \mathbb{P}(\Omega) = \mathbb{P}(\bigcup A_n) \neq \sum \mathbb{P}(A_n)$