

Exercise 1

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Exercise 2.

Part 2.1.

Let $Z \subseteq \mathbb{P}$ and let D_Z be as in the question, and let $x \in \mathbb{P}$ be any element. If x is incompatible with every element in Z then $x \in D_Z$, otherwise there is $p \in Z$ such that $x \parallel p$, let q be element that strengthening both x, p , because $p \in Z$ we have that $q \in D_Z$, so D_Z is dense.

Assume $x \in D_Z$ and y stronger than x , if there exists $p \in Z$ such that $y \parallel p$ then there is strengthening of both, q , this q is also stronger than x , in particular $x \not\perp p$ and so there exists $r \in Z$ that is weaker than x , in particular it is also weaker than y and hence $y \in D_Z$. Otherwise y is incompatible with every $p \in Z$ hence $p \in D_Z$.

Part 2.2.

Let $A \subseteq \mathbb{P}$ be maximal antichain.

Assume G is generic, from the previous part we know that D_A is dense, hence there is a $p \in D_A \cap G$, but from maximality of A there is no $p \in \mathbb{P}$ that is incompatible with all of A , hence $p \in D_A \cap G \implies p \in \{q \in \mathbb{P} \mid \exists r \in A (r < q)\} \cap G$, but G is closed downwards, so $A \cap G \neq \emptyset$.

Assume G intersects with every maximal antichain A , and let $D \subseteq \mathbb{P}$ be dense set. Let A' a maximal antichain in the restricted order to D (From AC, every poset has maximal antichain).

I claim that A' is a maximal antichain in \mathbb{P} . Indeed it is an antichain because if $p, q \in A'$ are compatible in \mathbb{P} , let r be a witness, and then there is some $r \leq t \in D$ that will witness it in D .

To see it is maximal, let $x \in \mathbb{P}$, because D is dense there exists some $p \in D$ such that $x \leq p$, then either $p \in A'$ or there exists some $q \in A'$ such that $p \parallel q$ from the maximality of A' , in either cases there exists some $y \in A'$ such that $x \parallel y$, in particular A' is maximal antichain in \mathbb{P} as well, so G intersects with A' which is a subset of D , so G intersects with D and hence it is generic.

Exercise 4.

Let \mathbb{P}, \mathcal{D} be as in the question.

Part 4.1.

Take $G, G_{\mathcal{D}}$ as in the question and let $A \subseteq \mathcal{D}$ be a maximal antichain of \mathcal{D} , in 2.2 I have shown that A is a maximal antichain of \mathbb{P} as well, hence from 2.2 we have $G \cap A = G \cap \mathcal{D} \cap A = G_{\mathcal{D}} \cap A \neq \emptyset$ and from 2.2 again we get that $G_{\mathcal{D}}$ is generic in $\cap D$.

Part 4.2.

Take $G, G_{\mathcal{D}}$ as in the question and let $A \subseteq \mathcal{D}$ be any dense set. For each $x \in A$ choose some element $y \in \mathcal{D}$ that is stronger, call the set of all such y as $A' \subseteq \mathcal{D}$. Notice that A' is dense in \mathcal{D} , indeed if $x \in \mathcal{D}$ then there exists some $y \in A$ stronger and hence $r \in A'$ that is stronger than both. Let $p \in G_{\mathcal{D}} \cap A'$, p must be stronger than some $q \in A$ by construction, hence $q \in G \cap A$ which means that G is generic.