Exercise 3

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Exercise 1.

Assume that Ω is star domain at ❀ $\in \mathbb{C}$, $T(z_1, z_2, z_3)^* \subseteq \Omega$, $z^\circ \in \Delta(z_1, z_2, z_3) \setminus T(z_1, z_2, z_3)^*$.

Let L be the line segment that starts at ❀ passes through z° (that is, continuing $I($❀$, z^\circ)$ in the direction of z°).

Because z° is inside of the circle, the line L must intersect (uniquely) with $\partial\Delta(z_1, z_2, z_3) = T(z_1, z_2, z_3)^*$ *after* passing through z° (and maybe one time before that), let $z^\bullet$ be that intersection.

Because Ω is star domain at ❀ we have that $I($❀$, z^\bullet)^* \subseteq \Omega$, but of course we have that $z^\circ \in I($❀$, z^\bullet)^*$.

Exercise 2.

Because γ is a finite sum of continuously differential curves, we can split the integral to finite sum of f over continuously differential curves (connecting at the endpoints), so it is enough to prove the required for γ a continuously differential curve.

We know that $\int_\gamma f(z)dz = \int_0^1 f(\gamma(z))\gamma'(z)dz$.

Because φ is strictly increasing, hence injective and we have that φ' is strictly positive, hence we can plug it in the formula for change of variables.

$$... = \int_0^1 f(\gamma(\varphi(z)))\gamma'(\varphi(z))\varphi'(z)dz = \int_0^1 f(\gamma \circ \varphi(z))(\gamma \circ \varphi)'(z)dz = \int_{\gamma\circ\varphi} f(z)dz$$

Exercise 3.

We defined $L(\gamma)$ the supremum of the length of the possible polygonal chain approximation.

For a given a polygonal chain characterized by the partition $P = [0 = p_0, \ldots, p_n = 1]$ we have $L(\gamma, P) = \sum_{i=0}^{n-1} |\gamma(p_{i+1}) - \gamma(p_i)| \leq \sum_{i=0}^{n-1} K|p_{i+1} - p_i| = K$, hence the supremum of all such approximations is $\leq K$.

Exercise 4.

First we note that $\int_{T(x,y,w)} = \int_{3I(x,y)} + \int_{\frac{1}{3}+3I(y,w)} + \int_{\frac{2}{3}+3I(w,x)}$, where $s + v \times I(a,b)$ is $I(a,b)$ accelerated by v starting at s.

Hence $\int_{T(x,y,z)} \overline{z}dz = \int_{3I(x,y)} f(z)dz + \int_{\frac{1}{3}+3I(y,w)} f(z)dz + \int_{\frac{2}{3}+3I(w,x)} f(z)dz$ is

$$\begin{aligned}
\int_{3I(x,y)} f(z)dz &= \int_0^{\frac{1}{3}} f((3I(x,y))(z))(3I(x,y))'(z)dz \\
\int_{\frac{1}{3}+\frac{3}{I}(y,w)} f(z)dz &= \int_{\frac{1}{3}}^{\frac{2}{3}} f\left(\left(\frac{1}{3}+3I(y,w)\right)(z)\right)\left(\frac{1}{3}+3I(y,w)\right)'(z)dz \\
\int_{\frac{2}{3}+\frac{3}{I}(w,x)} f(z)dz &= \int_{\frac{2}{3}}^{1} f\left(\left(\frac{2}{3}+3I(w,x)\right)(z)\right)\left(\frac{2}{3}+3I(w,x)\right)'(z)dz
\end{aligned}$$

Let $x = 0, y = a, z = ib, f(z) = \overline{z}$, and notice that $(I(0,a))'(t) = a, (I(a,ib))'(t) = ib - a, (I(ib,0))'(t) = -ib$, and the rest is a simple polynomial integration to show that the integral is indeed iab.

Exercise 5.

Part 5.1.

We remember that $\frac{\partial f}{\partial x} = u_x + iv_x, \frac{\partial f}{\partial y} = u_y + iv_y$ so plugging this in the operator:

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}(u_x + iv_x - iu_y + v_y) \\
&= \frac{1}{2}((u_x + v_y) + i(v_x - u_y)) \\
&\overset{\text{CR}}{=} \frac{1}{2}((u_x + u_x) + i(v_x + v_x)) = u_x + iv_x & = f'(z) \\
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}(u_x + iv_x + iu_y - v_y) \\
&= \frac{1}{2}((u_x - v_y) + i(v_x + u_y)) \\
&\overset{\text{CR}}{=} \frac{1}{2}((u_x - u_x) + i(v_x - v_x)) & = 0
\end{aligned}$$

Part 5.2.

Let $g = u' + iv' = u(x,-y) + iv(x,-y)$ Calculating $\frac{\partial g}{\partial x}(x,y) = u'_x + v'_x = u_x(x,-y) + iv_x(x,-y)$ and $\frac{\partial g}{\partial y}(x,y) = u'_y + v'_y = -u_y(x,-y) - iv_x(x,-y) = -(u_y(x,-y) + iv_x(x,-y))$ immediately shows that the 2 operators we define switch roles on g.

Exercise 6.

Part 6.1.

Obviously $f' \equiv 0$ implies that $u' = v' \equiv 0$, from the result on u, v as functions from $\mathbb{R}^2 \to \mathbb{R}$ we get that u, v are constants, but then $f = u + iv$ is also a constant.

Part 6.2.

If $f''\Omega \subseteq \mathbb{R}$, then in particular we have that $v \equiv 0$, hence $v_x = v_y \equiv 0$, but from Cauchy-Riemann we get that $u_x = v_y \equiv 0$ and that $u_y = -v_x \equiv 0$ which implies that $u' = v' \equiv 0$ and so $f' \equiv 0$.

Exercise 7.

Let $\gamma_R : [0,1] \to \mathbb{C} : t \mapsto R\exp(i\pi \cdot t)$ and look at the integral $\int_{\gamma_R} f(z)dz = \int_0^1 f(R\exp(i\pi \cdot t))i\pi R\exp(i\pi \cdot t)dt$

Setting $f(z) = \exp(iz)/z^2$ we get

$$\int_{\gamma_R} \exp(iz)/z^2 dz = \int_0^1 \frac{\exp(i(R\exp(i\pi \cdot t)))}{(R\exp(i\pi \cdot t))^2} i\pi R\exp(i\pi \cdot t)dt = \frac{i\pi}{R}\int_0^1 \frac{\exp(i(R\exp(i\pi \cdot t)))}{\exp(i\pi \cdot t)}dt$$

Noticing that $|\exp(i(R\exp(i\pi \cdot t)))|$ is bounded by M that is independent of R, so $\left|\int_{\gamma_R} \exp(iz)/z^2 dz\right| \leq \left|\frac{i\pi M}{R}\int_0^1 \frac{1}{\exp(i\pi \cdot t)}dt\right|$ which obviously goes to 0 as R grows.