# Exercise 1

Holo

Thursday $27^{\text{th}}$ April, 2023

**Exercise 2.**

**Part 2.1.**

Let $Z \subseteq \mathbb{P}$ and let $D_Z$ be as in the question, and let $x \in \mathbb{P}$ be any element. If $x$ is incompatible with every element in $Z$ then $x \in D_Z$, otherwise there is $p \in Z$ such that $x \| p$, let $q$ be element that strengthening both $x, p$, because $p \in Z$ we have that $q \in D_Z$, so $D_Z$ is dense.
Assume $x \in D_Z$ and $y$ stronger than $x$, if there exists $p \in Z$ such that $y \| p$ then there is strengthening of both, $q$, this $q$ is also stronger than $x$, in particular $x \not\perp p$ and so there exists $r \in Z$ that is weaker than $x$, in particular it is also weaker than $y$ and hence $y \in D_Z$. Otherwise $y$ is incompatible with every $p \in Z$ hence $p \in D_Z$.

**Part 2.2.**

Let $A \subseteq \mathbb{P}$ be maximal antichain.
Assume $G$ is generic, from the previous part we know that $D_A$ is dense, hence there is a $p \in D_A \cap G$, but from maximality of $A$ there is no $p \in \mathbb{P}$ that is incompatible with all of $A$, hence $p \in D_A \cap G \implies p \in \{q \in \mathbb{P} \mid \exists r \in A(r < q)\} \cap G$, but $G$ is closed downwards, so $A \cap G \neq \emptyset$.
Assume $G$ intersects with every maximal antichain $A$, and let $D \subseteq \mathbb{P}$ be dense set. Let $A'$ a maximal antichain in the restricted order to $D$ (From AC, every poset has maximal antichain).
I claim that $A'$ is a maximal antichain in $\mathbb{P}$. Indeed it is an antichain because if $p, q \in A'$ are compatible in $\mathbb{P}$, let $r$ be a witness, and then there is some $r \leq t \in D$ that will witness it in $D$.
To see it is maximal, let $x \in \mathbb{P}$, because $D$ is dense there exists some $p \in D$ such that $x \leq p$, then either $p \in A'$ or there exists some $q \in A'$ such that $p \| q$ from the maximality of $A'$, in either cases there exists some $y \in A'$ such that $x \| y$, in particular $A'$ is maximal antichain in $\mathbb{P}$ as well, so $G$ intersects with $A'$ which is a subset of $D$, so $G$ intersects with $D$ and hence it is generic.

**Exercise 4.**

Let Let $\mathbb{P}, \mathcal{D}$ be as in the question.

**Part 4.1.**

Take $G, G_{\mathcal{D}}$ as in the question and let $A \subseteq \mathcal{D}$ be a maximal antichain of $\mathcal{D}$, in 2.2 I have shown that $A$ is a maximal antichain of $\mathbb{P}$ as well, hence from 2.2 we have $G \cap A = G \cap \mathcal{D} \cap A = G_{\mathcal{D}} \cap A \neq \emptyset$ and from 2.2 again we get that $G_{\mathcal{D}}$ is generic in $\cap D$.

**Part 4.2.**

Take $G, G_{\mathcal{D}}$ as in the question and let $A \subseteq$ be any dense set.
For each $x \in A$ choose some element $y \in \mathcal{D}$ that is stronger, call the set of all such $y$ as $A' \subseteq \mathcal{D}$. Notice that $A'$ is dense in $\mathcal{D}$, indeed if $x \in \mathcal{D}$ then there exists some $y \in A$ stronger and hence $r \in A'$ that is stronger than both. Let $p \in G_{\mathcal{D}} \cap A'$, $p$ must be stronger than some $q \in A$ by construction, hence $q \in G \cap A$ which means that $G$ is generic.