Exercise 1

Yuval Paz

Saturday 2nd November, 2024

Exercise 1.

Part 1.1.

Let $A \subseteq \Omega$ with probability 0, and $B \subseteq \Omega$ any event with some probability α.
Let $B' = B \setminus A$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A \cup B') = \mathbb{P}(A) + \mathbb{P}(B') = \mathbb{P}(B')$.
We also have $\mathbb{P}(B) = \mathbb{P}((B \cap A) \cup B') = \mathbb{P}(B \cap A) + \mathbb{P}(B')$
From 1.2, $\mathbb{P}(B \cap A) \leq \mathbb{P}(A) \implies \mathbb{P}(B \cap A) = 0 \implies \mathbb{P}(B) = \mathbb{P}(B')$

Part 1.2.

Let $A \subseteq B$, and let $B' = B \setminus A$, then $\mathbb{P}(B) = \mathbb{P}(A \cup B') = \mathbb{P}(A) + \mathbb{P}(B) \geq \mathbb{P}(A)$ as $\mathbb{P}(B) \geq 0$

Part 1.3.

Let $(\Omega, \mathbb{P})$ be any discrete probability space, let $\text{❀} \notin \Omega$, and define $(\Omega \cup \{\text{❀}\}, \mathbb{P}^{\text{❀}})$ be discrete probability space defined as: $\mathbb{P}^{\text{❀}}(A) = \mathbb{P}(A \setminus \{\text{❀}\})$.

Clearly this is a probability space ($\mathbb{P}^{\text{❀}}(\Omega \cup \{\text{❀}\}) = \mathbb{P}(\Omega) = 1$, and given any countable set of disjointed subsets of $\Omega \cup \{\text{❀}\}$, at most one of them contains ❀, removing the flower from this specific set and looking at the σ-additivity of $\mathbb{P}$ gives the result)

It is also discrete, as if p is a discrete probability function inducing $\mathbb{P}$, then $p^{\text{❀}}$ defined as p on Ω and $p(\text{❀}) = 0$ will induce $\mathbb{P}^{\text{❀}}$.

In this probability space, let $A \subseteq \Omega$, then $A \subsetneq A \cup \{\text{❀}\}$ but $\mathbb{P}^{\text{❀}}(A) = \mathbb{P}^{\text{❀}}(A \cup \{\text{❀}\})$

Part 1.4.

If $\mathbb{P}(A \cap B) = \alpha \in [0, 1]$, the only way for the inequality to fail is for $\mathbb{P}(A) + \mathbb{P}(B) > 1 + \alpha$

Now let A', B' defined as in 1.1 and 1.2, then we have $\mathbb{P}(A) = \mathbb{P}(A') + \alpha \leq 1 \implies \mathbb{P}(A) \leq 1 - \alpha$, and similarly for B so $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A') + \mathbb{P}(B') + 2\alpha > 1 + \alpha \implies \mathbb{P}(A') + \mathbb{P}(B') + \alpha > 1$, but by the definition $\alpha = \mathbb{P}(A \cap B)$, and $A', B', A \cap B$ are all disjoints, so we get that $\mathbb{P}(A' \cup B' \cup (A \cap B)) > 1$, contradiction.

Part 1.5.

Let A', B' be as defined in 1.1 and 1.2.

We have $\mathbb{P}(A) = \mathbb{P}(A') + \mathbb{P}(A \cap B)$ and $\mathbb{P}(B) = \mathbb{P}(B') + \mathbb{P}(A \cap B)$

Notice that $B' \cap A' = \emptyset$, so adding the 2 equations we get $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A' \cup B') + 2\mathbb{P}(A \cap B) \implies \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B) = \mathbb{P}(A' \cup B')$

But $A' \cup B'$ is exactly $A \Delta B$, so we are done.

Exercise 2.

Let $\mathbb{P}$ be a probability function satisfying the conditions in the question.

Because $\mathbb{N}$ is countable, so every subset of $\mathbb{N}$, so $A = \cup_{n \in A}\{n\}$ is countable union of disjoint sets, hence $\mathbb{P}(A) = \sum_{n \in A} \mathbb{P}(\{n\})$, hence it is enough to show that there is a unique discrete probability function p on $\mathbb{N}$ satisfying $p(n) = 3p(n+1)$.

Notice that given 2 such discrete probability functions that agree on a single number must be equal.

Let $p(0) = \alpha$, by definition of discrete probability function we have $\sum_{n \in \mathbb{N}} \alpha/3^n = \alpha \cdot \sum_{n \in \mathbb{N}} 1/3^n = 1 \implies \alpha = \frac{1}{\sum_{n \in \mathbb{N}} 1/3^n}$, hence any 2 discrete probability functions satisfying $p(n) = 3p(n+1)$ must have the same value at 0, but this implies that they are equal.

$\mathbb{P}(\mathbb{N})$ must be 1, as $\mathbb{P}$ is a probability function, and (assuming $3\mathbb{N}$ means $\{3n \mid n \in \mathbb{N}\}$) $\mathbb{P}(3\mathbb{N}) = \sum_{n \in \mathbb{N}} \alpha/3^{3n}$

Exercise 3.

Define $I_n = \{i \in I \mid a(i) \in \left[\frac{1}{n+1}, \frac{1}{n}\right)\}$ (where we treat $\frac{1}{0}$ as $+\infty$) and $I = \bigcup I_n = \{i \in I \mid a(i) > 0\}$.

I is a countable union of sets, so if $|I| > \aleph_0$, there must be some $n \in \mathbb{N}$ such that $|I_n| \geq \aleph_0$ (I include 0 in $\mathbb{N}$)

But if $J \subseteq I_n$ is finite, then $\sum_J a(i) \geq \frac{|J|}{n+1}$, and because I_n is infinite, we can take J to be as big as we want.