Exercise 2

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Exercise 1.

Let $\mathbb{P}, A, f : A \to V^{\mathbb{P}}$ be as in the question, for each generic G let $\overset{\text{bonk!}}{G}$ be the unique $x \in A \cap G$, define $\uparrow : \mathbb{P} \times A \to 2^{\mathbb{P}}$, $g : V^{\mathbb{P}} \times A \to V^{\mathbb{P}}$ and $\sigma_{\cdot} : A \to V^{\mathbb{P}}$ as:

- $x \uparrow a$ is the set of all common strengthening of x and a
- $g(\tau, a) = \{(\pi, x) \mid \exists y (\pi, y) \in \tau \wedge x \in y \uparrow a\}$
- $\sigma_a = g(f(a), a)$

Notice that $(f(\overset{\text{bonk!}}{G}))_G = (\sigma_{\overset{\text{bonk!}}{G}})_G$, the $\subseteq$ direction is follows from the fact that every 2 elements in G have a common strengthening, indeed if $\tau \in f(\overset{\text{bonk!}}{G})$ is don't discarded by G then its right side is in G, in particular $\pi_2(\tau) \uparrow \overset{\text{bonk!}}{G} \neq \emptyset$ (where π_i is the projection function) and so there is some $\pi \in \sigma_{\overset{\text{bonk!}}{G}}$ such that $\pi_1(\pi) = \pi_1(\tau)$ and $\pi_2(\pi) \in G$.
The $\supseteq$ direction follows from the fact that G is closed downwards, if $\pi \in \sigma_{\overset{\text{bonk!}}{G}}$ is not discarded by G, then $\pi_2(\pi)$ comes from some $\tau \in f(\overset{\text{bonk!}}{G})$ with $\pi_1(\tau) = \pi_1(\pi)$ and $\pi_2(\tau) \leq \pi_2(\pi)$, which means that τ is also not discarded by G because it is closed downwards.

In addition if Q is a different generic ideal such that for some x we have $x \in (\sigma_{\overset{\text{bonk!}}{G}})_Q$ then $\overset{\text{bonk!}}{G} = \overset{\text{bonk!}}{Q}$ (in other words, $\sigma_{\overset{\text{bonk!}}{G}}$ and $\sigma_{\overset{\text{bonk!}}{H}}$ for G, H generics that don't have a common A-member don't interfere with one another) because any right side of an element of $\sigma_{\overset{\text{bonk!}}{G}}$ must have stronger tag than $\overset{\text{bonk!}}{G}$ and Q is closed downwards which means that $\overset{\text{bonk!}}{G} \in A \cap Q$ which by definition is equal to $\overset{\text{bonk!}}{Q}$, so we can let $\sigma_f = \bigcup_{a \in A} \sigma_a$.

Exercise 4.

Part 4.1.

For (2) $\to$ (1) in the book, notice that (1) is a special case of (2). For (2) $\to$ (3) notice that assuming (2) we have $\{r \mid r \Vdash^* \psi\} \supseteq \{r \mid r \geq p\}$, and the latter trivially dense above p.

For the atomic (1) $\to$ (2) direction, let $p \Vdash^* \tau_1 = \tau_2$ and $r > p$, I want to show that for all $(\pi_1, s_1) \in \tau_1$ the set $\{q \geq r \mid q \geq s_1 \implies \exists (\pi_2, s_2) \in \tau_2 (q \geq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$

is dense above r, indeed take an element s above r, then because $p \Vdash^* \tau_1 = \tau_2$ and that $s \geq r \geq p$ there exists some $t \in \{q \geq p \mid q \geq s_1 \implies \exists(\pi_2, s_2) \in \tau_2(q \geq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$ greater than s, but because $t \geq s \geq r$ we have that $t \in \{q \geq r \mid q \geq s_1 \implies \exists(\pi_2, s_2) \in \tau_2(q \geq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$, so that set is indeed dense above r. A symmetric argument will finish this proof.

Now assume $p \Vdash^* \tau_1 \in \tau_2$ and $r \geq p$, take $s \geq r \geq p$, there is some $t \in \{q \mid (\pi, x) \in \tau_2(q \geq x \wedge q \Vdash^* \pi = \tau_1)\}$ that is stronger than $s \geq r$, so the set is dense above r.

For (3) $\to$ (1) we just need to prove that if $\{r \mid D \text{ is dense above } r\}$ is dense above p, then D is dense above p, indeed take $q \geq p$ and $r \in \{r \mid D \text{ is dense above } r\}$ greater than q, then let $q' > r$, because D is dense above r there exists $t \in D$ such that $t \geq q' > r \geq q \geq p$, hence D is dense above D.

To finish all of the details of the proof of the lemma from the book we need to show (1) $\to$ (2) and (3) $\to$ (1) for $\neg$ and $\exists x$.

For (1) $\to$ (2), assume $p \Vdash^* \neg\varphi$ and $r \geq p$, then every $t \geq r$ is also stronger than p, hence does not force* the sentence φ, therefore $r \Vdash^* \neg\varphi$. Assume $p \Vdash^* \exists x\varphi(x)$ and take $r \geq p$ and $t \geq r$, because $t \geq p$ there exists $q \geq t$ such that $\exists\sigma \in V^{\mathbb{P}}(q \Vdash^* \varphi(\sigma))$, in particular $\{q \mid \exists\sigma \in V^{\mathbb{P}}(q \Vdash^* \varphi(\sigma))\}$ is dense above r.

For (3) $\to$ (1) for the $\neg\varphi$ case, assume $p \nVdash^* \neg\varphi$ then there is some $r \geq p$ that forces* φ, by the induction hypothesis we get that every $t \geq p$ also forces* φ, a contradiction. The $\exists x\varphi(x)$ case follows from the same fact as the atomic case, that if $\{r \mid D \text{ is dense above } r\}$ is dense above p, then D is dense above p

Part 4.2.

Assume $p \Vdash^* \varphi$, we want to prove $p \Vdash^* \neg\neg\varphi$, in particular we want to show:

$$\begin{aligned} & \forall q \geq p(\neg q \Vdash \neg\varphi) \\ \iff & \forall q \geq p(\neg(\forall r \geq q(\neg r \Vdash^* \varphi))) \\ \iff & \forall q \geq p(\exists r \geq q(r \Vdash^* \varphi)) \end{aligned}$$

But any $r \geq q$ will witness it is true, as $r \geq q \geq p$ and from (4.1) $r \Vdash^* \varphi$.

Part 4.3.

Let D_ψ be as in the question and let $p \in \mathbb{P}$ be any term, if there is $q \geq p$ such that $q \Vdash^* \psi$ then we are done as $p \leq q \in D_\psi$. Otherwise we have that $p \Vdash^* \neg\psi$ by definition, so $p \in D_\psi$.

So assume $p \Vdash^* \neg\neg\psi$, the set D_ψ is dense above p, by definition there is no $q \geq p$ that forces* $\neg\psi$, so $\{q \mid q \Vdash^* \psi\} = D_\psi \cap \{q \mid q \geq p\}$ is dense above p, by (4.1) this is equivalent to $p \Vdash \psi$.

Exercise 6.

Assume $(p \Vdash^* \varphi(\overline{\tau}))^M$, in particular we have that for every G a generic containing p we have $\exists q \in G(q \Vdash^* \varphi(\overline{\tau}))^M$, by theorem 3.5 in the book we have $M[G] \models \varphi(\overline{\tau}_G)$, hence by definition $p \Vdash_{\mathbb{P}}^M \varphi(\overline{\tau})$

Now assume $p \Vdash_{\mathbb{P}}^{M} \varphi(\overline{\tau})$ and let $r \geq p$ and G be a $\mathbb{P}$-generic contains r, by definition $M[G] \models \varphi(\overline{\tau}_G)$, by theorem 3.5 in the book there exists some $t \in G$ such that $(t \Vdash^{*} \varphi(\overline{\tau}))^{M}$, let $q \in G$ be stronger than r and t, by (4.1) we know that $(q \Vdash^{*} \varphi(\overline{\tau}))^{M}$, hence $\{q \mid q \Vdash^{*} \varphi(\overline{\tau})\}$ is (dense above $p)^{M}$, which implies that $(p \Vdash^{*} \varphi(\overline{\tau}))^{M}$

To get the form we had in class all we need to do is "concat" this result to question theorem 3.5, if G is a generic and $M[G] \models \varphi(\overline{\tau}_G) \iff \exists p \in G\ (p \Vdash^{*} \varphi(\overline{\tau}))^{M} \iff \exists p \in G\ p \Vdash_{\mathbb{P}}^{M} \varphi(\overline{\tau})$.

Exercise 7.

Part 7.1.

Let D be the set $\{p \mid \exists\sigma \in V^{\mathbb{P}}(p \Vdash \psi(\sigma, \overline{\tau}))\}$, we know this set is dense above $0_{\mathbb{P}}$, which means it is just dense.

Let $A \subseteq D$ be a maximal anti-chain, and for each $x \in A$ let $f(x)$ be a name that is a witness of $x \in D$, let σ_f be the name from exercise 1.

For every generic (= for every generic contains $0_{\mathbb{P}}$) we have that $M[G] \models \psi((\sigma_f)_G, \overline{\tau}_G)$, which by definition means $0_{\mathbb{P}} \Vdash \psi(\sigma_f, \overline{\tau})$.

Part 7.2.

Let $\psi(x, y, w, z) = (x \text{ is a function from } w \text{ to } z) \wedge (y \text{ is a function from } w \text{ to } z \implies x = y)$, notice that for every y', w', z' such that $(z' = \emptyset \implies w' = \emptyset)$ we have that $\exists x\psi(x, y', w', z')$ is tautology.

Let G be a generic ideal and $f : \alpha \to \beta$ function in $M[G]$, let τ be the name that G interpret as f, because f is a function we have that either $\beta \neq 0$ or both α and β are 0, in particular for every Q a generic ideal we have that $M[G] \models \exists x\psi(x, \tau_Q, \check{\alpha}_Q, \check{\beta}_Q)$, so $0_{\mathbb{P}} \Vdash \exists x\psi(x, \tau, \check{\alpha}, \check{\beta})$, from (7.1) we have that there is some name σ such that $0_{\mathbb{P}} \Vdash \psi(\sigma, \tau, \check{\alpha}, \check{\beta})$.

Clearly $0_{\mathbb{P}} \Vdash \sigma : \check{\alpha} \to \check{\beta}$ and because $\tau_G = f$ is a function from α to β we have that $f = \tau_G = \sigma_G$.