Exercise 0

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Exercise 1. *Trees*

Part 1.1.

We shall prove the general fact that transitivity is absolute between transitive models of ZFC, and because well-foundness of trees correspond to well-foundness of their reverse partial order we will finish. The upward direction is the interesting one, so let $<\in M \subseteq N$ be (well-founded)^M partial order on $X \in M$.

Let r be the rank function of $<$ in M.

Assume that $<$ is not well-founded in N, that is: there exists $(x_i \in X \mid i \in \omega) \in N$ $<$-descending sequence, in particular N can talk about the r-image sequence $(r(x_i) \in Ord^M \mid i \in \omega)$, this is a descending sequence of M-ordinals.

We shall show that $Ord^M \subseteq Ord^N$ and hence reach a contradiction. Look at $\min(Ord^M \setminus Ord^N)$, this is an ordinal in M, hence it is a set in N, but N knows that it is a transitive set of transitive sets, which is impossible as it is not an ordinal in N.

Part 1.2.

Let W be any transitive class of ZFC such that $M, N \in W$ and $W \models M$ is countable.

Enumerate $M = \{m_i \mid i \in \omega\}$ and define T to be the tree of all $n = (n_i) \in N^{<\omega}$ such that for all φ in the language of M, N with less than $|n| + 1$ parameters $M \models \varphi(m_0, \ldots m_k) \iff N \models \varphi(n_0, \ldots, n_k)$.

Note that a branch of T correspond to an elementary embedding $M \to N$ and vice versa, furthermore because W is a transitive class of ZFC we have $W^{<\omega} \subseteq W$, in particular $T \in W$.

From the previous part, if T is ill-founded in V (which it is, as witnessed by the branch $(j(m_i))$) it is ill-founded in W as well, in particular there is a branch of T in W which witness $\tilde{j} : M \prec N$.

To finish the proof just note that $L[A]$ is a transitive class satisfying all of conditions from above.

Exercise 2. *Ineffable Cardinals*

Part 2.1.

Assume κ is not strongly inaccessible, because it is regular we have that κ is not a strong limit. Let $\lambda < \kappa \leq 2^{\lambda}$ and let $\overline{A} = (A_i \in \mathcal{P}(\lambda) \mid i \in \kappa)$ be arbitrary injective sequence with $A_i = i$ for $i < \lambda$.

This sequence satisfy $A_i \subseteq i$, in particular let ❀ $\subseteq \kappa$ be such that ❀$^{\overline{A}} = \{\alpha \mid \alpha \cap$ ❀ $= A_\alpha\}$ is stationary.

But because each A_i is bounded by λ we must have that $\alpha \mapsto$ ❀ $\cap\, \alpha$ is constant above λ, but we started with injective sequence $\overline{A}$, contradiction.

To see κ is Mahlo, first note that we can replace the sequence in the definition of ineffable with a sequence indexed by a club instead of κ itself, so let $\overline{A} = (A_\eta \subseteq \eta \mid \eta \in \kappa \cap Card)$ be a sequence such that A_η is cofinal subset of η with cardinality $\operatorname{cof}(\eta)$.

Let ❀ $\subseteq \kappa$ is such that ❀$^{\overline{A}}$ is stationary, if the regular cardinals in ❀$^{\overline{A}}$ are not stationary then $\operatorname{cof}\restriction_{❀^{\overline{A}} \cap Singular}$ is regressive over a stationary set, hence constant λ over a stationary set ❀$^{\star} \subseteq$ ❀, that is impossible as $|A_\eta| = \operatorname{cof}(\eta) = \lambda$ for all $\eta \in$ ❀* but the λ^{+}-th element of $\overline{A} \restriction$ ❀* must have cardinality at least λ^{+}.

To finish the proof we just note that the strong limit cardinals are a club in any strong limit cardinal, and in particular in κ, and intersecting this club with the stationary set of regular cardinals results with a stationary set of inaccessible cardinals.

Part 2.2.

Lemma 2.1. *If κ is ineffable and $\overline{A} = (A_\alpha \subseteq \alpha^2 \mid \alpha \in \kappa)$ is any sequence, then there is* ❀ $\subseteq \kappa^2$ *such that* ❀$^{\overline{A}}$ *is stationary.*

Proof. We shall restrict ourselves to the club of cardinals, so WLOG $\operatorname{dom}\left(\overline{A}\right) = Card$. Let $G : Ord^2 \to Ord$ be Godel's pairing function, so $G \restriction \eta^2$ is the pairing function on η^2, and let $\overline{B} = (B_\alpha)$ defined as $B_\alpha = G(A_\alpha)$, there is a set ❀* that witness the fact κ is ineffable over $\overline{A}$, and ❀ $= G^{-1\prime\prime}$❀* is the set we wanted. $\square$

Assume that κ is Ineffable and and let $T \subseteq 2^{<\kappa}$ be a slim tree, and let B be the set of branches of T.

Let $\overline{B^\alpha} = (B^\alpha_\beta \mid \beta \in \alpha)$ be enumeration of $\{b \restriction \alpha \mid b \in B\}$ and encode each $\overline{B^\alpha}$ with $L_\alpha = \{(x, \beta) \mid B^\alpha_\beta(x) = 1\}$ and define the sequence $\overline{L} = (L_\alpha)$. From the lemma we have ❀ $\subseteq \kappa^2$ such that ❀$^{\overline{L}}$ is stationary.

For $\nu < \kappa$ define the following $b_\nu : \kappa \to 2$, as

$$b_\nu(x) = \begin{cases} 1 & \text{if } (x, \nu) \in \text{❀} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Let $B' = \{b_\nu \mid \nu \in \kappa\}$, we shall show that $B \subseteq B'$ and hence restrict the size of B to κ.

Assume $b \in 2^\kappa \setminus B'$, define $g : \kappa \to \kappa$ be such that $g(x)$ is the least such that $g(x) > g(z)$ for $z < x$ and $b \neq b_x$ (that is, $b \restriction g(x) \neq b_x \restriction g(x)$). This is a normal function, hence the set C of fixed points is a club in κ.

Let $\alpha \in C \cap \text{❀}^{\overline{L}}$, because $\alpha \in C$ we have that for all $\beta < \alpha$ $(b \restriction \alpha \neq b_\beta \restriction \alpha)$ but $b_\beta \restriction \alpha = B^\alpha_\beta$, in particular $b \restriction \alpha \notin \operatorname{range}\left(\overline{B_\alpha}\right)$, in other words $b \restriction \alpha \notin T$ hence $b \notin B$.

Part 2.3.

First we note that κ is regular, indeed if $\operatorname{cof}(\kappa) = \nu < \kappa$ let $(x_\alpha \mid \alpha \in \nu)$ be such witness, this witness is in $H(\mu)$ hence there exists some $\overline{y} = (y_\alpha \mid \alpha \in \nu) \in M$ that is cofinal in κ, but $j(\overline{y}) = \overline{y}$ as it is a short sequence of small ordinals, in particular $\kappa = \sup \operatorname{range}(\overline{y}) = \sup \operatorname{range}(j(\overline{y})) = j(\kappa)$.

Now, let $\overline{A} \in M$ be any sequence such that $A_\alpha \subseteq \alpha$ for all $(\alpha \in \kappa)^M$.

Let $\overline{B} = j(\overline{A})$ and note that $B_\alpha = A_\alpha$ for $(\alpha < \kappa)^M$. Let $\text{❀} = B_\kappa$ and let $C \in M$ be a $(\text{club of } \kappa)^M$, in particular $j(C)$ is a club of $j(\kappa)$ and $j(\text{❀}^{\overline{A}}) = j(\text{❀})^{\overline{B}}$.

Because $j(C)$ is a club and $C \subseteq j(C)$ we must have $\kappa \in j(C)$ as it is a limit point, and furthermore $x \in \text{❀} \iff x \in j(\text{❀})$ for all $(x \in \kappa)^M$, in particular $j(\text{❀}) \cap \kappa = \text{❀} \implies \kappa \in j(\text{❀})^{\overline{B}} = j(\text{❀}^{\overline{A}})$, combining the 2 we get $j(\text{❀}^{\overline{A}} \cap C) \neq \emptyset \implies \text{❀}^{\overline{A}} \cap C \neq \emptyset$, because C was arbitrary we have $(\text{❀}^{\overline{A}} \text{ is stationary})^M$.

In particular $M \models \kappa$ is ineffable, by elementary $H(\mu)$ also thinks so, but every sequence $(Q_\alpha \subseteq \alpha \mid \alpha \in \kappa)$ and every κ-club are in $H(\mu)$, so V agrees with $H(\mu)$ about stationary sets and hence about ineffability for cardinals bellow μ.

Part 2.4.

Let $\overline{M} = (M_\alpha)_{\alpha \in \kappa \cap Card}$ be a list of structures in the language $\mathcal{L}$ and assume WLOG that the domain of each M_α is an ordinal. We shall further assume that the domain of each M_α is exactly α (if the domain is not a arbitrary we can just attach to each model an isomorphism f_α into α to get a sequence $\overline{N}$ of models and work with $\overline{N}$), and show separately at the end what happens if the domain of M_α is less than α.

Encode each M_α in $\alpha^{<\omega}$, just like Lemma 2.1 we have $\text{❀} \subseteq \kappa^{<\omega}$ stationary such that $\text{❀}^{\overline{M}}$ is stationary. In particular, ❀ can be decoded as a model M_κ of $\mathcal{L}$ such that for each $\alpha \in \text{❀}^{\overline{M}}$ we have that M_α is substructure of M_κ.

Let $(p_{\alpha+1}(x))_{\alpha \in \kappa}$ be enumeration of $\varphi(x, \overline{a})$ the $\mathcal{L}$-formulaes with parameters from M_κ and 1 free variable such that $M_\kappa \models \exists x \varphi(x, \overline{a})$ with the property that if $p_{\alpha+1}(x) = \varphi(x, \overline{a})$ then $\alpha + 1 < |\max \overline{a}|^+$.

Define $g : \kappa \to \kappa$ as follows: for $x = \alpha + 1$ let $g(x)$ be the first $\beta > g(\alpha)$ such that $\exists y \in M_\beta (M_\kappa \models p_x(y))$. Otherwise let $g(x) = \sup_{w<x} g(w)$. g is a normal function hence C the set of fixed points of g is a club.

Let $\alpha \in C, \overline{a} \in M_\alpha^{<\omega}$ and $\varphi(x, y)$ such that $M_\kappa \models \exists x \varphi(x, \overline{a})$, we know that $\varphi(x, \overline{a}) = p_{\beta+1}(x)$ for some β with $\beta + 1 < |\max \overline{a}|^+ \leq \alpha$, in particular $g(\beta + 1) < g(\alpha) = \alpha$ as $\alpha \in C$.

By construction there is $\eta < \alpha$ such that $\exists y \in M_\eta (M_\kappa \models p_{\beta+1}(y))$ so $\exists y \in M_\alpha (M_\kappa \models p_{\beta+1}(y))$, in other words $\exists y \in M_\alpha (M_\kappa \models \varphi(y, \overline{a}))$.

By Tarski–Vaught criterion if $\alpha \in ❀^{\overline{M}}$ then $M_\alpha \prec M_\kappa$ is elementary.

Now if $C \cap ❀^{\overline{M}} \ni \alpha < \beta < \kappa$ then $M_\alpha \prec M_\kappa, M_\beta \prec M_\kappa$ and $M_\alpha \subseteq M_\beta$. Now let $\overline{a} \in M_\alpha$ such that $\exists y \in M_\alpha (M_\kappa \models \varphi(y, \overline{a}))$, in particular $\exists y \in M_\alpha (M_\beta \models \varphi(y, \overline{a}))$, henceforth $M_\alpha \prec M_\beta$ if.

Finally, we shall show that the limit steps are direct limit of the previous steps, but that is obvious as our embeddings are the inclusion functions.

To handle the case that $|M_\alpha| \neq \alpha$ for all α note that either $|M_\alpha| < \alpha$ on a stationary set $❀^\star$, or $|M_\alpha| = \alpha$ on a club, in the later case the above proof works by restricting our $\overline{M}$ to that club.

Assume $|M_\alpha| < \alpha$ for all $\alpha \in ❀^\star$ stationary. By Fodor we can assume WLOG that $|M_\alpha| = \eta$ for all $\alpha \in ❀^\star$. Because κ is uncountable strong limit, our language is countable and the club filter on κ is $2^\eta \times \aleph_0$-complete, we can WLOG assume that $M_\alpha \cong M_\beta$ for all $\alpha, \beta \in ❀^\star$. Like before we can encode each M_α to have domain η, again we note that the club filter on κ is 2^η-complete to be able to assume WLOG that $M_\alpha = M_\beta$ for all $\alpha, \beta \in ❀^\star$, let $M_\kappa = M_\alpha$ for some $\alpha \in ❀^\star$ to get the system we wanted.