Compactness and Ramsey's theorem

Holo

Thursday 2nd March, 2023

Ramsey theory

A branch of combinatorics, touches (set theoretical) infinitary combinatorics, graph theory and number theory

Deals with questions of the form

how large does the structure have to be to guarantee that there exists substructure with a given property?

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- Schur's theorem - for every $n \in \mathbb{N}$ there exists a number $c(n)$ such that for every colouring of $[c(n)] = \{1, \ldots, c(n)\}$ there exists monochromatic $a, b, c \in [c(n)]$ such that $a + b = c$

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- Hindman's theorem - If X is a set of all finite sums of some infinite subset of $\mathbb{N}$, and $X = A \sqcup B$, then either A or B are the set of all finite sums of some infinite subset of $\mathbb{N}$

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To be able to state the theorem nicely we first need to define 2 notations:

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Let A be a set and α be cardinal number, $[A]^\alpha$ denotes the set of subsets of A of size α, that is: $[A]^\alpha = \{X \in \mathcal{P}(A) \mid |X| = \alpha\}$

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- $[\mathbb{N}]^\omega$ is the set of all infinite sets of natural numbers

Definition (Ramsey's arrow notation)

Let κ, λ, m be cardinals, and let n be a natural number then

$$\kappa \to (\lambda)^n_m$$

Is the statement: for every colouring of $[\kappa]^n$ into m colours, there exists a $S \subseteq \kappa$ such that $|S| = \lambda$ and $[S]^n$ is monochromatic

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Example:

- Theorem on friends and strangers: $6 \to (3)^2_2$ - Let $|X| = 6$, then $[X]^2$ is K_6, we label the **edges** of K_6 with the 2 colours "friends", "strangers", then there exists a subset of X, $|Y| = 3$, such that $[Y]^2$ is all "friends" or "strangers", that it, either everyone from Y are friends with each other, or non of them know each other

Theorem (Ramsey's theorem)

Infinitary version: $\forall n, m \in \omega;\ \aleph_0 \to (\aleph_0)^n_m$

Finitary version: $\forall n, m, b \in \omega \exists a \in \omega;\ a \to (b)^n_m$

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Finitary Ramsey's theorem should be interpret in the "opposite" direction, "Given a finite set B, there exists a finite set A such that every partition of $[A]^n$ into m colours, there exists a homogeneous subset of A, C, such that $|C| = |B|$"

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Such cardinal cannot be proven to exists using ZFC, and it is called "weakly-compact cardinal"

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We will use the "ultrafilters lemma" to prove the infinitary version, and then use the "compactness theorem"(which is equivalent to the "ultrafilters lemma") to show that the infinitary case implies the finitary case.

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It is worth noting that the special case where $n = 2$ is provable in ZF, and the general theorem is weaker than the ultrafilters lemma.

Ultrafilters

Filters is one of the most useful tools in mathematics, it is used in set theory, topology, measure theory, and pretty much anywhere that we look at family of subsets of a specific set, one can also argue that they appear in boolean algebra and any place that deals with ideals.

Definition (filter)

Let X be a set, a set $\mathcal{F} \subsetneq \mathcal{P}(X)$ is a **filter** if:

- $\varnothing \notin \mathcal{F}, X \in \mathcal{F}$
- $A \subseteq B \subseteq X$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$
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The last property is the reason I think about it as "huge" and not "big", intersection of 2 $\mathcal{F}$-huge sets is in itself $\mathcal{F}$-huge.

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- The Fréchet filter on X: Let X be an infinite set, then the set $\mathcal{F}_X$ of cofinite subsets of X is a filter

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It turns out that the question cannot be solved in ZF, and to solve it in ZFC we need to following lemma

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This lemma deserves a full course by itself, but it is only useful when combined with the following result:

Lemma

An ultrafilter $\mathcal{U}$ on X is nonprincipal ultrafilter if and only if the Fréchet filter on X is a subset of $\mathcal{U}$, that is $\mathcal{F}_X \subseteq \mathcal{U}$, equivalently, it is a nonprincipal ultrafilter if and only if it does not contains any finite subset of X

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Corollary

For any infinite set X there exists a nonprincipal ultrafilter on X

Proof of infinitary case

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To give ourselves a better intuition let's remember the definition of $\omega \to (\omega)^2_m$

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In particular, remember that $[X]^2$ is exactly the set of edges of the complete graph with X as vertices.
In other words: given colouring on the edges of the complete graph on ω, there exists an infinite complete subgraph that all of it's edges are the same colour.

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$A_i(v) = \{u \in V \mid c(\{v,u\}) = i\}$, that is $A_i(v)$ is the set of all vertices that the edge between them and v is i-coloured.
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Letting $J = \{v_p \mid p \in \omega\}$ finishes the proof. ∎

Proof of finitary case

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To avoid using model theoretical or formal logic terms, we will not use the compactness theorem, but we will use the following theorem, which follows from the compactness theorem:

Theorem (The axiom of dependent choice for finite sets)

If A is a set, and $<_A$ is a relation on A such that $\{b \in A \mid a < b\}$ is finite for all $a \in A$, then there exists a choice sequence for $<_A$. That is, a sequence x_i such that $x_i <_A x_{i+1}$ for all $i \in \mathbb{N}$

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Exercise for those who are comfortable with the compactness theorem: prove the theorem.

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Define C_k^1 to be the set of colourings in C_k that can be extended into a colouring in C_{k+1}. Because C_{k+1} is non empty, so is C_k^1.

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Continue with this construction till we have C_k^p for all $k, p \in \mathbb{N}$.

Notice how $C_k^p \supseteq C_k^{p+1}$ and $|C_k^p|$ is finite for all $k, p \in \mathbb{N}$, hence $C_k^\omega = C_k \cap C_k^1 \cap C_k^2 \cap \cdots$ is non-empty finite set.

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Define the order $<_G$ on $C_1^\omega \cup C_2^\omega \cup \cdots$ with $x <_G y$ when there exists ℓ such that $x \in C_\ell^\omega, y \in C_{\ell+1}^\omega$ and y extends x.

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By the axiom of dependent choice for finite sets, there exists a sequence $g_0, g_1, \ldots$ such that g_{q+1} extends g_q for all q, let G be the colouring that is the union of this sequence. But G has no homogeneous subset of size b, which is contradiction to the infinitary version of Ramsey's theorem, hence the infinitary version implies the finitary version ∎