Exercise 1

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Complex Numbers

Exercise 1.

We know that $(\rho e^{i\varphi})^n = \rho^n e^{in\varphi}$, in particular we have $(\rho e^{i\varphi})^n = re^{i\theta}$ hence

$$\begin{aligned} \rho^n &= r && \implies \rho = r^{1/n} \\ n\varphi &\equiv \theta \pmod{2\pi} && \implies \varphi = (\theta + 2\pi k)/n,\ 0 \le k < n \end{aligned}$$

(notice that the first row is legal because we know that r, ρ both should be positive reals).

In particular there are n solutions.

Exercise 2.

Part 2.1.

$1 = 1e^{i0}$, plugin in the solutions from (1) we get

$$\begin{aligned} &\rho = 1 \\ &\varphi = 2\pi k/6,\ 0 \le k < 6 \end{aligned}$$

Part 2.2.

$-1 = 1e^{i\pi}$, hence:

$$\begin{aligned} &\rho = 1 \\ &\varphi = (\pi + 2\pi k)/4 = 3\pi k/4,\ 0 \le k < 4 \end{aligned}$$

Part 2.3.

$-1+i\sqrt{3} = |-1+i\sqrt{3}|e^{i\arg(-1+i\sqrt{3})} = 2e^{i2\pi/3}$ hence:

$$\rho = 2^{1/4}$$
$$\varphi = (2\pi/3 + 2\pi k)/4,\ 0 \le k < 4$$

Exercise 3.

Part 3.1.

$$\frac{1}{6+2i} = \frac{\overline{6+2i}}{(6+2i)(\overline{6+2i})} = \frac{6}{36+4} + i\frac{-2}{36+4} = \frac{3}{20} + i\frac{-1}{20}$$

Part 3.2.

$$\frac{(2+i)(3+2i)}{1-i} = \frac{(2+i)(3+2i)(\overline{1-i})}{(1-i)(\overline{1-i})} = \frac{-3+11i}{2} = -\frac{3}{2} + i\frac{11}{2}$$

Part 3.3.

$$-\frac{1}{2} + i\frac{\sqrt{3}}{2} = 1e^{i2\pi/3} \implies (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^4 = e^{i8\pi/3} = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Part 3.4.

$$-1+i0,\ 0+i(-1),\ 1+i0,\ 0+i1$$

Exercise 4.

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 = a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ &= (ac-bd)^2 + (ad+bc)^2\end{aligned}$$

Closed and Open Sets

Exercise 1.

Let $\{❀_i\}_{i\in I}$ be a family of open sets, and let $x \in \bigcup_{i\in I} ❀_i$.

By definition there is some $i \in I$ such that $x \in ❀_i$, because U_i is open there must be some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq ❀_i$, in particular $B_\epsilon(x) \subseteq \bigcup_{i\in I} ❀_i$, hence $\bigcup_{i\in I} ❀_i$ is open.

Let $\{❀_j\}_{j\in J}$ be a finite family of open sets, and let $x \in \bigcap_{j\in J} ❀_j$, by definition there exists ϵ_j for each $j \in J$ such that $B_{\epsilon_j}(x) \subseteq ❀_j$. Because J is finite, the set $\{\epsilon_j\}$ has a minimum, let ϵ be this minimum and it is clear that $B_\epsilon(x) \subseteq \bigcap_{j\in J} ❀_j$

Exercise 2.

Assume the contrary, that $f^{-1}(U)$ is not open, in particular there exists $x \in f^{-1}(U)$ that witness it.

Because f is continuous we know that for every ϵ there is some δ such that $x' \in B_\delta(x) \implies f(x') \in B_\epsilon(f(x))$. Let ϵ be sufficiently small so that $B_\epsilon(f(x)) \subseteq U$ (it exists because U is open), and let δ be as above, but the above can be restated as $f''B_\delta(x) \subseteq B_\epsilon(f(x))$, in particular $B_\delta(x) \subseteq f^{-1}(U)$, but this contradict the fact that x is witness of the failure of $f^{-1}(U)$ to be open.

Exercise 3.

Assume that $\lim x_n = x \notin C$, we must have then that $x \in \mathbb{R}^n \setminus C$, which is open.

Let ϵ be such that $B_\epsilon(x) \subseteq \mathbb{R}^n \setminus C$, we know that $B_\epsilon(x) \cap C = \emptyset$, so for all $n \in \omega$ we have $x_n \notin B_\epsilon(x)$, which contradict the fact that they converge to x.