Exercise 4

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Exercise 1.

Part 1.1.

Let p be a polynomial of degree k and large coefficient of M.

We know that the radius of convergence is $\limsup \frac{1}{\sqrt[n]{|p(n)a_n|}}$ and we want to show it is equal to $\limsup \frac{1}{\sqrt[n]{|a_n|}}$, for that we will show that $\limsup \sqrt[n]{|p(n)a_n|} = \limsup \sqrt[n]{|a_n|}$.

Indeed $\limsup \sqrt[n]{|p(n)a_n|} = \limsup \sqrt[n]{|p(n)||a_n|} = \limsup \sqrt[n]{|p(n)|}\sqrt[n]{|a_n|}$, but $\limsup \sqrt[n]{|p(n)|} = \limsup \sqrt[n]{|M|}\sqrt[n]{|n^k|} = \limsup \sqrt[n]{|M|} \limsup \sqrt[n]{|n^k|} = \limsup \sqrt[n]{|n|}^k = (\limsup \sqrt[n]{|n|})^k = 1^k = 1$.

Part 1.2.

We know that $|\frac{a_n}{\sqrt{n!}}| = \frac{|a_n|}{\sqrt{n!}}$.

Furthermore, we have that the limsup of $\sqrt[n]{|a_n|}$ is bounded (as the series has positive convergence radius) and (because we know that $\exp(z)$ has a radius of convergence ∞) that $\sqrt[n]{\frac{1}{\sqrt{n!}}}$ goes to 0, so the convergence radius of the new series is infinity.

Let $t \in \omega$ be an index such that $\sup_{p>t} \sqrt[p]{|a_p|} < \infty$, the define the sequence $b_k = \sup_{p>t+k} \sqrt[p]{|a_p|}$.

Clearly $b_k \xrightarrow[k\to\infty]{} \frac{1}{r}$ and furthermore we have that $b_k^2 = \sup_{p>t+k} \sqrt[p]{|a_p|} \sup_{p>t+k} \sqrt[p]{|a_p|} = \sup_{p>t+k} \sqrt[p]{|a_p|}\sqrt[p]{|a_p|} = \sup_{p>t+k} \sqrt[p]{|a_p|^2}$, so $\limsup \sqrt[n]{|a_n|^2} = \lim b_k^2$. From arithmetic of limits we have $b_k^2 \xrightarrow[k\to\infty]{} \frac{1}{r^2}$, so the convergence radius is r^2.

Exercise 2.

Part 2.1.

Let's F be as in the question, and calculate $F'(z) = \lim_{z_1\to z} \frac{F(z_1)-F(z)}{z_1-z} = \lim_{z_1\to z} \frac{\int_{C(z_0,r)} \frac{1}{w-z_1} - \frac{1}{w-z} dw}{z_1-z}$, looking at $\frac{1}{w-z_1} - \frac{1}{w-z}$ we have $\frac{1}{w-z_1} - \frac{1}{w-z} = \frac{(w-z)-(w-z_1)}{(w-z_1)(w-z)} = \frac{z_1-z}{(w-z_1)(w-z)}$ so:

$$\begin{aligned} F'(z) \qquad &= \lim_{z_1 \to z} \frac{\int_{C(z_0,r)} \frac{1}{w-z_1} - \frac{1}{w-z} dw}{z_1 - z} & &= \lim_{z_1 \to z} \frac{\int_{C(z_0,r)} \frac{z_1 - z}{(w-z_1)(w-z)} dw}{z_1 - z} \\ &= \lim_{z_1 \to z} \int_{C(z_0,r)} \frac{1}{(w-z_1)(w-z)} dw & &= \int_{C(z_0,r)} \frac{1}{(w-z)^2} dw \end{aligned}$$

And it is clear that $g(z,w) = \frac{1}{(w-z)^2}$ satisfy the conditions we want.

Part 2.2.

For a fixed z we have that $D_w = -\frac{1}{w-z} = f(w,z)$ at $\mathbb{C} \setminus \{z\}$, so let $G(w,z)$ is an anti-derivative of g for a fixed z.

Because $\mathbb{C} \setminus \{z\}$ is open and g has a global anti-derivative in there, we know that the integral over any closed loop contained in $\mathbb{C} \setminus \{z\}$ is 0, hence $F'(z)$, which is defined as such integral, is 0

Part 2.3.

$F(z_0) = \int_{C(z_0,r)} \frac{1}{w-z_0} dw = \int_0^{2\pi} \frac{D_t(z_0 + re^{it})}{(z_0 + re^{it}) - z_0} dt = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = i \int_0^{2\pi} dt = 2\pi i.$

Because $F' \equiv 0$, we know that F is a constant on any connected component of it's domain, in particular $F \equiv F(z_0)$ in $B_r(z_0)$.

Exercise 3.

We know that $\exp(z) = \sum_{n=0}^{\infty} \frac{\exp^{(n)}(z_0)}{n!}(z - z_0)^n \overset{\text{Taylor Series}}{=} \sum_{n=0}^{\infty} \frac{\exp(z_0)}{n!}(z - z_0)^n = \exp(z_0) \frac{\exp(z_0)}{n!}(z - z_0)^n = \exp(z_0)\exp(z - z_0)$.

So given $w, w' \in \mathbb{C}$ and letting $z = w + w'$ and $z_0 = w'$ we get $\exp(w + w') = \exp(z) = \exp(z_0)\exp(z - z_0) = \exp(w')\exp(w + w' - w') = \exp(w')\exp(w)$

Exercise 4.

Part 4.1.

The function is not defined precisely for $z \in \mathbb{C}$ such that $i(\exp(iz) + \exp(-iz)) = 0 \iff \exp(iz) + \exp(-iz) = 0 \iff \exp(iz) = -\exp(-iz)$.

Let $z = x + iy$: $\exp(iz) = -\exp(-iz) \iff \overset{\text{LHS}}{\exp(-y)\exp(ix)} = \overset{\text{RHS}}{-\exp(y)\exp(-ix)}$.

If $\exp(-y) \neq \exp(y)$ then the absolute value of the LHS and RHS will be different, hence they will be different, but $\exp(-y) = \exp(y) \iff y = 0$ for a real y, the function is not defined on z iff $z \in \mathbb{R}$ and $\exp(iz) = -\exp(-iz)$.

We know that $\exp(ix) = \cos(x) + i\sin(x)$, $-\exp(-ix) = -(\cos(-x) + i\sin(-x)) = -(\cos(x) - i\sin(x)) = i\sin(x) - \cos(x)$, comparing the 2 sides gives us that tan is not defined precisely on $z \in \mathbb{R}$ such that $\cos(z) = -\cos(z) \iff \cos(z) = 0 \iff z = \frac{\pi}{2} + n\pi$ for some integer n.

Part 4.2.

Let $z \in \mathrm{dom}\,(\tan)$, and let $r > r' \in \mathbb{R}_{>0}$ such that $B_r(z) \subseteq \mathrm{dom}\,(\tan)$, on $B_{r'}(z)$, we have that tan is analytic, so it's Taylor series around z has convergence radius $\geq r'$, so we have that the radius of convergence is $\sup\{r' \mid \exists r > r' \; s.t. \; B_r(z) \subseteq \mathrm{dom}\,(\tan)\}$, but this is exactly the minimum distance between z and $\{\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\}$ (to find the particular n for a given z, we can reduce the problem to minimize the distance of $\Re(z)$ from that set)

Exercise 5.

We know that f is analytic over $\mathbb{C}$, so the integral $\int_{\gamma_R} f = 0$ for all R, this integral is also equal to the sum of the integrals over the sides of the rectangle.

Let's calculate $\lim_{R\to\infty} \int_{I(R,R+i\xi)} f$, we have that $|\int_{I(R,R+i\xi)} f| \leq \int_{I(R,R+i\xi)} |f| \leq \int_{I(R,R+i\xi)} \sup_{z \in I(R,R+i\xi)^*} |f| = |\xi| \sup_{z\in I(R,R+i\xi)^*} |f|$.

Now for $z \in I(R, R+i\xi)^*$ we have $z = R + i\eta$ for $\eta \in [0, \xi]$, so $|f(z)| = |f(R+i\eta)| = |\exp(-(R+i\eta)^2/2)| = |\exp(-(R^2 + 2iR\eta - \eta^2)/2)| = |\exp((\eta^2 - R^2)/2)\exp(iR\eta)| = |\exp((\eta^2 - R^2)/2)| \leq |\exp((\xi^2 - R^2)/2)| \longrightarrow 0$, so the integral goes to 0.

Similar calculations happens for the integral on $I(-R + i\xi, -R)$.

So $0 = \lim\limits_{R\to\infty} \int_{\gamma_R} f = \lim\limits_{R\to\infty} \int_{I(-R,R)} f + \lim\limits_{R\to\infty} \int_{I(R+i\xi,-R+i\xi)} f$, hence $-\sqrt{2\pi} = -\lim\limits_{R\to\infty} \int_{I(-R,R)} f = \lim\limits_{R\to\infty} \int_{I(R+i\xi,-R+i\xi)} f$

Now we shall write $\int_{I(R+i\xi,-R+i\xi)} f$ explicitly:

$I(R+i\xi, -R+i\xi)(t) = \nu(t) = i\xi + R - tR$, $\nu'(t) = -R$, so $\int_{I(R+i\xi,-R+i\xi)} f = \int_0^2 f(\nu(t))\nu'(t)dt = \int_0^2 -R\exp(-(i\xi + R - tR)^2/2)dt$

Let $u = R - Rt$ and we get

$$\begin{aligned}
-\sqrt{2\pi} \quad & = \lim_{R\to\infty} \int_\nu f \\
& = \lim_{R\to\infty} \int_0^2 -R\exp(-(i\xi + R - tR)^2/2)dt \\
& = \int_\infty^{-\infty} \exp(-(i\xi + u)^2/2)du \\
& = \int_\infty^{-\infty} \exp((\xi^2 - 2i\xi u - u^2)/2)du \\
& = \int_\infty^{-\infty} \exp(\xi^2/2)\exp(-i\xi u)\exp((-u^2)/2)du \\
& = -\exp(\xi^2/2)\int_{-\infty}^{\infty} \exp(i\xi u)\exp((-(-u)^2)/2)du \\
& = -\exp(\xi^2/2)\int_{-\infty}^{\infty} \exp(i\xi u)f(u)du \\
& = -\exp(\xi^2/2)\hat{f}(\xi) \\
& = -\frac{\hat{f}(\xi)}{f(\xi)}
\end{aligned}$$

Hence we have $\hat{f}(\xi) = \sqrt{2\pi} f(\xi)$

Exercise 6.

Part 6.1.

f is analytic on the disk $\overline{B_1(0)}$ means by definition that there exists an open set $B \supseteq \overline{B_1(0)}$ on which f is analytic.

So by Cauchy formula we have $f'(0) = \frac{1}{2\pi i} \int_{C(0,1)} \frac{f(z)}{z^2} dz$.

With $C(0,1)(t) = \exp(it)$, now we can define $R(0,1) = -\exp(it) = C(0,1)(t+\pi)$. The integral over $C(0,1)$ and $R(0,1)$ are equal, as they are just moving the starting point, so $f'(0) = \frac{1}{2\pi i} \int_{R(0,1)} \frac{f(z)}{z^2} dz = \int_0^{2\pi} \frac{f(-\exp(it))}{(-\exp(it))^2} \cdot (-i \exp(it)) dt = \frac{1}{2\pi i} \int_{C(0,1)} \frac{-f(-z)}{(-z)^2} dz$