

# Exercise 5

Holo

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## Exercise 1. *Suslin trees*

By definition Suslin trees satisfy c.c.c.

Let  $T$  be a Suslin tree, for each  $\alpha \in \omega_1$  let  $D_\alpha$  be the set of nodes in  $T$  whose rank is greater or equal to  $\alpha$  (these sets are dense by the virtue of how we defined them, if we exclude the condition that each node has arbitrarily large extension we just add to  $D_\alpha$  all of the leaves of smaller rank), and  $\mathfrak{D}$  be the set of all  $D_\alpha$ , let  $G$  be  $\mathfrak{D}$ -generic for  $T$ .

Because  $T$  is a tree, 2 elements are compatible if and only if they are comparable, in particular  $G$  is a chain. Assume it is not maximal, let  $t \in T$  be element that is comparable with all of  $G$ , because  $G$  is closed downwards  $t$  must be above all of  $G$ , let  $\alpha$  be the rank of  $t$  and  $t' \in D_{\alpha+1} \cap G$ , we have that  $t, t'$  are comparable and the rank of  $t'$  is greater so  $t < t'$ , contradiction.

In addition  $\{\bigcup G \restriction \alpha \mid \alpha \in \omega_1\} = G$ , but it is clearly of cardinality  $\aleph_1$ , contradiction to the fact  $T$  is Suslin.

## Exercise 2. *Almost disjoint families*

### Part 2.1.

Let  $A_i$  be the powers of the  $i^{\text{th}}$  prime, this gives a collection of  $\aleph_0$ -many disjoint infinite subsets of  $\mathbb{N}$ .

For each  $\alpha \in \omega_1 \setminus \omega$  let  $f_\alpha : \alpha \rightarrow \omega$  be a bijection, assume that  $A_\beta$  is defined for each  $\beta < \alpha$  and that  $A_\beta$  are all infinite and that  $\{A_\beta\}$  is almost disjoint family.

Let  $\{B_i\}_{i \in \omega}$  be the set  $\{A_\beta\}$  mapped to order type  $\omega$  using  $f_\alpha$ .

Let  $a_i$  be an element from  $B_i \setminus (\bigcup_{j < i} B_j)$ , because  $B_i$  is infinite for each  $i \in \omega$  and  $B_i \cap (\bigcup_{j < i} B_j)$  is finite,  $a_i$  is well defined for each  $i \in \omega$ , furthermore for each  $i \in \omega$  we have that  $\{a_i\}_{i \in \omega} \cap B_j \subseteq \{a_i\}_{i \leq j}$  a finite set, so letting  $A_\alpha = \{a_i\}_{i \in \omega}$  will work.

### Part 2.2.

Given  $\vec{a}, A$ , let  $(s, F), (s, F') \in \mathbb{Q}(\vec{a}, A)$  we have that  $(s, F \cup F') \geq (s, F), (s, F')$  trivially, and because  $s \in [\omega]^{<\omega}$  which is countable, for given uncountably many conditions, there will be some elements with the same left-side, and any 2 such elements are compatible, so the forcing notion is c.c.c

Let  $a^G$  be as in the question, to see that  $a^G$  is infinite we claim that if  $(s, F)$  is any condition, then there exists  $(s', F) > (s, F)$  such that  $|s'| > |s|$ , this would imply that  $\{(s, F) \mid |s| > |a^G|\}$  is dense set in  $M$  that is disjoint from  $G$ .

Let  $(s, F)$  by any condition, note that the set of finite subsets and cofinite subsets of  $\omega$  is countable, so there exists  $a_\alpha \in \vec{a}$  that is infinite-cofinite, which has infinite intersection with each cofinite set, so each  $a_\alpha$  is cofinite.

In addition if  $\{b_i\}_{i \in I}$  is almost disjoint family and  $J \subseteq I$  is finite we have that  $\{b_i\}_{i \in I \setminus J} \cup \{\bigcup_{j \in J} b_j\}$  is also almost disjoint family.

Hence we can conclude that  $B = \{a_\alpha\}_{\alpha \in \omega_1 \setminus F} \cup \{\bigcup_{\alpha \in F} a_\alpha\}$  is almost disjoint, from the previous fact we get that  $\bigcup_{\alpha \in F} a_\alpha$  is cofinite, let  $k \in \omega \setminus \bigcup_{\alpha \in F} a_\alpha$  and we get that  $(s \cup \{k\}, F) > (s, F)$ . We had used only  $\aleph_0$  many dense sets to show this ( $D_n = \{(s, F) \mid |s| > n\}$ ).

Let  $j \notin A$  and assume  $a^G \cap a_j$  is finite, similarly to above, we can see that  $\{(s, F) \mid |s \cap a_j| > |a^G \cap a_j|\}$  is dense, indeed we saw before that  $B$  is almost disjoint, and because  $F \subseteq A$  it means that  $a_j \cap \bigcup_{\alpha \in F} a_\alpha$  is finite, in particular  $a_j \setminus \bigcup_{\alpha \in F} a_\alpha$  is not empty, and we can choose a  $k$  from there. We had used  $|\omega_1 \setminus A| \times \aleph_0 \leq \aleph_1 \times \aleph_0 = \aleph_1$  many dense sets for this (to be precise, we can remove the reference to  $a^G$  and use the dense sets  $D_{j,n} = \{(s, F) \mid |s \cap a_j| > n\}$ ).

Lastly, let  $j \in A$ , we want to show that  $a^G \cap a_j$  is finite, to do this we will show that there exists some finite  $a \subseteq a_j$  such that for each  $(s, F) \in G$  we have that  $s \cap a_j \subseteq a$  and conclude that  $a^G \cap a_j \subseteq a$  is finite.

Indeed, if for every  $(s, F) \in G$  we have that  $s \cap a_j = \emptyset$  we are done, so let  $(s', F') \in G$  be some fixed condition such that  $s' \cap a_j \neq \emptyset$ .

Because  $D^j = \{(s, F) \mid j \in F\}$  is dense (because we can always strengthen  $(s, F)$  to  $(s, F \cup \{j\})$ ) there must be some  $(z, W) \in G$  with  $j \in W$ , WLOG assume  $(z, W) > (s', F')$  (if not, just take the common strengthening), and let  $a = z \cap a_j \supseteq s' \cap a_j \neq \emptyset$ .

Assume  $(s, F) \in G$  such that there exists  $k \in s \cap a_j \setminus a$  and let  $(t, Q)$  be common strengthening of  $(s, F), (z, W)$ . Because  $(t, Q) \geq (s, F)$  we must have that  $k \in t$ , in particular  $k \in (t \setminus z) \cap a_j \neq \emptyset$ , but  $j \in W$ , so  $(t, Q) \not\geq (z, W)$ , contradiction. For this argument we used  $|A| \leq \aleph_1$  many dense sets.

### Part 2.3.

Assume  $\mathbf{MA}_{\omega_1}$  and fix some  $\vec{a}$  almost disjoint sequence as in the previous parts, for each  $A \subseteq \omega_1$  let  $\mathfrak{D}_A$  be the set of all dense sets we used in the previous part and let  $G_A$  be  $\mathfrak{D}_A$ -generic for  $\mathbb{Q}(\vec{a}, A)$  (note that  $|\mathfrak{D}_A| \leq \aleph_0 + \aleph_1 + \aleph_1 = \aleph_1$  and that  $\mathbb{Q}(\vec{a}, A)$  is c.c.c.).

Define the function  $f : 2^{\aleph_1} \rightarrow 2^{\aleph_0}$  as  $f(A) = a^{G_A}$ , because  $A$  is recoverable from  $a^{G_A}$  alone (using  $\vec{a}$  as a parameter, in particular we don't need to know what  $G_A$  is),  $f$  must be injective, hence  $2^{\aleph_1} \leq 2^{\aleph_0}$ , and the other direction is trivial.