Exercise 1

Yuval Paz

Saturday 2nd November, 2024

Exercise 1.

Part 1.1.

We need to show that $\mathcal{A}$ is closed under finite unions, it is enough to show that $\mathcal{A}$ is closed under union of size 2.

Let $Z, Y \in \mathcal{A}$, then $Z \cup Y = X \setminus ((X \setminus Y) \setminus Z) \in \mathcal{A}$

Part 1.2.

First let's note that $X \in \mathcal{A}_0$ hence $X \in \mathcal{A}$.

Now given $Z, Y \in \mathcal{A}$ there exists some n such that $Z, Y \in \mathcal{A}_n$ hence $Z \setminus Y \in \mathcal{A}_n$ and so $Z \setminus Y \in \mathcal{A}$ and we are done.

Part 1.3.

Note that given a set X and a finite family F over X, $\sigma(F)$ is finite.

So let $X = \mathbb{N}$, and recursively define:

- $\mathcal{A}_0 = \sigma(\emptyset)$
- $\mathcal{A}_{n+1} = \sigma(\mathcal{A}_n \cup \{n\})$
- $\mathcal{A} = \bigcup \mathcal{A}_n$

Because $\mathcal{A}$ contains all of the singletons, if it was a σ-algebra it would contains the set of even numbers, but clearly no $\mathcal{A}_n$ contains that set, so $\mathcal{A}$ is not a σ-algebra

Exercise 2.

Part 2.1.

For every $x \in U$ let I_x be an interval containing x and a subset of U (such interval exists by definition).

Given $x, y \in U$, we say $x \sim y$ if there exists $\{x, y\} \subseteq J \subseteq U$ such that $\bigcup_{z \in J} I_z$ is an interval.

This is clearly a equivalence relation, furthermore $\forall x \forall z \in I_x$ $(x \sim z)$. Let I be an equivalent class, and $z \in I$, then $I_z \subseteq I$, hence I is open.

Furthermore, if $x \sim y$ and $J \subseteq U$ witness that, then $x \sim w$ for every w in $\bigcup_{z \in J} I_z$.

If I is not an interval it means that there exists $a < b < c$ such that $a, c \in I$ but $b \notin I$, but that is impossible, from the previous sentence.

Part 2.2.

Every family of disjoint intervals in $\mathbb{R}$ is at most countable, indeed every interval contains some rational number and there are only countably many rational numbers.

Part 2.3.

Let $\mathcal{I}$ be the collection of open intervals and τ the collection of open sets, then from the previous 2 parts we have that

$$\mathcal{I} \subseteq \tau \subseteq \sigma(\mathcal{I}) \implies \sigma(\mathcal{I}) \subseteq \sigma(\tau) = \mathcal{B}(\mathbb{R}) \subseteq \sigma(\sigma(\mathcal{I})) = \sigma(\mathcal{I})$$

Exercise 3.

$(a) \implies (b)$: trivial.

$(b) \implies (c)$: assume (b), and let (A_n) be an $\subseteq$-increasing sequence of sets in $\mathcal{M}$, then $B_n = A_n \setminus (\bigcup_{k<n} A_k)$ is a sequence of disjoint subsets in $\mathcal{M}$ (it is in $\mathcal{M}$ because $\mathcal{M}$ is an algebra), from the assumption $\bigcup A_n = \bigcup B_n \in \mathcal{M}$

$(c) \implies (a)$: assume (c) and let (A_n) be arbitrary sequence of elements from $\mathcal{M}$, then $B_n = \bigcup_{k<n} A_k$ is an increasing sequence of elements from $\mathcal{M}$ (it is in $\mathcal{M}$ because $\mathcal{M}$ is an algebra), from the assumption $\bigcup A_n = \bigcup B_n \in \mathcal{M}$

Exercise 4.

Given $Z, Y \in \mathcal{M}_1$, then $Z = f^{-1}(r), Y = f^{-1}(t)$ for $r, t \in \mathcal{M}_2$.

- $X_1 = f^{-1}(X_2) \in \mathcal{M}_1$
- $Z \setminus Y = f^{-1}(r) \setminus f^{-1}(t) = f^{-1}(r \setminus t) \in \mathcal{M}_1$.
- Let $(A_n) = (f^{-1}(B_n))$ be a sequence of elements from $\mathcal{M}_1$, then $\bigcup A_n = f^{-1}(\bigcup B_n) \in \mathcal{M}_1$

Exercise 5.

Part 5.1.

Let A be the set of atoms in $\mathcal{M}$ (the set of non-empty sets that are $\subseteq$-minimal).

Every element in $\mathcal{M}$ is either disjoint or a superset of every atom, in particular all of the atoms are disjoint.

If A is infinite we are done. Otherwise $X \setminus \bigcup A \in \mathcal{M}$. Because $\mathcal{M}$ is infinite and 2^A is finite, then $X \setminus \bigcup A \neq \emptyset$.

Define the following binary tree:

- Define $T_\Lambda = X \setminus \bigcup A$ (Λ is the empty sequence)
- Given a finite binary sequence τ for which T_τ is already defined, let $\emptyset \neq A \subsetneq T_\tau$ be an element of $\mathcal{M}$ (it exists because T_τ is not an atom), then define $T_{\tau\frown\{0\}} = A, T_{\tau\frown\{1\}} = T_\tau \setminus A$
- Define T be the tree $(\{T_\tau \mid \tau \in 2^{<\mathbb{N}}\}, \subseteq)$

Notice that given $\tau, \sigma \in 2^{<\mathbb{N}}$, if τ is not an initial segment of σ and vice versa, then $T_\tau \cap T_\sigma = \emptyset$.

So $\{T_\tau \mid \tau \in 1^{<\mathbb{N}} \times \{1\}\}$ ($1^{<\mathbb{N}} \times \{1\} = \{1, 01, 001, 0001, \ldots\}$) is an infinite set of disjoint elements from $\mathcal{M}$.

Part 5.2.

If A is infinite set of disjoint elements from $\mathcal{M}$ then it contains a countable infinite subset B, so $\{\bigcup J \mid J \subseteq B\} \subseteq \mathcal{M}$ but the former has cardinality $2^{\aleph_0} = \mathfrak{c} > \aleph_0$