Exercise 1

Holo

Saturday 29$^{\text{th}}$ July, 2023

Exercise 1.

Let D be a dense set in $\mathbb{P}$ and G be as in the question.

Let $D\uparrow$ be the upwards closure of D, $D\uparrow$ is an open dense.

Let $x \in D \uparrow \cap G$, by definition there is some $y \in D$ such that $x \geq y$, and because G is closed downwards we have that $y \in D \cap G$ and hence G is generic.

Exercise 2.

Part 2.1.

Let $Z \subseteq \mathbb{P}$ and let D_Z be as in the question, and let $x \in \mathbb{P}$ be any element. If x is incompatible with every element in Z then $x \in D_Z$, otherwise there is $p \in Z$ such that $x \| p$, let q be element that strengthening both x, p, because $p \in Z$ we have that $q \in D_Z$, so D_Z is dense.

Assume $x \in D_Z$ and y stronger than x, if there exists $p \in Z$ such that $y \| p$ then there is strengthening of both, q, this q is also stronger than x, in particular $x \not\perp p$ and so there exists $r \in Z$ that is weaker than x, in particular it is also weaker than y and hence $y \in D_Z$. Otherwise y is incompatible with every $p \in Z$ hence $p \in D_Z$.

Part 2.2.

Let $A \subseteq \mathbb{P}$ be maximal antichain.

Assume G is generic, from the previous part we know that D_A is dense, hence there is a $p \in D_A \cap G$, but from maximality of A there is no $p \in \mathbb{P}$ that is incompatible with all of A, hence $p \in D_A \cap G \implies p \in \{q \in \mathbb{P} \mid \exists r \in A(r < q)\} \cap G$, but G is closed downwards, so $A \cap G \neq \emptyset$.

Assume G intersects with every maximal antichain A, and let $D \subseteq \mathbb{P}$ be dense set. Let A' a maximal antichain in the restricted order to D (From AC, every poset has maximal antichain).

I claim that A' is a maximal antichain in $\mathbb{P}$. Indeed it is an antichain because if $p, q \in A'$ are compatible in $\mathbb{P}$, let r be a witness, and then there is some $r \leq t \in D$ that will witness it in D.

To see it is maximal, let $x \in \mathbb{P}$, because D is dense there exists some $p \in D$ such that

$x \leq p$, then either $p \in A'$ or there exists some $q \in A'$ such that $p \| q$ from the maximality of A', in either cases there exists some $y \in A'$ such that $x \| y$, in particular A' is maximal antichain in $\mathbb{P}$ as well, so G intersects with A' which is a subset of D, so G intersects with D and hence it is generic.

Exercise 3.

Let G be a generic, and let G' be the complement of G in $\mathbb{P}$.

Let $x \in \mathbb{P}$ be any element and let $r, t > x$ be 2 incompatible elements, because they are incompatible at most one of them is in G, so at least one of them is in G', hence G' is dense and we get that $G' \cap G$ is nonempty, contrary to the definition of G'.

Exercise 4.

Let Let $\mathbb{P}, \mathcal{D}$ be as in the question.

Part 4.1.

Take $G, G_{\mathcal{D}}$ as in the question and let $A \subseteq \mathcal{D}$ be a maximal antichain of $\mathcal{D}$, in 2.2 I have shown that A is a maximal antichain of $\mathbb{P}$ as well, hence from 2.2 we have $G \cap A = G \cap \mathcal{D} \cap A = G_{\mathcal{D}} \cap A \neq \emptyset$ and from 2.2 again we get that $G_{\mathcal{D}}$ is generic in $\cap D$.

Part 4.2.

Take $G, G_{\mathcal{D}}$ as in the question and let $A \subseteq$ be any dense set.
For each $x \in A$ choose some element $y \in \mathcal{D}$ that is stronger, call the set of all such y as $A' \subseteq \mathcal{D}$. Notice that A' is dense in $\mathcal{D}$, indeed if $x \in \mathcal{D}$ then there exists some $y \in A$ stronger and hence $r \in A'$ that is stronger than both. Let $p \in G_{\mathcal{D}} \cap A'$, p must be stronger than some $q \in A$ by construction, hence $q \in G \cap A$ which means that G is generic.