Complement Like Operator

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1 Introduction

Given a set X, we can uniquely identify the complement operator $^c : \mathcal{P}(X) \to \mathcal{P}(X)$ using 3 properties:

1. for all $a \subseteq X$ we have $a^c \cap a = \emptyset$
2. for all $a \subseteq X$ we have $a^c \cup a = X$
3. for all $a \subseteq X$ we have $(a^c)^c = a$

We wish to explore "complement like operators", an operator $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 2 out of those 3 properties:

Definition 1.1. $\blacksquare$-complement is an operator $\mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 1 and 2

Definition 1.2. $\bullet$-complement is an operator $\mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 1 and 3

Definition 1.3. $\star$-complement is an operator $\mathcal{P}(X) \to \mathcal{P}(X)$ that satisfy only 2 and 3

Lemma 1.4. *There are no $\blacksquare$ operator*

Proof. We will now show that property 1 and 2 implies property 3:
Let * be $\blacksquare$ operator, By property 1 we have $x \in a^* \subseteq X$ implies $x \notin a$, so $a^* \subseteq a^c$. Similarly by property 2 we have $x \notin a$ implies $x \in a^*$ so $a^c \subseteq a^*$, hence $a^c = a^*$, thus property 3 also holds. $\square$

So now we only need to consider $\star$ and $\bullet$ operators. The following theorems will justify only considering one of those 2 operators:

Theorem 1.5. *There is bijection between the set of all $\bullet$ operators and the set of all $\star$ operators on $\mathcal{P}(X)$.*

Proof. Let * be a $\star$ operator, then $^* \to ^{c*c}$.

This is indeed a $\bullet$ operator: $(A^{c*c})^{c*c} = A^{c*cc*c} = A^{c**c} = A^{cc} = A$ and $x \in A^{c*c} \implies x \notin A^{c*} \implies x \notin A^{cc} = A \implies A \cap A^{c*c} = \emptyset$.

Moreover, this map is also an inverse of itself: $^{c(c*c)c} = ^*$ $\square$

From here on we will only consider $\star$-complement operators.

2 Properties of $\star$-complement operator

One can ask, is there exists a $\star$-complement operator? i.e. does there exists a set X with $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $a^* \cup a = X$ and $a^{**} = a$ for all $a \subseteq X$?

For explicit example consider $X = \mathbb{N}$, now we will construct $A \in \mathcal{P}(\mathcal{P}(X))$ like so:

1. let A_0 =the set of even natural numbers
2. If n is a natural number then:

$$A_n = A_{n-1} \cup \{\min(x \in \mathbb{N} \mid x \notin A_{n-1})\}$$

$$A_{-n} = A_{-n+1} \setminus \{\min(A_{-n+1})\}$$

Now $A = \{A_j\}_{j \in \mathbb{Z}}$.

It is easy to see that $A_j \subsetneq A_{j+1}$ for all $j \in \mathbb{Z}$, then we can define $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

$$a^* = \begin{cases} a^c & a \notin A \\ A^c_{j-1} & a = A_j \in A \end{cases}$$

This * is a $\star$-complement operator.

So, it is worth talking about $\star$-complement operators. The sharp ones may notice how $(A, \subsetneq) \cong (\mathbb{Z}, <)$, this relation is in fact the entire classification of *.

Definition 2.1. $\operatorname{cl}(A, f)$ is the closure of A under f: $\operatorname{cl}(A, f) = \bigcup_{k \in \omega} \{f^k[A]\}$

Definition 2.2. $\operatorname{clf}(A, f) = \operatorname{cl}(A, f) \cup \operatorname{cl}(A, f^{-1})$

Lemma 2.3. *If * is $\star$-complement operator on $\mathcal{P}(X)$, then for each $a \subseteq X$ we have $clf(a, ^{*c}) = \{a\}$ or $(clf(a, ^{*c}), \subsetneq) \cong (\mathbb{Z}, <)$ and there exists at least one $a \subseteq X$ is the latter. In addition, if a is finite or co-finite then $a^* = a^c$*

Proof. Let ^{*c} be such operator, and assume $\operatorname{clf}(a, ^{*c}) \neq \{a\}$, because $a^* \supsetneq a^c$ we have $a^{*c} \subsetneq a$, if $(a^{*c})^{*c} = a^{*c}$ then $(a^{*c})^* = a^*$ so $a^{*c} = a$, $a^{*c*c} \neq a$ as well because $a^{*c*c} \subsetneq a^{*c} \subsetneq a$, continuing it for both direction and that finish the proof of the first part(because ^{*c} is a bijective map this process will work for inverse as well).

Assume that a is finite(resp. co-finite) and $a^* \neq a^c$ then $\operatorname{clf}(a, ^{*c})$ has a minimum(resp. maximum), and hence is not isomorphic to $\mathbb{Z}$ $\square$

In fact, $\operatorname{clf}(a, ^{*c})$ can also be seen as the equivalence class $[a]_{\sim^\star_*}$ where $a \sim^\star_* b \iff \exists n \in \mathbb{Z}\ (A^{(*c)^n} = B)$, so $\{C \mid \exists a \subseteq X\ (\operatorname{clf}(a, ^{*c}) = C)\}$ is a partition of $\mathcal{P}(X)$.

Lemma 2.4. *If P is a partition of $\mathcal{P}(X)$ such that if $p \in P$ then either $|p| = 1$ or $(p, \subsetneq) \cong (\mathbb{Z}, <)$, and at least one $p \in P$ is the latter, then there exists $\star$-complement operator, *, such that $\{C \mid \exists a \subseteq X\ (clf(a, ^{*c}) = C)\} = P$.*

Proof. If $p \in P$ is such that $|p| = 1$ then $a^* = a^c$ for the $a \in p$.

If not, then $p = \{p_k\}_{k \in \mathbb{Z}} = \{\cdots, p_{-1} \subsetneq p_0 \subsetneq p_1 \cdots\}$, so let $p^*_k = p^c_{k-1}$. $\square$

Now we ets X we have $\star$-complement operator on $\mathcal{P}(X)$.

Theorem 2.5. *The following are equivalent:*

1. *There exists a $\star$-complement operator on $\mathcal{P}(X)$*
2. *There exists $\mathbb{Z}$-chain to $\mathcal{P}(X)$(ordered by $\subsetneq$)*
3. *X is countable union of infinite disjoint sets*
4. *$\mathcal{P}(X)$ is Dedekind infinite*

Proof. (1) $\iff$ (2) is clear by lemma 2.1 and lemma 2.2

Assume (2) let $\{P_j\}_{j\in\mathbb{Z}}$ be a chain of subsets of X, then define $C_j = P_j \setminus \bigcup_{k<j} P_k$, then $\{\bigcup_{j\in\omega} C_{\langle k,j\rangle}\}_{k\in\omega} \cup \{X \setminus \bigcup_{i\in\omega} P_i\}$ is a countable family of infinite disjoint sets whose union is X, where $\langle\cdot,\cdot\rangle : \omega^2 \to \omega$ is a pairing function.

Assume (3), let $\{C_i\}_{i\in\mathbb{Z}}$ be countable family of infinite disjoint sets whose union is X, let $P_i = \bigcup_{k<i} C_k$ to get (2).

(3) $\implies$ (4) is trivial and (4) $\implies$ (3) is due to Tarski[1]: Let X be a set such that $\mathcal{P}(X)$ is Dedekind-infinite, then let $(X_i)_{i\in\omega}$ be a sequence of subsets of X, and define the function $F : X \to \mathcal{P}(\omega)$:

Let $a \in X$, then define $F(a)_n$ for $n \in \omega$ recursively: let $F(a)_n$ be the minimal $k \in \omega$ such that $\bigcup_{i\leq n} X_{F(a)_i} \subsetneq \bigcup_{i<n} X_{F(a)_i}$ and $a \in X_k$, if not such k exists, let $F(a)_n = F(a)_{n-1}$, let $F(a) = \{F(a)_n\}_{n\in\omega}$.

If $F(a)$ is infinite we can use similar method as in the proof of (2) $\implies$ (3), if $F(a)$ is finite for all a we will note that $a \sim b \iff F(a) = F(b)$ is a equivalence relation, hence the underline equivalence classes are partition which is infinite and with injection to the set of finite subsets of ω, $\mathcal{P}_{<\omega}(\omega)$.

And because $|\mathcal{P}_{<\omega}(\omega)| = \aleph_0$, so does the partition of X, X is a countable union of infinite disjoint sets. $\square$

Definition 2.6. Δ_1-finite set is a set that is not disjoint union of 2 infinite sets

Definition 2.7. Amorphous set is an infinite Δ_1-finite set.

Theorem 2.8. *It is not provable in ZF that there are no amorphous sets.*

Proof. The proof can be found at Lévy[2] theorem 11. $\square$

Corollary 2.9. *It is not provable in ZF that if X is infinite set, then $\mathcal{P}(X)$ has a $\star$-complement operator.*

3 $\star$-strong complement operator

Now that we have shown some properties of $\star$-complement, we can ask "how far" can $\star$-complement be from the complement?

By lemma 2.1 we know that a is finite or co-finite implies any $\star$-complement of it must be the complement, can there exists a $\star$-complement where the other direction is also true? I.e. the $\star$ complement of a is the complement if and only if a is finite or co-finite?

Assuming some form of choice, in particular, assuming $\mathbb{R}$ can be well ordered, we can construct such operator:

Let $\mathcal{ICI}(X)$ be the set of infinite co-infinite subsets of X, let $<$ be well ordering of $\mathcal{ICI}(\mathbb{N})$ of order type $|\mathbb{R}|$, one can show that $A_{\subsetneq} = \{B \in \mathcal{ICI}(\mathbb{N}) \mid A \subsetneq B\}$ has the cardinality of $\mathbb{R}$ for all $A \in \mathcal{ICI}(\mathbb{N})$
Now assume we defined A^* for all $\mathcal{ICI}(\mathbb{N}) \ni A < B$, then let $B^* = \min(C \in B^c_{\subsetneq} \setminus \{A, A^*\}_{A<B})$, we can do it because the set of $A < B$ is of cardinality $< \mathbb{R}$.
For $A \notin \mathcal{ICI}(\mathbb{N})$ let $A^* = A^c$, and we are done.

The above suggest that there is a point to define $\star$-strong complement.

Definition 3.1. An operator $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is a $\star$-strong complement if it is a $\star$-complement and for all $a \subseteq X$ we have $a^* = a^c \iff a \notin \mathcal{ICI}(X)$.

One point we should be careful about in this definition is the fact that * is not the complement operator because it is $\star$-complement, and not because of $a^* = a^c \iff a \notin \mathcal{ICI}(X)$.
It is possible to define the following:

Definition 3.2. An operator $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is a $\star$-strong like complement if:

1. for all $a \subseteq X$ we have $a^c \cup a = X$
2. for all $a \subseteq X$ we have $(a^c)^c = a$
3. for all $a \subseteq X$ we have $a^* = a^c \iff a \notin \mathcal{ICI}(X)$

And contrary to what one may think, this definition is not equivalent to $\star$-strong complement in general, in fact we have the following:

Lemma 3.3. *Every $\star$-strong like complement on $\mathcal{P}(X)$ is a $\star$-complement on $\mathcal{P}(X)$ if and only if X is not amorphous set.*

But interestingly enough the following also hold:

Lemma 3.4. *For all infinite sets X there is $\star$-strong like complement on $\mathcal{P}(X)$ implies that for all infinite sets X there is $\star$-strong complement on $\mathcal{P}(X)$*

So even so the existence of a $\star$-strong like complement does not implies the existence of $\star$-strong complement, if all sets has $\star$-strong like complement then all sets has $\star$-strong complement.

Proof of lemma 3.2:

Proof. By lemma 3.1 we only need to show that there are no amorphous sets.

Let X be amorphous, then $2 \times X$ is not amorphous, by assumption we have $\star$-strong like complement on $\mathcal{P}(2 \times X)$, by lemma 3.1 we have $\star$-strong complement, by theorem 2.3 we have that $2 \times X = A_0 \cup A_1 \cup A_2$ for A_0, A_1, A_2 are infinite disjoint sets.

We have $A_0 \cap \{i\} \times X$ is infinite for some $i \in \{0, 1\}$, then both $A_1 \cap \{i\} \times X$ and $A_2 \cap \{i\} \times X$ are finite, hence both $A_1 \cap \{1-i\} \times X$ and $A_2 \cap \{1-i\} \times X$ are infinite, so we get $\pi_1 (A_1 \cap \{1-i\} \times X), \pi_1 (A_2 \cap \{1-i\} \times X)$ are 2 infinite disjoint subsets of X, which is contradiction of X being amorphous. $\square$

Theorem 3.5. *The following are equivalent:*

1. *There exists a $\star$-strong complement operator on $\mathcal{P}(X)$*
2. *There exists a partition of $\mathcal{ICI}(X)$ to $\mathbb{Z}$-chains(ordered by $\subsetneq$)*

Proof. (1) $\Longrightarrow$ (2): By lemma 2.1 we have $\mathrm{clf}(a, {}^{*c})$ is either $\mathbb{Z}$-chain or $\{a\}$ for all $a \in \mathcal{ICI}(X)$, if $\mathrm{clf}(a, {}^{*c}) = \{a\}$ then $a^{*c} = a \implies a^* = a^c \implies a \notin \mathcal{ICI}(X)$.

(2) $\Longrightarrow$ (1): Let P be partition of $\mathcal{P}(X)$ be the partition of $\mathcal{ICI}(X)$ plus singletons to the finite and co-finite elements. Let * be the operator from lemma 2.2, if * is not $\star$-strong complement, then take $a \in \mathcal{ICI}(X)$ such that $a^* = a^c$, then $\mathrm{clf}(a, {}^{*c}) = \{a\}$, this set is in the partition of $\mathcal{ICI}(X)$ so $(\{a\}, \subsetneq) \cong (\mathbb{Z}, <)$, contradiction. $\square$

4 Relation to Axiom of Choice

There are several questions we can ask about the $\star$-complement operator and AC:

1. Does the existence of a $\star$-strong complement is provable in ZF?
2. How strong exactly is the axiom "There exists a $\star$-complement operator on the power set of every infinite set"?
3. How strong exactly is the axiom "There exists a $\star$-strong complement operator on the power set of every infinite set"?

Theorem 2.3 is answers (2), that axiom is equivalent to:

$$\forall X \ (X \text{ is infinite} \implies \mathcal{P}(X) \text{ is Dedekind infinite})$$

If we to borrow definitions from Truss[3], then we have the following finiteness definition:

Definition 4.1. A cardinality κ is Δ_4-finite there is no surjective function from it to ω.

And, if we look at $\mathcal{P}(X)$ has a $\star$-complement operator as a finiteness definition, we have that it is equivalent to

$$\omega = \Delta_4$$

easily by theorem 2.3 form (3)

References

[1] Alfred Tarski. Sur les ensembles finis. *Fundamenta Mathematicae*, 6:45–95, 1924.

[2] John Truss. The independence of various definitions of finiteness. *Fundamenta Mathematicae*, 46:1—-13, 1958.

[3] John Truss. Classes of Dedekind Finite Cardinals. *Fundamenta Mathematicae*, 84:187–208, 1974.