Exercise 4

Holo

Tuesday 1^{st} August, 2023

Exercise 1. *Dense projections and Quotient posets*

Part 1.1.

Let $\pi, \mathbb{P}, \mathbb{Q}$ and $\Gamma^{\mathbb{P}}_{\mathbb{Q}}$ be as in the question, let G be any M-generic subset of $\mathbb{P}$ and let $\Gamma = (\Gamma^{\mathbb{P}}_{\mathbb{Q}})_G, \Gamma' = (\check{\pi}''\Gamma_{\mathbb{P}})_G$.

Let $p, q \in \Gamma$ and $p', q' \in \Gamma'$ be strengthening of p, q respectively, let $t, r \in G$ be such that $\pi(t) = p', \pi(r) = q'$, because G is generic we have $t, r < w \in G$, and because π is a weak homomorphism, $\pi(w) > q', p'$ hence $\pi(w) > q, p$, and almost by definition $\pi(w) \in \Gamma$.

Because Γ is closed downwards by definition, we only left to show that it intersect with every maximal anti-chain of $\mathbb{Q}$ that is in M, let A be such maximal anti-chain. For each $x \in \mathbb{Q}$ let $x^{\dagger}$ be a maximal anti-chain of $\{y \in \pi''\mathbb{P} \mid y \geq x\}$, and let $A^{\dagger} = \bigcup\{y^{\dagger} \mid y \in A\}$, because $\pi''\mathbb{P}$ is dense, we have that $A^{\dagger}$ is a maximal anti-chain as well.

Let $A^{\ddagger}$ be the inverse image of $A^{\dagger}$ by π, because π preserve incompatibility we must have that $A^{\ddagger}$ contains a maximal anti-chain, let $t \in A^{\ddagger} \cap G$, $\pi(t) \in A^{\dagger} \cap \Gamma$. By construction of $A^{\dagger}$ there exists some $p \in A$ such that $\pi(t) > p$, and because Γ is closed downwards we have that $p \in A \cap \Gamma$, hence $M[G] \vDash$ "Γ is generic over M for $\mathbb{Q}$".

Finally because G was arbitrary we can conclude that $0_{\mathbb{P}}$ forces that.

Part 1.2.

Let $G \subseteq \mathbb{P}$ be generic.

Notice that H is exactly Γ from (1.1), so H is generic over M for $\mathbb{Q}$.

Because being downwards closed is downwards absolute (from $\mathbb{P}$ to $\mathbb{P}/_{\pi}H$) and every 2 elements having a common extension in the subset is absolute (again from $\mathbb{P}$ to $\mathbb{P}/_{\pi}H$) we only need to care about it intersecting every maximal anti-chain of $\mathbb{P}/_{\pi}H$ in $M[H]$.

Let $p, q \in \mathbb{P}/_{\pi}H$, by definition $\pi(p), \pi(q) \in H \implies \pi(p) \| \pi(q) \implies p \| q$, the last implication is because π is homomorphism and the one before that is because H is a generic filter.

Let $A \subseteq \mathbb{P}/_{\pi}H$ be any maximal anti-chain in $M[H]$, that is $A = \{p\}$ for some $p \in \mathbb{P}/_{\pi}H$ (as any 2 elements of the quotient poset are compatible), so $A \in M$. Extend A to A' a maximal anti-chain in $\mathbb{P}$, because G is M generic for $\mathbb{P}$ we know that $G \cap A' \neq \emptyset$, but $G \subseteq \mathbb{P}/_{\pi}H$ so $G \cap A' = A \implies G \cap A = A \neq \emptyset$.

For the other direction, let G be any subset of $\mathbb{P}$ such that H is M-generic for $\mathbb{Q}$ and G is $M[H]$-generic for $\mathbb{P}/_\pi H$.

Similarly to before, every 2 elements in $\mathbb{P}/_\pi H$ are compatible, in particular any 2 elements in G are compatible, as $G \subseteq \mathbb{P}/_\pi H$.

Let $p \in G$ and $q < p$, because π is a weak homomorphism we have that $\pi(q) < \pi(p) \in H$, so $\pi(q) \in H \implies q \in \mathbb{P}/_\pi H$, because $q < p \in \mathbb{P}/_\pi H$ and G is closed downwards in $\mathbb{P}/_\pi H$ we must have that $q \in G$.

Lastly, let $p, q \in \mathbb{P}/_\pi H$, let $D_{p,q} = \{r \in \mathbb{P} \mid r \geq p, q \vee r \perp p \vee r \perp q\}$ and $D' = \pi'' D_{p,q}$, because $D_{p,q}$ is dense and π is dense homomorphism we know that D' is dense hence intersecting H, let $r \in D_{p,q}$ be such that $\pi(r) \in D' \cap H$. Because $\pi(r) \in H$ we have that $r \in \mathbb{P}/_\pi H$ but because every 2 elements of $\mathbb{P}/_\pi H$ are compatible in $\mathbb{P}$, we have that $r \| p$ and $r \| q$, by construction of $D_{p,q}$ we get that $r \geq p, q$.

That means that every 2 elements in $\mathbb{P}/_\pi H$ are compatible in $\mathbb{P}/_\pi H$, so the only generic filter is $\mathbb{P}/_\pi H$, so $G = \mathbb{P}/_\pi H$.

Now take A to be any maximal anti-chain, $\pi'' A$ is a maximal anti-chain as well, hence $\{\pi(p)\} = H \cap \pi'' A$ hence $G \cap A = \mathbb{P}/_\pi H \cap A = \{p\} \neq \emptyset$

Exercise 2. *Collapse Criterion*

First note that $\mathbb{Q}/q$ absorbs $\mathbb{Q}$, indeed for each M-generic for $\mathbb{Q}/q$ we can take the downwards closure and get an M-generic for $\mathbb{Q}$.

Therefore $0_{\mathbb{Q}/q} \Vdash |\check{\omega}| = |\check{\delta}|$, so $\mathbb{Q}/q$ is not δ.c.c. and has a maximal anti-chain of size δ.

Let $\dot{g}$ be a $\mathbb{Q}$ name that $0_\mathbb{Q} \Vdash$ "$\dot{g} : \omega \to \Gamma_\mathbb{Q}$ is surjective"

We will construct recursively an embedding from $\delta^{<\omega}$ to $\mathbb{Q}$ as a tree T:

- $T_\Lambda = 0_\mathbb{Q}$
- Let $t \in \delta^{n+1}$ and assume T_t is defined, let $C_t = \{c_t^{(\alpha)} \mid \alpha \in \delta\}$ be a maximal anti-chain of size δ in $\mathbb{Q}/T_t$ such that for each $c \in C_t$ we have that c decides the value of $\dot{g}(\check{n})$ and let $T_{t \frown \{i\}} = c_t^{(i)}$

Note that the anti-chain above is always well defined, let $p \in \mathbb{Q}$ be any element and let $A \subseteq \mathbb{Q}/p$ be any maximal anti-chain that decides $\dot{g}(\check{n})$, let $q \in A$ be any element and let $B \subseteq \mathbb{Q}/q$ of cardinality δ, then $A' = A \setminus q \cup B$ is a maximal anti-chain of size δ that decides $\dot{g}(\check{n})$.

This map is clearly an injective homomorphism, to see that it is dense let $p \in \mathbb{Q}$ be any element, remember that $p \Vdash$ "$\dot{g} : \omega \to \Gamma_\mathbb{Q}$ is surjective and $\check{p} \in \Gamma_\mathbb{Q} = \text{range}\,(\dot{g})$" which implies $p \Vdash \exists n \in \check{\omega}\ \dot{g}(n) = \check{p}$ in particular there exists a name $\dot{n}$ such that $p \Vdash \dot{g}(\dot{n}) = \check{p}$, let m be a real natural such that $p \Vdash \check{m} = \dot{n}$, thus $p \Vdash \dot{g}(\check{m}) = \check{p}$.

The set $\text{Lev}_T(m+1)$ is clearly a maximal anti-chain, so either $p \in \text{Lev}_T(m+1)$, in which case we are done, or there exists $q \in \text{Lev}_T(m+1)$ that is compatible with p, because p, q are compatible and both decide $\dot{g}(\check{m})$ they must agree with one another, that is $q \Vdash \dot{g}(\check{m}) = \check{p}$, in particular $q \Vdash \check{p} \in \text{range}\,(\dot{g}) = \Gamma_\mathbb{Q}$ therefore p is inside every generic G that q is in, in other words $p < q$.

Finally, let G be M-generic for $\mathbb{Q}$, and let H' be the intersection of G with the image of the tree, it is simple to see that H' is M-generic over T let H'' be the pre-image of H' into $\delta^{<\omega}$, and let H be the downwards closure of H'' into an M-generic for $\mathrm{Coll}(\omega, \delta)$.

To see that $M[G] = M[H]$ first note that because H is definable in $M[G]$ (with parameters) it means that $M[H] \subseteq M[G]$, to see the other direction, let $H^\dagger$ be the restriction of H to $\delta^{<\omega}$, let $H^\ddagger$ be the image of $H^\dagger$ under the embedding to T, we can verify that G is exactly the downward closure of $H^\ddagger$.

First because $H^\ddagger \subseteq G$ the downwards closure of $H^\ddagger$ is clearly a subset of G, and the downwards closure is M-generic for $\mathbb{Q}$ (as T is dense).

Let $p \in G$ be any element, extend $\{p\}$ to a maximal anti-chain A, A intersects with the downwards closure of $H^\ddagger$, but p is compatible with every element of the downwards closure of $H^\ddagger$, so the intersection must be $\{p\}$, so p is in the downwards closure of $H^\ddagger$.

So we got that G is definable in $M[H]$ (with parameters) and hence $M[G] \subseteq M[H]$ and we are done.

Exercise 3. *Finishing the proof of Solovay's Theorem*

Part 3.1.

Notice that $Coll(\omega, < \kappa)$ is exactly the forcing product of $Coll(\omega, \alpha)$ for $\alpha < \kappa$ with finite support.

To see that the forcing is κ.c.c let A be a family of conditions of cardinality κ, by the sunflower lemma we may assume that the places where 2 conditions has nontrivial condition is a constant finite r with $m = \max(r) < \kappa$, but then we have that the compatibility of the conditions depends only on the product up to $m + 1$, which has cardinality $< \kappa$ and hence κ.c.c.

Part 3.2.

Clearly $Coll(\omega, < \alpha) \times Coll(\omega, [\alpha, \kappa)) \cong Coll(\omega, < \kappa)$ as witness by concatenation, or in the other direction, as witness by splitting conditions at α.

Let G be generic in $Coll(\omega, < \kappa)$, and let $G' \subseteq Coll(\omega, < \alpha) \times Coll(\omega, [\alpha, \kappa))$ be image of G under the isomorphism above. We have shown in the lecture that a subset of a product forcing notion $H \times K \subseteq \mathbb{P} \times \mathbb{Q}$ is generic if and only if H is $\mathbb{P}$-generic over V, and K is $\mathbb{Q}$-generic over $M[H]$, which is exactly the situation the question asks for.

Part 3.3.

Let $G = G_{<\kappa}$ for ease.

First we notice that if $X \in M[G]$ such that X is a bounded subset of κ, then there exists some $\alpha < \kappa$ such that $X \in M[G_{<\alpha}]$. Indeed if X is as such, let $\tau = \{\{\alpha\} \times A_\alpha\}_{\alpha \in \sup X}$ be a nice name of X, because κ is regular and satisfy κ.c.c, we must have that τ is some $Coll(\omega, < \alpha)$ for some $\alpha < \kappa$, which means that $X = \tau_G = \tau_{G_{<\alpha}} \in M[G_{<\alpha}]$. In particular, if $X \in M$ is any set, we can encode biject it into an ordinal, and decode the bijection in $M[G]$, so any subset of X of cardinality $< \kappa$ first appear in some bounded stage.

Rewording the above we get it neatly: if $A \in M$ and $M[G] \models B \subseteq A \wedge |B| = \aleph_0$ then there exists $\alpha < \kappa$ such that $B \in M[G_{<\alpha}]$.

Now let $\mathbb{Q}$ be as in the question and let α be the first such ordinal and let $\lambda = \max(|\alpha|, |\mathbb{Q}|)^+$. Clearly we have that $M[H] \models$ "λ is uncountable". Because $|\mathbb{Q} \times (Coll(\omega, \lambda))^{M[H]}| = \lambda$ and it collapses λ to $\aleph_0$, by exercise 2 we have in M that $\mathbb{Q} \times (Coll(\omega, \lambda))^{M[H]} \cong Coll(\omega, \lambda) = Coll(\omega, < \lambda + 1)$.

Let $K \subseteq (Coll(\omega, \lambda))^{M[H]}$ such that $H \times K \cong G_{<\lambda+1}$, but this means that K is $(Coll(\omega, \lambda))^{M[H]}$-generic over $M[H]$. From the previous part we have that $M[G] = M[H][K][G_{[\lambda+1,\kappa)}]$.

Lastly we note that $(Coll(\omega, \lambda))^{M[H]}$ is a superset of $Coll(\omega, \lambda) = Coll(\omega, \lambda + 1)$, so any generic on the former will be generic for the latter, in particular K is such. So from the previous part again we get that there is a $Coll(\omega, < \kappa)$-generic $G^{\mathbb{Q}}$ (that comes from $K \times G_{[\lambda+1,\kappa)}$).

To see that $G^{\mathbb{Q}}$ is really generic over $M[H]$ note that K is generic over $M[H]$ and $G_{[\lambda+1,\kappa)}$ is generic over $M[H][K]$.