

Complement Like Operator

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1 Introduction

Given a set X , we can uniquely identify the complement operator $^c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ using 3 properties:

1. for all $a \subseteq X$ we have $a^c \cap a = \emptyset$
2. for all $a \subseteq X$ we have $a^c \cup a = X$
3. for all $a \subseteq X$ we have $(a^c)^c = a$

We wish to explore "complement like operators", an operator $*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ that satisfy only 2 out of those 3 properties:

Definition 1.1. ■-complement is an operator $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ that satisfy only 1 and 2

Definition 1.2. ●-complement is an operator $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ that satisfy only 1 and 3

Definition 1.3. ★-complement is an operator $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ that satisfy only 2 and 3

Lemma 1.4. *There are no ■ operator*

Proof. We will now show that property 1 and 2 implies property 3:

Let $*$ be ■ operator, By property 1 we have $x \in a^* \subseteq X$ implies $x \notin a$, so $a^* \subseteq a^c$. Similarly by property 2 we have $x \notin a$ implies $x \in a^*$ so $a^c \subseteq a^*$, hence $a^c = a^*$, thus property 3 also holds. \square

So now we only need to consider ★ and ● operators. The following theorems will justify only considering one of those 2 operators:

Theorem 1.5. *There is bijection between the set of all ● operators and the set of all ★ operators on $\mathcal{P}(X)$.*

Proof. Let $*$ be a ★ operator, then $*$ \rightarrow^{c*c} .

This is indeed a ● operator: $(A^{c*c})^{c*c} = A^{c*cc*c} = A^{c**c} = A^{cc} = A$ and $x \in A^{c*c} \implies x \notin A^{c*} \implies x \notin A^{cc} = A \implies A \cap A^{c*c} = \emptyset$.

Moreover, this map is also an inverse of itself: $c(c*c)^c = *$ \square

From here on we will only consider ★-complement operators.

2 Properties of \star -complement operator

One can ask, is there exists a \star -complement operator? i.e. does there exists a set X with $*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that $a^* \cup a = X$ and $a^{**} = a$ for all $a \subseteq X$?

For explicit example consider $X = \mathbb{N}$, now we will construct $A \in \mathcal{P}(\mathcal{P}(X))$ like so:

1. let A_0 = the set of even natural numbers
2. If n is a natural number then:

$$A_n = A_{n-1} \cup \{\min(x \in \mathbb{N} \mid x \notin A_{n-1})\}$$

$$A_{-n} = A_{-n+1} \setminus \{\min(A_{-n+1})\}$$

Now $A = \{A_j\}_{j \in \mathbb{Z}}$.

It is easy to see that $A_j \subsetneq A_{j+1}$ for all $j \in \mathbb{Z}$, then we can define $*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as follows:

$$a^* = \begin{cases} a^c & a \notin A \\ A_{j-1}^c & a = A_j \in A \end{cases}$$

This $*$ is a \star -complement operator.

So, it is worth talking about \star -complement operators. The sharp ones may notice how $(A, \subsetneq) \cong (\mathbb{Z}, <)$, this relation is in fact the entire classification of $*$.

Definition 2.1. $\text{cl}(A, f)$ is the closure of A under f : $\text{cl}(A, f) = \bigcup_{k \in \omega} \{f^k[A]\}$

Definition 2.2. $\text{clf}(A, f) = \text{cl}(A, f) \cup \text{cl}(A, f^{-1})$

Lemma 2.3. *If $*$ is \star -complement operator on $\mathcal{P}(X)$, then for each $a \subseteq X$ we have $\text{clf}(a, {}^{*c}) = \{a\}$ or $(\text{clf}(a, {}^{*c}), \subsetneq) \cong (\mathbb{Z}, <)$ and there exists at least one $a \subseteq X$ is the latter. In addition, if a is finite or co-finite then $a^* = a^c$*

Proof. Let *c be such operator, and assume $\text{clf}(a, {}^{*c}) \neq \{a\}$, because $a^* \supsetneq a^c$ we have $a^{*c} \subsetneq a$, if $(a^{*c})^{*c} = a^{*c}$ then $(a^{*c})^* = a^*$ so $a^{*c} = a$, $a^{*c*} \neq a$ as well because $a^{*c*} \subsetneq a^{*c} \subsetneq a$, continuing it for both direction and that finish the proof of the first part (because *c is a bijective map this process will work for inverse as well).

Assume that a is finite (resp. co-finite) and $a^* \neq a^c$ then $\text{clf}(a, {}^{*c})$ has a minimum (resp. maximum), and hence is not isomorphic to \mathbb{Z} \square

In fact, $\text{clf}(a, {}^{*c})$ can also be seen as the equivalence class $[a]_{\sim_*}$ where $a \sim_* b \iff \exists n \in \mathbb{Z} (A^{(*)^n} = B)$, so $\{C \mid \exists a \subseteq X (\text{clf}(a, {}^{*c}) = C)\}$ is a partition of $\mathcal{P}(X)$.

Lemma 2.4. *If P is a partition of $\mathcal{P}(X)$ such that if $p \in P$ then either $|p| = 1$ or $(p, \subsetneq) \cong (\mathbb{Z}, <)$, and at least one $p \in P$ is the latter, then there exists \star -complement operator, $*$, such that $\{C \mid \exists a \subseteq X (\text{clf}(a, {}^{*c}) = C)\} = P$.*

Proof. If $p \in P$ is such that $|p| = 1$ then $a^* = a^c$ for the $a \in p$.

If not, then $p = \{p_k\}_{k \in \mathbb{Z}} = \{\dots, p_{-1} \subsetneq p_0 \subsetneq p_1 \dots\}$, so let $p_k^* = p_{k-1}^c$. \square

Now we ets X we have \star -complement operator on $\mathcal{P}(X)$.

Theorem 2.5. *The following are equivalent:*

1. *There exists a \star -complement operator on $\mathcal{P}(X)$*
2. *There exists \mathbb{Z} -chain to $\mathcal{P}(X)$ (ordered by \subsetneq)*
3. *X is countable union of infinite disjoint sets*
4. *$\mathcal{P}(X)$ is Dedekind infinite*

Proof. (1) \iff (2) is clear by lemma 2.1 and lemma 2.2

Assume (2) let $\{P_j\}_{j \in \mathbb{Z}}$ be a chain of subsets of X , then define $C_j = P_j \setminus \bigcup_{k < j} P_k$, then $\{\bigcup_{j \in \omega} C_{\langle k, j \rangle}\}_{k \in \omega} \cup \{X \setminus \bigcup_{i \in \omega} P_i\}$ is a countable family of infinite disjoint sets whose union is X , where $\langle \cdot, \cdot \rangle : \omega^2 \rightarrow \omega$ is a pairing function.

Assume (3), let $\{C_i\}_{i \in \mathbb{Z}}$ be countable family of infinite disjoint sets whose union is X , let $P_i = \bigcup_{k < i} C_k$ to get (2).

(3) \implies (4) is trivial and (4) \implies (3) is due to Tarski[1]: Let X be a set such that $\mathcal{P}(X)$ is Dedekind-infinite, then let $(X_i)_{i \in \omega}$ be a sequence of subsets of X , and define the function $F : X \rightarrow \mathcal{P}(\omega)$:

Let $a \in X$, then define $F(a)_n$ for $n \in \omega$ recursively: let $F(a)_n$ be the minimal $k \in \omega$ such that $\bigcup_{i < n} X_{F(a)_i} \subsetneq \bigcup_{i < n} X_{F(a)_i}$ and $a \in X_k$, if not such k exists, let $F(a)_n = F(a)_{n-1}$, let $\bar{F}(a) = \{F(a)_n\}_{n \in \omega}$.

If $F(a)$ is infinite we can use similar method as in the proof of (2) \implies (3), if $F(a)$ is finite for all a we will note that $a \sim b \iff F(a) = F(b)$ is a equivalence relation, hence the underline equivalence classes are partition which is infinite and with injection to the set of finite subsets of ω , $\mathcal{P}_{<\omega}(\omega)$.

And because $|\mathcal{P}_{<\omega}(\omega)| = \aleph_0$, so does the partition of X , X is a countable union of infinite disjoint sets. \square

Definition 2.6. Δ_1 -finite set is a set that is not disjoint union of 2 infinite sets

Definition 2.7. Amorphous set is an infinite Δ_1 -finite set.

Theorem 2.8. *It is not provable in ZF that there are no amorphous sets.*

Proof. The proof can be found at Lévy[2] theorem 11. \square

Corollary 2.9. *It is not provable in ZF that if X is infinite set, then $\mathcal{P}(X)$ has a \star -complement operator.*

3 \star -strong complement operator

Now that we have shown some properties of \star -complement, we can ask "how far" can \star -complement be from the complement?

By lemma 2.1 we know that a is finite or co-finite implies any \star -complement of it must be the complement, can there exists a \star -complement where the other direction is also true? I.e. the \star complement of a is the complement if and only if a is finite or co-finite?

Assuming some form of choice, in particular, assuming \mathbb{R} can be well ordered, we can construct such operator:

Let $\mathcal{ICI}(X)$ be the set of infinite co-infinite subsets of X , let $<$ be well ordering of $\mathcal{ICI}(\mathbb{N})$ of order type $|\mathbb{R}|$, one can show that $A_{\subsetneq} = \{B \in \mathcal{ICI}(\mathbb{N}) \mid A \subsetneq B\}$ has the cardinality of \mathbb{R} for all $A \in \mathcal{ICI}(\mathbb{N})$.
Now assume we defined A^* for all $\mathcal{ICI}(\mathbb{N}) \ni A < B$, then let $B^* = \min(C \in B_{\subsetneq}^c \setminus \{A, A^*\}_{A < B})$, we can do it because the set of $A < B$ is of cardinality $< \mathbb{R}$.
For $A \notin \mathcal{ICI}(\mathbb{N})$ let $A^* = A^c$, and we are done.

The above suggest that there is a point to define \star -strong complement.

Definition 3.1. An operator $*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a \star -strong complement if it is a \star -complement and for all $a \subseteq X$ we have $a^* = a^c \iff a \notin \mathcal{ICI}(X)$.

One point we should be careful about in this definition is the fact that $*$ is not the complement operator because it is \star -complement, and not because of $a^* = a^c \iff a \notin \mathcal{ICI}(X)$.

It is possible to define the following:

Definition 3.2. An operator $*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a \star -strong like complement if:

1. for all $a \subseteq X$ we have $a^c \cup a = X$
2. for all $a \subseteq X$ we have $(a^c)^c = a$
3. for all $a \subseteq X$ we have $a^* = a^c \iff a \notin \mathcal{ICI}(X)$

And contrary to what one may think, this definition is not equivalent to \star -strong complement in general, in fact we have the following:

Lemma 3.3. *Every \star -strong like complement on $\mathcal{P}(X)$ is a \star -complement on $\mathcal{P}(X)$ if and only if X is not amorphous set.*

But interestingly enough the following also hold:

Lemma 3.4. *For all infinite sets X there is \star -strong like complement on $\mathcal{P}(X)$ implies that for all infinite sets X there is \star -strong complement on $\mathcal{P}(X)$*

So even so the existence of a \star -strong like complement does not implies the existence of \star -strong complement, if all sets has \star -strong like complement then all sets has \star -strong complement.

Proof of lemma 3.2:

Proof. By lemma 3.1 we only need to show that there are no amorphous sets.

Let X be amorphous, then $2 \times X$ is not amorphous, by assumption we have \star -strong like complement on $\mathcal{P}(2 \times X)$, by lemma 3.1 we have \star -strong complement, by theorem 2.3 we have that $2 \times X = A_0 \cup A_1 \cup A_2$ for A_0, A_1, A_2 are infinite disjoint sets.

We have $A_0 \cap \{i\} \times X$ is infinite for some $i \in \{0, 1\}$, then both $A_1 \cap \{i\} \times X$ and $A_2 \cap \{i\} \times X$ are finite, hence both $A_1 \cap \{1 - i\} \times X$ and $A_2 \cap \{1 - i\} \times X$ are infinite, so we get $\pi_1(A_1 \cap \{1 - i\} \times X), \pi_1(A_2 \cap \{1 - i\} \times X)$ are 2 infinite disjoint subsets of X , which is contradiction of X being amorphous. \square

Theorem 3.5. *The following are equivalent:*

1. *There exists a \star -strong complement operator on $\mathcal{P}(X)$*
2. *There exists a partition of $\mathcal{ICI}(X)$ to \mathbb{Z} -chains(ordered by \subsetneq)*

Proof. (1) \implies (2): By lemma 2.1 we have $\text{clf}(a, {}^{*c})$ is either \mathbb{Z} -chain or $\{a\}$ for all $a \in \mathcal{ICI}(X)$, if $\text{clf}(a, {}^{*c}) = \{a\}$ then $a^{*c} = a \implies a^* = a^c \implies a \notin \mathcal{ICI}(X)$.

(2) \implies (1): Let P be partition of $\mathcal{P}(X)$ be the partition of $\mathcal{ICI}(X)$ plus singletons to the finite and co-finite elements. Let $*$ be the operator from lemma 2.2, if $*$ is not \star -strong complement, then take $a \in \mathcal{ICI}(X)$ such that $a^* = a^c$, then $\text{clf}(a, {}^{*c}) = \{a\}$, this set is in the partition of $\mathcal{ICI}(X)$ so $(\{a\}, \subsetneq) \cong (\mathbb{Z}, <)$, contradiction. \square

4 Relation to Axiom of Choice

There are several questions we can ask about the \star -complement operator and AC:

1. Does the existence of a \star -strong complement is provable in ZF?
2. How strong exactly is the axiom "There exists a \star -complement operator on the power set of every infinite set"?
3. How strong exactly is the axiom "There exists a \star -strong complement operator on the power set of every infinite set"?

Theorem 2.3 is answers (2), that axiom is equivalent to:

$$\forall X (X \text{ is infinite} \implies \mathcal{P}(X) \text{ is Dedekind infinite})$$

If we to borrow definitions from Truss[3], then we have the following finiteness definition:

Definition 4.1. A cardinality κ is Δ_4 -finite there is no surjective function from it to ω .

And, if we look at $\mathcal{P}(X)$ has a \star -complement operator as a finiteness definition, we have that it is equivalent to

$$\omega = \Delta_4$$

easily by theorem 2.3 form (3)

References

- [1] Alfred Tarski. Sur les ensembles finis. *Fundamenta Mathematicae*, 6:45–95, 1924.
- [2] John Truss. The independence of various definitions of finiteness. *Fundamenta Mathematicae*, 46:1—13, 1958.
- [3] John Truss. Classes of Dedekind Finite Cardinals. *Fundamenta Mathematicae*, 84:187–208, 1974.