Exercise 1

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Exercise 1.

Part 1.1.

Let $A \subseteq \Omega$ with probability 0, and $B \subseteq \Omega$ any event with some probability α.
Let $B' = B \setminus A$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A \cup B') = \mathbb{P}(A) + \mathbb{P}(B') = \mathbb{P}(B')$.
We also have $\mathbb{P}(B) = \mathbb{P}((B \cap A) \cup B') = \mathbb{P}(B \cap A) + \mathbb{P}(B')$
From 1.2, $\mathbb{P}(B \cap A) \leq \mathbb{P}(A) \implies \mathbb{P}(B \cap A) = 0 \implies \mathbb{P}(B) = \mathbb{P}(B')$

Part 1.2.

Let $A \subseteq B$, and let $B' = B \setminus A$, then $\mathbb{P}(B) = \mathbb{P}(A \cup B') = \mathbb{P}(A) + \mathbb{P}(B) \geq \mathbb{P}(A)$ as $\mathbb{P}(B) \geq 0$

Part 1.3.

Let $(\Omega, \mathbb{P})$ be any discrete probability space, let $\text{❀} \notin \Omega$, and define $(\Omega \cup \{\text{❀}\}, \mathbb{P}^{\text{❀}})$ be discrete probability space defined as: $\mathbb{P}^{\text{❀}}(A) = \mathbb{P}(A \setminus \{\text{❀}\})$.

Clearly this is a probability space ($\mathbb{P}^{\text{❀}}(\Omega \cup \{\text{❀}\}) = \mathbb{P}(\Omega) = 1$, and given any countable set of disjointed subsets of $\Omega \cup \{\text{❀}\}$, at most one of them contains ❀, removing the flower from this specific set and looking at the σ-additivity of $\mathbb{P}$ gives the result)

It is also discrete, as if p is a discrete probability function inducing $\mathbb{P}$, then $p^{\text{❀}}$ defined as p on Ω and $p(\text{❀}) = 0$ will induce $\mathbb{P}^{\text{❀}}$.

In this probability space, let $A \subseteq \Omega$, then $A \subsetneq A \cup \{\text{❀}\}$ but $\mathbb{P}^{\text{❀}}(A) = \mathbb{P}^{\text{❀}}(A \cup \{\text{❀}\})$

Part 1.4.

If $\mathbb{P}(A \cap B) = \alpha \in [0, 1]$, the only way for the inequality to fail is for $\mathbb{P}(A) + \mathbb{P}(B) > 1 + \alpha$

Now let A', B' defined as in 1.1 and 1.2, then we have $\mathbb{P}(A) = \mathbb{P}(A') + \alpha \leq 1 \implies \mathbb{P}(A) \leq 1 - \alpha$, and similarly for B so $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A') + \mathbb{P}(B') + 2\alpha > 1 + \alpha \implies \mathbb{P}(A') + \mathbb{P}(B') + \alpha > 1$, but by the definition $\alpha = \mathbb{P}(A \cap B)$, and $A', B', A \cap B$ are all disjoints, so we get that $\mathbb{P}(A' \cup B' \cup (A \cap B)) > 1$, contradiction.

Part 1.5.

Let A', B' be as defined in 1.1 and 1.2.

We have $\mathbb{P}(A) = \mathbb{P}(A') + \mathbb{P}(A \cap B)$ and $\mathbb{P}(B) = \mathbb{P}(B') + \mathbb{P}(A \cap B)$

Notice that $B' \cap A' = \emptyset$, so adding the 2 equations we get $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A' \cup B') + 2\mathbb{P}(A \cap B) \implies \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B) = \mathbb{P}(A' \cup B')$

But $A' \cup B'$ is exactly $A \Delta B$, so we are done.

Exercise 2.

Let $\mathbb{P}$ be a probability function satisfying the conditions in the question.

Because $\mathbb{N}$ is countable, so every subset of $\mathbb{N}$, so $A = \cup_{n \in A}\{n\}$ is countable union of disjoint sets, hence $\mathbb{P}(A) = \sum_{n \in A} \mathbb{P}(\{n\})$, hence it is enough to show that there is a unique discrete probability function p on $\mathbb{N}$ satisfying $p(n) = 3p(n+1)$.

Notice that given 2 such discrete probability functions that agree on a single number must be equal.

Let $p(0) = \alpha$, by definition of discrete probability function we have $\sum_{n \in \mathbb{N}} \alpha/3^n = \alpha \cdot \sum_{n \in \mathbb{N}} 1/3^n = 1 \implies \alpha = \frac{1}{\sum_{n \in \mathbb{N}} 1/3^n}$, hence any 2 discrete probability functions satisfying $p(n) = 3p(n+1)$ must have the same value at 0, but this implies that they are equal.

$\mathbb{P}(\mathbb{N})$ must be 1, as $\mathbb{P}$ is a probability function, and (assuming $3\mathbb{N}$ means $\{3n \mid n \in \mathbb{N}\}$) $\mathbb{P}(3\mathbb{N}) = \sum_{n \in \mathbb{N}} \alpha/3^{3n}$

Exercise 3.

Define $I_n = \{i \in I \mid a(i) \in \left[\frac{1}{n+1}, \frac{1}{n}\right)\}$ (where we treat $\frac{1}{0}$ as $+\infty$) and $I = \bigcup I_n = \{i \in I \mid a(i) > 0\}$.

I is a countable union of sets, so if $|I| > \aleph_0$, there must be some $n \in \mathbb{N}$ such that $|I_n| \geq \aleph_0$ (I include 0 in $\mathbb{N}$)

But if $J \subseteq I_n$ is finite, then $\sum_J a(i) \geq \frac{|J|}{n+1}$, and because I_n is infinite, we can take J to be as big as we want.

Exercise 4.

All of the questions are about countable spaces, so when defining the probability function it is enough to only specify it on the singletons, and from σ-additivity the rest follows.

Part 4.1.

Ω will be $[6]^{10}$ modulo $\sim$ where $a \sim b$ if there exists a permutation on $[6]$ such that $\tau(a) = b$ (alternatively multisets of size 10 of elements from $[6]$) and define $\mathbb{P}(\{a\}) = \frac{|a|}{6^{10}}$ (remember that a is a set of elements in $[6]^{10}$ that is an equivalent class, so it's size is the amount of ways to get a specific set of results).

Let $\Omega' = [6]^{10}$ with uniform distribution $\mathbb{P}'$.

Now let $A = \{x \in \Omega \mid 1 \in x\}$ (here we view x as a multiset), by construction, $\mathbb{P}(A) = \mathbb{P}'(\bigcup A) = \mathbb{P}'(A')$ where A' is the set of elements from $[6]^{10}$ that contains 1 at some point. Looking at $\mathbb{P}([6]^{10} \setminus A') = (\frac{5}{6})^{10}$ and so $\mathbb{P}(A) = 1 - \mathbb{P}'([6]^{10} \setminus A') = 1 - (\frac{5}{6})^{10}$

Let $B = \{x \in \Omega \mid |\{z \in x \mid z = 1\}| = 1\}$ (here we view x as a multiset), as before this transform into B' the set of sequences with exactly one 1, and $\mathbb{P}(B) = \mathbb{P}'(B')$.

$\mathbb{P}'(B')$ is $\frac{1}{6} \cdot \left(\frac{5}{6}\right)^9 \cdot 10$ (the probability of rolling 1 first and then 9 other numbers, times 10 (getting 1 in any position instead of only the first place)).

Part 4.2.

$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ (the set of possible sums), and $\mathbb{P}(\{a\}) = \frac{|\{x \in [6]^2 | \pi_0(x) + \pi_1(x) = a\}|}{6^2}$ where π_i is the projection to the i^{th} index.

We can count $\mathbb{P}(\{2\}) = \frac{1}{36}, \mathbb{P}(\{3\}) = \frac{2}{36}, \mathbb{P}(\{4\}) = \frac{3}{36}, \mathbb{P}(\{5\}) = \frac{4}{36}, \mathbb{P}(\{6\}) = \frac{5}{36}$ and so $\mathbb{P}(\{7, 8, 9, 10, 11, 12\}) = 1 - \mathbb{P}(\{2, 3, 4, 5, 6\}) = 1 - (\sum_{i=1}^{5} \frac{i}{36})$

Part 4.3.

Assuming people's birth month is completely uniform, this probability space is the same as 4.1 but instead of dice with 6 sides, it is dice with 12 sides.

Similarly to 4.1 let $A = \{x \in \Omega \mid \exists n(x(n) > 1)\}$ (where x is viewed as a multiset, and $x(n)$ is the count of appearance of n in x) and let A' the corresponding event in Ω'.

Note that $\mathbb{P}'([12]^{10} \setminus A')$ is exactly $\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdots 3}{12^{10}}$, so $\mathbb{P}(A) = 1 - \frac{12!}{2 \cdot 12^{10}}$

Part 4.4.

Similarly to 4.3, we can view each basket as a label in a die, so the question can be modelled as 4.1 but with dice having 8 dice and rolling 12 dice instead of 10.

Let $A = \{x \in \Omega \mid \forall n \in [8](n \in x)\}$ where x viewed as a multiset.

As usual let A' be the corresponding event in Ω'.

Using Inclusion-Exclusion we have that

$$\mathbb{P}'([8]^{12} \setminus A') = \frac{\binom{8}{1}7^{12} - \binom{8}{2}6^{12} + \binom{8}{3}5^{12} - \binom{8}{4}4^{12} + \binom{8}{5}3^{12} - \binom{8}{6}2^{12} + \binom{8}{7}1^{12} - \binom{8}{8}0^{12}}{8^{12}}$$

And $\mathbb{P}(A) = 1 - \mathbb{P}'([8]^{12} \setminus A')$

Part 4.5.

Let $\Omega = [[13] \times \{\clubsuit, \blacklozenge, \spadesuit, \heartsuit\}]^4$ (where $[x]^n$ is the set of n-sized subset of x) with uniform probability ($\mathbb{P}(\{a\}) = \frac{1}{\binom{52}{4}}$)

There are $\binom{13}{4}$ many elements in Ω that are all of Diamond suit, so from uniformity $\mathbb{P}(\{x \in \Omega \mid \forall z \in x(\pi_1(z) = \blacklozenge)\}) = \frac{\binom{13}{4}}{\binom{52}{4}}$ (where π_i is the projection to the i^{th} index)

Part 4.6.

Let $\Omega = \{M, F\} \times \{x \in \{n \in \mathbb{N} \mid n \neq 0\}^{<\omega} \mid \sum x = 16\}$ with uniform distribution. ($X^{<\omega}$ denotes the finite sequences whose range is in X)

The right element each element of Ω is indicate how many people of the same gender sit in a row, and the left index indicate if a Male or a Female sits first.

We want to calculate $\mathbb{P}(\{x \in \Omega \mid \operatorname{range}(\pi_1(x)) = \{1\}\})$ (where π_i is the projection to the i^{th} index), clearly $|\{x \in \Omega \mid \operatorname{range}(\pi_1(x)) = \{1\}\}| = 2$.

$|\{x \in \{n \in \mathbb{N} \mid n \neq 0\}^{<\omega} \mid \sum x = 16\}|$ is well known to be 2^{15}, hence $|\Omega| = 2^{16}$ and so $\mathbb{P}(\{x \in \Omega \mid \operatorname{range}(\pi_1(x)) = \{1\}\}) = \frac{1}{2^{15}}$

Part 4.7.

Let $\Omega = [[n]]^k \times [[n]]^m$ (where $[x]^p$ is the p-sized subset of x) with uniform probability.

The left index indicate the winning tickets, and the right index indicate the tickets a person got.

We want to calculate the probability of $A = \{x \in \Omega \mid \pi_0(x) \cap \pi_1(x) \neq \emptyset\}$ where π_i is the projection to the i^{th} index.

Now fixing $w \in [[n]]^k$, we have $k \cdot \binom{n-1}{m-1}$ many elements $b \in [[n]]^m$ such that $(w, b) \in A$.

So the size of A is $\binom{n}{k} \cdot k \cdot \binom{n-1}{m-1}$, and $|\Omega| = \binom{n}{k} \cdot \binom{n}{m}$ so $\mathbb{P}(A) = k \cdot \frac{\binom{n-1}{m-1}}{\binom{n}{m}}$

Exercise 5. *Bonus*

Part 5.1.

If $X \in \mathcal{F}$ then either X is finite, hence X^c is co-finite, or X is co-finite, hence X^c is finite.

For finite unions note that either all elements of the unions are finite, and hence the union is finite, or there exists at least 1 cofinite set in there, but any superset of a cofinite set is cofinite, hence the union is in $\mathcal{F}$ in this case as well.

Part 5.2.

$\mathbb{P}(\emptyset) = 0$ because $|\emptyset| = 0$ and $\mathbb{P}(\Omega) = 1$ because $|X| = \aleph_0$ by definition.

Given $(A_n)_{n<k}$ a finite sequence of disjoint elements from $\mathcal{F}$, either all of A_n are finite, and then $\bigcup A_n$ is finite and hence additivity present, or exactly one of A_n is cofinite (as there are no 2 disjoint cofinite subsets of Ω), and then $\bigcup A_n$ is cofinite, hence has probability 1, and the additivity presented in this case as well.

Part 5.3.

WLOG we may assume $\Omega = \mathbb{N}$ and then we can let $A_n = \{n\}$, each A_n has probability 0, but $1 = \mathbb{P}(\Omega) = \mathbb{P}(\bigcup A_n) \neq \sum \mathbb{P}(A_n)$