Cantor-Bernstein implies LEM

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Definition 0.1. O is called Binary-Sequentially-Constructive(BSC) if:

$$\forall p \in 2^O \, (\exists x \in O \; p(x) = 0) \vee p \equiv 1$$

(This is my own naming , I don't know how those sets are usually called)

Lemma 0.2. *If O is BSC and there exists surjective function $F : O \to A + B$ then A is empty or inhabited*

Proof. Define $f \in 2^O$

$$f(x) = \begin{cases} 0, & \exists a \in A \; F(x) = \mathsf{inl(a)} \\ 2, & \exists b \in B \; F(x) = \mathsf{inr(b)} \end{cases}$$

Because O is BSC then either $\exists x \; f(x) = 0 \implies \exists a \in A$ or $\forall x \; f(x) = 1 \implies A = \emptyset$, the last implication is by the fact that F is surjective. $\square$

Definition 0.3.

$$\mathbb{N}_\infty = \left\{ p \in 2^{\mathbb{N}} \mid \forall n \in \mathbb{N} \, (p(n) = 1 \implies \forall m < n (p(m) = 1)) \right\}.$$

for $n \in \mathbb{N}$

$fn \in \mathbb{N}_\infty$

$$fn(m) = \begin{cases} 1 & m < n \\ 0 & \text{Otherwise} \end{cases}$$

$f\omega \in \mathbb{N}_\infty$

$$f\omega \equiv 1$$

$fS \in \mathbb{N}_\infty^{\mathbb{N}_\infty}$

$$fS(p)(k) = \begin{cases} 1 & k = 0 \\ p(n) & k = n + 1 \end{cases}$$

Lemma 0.4. *fS is injective, and $\forall p \in \mathbb{N}_\infty \; fS(p) \not\equiv f0$*

Proof.
If $p \in \mathbb{N}_\infty$ then $fS(p)(0) = 1 \neq 0 = f0(0)$.
If $fS(p) = fS(q)$ then $\forall m \; p(m) = fS(p)(m+1) = fS(q)(m+1) = q(m)$ $\square$

Lemma 0.5. *If* $P \in 2^{\mathbb{N}_\infty}$ *and* $P(f\omega) = 1$*, and* $\forall n \in \mathbb{N}\,(P(fn) = 1)$ *then* $P \equiv 1$

Proof.
Let P be such function, and let $p \in \mathbb{N}_\infty$, it is enough to prove that $P(p) \neq 0$.
Assume that $P(p) = 0$, // Also assume $\forall k\,(k < n \implies p(k) = 0) \wedge p(n) = 0$, then $p = fn$(not sure how to prove this), but $0 = P(p) = P(fn) = 1$, so $p = f\omega$, but $0 = P(p) = P(f\omega) = 1$, contradiction. $\square$

Lemma 0.6. *There exists a function* $e \in \mathbb{N}_\infty^{\left(2^{\mathbb{N}_\infty}\right)}$ *such that* $P(e(P)) = 1 \implies P \equiv 1$

Proof.
Define for $Q \in 2^{\mathbb{N}_\infty}$

$$e(Q)(n) = \begin{cases} 1 & \forall k \leq n\; Q(fk) = 1 \\ 0 & \text{Otherwise} \end{cases}$$

By induction one can prove that $Q(fk) = 1$ for all $k \in \mathbb{N}$, and it is obvious that $Q(f\omega) = 1$, thus by lemma 0.3 $Q \equiv 1$. $\square$

Noticing that if $Q \in 2^{\mathbb{N}_\infty}$ then either $Q(e(Q)) = 0$, in which case $\exists p \in \mathbb{N}_\infty\; Q(p) = 0$, or $Q(e(Q)) = 1$, then by Lemma 0.4 $Q \equiv 1$, so $\mathbb{N}_\infty$ is BSC.

Theorem 0.7. $\mathsf{CB} \implies \mathsf{LEM}$

Proof. Let pr be some proposition, and $A = \{0 \mid pr\} \subseteq 1$ and define:
$f : \mathbb{N}_\infty \to A + \mathbb{N}_\infty$ with $f(p) = \mathsf{inr(p)}$
$g : A + \mathbb{N}_\infty \to \mathbb{N}_\infty$ with $g(\mathsf{inl(0)}) = \mathsf{f0}$ and $g(\mathsf{inr(p)}) = \mathsf{fS(p)}$.

Both are injective, so by CB there exists bijection $h : \mathbb{N}_\infty \to A + \mathbb{N}_\infty$, in particular, h is surjective, so by Lemma 0.1, either $A = \emptyset$, which implies $\neg pr$, or A is inhabited, which implies pr. $\square$