Exercise 5

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Sunday 11^{th} February, 2024

Exercise 1.

Notice that $-\cos(x) = \sin^{(3)}(x)$, in particular, from Cauchy formula, we have:

$$-1 = -\cos(0) = \sin^{(3)}(x) = \frac{3!}{2\pi i}\int_{C(0,r)} \frac{\sin(\xi)}{(\xi - 0)^{3+1}} d\xi$$

Solving the equation we get that the integral is equal to $-\frac{\pi i}{3}$

Exercise 2.

Because f is entire it satisfy Cauchy formula for any $z \in \mathbb{C}, r \in \mathbb{R}^+, n \in \mathbb{N}$:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C(z,r)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

Using our favourite inequality we get:

$$\begin{aligned}
|f^{(\lfloor d \rfloor)}(z)| &\leq \frac{\lfloor d \rfloor !}{2\pi}\int_{C(z,r)} \left|\frac{f(\xi)}{(\xi - z)^{\lfloor d \rfloor + 1}}\right| d\xi \\
&\leq \frac{|C|\lfloor d \rfloor !}{2\pi}\int_{C(z,r)} \frac{|z|^{\lfloor d \rfloor} + 1}{r^{\lfloor d \rfloor + 1}} d\xi \\
&= \frac{|C|\lfloor d \rfloor !}{2\pi r^{\lfloor d \rfloor + 1}}\int_{C(z,r)} |z|^{\lfloor d \rfloor} + 1 d\xi \\
&= \frac{|C|\lfloor d \rfloor !}{r^{\lfloor d \rfloor + 1}}(|r + z|^{\lfloor d \rfloor} + 1) && \xrightarrow{r \to \infty} 0
\end{aligned}$$

In particular $f^{(\lfloor d \rfloor)} \equiv 0$, taking the anti-derivative $\lfloor d \rfloor$ times gives the desired result

Exercise 4.

Let f be an entire function satisfying $f(z) = f(z+i) = f(z+1)$, then $f''\{a + ib \mid a, b \in [0,1]\} = f''\mathbb{C}$, that is because given $z = x + iy$ then $f(x+iy) = f((x - \lfloor x \rfloor) + iy) = f((x - \lfloor x \rfloor) + i(y - \lfloor y \rfloor))$.

But $f''\{a + ib \mid a, b \in [0,1]\}$ is bounded, hence f is bounded, hence constant.

Exercise 5.

Assume both f, g are not the identity 0 and Ω connected.

Let $x_0 \in \Omega$ be such that $f(x_0) \neq 0$, in particular (as f is continuous) we have that there exists $\epsilon > 0$ such that $f(z) \neq 0$ on $B_\epsilon(x_0)$.

By the assumption we have that on that ball we have $f(z)g(z) = 0 \implies g(z) = 0$, but if g is constant on an open set in a connected set, it is constant on the whole set, hence g is constant everywhere (and because it is continuous, it is constant 0).

To see that the connected assumption is necessary, let Ω_0, Ω_1 be 2 disjoint open sets, and let:

$$f(z) = \begin{cases} 0, & z \in \Omega_0 \\ 1, & z \in \Omega_1 \end{cases}$$

$$g(z) = \begin{cases} 1, & z \in \Omega_0 \\ 0, & z \in \Omega_1 \end{cases}$$

Exercise 6.

Let's rearrange our equation to get $(f'g - g'f)(1/n) = 0$..

Now assume that $h = f'g - g'f \not\equiv 0$ and let $k \in \mathbb{N}$ be the first such that $h^{(k)}(0) \neq 0$, it exists as h analytic.

Because h is analytic, it equals to it's own Taylor series, in which the first k terms disappear, so $h(z) = z^k p(z)$ for analytic p with $p(0) \neq 0$.

But because p is continuous, it has a neighborhood Ω in around 0 in which it is never 0 in there, in particular $h(z) \neq 0$ for all $z \in \Omega \setminus \{0\}$, which is impossible as there exists a natural n such that $\frac{1}{n} \in \Omega$.

This means that $h \equiv 0$, which implies that $\left(\frac{f}{g}\right)' = 0$, hence $\frac{f}{g}$ is constant.

Exercise 7.

Let f, g be as in the question.

We already saw that g is analytic, it exhibit a local maxima on compact $\Omega \subseteq B_1(0)$ only on $\partial\Omega$.

For $r \in (0,1)$, let $\Omega_r = \overline{B_r(0)}$, then $\max_{\Omega_r}(|g(z)|) = \max_{\partial\Omega_r}(|g(z)|) = \max_{|z|=r}(|g(z)|) = \max_{|z|=r}\left(\frac{|f(z)|}{|z|}\right) \leq \frac{1}{r}$.

Let $r \to 1$, and we get that $|g(z)| \leq 1$ on the whole unit ball, and we are done.

Exercise 8.

We have that:

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a+b\cos(\theta)}d\theta \qquad &= \int_0^{2\pi} \frac{1}{a+b\frac{\exp(iz)+\exp(-iz)}{2}}dz \\
&=2\int_0^{2\pi} \frac{1}{2a+b(\exp(iz)+\exp(-iz))}dz \\
&=2\int_0^{2\pi} \frac{\exp(iz)}{2a\exp(iz)+b\exp(2iz)+b}d\xi \\
&=\frac{2}{i}\int_{C(0,1)} \frac{1}{b\xi^2+2a\xi+b}d\xi \\
&=\frac{2}{i}\int_{C(0,1)} \frac{1}{(\xi-\xi_0)(\xi-\xi_1)}d\xi \\
&\quad \text{Where } \xi_0 = \frac{-a+\sqrt{a^2-b^2}}{b}, \xi_1 = \frac{-a-\sqrt{a^2-b^2}}{b} \\
&=\frac{2}{i}\int_{C(0,1)} \frac{b}{2\sqrt{a^2-b^2}}\frac{1}{\xi-\xi_0} - \frac{b}{2\sqrt{a^2-b^2}}\frac{1}{\xi-\xi_1}d\xi \\
&=\frac{b}{\sqrt{a^2-b^2}i}\int_{C(0,1)} \frac{1}{\xi-\xi_0} - \frac{1}{\xi-\xi_1}d\xi
\end{aligned}$$

Only x_0 is in the unit circle, so the integral of the x_1 component is zero and the final answer is $\frac{2\pi b}{\sqrt{a^2-b^2}}$.