Exercise 5

Holo

Tuesday 1^{st} August, 2023

Exercise 1. *Suslin trees*

By definition Suslin trees satisfy c.c.c.

Let T be a Suslin tree, for each $\alpha \in \omega_1$ let D_α be the set if nodes in T whose rank is greater or equal to α (these sets are dense by the virtue of how we defined them, if we exclude the condition that each node has arbitrarily large extension we just add to D_α all of the leaves of smaller rank), and $\mathfrak{D}$ be the set of all D_α, let G be $\mathfrak{D}$-generic for T.

Because T is a tree, 2 elements are compatible if and only if they are comparable, in particular G is a chain. Assume it is not maximal, let $t \in T$ be element that is comparable with all of G, because G is closed downwards t must be above all of G, let α be the rank of t and $t' \in D_{\alpha+1} \cap G$, we have that t, t' are comparable and the rank of t' is greater so $t < t'$, contradiction.

In addition $\{\bigcup G \restriction \alpha \mid \alpha \in \omega_1\} = G$, but it is clearly of cardinality $\aleph_1$, contradiction to the fact T is Suslin.

Exercise 2. *Almost disjoint families*

Part 2.1.

Let A_i be the powers of the $i^{\mathbf{th}}$ prime, this gives a collection of $\aleph_0$-many disjoint infinite subsets of $\mathbb{N}$.

For each $\alpha \in \omega_1 \setminus \omega$ let $f_\alpha : \alpha \to \omega$ be a bijection, assume that A_β is defined for each $\beta < \alpha$ and that A_β are all infinite and that $\{A_\beta\}$ is almost disjoint family.

Let $\{B_i\}_{i\in\omega}$ be the set $\{A_\beta\}$ mapped to order type ω using f_α.

Let a_i be an element from $B_i \setminus (\bigcup_{j<i} B_j)$, because B_i is infinite for each $i \in \omega$ and $B_i \cap (\bigcup_{j<i} B_j)$ is finite, a_i is well defined for each $i \in \omega$, furthermore for each $i \in \omega$ we have that $\{a_i\}_{i\in\omega} \cap B_j \subseteq \{a_i\}_{i\leq j}$ a finite set, so letting $A_\alpha = \{a_i\}_{i\in\omega}$ will work.

Part 2.2.

Given $\vec{a}, A$, let $(s, F), (s, F') \in \mathbb{Q}(\vec{a}, A)$ we have that $(s, F \cup F') \geq (s, F), (s, F')$ trivially, and because $s \in [\omega]^{<\omega}$ which is countable, for given uncountably many conditions, there will be some elements with the same left-side, and any 2 such elements are compatible, so the forcing notion is c.c.c

Let a^G be as in the question, to see that a^G is infinite we claim that if (s, F) is any condition, then there exists $(s', F) > (s, F)$ such that $|s'| > |s|$, this would imply that $\{(s, F) \mid |s| > |a^G|\}$ is dense set in M that is disjoint from G.

Let (s, F) by any condition, note that the set of finite subsets and cofinite subsets of ω is countable, so there exists $a_\alpha \in \vec{a}$ that is infinite-coinfinite, which has infinite intersection with each cofinite set, so each a_α is coinfinite.

In addition if $\{b_i\}_{i\in I}$ is almost disjoint family and $J \subseteq I$ is finite we have that $\{b_i\}_{i\in I\setminus J} \cup \{\bigcup_{j\in J} b_j\}$ is also almost disjoint family.

Hence we can conclude that $B = \{a_\alpha\}_{\alpha\in\omega_1\setminus F} \cup \{\bigcup_{\alpha\in F} a_\alpha\}$ is almost disjoint, from the previous fact we get that $\bigcup_{\alpha\in F} a_\alpha$ is coinfinite, let $k \in \omega \setminus \bigcup_{\alpha\in F} a_\alpha$ and we get that $(s\cup\{k\}, F) > (s, F)$. We had used only $\aleph_0$ many dense sets to show this ($D_n = \{(s, F) \mid |s| > n\}$).

Let $j \notin A$ and assume $a^G \cap a_j$ is finite, similarly to above, we can see that $\{(s, F) \mid |s\cap a_j| > |a^G \cap a_j|\}$ is dense, indeed we saw before that B is almost disjoint, and because $F \subseteq A$ it means that $a_j \cap \bigcup_{\alpha\in F} a_\alpha$ is finite, in particular $a_j \setminus \bigcup_{\alpha\in F} a_\alpha$ is not empty, and we can choose a k from there. We had used $|\omega_1 \setminus A| \times \aleph_0 \leq \aleph_1 \times \aleph_0 = \aleph_1$ many dense sets for this (to be precise, we can remove the reference to a^G and use the dense sets $D_{j,n} = \{(s, F) \mid |s \cap a_j| > n\}$).

Lastly, let $j \in A$, we want to show that $a^G \cap a_j$ is finite, to do this we will show that there exists some finite $a \subseteq a_j$ such that for each $(s, F) \in G$ we have that $s \cap a_j \subseteq a$ and conclude that $a^G \cap a_j \subseteq a$ is finite.

Indeed, if for every $(s, F) \in G$ we have that $s\cap a_j = \emptyset$ we are done, so let $(s', F') \in G$ be some fixed condition such that $s' \cap a_j \neq \emptyset$.

Because $D^j = \{(s, F) \mid j \in F\}$ is dense (because we can always strengthen (s, F) to $(s, F\cup\{j\})$) there must be some $(z, W) \in G$ with $j \in W$, WLOG assume $(z, W) > (s', F')$ (if not, just take the common strengthening), and let $a = z \cap a_j \supseteq s' \cap a_j \neq \emptyset$.

Assume $(s, F) \in G$ such that there exists $k \in s \cap a_j \setminus a$ and let (t, Q) be common strengthening of $(s, F), (z, W)$. Because $(t, Q) \geq (s, F)$ we must have that $k \in t$, in particular $k \in (t \setminus z) \cap a_j \neq \emptyset$, but $j \in W$, so $(t, Q) \not\geq (z, W)$, contradiction. For this argument we used $|A| \leq \aleph_1$ many dense sets.

Part 2.3.

Assume MA_{ω_1} and fix some $\vec{a}$ almost disjoint sequence as in the previous parts, for each $A \subseteq \omega_1$ let $\mathfrak{D}_A$ be the set of all dense sets we used in the previous part and let G_A be $\mathfrak{D}_A$-generic for $\mathbb{Q}(\vec{a}, A)$ (note that $|\mathfrak{D}_A| \leq \aleph_0 + \aleph_1 + \aleph_1 = \aleph_1$ and that $\mathbb{Q}(\vec{a}, A)$ is c.c.c).

Define the function $f : 2^{\aleph_1} \to 2^{\aleph_0}$ as $f(A) = a^{G_A}$, because A is recoverable from a^{G_A} alone (using $\vec{a}$ as a parameter, in particular we don't need to know what G_A is), f must be injective, hence $2^{\aleph_1} \leq 2^{\aleph_0}$, and the other direction is trivial.

Exercise 3. *Finishing the proof of Solovay's Theorem*

Part 3.1.

Notice that $Coll(\omega, < \kappa)$ is exactly the forcing product of $Coll(\omega, \alpha)$ for $\alpha < \kappa$ with finite support.

To see that the forcing is κ.c.c let A be a family of conditions of cardinality κ, by the sunflower lemma we may assume that the places where 2 conditions has nontrivial condition is a constant finite r with $m = \max(r) < \kappa$, but then we have that the compatibility of the conditions depends only on the product up to $m + 1$, which has cardinality $< \kappa$ and hence κ.c.c.

Part 3.2.

Clearly $Coll(\omega, < \alpha) \times Coll(\omega, [\alpha, \kappa)) \cong Coll(\omega, < \kappa)$ as witness by concatenation, or in the other direction, as witness by splitting conditions at α.

Let G be generic in $Coll(\omega, < \kappa)$, and let $G' \subseteq Coll(\omega, < \alpha) \times Coll(\omega, [\alpha, \kappa))$ be image of G under the isomorphism above. We have shown in the lecture that a subset of a product forcing notion $H \times K \subseteq \mathbb{P} \times \mathbb{Q}$ is generic if and only if H is $\mathbb{P}$-generic over V, and K is $\mathbb{Q}$-generic over $M[H]$, which is exactly the situation the question asks for.

Part 3.3.

Let $G = G_{<\kappa}$ for ease.

First we notice that if $X \in M[G]$ such that X is a bounded subset of κ, then there exists some $\alpha < \kappa$ such that $X \in M[G_{<\alpha}]$. Indeed if X is as such, let $\tau = \{\{\alpha\} \times A_\alpha\}_{\alpha \in \sup X}$ be a nice name of X, because κ is regular and satisfy κ.c.c, we must have that τ is some $Coll(\omega, < \alpha)$ for some $\alpha < \kappa$, which means that $X = \tau_G = \tau_{G_{<\alpha}} \in M[G_{<\alpha}]$. In particular, if $X \in M$ is any set, we can encode biject it into an ordinal, and decode the bijection in $M[G]$, so any subset of X of cardinality $< \kappa$ first appear in some bounded stage.

Rewording the above we get it neatly: if $A \in M$ and $M[G] \models B \subseteq A \wedge |B| = \aleph_0$ then there exists $\alpha < \kappa$ such that $B \in M[G_{<\alpha}]$.

Now let $\mathbb{Q}$ be as in the question and let α be the first such ordinal and let $\lambda = \max(|\alpha|, |\mathbb{Q}|)^+$. Clearly we have that $M[H] \models$ "λ is uncountable". Because $|\mathbb{Q} \times (Coll(\omega, \lambda))^{M[H]}| = \lambda$ and it collapses λ to $\aleph_0$, by exercise 2 we have in M that $\mathbb{Q} \times (Coll(\omega, \lambda))^{M[H]} \cong Coll(\omega, \lambda) = Coll(\omega, < \lambda + 1)$.

Let $K \subseteq (Coll(\omega, \lambda))^{M[H]}$ such that $H \times K \cong G_{<\lambda+1}$, but this means that K is $(Coll(\omega, \lambda))^{M[H]}$-generic over $M[H]$. From the previous part we have that $M[G] = M[H][K][G_{[\lambda+1,\kappa)}]$.

Lastly we note that $(Coll(\omega, \lambda))^{M[H]}$ is a superset of $Coll(\omega, \lambda) = Coll(\omega, \lambda + 1)$, so any generic on the former will be generic for the latter, in particular K is such. So from the previous part again we get that there is a $Coll(\omega, < \kappa)$-generic $G^{\mathbb{Q}}$ (that comes from $K \times G_{[\lambda+1,\kappa)}$).

To see that $G^{\mathbb{Q}}$ is really generic over $M[H]$ note that K is generic over $M[H]$ and $G_{[\lambda+1,\kappa)}$ is generic over $M[H][K]$.