Exercise 3

Holo

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Exercise 1. *Nice Names*

Part 1.1.

Let $f : \omega \to 2^{\mathbb{P}}$ be a function defined as $f(n) = \{p \in \mathbb{P} \mid p \Vdash \check{n} \in \sigma\}$ and let A_n be some maximal anti-chain of $f(n)$.

We will see that the nice name $\sigma^* = \bigcup_{n<\omega} \{\check{n}\} \times A_n$ is a nice name such that $0_{\mathbb{P}} \Vdash \sigma = \sigma^*$, or equivalently that $M[\text{❀}] \models \sigma_{\text{❀}} = \sigma^*_{\text{❀}}$ for all generic ideals ❀.

Fix some ❀ and $n \in \omega$ a natural, assume $M[\text{❀}] \models n \in \sigma_{\text{❀}}$, I claim that $\text{❀} \cap A_n \neq \emptyset$, first remember that $g(n) = \{p \in \mathbb{P} \mid p \Vdash \check{n} \in \sigma \vee p \Vdash \check{n} \notin \sigma\} \supseteq f(n)$ is dense in $\mathbb{P}$, so extend A_n into B_n a maximal anti-chain in $g(n)$, because B_n is maximal anti-chain in a dense set, it is also maximal anti-chain in $\mathbb{P}$.

Let $p \in \text{❀} \cap B_n$, if $p \notin A_n$ it means that $p \Vdash \check{n} \notin \sigma$ which is false, hence $p \in \text{❀} \cap A_n$ which means by definition that $M[\text{❀}] \models n \in \sigma^*_{\text{❀}}$.

The direction of $M[\text{❀}] \models n \notin \sigma^*_{\text{❀}} \implies M[\text{❀}] \models n \notin \sigma_{\text{❀}}$ is just the contrapositive of the previous case.

The directions of $M[\text{❀}] \models n \in \sigma^*_{\text{❀}} \implies M[\text{❀}] \models n \in \sigma_{\text{❀}}$ and the contrapositive $M[\text{❀}] \models n \notin \sigma_{\text{❀}} \implies M[\text{❀}] \models n \notin \sigma^*_{\text{❀}}$ are directly from the definition of σ^*.

Part 1.2.

Let $\mathbb{P} = \mathrm{Add}\,(\omega, \omega_2)$, and note that $|\aleph_2| \leq |\mathbb{P}| \leq |[\aleph_0 \times \aleph_2 \times 2]^{<\omega}| = |[\aleph_2]^{<\omega}| = \aleph_2$.

Let $\mathcal{A}$ be the set of anti-chains of $\mathbb{P}$, because $\mathbb{P}$ is c.c.c. we have that $\aleph_2 = |\mathbb{P}| \leq |\mathcal{A}| \leq |[\aleph_2]^{\leq\omega}| = |[\aleph_2]^{<\omega} \cup [\aleph_2]^{\omega}| = \aleph_2 + |[\aleph_2]^{\omega}| = |[\aleph_2]^{\omega}| \leq \aleph_2^{\aleph_0} = \left(2^{\aleph_1}\right)^{\aleph_0} = 2^{\aleph_1 \times \aleph_0} = 2^{\aleph_1} = \aleph_2$

Notice that a function that sends $f : \omega \to \mathcal{A}$ to $\bigcup_{n<\omega} \{\check{n}\} \times f(n)$ is a bijection from the nice names to ${}^{\aleph_0}\mathcal{A}$ so the cardinality of the set of nice names is exactly $|\mathcal{A}|^{\aleph_0} = \aleph_2^{\aleph_0} = \aleph_2$

Part 1.3.

Let F be a bijection from the nice $\mathbb{P}$-names of M to $\aleph_2$, because F, $\text{❀} \in M[\text{❀}]$ and $M[\text{❀}] \models AC$ we can define inside of $M[\text{❀}]$ a function that for each $f \in 2^{\aleph_0}$ chooses some $\sigma \in \mathrm{dom}\,(F)$ such that $\sigma_{\text{❀}} = f$ and sends it to $F(\sigma)$, this is an injective function because F is injective and given $a \neq b \in G[\text{❀}]$ they are not evaluated from the same $\mathbb{P}$ name.

We have shown in class that $M[\text{❀}] \models 2^{\aleph_0} \geq \aleph_2$ and so because $M[\text{❀}]$ satisfy Cantor–Bernstein we have $M[\text{❀}] \models 2^{\aleph_0} = \aleph_2$.

Exercise 2. *Higher Closure*

Let $M, \kappa, \mathbb{P}$ be as in the question and G be a generic.

Let $\vec{a} = (a_i)_{i\in\lambda} \in M[G]$ be a sequence of ordinals of length $\lambda < \kappa$, we want to show it is in M. Let τ be a name such that $\tau_G = \vec{a}$.

I claim that $D = \{p \in \mathbb{P} \mid \forall \alpha \in \lambda \exists \beta (p \Vdash \tau_{\check{\alpha}} = \check{\beta})\}$ is dense.

Let $q \in \mathbb{P}$ be any element, in M construct the sequence:

- p_0 be some element $\geq q$ that decides the value of $\tau_{\check{0}}$
- $p_{\alpha+1}$ be some element $\geq p_\alpha$ that decides the value of $\tau_{\widecheck{\alpha+1}}$
- For limit $\beta < \lambda$, let t_β be an upper bound of the sequence $(p_\alpha)_{\alpha<\beta}$, and let p_β be an element above t_β that decides the value of $\tau_{\check{\beta}}$
- p_λ be an upper bound to the sequence $(p_\alpha)_{\alpha<\lambda}$

Because $\lambda < \kappa$ and $\mathbb{P}$ is $(\kappa\text{-closed})^M$, p_λ is well defined and because $p_\lambda \in D$ we have that D is dense.

Let $p \in G \cap D$.

Now in M define b_α to be the unique ordinal β such that $p \Vdash \tau_{\check{\alpha}} = \check{\beta}$, because $p \in G$ we have that $\vec{b} = \vec{a}$ and we are done.

Now let $\lambda < \kappa$ be a cardinal in M, if it is not a cardinal in $M[G]$ it means that there is some $\eta < \lambda$ and a new bijective sequence $f : \eta \to \lambda$, but $f \in (Ord^{<\kappa})^{M[G]} = (Ord^{<\kappa})^M \subseteq M$, contradiction.

Exercise 3. *Higher Chain Condition*

Let σ, X be as in the question, and let $x \in X$ and let D_x be the set of $p \in \mathbb{P}$ that decides the value of $\sigma(\check{x})$, this set is dense.

Let A_x be a maximal antichain subset of D_x. Let $F(x)$ be the set of values en element of A_x decides $\sigma(\check{x})$ to be. Because we have κ.c.c, $|F(x)| < \kappa$.

Now let G be any generic, because G intersect each of A_x, $\sigma(\check{x})_G \in F(x)$ for any $x \in X$.

To see that $\mathbb{P}$ preserves all cardinals $\geq \kappa$, it is enough to show that it preserve all cardinals $\geq \kappa$ whenever κ is regular from the following lemma:

Lemma 3.1. *If $\mathbb{P}$ is κ.c.c for a singular κ, then there exists some regular $\lambda < \kappa$ such that $\mathbb{P}$ is also λ.c.c.*

Now assume κ is regular.

Let $\lambda \geq \kappa$ be the first cardinal that is not being preserved, it means that there exists some generic G and an ordinal $\eta < \lambda$ such that $M[G] \models |\lambda| = |\eta|$, because this would imply $M[G] \models |\zeta| = |\eta|$ for any cardinal in $[|\eta|, |\lambda|]$, we must have that $\lambda = |\eta|^+$ (otherwise we would contradict the minimality of λ), in particular λ is regular.

Now we must have a new sequence $f \in M[G]$ that witness the collapse, let σ be a $\mathbb{P}$ name such that $0_{\mathbb{P}} \Vdash$ "$\sigma : \eta \to \lambda$ is a function" and $\sigma_G = f$. From the claim above we must have $F : \eta \to [\lambda]^{<\kappa}$ such that $0_{\mathbb{P}} \Vdash \forall x \in \check{\eta}(\sigma(x) \in F(x))$ so in $M[G]$ we have that $\sup f''\eta \leq \sup \bigcup F''\eta < \lambda$ (the last inequality comes from the fact λ is regular and that $\kappa \leq \lambda$), hence f is not surjective, contradiction.

Proof of lemma 3.1. Let $\mathbb{P}$ be κ.c.c for a singular κ, in particular κ is limit.

Let $c : \mathbb{P} \to Card$ defined as $c(x)$ is the supremum of the possible sizes of antichains above x. Because c is weakly downwards monotonic into the cardinals, above every x there exists some y such that $c(y) = c(z)$ for every $z \geq y$. Call those elements c-minimal elements. Note that because the c-minimal elements form an open dense set, there must exists an antichain consist only of c-minimal elements.

Let C be a maximal elements of c-minimal elements, then $\sup_{x \in C} c(x) = \kappa$. Indeed let A be an antichain of cardinality $\lambda < \kappa$, for each $a \in A$ let $s(a)$ be a common strengthening of a and some element of C, and for each $x \in C$ let $U(x) = |\{s(a) \geq x \mid a \in A\}| \leq c(x)$, then we have that $\sup_{x \in C} c(x) \geq \sup_{x \in C} U(x) = |A| = \lambda$. Because κ is limit and λ is arbitrary, we are done.

We can also see that there exists some c-minimal x such that $c(x) = \kappa$, let C be as above, if we don't have such x, then $|C| \geq \mathrm{cof}(\kappa)$. Let $(c_i \mid i \in |C|)$ be well ordering of C and let $s_i = c(c_i)$, let $s_{\beta(i)}$ be a cofinal subsequence. Above each $c_{\beta(i+1)}$ let C_{i+1} be a maximal antichain of cardinality $s_{\beta(i)} < s_{\beta(i+1)} = c(c_{\beta(i+1)})$ (we don't care about the limit case), the union of all C_{i+1} is an antichain of cardinality κ, extend it to a maximal antichain and we have a contradiction.

Lastly, let x be a c-minimal element, then $c(x)$ is regular, and hence from the previous observation we get to a contradiction.

Assume the contrary and let A be a maximal antichain above x of cardinality $\geq \mathrm{cof}(c(x))$, note $c(x) = c(y)$ for all $y \in A$. Because $c(x)$ is singular, it is a limit, so choose an unbounded sequence indexed by A, and above each $y \in A$ let A_y be a maximal antichain corresponds to the unbounded sequence (technically, each A_y may be bigger than the element in the sequence), the union of all A_y is a maximal antichain above x of cardinality $c(x)$, contradiction. □

Exercise 4. *Generalized Cohen forcing*

Part 4.1.

Assume κ is regular, and that $(p_i \mid i \in \lambda)$ is a sequence from $\mathrm{Add}(\kappa, X)$ of length $\lambda < \kappa$, note that $\bigcup p_i$ is a partial function from $\kappa \times X \to 2$ and that $|\bigcup p_i| \leq \sum |p_i| \leq \sum_{i \in \lambda} \sup(|p_i|) = \lambda \times \sup(|p_i|) < \kappa$, where the second inequality used the fact that κ is regular. Therefore $\bigcup p_i \in \mathrm{Add}(\kappa, X)$ is an upper bound to the sequence.

Part 4.2.

First we notice that $\kappa^{\mathrm{cof}(\kappa)} > \kappa$, so κ must be regular.

Let $A \subseteq \mathrm{Add}(\kappa, X)$ be a subset of cardinality κ^+, and we will find 2 elements that are compatible (we will actually find κ^+ many mutually compatible elements).

First note that if we have a family X of κ^+ elements whose intersection of domains is constant, we are done because if D is the common domain, then every element of X extends a sequence whose domain is D, and if 2 elements extends the same sequence, then they are compatible, but there are only $\kappa^{|D|} = \kappa < \kappa^+$ many such sequences from our question assumption.

So let B be the set of domains of elements from A. Instead of elements in B be a subset of $\kappa \times X$, we can view them as subsets of $|\bigcup_{b\in B} b| \leq \sum_{\alpha\in|B|} \kappa = \kappa^+$, which in turn we can view as strictly increasing sequences in $|\kappa^+|^{<\kappa}$.

I want to claim that $\bigcup_{b\in B} \operatorname{range}(b)$ is unbounded.

Otherwise we would have that $B \subseteq |\sup \bigcup_{b\in B} \operatorname{range}(b)|^{<\kappa} = \kappa^{<\kappa} = \kappa$, which is impossible as $|B| > \kappa$. To see that $\kappa^{<\kappa} = \kappa$ note that if $x \in \kappa^{<\kappa}$ then, because κ is regular, $x \in \alpha^\beta$ for some $\alpha, \beta < \kappa$, in particular $\kappa^{<\kappa} = \bigcup_{\beta<\kappa} \bigcup_{\alpha<\kappa} \alpha^\beta$ and $|\bigcup_{\beta<\kappa} \bigcup_{\alpha<\kappa} \alpha^\beta| \leq \sum_{\beta<\kappa} \sum_{\alpha<\kappa} \kappa^\beta = \sum_{\beta<\kappa} \sum_{\alpha<\kappa} \kappa = \sum_{\beta<\kappa} \kappa^2 = \sum_{\beta<\kappa} \kappa = \kappa^2 = \kappa$.

Because $\bigcup_{b\in B} \operatorname{range}(b)$ is unbounded, κ^+ is regular and $\bigcup_{b\in B} \operatorname{range}(b) = \bigcup_{\alpha<\kappa} \{b(\alpha) \mid b \in B\}$, there must exists some $\alpha < \kappa$ such that $\{b(\alpha) \mid b \in B\}$ is unbounded.

Let α_0 be the first such α. Define recursively over $\beta \in \kappa^+$:

- b_0 be an arbitrary element from B
- Assume b_α is defined for every $\alpha < \beta$, because $\bigcup_{\alpha<\beta} \operatorname{range}(b_\alpha)$ is $< \kappa^+$ union of $< \kappa^+$ sets, its supremum, γ, is $< \kappa^+$, let b_β be an element from B such that $b_\beta(\alpha_0) > \gamma$

Let B' be the set of b_β.

Notice that for each $\alpha < \alpha_0$, we have that $\{g(\alpha) \mid g \in B'\}$ is bounded (by the minimality of α_0), so $\bigcup_{\alpha<\alpha_0} \{g(\alpha) \mid g \in B'\}$ is also bounded by β_0, as κ^+ is regular, let $B'' = \{g \in B' \mid g(\alpha_0) > \beta_0\}$.

Now given $g, f \in B''$, the intersection of their ranges must come from before α_0, and each element $f \in B''$ extends a function from $\beta_0^{\alpha_0}$, but there are only $|\beta_0^{\alpha_0}| = \kappa$ many such functions, so there exists κ^+ many functions from B'' that extends the same function from $\beta_0^{\alpha_0}$ and hence has constant intersection of ranges.

Part 4.3.

Note that from the first part, $\operatorname{Add}(\kappa, \lambda)$ preserve cardinals bellow κ, so we only need to show that κ satisfy the conditions of the previous parts.

If $\kappa = \aleph_0$, it is trivial, if $\kappa = \eta^+$ then $\eta < 2^\eta \leq \kappa$ hence $2^\eta = \kappa$ and we have that $\kappa^\mu = (2^\eta)^\mu = 2^{\eta\times\mu} = 2^\eta = \kappa$ for every $\mu \leq \eta$.

Assume κ is weakly inaccessible, using similar trick as before we have: $k^{<\kappa} = |\bigcup_{\beta<\kappa} \bigcup_{\alpha<\kappa} \alpha^\beta| \leq \sum_{\beta<\kappa} \sum_{\alpha<\kappa} |\alpha^\beta| \leq \sum_{\beta<\kappa} \sum_{\alpha<\kappa} (2^{|\alpha|})^{|\beta|} = \sum_{\beta<\kappa} \sum_{\alpha<\kappa} 2^{|\alpha|\times|\beta|} = \sum_{\beta<\kappa} \sum_{\alpha<\kappa} 2^{\max(|\alpha|,|\beta|)} \leq \sum_{\beta<\kappa} \sum_{\alpha<\kappa} \kappa = \kappa$ (In fact, the condition of this question, the condition of the previous question, and $k^{<\kappa} = \kappa$, are all equivalent).

Exercise 5. *Finite Continuum Patterns*

In an identical manner to the countable case, if σ is a $\mathbb{P}$-name of a subset of κ we can define $f : \kappa \to 2^{\mathbb{P}}$ by $f(\alpha) = \{p \in \mathbb{P} \mid p \Vdash \check{\alpha} \in \sigma\}$, let A_α be a maximal antichain of $f(\alpha)$ and define the generalized nice name of σ to be $\bigcup_{\alpha<\kappa}\{\check{\alpha}\} \times A_\alpha$.

Again, in a similar manner to the countable case, if GCH holds and $\mathbb{P} = \mathrm{Add}\,(\kappa, \lambda)$ for $\lambda > \kappa$ we have at most $[\lambda]^{<\lambda} < \lambda^{<\lambda} = \lambda$ antichains (the proof of last equality appears at the end of the last part of the previous question).

Hence the set of generalized $\mathbb{P}$ nice names has cardinality at most $\lambda^\kappa = \lambda$ (this equality again comes from the end of the last part of the previous question).

This implies that if $f : m \to \omega$ is any monotonic function satisfy $f(n) \geq n + 1$ for all $n \in \omega$, let M_{-1} be transitive countable model of $\mathsf{ZFC} + \mathsf{V} = \mathsf{L}$, let M_n be $M_{n-1}[G]$ for a generic of $\mathrm{Add}\left(\aleph_{m-1-n}, \aleph_{f(m-1-n)}\right)$, to see that M_{m-1} satisfy what we want, note that because at M_n we used $\aleph^+_{m-1-n}$.c.c forcing and $\aleph_{m-1-n}$-closed forcing, in particular we preserve the cardinality of all previous stages, and we didn't break the fact that $\aleph_{m-1-n-1}^{<\aleph_{m-1-n-1}}$, in particular $\mathrm{Add}\left(\aleph_{m-1-n-1}, \aleph_{f(m-1-n-1)}\right)$ still satisfy the conditions of the previous exercise.

Exercise 6. *Automorphisms of posets*

Part 6.1.

Because $\pi(0) = 0$ and all of the tags (recursively) of $\check{x}$ are 0, $\pi(\check{x}) = \check{x}$

Part 6.2.

We can note that if $\pi \in V$ we have the simple proof that if G is a generic then $\pi''G$ is generic and given $\mathbb{P}$-name a we have $a_G = \pi(a)_{\pi''G}$, and because $\pi \in M[G], M[\pi''G]$ we have $M[G] = M[\pi''G]$, so given $p \Vdash \varphi(a)$, we have that for each generic $G \ni \pi(p)$ we have $M[G] \models \varphi(a_G)$, so $\pi(p) \Vdash \varphi(\pi(a))$. But if $\pi \notin M$ we need to go through a syntactic proof:

First note that the image of a dense set under π is again dense, and if D is dense above p, then $\pi''D$ is dense above $\pi(p)$.

We will prove the problem by induction. Note that the $\implies$ direction is enough because π^{-1} is also an automorphism:

For $p \Vdash \tau = \sigma$ we have that:

For any $(p', z) \in \tau$ we have that $\{q \geq p \mid q \geq p' \implies \exists(q', w) \in \sigma, q \geq q' \wedge q \Vdash z = w\}$ is dense above p and similarly when swapping τ, σ.

But of course using π we have:

For any $(\pi(p'), \pi(z)) \in \pi(\tau)$ we have that $\{q \geq \pi(p) \mid q \geq \pi(p') \implies \exists(q', w) \in \pi(\sigma), q \geq q' \wedge q \Vdash z = w\} = \pi''\{q \geq p \mid q \geq p' \implies \exists(q', w) \in \sigma, q \geq q' \wedge q \Vdash z = w\}$ is dense above $\pi(p)$. And similarly when swapping τ, σ.

Hence $\pi(p) \Vdash \pi(\tau) = \pi(\sigma)$

For $p \Vdash \tau \in \sigma$ we have that:

$\{q \geq p \mid \exists(p', z) \in \sigma, q \geq p' \wedge q \Vdash z = \tau\}$ is dense.

Just like before, using π we get:

$\{q \geq \pi(p) \mid \exists(p', z) \in \pi(\sigma), q \geq p' \wedge q \Vdash z = \pi(\tau)\} = \pi''\{q \geq p \mid \exists(p', z) \in \sigma, q \geq p' \wedge q \Vdash z = \tau\}$ is dense and hence $\pi(p) \Vdash \pi(\tau) \in \pi(\sigma)$.

The disjunction case is trivial.

$p \Vdash \neg\varphi$ if and only if (there is no $q \geq p$ such that $q \Vdash \varphi$), from the induction assumption it is if and only if (there is no $q \geq \pi(p)$ such that $q \Vdash \pi(\varphi)$) if and only if $\pi(p) \Vdash \pi(\varphi)$ (where $\pi(\varphi)$ means using π on all of the parameters).

And lastly $p \Vdash \exists\varphi(x)$ if and only if the set $\{q \geq p \mid \exists x(q \Vdash \varphi(x))\}$ is dense above p, which implies that the set $\{q \geq \pi(p) \mid \exists x(\pi(p) \Vdash \varphi(\pi(x)))\} = \pi''\{q \geq p \mid \exists x(q \Vdash \varphi(x))\}$ is dense which happens if and only if $\pi(p) \Vdash \exists x \pi(\varphi)(x)$

Exercise 7. *Homogeneous Posets*

Part 7.1.

For 2 partial function p, q let $K(p,q) = \mathrm{dom}\,(p) \cap \mathrm{dom}\,(q)$ and $p^{(q)} = p \cup (q \restriction \mathrm{dom}\,(q) \setminus K(p,q))$.

Notice that $p^{(q)} \geq p$ and that if p, q are comparable then $p^{(q)} = \max(p,q)$.

Now let $\mathbb{P}$ be a poset of partial functions ordered by inclusion and $\pi : \mathbb{P} \to \mathbb{P}$ be a bijection that is bit-wise, that is $\mathrm{dom}\,(p) = \mathrm{dom}\,(\pi(p))$ and $\pi(p)(n)$ depends only on $p(n)$ and n for $n \in \mathrm{dom}\,(p)$, then we have that π is an automorphism.

Indeed let $\tau(n, p(n))$ be $(n, \pi(p)(n))$, because π is a bijection so is τ, and let $p \subseteq q$, if $(a,b) \in p$ then $(a,b) \in q$ so $\tau(a,b) \in \pi(p)$ and $\tau(a,b) \in \pi(q)$, and if $(a,b) \in \pi(p)$ then $\tau^{-1}(a,b) \in p$ so $\tau(\tau^{-1}(a,b)) = (a,b) \in \pi(q)$ so π is order-preserving.

Now fix some $p, q \in \mathbb{P}$ and lets define the automorphism $\pi_p^q : \mathbb{P} \to \mathbb{P}$ that swaps $p^{(q)}$ and $q^{(p)}$ by swapping $(n, p(n))$ with $(n, q(n))$ for all $n \in K(p,q)$ and let it not change any other pair.

Indeed π_p^q is bit-wise, $\pi_p^q(t)(n) = t(n)$ if $n \notin K(p,q)$ or $t(n) \notin \{p(n), q(n)\}$, otherwise if $t(n) = p(n)$ let $p(t)(n) = q(n)$ and vice versa.

To see it swaps $p^{(q)}$ with $q^{(p)}$ notice that $p^{(q)}, q^{(p)}$ have the domain of of $\mathrm{dom}\,(p) \cup \mathrm{dom}\,(q)$ and they agree on their domain apart from (maybe) $K(p,q)$, so let $n \in K(p,q)$ and we get that $p^{(q)}(n) = p(n)$, so $\pi_p^q(p^{(q)})(n) = q(n)$ by definition, and vice versa.

Because $\mathrm{Add}\,(\kappa, 1)$ and $Col(\omega, \lambda)$ are posets of partial functions ordered by inclusion we are done.

Part 7.2.

Let G be any generic we know that $\varphi(\overline{\check{x}_G})$ either holds in $M[G]$ or its negation holds, WLOG assume it holds, and take $p \in G$ such that $p \Vdash \varphi(\overline{\check{x}})$.

Let $q \in \mathbb{P}$ be any element, and let π be an automorphism that sends $r \geq p$ to $\underline{t \geq q}$, because $r \geq p$, it also forces $\varphi(\overline{\check{x}})$ and so from problem (6.2) we have that $t \Vdash \varphi(\overline{\pi(\check{x})})$ and from (6.1) we can conclude that $t \Vdash \varphi(\overline{\check{x}})$, so $\{p \in \mathbb{P} \mid p \Vdash \varphi(\overline{\check{x}})\}$ is dense above $0_{\mathbb{P}}$, hence $0_{\mathbb{P}}$ also forces that.