Exercise 2

Holo

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Exercise 1.

Let $\mathbb{P}, A, f : A \to V^{\mathbb{P}}$ be as in the question, for each generic G let bonk_G be the unique $x \in A \cap G$, define $\uparrow : \mathbb{P} \times A \to 2^{\mathbb{P}}$, $g : V^{\mathbb{P}} \times A \to V^{\mathbb{P}}$ and $\sigma_{\cdot} : A \to V^{\mathbb{P}}$ as:

- $x \uparrow a$ is the set of all common strengthening of x and a
- $g(\tau, a) = \{(\pi, x) \mid \exists y (\pi, y) \in \tau \wedge x \in y \uparrow a\}$
- $\sigma_a = g(f(a), a)$

Notice that $(f(\text{bonk}_G))_G = (\sigma_{\text{bonk}_G})_G$, the $\subseteq$ direction is follows from the fact that every 2 elements in G have a common strengthening, indeed if $\tau \in f(\text{bonk}_G)$ is don't discarded by G then its right side is in G, in particular $\pi_2(\tau) \uparrow \text{bonk}_G \neq \emptyset$ (where π_i is the projection function) and so there is some $\pi \in \sigma_{\text{bonk}_G}$ such that $\pi_1(\pi) = \pi_1(\tau)$ and $\pi_2(\pi) \in G$.
The $\supseteq$ direction follows from the fact that G is closed downwards, if $\pi \in \sigma_{\text{bonk}_G}$ is not discarded by G, then $\pi_2(\pi)$ comes from some $\tau \in f(\text{bonk}_G)$ with $\pi_1(\tau) = \pi_1(\pi)$ and $\pi_2(\tau) \leq \pi_2(\pi)$, which means that τ is also not discarded by G because it is closed downwards.

In addition if Q is a different generic ideal such that for some x we have $x \in (\sigma_{\text{bonk}_G})_Q$ then $\text{bonk}_G = \text{bonk}_Q$ (in other words, σ_{bonk_G} and σ_{bonk_H} for G, H generics that don't have a common A-member don't interfere with one another) because any right side of an element of σ_{bonk_G} must have stronger tag than bonk_G and Q is closed downwards which means that $\text{bonk}_G \in A \cap Q$ which by definition is equal to bonk_Q, so we can let $\sigma_f = \bigcup_{a \in A} \sigma_a$.

Exercise 2. *The relation $p \Vdash^* \tau_1 = \tau_2$*

Part 2.1.

To fix the circular definition we need to add the base case of $p \Vdash^* \emptyset = \emptyset$ for every $p \in \mathbb{P}$

Part 2.2.

We will use induction on the $\mathbb{P}$-rank of τ_1, τ_2.

Let $p \in \mathbb{P}$ be some element, G a generic such that $p \in G$, and τ_1, τ_2 be $\mathbb{P}$-names such that $p \Vdash^* \tau_1 = \tau_2$.

We will show that $(\tau_1)_G \subseteq (\tau_2)_G$ and from symmetry we would be done.

Let $\pi_G \in (\tau_1)_G$ where $(\pi, s) \in \tau_1$ and $s \in G$. Choose $r \geq p, s$ in G.

Because $\{q \geq r \mid q \geq s \implies \exists(\sigma, t) \in \tau_2(q \geq t \wedge q \Vdash^* \sigma = \pi)\} = \{q \geq r \mid \exists(\sigma, t) \in \tau_2(q \geq t \wedge q \Vdash^* \sigma = \pi)\}$ (this equality is possible because we switched from p to r) is dense above r (as $r \geq p$ implies $r \Vdash^* \tau_1 = \tau_2$), let r' be an element from that set intersect G, and let (σ, t) witnesses, because $r' \Vdash^* \sigma = \pi$, and the $\mathbb{P}$-rank of σ and π is lower than those of τ_i, and because $r' \in G$, we can use the induction hypothesis and get $\sigma_G = \pi_G$, in particular we have $\pi_G = \sigma_G \in (\tau_2)_G$ and we are done.

Now assume $(\tau_1)_G = (\tau_2)_G$ and let D be the set of $p \in P$ that forces* either $\tau_1 = \tau_2$, or it's negation (where "it's negation" means that it is not force* by any stronger condition).

This set is dense, indeed if $\neg p \Vdash^* \tau_1 = \tau_2$, then (WOLG) there exists some $(\pi, s) \in \tau_1$ such that $\{q \geq p \mid q \geq s \implies \exists(\sigma, t) \in \tau_2(q \geq t \wedge q \Vdash^* \sigma = \pi)\}$ is not dense above p. Let $r \geq p$ be a witness, we must have that everything above r does not have the above set as dense above it, in particular r force* the negation of $\tau_1 = \tau_2$.

Now we only need to show that if $p \in G$, p does not force* the negation of $\tau_1 = \tau_2$.

Indeed if $p \in G$ is such element, let $(\pi, s) \in \tau_1$ be such that $s \in G$ and there is no $(\sigma, t) \in \tau_2$ $(t \in G)$ and $r \geq p$ such that $r \Vdash^* \sigma = \pi$.

Because $\pi_G \in (\tau_1)_G$, there exists some $(\sigma', t') \in \tau_2$ such that $t' \in G$ and $\sigma'_G = \pi_G$, because the rank of σ'_G, π_G is smaller than those of τ_i, we can use the induction hypothesis and find $q \in G$ such that $q \Vdash^* \sigma' = \pi$, by taking $q' \geq q, r$ we get a contradiction as $q' \Vdash^* \sigma' = \pi$ as well, which never happen as $q' \geq r$.

Exercise 4.

Part 4.1.

For (2) $\to$ (1) in the book, notice that (1) is a special case of (2). For (2) $\to$ (3) notice that assuming (2) we have $\{r \mid r \Vdash^* \psi\} \supseteq \{r \mid r \geq p\}$, and the latter trivially dense above p.

For the atomic (1) $\to$ (2) direction, let $p \Vdash^* \tau_1 = \tau_2$ and $r > p$, I want to show that for all $(\pi_1, s_1) \in \tau_1$ the set $\{q \geq r \mid q \geq s_1 \implies \exists(\pi_2, s_2) \in \tau_2(q \geq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$ is dense above r, indeed take an element s above r, then because $p \Vdash^* \tau_1 = \tau_2$ and that $s \geq r \geq p$ there exists some $t \in \{q \geq p \mid q \geq s_1 \implies \exists(\pi_2, s_2) \in \tau_2(q \geq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$ greater than s, but because $t \geq s \geq r$ we have that $t \in \{q \geq r \mid q \geq s_1 \implies \exists(\pi_2, s_2) \in \tau_2(q \geq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$, so that set is indeed dense above r. A symmetric argument will finish this proof.

Now assume $p \Vdash^* \tau_1 \in \tau_2$ and $r \geq p$, take $s \geq r \geq p$, there is some $t \in \{q \mid (\pi, x) \in \tau_2(q \geq x \wedge q \Vdash^* \pi = \tau_1)\}$ that is stronger than $s \geq r$, so the set is dense above r.

For (3) $\to$ (1) we just need to prove that if $\{r \mid D \text{ is dense above } r\}$ is dense above p, then D is dense above p, indeed take $q \geq p$ and $r \in \{r \mid D \text{ is dense above } r\}$ greater than q, then let $q' > r$, because D is dense above r there exists $t \in D$ such that $t \geq q' > r \geq q \geq p$, hence D is dense above D.

To finish all of the details of the proof of the lemma from the book we need to show (1) $\to$ (2) and (3) $\to$ (1) for $\neg$ and $\exists x$.

For (1) $\to$ (2), assume $p \Vdash^* \neg\varphi$ and $r \geq p$, then every $t \geq r$ is also stronger than p, hence does not force* the sentence φ, therefore $r \Vdash^* \neg\varphi$. Assume $p \Vdash^* \exists x\varphi(x)$ and take $r \geq p$ and $t \geq r$, because $t \geq p$ there exists $q \geq t$ such that $\exists\sigma \in V^{\mathbb{P}}(q \Vdash^* \varphi(\sigma))$, in particular $\{q \mid \exists\sigma \in V^{\mathbb{P}}(q \Vdash^* \varphi(\sigma))\}$ is dense above r.

For (3) $\to$ (1) for the $\neg\varphi$ case, assume $p \nVdash^* \neg\varphi$ then there is some $r \geq p$ that forces* φ, by the induction hypothesis we get that every $t \geq p$ also forces* φ, a contradiction. The $\exists x\varphi(x)$ case follows from the same fact as the atomic case, that if $\{r \mid D \text{ is dense above } r\}$ is dense above p, then D is dense above p

Part 4.2.

Assume $p \Vdash^* \varphi$, we want to prove $p \Vdash^* \neg\neg\varphi$, in particular we want to show:

$$\begin{aligned} &\forall q \geq p(\neg q \Vdash \neg\varphi) \\ \iff &\forall q \geq p(\neg(\forall r \geq q(\neg r \Vdash^* \varphi))) \\ \iff &\forall q \geq p(\exists r \geq q(r \Vdash^* \varphi)) \end{aligned}$$

But any $r \geq q$ will witness it is true, as $r \geq q \geq p$ and from (4.1) $r \Vdash^* \varphi$.

Part 4.3.

Let D_ψ be as in the question and let $p \in \mathbb{P}$ be any term, if there is $q \geq p$ such that $q \Vdash^* \psi$ then we are done as $p \leq q \in D_\psi$. Otherwise we have that $p \Vdash^* \neg\psi$ by definition, so $p \in D_\psi$.

So assume $p \Vdash^* \neg\neg\psi$, the set D_ψ is dense above p, by definition there is no $q \geq p$ that forces* $\neg\psi$, so $\{q \mid q \Vdash^* \psi\} = D_\psi \cap \{q \mid q \geq p\}$ is dense above p, by (4.1) this is equivalent to $p \Vdash \psi$.

Exercise 5. *Main $\Vdash^*$ theorem*

We have proven in Exercise 2 the base $=$ case.

Very similarly to the $=$ case, suppose that $G \ni p \Vdash^* \tau_1 \in \tau_2$, because $\{q \geq p \mid \exists(\pi, s) \in \tau_2(q \geq s \wedge q \Vdash^* \pi = \tau_1)\}$ is dense above p, let $q \in G, (\pi, s)$ from that set.

From the the $=$ case we have that $\pi_G = (\tau_1)_G$ and because $q \geq s$ we have that $s \in G$, in particular $(\tau_1)_G = \pi_G \in (\tau_2)_G$

For the other direction assume $(\tau_1)_G \in (\tau_2)_G$, and let $(\pi, s) \in \tau_2$ such that $s \in G$ and $(\tau_1)_G = \pi_G$, from the $=$ case, we have $p \in G$ such that $p \Vdash^* \tau_1 = \pi$, let $r \in G$ be a common strengthening of p and s, so $r \Vdash^* \tau_1 \in \tau_2$.

The $\wedge, \neg$ cases are trivial.

Assume $G \ni p \Vdash^* \exists x\varphi(x, \tau)$, that is $\{r \geq p \mid \exists\sigma \in V^{\mathbb{P}}(r \Vdash^* \varphi(\sigma, \tau))\}$ is dense above p, and let $r \in G$ be an element of the above set with witness σ', we have that $r \Vdash^* \varphi(\sigma, \tau)$, from the induction hypothesis $M[G] \models \varphi(\sigma_G, \tau_G)$ which implies $M[G] \models \exists x\varphi(x, \tau_G)$.

Assume $M[G] \models \exists x \varphi(x, \tau_G)$, let σ_G be a witness, and let σ be its name.

Because $M[G] \models \varphi(\sigma_G, \tau_G)$, by the induction hypothesis we have $p \in G$ such that $p \Vdash^* \varphi(\sigma, \tau)$, in particular $\{q \geq p \mid \exists x \in V^{\mathbb{P}}(q \Vdash^* \varphi(x, \tau))\} \subseteq \{q \geq p \mid q \Vdash^* \varphi(\sigma, \tau)\} = \{q \geq p\}$ is dense above p, and hence $p \Vdash^* \exists x \varphi(x, \tau)$

Exercise 6.

Assume $(p \Vdash^* \varphi(\overline{\tau}))^M$, in particular we have that for every G a generic containing p we have $\exists q \in G(q \Vdash^* \varphi(\overline{\tau}))^M$, by theorem 3.5 in the book we have $M[G] \models \varphi(\overline{\tau}_G)$, hence by definition $p \Vdash^M_{\mathbb{P}} \varphi(\overline{\tau})$

Now assume $p \Vdash^M_{\mathbb{P}} \varphi(\overline{\tau})$ and let $r \geq p$ and G be a $\mathbb{P}$-generic contains r, by definition $M[G] \models \varphi(\overline{\tau}_G)$, by theorem 3.5 in the book there exists some $t \in G$ such that $(t \Vdash^* \varphi(\overline{\tau}))^M$, let $q \in G$ be stronger than r and t, by (4.1) we know that $(q \Vdash^* \varphi(\overline{\tau}))^M$, hence $\{q \mid q \Vdash^* \varphi(\overline{\tau})\}$ is (dense above $p)^M$, which implies that $(p \Vdash^* \varphi(\overline{\tau}))^M$

To get the form we had in class all we need to do is "concat" this result to question theorem 3.5, if G is a generic and $M[G] \models \varphi(\overline{\tau}_G) \iff \exists p \in G\ (p \Vdash^* \varphi(\overline{\tau}))^M \iff \exists p \in G\ p \Vdash^M_{\mathbb{P}} \varphi(\overline{\tau})$.

Exercise 7.

Part 7.1.

Let D be the set $\{p \mid \exists \sigma \in V^{\mathbb{P}}(p \Vdash \psi(\sigma, \overline{\tau}))\}$, we know this set is dense above $0_{\mathbb{P}}$, which means it is just dense.

Let $A \subseteq D$ be a maximal anti-chain, and for each $x \in A$ let $f(x)$ be a name that is a witness of $x \in D$, let σ_f be the name from exercise 1.

For every generic (= for every generic contains $0_{\mathbb{P}}$) we have that $M[G] \models \psi((\sigma_f)_G, \overline{\tau}_G)$, which by definition means $0_{\mathbb{P}} \Vdash \psi(\sigma_f, \overline{\tau})$.

Part 7.2.

Let $\psi(x, y, w, z) = (x$ is a function from w to $z) \wedge (y$ is a function from w to $z \implies x = y)$, notice that for every y', w', z' such that $(z' = \emptyset \implies w' = \emptyset)$ we have that $\exists x \psi(x, y', w', z')$ is tautology.

Let G be a generic ideal and $f : \alpha \to \beta$ function in $M[G]$, let τ be the name that G interpret as f, because f is a function we have that either $\beta \neq 0$ or both α and β are 0, in particular for every Q a generic ideal we have that $M[G] \models \exists x \psi(x, \tau_Q, \check{\alpha}_Q, \check{\beta}_Q)$, so $0_{\mathbb{P}} \Vdash \exists x \psi(x, \tau, \check{\alpha}, \check{\beta})$, from (7.1) we have that there is some name σ such that $0_{\mathbb{P}} \Vdash \psi(\sigma, \tau, \check{\alpha}, \check{\beta})$.

Clearly $0_{\mathbb{P}} \Vdash \sigma : \check{\alpha} \to \check{\beta}$ and because $\tau_G = f$ is a function from α to β we have that $f = \tau_G = \sigma_G$.

Exercise 8.

Let $M, \mathbb{P}, G$ as in the question, because $M[G]$ is transitive, it satisfy Extensionality and Foundation.

Because it contains all ordinals it also satisfy Infinite and Emptyset.

Let $x \in M[G]$ with name τ, for each $(\pi, s) \in \tau$ and for each $(\sigma, t) \in \pi$ take all of the names (σ, r) for $r \geq t, s$, the set of all such pairs is a name for $\bigcup x$.

For replacement, let $\varphi(x, y, a)$ be functional formula over A for $a \in M[G]$ parameter with τ name. Let π be the name of A.

Let $p \in G$ that forces that $\varphi(x, y, \tau)$ is functional over π.

For each $(\sigma, s) \in \pi$, $s \in G$ we have $p \Vdash \exists x \varphi(x, \sigma, \tau)$ so for each such $(\sigma, s) \in \pi$ (regardless of s) if $p \Vdash \exists x \varphi(x, \sigma, \tau)$ let $x_{(\sigma,s)}$ be a single name that witness that (exists from name integration) and let $A_{(\sigma,s)}$ be the set $(x_{(\sigma,s)}, t)$ for all $t \geq s, p$, if p does not force that, let $A_{(\sigma,s)}$ be empty.

(Note that we can replace $x_{(\sigma,s)}$ with the set of all such witnesses over a bounded rank to avoid using AC. This works because φ is functional, so all of those names translate to the same set in $M[G]$)

The union of all those $A_{(\sigma,s)}$ will witness that the image of φ is a set in $M[G]$.

Because $|M[G]| > 1$ and we have Emptyset and replacement, we immediately get Pairing and Separation.

For AC, if X is any set with a name π, let R be a well ordering of the name π, we have $R, \pi, G \in M[G]$ so define in $M[G]$ be function that sends (σ, s) to (σ_G, s), then R induce a well ordering on $\{(\sigma_G, s) \mid (\sigma, s) \in \pi\}$ in $M[G]$, which can be used to define a well ordering on X.

All we left to do is to show that the Powerset exists. Let $X \in M[G]$ with name π, let $(A \subseteq X)^{M[G]}$ with name τ, by name integration we may assume $0 \Vdash \tau \subseteq \pi$.

Let $\{(\sigma, s) \mid (\sigma, p) \in \pi, s \geq p, s \Vdash \sigma \in \tau\}$ is a name for A as well.

This gives a set of names that (under G) contains all subsets of X. By separation we are done.