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POWERS OF REGULAR CARDINALS

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§ 1. Introduction

Recently, Paul J.Cohen has shown[1] that Cantor's continuum hypothesis $2^{\aleph_0} = \aleph_1$ is independent of the axioms of Gödel-Bernays set theory, including the axioms of choice and foundation[2]; *a fortiori*, the generalized continuum hypothesis $(\alpha)[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$ is also independent of these axioms. We wish to consider the ways in which the generalized continuum hypothesis can be violated, i.e., to determine the forms of the function G defined by $(\alpha)[2^{\aleph_\alpha} = \aleph_{G`\alpha}]$ which are allowed by the axioms of Σ_*.

Let M be a model of Σ_*. An *extension* N of M will be a model

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[1] The Independence of the Axiom of Choice, mimeographed, Stanford University, 1963; The Independence of the Continuum Hypothesis, Proc. Nat. Acad. Sci. USA 50 (1963) 1143-1148, and 51 (1964) 105-110. (Cohen gives the proof for Zermelo-Fraenkel set theory, rather than for Gödel-Bernays set theory.)

[2] The system of set theory considered here will be that of K.Gödel, The Consistency of the Continuum Hypothesis (Princeton University Press, Princeton, 1940). The system consisting of the axioms of groups A, B, C, and D will be called "Σ"; by "Σ_*", we will mean the system obtained by adding axiom E, the class form of the axiom of choice, to Σ.

of Σ_* in which M is embedded as a complete submodel[1] whose ordinal numbers are precisely those of N. We will say that cardinals are *absolute* in the extension if the cardinals of N are precisely those of M.

We choose a particular countable model M of Σ_* in which the generalized continuum hypothesis holds. Let G be a function[2] in M which takes ordinals into ordinals; we consider conditions for the existence of an extension N of M in which cardinals are absolute and $(\alpha)[2^{\aleph_\alpha} = \aleph_{G`\alpha}]$.

König's theorem[3] implies that for any cardinal $\aleph_\alpha$, $2^{\aleph_\alpha}$ is of cofinality greater than $\aleph_\alpha$, i.e., $2^{\aleph_\alpha}$ is not the sum of $\aleph_\alpha$ smaller cardinals. Since a pair of cardinals cofinal in M are also cofinal in any extension in which cardinals are absolute, $\aleph_{G`\alpha}$ must be of cofinality greater than $\aleph_\alpha$ for each α.

A second requirement on the function G is given by the observation that if $\alpha \leq \beta$, then $2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$; clearly, we must have $G`\alpha \leq G`\beta$ if $\alpha \leq \beta$. The theorem below states that for *regular* cardinals, the function G can be chosen arbitrarily subject only to these two conditions. In the models we obtain, power sets of singular cardinals have the smallest cardinality allowed by König's theorem; whether or not there is more flexibility in the choice of power sets for singular cardinals remains an open question.

Theorem 1. *Let M be a countable model of Σ_*, and let G be a function in M such that:*

(i) *$\alpha \leq \beta$ implies $G`\alpha \leq G`\beta$;*

(ii) *$\aleph_{G`\alpha}$ is not cofinal with any cardinal less than or equal to $\aleph_\alpha$.*

[1] I.e., the sets of M form a class in N; furthermore, the ϵ-relation of M is the same as that of N, and the elements of sets of M are again sets of M. See J.C.Sheperdson, Inner Models for Set Theory – Part I, J. Symbolic Logic 16 (1951) 161-190.

[2] Defined by a *class* of M.

[3] Let $A = \{a_t | t \in T\}$ and $B = \{b_t | t \in T\}$ be sets of cardinals such that for each t, $a_t < b_t$. Then $\sum_{t \in T} a_t < \prod_{t \in T} b_t$. See A.Fraenkel, Abstract Set Theory (North-Holland, Amsterdam, 1961) pp. 98-100.

Then there is an extension N of M in which cardinals are absolute and $2^{\aleph_\alpha} = \aleph_{G`\alpha}$ for regular cardinals $\aleph_\alpha$.

We have chosen to state our results as a theorem on extension of models. If the function G can be described by an appropriate axiom (e.g., $G`\alpha = \aleph_\alpha + 1$), the theorem could be recast as an assertion about the relative consistency of two axiom systems. The proof given here is formalizable in first-order arithmetic; using the procedure outlined by Cohen [1], it could presumably be given in recursive arithmetic. The problem of which cardinals and ordinals can be defined "by an appropriate axiom" is, however, still open to study [2].

We will briefly describe Cohen's construction and the manner in which we will modify it. Suppose that we wish to add a new subset of $\aleph_0$ to our model M. We consider the properties of such a set a which can be described by sets of the model M. It is clear that the intersection of a with a finite set of integers can be so described; Cohen's idea is to determine the new set in such a way that the *only* properties of the set a given by sets of M are its intersections with finite sets. To determine such a set a, Cohen introduces the sets of the new model as a transfinite ramified hierarchy depending, of course, on a. He then introduces by transfinite induction the notion of a statement about the new model being "forced" by a "set of conditions" – i.e., by a set of M describing the intersection of a with a finite set of integers. Finally, the set a is determined by an increasing sequence of sets of conditions such that every statement about the new model or its negation is eventually forced.

[1] The Independence of the Continuum Hypothesis II, Proc. Nat. Acad. Sci. USA 51 (1964) 105-110.

[2] One attempt in this direction is given by A.Hajnal in On a consistency theorem connected with the generalized continuum problem, Acta Math. Acad. Sci. Hungaricae 12 (1961) 321-376. The concept of an "ordinal number absolutely definable in the weak sense" is described there; an appropriate extension should yield a sufficient condition for a function G to be defined "by an appropriate axiom".

In the model constructed below, "generic" subsets of certain cardinal numbers are constructed by the method of forcing. The model so constructed is made to have the desired properties by allowing more complicated properties of the new sets – such as their intersections with countable sets of the model M – to be expressable by sets of the model M. The notion of forcing is still defined in such a way that *only* properties of a certain kind – namely, those described by "sets of conditions" – can be described by sets of the model M; as in the Cohen argument, the apparent circularity of this idea is removed by the use of a ramified hierarchy. Thus, in the construction below, the definition of forcing is essentially the same as in Cohen's paper (except as required by the fact that we introduce a proper class of generic sets), while a somewhat complicated notion of "set of conditions" is used to ensure that the extension has the desired properties.

Sections 2 through 5 will be devoted to the proof of theorem 1. The model N is constructed by a procedure similar to that of Gödel's consistency proof[1]; in section 2, we introduce a language L to describe the construction procedure. In section 3, we give the forcing construction in detail; in section 4, we prove that the resulting model is indeed a model of Σ_*. The proofs in these two sections follow the proofs in Cohen's paper, except for modifications to allow non-standard models and to allow a proper class of "generic" sets.

In section 5, we present the proof that cardinals are absolute in the extension from M to N. Ideas due to Cohen and Solovay are used here. Solovay has shown[2] that one can introduce "generic" subsets of any regular cardinal $\aleph_\alpha$ in such a way that no subsets of smaller cardinals are introduced. The method is to allow sets of

[1] K.Gödel, Consistency-Proof for the Generalized Continuum Hypothesis, Proc. Nat. Acad. Sci. USA 25 (1939) 220-224; K.Gödel, The Consistency of the Continuum Hypothesis (Princeton, 1940). We will refer to the latter as the "Gödel monograph".

[2] Independence results in the theory of cardinals (abstract), Notices Amer. Math. Soc. 10 (1963) 595.

conditions to be of any cardinality less than $\aleph_\alpha$ and then to show that subsets of smaller cardinals have already been determined at some finite stage in the forcing construction. We use a similar idea to show that subsets of $\aleph_\alpha$ can be constructed from the "generic" subsets of cardinals $\leq \aleph_\alpha$.

We list the axioms of Σ_* below. Rather than giving the axioms in symbolic form, we give their intuitive content in English. The reader is referred to the Gödel monograph for a precise statement of the system. The primitive notions of the system are: class, set, and the membership relation ϵ. The axioms are divided into five groups, A, B, C, D and E.

Group A

1. Every set is a class.
2. Members of classes are sets.
3. Axiom of extensionality: two classes are equal if they have the same members.
4. Axion of pairing: if x and y are sets, there is a set which contains x and y as its sole members.

Group B

Group B consists of eight axioms which are sufficient to prove that any collection of sets defined in terms of other classes by a formula involving no bound class variables is a class.

Group C

1. Axiom of infinity: there is a non-empty set with the property that each of its elements is a proper subset of another element.
2. Axiom of union: the sum-class of a set is a set.
3. Axiom of power-set: for any set x, there is a set whose elements are precisely the subsets of x.
4. Axiom of replacement: the image of a set under a function is a set.

Axiom D

Axiom of foundation: every non-void class A contains a set x such that A and x are disjoint.

Axiom E

Axiom of choice: there is a function which assigns to each non-empty set x an element of x.

The reader is referred to the Gödel monograph for any terminology not defined here. By a function, we mean a *class* A of ordered pairs $\langle xy \rangle$ such that if A contains $\langle xy \rangle$ and $\langle x_1 y \rangle$, then $x = x_1$. We adopt Gödel's definition of cardinal and ordinal numbers, whereby an ordinal is the set of all smaller ordinals and a cardinal is the smallest ordinal of a given cardinality [1].

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§ 2. A ramified language

Let M be a fixed countable model of Σ_* in which the generalized continuum hypothesis holds, and let G be a function satisfying the hypothesis of theorem 1. We will use a construction procedure similar to that of Gödel's consistency proof to obtain a model N containing subsets a^α_η of regular cardinals $\aleph_\alpha$ which are not already in the model M. Cohen's method of forcing is used to construct "generic" sets a^α_η such that the resulting model is indeed a model of Σ_*.

The elements of the model N will be constructed in a ramified

[1] In the Gödel monograph, an ordinal number is an ordinal which is a set, and not a proper class. We will frequently use the term "ordinal" when we mean "ordinal number". A similar remark applies to "cardinal" and "cardinal number".

hierarchy, with a level for each ordinal of M[1]; at level α, we introduce as sets all collections of sets of lower level definable in terms of the ϵ-relation and the relation A given by

$$A(\gamma, \alpha, \eta) \equiv \gamma \in a_\eta^\alpha .$$

We introduce a language L which will be used to describe the ramified hierarchy[2]. L will be a many-sorted predicate calculus with functional constants ϵ and A. There will be *ranked variables* v_i^α for each ordinal α and each integer i of M and there will be [*unranked*] *variables* v_i for each integer i of M[3]. We will use the letters $x^\alpha, y^\alpha, \ldots$, to stand for ranked variables and the letters $x, y, \ldots$, to stand for unranked variables. In addition, there will be *constants* $\underline{S}$, $\underline{T}$, ..., corresponding to classes S, T, ..., of the model M. If s is a set of rank β, we will say that $\underline{s}$ is a *set constant* of rank β.

Definition 1. We give an inductive definition of *ranked formula* and of *abstraction term*. The definition is to be given in the model M[4].

[1] The construction in Cohen's paper follows the procedure of the Gödel monograph, where the sets of the new model are defined by transfinite induction so that one set is introduced for each ordinal number. The idea of using a ramified hierarchy, as Gödel did in his original proof, is found in the mimeographed papers of Feferman and Lévy. Feferman credits the idea to Dana Scott.

[2] The language L is similar to the language $\mathcal{L}^*$ of Lévy's mimeographed notes. We have added conditions on the form of the formula $\Phi(x^\alpha)$ which may be used to form an abstraction term $\hat{x}^\alpha \Phi(x^\alpha)$ to correspond more closely to the idea of a ramified hierarchy and to simplify some of our proofs.

[3] Note that if the integers of the model are not well-ordered, we have variables v_ζ^α even for "non-standard" integers ζ.

[4] We intend that, if the integers of M are not well-ordered, then ranked "formulas" of the form $(\exists x_1^\alpha)(\exists x_2^\alpha) \ldots (\exists x_\zeta^\alpha)\Psi$ are to be allowed, even for "non-standard" integers ζ. The possibility of using this technique to avoid the assumption that M is a well-founded model was pointed out by R.Solovay. Alternatively, the use of ranked formulas could be completely avoided by using a construction procedure similar to that of the Gödel monograph and defining forcing only for elementary statements $F_\alpha \in F_\beta$ and $F_\alpha \notin F_\beta$.

1° If u, v, and w, are abstraction terms, set constants or ranked variables, then $A(u, v, w)$ and $u \in v$ are ranked formulas;
2° If Φ and Ψ are ranked formulas, then $\sim \Phi$, $\Phi \vee \Psi$, and $(\exists x^{\alpha})\Phi$ are ranked formulas;
3° If Φ is a ranked formula containing no free variables other than x^{α}, no occurrences of $(\exists y^{\beta})\Psi$ with $\beta > \alpha$, and no occurrences of abstraction terms $\hat{y}^{\beta}\Psi$ or set constants of rank β with $\beta \geq \alpha$, then $\hat{x}^{\alpha}\Phi$ is an abstraction term.

We will refer to abstraction terms and to *set* constants as *constant terms*. In the intended interpretation of L, the constant terms will play the role of individual constants; each set of the model to be constructed will be denoted by such a term.

Definition 2. The *rank* of a variable, abstraction term, or set constant is given by:
(i) $\rho(x^{\alpha}) = \alpha$;
(ii) $\rho(\hat{x}^{\alpha}\Phi) = \alpha$;
(iii) $\rho(\underset{\sim}{s})$ is the rank of the set s.

Definition 3. We now define [*unranked*] *formulas* of the language L. The definition is to take place in the metalanguage rather than in M[1].
1° If u, v, and w are variables or constant terms, then $u \in v$ and $A(u, v, w)$ are formulas. Furthermore, if $\underset{\sim}{S}$ is any constant, then $u \in \underset{\sim}{S}$ is a formula.
2° If Φ and Ψ are formulas, then $\sim \Phi$, $\Phi \vee \Psi$, $(\exists x)\Phi$, and $(\exists x^{\alpha})\Phi$ are formulas.

Lemma 1. *There is an assignment of sets of M to ranked formulas of L such that the collection of sets assigned to ranked formulas is*

[1] So that the length of a formula will be finite in the "usual" sense. Recall that this was not necessarily the case for ranked formulas.

a class of the model M and the usual syntactical operations (forming negations, substitution, etc.) are represented by functions of M.

Proof. We define a set 'u' corresponding to each term u and a set "Φ" corresponding to each ranked formula Φ:

$$\text{`}\underline{s}\text{'} = \langle 0, s\rangle$$
$$\text{`}v_i^\alpha\text{'} = \langle 1, i, \alpha\rangle$$
$$\text{`}\hat{v}_i^\alpha\Phi\text{'} = \langle 2, i, \alpha, \text{“}\Phi\text{”}\rangle$$
$$\text{“}A(u, v, w)\text{”} = \langle 3, \text{`}u\text{'}, \text{`}v\text{'}, \text{`}w\text{'}\rangle$$
$$\text{“}u \in v\text{”} = \langle 4, \text{`}u\text{'}, \text{`}v\text{'}\rangle$$
$$\text{“}\sim \Phi\text{”} = \langle 5, \text{“}\Phi\text{”}\rangle$$
$$\text{“}\Phi \vee \Psi\text{”} = \langle 6, \text{“}\Phi\text{”}, \text{“}\Psi\text{”}\rangle$$
$$\text{“}(\exists v_i^\alpha)\Phi\text{”} = \langle 7, i, \alpha, \text{“}\Phi\text{”}\rangle \cdot$$

It should be clear that the required classes and functions are definable in M by transfinite induction. We do not carry out the procedure in detail, since it is quite similar to well-known methods for arithmetization of syntax.

Propositional connectives other than $\sim$ and $\vee$, as well as the universal quantifier, are to be introduced by definition. We adopt the following definitions:

$$\Phi \;\&\; \Psi \rightarrow \sim [\sim \Phi \;.\!\vee\!.\; \sim \Psi]$$
$$\Phi \supset \Psi \rightarrow \sim \Phi \;.\!\vee\!.\; \Psi$$
$$\Phi \equiv \Psi \rightarrow [\Phi \supset \Psi] \;\&\; [\Psi \supset \Phi]$$
$$(x^\alpha)\Phi \rightarrow \sim (\exists x^\alpha) \sim \Phi$$
$$(x)\Phi \rightarrow \sim (\exists x) \sim \Phi \,.$$

The equality relation is introduced (for unranked formulas) by the following definition:

$$u = v \rightarrow (x)[x \in u \,.\!\equiv\!.\, x \in v] \;.$$

In addition, we introduce the expression $u \cong v$, where u and v are constant terms, by

$$u \cong v \rightarrow (x^\gamma)[x^\gamma \in u \,.\!\equiv\!.\, x^\gamma \in v] \;,$$

where $\gamma = \max\,[\rho(u), \rho(v)]$. We will show later that $u \cong v$ holds if and only if $u = v$ holds in the model to be defined.

We will frequently make use of various notations defined in the Gödel monograph. In particular, if we write an unranked formula containing normal notations, operations, and variables, these are to be eliminated by the procedure described on pp. 9-13 of the Gödel monograph. Thus, it will be meaningful to speak of formulas $\Phi(\langle xy\rangle)$ of L. We note that constants $\underline{S}$, where S is a proper class of M, occur in formulas of L only in the context $u \in \underline{S}$; clearly, they could be introduced by definition in other contexts, if desired.

We will make one additional convention. If we refer to a formula $\Phi(x_1, ..., x_j, y_1^\alpha, ..., y_k^\beta)$, we intend that the formula in question is to have no free variables other than those displayed.

§ 3. The forcing argument

In this section, we apply Cohen's method of forcing to construct "generic" subsets a_η^α of regular cardinals $\aleph_\alpha$ which can be added to our model M to form a new model of Σ_*. The method is to allow the sets a_η^α to be "partially described" in the original model; we must be able to say enough about them in M to prove the axioms of replacement and power-set, but not so much that they are already sets of M.

The partial descriptions of the sets a_η^α are to be given by "sets of conditions". These will be certain sets of M which encode the fact that certain ordinals are or not elements of the a_η^α's.

Definition 4. A *set of conditions* is a set q of the model M of quadruples $\langle 0\gamma\alpha\eta\rangle$ and $\langle 1\gamma\alpha\eta\rangle$ with the following properties:

(i) $q = \bigcup_{\alpha \in \mathrm{Reg}} q^\alpha$, where Reg is the class of ordinals α such that $\aleph_\alpha$ is regular, and

(ii) q^α is a set of quadruples $\langle i\gamma\alpha\eta\rangle$, $i < 2$, $\gamma < \aleph_\alpha$, and $\eta < \aleph_{G'\alpha}$.

(iii) For regular $\aleph_\alpha$, $\bigcup_{\beta \leq \alpha} q^\beta$ is of cardinality $< \aleph_\alpha$.

(iv) For no γ, α, and η, does q contain both $\langle 0\gamma\alpha\eta\rangle$ and $\langle 1\gamma\alpha\eta\rangle$.

It is clear that the collection of all sets of conditions forms a proper class in the model M. Thus, we make the following definition:

Definition 5. *Sc* is the class of all sets of conditions.

Definition 6. p' is an *extension* of a set of conditions p if p' is a set of conditions and $p \subseteq p'$.

We will give a definition of the relation $p \Vdash \Phi$, "*p forces* Φ", between sets of conditions p and statements Φ of L. The definition is given in two parts: first by transfinite induction in M for ranked statements Φ, then by a simple induction in the metalanguage on the length of statements for unranked statements Φ. For the first part of the definition, we will assign an ordinal ord(Φ) to each ranked statement Φ and an ordinal rank(p) to each set of conditions p. The definition will then proceed by transfinite induction on $\max\{\mathrm{ord}(\Phi), \mathrm{rank}(p)\} + \mathrm{ord}(\Phi)$.

Definition 7. For a ranked statement Φ, we set:

$$\mathrm{ord}(\Phi) = \omega^2 \cdot \alpha + \omega \cdot t + l\,,$$

where α is the least ordinal such that Φ contains no variable of rank

$> \alpha$ and no constant term of rank $\geq \alpha$; $t = 0$ if[1] Φ contains no subformula of the form $v \in u$, where v is a constant term of rank α, and no subformula $A(u, v, w)$ other than inside an abstraction term; otherwise, $t = 1$; l is the length of the formula Φ. ($u \in v$ and $A(u, v, w)$ have length 1.)

Definition 8. The *rank* of a set of conditions p, rank(p) is the supremum of the ranks of its elements, where $\langle i\gamma\alpha\eta\rangle$ is said to have rank $\max(\gamma, \alpha, \eta)$.

Definition 9. $p \Vdash \Phi$ is defined (for ranked Φ) in terms of $p' \Vdash \Psi$, $\mathrm{ord}(\Psi) < \mathrm{ord}(\Phi)$ and $\mathrm{rank}(p') \leq \max\{\mathrm{ord}(\Phi), \mathrm{rank}(p)\}$, as follows:

1° $p \Vdash \sim \Phi$ if there is no set of conditions p', $\mathrm{rank}(p') \leq \mathrm{ord}(\Phi)$, such that p' is compatible with p and $p' \Vdash \Phi$. (Sets of conditions p and p' are said to be *compatible* if their union $p \cup p'$ is a set of conditions.)

2° $p \Vdash \Phi \vee \Psi$ if either $p \Vdash \Phi$ or $p \Vdash \Psi$ (or both).

3° $p \Vdash (\exists x^\alpha)\Phi(x^\alpha)$ if $p \Vdash \Phi(u)$ for some constant term u, $\rho(u) < \alpha$.

4° $p \Vdash u \in \underline{s}$ if $p \Vdash u \cong \underline{t}$ for some $t \in s$.

5° $p \Vdash u \in \hat{x}^\alpha\Phi(x^\alpha)$ if for some constant term u', $\rho(u') < \alpha$, $p \Vdash u \cong u'$, and $p \Vdash \Phi(u')$.

6° $p \Vdash A(u, v, w)$ if there exist ordinals γ, α, and η, such that $\gamma \leq \rho(u)$, $\alpha \leq \rho(v)$, $\eta \leq \rho(w)$, $p \Vdash u \cong \underline{\gamma}$, $p \Vdash v \cong \underline{\alpha}$, $p \Vdash w \cong \underline{\eta}$, and p contains $\langle 0\gamma\alpha\eta\rangle$.

7° $p \Vdash \Phi$ only as required by 1°–6° above.

In connection with the above definition, we recall that the expression $u \cong v$ was defined above as follows:

[1] The case $t = 0$ corresponds to a formula of type $\mathfrak{R}$ in Cohen, The Independence of the Continuum Hypothesis, Proc. Nat. Acad. Sci. USA 50 (1963) 1143-1148. This slight perturbation of the ordering causes no trouble, but permits a simpler definition of the equality relation.

$$u \cong v \rightarrow (x^\gamma)[x^\gamma \in u \;.\equiv .\; x^\gamma \in v] \;,$$

where $\gamma = \max\{\rho(u), \rho(v)\}$. Thus, in the definition of $\mathrm{ord}(u \cong v)$, $t = 0$.

Clause 1° of the above definition is the formal equivalent of the assertion that no properties of the generic sets are to be expressed by sets of M other than those expressed by sets of conditions. The other clauses correspond to the definition of validity and to the intended meaning of the predicate A.

Definition 10. $p \Vdash \Phi$ is defined for unranked Φ by induction on the length of the statement Φ.
1° $p \Vdash u \in v$, $p \Vdash A(u, v, w)$ if so required by definition 9.
2° $p \Vdash u \in \underline{S}$ if for some $t \in S$, $p \Vdash u \cong \underline{t}$.
3° $p \Vdash \sim \Phi$ if there is no extension p' of p such that $p' \Vdash \Phi$.
4° $p \Vdash \Phi \vee \Psi$ if $p \Vdash \Phi$ or $p \Vdash \Psi$ (or both).
5° $p \Vdash (\exists x^\alpha)\Phi(x^\alpha)$ if, for some constant term u, $p \Vdash \Phi(u)$.
6° $p \Vdash \Phi$ only as required by 1°–5° above.

Lemma 2. *There is a class in the model M whose elements are the pairs $\langle p, \text{“}\Phi\text{”}\rangle$ such that Φ is a ranked statement and $p \Vdash \Phi$.*

Proof. We use the metatheorem on definition by transfinite induction[1] to define a function Fc such that $Fc`\alpha$ is the set of pairs $\langle p, \text{“}\Phi\text{”}\rangle$ such that $p \Vdash \Phi$ and $\max\{\mathrm{rank}(p), \mathrm{ord}(\Phi)\} + \mathrm{ord}(\Phi) \leq$ $\leq \alpha$. The desired class is then the union of the range of $Fc$. Definition 9 above yields a definition of $Fc`\alpha$ in terms of the restriction of Fc to ordinals less than α by a normal propositional function; the metatheorem then gives the function Fc.

Using lemma 2, we prove the following result, which is essential to the forcing argument. It is the justification for using definition

[1] Gödel monograph, theorem 7.5.

by transfinite induction to define classes of M in terms of the forcing relation for a given unranked formula $\Phi(x_1, ..., x_n)$.

Lemma 3. *Let $\Phi(x_1, ..., x_n)$ be an unranked formula of L. There is a class of the model M whose elements are the $(n + 1)$-tuples $\langle p, `u_1', ..., `u_n' \rangle$ such that $p \Vdash \Phi(u_1, ..., u_n)$.*

Proof. By induction on the length of the formula Φ.

Case 1. $\Phi(x_1, ..., x_n)$ is an atomic formula. Then either $\Phi(u_1, ..., u_n)$ is a ranked formula, and the result follows from the preceding lemma, or it is of the form $v \in \underline{S}$, and $p \Vdash v \in \underline{S} \Longleftrightarrow (\exists x)[x \in S .\&. p \Vdash v \cong \underline{x}]$.

Case 2. $\Phi(x_1, ..., x_n)$ is $\Psi(x_1, ..., x_n) \vee \Upsilon(x_1, ..., x_n)$. Let the classes C and D satisfy the lemma for $\Psi(x_1, ..., x_n)$ and $\Upsilon(x_1, ..., x_n)$, respectively. Then $C \cup D$ is the required class for $\Phi(x_1, ..., x_n)$.

Case 3. $\Phi(x_1, ..., x_n)$ is $(\exists y)\Psi(y, x_1, ..., x_n)$. If C satisfies the lemma for $\Psi(x_0, x_1, ..., x_n)$, then $\{\langle yz \rangle | (\exists x)[\langle yxz \rangle \in C]\}$ is the required class.

Case 4. $\Phi(x_1, ..., x_n)$ is $(\exists y^\alpha)\Psi(y^\alpha, x_1, ..., x_n)$. Similar to case 3.

Case 5. $\Phi(x_1, ..., x_n)$ is $\sim \Psi(x_1, ..., x_n)$. If C satisfies the lemma for $\Psi(x_1, ..., x_n)$ and $Ct = (`u' | u$ is a constant term), then

$$\{\langle yz \rangle | y \in Sc .\&. z \in Ct^n .\&. \sim (\exists w)[y \subseteq w .\&. \langle wz \rangle \in C]\}$$

is the required class.

The "usual" definition of $p \Vdash \sim \Phi$ for both ranked and unranked statements is that no extension of p forces Φ. Since the sets of conditions constitute a proper class in M, we have given a slightly different definition in order to use the metatheorem on definition by transfinite induction. We now prove (lemma 5) that, in fact, $p \Vdash \sim \Phi$ if and only if no extension of p forces Φ. We prove a preliminary result, since it will be needed later.

Lemma 4. *If Φ is a ranked formula and $p \Vdash \Phi$, then $p' \Vdash \Phi$ for any extension p' of p.*

Proof. The proof is by transfinite induction on $\operatorname{ord}(\Phi)$. Assume that the lemma has been proved for formulas Ψ such that $\operatorname{ord}(\Psi) < \operatorname{ord}(\Phi)$.

Case 1. Φ is $\sim \Psi$. If p' does not force Φ, then there is a set of conditions p'', $\operatorname{rank}(p'') \leq \operatorname{ord}(\Psi)$, such that p'' is compatible with p' and $p'' \Vdash \Psi$. But then, p'' is compatible with p, so p does not force Φ.

Case 2. Φ is $\Psi \vee \Upsilon$. If $p \Vdash \Phi$, then either $p \Vdash \Psi$ or $p \Vdash \Upsilon$. By the induction hypothesis, the latter implies that either $p' \Vdash \Psi$ or $p' \Vdash \Upsilon$, hence $p' \Vdash \Phi$.

Case 3. Φ is $(\exists x^\alpha)\Psi(x^\alpha)$, $u \in v$, or $A(u, v, w)$. Similar to case 2.

Lemma 5. $p \Vdash \sim \Phi \iff$ *no extension p' of p forces Φ.*

Proof. It suffices to show that if $p \Vdash \Phi$, then some subset p' of p, $\operatorname{rank}(p') \leq \operatorname{ord}(\Phi)$, also forces Φ. This is proved by transfinite induction on $\operatorname{ord}(\Phi)$.

Case 1. Φ is $\sim \Psi$. Let p' be the set of members of p of rank $\leq \operatorname{ord}(\Psi)$. If p'' is of rank $\leq \operatorname{ord}(\Psi)$ and p'' is compatible with p', then p'' is also compatible with p. It follows that $p \Vdash \Phi$ implies $p' \Vdash \Phi$.

Case 2. Φ is $\Psi \vee \Upsilon$. If $p \Vdash \Phi$, then either $p \Vdash \Psi$ or $p \Vdash \Upsilon$. Hence, by the induction hypothesis, $p \Vdash \Phi$ implies that for some subset p' of p of rank $\leq \operatorname{ord}(\Phi)$, either $p' \Vdash \Psi$ or $p' \Vdash \Upsilon$. This, in turn, implies that $p' \Vdash \Phi$.

Case 3. Φ is $(\exists x^\alpha)\Psi(x^\alpha)$ or $u \in v$. Similar to case 2.

Case 4. Φ is $A(u, v, w)$. Assume that $p \Vdash A(u, v, w)$. Then there are ordinals γ, α, and η, *less than* $\operatorname{ord}(\Phi)$ such that p contains $\langle 0\gamma\alpha\eta\rangle$ and p forces $u \cong \underline{\gamma}$, $v \cong \underline{\alpha}$, and $w \cong \underline{\eta}$. By the induction hypothesis, there are subsets of p of rank less than $\operatorname{ord}(\Phi)$ which force $u \cong \underline{\gamma}$, $v \cong \underline{\alpha}$, and $w \cong \underline{\eta}$, respectively. By lemma 4, the union of these three subsets and $\{\langle 0\gamma\alpha\eta\rangle\}$ also forces Φ.

Following Cohen, we prove the three basic properties of the forcing relation. The first two are immediate consequences of the definition of $p \Vdash \sim \Phi$ and lemma 5.

Lemma 6. *No set of conditions forces both a statement Φ and its negation $\sim \Phi$.*

Lemma 7. *Let p be a set of conditions and let Φ be a statement of L. Then there is an extension p' of p such that either $p' \Vdash \Phi$ or $p' \Vdash \sim \Phi$.*

Lemma 8. *Let $p \Vdash \Phi$ and let p' be an extension of p. Then $p' \Vdash \Phi$.*

Proof. The proof for ranked statements Φ has already been given in lemma 4 above. The proof for unranked statements proceeds by induction on the length of the statement Φ.

Case 1. Φ is an atomic formula. Then either Φ is a ranked formula, and lemma 4 applies, or Φ is of the form $u \in \underline{S}$, and the proof follows as in case 1 of lemma 3.

Case 2. Φ is $\sim \Psi$. Since any extension of p' is *a fortiori* an extension of p, $p \Vdash \Phi$ implies $p' \Vdash \Phi$, as required.

Case 3. Φ is $(\exists x^\alpha)\Psi(x^\alpha)$, $(\exists x)\Psi(x)$, or $\Psi \vee \Upsilon$. Similar to case 2 in the proof of lemma 4.

The forcing relation, as defined above, has one rather troublesome property. Namely, it is not "deductively closed" [1]. We introduce a notion of weak forcing, due to Feferman and others, which does not have this drawback. The intention is that p will weakly force Φ if Φ is true in all models obtained from complete sequences of sets of conditions which contain p as a term.

Definition 11. $p \Vdash^* \Phi$, *p weakly forces Φ*, if $p \Vdash \sim \sim \Phi$.

[1] For example, the void set of conditions does not force $A(\underline{0}, \underline{0}, \underline{0}) \vee \sim A(\underline{0}, \underline{0}, \underline{0})$.

The following properties of the weak forcing relation can be verified by direct computation. (Recall that $\Phi \,\&\, \Psi$, $\Phi \equiv \Psi$, $(x)\Psi$, $u \cong v$, etc., are of the form $\sim \Upsilon$.)

Lemma 9. The weak forcing relation has the following properties:

(i) $p \Vdash^* \Phi \Longleftrightarrow$ *no extension of* p *forces* $\sim \Phi$.

(ii) $p \Vdash \Phi \Rightarrow p \Vdash^* \Phi$.

(iii) $p \Vdash^* \sim \Phi \Longleftrightarrow p \Vdash \sim \Phi$.

(iv) *If* Φ *is of the form* $\Psi \,\&\, \Upsilon$, $\Psi \equiv \Upsilon$, $(x)\Psi$, $(x^\alpha)\Psi$, $u \cong v$, *or* $u = v$, *then* $p \Vdash \Phi \Longleftrightarrow p \Vdash^* \Phi$.

(v) $p \Vdash^* (x)\Phi(x) \Longleftrightarrow p \Vdash \Phi(u)$ *for all constant terms* u.

(vi) $p \Vdash^* (x^\alpha)\Phi(x^\alpha) \Longleftrightarrow p \Vdash^* \Phi(u)$ *for all constant terms* u *of rank less than* α.

(vii) $p \Vdash^* \Phi \equiv \Psi$ *and* $p \Vdash^* \Phi \Rightarrow p \Vdash^* \Psi$.

(viii) $p \Vdash^* \Phi \equiv \Psi$ *and* $p \Vdash^* \Psi \Rightarrow p \Vdash^* \Phi$.

We have introduced two expressions, $u \cong v$ and $u = v$, which will both represent the equality relation in the model to be constructed. Before that model can be defined, we must prove that our two equality symbols have certain properties.

Lemma 10. $p \Vdash u \cong v \Longleftrightarrow p \Vdash u = v$.

Proof. We proof only the implication in the forward direction; the converse is proved by a straightforward computation. The proof is by transfinite induction on $\gamma = \max\{\rho(u), \rho(v)\}$.

Assume that $p \Vdash u \cong v$. It suffices to show that there is no extension p' of p such that either $p' \Vdash w \in u$ and $p' \Vdash \sim w \in v$ or else $p' \Vdash w \in v$ and $p' \Vdash \sim w \in u$. We prove only the former; the latter is proved similarly.

Let $p \subseteq p'$ and $p' \Vdash w \in u$. Then, there is a term w', $\rho(w') < \rho(u)$, such that $p' \Vdash w \cong w'$ and $p' \Vdash w' \in u$. Since $\rho(w') < \gamma$, $p' \Vdash^* w' \in v$ by lemma 9. Let p'' be an extension of p' such that $p'' \Vdash w' \in v$. We consider two cases.

Case 1. $\rho(w') < \rho(v)$. Then $p'' \Vdash w \in v$ by definition.

Case 2. $\rho(w') \geq \rho(v)$. Then there is a term w'', $\rho(w'') < \rho(v)$, such that $p'' \Vdash w' \cong w''$ and $p'' \Vdash w'' \in v$. By the induction hypothesis, $p'' \Vdash w' = w''$. A straightforward computation now yields $p'' \Vdash w \cong w''$, so $p'' \Vdash w \in v$.

Lemma 11. *For any set of conditions p,*
(i) $p \Vdash u = u$.
(ii) $p \Vdash u = v \Longleftrightarrow p \Vdash v = u$.
(iii) $p \Vdash u = v$ *and* $p \Vdash v = w \Rightarrow p \Vdash u = w$.

Lemma 12. *For any set of conditions p,*
(i) $p \Vdash u \in w$ *and* $p \Vdash u = v \Rightarrow p \Vdash v \in w$.
(ii) $p \Vdash w \in u$ *and* $p \Vdash u = v \Rightarrow p \Vdash^* w \in v$.

Proof. Proposition (i) follows immediately from the definition of forcing and lemma 10. Proposition (ii) follows by lemma 9.

Lemma 13. If $p \Vdash u = \alpha$, *then* $\rho(u) \geq \alpha$.

Proof. Let u be a term of smallest rank for which the lemma is false, and let $p \Vdash u = \underline{\alpha}$, $\rho(u) < \alpha$. Let β be any ordinal less than α. $p \Vdash \underline{\beta} \in \underline{\alpha}$, hence $p \Vdash^* \underline{\beta} \in u$. Thus, there is a set of conditions p' such that $p' \Vdash \underline{\beta} \in u$. But then, $p' \Vdash v = \underline{\beta}$ and $p' \Vdash v \in u$ for some term v, $\rho(v) < \rho(u)$. By the choice of u, $\rho(v) \geq \beta$. Hence, $\rho(u) > \beta$ for all $\beta < \alpha$, so $\rho(u) \geq \alpha$.

Corollary 13.1. *If* $p \Vdash A(u, v, w)$ *and* $p \Vdash u = u'$, $p \Vdash v = v'$, *and* $p \Vdash w = w'$, *then* $p \Vdash A(u', v', w')$.

Proof. Lemma 13 shows that the restrictions $\gamma \leq \rho(u')$, $\alpha \leq \rho(v')$, and $\eta \leq \rho(w')$ in the definition of $p \Vdash A(u', v', w')$ are automatically satisfied. The proof then follows as in lemma 12.

Definition 12. A sequence of sets of conditions $p^{(0)} \subseteq p^{(1)} \subseteq \ldots$

is said to be *complete* [1] if for every class C of sets of conditions such that every set of conditions has an extension in C, $p^{(k)} \in C$ for some k.

Lemma 14. *Let $p^{(0)} \subseteq p^{(1)} \subseteq \ldots$ be a complete sequence of sets of conditions. Then every (ranked or unranked) statement or its negation is eventually forced by some $p^{(k)}$.*

Proof. Given Φ, the collection of sets of conditions such that either $p \Vdash \Phi$ or $p \Vdash \sim \Phi$ is a class by lemmas 2 and 3. By lemma 7, every set of conditions can be extended to a set of conditions in this class.

Lemma 15. *There exists a complete sequence of sets of conditions.*

Proof. Let $C_0, C_1, \ldots,$ be an enumeration of all classes of sets of conditions. (There are countably many because M is a countable model.) Let $p^{(0)}$ be any set of conditions, and let $p^{(n+1)} \in C_n$ if such an extension of $p^{(n)}$ exists; otherwise, let $p^{(n+1)} = p^{(n)}$.

We now choose a complete sequence of sets of conditions $p^{(0)} \subseteq p^{(1)} \subseteq \ldots$ and define a corresponding model N.

Definition 13. [*Definition of the model N.*]

(i) Let $\Phi(x)$ be an unranked formula of L. The collection all constant terms v such that for some k, $p^{(k)} \Vdash \Phi(v)$ will be a *class* of the model N; we will denote this class by $\hat{x}\Phi(x)$.

(ii) *Sets* of the model N will be classes of the form $\hat{x}[x \in u]$, where u is a constant term of L.

(iii) The ϵ-*relation* is defined as follows: $\hat{x}\Phi(x) \in \hat{y}\Psi(y)$ will hold

[1] In Feferman's notes, such a sequence is *complete* if every statement Φ or its negation $\sim \Phi$ is forced by some $p^{(k)}$. Lemma 14 states that a sequence complete in our sense is also complete in Feferman's sense. In many cases the two definitions are equivalent; on the other hand, our definition will simplify the later work.

if $\hat{x}\Phi(x)$ is the same as $\hat{x}[x \in u]$ for some constant term u and $p^{(k)} \Vdash \Psi(u)$ for some k.

(iv) The *relation* $A(\hat{x}\Phi(x), \hat{y}\Psi(y), \hat{z}\Upsilon(z))$ will hold if for some γ, α, and η such that $\langle 0\gamma\alpha\eta\rangle$ is contained in some $p^{(k)}$, $\hat{x}\Phi(x)$ is $\hat{x}[x \in \underline{\gamma}]$, $\hat{y}\Psi(y)$ is $\hat{y}[y \in \underline{\alpha}]$, and $\hat{z}\Upsilon(z)$ is $\hat{z}[z \in \underline{\eta}]$.

(v) If W is a constant term or constant of L, then W *denotes* the class $\hat{x}[x \in W]$.

(vi) Individual variables $x, y, \ldots$ range over all sets of N; ranked variables of rank α range over sets of the form $\hat{x}[x \in u]$ where u is a constant term of rank less than α.

Definition 14. We will say that the ranked or unranked statement Φ is *eventually forced* if for some k, $p^{(k)} \Vdash \Phi$.

Our next task is to prove the lemma that a statement is true in N if an only if it is eventually forced. It is precisely this fact which allows Cohen to reduce questions about the extension to questions which can be asked in the model M. By lemma 16, every statement or its negation is eventually forced.

Lemma 16. *$\hat{x}\Phi(x)$ is the same as $\hat{y}\Psi(y)$ if and only if $(x)[\Phi(x) \equiv \equiv \Psi(x)]$ is eventually forced. Hence, if U and W are constants or constant terms, then $U = W$* (i.e., $\hat{x}[x \in U]$ *and* $\hat{y}[y \in W]$ *are identical*) *if and only if $U = W$ is eventually forced* (i.e., $(x)[x \in U .\equiv. .\equiv. x \in W]$ *is eventually forced*).

Proof. If $\hat{x}\Phi(x)$ and $\hat{y}\Psi(y)$ are identical, then $\Phi(w) \equiv \Psi(w)$ is eventually forced for every constant term w. If no $p^{(k)}$ forces $(x)[\Phi(x) \equiv \Psi(x)]$, then some $p^{(k)}$ must force $(\exists x) \sim [\Phi(x) \equiv \Psi(x)]$, since $p^{(0)}, p^{(1)}, \ldots$ is complete. But then, $p^{(k)} \Vdash \sim [\Phi(w) \equiv \Psi(w)]$ for some w; hence some $p^{(j)}$ forces both $[\Phi(w) \equiv \Psi(w)]$ and $\sim [\Phi(w) \equiv \Psi(w)]$, contrary to lemma 6.

If $p^{(k)} \Vdash (x)[\Phi(x) \equiv \Psi(x)]$, then $p^{(k)} \Vdash \Phi(w) \equiv \Psi(w)$ for every constant term w. Thus, if $j \geq k$ and $p^{(j)} \Vdash \Phi(w)$, then $p^{(j)} \Vdash^* \Psi(w)$ so $\Psi(w)$ is eventually forced. Thus, $\hat{x}\Phi(x) \subseteq \hat{y}\Psi(y)$; similarly, $\hat{y}\Psi(y) \subseteq \hat{x}\Phi(x)$.

We will write $\hat{x}\Phi(x) = \hat{y}\Psi(y)$ to indicate that $\hat{x}\Phi(x)$ and $\hat{y}\Psi(y)$ are identical. Furthermore, we will write W instead of $\hat{x}[x \in W]$.

Lemma 17. *An unranked statement Φ of L is true in the model N if and only if Φ is eventually forced.*

Proof. By induction on the length of the formula Φ.

Case 1. Φ is $u \in W$. If $u = \hat{x}[x \in v]$, then $u = v$ is eventually forced. Hence, by lemma 9, $u \in W$ is eventually forced if and only if $v \in W$ is eventually forced. (Note that if W is $\hat{x}\Psi(x)$, then $u \in W$ is eventually forced if and only if $\Psi(u)$ is eventually forced, by the definition of $\hat{x}\Psi(x)$.)

Case 2. Φ is $A(u, v, w)$. $A(u, v, w)$ is eventually forced only if for some γ, α, and η, some $p^{(k)}$ contains $\langle 0\gamma\alpha\eta \rangle$ and $u = \underline{\gamma}$, $v = \underline{\alpha}$, and $w = \underline{\eta}$ are eventually forced. The result follows by the preceding lemma.

Case 3. Φ is $\sim \Psi$. Since $p^{(0)}, p^{(1)}, \ldots$ is complete, Φ is eventually forced $\Longleftrightarrow$ Ψ is not eventually forced. Hence,

Φ is true $\Longleftrightarrow$ Ψ is false
$\Longleftrightarrow$ Ψ is not eventually forced
$\Longleftrightarrow$ Φ is eventually forced.

Case 4. Φ is $\Psi \vee \Upsilon$. Then,

Φ is true $\Longleftrightarrow$ either Ψ or Υ is true
$\Longleftrightarrow$ either Ψ or Υ is eventually forced
$\Longleftrightarrow$ Φ is eventually forced.

Case 5. Φ is $(\exists x)\Psi(x)$. Then,

Φ is true $\Longleftrightarrow$ for some u, $\Psi(u)$ is true
$\Longleftrightarrow$ for some u, $\Psi(u)$ is eventually forced
$\Longleftrightarrow$ $(\exists x)\Psi(x)$ is eventually forced
$\Longleftrightarrow$ Φ is eventually forced.

Case 6. Φ is $(\exists x^{\alpha})\Psi(x^{\alpha})$. Similar to case 5.

Thus, we have characterized the true sentences in the model N in terms of the forcing relation. It is easy to see that the weak forcing relation has the intended property:

Lemma 18. *If Φ is an unranked statement of L, then $P \Vdash^* \Phi$ if and only if Φ is true in all models obtained by the above construction from complete sequences of sets of conditions in which p occurs.*

Proof. If $p \Vdash^* \Phi$, then $\sim \Phi$ cannot be eventually forced, so Φ must be eventually forced. If p does not weakly force Φ, then there is an extension p' of p which forces $\sim \Phi$; hence, Φ is false in a model obtained from a complete sequence of sets of conditions in which both p and p' occur.

Corollary 18.1. *If $\Phi(x)$ is a ranked formula of "standard" finite length, then $u \in \hat{x}^\alpha \Phi(x^\alpha)$ is true in N if and only if $u = u'$ for some u' of rank less than α and $\Phi(u)$ is true in N.*

Before we go on to prove that the axioms of Σ_* holds in N, we show that the model M can be embedded in the model N as a complete submodel. This can be done even before the axioms have been proved to hold because we have provided constants for the classes of M.

Lemma 19. *The mapping ψ of M into N given by $\psi(S) = \underline{S}$ is an isomorphism with respect to the $\in$-relation.*

Proof. We must show first that if $p \Vdash \underline{s} = \underline{t}$ for sets s and t of M, then, in fact, $s = t$. The proof of lemma 13, with only minor modifications, can be used to prove that if $p \Vdash \underline{w} \in \underline{t}$, then $w \in t$. We now assume that the preliminary result is true for sets s' and t' of rank less than the maximum of the ranks of s and t, and prove it for s and t. If s and t are not the same set, there is some set w which is contained in one and not the other. By lemma 11, we may assume that w is an element of s but not of t. By the definition of forcing, $p \Vdash \underline{w} \in \underline{s}$ for any p. If $p \Vdash \underline{w} \in \underline{t}$, then for some $w' \in t$, $p \Vdash \underline{w} = \underline{w}'$. By the induction hypothesis, $w = w'$, so $w \in t$, contrary to the assumption. Thus, $\underline{w} \in \underline{t}$ can never be forced; it is a straightforward computation to show that $\underset{\sim}{s} = \underset{\sim}{t}$ can never be forced either.

We return to the proof of the lemma. By the definition of forcing and the result above, $p \Vdash \underset{\sim}{t} \in \underset{\sim}{S}$ if and only if $t \in S$. Hence, if S and T are distinct classes in M, $\underset{\sim}{S}$ and $\underset{\sim}{T}$ denote distinct classes in N. It is clear that $\underset{\sim}{t} \in \underset{\sim}{S}$ is true if and only if $t \in S$ is true in M. If T is a proper class, then there is no u such that $\underset{\sim}{T} = u$ and $u \in \underset{\sim}{S}$, for then $\underset{\sim}{T} = \underset{\sim}{t}'$ for some t', which is impossible.

From now on, we will identify the model M with its isomorphic image in N. (It will follow from lemma 32 below that the mapping ψ takes proper classes into proper classes.) The model M is a *complete* inner model of N, i.e., if $u \in \underset{\sim}{S}$, then $u = \underset{\sim}{t}$ for some $t \in S$, as can be seen by lemma 17 and the definition of forcing.

§ 4. Proof of the axioms in the model

We now prove that the model N constructed above is a model of Gödel-Bernays set theory and the axiom of choice. As in Cohen's construction, the difficult axioms are the axioms of power set and replacement (C4 and C3). Indeed, if we had chosen to work with well-founded models, the other axioms would have been "inherited" from the universe.

Lemma 20. *The axioms of group A hold in the model N.*

Proof. A1. Sets are classes by definition. A2. By definition, $\hat{x}\Phi(x) \in \hat{y}\Psi(y)$ holds only if $\hat{x}\Phi(x)$ is $\hat{x}[x \in u]$ for some u, i.e., if $\hat{x}\Phi(x)$ is a set. A3. The axiom of extensionality follows immediately from the definition of the model N and lemma 17. A4. If u and v are abstraction terms of rank at most γ, then $\hat{x}^{\gamma+1}[x^{\gamma+1} \cong u \,.\!\vee\!.\, x^{\gamma+1} \cong v]$ is the unordered pair of the sets u and v.

Axiom A4 justifies the introduction of ordered pairs in terms of unordered pairs, $\langle xy \rangle = \{\{x\}\{xy\}\}$. We recall that normal operations, etc., are to be used as abbreviations in writing unranked formulas; an unranked formula so abbreviated is to be obtained as in the Gödel monograph.

Lemma 21. *The axioms of group B hold in the model N.*

Proof. Each of the axioms of group B is of the form $(X_1) \ldots (X_k)(\mathrm{E}Y)(z)[z \in Y \,.\!\equiv\!.\, \Phi(X_1, \ldots, X_k, z)]$, with $k = 0, 1$, or 2. For classes $\hat{x}_1 \Psi_1(x_1), \ldots, \hat{x}_k \Psi_k(x_k)$, the required class Y is $\hat{z}\Phi(\hat{x}_1 \Psi_1(x_1), \ldots, \hat{x}_k \Psi_k(x_k), z)$ where $\Phi(\hat{x}_1 \Psi_1(x_1), \ldots, \hat{x}_k \Psi_k(x_k), z)$ is the unranked formula obtained by replacing each subformula $u \in X_i$ of $\Phi(X_1, \ldots, X_k, z)$ by $\Psi_i(u)$.

Lemma 22. *Axioms C1 and C2 hold in the model N.*

Proof. C1. The set $\underline{\omega}$ is a set of the form required by the axiom of infinity. C2. For any constant term u of rank γ, the generalized union of the set u is the set $\hat{x}^{\gamma+1}(\mathrm{E}y^{\gamma})[x^{\gamma+1} \in y^{\gamma} \,.\&.\, y^{\gamma} \in u]$.

Lemma 23. *Axiom D (the axiom of foundation holds in the model N.*

Proof. Let $\hat{x}\Phi(x)$ be a non-empty class. We must show that for some u, $\Phi(u)$ is true and $(\exists x)[x \in u \,.\&.\, \Phi(x)]$ is false. Let p be any set of conditions which forces $(\exists x)\Phi(x)$. Let u be a term of least rank such that some extension of p forces $\Phi(u)$, and let p' be such an extension. Then $p' \Vdash \Phi(u)$; but, by the choice of u, no extension of p' can force both $\Phi(w)$ and $w \in u$. Hence, $p' \Vdash \sim (\exists x)[x \in u \,.\&.\, \Phi(x)]$. Since any set of conditions can be extended in this way, some $p^{(k)}$ is obtained which forces the same. Hence, axiom D is true for the class $\hat{x}\Phi(x)$.

In the original Cohen construction, the proof of the axioms of replacement and power set makes use of the fact that the sets of conditions form a set in the original model. We will verify these axioms by showing that certain questions about the model N can be reduced to consideration of a *set* of sets of conditions.

Definition 15. We define classes Γ_α and Δ_α as follows:
(i) $\Gamma_\alpha = \cup\{2 \times \aleph_\beta \times \{\beta\} \times \aleph_{G'\beta} \mid \beta \in \mathrm{Reg} .\&. \beta \leq \alpha\}$,
(ii) $\Delta_\alpha = \cup\{2 \times \aleph_\beta \times \{\beta\} \times \aleph_{G'\beta} \mid \beta \in \mathrm{Reg} .\&. \beta > \alpha\}$,
where Reg is the class of ordinals β such that $\aleph_\beta$ is regular.

Thus, Γ_α consists of those quadruples $\langle i\gamma\alpha\eta\rangle$ which can occur in sets of conditions and for which $\beta \leq \alpha$; and Δ_α consists of such quadruples for which $\beta > \alpha$. Any set of conditions can be written in the form $p \cup q$, where $p \subseteq \Gamma_\alpha$ and $q \subseteq \Delta_\alpha$. In this case, p contains information about the a^β_η's with $\beta \leq \alpha$ and q contains information about the a^β_η's with $\beta > \alpha$.

We note that Γ_α is a set of M. If $\aleph_\alpha$ is a regular cardinal and $p \subseteq \Gamma_\alpha$ is a set of conditions, then $\bar{\bar{p}} < \aleph_\alpha$. On the other hand, if $q_\beta \subseteq \Delta_\alpha$ for $\beta < \aleph_\alpha$ and $q_\beta \subseteq q_\delta$ for $\beta \leq \delta$, then $\bigcup_{\beta < \aleph_\alpha} q_\beta$ is again a set of conditions.

Definition 16. Sets of conditions p and q are said to be *compatible* if their union is again a set of conditions, i.e., if it is not the case that one of them contains a quadruple $\langle 0\gamma\alpha\eta\rangle$ while the other contains $\langle 1\gamma\alpha\eta\rangle$.

The next lemma is the major tool used to show that cardinals are absolute in the extension from M to N. We prove it at this point, since we will make use of it in the proofs of the remaining axioms[1].

[1] Axiom E must be used in the proof to select the sets q_α. The use of axiom E at this point seems to be essential to the proof of the replacement axiom in the model; if we did not insist that the relation A be represented by a class, the axiom of choice for sets would suffice.

Lemma 24. *Let $\aleph_\alpha$ be a regular cardinal of M, and let q be a set of conditions, $q \subseteq \Delta_\alpha$, and let Φ be a statement of L. Then there is an extension $\bar{q} \subseteq \Delta_\alpha$ of q and a set Π of sets of conditions such that:*
(i) $\overline{\overline{\Pi}} \leq \aleph_\alpha$.
(ii) $p \in \Pi \Rightarrow$ *either* $p \cup \bar{q} \Vdash \Phi$ *or* $p \cup \bar{q} \Vdash \sim \Phi$.
(iii) *If p' is any set of conditions, there is some $p \in \Pi$ compatible with p'.*
(iv) $p \in \Pi \Rightarrow p \subseteq \Gamma_\alpha$.

Proof. We will construct sets p_μ and q_μ for $\mu < \aleph_\alpha \cdot \aleph_\alpha$ in $\aleph_\alpha$ stages.

Stage 0. Let $p_0 \cup q_0$ be any extension of q which forces either Φ or $\sim \Phi$. We set $p_\mu = p_0$ and $q_\mu = q_0$ for $\mu < \aleph_\alpha$.

Stage $\mu > 0$. Let $\{p_\nu^0 \mid \aleph_\alpha \cdot \mu \leq \nu < \aleph_\alpha \cdot (\mu + 1)\}$ be the set of all sets of conditions p such that $\langle i\gamma\beta\eta \rangle \in p$ only if either $\langle 0\gamma\beta\eta \rangle \in p_\lambda$ or $\langle 1\gamma\beta\eta \rangle \in p_\lambda$ for some $\lambda < \aleph_\alpha \cdot \mu$. (Since the generalized continuum hypothesis holds in M, there are at most $\aleph_\alpha$ such sets of conditions.) We define $p_\nu \cup q_\nu (\aleph_\alpha \cdot \mu \leq \nu\ \aleph_\alpha \cdot (\mu + 1))$ to be an extension of $p_\nu^0 \cup \left(\bigcup_{\lambda < \nu} q_\lambda \right)$ which forces either Φ or $\sim \Phi$.

Finally, we let $\bar{q}$ be the union of the q_μ's and we let Π be the set of all p_μ's for $\mu < \aleph_\alpha \cdot \aleph_\alpha$. It is clear from the construction that $\bar{q} \subseteq \Delta_\alpha$ and that conditions (i), (ii), and (iv) are satisfied. We must show that (iii) is also satisfied.

Let p' be any set of conditions. Since $p \subseteq \Gamma_\alpha$, we may as well assume that $p' \subseteq \Gamma_\alpha$ also. Thus, $\overline{\overline{p'}} < \aleph_\alpha$. We associate with each element $\langle i\gamma\beta\eta \rangle$ of p' the first ordinal μ such that $\langle 1 - i, \gamma, \beta, \eta \rangle$ is an element of some p_ν constructed at stage μ, if such a μ exists. Since $\overline{\overline{p'}} < \aleph_\alpha$, there is some $\bar{\mu} < \aleph_\alpha$ greater than all such μ's. But then, for some ν, $\aleph_\alpha \cdot \bar{\mu} \leq \nu < \aleph_\alpha \cdot (\bar{\mu} + 1)$,

$$\begin{aligned} \langle i\gamma\beta\eta \rangle \in p' \Longleftrightarrow \text{ either } & \langle i\gamma\beta\eta \rangle \in p_\nu^0 \\ & \text{or no } p_\lambda \text{ contains } \langle 1 - i, \gamma, \beta, \eta \rangle . \end{aligned}$$

Hence, p_ν^0 is compatible with p' and p_ν contains no quadruples not already in p_ν^0 which conflict with p'. Hence, p_ν and p' are compatible.

Lemma 25. *Let β be any ordinal of M, let q be a set of conditions, let $\Phi(x_1, ..., x_n)$ be an unranked formula of L, and let $u_{i\mu}$, $1 \leq i \leq n$ and $\mu < \aleph_\beta$ be constant terms. There is an extension $\bar{q}$ of q and a set Π of sets of conditions such that:*

(i) $\overline{\overline{\Pi}} \leq \aleph_\beta$.

(ii) *If q' is an extension of $\bar{q}$, and $\mu < \aleph_\beta$, there is some $p \in \Pi$ compatible with q' such that either $p \cup \bar{q} \Vdash \Phi(u_{1\mu}, ..., u_{n\mu})$ or $p \cup \bar{q} \Vdash \sim \Phi(u_{1\mu}, ..., u_{n\mu})$.*

(iii) $p \in \Pi \Rightarrow p \subseteq \Gamma_\beta$.

Proof. In order to simplify our notation, we let Φ_μ be the statement $\Phi(u_{1\mu}, ..., u_{n\mu})$.

We first assume that $\aleph_\beta$ is a regular cardinal and that $q \subseteq \Delta_\beta$. We will construct an extension $\bar{q}$ of q satisfying the conclusion of the lemma such that $\bar{q} \subseteq \Delta_\beta$ as well.

We apply the preceding lemma $\aleph_\beta$ times to define sets q_μ and Π_μ for $\mu < \aleph_\beta$. At stage $\mu < \aleph_\beta$, we construct (by lemma 24) an extension q_μ of $q \cup \left(\bigcup_{\nu<\mu} q_\nu \right)$ and a set Π_μ such that:

(i) $\overline{\overline{\Pi}}_\mu \leq \aleph_\beta$.

(ii) $p \in \Pi_\mu \Rightarrow p \cup q_\mu \Vdash \Phi_\mu$ or $p \cup q_\mu \Vdash \sim \Phi_\mu$.

(iii) If q' is an extension of q_μ, there is some $p \in \Pi_\mu$ compatible with q'.

(iv) $p \in \Pi_\mu \Rightarrow p \subseteq \Gamma_\beta$.

We set $\bar{q} = \bigcup_{\mu<\aleph_\beta} q_\mu$ and $\Pi = \bigcup_{\mu<\aleph_\beta} \Pi_\mu$. Since $q_\mu \subseteq \Delta_\beta$ for each μ, $\bar{q}$ is a set of conditions and $\bar{q} \subseteq \Delta_\beta$. If q' is an extension of $\bar{q}$ and $\mu < \aleph_\beta$, then (iii) above permits us to choose $p \in \Pi_\mu$ such that (ii) is satisfied for $\Phi(u_{1\mu}, ..., u_{n\mu}) = \Phi_\mu$.

Now let us drop the assumption that $q \subseteq \Delta_\beta$ and assume only that $\aleph_\beta$ is regular. Let $\bar{q}$ and Π be constructed as above with $q \cap \Delta_\beta$

in place of q. Since $\bar{q} \subseteq \Delta_\beta$, $q \cup \bar{q}$ is a set of conditions. Since an extension of $q \cup \bar{q}$ is also an extension of $\bar{q}$, the conclusion of the lemma is satisfied by $q \cup \bar{q}$ and Π.

Finally, we prove the lemma for the case that $\aleph_\beta$ is a singular cardinal. Let $\aleph_\beta = \sup_{\mu<\beta'} \aleph_{\alpha_\mu+1}$, where $\beta' < \aleph_\beta$. We apply the preceding construction to define sets Π_μ and q_μ for each $\mu < \beta'$, such that

(i) $\bar{\bar{\Pi}}_\mu \leq \aleph_{\alpha_\mu+1}$ and q' is an extension of q_μ, then there is some $p \in \Pi_\mu$ compatible with q' such that $p \cup q_\mu \Vdash \Phi_\mu$ or $p \cup q_\mu \Vdash$ $\Vdash \sim \Phi_\mu$.

(iii) $p \in \Pi_\mu \Rightarrow p \subseteq \Gamma_{\alpha_\mu+1}$.

(iv) $q_\mu - \left(q \cup \left(\bigcup\limits_{\nu<\mu} q_\nu\right)\right) \subseteq \Delta_{\alpha_\mu+1}$.

By the cardinality restriction on q_μ implied by (iv) above, the union of the q_μ's is a set of conditions. It is now easy to see that $\bar{q} = \bigcup\limits_{\mu<\beta'} q_\mu$ and $\Pi = \bigcup\limits_{\mu<\beta'} \Pi_\mu$ are the required sets.

We will now show that the axiom of replacement holds in our model N. In the Cohen proof, "bounds" are computed for quantifiers in certain unranked statements by showing that for any statement $(\exists x)\Phi(x)$ there is an ordinal γ such that for every set of conditions p,

$$p \Vdash (\exists x)\Phi(x) \Longleftrightarrow p \Vdash (\exists x^\gamma)\Phi(x^\gamma) .$$

In the present case, we do not know how to compute such a uniform bound. Instead, we use the preceding lemma to show that any set of conditions can be extended to force the existence of an appropriate bound for a set of such statements.

Lemma 26. *Let q be a set of conditions and let $\Phi(x_1^\gamma, ..., x_n^\gamma)$ be an unranked formula. Then there is an extension $\bar{q}$ of q and an ordinal δ such that*

$$\bar{q} \Vdash (x_1^\gamma) \ldots (x_n^\gamma)[(\exists y)\Phi(x_1^\gamma, \ldots, x_n^\gamma, y) .\equiv. (\exists y^\delta)\Phi(x_1^\gamma, \ldots, x_n^\gamma, y^\delta)] .$$

Proof. By lemma 25, there is an extension $\bar{q}$ of q and a set Π of sets of conditions such that for any terms $u_1, \ldots, u_n$ of rank less than γ, if q' is an extension of $\bar{q}$ and $q' \Vdash (\exists y)\Phi(u_1, \ldots, u_n, y)$, there is some $p \in \Pi$ such that p is compatible with q' and $p \cup \bar{q} \Vdash$ $\Vdash (\exists y)\Phi(u_1, \ldots, u_n, y)$. (In the application of lemma 25, $\aleph_\beta$ is to be chosen so that there are no more than $\aleph_\beta$ terms of rank less than γ.)

For each $p \in \Pi$ and each $u_1, \ldots, u_n$ of rank less than γ, we let $\beta(p, u_1, \ldots, u_n)$ be the least ordinal β such that $p \cup \bar{q} \Vdash$ $(\exists y)\Phi(u_1, \ldots, u_n, v)$ for some term v of rank β, if such a β exists; otherwise, we let $\beta(p, u_1, \ldots, u_n) = 0$. By lemma 3, the correspondence so defined is represented by a class of the model M. Hence, by the axiom of replacement in the model M, the collection of all such ordinals $\beta(p, u_1, \ldots, u_n)$ forms a set in the model M. Thus, we can choose an ordinal δ greater than all of the ordinals $\beta(p, u_1, \ldots, u_n)$.

It is now easy to see that if q' is an extension of $\bar{q}$ which forces $(\exists y)\Phi(u_1, \ldots, u_n, y)$, then there is a further extension $p \cup q'$ which forces $(\exists y^\delta)\Phi(u_1^\gamma, \ldots, u_n^\gamma, y^\delta)$. That $\bar{q}$ and δ have the required property follows by a straightforward computation.

Corollary 26.1. *Let $\Phi(x_1^\gamma, \ldots, x_n^\gamma, y)$ be an unranked formula. Then there is an ordinal δ such that*

$$(x_1^\gamma) \ldots (x_n^\gamma)[(\exists y)\Phi(x_1^\gamma, \ldots, x_n^\gamma, y) .\equiv. (\exists y^\delta)\Phi(x_1^\gamma, \ldots, x_n^\gamma, y^\delta)]$$

is true in the model N.

Lemma 27. *Let $\Phi(x_1^\gamma, \ldots, x_n^\gamma)$ be an unranked formula. Then there is a formula $\Phi'(x_1^\gamma, \ldots, x_n^\gamma)$ which is both a ranked and an unranked* *formula such that*

$$(x_1^\gamma) \dots (x_n^\gamma)[\Phi(x_1^\gamma, \dots, x_n^\gamma) .\equiv. \Phi'(x_1^\gamma, \dots, x_n^\gamma)]$$

is true in N.

Proof. By induction on the length of the formula $\Phi(x_1^\gamma, \dots, x_n^\gamma)$.

Basis. Since an atomic formula contains no unranked quantifiers, we can let $\Phi'(x_1^\gamma, \dots, x_n^\gamma)$ be $\Phi(x_1^\gamma, \dots, x_n^\gamma)$; except that if $\Phi(x_1^\gamma, \dots, x_n^\gamma)$ is $u \in \underline{S}$, we let $\Phi'(x_1^\gamma, \dots, x_n^\gamma)$ be $u \in \underline{t}$, where t is the subset of S consisting of sets of rank at most $\rho(u)$.

Induction. We consider four cases.

Case 1. $\Phi(x_1^\gamma, \dots, x_n^\gamma)$ is $\sim \Psi(x_1^\gamma, \dots, x_n^\gamma)$. By the induction hypothesis,

$$(x_1^\gamma) \dots (x_n^\gamma)[\Psi(x_1^\gamma, \dots, x_n^\gamma) .\equiv. \Psi'(x_1^\gamma, \dots, x_n^\gamma)]$$

is true in N. Hence, we can let $\Phi'(x_1^\gamma, \dots, x_n^\gamma)$ be $\sim \Psi'(x_1^\gamma, \dots, x_n^\gamma)$.

Case 2. $\Phi(x_1^\gamma, \dots, x_n^\gamma)$ is $\Psi(x_1^\gamma, \dots, x_n^\gamma) \vee \Upsilon(x_1^\gamma, \dots, x_n^\gamma)$. Similar to case 1.

Case 3. $\Phi(x_1^\gamma, \dots, x_n^\gamma)$ is $(\exists y)\Psi(x_1^\gamma, \dots, x_n^\gamma, y)$. By corollary 26.1, there is an ordinal δ such that

$$(x_1^\gamma) \dots (x_n^\gamma)[(\exists y)\Psi(x_1^\gamma, \dots, x_n^\gamma, y) .\equiv. \\ (\exists y^\delta)\Psi(x_1^\gamma, \dots, x_n^\gamma, y^\delta)]$$

is true in N. We continue as in case 4, below.

Case 4. $\Phi(x_1^\gamma, \dots, x_n^\gamma)$ is $(\exists y^\delta)\Psi(x_1^\gamma, \dots, x_n^\gamma, y^\delta)$. Let $\zeta = \max(\gamma, \delta)$. By the induction hypothesis, there is a formula $\Psi'(x_1^\zeta, \dots, x_n^\zeta, y^\zeta)$ containing no unranked variables such that

$$(x_1^\zeta) \dots (x_n^\zeta)(y^\zeta)[\Psi(x_1^\zeta, \dots, x_n^\zeta, y^\zeta) .\equiv. \\ \Psi'(x_1^\zeta, \dots, x_n^\zeta, y^\zeta)]$$

is true in N. It follows that

$$(x_1^\gamma) \dots (x_n^\gamma)[(\exists y^\delta)\Psi(x_1^\gamma, \dots, x_n^\gamma, y^\delta) .\equiv. (\exists y^\delta)\Psi'(x_1^\gamma, \dots, x_n^\gamma, y^\delta)]$$

is also true in N.

Lemma 28. *The axiom of replacement (axiom C4) holds in N.*

Proof. Let u be a term of rank γ, and let $\hat{x}\Phi(x)$ be a class for which $\mathfrak{U}\mathrm{n}(\hat{x}\Phi(x))$ holds – i.e., a class which contains at most one pair $\langle yx\rangle$ for any set x. We must show that there is a set of N which contains precisely those sets y such that for some $x \in u$, $\hat{x}\Phi(x)$ contains $\langle yx\rangle$.

By corollary 26.1, there is an ordinal δ such that

$$(x^\gamma)[(\exists y)\Phi(\langle yx^\gamma\rangle) .\equiv. (\exists x^\delta)\Phi(\langle y^\delta x^\gamma\rangle)]$$

holds in N. Hence, if $\langle yx\rangle \in \hat{x}\Phi(x)$ for some $x \in u$, then y is denoted by a term of rank less than δ.

By lemma 27, there is a formula $\Phi'(y^\zeta, x^\zeta)$, where $\zeta = \max(\gamma, \delta)$, such that

$$(x^\zeta)(y^\zeta)[\Phi(\langle y^\zeta x^\zeta\rangle) .\equiv. \Phi'(y^\zeta, x^\zeta)]$$

holds in N and such that $\Phi'(y^\zeta, x^\zeta)$ contains no unranked variables. It follows that

$$(x^\gamma)(y^\delta)[\Phi(\langle y^\delta x^\gamma\rangle) .\equiv. \Phi'(y^\delta, x^\gamma)]$$

holds in N. Thus, a set y is in the required set if and only if it is denoted by some term v of rank less than δ such that

$$(\exists x^\gamma)[x^\gamma \in u .\&. \Phi'(v, x^\gamma)]$$

is true in N. Thus, the required set is

$$\hat{y}^{\xi}(\exists x^{\gamma})(z^{\delta})[y^{\xi} = z^{\delta} \,.\&.\, y^{\xi} \in u \,.\&.\, \Phi'(y^{\xi}, x^{\gamma})]$$

where $\xi = \max(\gamma, \delta) + 1$.

Before proving the power set axiom, we give two rather easy lemmas, whose content is that the construction of the model N can be described within that model. We make use of the fact that the universe of the model M is a class of the model N.

Lemma 29. *There is a class Q of the model N such that $x \in Q$ if and only if x is a set of conditions compatible with all $p^{(k)}$ in the complete sequence used to define N.*

Proof. Let Q be the class:

$$\hat{x}[x \in S \,.\&.\, (y)(z)(z_1)[\langle \underline{0} y z z_1\rangle \in x \,.\supset.\, A(y, z, z_1)] \;.\&.\; (y)(z)(z_1)[\langle \underline{1} y z z_1\rangle \in x \,.\supset.\, \sim A(y, z, z_1)]] \,.$$

Lemma 30. *There is a class* Den *of the model N such that $\langle yx\rangle \in$ $\in$* Den *if and only if x is a constant term and y is the set denoted by it.* (I.e., Den *is the class of pairs $\langle yx\rangle$ such that for some ranked term u denoting y, x is the set 'u' of lemma 1.)*

Proof. We note first that the relation $\mathfrak{Eps}$ ('u', 'v'), that $u \in v$ is true in N, can be defined by a normal propositional function, namely:

$$(\exists x)[x \in Q \,.\&.\, x \Vdash u \in v] \,.$$

We now define Den' 'u' by transfinite induction on the rank of the term u:

$$\text{Den}`\,`u\text{'} = \{\text{Den}`\,`v\text{'} \mid \rho(v) < \rho(u) \,.\&.\, \mathfrak{Eps}\,(`v\text{'}, `u\text{'})\} \,.$$

Cohen's proof that the power set axiom holds in the extension follows the proof in the last chapter of the Gödel monograph, that $V = L$ implies the generalized continuum hypothesis. We follow, instead, an elegant proof due to R.Solovay which avoids Gödel's argument.

Solovay's proof was to define a function in M taking sets of pairs $\langle pv \rangle$, where p is a set of conditions and v is a constant term of rank $< \rho(u)$, into constant terms and to show that every subset of u was denoted by a term in the range of the function. Since the class of such pairs is no longer a set, we modify Solovay's argument using lemma 25.

Lemma 31. *The power set axiom (axiom C3) holds in the model N.*

Proof. Let u be a constant term of rank at most $\aleph_\alpha$. We will show that the collection of all subsets of the set denoted by u is a set in N.

Let s be a set in M of pairs $\langle pv \rangle$ such that $p \subseteq \Gamma_\alpha$ and v is a constant term of rank less than $\aleph_\alpha$. We define $K`s$ as follows: if there is a set of conditions $\bar{q}$ compatible with all $p^{(k)}$'s – i.e., $\bar{q} \subseteq Q$ – and a constant term w such that

(i) $\bar{q} \Vdash w \subseteq u$.

(ii) For every constant term v of rank less than $\aleph_\alpha$ and every extension q' of $\bar{q}$,

$$q' \Vdash^* v \in w \Rightarrow (q' \cap \Gamma_\alpha) \cup \bar{q} \Vdash^* v \in w\,.$$

(iii) $s = \{\langle pv \rangle \mid p \in Sc \,.\&.\, p \subseteq \Gamma_\alpha \,.\&.\, \rho(v) < \aleph_\alpha \,.\&.\, p \cup \bar{q} \Vdash^* v \in w\}$, then $K`s$ = Den` 'w'; otherwise, $K`s = 0$.

We claim that K is a function mapping the set of sets s of the form described onto the power class of u; as soon as this has been shown, it follows by the axiom of replacement (already proved above) that the power class of u is a set.

We show first that the class K is indeed a function. Suppose that conditions (i) to (iii) above are satisfied both by w_0 and $\bar{q}_0$ and by w_1 and $\bar{q}_1$; we claim that Den' 'w_0' = Den' 'w_1'. Assume that $v \in w_0$ is true in N. Then there is some $p^{(k)}$ such that $p^{(k)} \Vdash v \in w_0$. We may assume that $p^{(k)}$ extends both $\bar{q}_0$ and $\bar{q}_1$. By (ii) above, $(p^{(k)} \cap \Gamma_\alpha) \cup \bar{q}_0 \Vdash^* v \in w_0$. Hence, by the choice of w_0 and w_1, $(p^{(k)} \cap \Gamma_\alpha) \cup \bar{q}_1 \Vdash^* v \in w_1$, so $v \in w_1$ is also true in N. Similarly, $v \in w_1$ implies $v \in w_0$.

We must show that every subset of u is of the form K's for some set s of the kind described. Let w denote a subset of u, and let q be a set of conditions such that $q \Vdash w \subseteq u$. By lemma 25, there is an extension $\bar{q}$ of q and a set Π of sets of conditions $p \subseteq \Gamma_\alpha$ such that if $\bar{q} \subseteq q'$ and $\rho(v) < \aleph_\alpha$, there is some $p \in \Pi$ compatible with q' for which $p \cup \bar{q}$ forces either $v \in w$ or $\sim v \in w$. If $q' \Vdash^* v \in w$ and $(q' \cap \Gamma_\alpha) \cup \bar{q}$ does not weakly force $v \in w$, then there is an extension q'' of $(q' \cap \Gamma_\alpha) \cup \bar{q}$ which forces $\sim v \in w$. Let p be an element of Π compatible with q'' such that either $p \cup \bar{q} \Vdash v \in w$ or $p \cup \bar{q} \Vdash \sim v \in w$. Since $p \cup q''$ is an extension of $p \cup \bar{q}$, we must have $p \cup \bar{q} \Vdash \sim v \in w$. On the other hand, p is compatible with q'; hence, $p \cup \bar{q} \Vdash v \in w$, which contradicts lemma 6. Hence, if

$$s = \{\langle pv\rangle \mid p \in Sc \;.\&.\; p \subseteq \Gamma_\alpha \;.\&.\; \rho(v) < \aleph_\alpha \;.\&. \\ p \cup \bar{q} \Vdash^* v \in w\},$$

then $w = K$'s.

Lemma 32. *The ordinal numbers of N are precisely the sets denoted by the constants $\underline{\alpha}$.*

Proof. By the axiom of foundation, a set is an ordinal if it is complete – i.e., if elements of it are subsets of it – and if distinct elements of it are comparable under the ϵ-relation [1]. It is clear from the definition of forcing that the constants $\underline{\alpha}$ denote ordinals.

[1] See the Gödel monograph, p. 22.

Now, suppose that there is an ordinal ζ of N not denoted by such a constant. If ζ is less than some $\underline{\alpha}$, then $\zeta = \underline{\beta}$ for some $\beta < \underline{\alpha}$. Thus ζ is greater than all $\underline{\alpha}$, so $\underline{\alpha} \in \zeta$ for all ordinals α of M. Let ζ be denoted by an abstraction term of rank γ. Then $\underline{\gamma} \in \zeta$, which is impossible by the definition of forcing and lemmas 13 and 17.

Corollary 32.1. *The ordinals of N are precisely the ordinals of M; in particular, if M is a well-founded model, then so is N.*

Corollary 32.2. *The sets of M are well-ordered by a class in the model N.*

Proof. Since axiom E holds in M, there is a one-to-one function which maps the universe of M onto the ordinals of M; by the lemma, the same function maps the universe of M onto the ordinals of N.

Lemma 33. *Axiom E (the class form of the axiom of choice) holds in N.*

Proof. By corollary 32.2, the sets of M are well-ordered in the extension. Hence, we can define a function C such that $C`x$ is the first set $y$ of $M$ (in some chosen well-ordering) for which $x = \mathrm{Den}`y$. The function C is a one-to-one mapping of the universe into a well-ordered class; hence, a well-ordering of the universe of M induces a well-ordering of the universe of N.

This completes the proof that the model N is a model of Σ_* in which M is embedded as a complete submodel.

§ 5. Cardinals and cardinal powers

Now that the model N has been constructed and shown to be a model of Σ_*, we must show that the cardinal numbers of N have the properties required by theorem 1. We must first know exactly what the cardinals are; following Cohen, we prove that they are precisely the cardinals of the model M.

Cohen proves that cardinals are absolute by showing that if a constant term denotes a function from ordinals to ordinals, then its value at any point must lie in a certain set of the model M of cardinality $\aleph_0$. Thus, if two ordinals are of the same cardinality in N, they must also be of the same cardinality in M.

Our proof is based on Cohen's. The construction already given in lemma 24 consists of applying Cohen's construction to sets of conditions $p \subseteq \Gamma_\alpha$ while simultaneously extending sets of conditions $q \subseteq \Delta_\alpha$. The result is that the value of a function on $\aleph_\alpha$ is in a set of M of cardinality $\aleph_\alpha$, where the latter set is determined by the extension $q \subseteq \Delta_\alpha$.

Since no extra effort is involved, we prove a stronger result, that the concept of cofinality is absolute in the extension. The function Cf is defined as follows: $Cf`\beta$ is the smallest cardinal $\aleph_\delta$ such that β is the union of $\aleph_\delta$ smaller ordinals.

Lemma 34. *If $\aleph_\beta$ is of cofinality greater than $\aleph_\alpha$ in the model M, then the same holds in the extension N* [1]*. Thus, $Cf_N = Cf_M$.*

Proof. Suppose not. Then there is a constant term u which denotes a function mapping $\aleph_\alpha$ into $\aleph_\beta$ such that $\mathfrak{Lim}_{\gamma<\aleph_\alpha} u`\gamma = \aleph_\beta$. By lemma 25, if q is any set of conditions, there is an extension $\bar{q}$ of q and a set Π, $\bar{\bar{\Pi}} = \aleph_\alpha$, such that if $\bar{q} \subseteq q'$ and $\gamma < \aleph_\alpha$, then there is some $p \in \Pi$ compatible with $\bar{q}$ which forces either $(\exists x^{\aleph_\beta})[\langle x^{\aleph_\beta}, \gamma\rangle \in u]$ or $\sim (\exists x^{\aleph_\beta})[\langle x^{\aleph_\beta}, \gamma\rangle \in u]$. Since $p^{(0)}, p^{(1)}, \ldots,$ is complete, some $p^{(k)}$ is obtained in this manner. For each $p \in \Pi$ and $\gamma < \aleph_\alpha$, we let $\delta(p, \gamma)$ be the least ordinal $\eta < \aleph_\beta$ such that $p \cup p^{(k)} \Vdash \langle w, \gamma\rangle \in u$ for some term w of rank η, if such an ordinal exists; otherwise, we let $\delta(p, \gamma) = 0$.

[1] Until we have shown that cardinals are absolute, "$\aleph_\alpha$" will denote that ordinal which is the α'th cardinal of the model M.

The ordinals $\delta(p, \gamma)$ form a subset of $\aleph_\beta$ of cardinality $\aleph_\alpha$ in the model M. Since $Cf_M`\aleph_\beta > \aleph_\alpha$, there is an ordinal $\delta < \aleph_\beta$ greater than all $\delta(p, \gamma)$'s. Clearly, for each $\gamma < \aleph_\alpha$, $u`\gamma < \delta$ in N.

Corollary 34.1. *Cardinals are absolute in the extension from M to N.*

Proof. Clearly, cardinals of N are also cardinals of M. If $\aleph_\alpha$ is not a cardinal of N, then there is a one-to-one function in N mapping $\aleph_\alpha$ onto $\aleph_\beta$ for some $\beta < \alpha$. It follows that $Cf_N` \aleph_{\beta+1} \leq \aleph_\beta$, which contradicts the preceding lemma.

We will now show that $2^{\aleph_\alpha} = \aleph_{G`\alpha}$ in the model $N$. It will turn out that if $\aleph_\alpha$ is a singular cardinal, then $2^{\aleph_\alpha}$ will be the smallest cardinal allowed by König's theorem. Thus, in case $\aleph_\alpha$ is not a regular cardinal, we assume that $\aleph_{G`\alpha}$ is the smallest cardinal not cofinal with $\aleph_\alpha$ which is $\geq \aleph_{G`\beta}$ for all $\beta < \alpha$. The proof below will show that $2^{\aleph_\alpha} = \aleph_{G`\alpha}$ in N for all cardinals $\aleph_\alpha$.

The calculation of $2^{\aleph_0}$ given in the Cohen paper makes use of the same argument of Gödel that was used to prove the power set axiom. We give, instead, a proof based on Solovay's proof of the power set axiom; the details of this proof are much simpler, since one does not need to construct inner models with the model N.

Lemma 35. *Let u denote a subset of $\aleph_\alpha$ in N. Then there is some $p^{(k)}$ in the complete sequence used to define N and some set Π of sets of conditions such that:*

(i) $p \in \Pi \Longleftrightarrow p \subseteq \Gamma_\alpha$.

(ii) $\overline{\overline{\Pi}} \leq \aleph_\alpha$.

(iii) $p^{(k)} \Vdash u \subseteq \aleph_\alpha$.

(iv) *If q' is an extension of $p^{(k)}$ and $\gamma < \aleph_\alpha$, there is some $p \in \Pi$ compatible with q' such that either $p \cup p^{(k)} \Vdash \underline{\gamma} \in u$ or $p \cup p' \Vdash \sim \underline{\gamma} \in u$.*

Proof. Let q be any set of conditions such that $q \Vdash u \subseteq \aleph_\alpha$. By lemma 25, there is an extension $\bar{q}$ of q and a set Π of sets of conditions such that (i) to (iv) above are satisfied with $p^{(k)}$ replaced by $\bar{q}$. Since any set of conditions can be extended in this way, some $p^{(k)}$ can be so obtained.

Lemma 36. $2^{\aleph_\alpha} = \aleph_{G`\alpha}$ *in the model N.*

Proof. We first show that $2^{\aleph_\alpha} \geq \aleph_{G`\alpha}$. For regular cardinals $\aleph_\alpha$, this is clear, since the sets $a_\eta$, $\eta < \aleph_{G`\alpha}$ are distinct subsets of $\aleph_\alpha$. (Any set of conditions can be extended to force $\gamma' \in a_\xi$ and $\sim \gamma' \in a_\eta$, provided $\xi \neq \eta$.) For singular cardinals $\aleph_\alpha$, $\aleph_{G`\alpha}$ is the first cardinal greater than or equal to $\aleph_{G`\beta}$ for all $\beta < \alpha$ which is not confinal with $\aleph_\alpha$. Clearly, $2^{\aleph_\alpha} \geq 2^{\aleph_\beta} \geq \aleph_{G`\beta}$ for all $\beta < \alpha$. By König's theorem, $Cf`2^{\aleph_\alpha} > \aleph_\alpha$; hence, $2^{\aleph_\alpha} \geq \aleph_{G`\alpha}$.

We must show that $2^{\aleph_\alpha} \leq \aleph_{G`\alpha}$. Let Σ be the set (in M) of all pairs $\langle \Pi, s\rangle$ such that

(i) Π is a set of sets of conditions.
(ii) $p \in \Pi \Rightarrow p \subseteq \Gamma_\alpha$.
(iii) $\bar{\bar{\Pi}} \leq \aleph_\alpha$.
(iv) $s \subseteq \Pi \times \aleph_\alpha$ and s is a set of M.

We first show[1] that $\bar{\bar{\Sigma}} = \aleph_{G`\alpha}$. It is clear that $\bar{\bar{\Gamma}}_\alpha \leq \aleph_\alpha \cdot \aleph_\alpha \cdot \aleph_{G`\alpha} = \aleph_{G`\alpha}$; hence, there are only $\aleph_{G`\alpha}^{\aleph_\alpha}$ sets Π. Since $Cf`\aleph_{G`\alpha} > \aleph_\alpha$, $\aleph_{G`\alpha}^{\aleph_\alpha} = \aleph_{G`\alpha}$. For each set Π, there are at most $2^{\aleph_\alpha \cdot \aleph_\alpha} = \aleph_{\alpha+1}$ subsets of $\Pi \times \aleph_\alpha$ in the model M. Hence, $\bar{\bar{\Sigma}} \leq \aleph_{G`\alpha} \cdot \aleph_\alpha = \aleph_{G`\alpha}$; it is easy to see that, in fact, equality holds.

Thus, to complete the proof of the lemma, we must exhibit a function in N which maps the set Σ onto the power set of $\aleph_\alpha$. If $\langle \Pi, s\rangle \in \Sigma$, we set $K`\langle \Pi, s\rangle = y$ if for some constant term w and some set of conditions $\bar{q}$,

(i) $\bar{q} \in Q$[2].
(ii) If $\bar{q} \subseteq q'$ and $\gamma < \aleph_\alpha$, there is some $p \in \Pi$ compatible with q' which forces either $\gamma \in w$ or $\sim \gamma \in w$.

[1] All calculations take place in M.
[2] The class Q was defined in lemma 29.

(iii) $s = \{\langle p\gamma\rangle \in \Pi \times \aleph_\alpha | p \cup \bar{q} \Vdash \underline{\gamma} \in w\}$.
(iv) y = Den' 'w'.
If no such w and $\bar{q}$ exist, we set $K`\langle \Pi, s\rangle = 0$.

We must show that the function K is well defined and that its range includes all subsets of $\aleph_\alpha$. Suppose that conditions (i) to (iv) are satisfied both by w_0 and $\bar{q}_0$ and by w_1 and $\bar{q}_1$; we claim that Den' 'w_0' and Den' 'w_1' are equal. Let $p^{(j)}$ be a set of conditions in the complete sequence used to define N which extends both $\bar{q}_0$ and $\bar{q}_1$ and which forces $\underline{\gamma} \in w_0$. If q' is an extension of $p^{(j)}$ which forces $\sim \underline{\gamma} \in w_1$, we choose $p \in \Pi$ compatible with q' such that $p \cup \bar{q}_0$ forces either $\underline{\gamma} \in w_0$ or $\sim \underline{\gamma} \in w_0$. Since $p \cup \bar{q} \subseteq p \cup p^{(j)}$, we must have $p \cup \bar{q}_0 \Vdash \underline{\gamma} \in w_0$. But then, $\langle p\gamma\rangle \in s$, so $p \cup \bar{q}_1 \Vdash \underline{\gamma} \in w_1$. Hence, since $p \cup \bar{q}_1 \subseteq p \cup q'$, $p \cup q'$ forces both $\underline{\gamma} \in w_1$ and $\sim \underline{\gamma} \in w_1$, contrary to lemma 6. Therefore, $p^{(j)} \Vdash^* \underline{\gamma} \in w_1$. Hence, $\gamma \in$ Den' 'w_0' $\Rightarrow \gamma \in$ Den' 'w_1'; similarly, $\gamma \in$ Den 'w_1' $\Rightarrow$ $\gamma \in$ Den' 'w_0'. Therefore, Den' 'w_0' = Den' 'w_1'; and K is a well-defined function.

Lemma 35 states that for any term u denoting a subset of $\aleph_\alpha$, there is a set Π and a set of conditions $p^{(k)}$ such that $\overline{\overline{\Pi}} < \aleph_\alpha$ and $p \in \Pi \Rightarrow p \subseteq \Gamma_\alpha$ and $p^{(k)}$ satisfies (i) to (iii) above (with $\bar{q}$ replaced by $p^{(k)}$). We set $s = \{\langle p\gamma\rangle \in \Pi \times \aleph_\alpha | p \cup p^{(k)} \Vdash \underline{\gamma} \in u\}$; it is clear that Den' 'u' = $K\langle \Pi, s\rangle$.

The proof of theorem 1 is now complete.

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