Annals of Mathematical Logic 8 (1975) 1–184
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AN INDEPENDENCE RESULT CONCERNING THE AXIOM OF CHOICE *

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Received 7 January 1974

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* This paper is essentially a transcript of my Ph.D. thesis, submitted at the Hebrew University of Jerusalem, in June, 1973, with some simplification of the major sections. An earlier manuscript (Spring, 1972) contained my thesis' results based on the assumption of an inaccessible cardinal. Since the role of an inaccessible was later found to be dispensable it was eliminated in the final work.

I would like to express my deepest thanks to Professor Azriel Lévy for making this thesis possible and for his keen interest in the various stages of its development. I would also like to thank both Professors Azriel Lévy and Robert Solovay who read the final drafts of my thesis, uncovered numerous mistakes and offered many valuable suggestions.

The paper was partially supported by NSF Grant GP-27964 during the final stage of preparation.

0. Preface

Cardinals of the form $2\mathfrak{m}$ are said to be *even*, and those of the form $\mathfrak{m}^2$ are said to be *square*. In the terminology of Ellentuck [4] cardinals $\mathfrak{m}$ which are equal to their own squares, i.e. $\mathfrak{m}^2 = \mathfrak{m}$ are called *idempotent* and those $\mathfrak{m}$ for which $2\mathfrak{m} = \mathfrak{m}$, are said to be *idemmultiple*. It is well known that the Axiom of Choice implies the idemmultiplicity and idempotency of all infinite cardinals. Moreover, Tarski [21] has shown that the Axiom of Choice and the seemingly much restricted assumption that all infinite cardinals are idempotent are in fact equivalent. It was natural to ask if the weaker assumption that all infinite cardinals are idemmultiple, which we call the *idemmultiple hypothesis*, also implies AC (AC is short for the Axiom of Choice). This question was raised by Tarski [21] in 1924. The problem has since been raised on a number of occasions [10, 12] and has up till now remained unsettled.

Significant results concerning even cardinals in the absence of AC were obtained by Bernstein [1], who showed that

$$(\forall \mathfrak{m}, \mathfrak{n})\,(2\mathfrak{m} = 2\mathfrak{n} \to \mathfrak{m} = \mathfrak{n})\,;$$

this was improved upon by Sierpiński [16, 18], who showed that

$$(\forall \mathfrak{m}, \mathfrak{n})\,(2\mathfrak{m} \leqslant 2\mathfrak{n} \to \mathfrak{m} \leqslant \mathfrak{n})\,.$$

In [22], Tarski proved the more general cancellation laws

$$(\forall k < \aleph_0)\,(\forall \mathfrak{m}, \mathfrak{n})\,(k\mathfrak{m} = k\mathfrak{n} \to \mathfrak{m} = \mathfrak{n})\,.$$

It has been shown by Sierpiński [19] that there are interesting theorems in mathematics proved with the use of AC that can be proven for sets with idemmultiple cardinality without choice.

Recently, work has been done by Halper and Howard [5] concerning various characterizations of idemmultiple cardinals, where it was also shown that certain strengthenings of the idemmultiple hypothesis imply the Axiom of Choice. They have also shown, independently of the author's work [6], that in set theory allowing urelements, the idemmultiple hypothesis does not imply AC.

In this work it is shown that the idemmultiple hypothesis together with the ordinary axioms of set theory do not imply AC, provided that set theory is consistent. This is done by constructing a standard model N of ZF + $\neg$AC in which every infinite cardinal is idemmultiple. The

Axiom of Choice is strongly violated in this model; it is shown that not even for a countable set of disjoint countable sets of reals does a choice set generally exist. On the other hand, various weak choice axioms hold in the model. Specifically, the axiom of order (denoted O) and the $\aleph_0$-multiple choice axiom, i.e., for every set x of disjoint sets there is a multiple choice function F such that for all $y \in x$, $F(y) \subsetneq y$ and $|F(y)| \leqslant \aleph_0$ (denoted $\aleph_0$-multiple choice) are shown to hold in N. Thus

$$\begin{aligned}\mathrm{Con(ZF)} \Rightarrow \mathrm{Con}(\mathrm{ZF} + \neg \mathrm{AC} &+ (\forall \mathfrak{m} \geqslant \aleph_0)(2\mathfrak{m} = \mathfrak{m}) \\ &+ \mathrm{O} + \aleph_0\text{-multiple choice}) .\end{aligned}$$

This also solves a problem raised by Azriel Lévy whether

$$\mathrm{Con(ZF)} \Rightarrow \mathrm{Con}(\mathrm{ZF} + \neg \mathrm{AC} + \aleph_0\text{-multiple choice})$$

(see also [9]).

0.10. Definition. Let $\mathfrak{m}, \mathfrak{n}, \mathfrak{l}$ be infinite cardinals; then

(i) $\mathfrak{n}$ is a 1-*successor* of $\mathfrak{m}$ iff $(\forall \mathfrak{l})\,(\mathfrak{m} < \mathfrak{l} \leqslant \mathfrak{n} \to \mathfrak{l} = \mathfrak{n})$,

(ii) $\mathfrak{n}$ is a 2-*successor* of $\mathfrak{m}$ iff $(\forall \mathfrak{l})\,(\mathfrak{m} < \mathfrak{l} \to \mathfrak{l} \geqslant \mathfrak{n})$,

(iii) $\mathfrak{n}$ is a 3-*successor* of $\mathfrak{m}$ iff $(\forall \mathfrak{l})\,(\mathfrak{l} < \mathfrak{n} \to \mathfrak{n} \leqslant \mathfrak{m})$.

It has been shown by Tarski [23] that every cardinal has a 1-successor. Lévy has shown (see Truss [24]) that if every well-ordered set has a 2-successor, then the Axiom of Choice holds. Tarski and Truss have independently proven [24] that if every cardinal has a 3-successor, then for all infinite cardinals $\mathfrak{m}$, $2\mathfrak{m} = \mathfrak{m}$. The question arises if the reverse implication is true. The answer to this is negative, for it can be shown that in N there are cardinals that do not have a 3-successor. Thus the universal existence of 3-successors is not equivalent to the idemmultiple hypothesis in ZF. The proof of this will be given in another paper.

The model N is constructed by employing an iterated forcing technique. Although iterated forcing is not new [20], the particular method developed here, where the iteration is telescoped into a single Cohen extension, is of interest in itself. The particular kinds of innovations involved in the notion of a condition are, as far as we know, new in the literature, and we hope may be instructive to the reader. It is also hoped that methods similar to the one applied here, dealing with cardinalities in models, will be applicable in obtaining new results in the area of cardinal arithmetic without the Axiom of Choice.

1. Introduction

An analysis of the problem and a preliminary outline of the construction will now be given.

Consider a set A and a univalent function f mapping A onto $A \times 2$. Such an f imposes a partition on A and induces upon the components an almost tree-like structure, conceived in the following manner. Let $\hat{f}, \check{f}$ be defined on A as follows: if $f(a) = \langle b, \delta \rangle$, $\delta \in 2$, then $\hat{f}(a) = \delta$ and $\check{f}(a) = b$. Finite iterations of these functions are defined by

$$\check{f}^0(a) = a\ , \qquad \check{f}^n(a) = \check{f}(\check{f}^{n-1}(a))\ ;$$

$$\hat{f}^1(a) = \hat{f}(a)\ , \qquad \hat{f}^{m+1}(a) = \hat{f}(\check{f}^m(a))\ , \quad m \geqslant 0\ .$$

Two elements $a,b \in A$ are *f-equivalent* iff for some c and $m,n \geqslant 0$, $\check{f}^m(a) = c = \check{f}^n(b)$. This is clearly an equivalence relation, denoted $\equiv_f$, and the set of equivalence classes in A of $\equiv_f$ is denoted by R_f. $r \in R_f$ is said to contain a *circuit* if, for some $a \in r$ and $m > 0$, $\check{f}^m(a) = a$. $m > 0$ is said to be the *length* of the circuit if $m > 0$ is minimal, for which $\check{f}^m(a) = a$; and the elements

$$\check{f}^1(a), \check{f}^2(a), ..., \check{f}^m(a) = a = \check{f}^0(a)$$

are said to be the *circuit elements.* An $r \in R_f$ can contain at most one circuit because if $a, b \in r$ belong to different circuits, then for no $n \geqslant 0$ does $\check{f}^n(a) = b$ or $\check{f}^n(b) = a$, hence $a \not\equiv_f b$; a contradiction. A relation $\preccurlyeq_r$ is defined on $r \in R_f$ as follows:

$$a,b \in r\ , \quad a \preccurlyeq_r b \quad \text{iff for some } n \geqslant 0,\ \check{f}^n(b) = a\ .$$

If $r \in R_f$ does not contain a circuit, then $(r, \prec_r)$ is a tree-like partially ordered set without a root; and if r contains a circuit, then $(r, \preccurlyeq_r)$ is a tree-like structure with a circuit at its base. (See Fig. 2.) If $(r, \preccurlyeq_r)$ does not contain a circuit, it is called a non-rigid tree (see Fig. 1). A circuit is called *symmetric* if there is a non-trivial permutation π of the circuit elements preserving the relation f restricted to the circuit elements. In this case π is is extendible to an automorphism of the structure $(r, \preccurlyeq_r)$ preserving f. In Fig. 2, there is an example of a symmetric circuit; on the other hand, if $f(a) = (b, 1)$, $f(b) = (a, 0)$ in the figure,

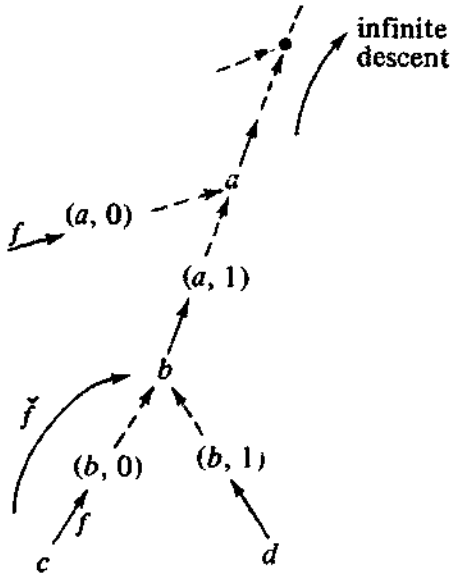

Fig. 1. A component without a circuit.

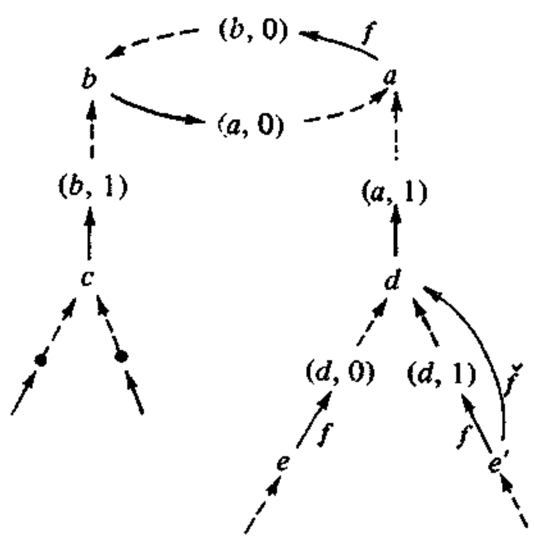

Fig. 2. A component with a symmetric circuit of length 2.

then the elements of the circuit (hence all elements of r) are definable in terms of f.

Further insight is gained by considering the following notion: to each $a \in r$, a real $[a]_f \in 2^\omega$ is associated by

$$[a]_f\,(n) = \hat{f}^{n+1}(a)\,, \quad n \in \omega\,.$$

(Henceforth reals are always thought of as sequences of 0's and 1's.)

It is clearly seen that if r contains a circuit, then for every $a \in r$, $[a]_f$ is a periodic real. If r does not contain a circuit, then, for all $a \in r$, $[a]_f$ is either periodic or non-periodic, depending upon the nature of f_r. (See Fig. 3.)

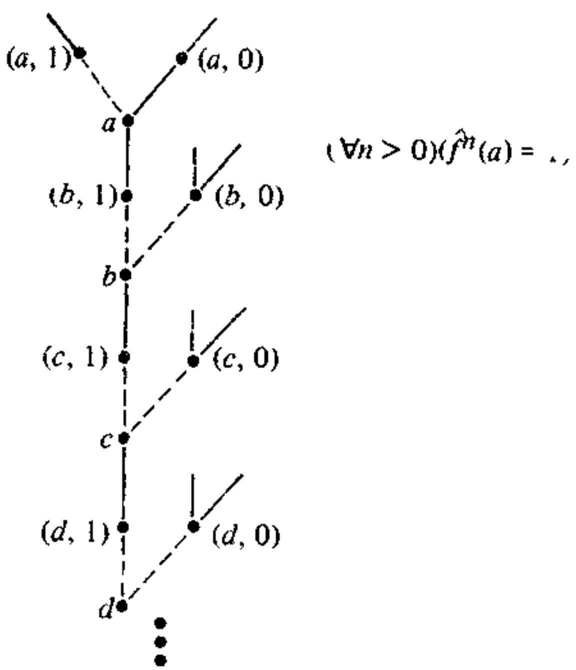

Fig. 3. A tree with associated periodic reals.

Clearly the components $r \in R_f$ are infinite and countable. If A is non-well-orderable, then R_f must be infinite. In a universe where all infinite sets are idemmultiple, R_f also decomposes into countable equivalence classes via a univalent function g mapping $R_f \times 2$ onto R_f. Hence R_f contains countable subsets.

In view of this analysis, it was naturally conceived that a countable set of primitive structures $(r, \preccurlyeq_r, f_r)$ should be adjoined to a standard model M of ZF + V = L for a proper choice of r and f_r, in the hope of obtaining a desired model.

Now if (r, f) is adjoined to M, where f is a generic function mapping r onto $r \times 2$ in such a way that $\langle r, \preccurlyeq_r\rangle$ is a non-rigid tree, then since the reals $[a]_f$, $a \in r$, are generic, we would have

$$[a]_f = [b]_f \quad \text{iff} \quad a = b \ ;$$

therefore the structure (r, f) is entirely reflected in $\{[a]_f \mid a \in r\}$, which will henceforth be referred to as a *tail.*

As was brought out in the above analysis, tails and related notions are basic in this work and the following notation and conventions concerning them will be adopted.

1.10. Definition. Let $a \in 2^\omega$, $l < \omega$, $t \in 2^l$; then by $\text{slash}_l(a)$ we denote the real $b \in 2^\omega$ for which $b(n) = a(l+n)$ for all $n < \omega$. By $\text{append}_t(a)$ we denote the real $b \in 2^\omega$ such that

$$b(n) = \begin{cases} t(n) & \text{for all } n < l \\ a(n-l) & \text{for } n \geqslant l \ . \end{cases}$$

1.20. Definition. Let $a \in 2^\omega$; the *tail of a*, denoted $K(a)$, is the set

$$\{x \mid (\exists l, m < \omega)\, (\text{slash}_l(x) = \text{slash}_m(a))\} \ .$$

It is immediate that:

1.30. Lemma. *For any* $a, b \in 2^\omega$, $K(a) = K(b)$ *or* $K(a) \cap K(b) = \emptyset$.

For non-periodic reals a there is a natural tree-like partial ordering of $K(a)$ defined as follows:

1.40. Definition. For $b, c \in K(a)$, a a non-periodic real,

$$b \preccurlyeq c \quad \text{iff} \quad (\exists l < \omega)\,(\mathrm{slash}_l(c) = b)\,.$$

Reals generic over a model M are always non-periodic. Henceforth we will be concerned only with non-periodic reals and the term "real" will always mean a non-periodic element of 2^ω unless otherwise stated.

It is immediate from Definition 1.20 that:

1.50. Lemma. *For any $a, b \in K$, where K is a tail, there are $l, m < \omega$, $t \in 2^m$ such that*

$$b = \mathrm{append}_t(\mathrm{slash}_l(a))\,.$$

Moreover, if l is minimal, then l, t are unique.

Note that the operation of interchanging a finite number of zero's and one's in a real a is obtained by an operation $\mathrm{append}_t(\mathrm{slash}_l(a))$ for appropriate t, l.

The following basic fact concerning tails is an obvious result of the preceding definitions.

1.60. Lemma. *If K is a tail, then there exists a univalent function f mapping $K \times 2$ onto K, definable from K.*

Proof. For every $x \in K$, $\delta < 2$, set $t_\delta = \{\langle 0, \delta\rangle\} \in 2^1$, and define

$$f(x, \delta) = \mathrm{append}_{t_\delta}(x)\,.$$

Clearly f is onto, since

$$f(\mathrm{slash}_1(x), x(0)) = x\ ;$$

that f is one–one is immediate.

In proving that all infinite cardinals in N are idemmultiple, essential use was made of the following more general fact.

A set s included in a tail K is said to be *dense in* K if for all $x \in K$ there is a $y \in s$ such that $x \preccurlyeq y$. In Lemma 9.44 we prove:

Lemma. *If s is dense in the tail K, then there exists a univalent function f_s mapping $s \times 2$ onto s, definable from s (not from elements of s).*

1.70. Notation. (i) We shall employ vector notation: a sequence $b_1, ..., b_n$ will be denoted by $\bar{b}$; and if the length of the sequence is relevant, we write $\bar{b}^n$.

(ii) If $b_i \in 2^\omega$, $1 \leq i \leq n$, the *tail of* $\bar{b}$ is defined as

$$K(b_1) \times K(b_2) \times ... \times K(b_n) ,$$

and is denoted by $K(\bar{b})$.

(iii) If f is a function or a general binary relation, we denote the domain of f by $\mathrm{dom}(f)$ and the range by $\mathrm{rng}(f)$.

1.80. Definition. A set x is said to be *idemmultiple* if x is equinumerous to $x \times 2$.

We return to our original motivation, and consider a Cohen extension $N_0 = M[J] \supseteq M$ of a standard countable model M of ZF+V = L, where J is an enumeration of a countable set of tails,

$$J = \{\langle i, K(a_i)\rangle \mid i \in \omega\} ,$$

a_i being generic over M. It is easily forced that

$$K(a_i) \cap K(a_j) = \emptyset, \quad i \neq j$$

(Lemma 4.71), and that there is no choice function in N_0 for the set $\{J(i) \mid i \in \omega\}$ (Lemma 5.50).

Some reflection shows that not only is $D = \bigcup\{J(i) \mid i \in \omega\}$ idemmultiple in N_0, but that also all infinite subsets of D are idemmultiple (this can be seen by first observing that all subsets of D which are not definable from reals a_i, i.e. only from J and ordinals, are open in the relative product topology of $\{0,1\}$, and that basic open sets are idem multiple). Although this fact is in some sense encouraging, the following partitions of D in N_0 are observed to be of non-idemmultiple cardinality in N_0.

1.90. Examples. (a) For a tail K and an element $x \in K$, define the *layer* of x as

$$\{y \mid y \in K, (\exists l \in \omega)\,(\mathrm{slash}_l(x) = \mathrm{slash}_l(y))\}\ .$$

The layer of x is denoted by $\mathrm{lr}(x)$.

The set of layers $B(K)$ of a tail K is a partition, and there is a natural ordering of type $\omega^* + \omega$ of $B(K)$ defined by

$$\mathrm{lr}(x) \leqslant \mathrm{lr}(y) \quad \text{iff} \quad (\exists l_1, l_2 \in \omega)\,(l_1 \leqslant l_2,\ \mathrm{slash}_{l_1}(x) = \mathrm{slash}_{l_2}(y))$$

$x, y \in K$.

It can be shown that $\{B(J(i) \mid i \in \omega\}$ is not idemmultiple in N_0 (Lemma 5.51).

(b) If $r_1, r_2 \in B(K)$, $r_1 = \mathrm{lr}(x)$, $r_2 = \mathrm{lr}(y)$ and $\mathrm{slash}_{l_1}(x) = \mathrm{slash}_{l_2}(y)$, then $|l_1 - l_2|$ is called the *distance* between r_1, r_2. It is easily seen that the definition is independent of the choice of the elements $x \in r_1$, $y \in r_2$. For $1 < m \in \omega$ and $x \in K$, define $B_m(x)$ to be the union of layers $r \in B(K)$ whose distance from $\mathrm{lr}(x)$ is a multiple of m, i.e.,

$$B_m(x) = \bigcup \{r \mid (\exists n \in \omega)\,(\mathrm{distance}\,(r, \mathrm{lr}(x)) = n \cdot m)\}\ .$$

Set $B_m(K) = \{B_m(x) \mid x \in k\}$. It is clear that $B_m(K)$ is a partition and that $|B_m(K)| = m$. It can be shown that $B_m(J(i))$ has no ordering in N_0 definable from J and ordinals. Moreover $\bigcup_{i \in \omega} \{B_m(J(i))\}$ is not idemmultiple in N_0 for every $m > 1$.

(c) The following partitions are worth keeping in mind. Assume $\lambda \subseteq \omega$ and $x \in K$; set

$$h_\lambda(x) = \{y \mid y \in K,\ \mathrm{lr}(x) = \mathrm{lr}(y),$$
$$(\forall i \in \omega)\,(y(i) \neq x(i) \to i \in \lambda)\},$$

and define $H_\lambda(K) = \{h_\lambda(x) \mid x \in K\}$. It is clear that $H_\lambda(K)$ is a partition of K. It can be shown that $\bigcup \{H_\lambda(J(i)) \mid i \in \omega\}$ is not idemmultiple in N_0 for every non-finite λ.

Although these examples are of a very particular kind, it turns out that partitions of D are fundamental in understanding the nature of all cardinals in N_0 and in subsequent models. From one of our main lemmas

(9.30) it will follow that all sets in N_0 can be decomposed in N_0 into a well-ordered sequence of sets of disjoint subsets each of which is definably equinumerous to a tail partition of some $K(\bar{a}_i)$. The analysis of cardinalities in the following models by the reduction to the study of such well-ordered sequences is an important element in this work.

The next step of the construction is to make cardinals of N_0 idemmultiple in an extension $N_1 \supsetneq N_0$ of N_0 while retaining $\neg$AC. This is done by adjoining, for every reasonably symmetric partition H of a tail $K(\bar{a}_i)$, a univalent generic function χ_H mapping H into a set of tails, and onto if H is infinite. Since the partitions H can be of a very varied nature, the choice of the functions χ_H in order to avoid AC is fairly intricate. A general discussion of this problem can be found in Definition 2.50(e). By observing partitions of the type H_λ (Example 1.90(c)) it is easily seen that, since new reals have been adjoined, there are new partitions of D in N_1 which can be shown to be non-idemmultiple in N_1.

The general plan is now clear; we come to consider a tower of models

$$M \subseteq N_0 \subsetneq \dots \subsetneq N_\alpha \subsetneq N_{\alpha+1} \subsetneq \dots,$$

where $N_{\alpha+1}$ is designed to make cardinals of N_α idemmultiple in $N_{\alpha+1}$ while retaining $\neg$AC. This is accomplished by adjoining functions χ_H mapping tail partitions H into a set of tails in a specified manner to avoid AC. That it suffices to consider partitions of the original tails only at each stage follows from the fact that partitions of the new tails induce, via the corresponding functions χ_H, equivalent partitions of the original tails. The question now is, what happens at limit stages? The fact that new reals evolve at countable limit stages, via certain sequences of functions χ_H, which induce new partitions of the type $H_\lambda(K)$, forces us to consider iterations of length $\kappa \in M$ such that κ is not cofinal with ω in N_κ. One of our key lemmas will tell us that the cofinality of cardinals is retained in N_κ and that any partition in $N_{\aleph_1^{(M)}}$ of a tail K is already obtained at some countable $^{(M)}$ stage. From this we will be able to derive our main result, that all infinite cardinals in $N_{\aleph_1^{(M)}}$ are idemmultiple.

The main themes of this work culminate in Sections 8, 9 and 10. The reader interested in obtaining an early apprehension of our method of proof is advised to proceed to these sections rapidly, omitting as many proofs in the first reading as feasible. Especially straightforward and uninteresting proofs (for instance those which are involved in the ranking of the ramified languages) appear starred (i.e. **Proof***). Sections 6 and 7 are quite standard. In Section 5 the symmetry properties of the models are studied. Sections 3 and 4 are occupied with the various aspects of our iteration technique. Section 2 deals with the basic notions involved in the model construction. Here the notion of a condition is most pertinent, embodying the main ideas behind the construction of the model and its properties.

2. Construction of the model N

2.10. Definition. Let ZF be the set theory formulated in the applied first order predicate calculus (without equality) with $\in$ as its only extralogical primitive symbol, and consisting of the axioms of extensionality, union, power set, infinity, replacement and foundation. Equality is taken as a defined relation and $u = v$ always stands for the formula $(\forall v)\,(w \in u \leftrightarrow w \in v)$. The language of ZF will be referred to as *the language of set theory.*

As is usually done in obtaining independence results, we will adopt additional axioms which provide us with a standard countable model of ZF + V = L. As remarked by Cohen [2], this could be avoided in a variety of ways yielding pure syntactic arguments, but would involve an unnatural and inconvenient exposition. Thus we will operate in the context of an enriched set theory ZFM similar to Lévy [11].

2.11. Definition. The language of ZFM is like the language of set theory but has an additional individual constant M. The axioms of ZFM are those of ZF, including all instances of the axiom scheme of replacement which contain the symbol M, together with the following additional axioms:

(a) *M is denumerable.*

(b) *M is transitive*, i.e. $(\forall u, v)\,(u \in v \wedge v \in M \rightarrow u \in M)$.

(c) $\varphi^{(M)}$ *for every theorem φ of* ZF,

where $\varphi^{(M)}$ is the formula obtained from φ by relativizing the quantifiers of φ to M in the usual way.

(d) *M satisfies the axiom of constructibility.*

Although the language of set theory has $\in$ as its only extralogical symbol, in practice we use an extended language which is obtained from the primitive language by introduction of new symbols for defined relations, operations, special variables, etc. Commonly used symbols having their usual meaning here will be employed without being explicitly defined. The resulting language will be referred to as *the extended language of set theory.*

(2.12) $\alpha, \beta, \gamma, \lambda, \eta, \mu, \nu, \xi, \zeta, \kappa$ will denote ordinals in M.

In the sequel we shall deal with certain designated objects and their properties, defined in the extended language of set theory, which are absolute with respect to M (and with respect to transitive models in general). In particular, the definitions of syntax and forcing will be seen to be definable in M and absolute. We do not intend giving an exhaustive list of all absoluteness properties implicit in this work, since the relevant facts can easily be established by standard procedures with which the reader is assumed to be acquainted, (see Lévy [11] and Halpern–Lévy [7] for details). However, the reader should convince himself of the absoluteness facts pointed out in this work.

As previously explained, the model N is conceived as being obtained from M by adding generic objects dealing with elementary tail partitions in $\aleph_1^{(M)}$ consecutive stages. Thus the ramified local language $\mathcal{L}$ which will be associated with the model N is also conceived in corresponding stages. Besides the ordinary ingredients of a ramified language. there will be sets of constants $\mathcal{X}_\alpha$, $\alpha \leqslant \aleph_1^{(M)}$, to denote the generic functions χ_ρ added up to and including the α^{th} stage, and sets of constants $\mathcal{I}_\alpha$ for denoting elements I_β, $\beta \leqslant \alpha$. Each I_β will collect all the generic objects added up to and including stage β, in such a manner that enumerations of these objects can be derived. The restriction of $\mathcal{L}$ to the constants in $\mathcal{X}_\alpha \cup \mathcal{I}_\alpha$ yields a language $\mathcal{L}_\alpha$ which will eventually be associated with an α^{th} stage model $M[I_\alpha]$ (i.e. the universe obtained from M by adjoining I_α and its transitive closure). $M[I_\alpha]$ will also be denoted by N_α. There will also be sets P_α of conditions concerning the α^{th} stage generic objects; and weak forcing relations $\Vdash_\alpha$ between conditions $p \in P_\alpha$ and sentences φ of the language $\mathcal{L}_\alpha$ which, as usual, will resemble the definition of truth. Although the language $\mathcal{L}_\alpha$ will be the part of $\mathcal{L}$ restricted to constants of $\mathcal{X}_\alpha \cup \mathcal{I}_\alpha$, $\mathcal{X}_\alpha$ and thus $\mathcal{L}_\alpha$, P_α and $\Vdash_\alpha$ will actually be defined in terms of $\mathcal{L}_\beta$, P_β, and $\Vdash_\beta$, $\beta < \alpha$, yielding a complicated recursive definition of everything concerned. The process is cumulative in the sense that

$$\mathcal{X}_\beta \subseteq \mathcal{X}_\alpha, \quad \mathcal{I}_\beta \subseteq \mathcal{I}_\alpha, \quad \mathcal{L}_\beta \subseteq \mathcal{L}_\alpha, \quad P_\beta \subseteq P_\alpha,$$

$$I_\beta \subseteq I_\alpha, \quad N_\beta \subseteq N_\alpha, \qquad \beta \leqslant \alpha \leqslant \aleph_1^{(M)} .$$

However, since the interpretation of terms and statements will vary from model to model, the corresponding forcing relations will not extend each other. On the other hand, relativized statements will retain their meanings in different models and this will be expressed via the forcing relations in the following way: if $p \in P_\beta$ and φ is a sentence of $\mathcal{L}_\beta$, $\beta \leqslant \alpha$, then

$$p \Vdash_\beta \varphi \quad \text{iff} \quad p \Vdash_\alpha \varphi^{(N_\beta)},$$

where $\varphi^{(N_\beta)}$ is the relativization of φ to $N_\beta = M[I_\beta]$ which is definable in $\mathcal{L}_\alpha$ from I_β.

The sets $\mathcal{X}_\alpha$, $\mathcal{G}_\alpha$, $\mathcal{L}_\alpha$, P_α and $\Vdash_\alpha$, $\alpha \leqslant \aleph_1^{(M)}$, are to be defined by induction on α; and assuming $\mathcal{X}_\beta$, $\mathcal{G}_\beta$, $\mathcal{L}_\beta$, P_β and $\Vdash_\beta$ defined for $\beta < \alpha$, we first describe the language $\mathcal{L}_\alpha$ on the basis of $\mathcal{X}_\alpha$, $\mathcal{G}_\alpha$ to be defined below.

2.20. Definition (A). The primitive symbols of $\mathcal{L}$ which are common to all the languages are chosen in M, and have their usual denotation in the extended language of set theory and meta-language. These symbols are:

(a) *Standard symbols*:

$\neg = \langle 0, 0\rangle$ (negation),

$\wedge = \langle 0, 1\rangle$ (conjunction),

$\forall = \langle 0, 2\rangle$ (universal quantifier),

$\in = \langle 0, 3\rangle$ (membership),

$) = \langle 0, 4\rangle$ (bracket),

$(= \langle 0, 5\rangle$ (bracket),

$v^i = \langle 1, i\rangle$ (v^i are called variables and we will denote arbitrary variables by the letters $u, v, w, u_1, v_1, w_1, \ldots$ with the understanding that different letters stand for distinct variables).

$\forall_\gamma = \langle 2, \gamma\rangle$, γ an ordinal in M (restricted universal quantification),

$\lambda_\gamma = \langle 3, \gamma\rangle$, γ an ordinal in M (restricted abstraction operators);

$\dot{s} = \langle 4, s\rangle$ for every set s in M.

(These symbols will be referred to as *set constants.*)

(b) *Special constants, predicates and operation symbols*:

(1) Individual constants $a_i = \langle 5, i\rangle$, $i < \omega$, which are to denote independent Cohen-generic reals added at the first stage. These constants are called *generic-real-constants.* $b_1, b_2, \ldots$ will denote arbitrary generic-

real-constants, with the understanding that different letters stand for distinct constants.

(2) As pointed out in Lemma 1.50, every real y in a tail $K(x)$ can be obtained from x by a composition of two elementary operations, i.e., $y = \text{append}_t(\text{slash}_l(x))$ for some choice of l, t; and if l is assumed minimal, then l, t are unique. Thus the following one place functional constants will suffice for our purposes:

$$^{(t,l)}\theta(\cdot) = \langle 8, t, l\rangle, \qquad t \in 2^n,\ n,l < \omega,$$

and will denote the operation $\text{append}_t(\text{slash}_l(\cdot))$. If $t = 0$, we write $^l\theta(\cdot)$ instead of $^{(0,l)}\theta(\cdot)$; and similarly if $l = 0$, $^t\theta(\cdot)$ stands for $^{(t,0)}\theta(\cdot)$. Let Γ denote the set of all these constants, i.e.,

$$\Gamma = \{\langle 8, t, l\rangle \mid t \in 2^n,\ l, n < \omega\}.$$

Clearly $\Gamma \in M$. Elements of Γ will be denoted by $\theta, \theta_1, \theta_2, \ldots$, where it is again understood that different symbols denote distinct constants.

(3) $K(\cdot) = \langle 9, 0\rangle$ is a one place functional constant for the tail operation (see Definition 1.20).

(4) In order to facilitate relativization of terms and formulas to the γ^{th} stage model, unary predicate symbols $N_\gamma(\cdot) = \langle 10, \gamma\rangle$, $\gamma \leqslant \aleph_1^{(M)}$ are introduced. This enables us to avoid formalization of syntax and its semantics in the languages at an early stage of the work. Note that a language $\mathcal{L}_\alpha$, $\alpha < \aleph_1^{(M)}$ will contain also the symbols $N_\beta(\cdot)$ for $\beta > \alpha$. These predicate constants however will turn out to be trivial in N_α (i.e. vacuously fulfilled) as will be seen from the forcing_α definition.

(B) Besides the symbols common to all languages described in (A), each language $\mathcal{L}_\alpha$ will have two sets of special individual constants $\mathcal{X}_\alpha$ and $\mathcal{I}_\alpha$. $\mathcal{I}_\alpha = \{I_\beta \mid \beta \leqslant \alpha\}$, where $I_\beta = \langle 11, \beta\rangle$. I_0 will denote an enumeration of the tails $K(a_i)$, $i \in \omega$; and I_β, $0 < \beta \leqslant \alpha$, will denote functions the effect of which will be to enumerate the generic functions assigned to the constants of $\mathcal{X}_\beta$. $\mathcal{X}_\alpha$ is yet to be defined below.

2.21. Definition. Certain finite sequences of primitive symbols of $\mathcal{L}_\alpha$ are known as *local formulas* and *terms* of $\mathcal{L}_\alpha$. These notions are simultaneously defined by induction together with a *ranking function*, rnk_α on terms as follows:

(a) Variables and individual constants of $\mathcal{L}_\alpha$ are terms of $\mathcal{L}_\alpha$.

(b) If σ is a variable or a term of $\mathcal{L}_\alpha$, then $K(\sigma)$ and $\theta(\sigma)$ are terms of $\mathcal{L}_\alpha$ for all $\theta \in \Gamma$.

(c) If σ and τ are variables or terms of $\mathcal{L}_\alpha$, then $\sigma \in \tau$ and $N_\alpha(\sigma)$ are local formulas of $\mathcal{L}_\alpha$, $\alpha \leqslant \aleph_1^{(M)}$.

(d) If φ and ψ are local formulas of $\mathcal{L}_\alpha$, so are (φ), $\wedge\varphi\psi$, $\neg\varphi$, and $(\forall_\alpha v)(\varphi)$, provided that if the variable v in the range of the quantifier occurs in an abstraction term, then $\alpha \leqslant \omega+1$.

(e) *Abstraction terms* are defined in such a way that free variables are allowed to occur in them. The purpose of this is to enable us to speak conveniently about the elementary tail partitions in the language $\mathcal{L}_\alpha$. For instance, if $\sigma(a_1, ..., a_n)$ is a term of $\mathcal{L}_\alpha$, then we want to have a simple term in $\mathcal{L}_\alpha$ for

$$\{\langle x_1, ..., x_n\rangle \mid \langle x_1, ..., x_n\rangle \in K(a_1, ..., a_n), \text{ and}$$
$$N_\alpha \models \sigma(x_1, ..., x_n) = \sigma(a_1, ..., a_n)\} .$$

This is easily done if we allow for the term $\sigma(v_1, ..., v_n)$ (see Definition 2.30 below). Noting the limited purpose for allowing free variables to occur in abstraction terms, we decide upon a bound for the abstraction operator by treating the variables as of level ω. Thus we define:

If $\varphi(w, v_1, ..., v_n)$ is a local formula of $\mathcal{L}_\alpha$ where w is a free variable occurring in φ, and γ is an ordinal in M such that:

(i) φ contains no occurrence of a symbol $\forall_\beta$, $\beta > \gamma$;

(ii) φ contains no occurrence of a term σ with $\mathrm{rnk}_\alpha(\sigma) \geqslant \gamma$;

(iii) if φ contains free variables other than w, then $\gamma > \omega$;

then $(\lambda_\gamma\, w)(\varphi)$ is an *abstraction term.*

Terms containing free variables will be called *variable terms,* otherwise they are called *constant terms.*

(f) The rank_α of terms of $\mathcal{L}_\alpha$ will be defined considering variables as terms of rank zero.

(i) $\mathrm{rnk}_\alpha(a_i) = \omega, \quad i \in \omega;$
$\mathrm{rnk}_\alpha(\theta(\sigma)) = \max(\omega, \mathrm{rnk}_\alpha(\sigma)), \quad \theta \in \Gamma;$
$\mathrm{rnk}_\alpha(K(\sigma)) = \max(\omega+1, \mathrm{rnk}_\alpha(\sigma)+1);$

(ii) $\mathrm{rnk}_\alpha(\lambda_\gamma\, u)(\varphi) = \gamma;$

(iii) for every $\sigma \in \mathfrak{X}_\alpha$, $\mathrm{rnk}_\alpha(\sigma) = \omega\cdot 2;$
for every $\sigma \in \mathcal{G}_\alpha$, $\mathrm{rnk}_\alpha(\sigma) = (\aleph_{\aleph_1})^{(M)};$

(iv) for every $s \in M$, $\mathrm{rnk}_\alpha(\dot{s}) =$ the *set rank of* s = Rank(s); where the set rank is defined as usual, i.e. first define $R(\beta)$ by induction,

$$R(\beta) = \bigcup_{\gamma<\beta} \mathcal{P}(R(\gamma)) ,$$

(where $\mathcal{P}(x)$ denotes the power set of x) then Rank(x) is taken to be the least β for which $x \subseteq R(\beta)$. That Rank(x) exists is a consequence of the axiom of foundation

The reason that rnk_α is indexed is because these functions will be extended to include formulas in their domain, and as will be seen, we may have $\mathrm{rnk}_\alpha(\varphi) \neq \mathrm{rnk}_\beta(\varphi)$, $\beta \neq \alpha$. With regard to terms, however, we obviously have:

2.22. Lemma. *If σ is a term of $\mathcal{L}_\beta$, $\beta < \alpha$, then* $\mathrm{rnk}_\alpha(\sigma) = \mathrm{rnk}_\beta(\sigma)$.

Thus with regard to term ranking we may drop the index.

2.23. Definition. The notion of a *global formula* of $\mathcal{L}_\alpha$ is defined as follows:

(i) If σ, τ are constant terms of $\mathcal{L}_\alpha$, then $\sigma \in \tau$ and $N_\beta(\sigma)$ are global formulas;

(ii) if φ, ψ are global formulas of $\mathcal{L}_\alpha$, then so are (φ), $\wedge\varphi\psi$, $\neg\varphi$ and $(\forall v)(\varphi)$, provided that the variables v that occur in the range of the quantifier $(\forall v)$ do not occur free in abstraction terms. Both local and global formulas are referred to, simply, as formulas. A formula without free variables is called a *sentence*, or a *statement*.

2.231. Notation. The set of formulas of $\mathcal{L}_\alpha$ is denoted by $\mathcal{F}_\alpha$, and the set of terms of $\mathcal{L}_\alpha$ is denoted by $\mathcal{T}_\alpha$. We also denote

$$\mathcal{F}_{\aleph_1}(M) = \mathcal{F} , \qquad \mathcal{T}_{\aleph_1}(M) = \mathcal{T} .$$

The set of constant terms in $\mathcal{L}_\alpha$ is denoted by $\mathcal{T}^c_\alpha$, and the set of sentences in $\mathcal{L}_\alpha$ is denoted by $\mathcal{F}^c_\alpha$.

2.24. Notation and terminology. The following abbreviations relating to other standard logical symbols will be used throughout this work. $\varphi \wedge \psi$ stands for $\wedge\varphi\psi$. (Note that by this definition, brackets could have been done away with altogether, see Lévy [9].)

For any integer m and formulas $\varphi, \psi, \varphi_1, ..., \varphi_m$,

$\varphi_1 \vee ... \vee \varphi_m$ is an abbreviation for $\neg(\neg\varphi_1 \wedge ... \wedge \neg\varphi_m)$ (compound disjunction);

$\varphi \to \psi$ is an abbreviation for $\neg(\varphi \wedge \neg\psi)$ (implication);

$\varphi \leftrightarrow \psi$ is an abbreviation for $\varphi \to \psi \wedge \psi \to \varphi$ (bi-implication);

$(\exists\varphi)$ abbreviates $\neg(\forall v)(\neg\varphi)$ (existential quantification);

$(\exists_\gamma v)(\varphi)$ abbreviates $\neg(\forall_\gamma v)(\neg\varphi)$ (restricted existential quantification).

2.241. Compound conjunction and disjunction will be abbreviated as follows: if $\varphi_1, ..., \varphi_m$ are formulas, then

$$\bigwedge_{i=1}^{m} \varphi_i = \varphi_1 \wedge ... \wedge \varphi_m, \qquad \bigvee_{i=1}^{m} \varphi_i = \varphi_1 \vee ... \vee \varphi_m.$$

As in the language of set theory, equality is taken as an abbreviation; thus if σ and τ are terms or variables, then $\sigma = \tau$ is an abbreviation for the formula $(\forall v)(v \in \sigma \leftrightarrow v \in \tau)$.

2.25. Convention. (a) We also use a bounded equality abbreviation as in Easton [3], Lévy [11]. If σ, τ are terms or variables, then $\sigma \simeq \tau$ stands for the ranked formula

$$(\forall_\gamma v)(v \in \sigma \leftrightarrow v \in \tau) \quad \text{where } \gamma = \max(\mathrm{rnk}(\sigma), \mathrm{rnk}(\tau)).$$

(b) If φ is a local formula, then $(\rangle\!\!\mathrm{I} v)(\varphi)$ stands for the terms $(\rangle\!\!\mathrm{I}_\gamma v)(\varphi)$ where γ is the smallest ordinal such that $(\rangle\!\!\mathrm{I}_\gamma v)(\varphi)$ is a well defined term. More generally, if φ is a formula, then by $(\rangle\!\!\mathrm{I} v)(\varphi)$ we denote the term $(\rangle\!\!\mathrm{I}_\gamma v)(\varphi')$ where γ is the smallest ordinal such that if φ' is obtained from φ by bounding its unrestricted quantifiers with γ, then $(\rangle\!\!\mathrm{I}_\gamma v)(\varphi')$ is a well defined term.

(c) Henceforth, $\sigma \notin \tau$, $\sigma \neq \tau$, and $\sigma \not\simeq \tau$ stand for $\neg(\sigma \in \tau)$, $\neg(\sigma = \tau)$, and $\neg(\sigma \simeq \tau)$ respectively.

2.26. Designation and abbreviation of terms in $\mathcal{L}$ for elementary notions. If $\tau_1, ..., \tau_n$ are terms or variables, then $\{\tau_1, ..., \tau_n\}$ denotes the term

$$(\text{\reflectbox{λ}}w)\,(w \simeq \tau_1 \vee \ldots \vee w \simeq \tau_n)\,.$$

The term for the ordered pair is defined by

$$\dot{\langle}\tau_1, \tau_2\dot{\rangle} = \dot{\{}\dot{\{}\tau_1\dot{\}}, \dot{\{}\tau_1, \tau_2\dot{\}}\dot{\}}\,.$$

The n-tuple is defined by induction; hence assuming it defined for $2 \leqslant m < n$, define

$$\dot{\langle}\tau_1, \ldots, \tau_n\dot{\rangle} = \dot{\langle}\dot{\langle}\tau_1, \ldots, \tau_{n-1}\dot{\rangle}, \tau_n\dot{\rangle}\,.$$

For intersection, union and difference we define

$$\sigma \mathbin{\dot{\cap}} \tau = (\text{\reflectbox{λ}}w)\,(w \in \sigma \wedge w \in \tau)\,,$$

$$\sigma \mathbin{\dot{\cup}} \tau = (\text{\reflectbox{λ}}w)\,(w \in \sigma \vee w \in \tau)\,,$$

$$\dot{\bigcup}\sigma = (\text{\reflectbox{λ}}w)\,(\exists u)\,(u \in \sigma \wedge w \in u)\,,$$

$$\sigma \mathbin{\dot{-}} \tau = (\text{\reflectbox{λ}}w)\,(w \in \sigma \wedge w \notin \tau)\,,$$

$$\sigma \mathbin{\dot{\subseteq}} \tau = (\forall w)\,(w \in \sigma \to w \in \tau)\,,$$

where σ and τ are terms or variables.

2.27. Notation. Let σ be a term of $\mathcal{L}$. On many occasions we will need to display the real-constants occurring in σ. This is done as follows: we write

$$\sigma = \sigma(a_{h_1}, \ldots, a_{h_n})$$

where $a_{h_1}, \ldots, a_{h_n}$ are all the real-constants occurring in σ in order of *increasing index*, (and where possibly $n = 0$). To make our notation less cumbersome we employ vector notation. If $x_1, \ldots, x_n$ is a finite sequence, we write simply $\bar{x}^n$ or even $\bar{x}$ when no confusion is possible. Also for example, $\bar{x}_1^n(\overline{x^1}^n)$ denotes $x_{1,1}, \ldots, x_{1,n}(x_1^1, \ldots, x_n^1)$. If the x's have a compound index, say $x_{h_1}, \ldots, x_{h_n}$, we write simply $\bar{x}_h^n$ (or $\bar{x}_h$) to denote it. Thus conforming to this notation we write $\sigma = \sigma(\bar{a}_h^n)$.

2.271. $f, g, h,$ and the corresponding superscript letters will denote finite sets of integers; but these letters with subscript index will denote integers. We make a further convention that if $w > i > j$, then $h_i > h_j$. Thus when we write $h = \{h_1, \ldots, h_n\}$, the h_i increase with increasing index; and when we write $\bar{a}_h$, h is identical with the set of indices. The sequence

$$\theta_1(a_{h_1}), \ldots, \theta_n(a_{h_n}), \quad \theta_i \in \Gamma ,$$

is denoted by $\bar{\theta} \cdot \bar{a}_h^n$ or $\bar{\theta} \cdot \bar{a}_n$ when no confusion is possible. The sole exception to this usage of vector notation is $\bar{\Gamma}^n$ which stands for $\Gamma \times \Gamma \times \ldots \times \Gamma$, n times and $\bar{\theta} \in \bar{\Gamma}^n$ is written instead of $\theta_i \in \Gamma$, $1 \leqslant i \leqslant n$. If $\sigma = \sigma(\bar{a}_h)$, then $\sigma(\bar{v})$ stands for the term obtained from σ by exchanging every occurrence of a_{h_i} by the variable v_i, $1 \leqslant i \leqslant n$. Also $\sigma(\bar{\theta} \cdot \bar{a}_h)$ denotes the term obtained from σ by replacing everywhere a_i with $\theta_i(a_{h_i})$, $1 \leqslant i \leqslant n$. (Note that we get legitimate terms.) In formulas, blocks of quantifiers of the form $(\forall v^1)(\forall v^2) \ldots (\forall v^n)$ and $(\forall_\alpha v^1)(\forall_\alpha v^2) \ldots (\forall_\alpha v^n)$ will be denoted by $\overline{\forall v^n}$ and $\overline{\forall_\alpha v^n}$ respectively; similarly for existential quantifiers.

2.3. Elementary tail partitions. Let $\sigma(\bar{a}_h) = \sigma(\bar{b}) \in \mathcal{T}^c_\alpha$, ($\bar{b}$ denotes $\bar{a}_h$, recall Definition 2.20(b)(1)); then in N_α, σ will determine an equivalence relation ${}^\alpha E_\sigma$ on $K(\bar{b})$ by

$$\bar{\theta}_1 \cdot \bar{b}\, {}^\alpha E_\sigma \bar{\theta}_2 \cdot \bar{b} \quad \text{iff} \quad N_\alpha \models \sigma(\bar{\theta}_1 \cdot \bar{b}) = \sigma(\bar{\theta}_2 \cdot \bar{b}) .$$

This equivalence relation induces a partition ${}^\alpha H_\sigma$ of $K(\bar{b})$ as recalled from the introduction. Such a partition is called an *elementary tail partition.* We now define terms in $\mathcal{L}_\alpha$ which will denote the elementary tail partitions and related notions that play a central role in this work – the intuitive content of which can be clearly seen.

2.30. Definition. Terms for the compound elementary tail operations are:

$$K(\bar{a}_h^n) = (\lambda w)\, (\overline{\exists_{\omega+1} v})\, (\overline{\exists_{\omega+2} u}) \Big(\bigwedge_{i=1}^{n} (v_i \in u_i \wedge \dot{\langle} i, u_i \dot{\rangle} \in I_0$$
$$\wedge\, w \simeq \dot{\langle} v_1, \ldots, v_n \dot{\rangle}) \Big) .$$

Let $\sigma(\bar{a}_n) = \sigma(\bar{b}) \in \mathcal{T}^c_\alpha$; then the term for ${}^\alpha E_\sigma$ is

$$E_\sigma = (\lambda w)\, (\overline{\exists_{\omega+1} v})\, (\overline{\exists_{\omega+1} u})\, (\dot{\langle} \bar{v} \dot{\rangle} \in K(\bar{b}) \wedge \dot{\langle} \bar{u} \dot{\rangle} \in K(\bar{b})$$
$$\wedge\, \sigma(\bar{u}) \simeq \sigma(\bar{v})$$
$$\wedge\, w \simeq \ddot{\langle} \dot{\langle} \bar{u} \dot{\rangle}, \dot{\langle} \bar{v} \dot{\rangle} \ddot{\rangle}).$$

For any $\bar{\theta} \in \bar{\Gamma}$, the equivalence class in $K(\bar{b})$ of $\bar{\theta} \cdot \bar{b}$ with respect to E_σ is denoted in $\mathcal{L}_\alpha$ by the term $H_\sigma(\bar{\theta} \cdot \bar{b})$, where

$$H_\sigma(\bar{u}^n) = (\lambda w)\,(\overline{\exists_{\omega+1} v})\,(\sigma(\bar{u}) \simeq \sigma(\bar{v}) \wedge \langle \dot{\bar{v}} \rangle \in K(\bar{b}) \wedge w \simeq \langle \dot{\bar{v}} \rangle).$$

The set of equivalence classes of ${}^\alpha E_\sigma$ will be denoted in $\mathcal{L}_\alpha$ by the term

$$H_\sigma = (\lambda w)\,(\overline{\exists_{\omega+1} u})\,(w \simeq H_\sigma(\bar{u}) \wedge \langle \dot{\bar{u}} \rangle \in K(\bar{b}))\,.$$

The functions ${}^{\alpha+1}\chi_\sigma$ to be added at the $\alpha + 1^{\text{th}}$ stage will map $K(\bar{a}_h)$ into a set of generic tails such in a way that if

$${}^\alpha H_\sigma(\bar{x}) = {}^\alpha H_\sigma(\bar{y})\,,$$

then ${}^{\alpha+1}\chi_\sigma(\bar{x})$, ${}^{\alpha+1}\chi_\sigma(\bar{y})$ have the same tail, and

$${}^{\alpha+1}\chi_\sigma(\bar{x}) = {}^{\alpha+1}\chi_\sigma(\bar{y}) \quad \text{iff} \quad {}^\alpha H_\sigma(\bar{x}) = {}^\alpha H_\sigma(\bar{y})\,.$$

2.31. Remark. Observe that no generic real constants occur in any of the above defined formulas!

2.4. $\mathcal{X}_\alpha$, prelude and definition. Every α in $\mathcal{T}^c_\alpha$ will determine an elementary tail partition in N_α; however, there will be many terms yielding the same partition. All the elementary tail partitions in N_α are included in a set in N_α, while all the terms of $\mathcal{L}_\alpha$ form a class in M, (and in N_α). In order that the $\mathcal{X}_\beta$ be sets in M, we will not supply symbols $\gamma \in \mathcal{X}_{\beta+1}$ pertaining directly to the terms of $\mathcal{L}_\beta$, but to associated objects relating to the Boolean universe associated with N_α. These objects will comprise a set in M and will sufficiently encode all elementary tail partitions to be encountered in N_α.

We are assuming the existence of $\mathcal{L}_\beta$, P_β and $\Vdash_\beta$, $\beta < \alpha$, where the elements of P_β are finite pieces of information and $\Vdash_\beta$ is a certain weak forcing relation between sentences φ of $\mathcal{L}_\beta$ and conditions $p \in P_\beta$, having the usual interpretation; i.e. $p \Vdash_\beta \varphi$ is understood as: "φ is true in all models N_β realizing $\mathcal{L}_\beta$ where the generic objects are compatible with p".

2.40. Definition. (a) $p \in P_\beta$ is said to be a *minimal*$_\beta$ *condition* for the sentence φ if $p \Vdash_\beta \varphi$, and for no $q \subset p$ does $q \Vdash_\beta \varphi$.

(b) $\|\varphi\|_\beta = \{P \mid P \text{ is minimal}_\beta \text{ for } \varphi\}$, φ a sentence of $\mathcal{L}_\alpha$.

$\|\varphi\|_\beta$ contains only the essential information involved in the truth of φ in a model. It will be shown in Lemma 6.10 below that $\|\varphi\|_\beta \in M$ is always countable in M. This important point will enable us to show that our process terminates in $\aleph_1^{(M)}$ stages.

$\|\varphi\|_\beta$ can also be viewed as the Boolean value of φ in the complete Boolean algebra associated with P_α. In N_β, $\sigma(\bar{b}) \in \mathcal{T}_\beta^c$ determines the equivalence relation E_σ on $K(\bar{b})$, while, in the associated Boolean universe, σ determines a β-Boolean equivalence relation on $K(\bar{b})$ which, roughly speaking, is

$$\{\langle\langle\bar{\theta}_1 \cdot \bar{b}, \bar{\theta}_2 \cdot \bar{b}\rangle, \|\sigma(\bar{\theta}_1 \cdot \bar{b}) = \sigma(\bar{\theta}_2 \cdot \bar{b})\|_\beta\rangle \mid \bar{\theta}_1, \bar{\theta}_2 \in \bar{\Gamma}\} .$$

The important thing here is that there is a set in M consisting of all these Boolean equivalence relations; and that for all $\sigma(\bar{b})$, $\tau(\bar{b}) \in \mathcal{T}_\beta^c$ with the same Boolean equivalence relation, we shall have ${}^\beta H_\sigma = {}^\beta H_\tau$. Aside from this idea, the Boolean universe and corresponding notions will not interest us; we now define the notions with which we will actually be concerned.

2.41. Definition. (a) For $\sigma(\bar{b}) \in \mathcal{T}_\beta^c$, $\beta < \alpha$ and $\bar{\theta}_1, \bar{\theta}_2 \in \bar{\Gamma}$ set:

$$[+\sigma, \bar{\theta}_1, \bar{\theta}_2] = \langle\beta, \langle\{\bar{\theta}_1 \cdot \bar{b}, \bar{\theta}_2 \cdot \bar{b}\}, \|\sigma(\bar{\theta}_1 \cdot \bar{b}) \simeq \sigma(\bar{\theta}_2 \cdot \bar{b})\|_\beta\rangle\rangle,$$

$$[-\sigma, \bar{\theta}_1, \bar{\theta}_2] = \langle\beta, \langle\{\bar{\theta}_1 \cdot \bar{b}, \bar{\theta}_2 \cdot \bar{b}\}, \|\sigma(\bar{\theta}_1 \cdot \bar{b}) \not\simeq \sigma(\bar{\theta}_2 \cdot \bar{b})\|_\beta\rangle\rangle ,$$

$$[\sigma, \bar{\theta}_1, \bar{\theta}_2] = \langle[+\sigma, \bar{\theta}_1, \bar{\theta}_2], [-\sigma, \bar{\theta}_1, \bar{\theta}_2]\rangle;$$

(b) $s_\sigma^\beta = \{[\sigma, \bar{\theta}_1, \bar{\theta}_2] \mid \bar{\theta}_1, \bar{\theta}_2 \in \bar{\Gamma}\}$ (this is the β-Boolean relation);
(c) $s_{\bar{b}}^\beta = \{s_\sigma^\beta \mid \sigma = \sigma(\bar{b})\}$;
(d) $s^\beta = \{s_\sigma^\beta \mid \sigma \in \mathcal{T}_\beta\} = \bigcup_{\bar{b}}\{s_{\bar{b}}^\beta\}$.

Note that we may have $s_\sigma^\beta = s_\tau^\beta$, $\sigma \neq \tau$, and that s^β is seen to be a set in M once it is known that P_β is a set in M and $\Vdash_\beta$ is definable in M, $\beta < \alpha$.

The following facts will be easily proven, though they comprise part of our induction hypothesis needed in the definition of a condition; they will show that the s_σ^β sufficiently encode the ${}^\beta H_\sigma$, $\beta < \alpha$, in M.

2.42. Induction hypothesis assumptions. For all $\beta < \alpha$: If

$$\sigma = \sigma(\bar{b}) = \sigma(\bar{a}_h) \in \mathcal{T}^c_\beta, \qquad \tau = \tau(\bar{b}) \in \mathcal{T}^c_\beta, \qquad s^\beta_\sigma = s^\beta_\tau ,$$

then $0 \Vdash_\beta H_\sigma \simeq H_\tau$. If $\bar{\theta} \in \bar{\Gamma}$, then

$$0 \Vdash_\beta H_\sigma(\bar{\theta} \cdot \bar{b}) \simeq H_\tau(\bar{\theta} \cdot \bar{b}) ;$$

$$p \Vdash_\beta H_\sigma(\bar{\theta}_1 \cdot \bar{b}) \simeq H_\sigma(\bar{\theta}_2 \cdot \bar{b})$$

iff

$$p \Vdash_\beta H_\tau(\bar{\theta}_1 \cdot \bar{b}) \simeq H_\tau(\bar{\theta}_2 \cdot \bar{b});$$

$$p \Vdash_\beta H_\sigma(\bar{\theta}_1 \cdot \bar{b}) \not\simeq H_\sigma(\bar{\theta}_2 \cdot \bar{b})$$

iff

$$p \Vdash_\beta H_\tau(\bar{\theta}_1 \cdot \bar{b}) \not\simeq H_\tau(\bar{\theta}_2 \cdot \bar{b}).$$

2.43. Definition of $\beta+1$ indices, $\beta<\alpha$. A $\beta+1$-*index* is a triple $e = (h, g, s)$ where $s \in s^\beta_{\bar{a}^n_h}$, $h = \{h_1, ..., h_n\}$ is a finite set of integers, and $g \subseteq h$.

Note that s contains the unique ordinal β in it, determining the *stage of the index*, $\beta+1$.

2.44. Definition of $\mathcal{X}_\alpha$. Let $\mathcal{X}_0 = \emptyset$. If α is a limit ordinal, we define $\mathcal{X}_\alpha = \bigcup_{\beta<\alpha} \mathcal{X}_\beta$. If $\alpha = \beta+1$, let $\mathcal{X}^*_\alpha$ be the set of symbols $\chi_e = \langle 12, e\rangle$ for every $\beta+1$-index e, and set $\mathcal{X}_\alpha = \mathcal{X}_\beta \cup \mathcal{X}^*_\alpha$.

That $\mathcal{X}_\alpha$ is a set in M results from the fact that the set of $\beta+1$ indices is a set in M. The cardinality and set rank of $\mathcal{X}_\alpha$ depend on those of the set of $\beta+1$ indices $\beta < \alpha$ and will be determined below.

Let $e_g = (h, g, s)$, where $s = s^\beta_\sigma \in s^\beta_{\bar{b}}$ for some $\sigma(\bar{b}) \in \mathcal{T}^c_\beta$ $(\bar{b} = \bar{a}_h)$; then $\bigcup_{g\subseteq h} \chi_{e_g}$ will be a function defined on an elementary tail such that if

$$(2.45) \quad \mathring{\chi}_{e_g} = \{\langle x, y\rangle \mid y \in \operatorname{rng}(\chi_{e_g}), x = \{\bar{z}_h \mid \bar{z}_h \in \operatorname{dom}(\chi_{e_g}), \chi_{e_g}(\bar{z}_h) = y\}\},$$

then $\bigcup_{g\subseteq h} \mathring{\chi}_{e_g}$ will be a univalent function mapping ${}^\beta H_\sigma$ into a set of generic tails such that $H_\sigma(\bar{\theta}_1 \cdot \bar{b})$, $H_\sigma(\bar{\theta}_2 \cdot \bar{b})$ are mapped into different tails if

$$H_\sigma(\bar{\theta}_1, \bar{b}) \neq H_\sigma(\bar{\theta}_2 \cdot \bar{b}) .$$

Other requirements on $\mathring{\chi}_{e_g}$ will also be necessary and will be clear from the following definition of a condition. The particular σ chosen for which $s = s^\beta_\sigma$ is irrelevant as can be seen from 2.42. I_α will be

$$\{\langle e, \chi_e\rangle \mid e \text{ a } \beta+1\text{-index}, \beta < \alpha\} \cup I_0 ,$$

and I_0 will be $\{\langle i, K(a_i)\rangle\}_{i\in\omega}$. Since the set of indices is well-ordered in M, we will be able to derive in N_α a well-ordering of all the $\mathring{\chi}_e$, e a $\beta+1$-index.

2.5. Conditions

2.50. Definition of P_α, the set of α^{th} stage conditions. P_α is defined by induction assuming $\mathcal{L}_\beta, P_\beta, \Vdash_\beta$ and their properties which were specified in 2.42 for $\beta < \alpha$. Elements of P_α are finite sets consisting of finite sequences called *preconditions*, that fulfill the specific requirements listed below. The preconditions are of three types:

(i) triples of the form $\langle i, j, \delta\rangle$, $i, j < \omega$, $\delta < 2$, (to assert that $a_i(j) = \delta$);

(ii) quadruples of the form $\langle e, \bar{\theta}, j, \delta\rangle$ where e is a $\beta+1$-index, $\beta < \alpha$, $\bar{\theta} \in \bar{\Gamma}$, $j < \omega$, $\delta < 2$, (to assert that $\chi_e(\bar{\theta}\cdot\bar{b})(j) = \delta$);

(iii) quintuples of the form $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle$ where e is a $\beta+1$-index, $\beta < \alpha$, $\bar{\theta}_1, \bar{\theta}_2 \in \bar{\Gamma}$ and $l_1, l_2 < \omega$, (to assert that

$$\mathrm{slash}_{l_1}(\chi_e(\bar{\theta}_1\cdot\bar{b})) = \mathrm{slash}_{l_2}(\chi_e(\bar{\theta}_2\cdot\bar{b})) ,$$

thereby giving $\chi_e(\bar{\theta}_1\cdot\bar{b})$ and $\chi_e(\bar{\theta}_2\cdot\bar{b})$ the same tail).

Type (i) preconditions are referred to as *0-stage preconditions*, and those of types (ii) and (iii) are referred to as $\beta+1$-*stage preconditions*.

2.51. If p is a set of preconditions, then $\hat{p}^\gamma$ will denote the set of all γ'-stage preconditions in p for $\gamma' < \gamma$. If e and $\bar{\theta}$ occur in a precondition of p, we say that *p mentions* $\chi_e(\bar{\theta}\cdot\bar{b})$, (where $\bar{b} = \bar{a}_h$, and h is the first component of e).

Conditions $p \in P_0$ are finite sets of preconditions of type (i) fulfilling the sole requirement that for no $i, j < \omega$ are both $\langle i, j, 0\rangle$ and $\langle i, j, 1\rangle$ in p.

If $\alpha > 0$ is a limit ordinal, we define $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$.

If $\alpha = \beta+1$ is a successor ordinal, then $P_{\beta+1}$ is defined as follows: $p \in P_{\beta+1}$ iff it is a finite set of γ-stage preconditions, $\gamma \leqslant \alpha$, fulfilling the following list of requirements:

(a) $\hat{p}^\beta$ is a condition of P_β. (Thus for the rest of the definition we may be concerned only with $\alpha+1$-stage preconditions of p.)

(b) p does not contain both $\langle e, \bar{\theta}, j, 0\rangle$ and $\langle e, \bar{\theta}, j, 1\rangle$.

(c) (1) If $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in p$ and for some $0 \leqslant k < \omega$, $\delta < 2$, $\langle e, \bar{\theta}_1, l_1+k, \delta\rangle \in p$, then also $\langle e, \bar{\theta}_2, l_2+k, \delta\rangle \in P$. (Thus it is clear that $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle$ means

$$\mathrm{slash}_{l_1}(\chi_e(\bar{\theta}_1 \cdot \bar{b})) = \mathrm{slash}_{l_2}(\chi_e(\bar{\theta}_2 \cdot \bar{b})),$$

and in fact it will be seen that such a p actually forces that statement.)

(2) If p contains $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle$ and $\langle e, \bar{\theta}_1, m_1, \bar{\theta}_2, m_2\rangle$, then $l_1 - l_2 = m_1 - m_2$.

If otherwise, then these two "clauses" combined would mean that two different slashings of $\chi_e(\bar{\theta}_1 \cdot \bar{b})$ yield the same real, i.e. $\chi_e(\bar{\theta}_1 \cdot \bar{b})$ is eventually periodic; but it is essential that it be generic.)

(3) If $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in p$, and $\langle e, \bar{\theta}_2, m_2, \bar{\theta}_3, m_3\rangle \in p$, then

$$\langle e, \bar{\theta}_1, l_1 + m_2 - k, \bar{\theta}_2, m_3 + l_2 - k\rangle \in P$$

for some k such that $k \geqslant \min(m_2, l_2)$.

(This is a "transitivity" requirement establishing the right relations between $\chi_e(\bar{\theta}_2 \cdot \bar{b})$ and $\chi_e(\bar{\theta}_3 \cdot \bar{b})$.) With regard to transitivity it is also necessary that,

(4) if $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in P$, then $\langle e, \bar{\theta}_2, l_2, \bar{\theta}_1, l_1\rangle \in p$.

For the rest of the definition assume $e = (h, g, s)$ where $\bar{a}_h = \bar{b}$ and $s = s^\beta_\sigma$ for $\sigma(\bar{b}) \in \mathcal{T}_\beta$. Because of assumptions in 2.42 the particular representative chosen from the class of terms $\tau \in \mathcal{T}_\beta$ for which $s^\beta_\tau = s$ will not affect the definition.

2.52. By $q \parallel_\beta \varphi$ we mean "$q \Vdash_\beta \varphi$ or $q \Vdash \neg\varphi$", in which case we say that q decides φ.

In the last two clauses (d), (e), we deal with the domain H_σ of $\mathring{\chi}_e$, (see (2.45)).

(d) If $\chi_e(\bar{\theta}_1 \cdot \bar{b})$ and $\chi_e(\bar{\theta}_2 \cdot \bar{b})$ are mentioned by p, then

$$\hat{P}^\beta \parallel_\beta H_\sigma(\bar{\theta}_1 \cdot \bar{b}) \simeq H_\sigma(\bar{\theta}_2 \cdot \bar{b});$$

and if $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in p$, then:

(1) If $\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}_1 \cdot \bar{b}) \not\simeq H_\sigma(\theta_2 \cdot \bar{b})$, then either $l_1 \neq l_2$ or $l_1 = l_2$; and for some $j < l_1$, $\delta < 2$,

$$\langle e, \bar{\theta}_1, j, \delta\rangle \in p, \qquad \langle e, \bar{\theta}_2, j, 1-\delta\rangle \in p;$$

(2) If $\hat{p}^{\beta} \Vdash_{\beta} H_{\sigma}(\bar{\theta}_1 \cdot \bar{b}) \simeq H_{\sigma}(\bar{\theta}_2 \cdot \bar{b})$, then $l_1 = l_2$; moreover $\langle e, \bar{\theta}_1, 0, \bar{\theta}_2, 0\rangle \in p$, (0 is the identity element of Γ). (Note that from this requirement it follows that $\langle e, \bar{\theta}_1, i, \delta\rangle \in P \Leftrightarrow \langle e, \bar{\theta}_2, i, \delta\rangle \in P$, $i < \omega$, $\delta < 2$; and by the transitivity requirement it also follows that for all $\bar{\theta}^* \in \bar{\Gamma}$,

$$\langle e, \bar{\theta}_1, l_1, \bar{\theta}^*, l^*\rangle \in p \Leftrightarrow \langle e, \bar{\theta}_2, l_2, \bar{\theta}^*, l^*\rangle \in P.)$$

(3) If $\chi_e(\bar{\theta}_1 \cdot \bar{b})$ and $\chi_e(\bar{\theta}_2 \cdot \bar{b})$ are mentioned by P, then for some l_1, l_2, $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in p$.

In (d) we stipulated when an arbitrary precondition of the form $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle$ can or cannot be added to a condition.

(e) The final requirement on conditions is designed to avoid choice in N by making sure that certain "nearly g-definable" elements of ${}^{\theta}H_{\sigma}$ are not mapped by $\mathring{\chi}_e$ into the same tail as the "non-nearly g-definable" elements. The idea behind requirement (e) is best conveyed by the following example.

2.53. Example. Assume that a model has been realized without further restrictions on the conditions. A sequence of generic functions $\{\langle i, \chi_i\rangle\}_{i\in\omega}$ is then foreseen to exist in $N_1 \supseteq N_0$, yielding a choice function on

$$\{K(a_i)\}_{i\in\omega} = \{I_0(i) \mid i \in \omega\}$$

in N_1. (It should be clear that if there is a choice function for $\{K(a_i)\}_{i\in\omega}$, then we have choice in the model.)

Let $\sigma_i(a_i) \in \mathcal{T}_0^c$, $i \in \omega$, be the terms:

$$(\lambda_{\omega} u)\,((u \in K(a_i) \wedge u \simeq a_i \wedge \langle \dot{\dot{0}}, \dot{\dot{0}}\rangle \in u)$$
$$\vee\,(u \in K(a_i) \wedge \langle \dot{\dot{0}}, \dot{\dot{1}}\rangle \in u \wedge \langle \dot{0}, \dot{\dot{1}}\rangle \in a_i))\,.$$

The term $\sigma_i(a_i)$ induces a partition H_{σ_i} on $K(a_i) = I_0(i)$, $i \in \omega$, such that $x \in H_{\sigma_i}$ iff either

$$x = \{y \mid y \in K(a_i) \wedge y(0) = 0\}\ ,$$

or $x = \{y\}$ with $y \in K(a_i)$ and $y(0) = 1$. Denote the component

$$\{y \mid y \in K(a_i) \wedge y(0) = 0\}$$

by r_i. Then r_i is definable. Let $h^i = \{i\}$, and $g^i = \emptyset$. Set

$$e_i = (h^i, g^i, s^0_{\sigma_i}) .$$

Since $|H_{\sigma_i}| = \aleph_0$, $\mathring{\chi}_{e_i}$ will map H_{σ_i} onto a tail. Now, to choose canonical elements from each $I_0(i)$, $i \in \omega$, (in N_1), set $t = \langle 0, 1\rangle \in 2^1$ and define the sets:

$$(\mathring{\chi}_{e_i})^{-1}\,(\text{append}_t(\mathring{\chi}_{e_i}(r_i))) \in H_{\sigma_i}$$

(see Fig. 4). Each such component is a singleton containing an element of $K(a_i)$; therefore a choice function would exist in N_1, since an enumeration of χ_{e_i} can be obtained from I_1.

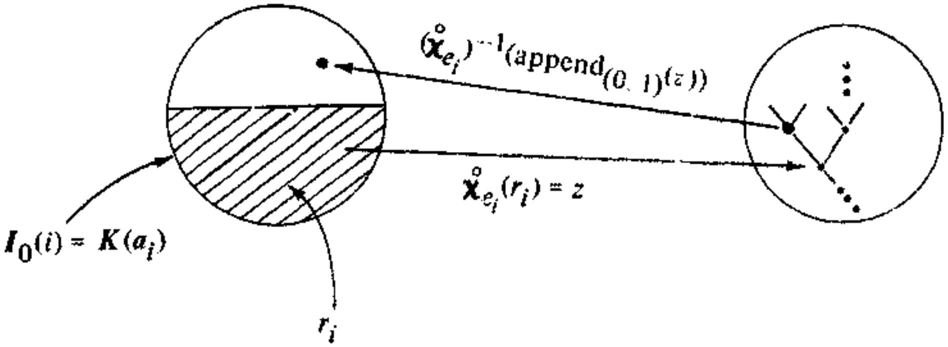

Fig. 4.

The same situation would arise if we had considered the following slightly more involved partitions of $K(a_i)$:

$$H_i = \{\{x\} \mid x \in K(a_i), x(0) = 0\}$$
$$\cup \{y \mid \text{for some } z \in B_2(K(a_i)), y = \{x \mid x \in z, x(0) = 1\}\},$$

(see Example 1.90 for definition of $B_2(K(a_i))$). In H_i there are no definable elements; but there are elements belonging to finite definable sets. These elements we call *nearly definable*. By considering the latter components, we can again show the existence of a choice function for $\{K(a_i)\}_{i\in\omega}$ by an argument similar to that of the previous example. In these examples, choice could be avoided either by mapping the nearly definable components of H_σ into a separate tail, or by mapping the components of H_σ with definable elements into a separate tail. It turns out that this idea works also in the general case; but we will have to deal with a more refined concept called *nearly* $(-l)$-*definable* where l is an integer. A set $x \in N_\beta$ will be nearly $(-l)$-definable$_\beta$, if for some finite set of integers h^* with $l \notin h^*$, there is an element $\bar{z} \in K(\bar{a}_{h^*})$ such that x is definable$_\beta$ from $\bar{z}$ by a formula of $\mathcal{L}_\beta$ in which real constants do not occur. A formulation of this and related notions is given in Definition 4.80 below. What is actually needed for requirement (e) is a formulation of this notion via forcing. This would seem to be complicated because it demands dealing with infinitely many formulas; but it is really quite simple.

2.54. Assume $q \in P_\beta$, and that $\tau \in \mathcal{T}^c_\beta$ is a constant term; we say that *q forces$_\beta$ τ to be nearly $(-l)$-definable$_\beta$* if for some constant term $\rho \in \mathcal{T}^c_\beta$ and some limited formula $\varphi(u_1, ..., u_m, v)$ of $\mathcal{L}_\beta$ without real constants, and some h such that $l \notin h$, we have

$$q \Vdash_\beta (\exists! v)\,(\varphi(\bar{a}_h, v) \wedge \text{“}\rho \text{ is finite”} \wedge \tau \in \rho \wedge \varphi(\bar{a}_h, \rho))$$

($(\exists! v)\,(\psi(v))$ stands for "there exists a unique v such that $\psi(v)$"; and "τ is finite" stands for some formula saying that there exists an integer i and a univalent mapping of τ onto i.) We say that "*q forces$_\beta$ τ to be not nearly $(-l)$-definable$_\beta$*" if there is no $q^* \supseteq q$ such that q^* forces$_\beta$ τ to be nearly $(-l)$-definable$_\beta$.

Finally, requirement (e) can be expressed as follows:

2.55. Assume that $e = (h, g, s)$, and that $\chi_e(\bar{\theta} \cdot \bar{a}_h)$ is mentioned by p; then if $l \in g$, $\bar{p}^\beta$ forces$_\beta$ $H_\sigma(\bar{\theta}(\bar{a}_h))$ to be nearly $(-l)$-definable$_\beta$, and if $l \in h - g$, then $\bar{p}^\beta$ forces$_\beta$ $H_\sigma(\bar{\theta} \cdot \bar{a}_h)$ to be not nearly $(-l)$-definable$_\beta$.

2.551. **Remark.** Observe that if $e_g = (h, g, s), g \subseteq h$ are $\beta + 1$ indices, then setting $\mathring{\chi}_s = \bigcup_{h \supseteq g} \mathring{\chi}_{e_g}$, we will have

$$\operatorname{dom}(\mathring{\chi}_{e_{g^*}}) \cap \operatorname{dom}(\mathring{\chi}_{e_{g^{**}}}) = \emptyset, \qquad g^* \neq g^{**}, \quad g^*, g^{**} \subseteq h$$

$$\operatorname{dom}(\mathring{\chi}_s) = \operatorname{dom}(\bigcup_{h \supseteq g} \mathring{\chi}_{e_g}) = {}^\beta H_\sigma .$$

$\mathring{\chi}_s$ is actually the function needed. The reason the functions $\mathring{\chi}_{e_g}$ are taken as primitives is because the notion of nearly $(-l)$-definable is not easily expressed in the language. To express this notion a formalization of syntax and satisfaction is needed in the language, which is delayed to a later part of the work. The distinction of the functions $\chi_{e_g}, g \subseteq h$, at the outset sidesteps the process, as can be clearly seen in Section 4.

We now add the last of the induction hypotheses needed to complete the definition.

2.56. Additional induction hypotheses. For $\beta < \alpha$ we assume:

(a) $\mathcal{F}_\beta \cup \mathcal{T}_\beta \subseteq M$,

(b) $P_\beta \subseteq R^{(M)}(\omega \cdot \beta) = R(\omega \cdot \beta) \cap M \in M$,

(c) if e is a $\beta + 1$-index, then $\mathrm{Rank}(e) \leqslant \omega(\beta + 1)$.

2.57. Corollary. *If e is a $\beta + 1$-index, $\beta < \alpha$, then we are assuming that*

$$\mathrm{rnk}_\gamma(\dot{e}) \leqslant \omega(\beta + 1) < \aleph_1^{(M)} \qquad \beta < \gamma \leqslant \alpha .$$

2.6. Forcing. We precede the inductive definition of $\Vdash_\alpha$ by extending the function rnk_α to include formulas. As usual the ranking has to account for the levels of constructibility, length of formulas and minor technicalities. In our case the situation is slightly more involved because of the predicate $N_\gamma(\cdot)$; therefore, the ranking will have to also account for the *stages* to which terms and formulas are relativized.

2.60. Definition. The *relativization* of formulas and terms of $\mathcal{L}$ to N_γ is defined by induction on length as follows:

(a) If σ is a variable or an individual constant, then $(\sigma)^{(N_\gamma)} = \sigma$. (Note that no provision has been made to exclude constants χ_e, I_β, where e is a $\beta + 1$-index, $\beta \geqslant \gamma$. Those cases will not cause any difficulties.)

(b) $(\sigma \in \tau)^{(N_\gamma)} = (\sigma)^{(N_\gamma)} \in (\tau)^{(N_\gamma)}$, and $(N_\beta(\sigma))^{(N_\gamma)} = N_\beta((\sigma)^{(N_\gamma)})$ where σ, τ are terms or variables;

$$(\psi_1 \wedge \psi_2)^{(N_\gamma)} = (\psi_1)^{(N_\gamma)} \wedge (\psi_2)^{(N_\gamma)}, \qquad (\neg\psi)^{(N_\gamma)} = \neg(\psi)^{(N_\gamma)},$$

$$((\psi))^{(N_\gamma)} = (\psi)^{(N_\gamma)}, \qquad (\forall v)(\psi)^{(N_\gamma)} = (\forall v)(N_\gamma(v) \to (\psi)^{(N_\gamma)}),$$

$$(\forall_\lambda v)(\psi)^{(N_\gamma)} = (\forall_\lambda v)(N_\gamma(v) \to (\psi)^{(N_\gamma)}), \qquad \lambda \in M.$$

(c)

$$(K(\tau))^{(N_\gamma)} = K((\tau)^{(N_\gamma)}), \qquad (\theta(\tau))^{(N_\gamma)} = \theta((\tau^{(N_\gamma)}), \quad \theta \in \Gamma;$$

$$(\forall_\lambda v)(\psi)^{(N_\gamma)} = (\forall_\lambda v)(N_\gamma(v) \wedge (\psi)^{(N_\gamma)}) .$$

Note that since $\exists$ is defined in terms of $\forall$, we have

$$(\exists v)(\psi)^{(N_\gamma)} = (\exists v)(N_\gamma(v) \wedge \neg\neg(\psi)^{(N_\gamma)})$$

instead of the usual $(\exists v)(N_\gamma(v) \wedge (\psi)^{(N_\gamma)})$; similarly for the restricted existential quantifier.

The *stage*$_\alpha$, and *order*$_\alpha$ of terms, quantifiers and formulas are now defined by induction.

2.61. Definition. (a) Let σ be a term of $\mathcal{L}_\alpha$, and let γ be the minimal ordinal for which $\sigma \in \mathcal{T}_\gamma$.

(i) If abstraction symbols and quantifiers do not occur in σ, then $\mathrm{stg}_\alpha(\sigma) = \gamma$; otherwise,

(ii) if σ is not of the form $(\tau)^{(N_\beta)}$, $\beta < \alpha$, then $\mathrm{stg}_\alpha(\sigma) = \alpha$; otherwise,

(iii) assume that

$$\sigma = (\dots(\tau)^{(N_{\beta_1})})^{(N_{\beta_2})}\dots)^{(N_{\beta_n})} ,$$

where τ is such that for no ρ and β is $\tau = (\rho)^{(N_\beta)}$; then

$$\mathrm{stg}_\alpha(\sigma) = \max(\min_{1 \leq i \leq n}(\beta_i), \gamma) .$$

(b) *The stage*$_\alpha$ *of a quantifier in a formula* is defined as follows: Let $\varphi = (\forall_\lambda v)(\psi)$. If ψ is not of the form $N_\beta(v) \to \psi^*$, then $\mathrm{stg}_\alpha((\forall_\lambda v), \varphi)$ is α; otherwise assume

$$\psi = N_{\beta_1}(v) \to (N_{\beta_2}(v) \to (\dots \to (N_{\beta_n}(v) \to \psi^*)\dots)) ,$$

where for no β, ψ^{**} is $\psi^* = N_\beta(v) \to \psi^{**}$; then

$$\mathrm{stg}_\alpha((\forall_\lambda v), \varphi) = \min_{1 \leq i \leq n}(\beta_i) .$$

2.6101. Lemma. (a) *The stage*$_\alpha$ *of the terms*, $\mathring{s}$, $s \in M$; $a_i, \theta(a_i), K(a_i)$, *and* I_0 *is zero*, $i \in \omega, \theta \in \Gamma$.

(b) $\mathrm{stg}_\alpha(I_\beta) = \beta$, $\beta \leqslant \alpha$, *and* $\mathrm{stg}_\alpha(\chi_e) = \beta + 1$, *if* e *is a* $\beta + 1$*-index*, $\beta < \alpha$.

Proof. Immediate from the definition.

2.6102. Remark. Our definition of the stage$_\alpha$ of a term is somewhat artificial, since we are demanding a uniform relativization of the quantifiers. The natural thing to do would be to define by induction the stage of a formula φ as the maximum of the stages of the quantifiers and the terms occurring in φ, and define the stage of an abstraction term $\sigma = (\lambda_\lambda v)(\psi)$, as follows:

if for no β and ψ^* is $\psi = N_\beta(v) \wedge \psi^*$, then $\mathrm{stg}_\alpha(\sigma) = \alpha$;
otherwise let

$$\psi = N_{\beta_1}(v) \wedge N_{\beta_2}(v) \wedge \ldots \wedge N_{\beta_n}(v) \wedge \psi^*,$$

where for no β and ψ^{**} is $\psi^* = N_\beta(v) \wedge \psi^{**}$, then take

$$\mathrm{stg}_\alpha(\sigma) = \max(\min_{1 \leqslant \beta_i \leqslant n}(\beta_i), \mathrm{stg}_\alpha(\psi^*)) .$$

The definition we gave is simpler, and a more refined notion will not be necessary.

We now define the order$_\alpha$ of terms, quantifiers and formulas.

2.6103. Definition. (a) If $\tau \in \mathcal{T}_\alpha$, then

$$\mathrm{ord}_\alpha(\tau) = (\mathrm{stg}_\alpha(\tau), \mathrm{rnk}(\tau)) .$$

(b) If $\varphi = (\forall_\lambda v)\, \psi \in \mathcal{F}_\alpha$, then $\mathrm{ord}_\alpha((\forall_\lambda v), \varphi) = (\mathrm{stg}_\alpha((\forall_\lambda v), \varphi), \lambda)$ and is said to be the *order*$_\alpha$ *of* $(\forall_\lambda v)$ *in* φ. (An unranked quantifier is to have no order.)

(c) Let $\prec$ be the left-lexicographic well-ordering of all pairs of ordinals; we then define the order$_\alpha$ of a ranked formula $\varphi \in \mathcal{L}_\alpha$ as follows:

$\mathrm{ord}_\alpha(\varphi)$ is the least (λ, η) such that $(\lambda, \eta) \succneqq \mathrm{ord}_\alpha(\sigma)$ for all $\sigma \in \mathcal{T}_\alpha$ occurring in φ and

$$(\lambda, \eta) \succeq \mathrm{ord}_\alpha((\forall_\gamma v), (\forall_\gamma v)(\psi))$$

for all well formed parts $(\forall_\gamma v)(\psi)$ of φ.

2.6104. Remark. The stage$_\alpha$ and order$_\alpha$ of an existential quantifier are taken to be those of the universal quantifier in the corresponding expanded forms.

2.611. Definition. For all local formulas $\varphi \in \mathcal{L}_\alpha$ define:

$$\mathrm{rnk}_\alpha(\varphi) = (\mathrm{ord}_\alpha(\varphi), i(\varphi), l(\varphi))$$

where $i(\varphi)$ and $l(\varphi)$ are integers such that

(a) $i(\varphi) < 2$, and $i(\varphi) = 0$ if $\mathrm{ord}_\alpha(\varphi) = (\lambda, n+1)$, and

(i) $\mathrm{ord}_\alpha(\varphi)$ is larger than the order$_\alpha$ of quantifiers occurring in φ;

(ii) φ does not contain subformulas of the form $\sigma \in \tau$ where either $\eta = \mathrm{rnk}_\alpha(\sigma) \geq \mathrm{rnk}_\alpha(\tau)$, or $(\lambda, \eta) = \mathrm{ord}_\alpha(\sigma) \geq \mathrm{ord}_\alpha(\tau)$;

(iii) no subformula of the form $N_\gamma(\sigma)$ occurs in φ other than in an abstraction term; otherwise $i(\varphi) = 1$.

(b) $l(\varphi)$ is the length of φ, where individual constants and abstraction terms are considered to be of length one; the predicates $N_\gamma(\cdot)$ and function symbols $K(\cdot)$, $\theta(\cdot)$ are also of length one. (Thus, for example, $\theta((\mathsf{X}_\gamma v)(\varphi))$ is of length 2.)

2.62. Lemma. *The class of all* $\mathrm{rnk}_\alpha(\varphi)$, $\varphi \in \mathcal{F}_\alpha$ *is left-lexicographically well-ordered, (i.e. from left to right).*

2.621. Denote this ordering by $\prec_\alpha$.

2.622. Remark. Note that $\prec_\alpha$ is *not a well-founded relation,* where a well-founded relation is defined as follows.

2.623. Definition. (a) A binary relation $\langle A, R\rangle$ is said to be *left* (*right*) *narrow,* if for any $a \in A$ the class of all $x \in A$ such that $x\ R\ a$ ($a\ R\ x$) is a set.

(b) A binary relation $\langle A, R\rangle$ is said to be *well-founded* if

(i) for all $B \subseteq A$ there is an $x \in B$ such that for no $y \in B$ does $y\ R\ x$ hold (i.e., there are no infinite R-descending sequences);

(ii) R is left narrow.

However, in order to define $\Vdash_\alpha$ it will only be necessary to resort to a subrelation of $\prec_\alpha$, which will be shown in Lemma 2.65 to be well-founded.

A bounded equality relation will now be defined between constant terms that takes into account the stage of the terms as well as the rank.

2.63. Definition. If $\sigma, \tau \in \mathcal{T}^c_\alpha$, then $\sigma \approxeq \tau$ denotes the sentence

$$(\forall_\gamma u)\,(N_\gamma(u) \to (\omega \in \sigma \leftrightarrow u \in \tau))\ ,$$

where $\lambda = \max(\mathrm{rnk}(\sigma), \mathrm{rnk}(\tau))$ and $\gamma = \max(\mathrm{stg}_\alpha(\tau))$.

2.631. In Definition 2.63, note that the order$_\alpha$ of the quantifier in $\sigma \approxeq \tau$ equals $\max(\mathrm{ord}_\alpha(\sigma), \mathrm{ord}_\alpha(\tau))$; thus

$$\mathrm{ord}_\alpha(\sigma \approxeq \tau) = (\gamma, \lambda + 1)\ .$$

On the other hand note that

$$\mathrm{ord}_\alpha(\sigma \simeq \tau) = (\alpha, \lambda + 1)$$

(i.e. a possibly large increase of order$_\alpha$ over $\mathrm{ord}_\alpha(\sigma)$, $\mathrm{ord}_\alpha(\tau)$), where $\sigma \simeq \tau = (\forall_\lambda u)\,(u \in \sigma \leftrightarrow u \in \tau)$, which is the formula customarily used (see Condition 2.25(a) and [11]).

2.632. In the following definition we will be using terms like $(\langle \dot{j}, \dot{a}_i \rangle)^{(N_0)}$ instead of simply the term $\langle \dot{j}, \dot{a}_i \rangle$. This is because $\langle \dot{j}, \dot{a}_i \rangle$ is not of stage 0 while $\dot{j}$, $\dot{a}_i$ and $(\langle \dot{j}, \dot{a}_i \rangle)^{(N_0)}$ are. Clearly $(\langle \dot{j}, \dot{a}_i \rangle)^{(N_0)}_\sim$ will be equal to $\langle \dot{j}, \dot{a}_i \rangle_\sim \in N_0$, and it will follow from Lemma 3.02 that 0 forces$_\alpha$ the equality.

2.64. Definition. The weak forcing relation $\Vdash_\alpha$ between $p \in P_\alpha$ and sentences $\psi \in \mathcal{T}^c_\alpha$ is defined by induction on $\mathrm{rnk}_\alpha(\psi)$ as follows:

(a) $p \Vdash_\alpha \sigma \in \dot{s}$, where $\dot{s}$ is a set constant iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists t \in s)\,[q \Vdash_\alpha \sigma \approxeq \dot{t}\,]\ .$$

(b) $p \Vdash_\alpha \sigma \in a_i$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\exists \delta < 2)\,[q \Vdash_\alpha \sigma \approxeq (\langle \dot{j}, \dot{\delta} \rangle)^{(N_0)}$$
$$\text{and } \langle i, j, \delta \rangle \in q\,]\ .$$

(c) $p \Vdash_\alpha \sigma \in {}^{(t,l)}\theta(\tau)$, where $t \in 2^m$ iff

$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists k < \omega)\,(\exists \delta < 2)\,[q \Vdash_\alpha \sigma \approx (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)}$,

where either $k < m$ and $t(k) = \delta$,
or $k \geq m$, $j = k - m + l$ and
$q \Vdash_\alpha (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \in \tau]$.

(d) $p \Vdash_\alpha \sigma \in K(\tau)$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \theta \in \Gamma)\,[q \Vdash_\alpha \sigma \approx \theta(\tau)]\ .$$

(e) $p \Vdash_\alpha \sigma \in \chi_e$, $e = (h, g, s)$, iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \bar{\theta}^n \in \bar{\Gamma}^n)\,(\exists l < \omega)\,(\exists \delta < 2)$$
$$[q \Vdash_\alpha \sigma \approx (\langle\langle \bar{\theta} \cdot \bar{a}_h\rangle, \langle \dot{l}, \dot{\delta}\rangle\rangle)^{(N_0)} \text{ and } \langle e, \bar{\theta}^n, l, \delta\rangle \in q]\,.$$

(f) $p \Vdash_\alpha \sigma \in I_\beta$, $\beta \leq \alpha$, iff either
(i) $\beta = 0$ and $(\forall r \supseteq p)(\exists q \supseteq r)(\exists i < \omega)[q \Vdash_\alpha \sigma \approx (\langle \dot{i}, K(a_i)\rangle)^{(N_0)}]$,

or

(ii) $0 < \beta$ and $(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \chi_e \in \mathfrak{X}_\beta)[q \Vdash_\alpha \sigma \approx (\langle \dot{e}, \chi_e\rangle)^{(N_\beta)}]$.

(g) $p \Vdash_\alpha N_\gamma(\sigma)$, iff $\mathrm{stg}_\alpha(\sigma) \leq \gamma$ or $\mathrm{stg}_\alpha(\sigma) > \gamma$ and

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \tau \in \mathcal{T}^c_\gamma)\,[\mathrm{stg}_\alpha(\tau) \leq \gamma,\ \mathrm{rnk}(\tau) \leq \mathrm{rnk}(\sigma) + \aleph_1^{(M)} \cdot 2$$
$$\text{and } q \Vdash_\alpha \sigma \approx \tau]\,.$$

(Note that possibly $\mathrm{rnk}(\tau) > \mathrm{rnk}(\sigma)$. Also note that for all $\gamma \geq \alpha$ and $\sigma \in \mathcal{T}^c_\alpha$, we will have $0 \Vdash_\alpha N_\gamma(\sigma)$, (since $\mathrm{stg}_\alpha(\sigma) \leq \gamma$).)

(h) $p \Vdash_\alpha (\varphi)$ iff $p \Vdash_\alpha \varphi$.

(i) $p \Vdash_\alpha \varphi_1 \wedge \varphi_2$ iff $p \Vdash_\alpha \varphi_1$ and $p \Vdash_\alpha \varphi_2$.

(j) $p \Vdash_\alpha \neg\varphi$ iff $\neg(\exists q \supseteq p)\,[q \Vdash_\alpha \varphi]$.

(k) $p \Vdash_\alpha (\forall_\lambda v)\,(\varphi(v))$ iff

$$(\forall \sigma \in \mathcal{T}^c_\alpha)\,[\mathrm{rnk}_\alpha(\sigma) < \lambda \text{ and } \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha((\forall_\lambda v), (\forall_\lambda v)\,(\varphi))$$
$$\text{implies } p \Vdash_\alpha \varphi(\sigma)]\,.$$

(l) $p \Vdash_\alpha \sigma \in \tau$, where $\tau = (\lambda_\lambda u)\,(\varphi(u))$, $\mathrm{rnk}(\sigma) < \lambda$ and $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\tau)$, iff $p \Vdash_\alpha \varphi(\sigma)$.

(m) $p \Vdash_\alpha \sigma \in \tau$, where τ is an abstraction term and either $\mathrm{rnk}(\sigma) \geqslant \mathrm{rnk}(\tau)$ or $\mathrm{ord}_\alpha(\sigma) \geqslant \mathrm{ord}_\alpha(\tau)$, iff

$$(\forall r \supseteq p)(\exists q \supseteq r)(\exists \rho \in \mathcal{T}^c_\alpha[\mathrm{rnk}_\alpha(\rho) < \mathrm{rnk}(\tau), \mathrm{ord}_\alpha(\rho) < \mathrm{ord}_\alpha(\tau),$$
$$\text{and } q \Vdash_\alpha \sigma \approxeq \rho \wedge \rho \in \tau].$$

2.65. Lemma. *$\Vdash_\alpha \subseteq P_\alpha \times \mathcal{T}^c_\alpha$ is a well-defined relation.*

Proof. It first has to be shown that $\Vdash_\alpha$ is being defined with respect to a well-founded relation in M (see Definition 2.623). To see this, observe that the class of all pairs

$$\langle\langle(\alpha_1, \beta_1), i_1, l_1\rangle, \langle(\alpha_2, \beta_2), i_2, l_2\rangle\rangle$$

in $\prec_\alpha$ (see 2.621) such that

$$\alpha_1, \alpha_2 \leqslant \aleph_1^{(M)}, \qquad \beta_1 < \beta_2 + \aleph_1^{(M)} \cdot 2$$

suffices for our definition. Denote this class by $\prec^*_\alpha$. It is immediately seen that $\prec^*_\alpha$ is well-founded in M.

2.651. In this respect it will be convenient to have the following well-founded order: $(\alpha_1, \beta_1) \prec^* (\alpha_2, \beta_2)$ iff either $\alpha_1 = \alpha_2$ and $\beta_1 \leqslant \beta_2$, or $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2 + \aleph_1^{(M)} \cdot 2$.

Henceforth, we will generally write $\mathrm{ord}_\alpha(\sigma) \prec^* \mathrm{ord}_\alpha(\tau)$ instead of $\mathrm{ord}_\alpha(\sigma) \leqslant \mathrm{ord}_\alpha(\tau)$ $(\mathrm{stg}_\alpha(\sigma) \leqslant \mathrm{stg}_\alpha(\tau))$ and $\mathrm{rnk}(\sigma) < \mathrm{rnk}(\tau) + \aleph_1^{(M)} \cdot 2$.

The proof that Definition 2.64 is recursive amounts to a direct check of the different cases in the forcing definition.

(a) If $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\dot{s})$, then also $\mathrm{rnk}(\sigma) < \mathrm{rnk}(\dot{s})$, because $\mathrm{stg}_\alpha(\dot{s}) = 0$; therefore we obtain a reduction in $\mathrm{ord}_\alpha(\psi)$. Otherwise there is a reduction of $i(\psi)$ with no increase of $\mathrm{ord}_\alpha(\psi)$, (see Remark 2.631).

(b) $\mathrm{stg}_\alpha(a_i) = 0$; thus, if $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(a_i)$, then there is a reduction in $\mathrm{ord}_\alpha(\psi)$, (since we also have $\mathrm{stg}_\alpha((\langle j, \delta\rangle)^{(N_0)} = 0)$; and if $\mathrm{ord}_\alpha(\sigma) \geqslant \mathrm{ord}_\alpha(a_i)$, then there is a reduction in $i(\psi)$ without an increase in $\mathrm{ord}_\alpha(\psi)$ (see also Definition 2.21(f)).

(c) If $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha({}^{(t,l)}\theta(\tau))$, then, since $\omega \leqslant \mathrm{rnk}({}^{(t,l)}\theta(\tau))$, we have $\mathrm{ord}_\alpha(\sigma \approxeq (\langle k, \delta\rangle)^{(N_0)}) < \mathrm{ord}_\alpha(\sigma \in {}^{(t,l)}\theta(\tau))$; otherwise there is a reduction in $i(\psi)$ with no increase in $\mathrm{ord}_\alpha(\psi)$. With regard to the remaining formula, consider the following cases: if $\mathrm{rnk}(\tau) < \omega$, then

$$\mathrm{ord}_\alpha((\langle\dot{j}, \dot{\delta}\rangle)^{(N_0)} \in \tau) < \mathrm{ord}_\alpha(\sigma \in {}^{(t,l)}\theta(\tau)) ,$$

(since also $\mathrm{stg}_\alpha(\theta(\tau)) = \mathrm{stg}_\alpha(\tau)$); if $\mathrm{rnk}(\tau) \geqslant \omega$, then, regardless of $\mathrm{ord}_\alpha(\sigma)$, we have

$$\mathrm{ord}_\alpha((\langle\dot{j}, \dot{\delta}\rangle)^{(N_0)} \in \tau) \leqslant \mathrm{ord}_\alpha(\sigma \in {}^{(t,l)}\theta(\tau)) ,$$

$$i((\langle\dot{j}, \dot{\delta}\rangle)^{(N_0)} \in \tau) \leqslant i(\sigma \in {}^{(t,l)}\theta(\tau)) ,$$

$$3 = l((\langle\dot{j}, \dot{\delta}\rangle)^{(N_0)} \in \tau) \nleqslant l(\sigma \in {}^{(t,l)}\theta(\tau)) = 4 .$$

These are all the possibilities.

(d) If $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(K(\tau))$ and $\mathrm{rnk}(\sigma) < \mathrm{rnk}(K(\tau))$, then there is a reduction in $\mathrm{ord}_\alpha(\psi)$ (see Definitions 2.21(f) and 2.611(ii)); otherwise, there is a reduction in $i(\psi)$.

(e) If $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\chi_e)$ and $\mathrm{rnk}(\sigma) < \mathrm{rnk}(\chi_e)$, then there is a reduction in the order_α of the formula; otherwise there is a reduction in $i(\psi)$ with no increase of $\mathrm{ord}_\alpha(\psi)$, (see Definition 2.21(f)).

(f) Case (i) is seen by considerations analogous to those of (e). To see case (ii), observe that

$$\mathrm{rnk}(\dot{e}) \leqslant \omega\cdot\beta < \aleph_1 < \mathrm{rnk}(I_\beta) ,$$

(by 2.56), and $\mathrm{stg}_\alpha(I_\beta) = \beta$; thus, if $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(I_\beta)$ and $\mathrm{rnk}(\sigma) < \mathrm{rnk}(I_\beta)$, there is a reduction of $\mathrm{ord}_\alpha(\psi)$; otherwise there is a reduction of $i(\psi)$.

(g) There is *no increase* in $\mathrm{ord}_\alpha(\psi)$; but there is a reduction of $i(\psi)$.

(h), (i) and (j) involve a direct reduction of length with no increase of the other parameters.

(k) Here we also have a reduction in length. Observe that our restrictions inhibit an increase of $\mathrm{ord}(\psi)$; moreover the value of $i(\psi)$ does not increase.

(l) In this case there is an immediate reduction of $\mathrm{ord}_\alpha(\psi)$.

(m) Here we have an immediate reduction of $i(\psi)$ while $\mathrm{ord}_\alpha(\psi)$ remains constant.

2.66. Definition. $\Vdash_\alpha$ is extended to include global sentences, by induction on the length of formulas as follows:

(a) $p \Vdash_\alpha (\forall u)(\varphi(u))$ iff $(\forall \sigma \in \mathcal{T}^c_\alpha)\ [p \Vdash_\alpha \varphi(\sigma)]$.

(b) $p \Vdash_\alpha \varphi \wedge \psi$ iff $p \Vdash_\alpha \varphi$ and $p \Vdash_\alpha \psi$.
(c) $p \Vdash_\alpha \neg\varphi$ iff $\neg(\exists q \supseteq p)\ [q \Vdash_\alpha \varphi]$.
(d) $p \Vdash_\alpha (\varphi)$ iff $p \Vdash_\alpha \varphi$.

We have considered all ways of generating global sentences from atomic sentences, and since all cases involve a reduction in length we have:

2.67. Lemma. *$\Vdash_\alpha \subseteq P_\alpha \times \mathcal{F}^c_\alpha$ is a well-defined relation on $P_\alpha \times \mathcal{F}^c_\alpha$.*

The following basic forcing lemmas are immediate from the definition, (see [2]).

2.68. Lemma *$p \Vdash_\alpha \varphi$ and $p \subseteq q \in P_\alpha$ imply $q \Vdash_\alpha \varphi$.*

2.69. Lemma. *For all $p \in P_\alpha$ and $\varphi \in \mathcal{F}^c_\alpha$, $p \not\Vdash_\alpha \varphi \wedge \neg\varphi$.*

2.70. Lemma. *For all $p \in P_\alpha$ and $\varphi \in \mathcal{F}^c_\alpha$, either $p \Vdash_\alpha \neg\varphi$ or there exists $q \supseteq p$, $q \Vdash_\alpha \varphi$.*

2.71. Definition. If $p \Vdash_\alpha \varphi$ or $p \Vdash_\alpha \neg\varphi$, we say p *decides*$_\alpha$ φ and write $p \parallel_\alpha \varphi$.

It is easily shown that our various induction hypotheses hold for $\beta = \alpha$. The following lemma shows that those of 2.42 hold; however we delay the proof till we have all elementary lemmas concerning forcing$_\alpha$ needed. (All the lemmas that will be needed in the proof will follow from the forcing definition for *fixed* α; i.e. not from other lemmas dealt with which concern *all ordinals* $\leqslant \aleph_1^{(M)}$. Therefore no circularity arises.)

2.72. Lemma. *If $\sigma = \sigma(\bar{b}), \tau(\bar{b}) \in \mathcal{F}^c_\alpha$, and $s^\alpha_\tau = s^\alpha_\sigma$, then $0 \Vdash_\alpha H_\sigma = H_\tau$; and if $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta} \in \bar{\Gamma}$, then*

$$0 \Vdash_\alpha H_\sigma(\bar{\theta} \cdot \bar{b}) = H_\tau(\bar{\theta} \cdot \bar{b})\,,$$

$$p \Vdash_\alpha H_\sigma(\bar{\theta}_1 \cdot \bar{b}) = H_\sigma(\bar{\theta}_2 \cdot \bar{b}) \quad \textit{iff} \quad p \Vdash_\alpha H_\tau(\bar{\theta}_1 \cdot \bar{b}) = H_\tau(\bar{\theta}_2 \cdot \bar{b});$$

$$p \Vdash_\alpha H_\sigma(\bar{\theta}_1 \cdot \bar{b}) \neq H_\sigma(\bar{\theta}_2 \cdot \bar{b}) \quad \textit{iff} \quad p \Vdash_\alpha H_\tau(\bar{\theta}_1 \cdot \bar{b}) \neq H_\tau(\bar{\theta}_2 \cdot \bar{b}).$$

2.721. Convention. On many occasions throughout this paper, when dealing with an $\alpha+1$-index $e = (h, g, s)$, we will have to refer to some $\sigma \in \mathcal{T}^c_\alpha$ such that $s = s^\alpha_\sigma$. In view of Lemma 2.72 it will be irrelevant which σ is chosen, and to fix things we shall always assume that σ is the first term of $\mathcal{T}^c_\alpha$, according to the canonic constructible well-ordering of M, such that $s = s^\alpha_\sigma$.

The hypotheses in 2.56 are now verified for $\beta = \alpha$.

2.73. Lemma. (a) $\mathcal{F}_\alpha \cup \mathcal{T}_\alpha \subseteq M$,

(b) $P_\alpha \subseteq R^{(M)}(\omega\cdot\alpha) \in M$,

(c) *if e is an $\alpha+1$-index, then* $\operatorname{Rank}(e) \leqslant \omega(\alpha+1)$.

Proof. If α is a limit ordinal, then, since

$$\mathcal{X}_\alpha = \bigcup_{\beta<\alpha} \mathcal{X}_\beta, \qquad \mathcal{G}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\beta,$$

it follows from the induction hypothesis and the definition of $\mathcal{L}_\alpha$ that

$$\mathcal{F}_\alpha \cup \mathcal{T}_\alpha = \bigcup_{\beta<\alpha} (\mathcal{F}_\beta \cup \mathcal{T}_\beta) \subseteq M .$$

Also,

$$P_\alpha = \bigcup_{\beta<\alpha} P_\beta \subseteq \bigcup_{\beta<\alpha} R^{(M)}(\omega\cdot\beta) = \bigcup_{\beta<\alpha} (R(\omega\cdot\beta) \cap M)$$

$$= (\bigcup_{\beta<\alpha} R(\omega\cdot\beta)) \cap M = R(\omega\cdot\alpha) \cap M = R^M(\omega\cdot\alpha) .$$

If α is a successor ordinal, then by assumption 2.56(c) all α-indices constitute a set in M; hence $\mathcal{X}_\alpha \in M$. It follows that all primitive symbols of $\mathcal{L}_\alpha$ and all finite strings of symbols are in M. Therefore $\mathcal{F}_\alpha \cup \mathcal{T}_\alpha \subseteq M$, and $P_\alpha \in M$. Since for an α-index e we are assuming $e \in R^{(M)}(\omega\cdot\alpha)$, it follows that also

$$P_\alpha \in R^{(M)}(\omega\cdot\alpha) = R(\omega\cdot\alpha) \cap M ,$$

(because $\omega\cdot\alpha$ is a limit ordinal). It now has to be shown that if e is an $\alpha+1$-index, then $\operatorname{Rank}(e) \leqslant \omega(\alpha+1)$. This is seen as follows: first it is observed that $\Vdash_\alpha$ is absolute, since the definition of $\Vdash_\alpha$ above could be given in the extended language of set theory by a formula

$$\Pi(w, w', u, v)$$

such that for all $x, y \in M$,

$$\Pi^{(M)}(\mathcal{X}_\alpha \cup \mathcal{G}_\alpha, P_\alpha, x, y) \quad \text{iff} \quad x \in P_\alpha,\ y \in \mathcal{G}^c_\alpha$$

is a ranked formula, and $x \Vdash_\alpha y$, (where $\Pi^{(M)}$ is the relativization of Π to M). (A detailed proof of this is a routine matter and is omitted here; see Lévy [11].) The notion of a "minimal condition for φ" is obviously absolute, and, since $\Vdash_\alpha$ is absolute, it follows that

$$\|\varphi\|_\alpha = \|\varphi\|_\alpha^{(M)} \in M\ ;$$

moreover we have

$$P_\alpha \supseteq \|\varphi\|_\alpha \subseteq R^{(M)}(\omega \cdot \alpha)\ .$$

Also $[\pm\sigma, \bar\theta_1, \bar\theta_2]_\alpha$, $[\sigma, \bar\theta_1, \bar\theta_2]_\alpha$, s^α_σ, $\alpha+1$-indices, symbols of $\mathcal{X}_{\alpha+1}$, and $P_{\alpha+1}$ are absolute and are easily seen to be in

$$R^{(M)}(\omega \cdot \alpha + \omega) = R^{(M)}(\omega(\alpha+1))\ .$$

Hence, $\mathrm{Rank}(e) \leqslant \omega\cdot(\alpha+1)$ for an $\alpha+1$-index, as was to be shown.

The definition of $\mathcal{L}_\alpha, P_\alpha, \Vdash_\alpha$ together with all auxiliary terms and notions has been completed for $\alpha \leqslant \aleph_1^{(M)}$.

Reasoning along the lines of Lemma 2.73, we can devise a formal proof of the standard absoluteness lemmas needed to show the existence of a generic model for ZF. We state these lemmas without proof which run along standard lines up to obvious details.

2.74. Lemma. (a) *There is a formula $\Pi(w, u, v)$ of the extended language of set theory such that for all $z, x, y \in M$, $\Pi^{(M)}(z, x, y)$ iff z is an ordinal, $z \leqslant \aleph_1^{(M)}$, $x \in P_z$, y is a local sentence of $\mathcal{L}_z$ and $x \Vdash_z y$.*

(b) *For every global sentence $\varphi(u_1, ..., u_m)$ of $\mathcal{L}_\alpha$, $\alpha \leqslant \aleph_1^{(M)}$, not containing constants, there is a formula*

$$\Pi_\varphi(u_1, ..., u_m, v, w)$$

in the M-language such that for all terms $\sigma_1, ..., \sigma_m \in \mathcal{Q}_\alpha$, and $x \in M$, M satisfies

$$\Pi_\varphi(\sigma_1, ..., \sigma_m, \alpha, x) \quad \text{iff} \quad x \Vdash_\alpha \varphi(\sigma_1, ..., \sigma_m)\ .$$

The following lemmas are well known from the theory of forcing (Lévy [11]) and will be used frequently in the sequel.

2.75. Lemma. (a) $p \Vdash_\alpha \varphi \vee \psi$ *iff* $(\forall r \supseteq p)(\exists q \supseteq r)[q \Vdash_\alpha \varphi$ *or* $q \Vdash_\alpha \psi]$.

(b) $p \Vdash_\alpha \varphi \to \psi$ *iff* $(\forall r \supseteq p)(\exists q \supseteq r)[q \Vdash_\alpha \neg\varphi$ *or* $q \Vdash_\alpha \psi]$.

(c) $p \Vdash_\alpha \varphi \leftrightarrow \psi$ *implies* $[p \Vdash_\alpha \varphi$ *iff* $p \Vdash_\alpha \psi]$.

(d) $p \Vdash_\alpha (\exists u)(\varphi(u))$ *iff* $(\forall r \supseteq p)(\exists q \supseteq r)(\exists \sigma \in \mathcal{L}_\alpha)[q \Vdash_\alpha \varphi(\sigma)]$.

(e) $p \Vdash_\alpha (\exists_\lambda u)(\varphi)$ *iff* $(\forall r \supseteq p)(\exists q \supseteq r)(\exists \sigma \in \mathcal{L}_\alpha)$

$$[\mathrm{rnk}(\sigma) < \lambda, \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\exists_\lambda u, \exists_\lambda u\varphi), q \Vdash_\alpha \varphi(\sigma)].$$

(f) $p \Vdash_\alpha \neg\neg\neg\varphi$ *iff* $p \Vdash_\lambda \neg\varphi$. (*It will be shown that even* $p \Vdash_\alpha \varphi$ *iff* $p \Vdash_\alpha \neg\neg\varphi$.)

2.76. Lemma. *For all* $p_\alpha \in P$, $\varphi, \psi \in \mathcal{T}^c_\alpha$, *if* $p \Vert_\alpha \varphi$ *and* $p \Vert_\alpha \psi$, *then*:

(a) $p \Vert_\alpha \neg\varphi$, $p \Vert_\alpha \varphi \wedge \psi$, $p \Vert_\alpha \varphi \vee \psi$, $p \Vert_\alpha \varphi \to \psi$, $p \Vert_\alpha \varphi \leftrightarrow \psi$;

(b) $p \Vdash_\alpha \varphi \vee \psi$ *iff* $[p \Vdash_\alpha \varphi$ *or* $p \Vdash_\alpha \psi]$;

(c) $p \Vdash_\alpha \varphi \leftrightarrow \psi$ *iff* $[p \Vdash_\alpha \varphi$ *iff* $p \Vdash_\alpha \psi]$.

If C is an n-ary sentential connective of the language of set theory (i.e. an operation which is an iteration of the primitive sentential connectives (hence any of the aforementioned sentential connectives)), and $\dot{C}$ is the corresponding sentential connective of $\mathcal{L}_\alpha$, then:

2.77. Lemma. (a) $p \Vert_\alpha \varphi_i$, $1 \leqslant i \leqslant n$, implies $p \Vert_\alpha \dot{C}(\varphi_1, ..., \varphi_n)$ *and*

$$p \Vdash_\alpha \dot{C}(\varphi_1, ..., \varphi_n) \quad \textit{iff} \quad C(p \Vdash_\alpha \varphi_1, ..., p \Vdash_\alpha \varphi_n)\,.$$

(b) *If* $C(\Phi_1, ..., \Phi_n)$ *is a tautology for all* $\Phi_1, ..., \Phi_n$, *then for all* $\varphi_1, ..., \varphi_n, \varphi_i$ *a sentence of* $\mathcal{L}_\alpha$, $1 \leqslant i \leqslant n$,

$$0 \Vdash_\alpha \dot{C}(\varphi_1, ..., \varphi_n)\,.$$

Since we have defined a weak forcing relation, we also have the following lemma:

2.78. Lemma. $p \Vdash_\alpha \varphi$ *iff* $p \Vdash_\alpha \neg\neg\varphi$, $\alpha \leqslant \aleph_1^{(M)}$.

Proof. The equivalence $p \Vdash_\alpha \neg\varphi$ iff $\Vdash_\alpha \neg\neg\neg\varphi$ was immediate from the forcing definition. The present equivalence follows by induction on $\mathrm{rnk}_\alpha(\varphi)$. Clearly $p \Vdash_\alpha \varphi$ implies $p \Vdash_\alpha \neg\neg\varphi$ and it remains to prove that $p \Vdash_\alpha \neg\neg\varphi$ implies $p \Vdash_\alpha \varphi$. We consider the different cases in Definition 2.54.

(a) $\varphi = \sigma \in \dot{s}$, $\dot{s}$ a set constant, then $p \Vdash_\alpha \neg\neg\varphi$ implies

$$(\forall p' \supseteq p)\,(\exists p'' \supseteq p')\,[p'' \Vdash_\alpha \varphi]\,;$$

hence let $r \supseteq p$ imply $(\exists p' \supseteq r)\,(p' \Vdash_\alpha \sigma \in \dot{s})$ implies

$$(\exists q \supseteq p')\,(\exists s' \in s)\,[q \Vdash_\alpha \dot{s}' \approx \sigma]\,, \quad q \supseteq r \supseteq p,$$

therefore $p \Vdash_\alpha \sigma \in \dot{s}$.

(b) If $\varphi = \sigma \in a_i$ and $p \Vdash_\alpha \neg\neg\varphi$ as before let $r \supseteq p$, then

$$(\exists p' \supseteq r)\,[p' \Vdash_\alpha \sigma \in a_i]\;;$$

implies

$$(\exists q \supseteq p')\,(\exists j < \omega)\,(\exists \delta < 2)\,[q \Vdash_\alpha \sigma \approx (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \text{ and } \langle i, j, \delta\rangle \in q]$$

implies

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\exists \delta < 2)\,[q \Vdash_\alpha \sigma \approx (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \text{ and } \langle i, j, \delta\rangle \in q]$$

implies

$$p \Vdash_\alpha \sigma \in a_i\,.$$

All cases (a) through (g) are shown in a completely analogous manner. Case (h) is trivial.

(i) $\varphi = \psi_1 \wedge \psi_2$ and $p \Vdash_\alpha \neg\neg\varphi$ implies

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,[q \Vdash_\alpha \psi_1 \wedge \psi_2]$$

implies $q \Vdash_\alpha \psi_1$ and $q \Vdash_\alpha \psi_2$, therefore

$$p \Vdash_\alpha \neg\neg\psi_1 \quad \text{and} \quad p \Vdash_\alpha \neg\neg\psi_2\,;$$

by the induction hypothesis we get

$$p \Vdash_\alpha \psi_1 \quad \text{and} \quad p \Vdash_\alpha \psi_2 \quad \text{implies} \quad p \Vdash_\alpha \psi_1 \wedge \psi_2\,.$$

(j) If $\varphi = \neg\psi$ and $p \Vdash_\alpha \neg\neg\varphi$, then by Lemma 2.75(f)

$$p \Vdash_\alpha \neg\neg\neg\psi \quad \text{implies} \quad p \Vdash_\alpha \neg\psi\,.$$

(k) If $\varphi = (\forall_\lambda v)(\psi(v))$ and $p \Vdash_\alpha \neg\neg\varphi$, then

$$(\forall r \supseteq p)(\exists q \supseteq r)$$
$$(\forall \sigma \text{ such that } \mathrm{rnk}(\sigma) < \lambda \text{ and } \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\forall_\lambda, v, \varphi))$$
$$[q \Vdash_\alpha \psi(\sigma)] \ ,$$

i.e., $p \Vdash_\alpha \neg\neg\psi(\sigma)$, and by the induction hypothesis $p \Vdash_\alpha \psi(\sigma)$. We have shown that for all $\sigma \in \mathcal{T}_\alpha$ with $\mathrm{rnk}(\sigma) < \lambda$ and $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\forall_\lambda v, \varphi)$ that $p \Vdash_\alpha \psi(\sigma)$; therefore $p \Vdash_\alpha (\forall_\lambda v)(\psi(v))$.

(l) $\varphi = \sigma \in \tau$, where $\tau = (\lambda_\lambda u)(\psi(u))$, $\mathrm{rnk}(\sigma) < \lambda$, and $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\tau)$; then $p \Vdash_\alpha \neg\neg\varphi$ implies

$$(\forall r \supseteq p)(\exists q \supseteq r)[q \Vdash_\alpha \psi(\sigma)] ;$$

which implies $p \Vdash_\alpha \neg\neg\psi(\sigma)$, and by the induction hypothesis $p \Vdash_\alpha \psi(\sigma)$; which implies $p \Vdash_\alpha \sigma \in \tau$.

(m) This case is shown in a manner analogous to case (a).

For arbitrary sentences of $\mathcal{L}_\alpha$ the proof proceeds by induction on length of formulas in a manner similar to that used in the corresponding local cases.

2.79. Corollary. *If $p \nVdash_\alpha \varphi$, then $(\exists q \supseteq p)(q \Vdash_\alpha \neg\varphi)$.*

Proof. By Lemma 2.78, $p \nVdash_\alpha \varphi$ iff $p \nVdash_\alpha \neg\neg\varphi$; hence there is a $q \supseteq p$ such that $q \Vdash_\alpha \neg\varphi$ as required.

2.80. Lemma. *For all $\sigma, \tau \in \mathcal{T}^c_\alpha$,*

(a) $0 \Vdash_\alpha \sigma = \sigma$,

(b) $p \Vdash_\alpha \sigma = \tau$ *implies* $p \Vdash_\alpha \tau = \sigma$,

(c) $p \Vdash_\alpha \rho = \sigma \wedge \sigma = \tau$ *implies* $p \Vdash_\alpha \rho = \tau$,

(d) $p \Vdash_\alpha \sigma = \tau$ *iff* $p \Vdash_\alpha \sigma \simeq \tau$, (see Lévy [11]).

2.801. Lemma. (a) $0 \Vdash_\alpha \sigma \approx \sigma$,

(b) $p \Vdash_\alpha \sigma \approx \tau$ *implies* $p \Vdash_\alpha \tau \approx \sigma$,

(c) $0 \Vdash_\alpha N_\beta(\sigma)$, *for all σ such that* $\mathrm{stg}_\alpha(\sigma) \leqslant \beta$.

(d) $p \Vdash_\alpha N_\beta(\sigma)$ *implies that* $p \Vdash_\alpha N_\gamma(\sigma)$ *for all* $\gamma \geqslant \beta$.

These are immediate from the definition. The transitivity of $\approx$ is shown in Lemma 2.811.

2.81. Lemma. *If* $p \Vdash_\alpha \sigma \in \tau$, *then for all* $r \supseteq p$ *there is some* $q \supseteq r$ *and some* $\sigma' \in \mathcal{T}^c_\alpha$ *such that*

$$\mathrm{rnk}(\sigma') < \mathrm{rnk}(\tau), \qquad \mathrm{ord}_\alpha(\sigma') < \mathrm{ord}_\alpha(\tau),$$

$$q \Vdash_\alpha \sigma' \approx \sigma \wedge \sigma' \in \tau .$$

Proof. If $\mathrm{rnk}(\sigma) < \mathrm{rnk}(\tau)$ and $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\tau)$, take $\sigma' = \sigma$; otherwise consider the different τ. If τ is an individual constant, then the result follows immediately from Definition 2.50 (cases (a), (b), (e) and (f)). If τ is an abstraction term, then the result is an immediate consequence of case (m) of Definition 2.50. There remain cases (c) and (d). Consider first case (c), i.e.,

$$p \Vdash_\alpha \sigma \in {}^{(t,l)}\theta(\rho), \qquad t \in 2^m .$$

Let $r \supseteq p$; then for some $q \supseteq r$ and $k < \omega$, $\delta < 2$,

$$q \Vdash_\alpha \sigma \approx (\langle \dot{K}, \dot{\delta} \rangle)^{(N_0)} .$$

The result now follows from the fact that

$$\mathrm{rnk}((\langle \dot{K}, \dot{\delta} \rangle)^{(N_0)}) < \omega \leqslant \mathrm{rnk}({}^{(t,l)}\theta(\rho)) .$$

Case (d) is also similar, for if $p \Vdash_\alpha \sigma \in K(\rho)$ and for some $q \supseteq r \supseteq p$, $\theta \in \Gamma$, $q \Vdash_\alpha \sigma \approx \theta(\rho)$, then $q \Vdash_\alpha \theta(\rho) \in K(\rho)$; and since $\mathrm{rnk}(\theta(\rho)) < \mathrm{rnk}(K(\rho))$, (see Definition 2.21(f)), and $\mathrm{stg}_\alpha(\theta(\rho)) = \mathrm{stg}_\alpha(K(\rho))$, the result also follows in this case.

2.811. Lemma.

(a) $p \Vdash_\alpha \sigma \approx \tau \wedge \tau \approx \rho$ *implies* $p \Vdash_\alpha \sigma \approx \rho$.

(b) $p \Vdash_\alpha \sigma \in \tau \wedge \sigma \approx \rho$ *implies* $p \Vdash_\alpha \rho \in \tau$.

(c) $p \Vdash_\alpha \sigma \in \tau \wedge \tau \approx \rho$ *implies* $p \Vdash_\alpha \sigma \in \rho$.

(d) $p \Vdash_\alpha N_\beta(\sigma) \wedge \sigma \approx \tau$ *implies* $p \Vdash_\alpha N_\beta(\tau)$.

(e) $p \Vdash_\alpha \varphi(\sigma) \wedge \sigma \approx \tau$ *implies* $p \Vdash_\alpha \varphi(\tau)$.

Proof*. We prove this lemma by a multiple induction on

$$\max(\mathrm{ord}_\alpha(\sigma), \mathrm{ord}_\alpha(\tau), \mathrm{ord}_\alpha(\rho), \mathrm{ord}_\alpha(\varphi)) = (\mu, \xi).$$

(a) Assuming

$$p \Vdash_\alpha (\forall_\lambda u)(N_\gamma(u) \to (\omega \in \sigma \leftrightarrow \omega \in \tau))$$
$$\wedge (\forall_\eta v)(N_\beta(v) \to (v \in \tau \leftrightarrow v \in \rho)),$$

where

$$\lambda = \max(\mathrm{rnk}(\sigma), \mathrm{rnk}(\tau)), \qquad \eta = \max(\mathrm{rnk}(\tau), \mathrm{rnk}(\rho)),$$

$$\gamma = \max(\mathrm{stg}_\alpha(\sigma), \mathrm{stg}_\alpha(\tau)), \quad \beta = \max(\mathrm{stg}_\alpha(\tau), \mathrm{stg}_\alpha(\rho)),$$

it has to be shown that $\sigma \approx \rho$. Let r, σ' be such that $\mathrm{rnk}(\sigma') < \lambda$, $\mathrm{ord}_\alpha(\sigma') < (\gamma, \lambda)$, $r \supseteq p$ and $r \Vdash_\alpha \sigma' \in \sigma$. Let $r' \supseteq r$ be such that $r \parallel_\alpha \sigma' \in \tau$ and $r' \parallel_\alpha \sigma' \in \rho$. By Lemma 2.801(c), $r \Vdash_\alpha N_\gamma(\sigma')$; hence $r' \Vdash_\alpha \sigma' \in \tau$. If $\eta \geqslant \lambda$ and $\beta \geqslant \gamma$, we immediately get $r' \Vdash_\alpha \sigma' \in \rho$. If not, then by Lemma 2.81 there exists a $\sigma'' \in \mathcal{T}_\alpha^c$ with $\mathrm{rnk}(\sigma'') < \mathrm{rnk}(\rho)$, $\mathrm{ord}_\alpha(\sigma'') < \mathrm{ord}_\alpha(\rho)$ and $r'' \supseteq r'$ such that

$$r'' \Vdash_\alpha \sigma'' \approx \sigma' \wedge \sigma'' \in \tau .$$

Then $r'' \Vdash_\alpha N_\beta(\sigma'')$. We can also assume that $r'' \parallel_\alpha \sigma'' \in \rho$; therefore $r'' \Vdash_\alpha \sigma'' \in \tau$ implies

$$r'' \Vdash_\alpha \sigma'' \in \rho \wedge \sigma'' \approx \sigma' .$$

If ρ is an abstraction term, say $\rho = (\lambda_\xi w)(\psi(w))$, then we have

$$r'' \Vdash_\alpha \psi(\sigma'') \wedge \sigma'' \approx \sigma';$$

and since $\max(\mathrm{ord}_\alpha(\sigma'), \mathrm{ord}_\alpha(\sigma''), \mathrm{ord}_\alpha(\psi(\sigma''))) < (\mu, \xi)$, we have by the induction hypothesis that $r'' \Vdash_\alpha \psi(\sigma')$; which implies $r'' \Vdash_\alpha \sigma' \in \rho$. If ρ is an individual constant, say $\rho = a_i$, then $r'' \Vdash_\alpha \sigma'' \in \rho$; implies that, for some $j < \omega$, $\delta < 2$, $q \supseteq r''$, that

$$q \Vdash_\alpha \sigma'' \approx (\langle \dot{j}, \dot{\delta} \rangle)^{(N_0)} \qquad \langle i, j, \delta \rangle \in q .$$

Thus since $\max(\mathrm{ord}_\alpha(\sigma'), \mathrm{ord}_\alpha(\sigma''), \mathrm{ord}_\alpha((\langle \dot{j}, \dot{\delta} \rangle)^{(N_0)})) < (\mu, \xi)$, we have by the induction hypothesis that

$$q \Vdash_\alpha \sigma' \approx (\langle \dot{j}, \dot{\delta} \rangle)^{(N_0)} ;$$

which implies that $q \Vdash_\alpha \sigma' \in \rho$. Similarly for the other individual and function constants. By a symmetric argument we can show that if $p \subseteq r \subseteq r' \Vdash_\alpha \sigma' \in \rho$, then for some $q \supseteq r'$, $q \Vdash_\alpha \sigma' \in \sigma$. Thus $p \Vdash_\alpha \sigma \approx \rho$ has been shown.

(b) is now obtained by a direct application of (a).

(c) Let

$$p \Vdash_\alpha \sigma \in \tau \wedge (\forall_\lambda u)\,(N_\gamma(u) \to (u \in \tau \leftrightarrow u \in \rho))\,,$$

where $\lambda = \max(\mathrm{rnk}(\tau), \mathrm{rnk}(\rho))$, $\gamma = \max(\mathrm{stg}_\alpha(\tau), \mathrm{stg}_\alpha(\rho))$, and let $p' \supseteq p$. Then by Lemma 2.81 for σ' with $\mathrm{rnk}(\sigma') < \mathrm{rnk}(\tau)$, $\mathrm{ord}_\alpha(\sigma') < \mathrm{ord}_\alpha(\tau) \leqslant (\gamma, \lambda)$ and some $r \supseteq p'$, we have

$$r \Vdash_\alpha \sigma \lessapprox \sigma' \wedge \sigma' \in \tau.$$

Let $q \supseteq r$ be such that $q \Vert_\alpha\ \sigma' \in \rho$, then since

$$q \Vdash_\alpha N_\gamma(\sigma') \wedge \sigma' \in \tau \ \text{ implies } \ q \Vdash_\alpha \sigma' \in \rho \wedge \sigma \lessapprox \sigma'\ ;$$

by (b) we now have $q \Vdash_\alpha \sigma \in \rho$. We have shown that for all $p' \supseteq p$, there exists $q \supseteq p'$ such that $q \Vdash_\alpha \sigma \in \rho$, i.e., $p \Vdash_\alpha \neg\neg\sigma \in \rho$; therefore by Lemma 2.78 $p \Vdash_\alpha \sigma \in \rho$ as required.

(d) If $p \Vdash_\alpha N_\beta(\sigma) \wedge \sigma \lessapprox \tau$, and $r \supseteq p$, then for some $q \supseteq r$ and $\rho \in \mathcal{T}^c_\alpha$ with

$$\mathrm{ord}_\alpha(\rho) \leqslant^* \mathrm{ord}_\alpha(\sigma), \qquad \mathrm{stg}_\alpha(\rho) \leqslant \beta,$$

we have $q \Vdash_\alpha \sigma \lessapprox \rho \wedge \sigma \lessapprox \tau$, (see Definition 2.64(g)). By (a) we now have $q \Vdash_\alpha \tau \lessapprox \rho$, thus $p \Vdash_\alpha N_\beta(\tau)$.

(e) is shown by induction on the length of φ. For φ an atomic formula the result follows from (b), (c) and (d); thus we consider only the following cases:

(i) Assume $p \Vdash_\alpha \neg\psi(\sigma) \wedge \sigma \lessapprox \tau$; if $p \nVdash_\alpha \neg\psi(\tau)$, then for some $q \supseteq p$, $q \Vdash_\alpha \psi(\tau)$; hence by the induction hypothesis $q \Vdash_\alpha \psi(\sigma)$; a contradiction.

(ii) $p \Vdash_\alpha \psi_1(\sigma) \wedge \psi_2(\sigma) \wedge \sigma \lessapprox \tau$, (where σ may possibly not occur in ψ_1 or ψ_2), iff

$$p \Vdash_\alpha \psi_1(\sigma) \wedge \sigma \lessapprox \tau \qquad p \Vdash_\alpha \psi_2(\sigma) \wedge \sigma \lessapprox \tau$$

and, by the induction hypothesis

$$p \Vdash_\alpha \psi_1(\tau) \quad \text{and} \quad p \Vdash_\alpha \psi_2(\tau) \quad \text{iff} \quad p \Vdash_\alpha \psi_1(\tau) \wedge \psi_2(\tau).$$

(iii) $p \Vdash_\alpha (\forall_\lambda u)\,(\psi(u, \sigma)) \wedge \sigma \lessapprox \tau$, ($u$ obviously does not occur in σ), iff for all $\rho \in \mathcal{T}^c_\alpha$ with $\mathrm{rnk}(\rho) < \alpha$,

$$\mathrm{ord}_\alpha(\rho) < \mathrm{ord}_\alpha(\forall_\lambda u, (\forall_\lambda u)(\psi)), \qquad p \Vdash_\alpha \psi(\rho, \sigma) \wedge \sigma \lessapprox \tau.$$

By the induction hypothesis, $p \Vdash_\alpha \psi(\rho, \tau)$; which implies $p \Vdash_\alpha (\forall_\lambda u)[\psi(u,\tau)]$, as required. The case $\varphi = (\psi(\sigma))$ is trivial.

2.8111. Lemma. *$p \Vdash_\alpha \sigma \approx \tau$ implies $p \Vdash_\alpha K(\sigma) \approx K(\tau)$ and $p \Vdash_\alpha \theta(\sigma) \approx \theta(\tau)$, for all $\theta \in \Gamma$.*

Proof*. Assume $p \Vdash_\alpha (\forall_\xi u)(N_\eta(u) \to (u \in \sigma \leftrightarrow u \in \tau))$, where

$$\xi = \max(\mathrm{rnk}(\sigma), \mathrm{rnk}(\tau)), \qquad \eta = \max(\mathrm{stg}_\alpha(\sigma), \mathrm{stg}_\alpha(\tau)).$$

We must show that

$$(\forall_{\xi^*} u)(N_\eta(u) \to (u \in \theta(\sigma) \leftrightarrow u \in \theta(\tau))),$$

where

$$\xi^* = \max(\omega, \xi) = \max(\mathrm{rnk}(\theta(\sigma)), \mathrm{rnk}(\theta(\tau))),$$

$$\eta = \max(\mathrm{stg}_\alpha(\sigma), \mathrm{stg}_\alpha(\tau)),$$

$$\theta(\cdot) = {}^{(t,l)}\theta(\cdot) \in \Gamma .$$

Denote the latter formula by φ. Let $\rho \in \mathcal{T}_\alpha^c$ be such that $\mathrm{rnk}(\rho) < \xi^*$, and $\mathrm{ord}_\alpha(\rho) < \mathrm{ord}_\alpha(\forall_{\xi^*} u, \varphi)$. Let $r \supseteq p$ be such that $r \Vdash_\alpha \rho \in \theta(\sigma)$ and $r \parallel_\alpha \rho \in \theta(\tau)$. By Lemma 2.76(c) we have to show that

$$r \Vdash_\alpha \rho \in \theta(\sigma) \quad \text{iff} \quad r \Vdash_\alpha \rho \in \theta(\tau),$$

(since $\mathrm{stg}_\alpha(\rho) \leqslant \eta$, $r \Vdash_\alpha N_\eta(\rho)$, by Lemma 2.801(c)). Assume $r \not\Vdash_\alpha \rho \in \theta(\tau)$; then by Lemma 2.79 there is an $r' \supseteq r$ such that $r' \Vdash_\alpha \rho \notin \theta(\tau)$. By Definition 2.64(c), for any $r'' \supseteq r'$, there is a $q \supseteq r''$ such that

$$q \Vdash_\alpha \rho \approx (\langle \dot{k}, \dot{c} \rangle)^{(N_0)}$$

where either $k < m$ and $t(k) = \delta$, or $k \geqslant m$ and $q \Vdash_\alpha (\langle \dot{k} - \dot{m} + l, \dot{\delta} \rangle^{(N_0)} \in \sigma$. If $k < m$, $t(k) = \delta$ we have a contradiction to $r' \Vdash_\alpha \rho \notin \theta(\tau)$. Thus assume

$$q \Vdash_\alpha (\langle \dot{k} - \dot{m} + l, \dot{\delta} \rangle)^{(N_0)} \in \sigma \wedge \rho \approx (\langle \dot{k}, \dot{\delta} \rangle)^{(N_0)}.$$

Now $q \Vdash_\alpha \approx \tau$; hence by Lemma 2.811(c),

$$q \Vdash_\alpha (\langle \dot{k} - \dot{m} + l, \dot{\delta} \rangle)^{(N_0)} \in \tau \wedge \rho \approx (\langle \dot{k}, \dot{\delta} \rangle)^{(N_0)} ,$$

which implies $r \Vdash_\alpha \rho \in \theta(\tau)$; this is a contradiction, since $r \subseteq r' \Vdash_\alpha \rho \notin \theta(\tau)$.

We now show that $p \Vdash_\alpha K(\sigma) \gtrapprox K(\tau)$, i.e.

$$p \Vdash_\alpha (\forall_{\xi^*} u)\,(N_\eta(u) \to (u \in K(\sigma) \leftrightarrow u \in K(\tau))),$$

where

$$\xi^* = \max(\mathrm{rnk}(K(\sigma)), \mathrm{rnk}(K(\tau)) = \max(\omega+1, \xi+1).$$

Thus we must show, for all $r \supseteq p$ and $\rho \in \mathcal{T}^c_\alpha$ such that $\mathrm{rnk}(\rho) < \xi^*$, $\mathrm{stg}_\alpha(\rho) \leqslant \eta$ and $r \parallel_\alpha \rho \in K(\sigma)$, $r \parallel_\alpha \rho \in K(\tau)$, that

$$r \Vdash_\alpha \rho \in K(\sigma) \quad \text{iff} \quad r \Vdash_\alpha \rho \in K(\tau).$$

Now, assume that $r' \supseteq r$, then by Definition 2.64, $r' \Vdash_\alpha \rho \in K(\sigma)$ implies that for some $\theta \in \Gamma$, $q \supseteq r'$, $q \Vdash_\alpha \rho \gtrapprox \theta(\sigma)$. By the first part of this lemma $q \Vdash_\alpha \theta(\sigma) \gtrapprox \theta(\tau)$; thus by Lemma 2.811(a)

$$q \Vdash_\alpha \rho \gtrapprox \theta(\tau) \quad \text{implies} \quad r \Vdash \rho \in K(\tau),$$

as required. The reverse direction is shown symmetrically.

2.8112. Lemma. *For all* $\eta \geqslant \alpha$, $0 \Vdash_\alpha (\sigma)^{(N_\eta)} \gtrapprox \sigma \wedge (\varphi)^{(N_\eta)} \to \varphi$; $\sigma \in \mathcal{T}^c_\alpha$, $\varphi \in \mathcal{F}^c_\alpha$.

Proof. This lemma is a simple result of Lemma 2.801(d). It is proved by induction on the rnk_α of formulas and the order_α of terms simultaneously. Since all cases are entirely trivial we omit the details.

2.812. Lemma. $p \Vdash_\alpha \sigma = \tau$ *iff* $p \Vdash_\alpha \sigma \gtrapprox \tau$.

Proof*. Obviously $p \Vdash_\alpha \sigma = \tau$ implies $p \Vdash_\alpha \sigma \gtrapprox \tau$. Assume

(2.813) $\quad (\forall_\lambda u)\,(N_\gamma(u) \to (u \in \sigma \leftrightarrow u \in \tau))$,

where $\lambda = \max(\mathrm{rnk}(\sigma), \mathrm{rnk}(\tau))$, $\gamma = \max(\mathrm{stg}_\alpha(\sigma), \mathrm{stg}_\alpha(\tau))$. It has to be shown that

$$p \Vdash_\alpha (\forall u)\,(u \in \sigma \leftrightarrow \omega \in \tau)\ ,$$

i.e., for all $\rho \in \mathcal{T}^c_\alpha$, $p \Vdash_\alpha \rho \in \sigma \leftrightarrow \rho \in \tau$. Let $r \supseteq p$ be such that $r \Vdash_\alpha \rho \in \sigma$ and $r \parallel_\alpha \rho \in \tau$; then by Lemma 2.76(c) we must show that

$$r \Vdash_\alpha \rho \in \sigma \quad \text{iff} \quad r \Vdash_\alpha \rho \in \tau\ .$$

If $r \Vdash_\alpha \rho \in \sigma$, then by Lemma 2.81 there is a $\rho^* \in \mathcal{T}^c_\alpha$ with $\mathrm{rnk}(\rho^*) <$

$\mathrm{rnk}(\sigma) \leqslant \lambda$, $\mathrm{stg}_\alpha(\rho^*) \leqslant \mathrm{stg}_\alpha(\sigma) \leqslant \gamma$ and a $q \supseteq r$ such that $q \Vdash_\alpha \rho^* \in \sigma \wedge \rho^* \approx \rho$. Now, by (2.813) we get $q \Vdash_\alpha \rho^* \in \tau \wedge \rho^* \approx \rho$ and by Lemma 2.811 we get $q \Vdash_\alpha \rho \in \tau$ as required. By a symmetric argument, if $p \subseteq r \Vdash_\alpha \rho \in \tau$, then for some $q \supseteq p$, $q \Vdash_\alpha \rho \in \sigma$. Hence $p \Vdash_\alpha \sigma = \tau$, as required.

The following lemma can now be shown by a direct induction on length of formulas as in Lemma 2.811(e), using Lemmas 2.812 and 2.811.

2.814. Lemma (Substitution of equals). *If* $\varphi(\sigma_1, ..., \sigma_n) \in \mathcal{L}^c_\alpha$ *and*

$$p \Vdash_\alpha \varphi(\sigma_1, ..., \sigma_n) \wedge \sigma_1 = \tau_1 \wedge ... \wedge \sigma_n = \tau_n ,$$

then $p \Vdash_\alpha \varphi(\tau_1, ..., \tau_n)$.

2.815. Remark. In view of Lemmas 2.80 and 2.812 we may freely interchange $\approx$ and $\simeq$ with $=$ and vice versa.

The following lemmas will be needed in the future.

2.816. Lemma. (a)(i) *If for all* $k, 1 \leqslant k \leqslant n$, *there is an* l, $1 \leqslant l \leqslant m$, *such that* $p \Vdash_\alpha \rho_k = \tau_l$, *and for all* i, $1 \leqslant i \leqslant m$, *there is a* j, $1 \leqslant j \leqslant n$, *such that* $p \Vdash_\alpha \tau_i = \rho_j$, *then*

$$p \Vdash_\alpha \dot{\{}\rho_1, ..., \rho_n\dot{\}} = \dot{\{}\tau_1, ..., \tau_m\dot{\}} .$$

(ii) $p \Vdash_\alpha \dot{\{}\rho_1, ..., \rho_n\dot{\}} = \dot{\{}\tau_1, ..., \tau_m\dot{\}}$ *implies that for all* $r \supseteq p$, *there there is a* $q \supseteq r$ *such that for any* k, $1 \leqslant k \leqslant n$, *there is an* l *such that* $q \Vdash_\alpha \rho_k = \tau_l$, *and for any* i, $1 \leqslant i \leqslant m$, *there is a* j, $1 \leqslant j \leqslant n$, *such that* $q \Vdash_\alpha \tau_i = \rho_j$.

(b)(i) $p \Vdash_\alpha \bigwedge_{i=1}^n (\rho_i = \tau_i)$ *implies that*

$$p \Vdash_\alpha \dot{\langle}\rho_1, ..., \rho_n\dot{\rangle} = \dot{\langle}\tau_1, ..., \tau_n\dot{\rangle} .$$

(ii) $p \Vdash_\alpha \dot{\langle}\rho_1, ..., \rho_n\dot{\rangle} = \dot{\langle}\tau_1, ..., \tau_n\dot{\rangle}$ *implies that for any* $r \supseteq p$ *there is a* $q \supseteq r$ *such that* $q \Vdash_\alpha \bigwedge_{i=1}^n (\rho_i = \tau_i)$.

Proof*. (a)(i). $p \Vdash_\alpha \dot{\{}\bar{\rho}^n\dot{\}} = \dot{\{}\bar{\tau}^m\dot{\}}$ if and only if for all $\sigma \in \mathcal{T}^c_\alpha$ and all $q \supseteq p$ such that $q \Vdash_\alpha \sigma \in \dot{\{}\bar{\rho}^n\dot{\}}$ and $q \Vdash_\alpha \sigma \in \dot{\{}\bar{\tau}^m\dot{\}}$, we have

(2.8161) $\quad q \Vdash_\alpha \sigma \in \dot{\{}\bar{\rho}^n\dot{\}} \Leftrightarrow q \Vdash_\alpha \sigma \in \dot{\{}\bar{\tau}^m\dot{\}}$.

Assume the left side of (2.8161). Then by Definitions 2.64(l), (m) and 2.26, $q \Vdash_\alpha \sigma = \rho_1 \vee \ldots \vee \sigma = \rho_n$. Hence for all $r \supseteq q$, there is a $q^* \supseteq r$ and a ρ_k such that $q^* \Vdash_\alpha \sigma = \rho_k$; and by assumption for some l, $q^* \Vdash_\alpha \rho_k = \tau_l$, thus by transitivity $q^* \Vdash_\alpha \sigma = \tau_l$; hence $q \Vdash_\alpha \sigma \in \dot{\{}\bar{\tau}^m\dot{\}}$. Assuming the right side of (2.8161), we show the left side by a symmetrical argument. Therefore $p \Vdash_\alpha \dot{\{}\bar{\rho}^n\dot{\}} = \dot{\{}\bar{\tau}^m\dot{\}}$ as required.

(ii) Assume $p \Vdash_\alpha \dot{\{}\bar{\rho}_1^n\dot{\}} = \dot{\{}\bar{\tau}^m\dot{\}}$. Then since

$$p \Vdash_\alpha \bigwedge_{k=1}^{n} (\rho_k \in \dot{\{}\bar{\rho}^n\dot{\}}) ,$$

we must also have $p \Vdash_\alpha \rho_k \in \dot{\{}\bar{\tau}^m\dot{\}}$, $1 \leqslant k \leqslant n$. For any $r \supseteq p$ there exists a $q \supseteq r$ such that $q \parallel_\alpha \rho_i = \tau_j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and since

$$q \Vdash_\alpha \bigwedge_{k=1}^{n} (\rho_k \in \dot{\{}\bar{\tau}^m\dot{\}}) ,$$

we must for any k, $1 \leqslant k \leqslant n$, have an l, $1 \leqslant l \leqslant m$, such that $q \Vdash_\alpha \rho_k = \tau_l$. Similarly for any $1 \leqslant i \leqslant m$ there is a j, $1 \leqslant j \leqslant n$, such that $q \Vdash_\alpha \tau_i = \rho_j$.

(b)(i) This fact is shown by finite induction on n. For $n = 1$ the result follows directly from (a)(i). For $n > 1$, we have by the induction hypothesis that

$$p \Vdash_\alpha \dot{\langle}\rho_1, \ldots, \rho_{n-1}\dot{\rangle} = \dot{\langle}\tau_1, \ldots, \tau_{n-1}\dot{\rangle} \wedge \rho_n = \tau_n .$$

Hence by two applications of (a)(i), we get

$$(2.8162)\quad p \Vdash_\alpha \dot{\{}\dot{\{}\dot{\langle}\rho_1, \ldots, \rho_{n-1}\dot{\rangle}, \rho_n\dot{\}}, \dot{\{}\rho_n\dot{\}}\dot{\}} = \\ = \dot{\{}\dot{\{}\dot{\langle}\tau_1, \ldots, \tau_{n-1}\dot{\rangle}, \tau_n\dot{\}}, \dot{\{}\tau_n\dot{\}}\dot{\}} ,$$

as required.

(ii) Assume $p \Vdash_\alpha \dot{\langle}\bar{\rho}_1^n\dot{\rangle} = \dot{\langle}\bar{\tau}^n\dot{\rangle}$, i.e., (2.8162) above. We consider the various possibilities, which follow from (a)(ii). For any $r \supseteq p$, let $q \supseteq r$ fulfill (a)(ii) with regard to (2.8162).

(1) If $q \Vdash_\alpha \dot{\{}\rho_n\dot{\}} = \dot{\{}\tau_n\dot{\}} \wedge \dot{\{}\dot{\langle}\rho_1, \ldots, \rho_{n-1}\dot{\rangle}, \rho_n\dot{\}} = \dot{\{}\dot{\langle}\tau_1, \ldots, \tau_{n-1}\dot{\rangle}, \tau\dot{\}}$, then from the first conjunct and (a) we get $q \Vdash_\alpha \rho_n = \tau_n$. From the second conjunct, either for some $q^* \supseteq q$,

$$q^* \Vdash_\alpha \dot{\langle}\rho_1, \ldots, \rho_{n-1}\dot{\rangle} = \dot{\langle}\tau_1, \ldots, \tau_{n-1}\dot{\rangle} ,$$

in which case the result follows from the induction hypothesis; or if not, then by (a) for some $q^* \supseteq q$,

$$q^* \Vdash_\alpha \langle \dot{\rho}_1, ..., \rho_{n-1}\dot{\rangle} = \tau_n \wedge \langle \dot{\tau}_1, ..., \tau_{n-1}\dot{\rangle} = \rho_n ,$$

and since $q \Vdash_\alpha \rho_n = \tau_n$ we have obtained a contradiction to our assumption (by transitivity).

(2) If $q \not\Vdash_\alpha \{\dot{\rho}_n\dot{\}} = \{\dot{\tau}_n\dot{\}}$, then for some $q^* \supseteq q$,

$$q^* \Vdash_\alpha \{\dot{\rho}_n\dot{\}} \neq \{\dot{\tau}_n\dot{\}} ;$$

hence by (a) we must have

$$\begin{aligned} q \subseteq q^* \Vdash_\alpha \{\dot{\rho}_n\dot{\}} &= \{\dot{\langle}\tau_1, ..., \tau_{n-1}\dot{\rangle}, \tau_n\dot{\}} \wedge \{\dot{\tau}_n\dot{\}} \\ &= \{\dot{\langle}\rho_1, ..., \rho_{n-1}\dot{\rangle}, \rho_n\dot{\}} . \end{aligned}$$

Hence again by (a)(ii), for some $q^{**} \supseteq q^*$ we must have $q^{**} \Vdash_\alpha \rho_n = \tau_n$. This is a contradiction; therefore only the first case in (1) is possible, and (b)(ii) holds as required.

2.817. Corollary. *If* $p \Vdash_\alpha \langle \dot{\bar{\rho}}_h \dot{\rangle} = \langle \dot{\bar{\tau}}_h \dot{\rangle}$ *and* $f \subseteq h$, *then*

$$p \Vdash_\alpha \langle \dot{\bar{\rho}}_f \dot{\rangle} = \langle \dot{\bar{\tau}}_f \dot{\rangle} .$$

Proof. By Lemma 2.816(b)(ii), for all $r \supseteq p$, there is a $q \supseteq r$ such that

$$q \Vdash_\alpha \bigwedge_{i \in h} (\rho_i = \tau_i) ;$$

hence

$$q \Vdash_\alpha \bigwedge_{i \in f} (\rho_i = \tau_i) ;$$

and by applying Lemma 2.816(b)(i), $q \Vdash_\alpha \langle \dot{\bar{\rho}}_f \dot{\rangle} = \langle \dot{\bar{\tau}}_f \dot{\rangle}$. By Lemma 2.78 $p \Vdash_\alpha \langle \dot{\bar{\rho}}_f \dot{\rangle} = \langle \dot{\bar{\tau}}_f \dot{\rangle}$, as required.

2.9. Realization. Generic models realizing a language $\mathcal{L}_\alpha$ are constructed as usual by choice of a generic set of conditions $Q \subseteq P_\alpha$, from which a valuation is defined.

2.90. Definition. (a) A set of conditions $Q \subseteq P_\alpha$ is said to be *complete* for $\mathcal{L}_\alpha$ if it decides every sentence in $\mathcal{L}_\alpha$.

(b) For any $Q \subseteq P_\alpha$ and sentence $\varphi \in \mathcal{L}_\alpha$, define $Q \Vdash_\alpha \varphi$ iff $(\exists p \in Q)(p \Vdash_\alpha \varphi)$, and $Q \parallel_\alpha \varphi$ iff $(\exists p \in Q)(p \parallel_\alpha \varphi)$; in this case Q is said to *decide* φ.

(c) $Q \subseteq P_\alpha$ is said to be *α-consistent* (or simply consistent) if for no $\varphi \in \mathcal{L}_\alpha$ does $Q \Vdash_\alpha \varphi$ and $Q \Vdash_\alpha \neg\varphi$.

2.901. Definition. A set of conditions $Q \subseteq P_\alpha$ is said to be *generic* for $\mathcal{L}_\alpha$ (or simply generic) if and only if:

(a) Q is complete for $\mathcal{L}_\alpha$,

(b) for all $p, q \in Q$ there is an $r \in Q$ such that $p \cup q \subseteq r$ (i.e. Q is α-consistent),

(c) $p \in Q$ and $q \subseteq p$ implies $q \in Q$.

By the basic properties of forcing and the fact that M is countable we can prove, in the usual way, via an enumeration of $\mathcal{L}_\alpha$:

2.902. Lemma. *For any $p \in P_\alpha$ there exists a generic set of conditions for $\mathcal{L}_\alpha$, $Q \subseteq P_\alpha$, with $p \in Q$.*

2.903. Definition. Let $Q \subseteq P_\alpha$ be a generic set of conditions for $\mathcal{L}_\alpha$. The function val_Q is defined on all constant terms of $\mathcal{L}_\alpha$ by induction on rank_α as follows: Let $\tau \in \mathcal{T}^c_\alpha$, then

$$(2.904)\qquad \mathrm{val}_Q(\tau) = \{\mathrm{val}_Q(\sigma) \mid \sigma \in \mathcal{T}^c_\alpha, \mathrm{rnk}(\sigma) < \mathrm{rnk}(\tau), \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\tau) \text{ and } Q \Vdash_\alpha \sigma \in \tau\}\,.$$

The model N_Q is now defined by:

$$N_Q = \{\mathrm{val}_Q(\tau) \mid \tau \in \mathcal{T}^c_\alpha\}\,.$$

The following two lemmas are obvious from Definition 2.903.

2.905. Lemma. *N_Q is a transitive set.*

2.906. Lemma. (a) *For all $s \in M$,* $\mathrm{val}_Q(\dot{s}) = s$;

(b) $M \subseteq N_Q$.

Proof. (a) follows easily from the definition of val_Q by induction on $\mathrm{rnk}(\dot{s}) = \mathrm{Rank}(s)$;

(b) hence also $M \subseteq N_Q$.

2.907. Definition. (a) val_Q is extended to include variable terms relative to an assignment x of values from N_Q to the free variables of the term as follows:

$\mathrm{val}_Q(\tau(u_1, ..., u_n), x) = t$ iff for all i, $1 \leqslant i \leqslant n$, there exist terms σ_i with $\mathrm{rnk}(\sigma_i) \leqslant \omega$ such that $x(u_i) = \mathrm{val}_Q(\sigma_i)$ and $t = \mathrm{val}_Q(\tau(\sigma_1, ..., \sigma_n))$; otherwise $t \neq \emptyset$. Clearly val_Q is a well-defined function.

(b) The binary function sat_Q defined on all formulas of $\mathcal{L}_\alpha$ relative to an assignment of values, x, to the free variables of φ is defined by induction on the length of φ in the usual manner, with the exception of the additional predicates N_β, $\beta \leqslant \aleph_1^{(M)}$. In the following, σ, τ stand for terms or variables; if σ is a variable, $\mathrm{val}_Q(\sigma, x)$ means $x(u)$.

(1) $$\mathrm{sat}_Q(\sigma \in \tau, x) = \begin{cases} 1 & \text{if } \mathrm{val}_Q(\sigma, x) \in \mathrm{val}_Q(\tau, x); \\ 0 & \text{otherwise.} \end{cases}$$

(2) $$\mathrm{sat}_Q(\neg\varphi, x) = 1 - \mathrm{sat}_Q(\varphi, x).$$

(3) $$\mathrm{sat}_Q(\varphi \wedge \psi, x) = \mathrm{sat}_Q(\varphi, x) \cdot \mathrm{sat}_Q(\psi, x).$$

(4) $$\mathrm{sat}_Q((\forall_\lambda u)(\varphi(u)), x) = \begin{cases} 1 & \text{if for all } \sigma \text{ such that } \mathrm{rnk}(\sigma) < \lambda \\ & \text{and } \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\forall_\lambda u, (\forall_\lambda u)(\varphi)), \\ & \text{we have } \mathrm{sat}_Q(\varphi(u), x \cup \{\langle u, \mathrm{val}_Q(\sigma)\rangle\}) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

(5) $$\mathrm{sat}_Q((\forall u)(\varphi(u)), x) = \begin{cases} 1 & \text{if for all } \sigma \text{ we have} \\ & \mathrm{sat}_Q(\varphi(u), x \cup \{\langle u, \mathrm{val}_Q(\sigma)\rangle\}) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

(6) $$\mathrm{sat}_Q(N_\beta(\sigma), x) = \begin{cases} 1 & \text{if for some } \tau \text{ such that } \mathrm{stg}_\alpha(\tau) \leqslant \beta \text{ and} \\ & \mathrm{ord}_\alpha(\tau) \leqslant^* \mathrm{ord}_\alpha(\sigma), \text{ we have} \\ & \mathrm{val}_Q(\sigma, x) = \mathrm{val}_Q(\tau, x); \\ 0 & \text{otherwise.} \end{cases}$$

(7) $$\mathrm{sat}_Q((\varphi), x) = \mathrm{sat}_Q(\varphi, x).$$

It is clear that val_Q and sat_Q are well-defined and that N_Q is a transitive model of the $\in$-relation. Moreover it is seen that, for formulas of the language of set theory, sat_Q coincides with the usual definition of satisfaction.

2.9071. Notation. If $\varphi \in \mathcal{T}^c_\alpha$, denote

$$N_Q \models \varphi \quad \text{iff} \quad \mathrm{sat}_Q(\varphi) = 1;$$

if $\varphi \in \mathcal{L}_\alpha$ and x is an assignment to the free variables of φ,

$$N_Q \models \varphi[x] \quad \text{iff} \quad \mathrm{sat}_Q(\varphi, x) = 1.$$

2.908. Lemma. $Q \Vdash_\alpha \sigma = \tau$ *iff* $Q \Vdash_\alpha \sigma \approx \tau$ *iff* $\mathrm{val}_Q(\sigma) = \mathrm{val}_Q(\tau)$, $\sigma, \tau \in \mathcal{T}^c_\alpha$.

Proof. By induction on $\max(\mathrm{ord}_\alpha(\sigma), \mathrm{ord}_\alpha(\tau))$, using Lemmas 2.81, 2.811 and 2.812 as in Lévy [9]. We omit the details.

The following basic lemma can now be proved.

2.909. Lemma. $Q \Vdash_\alpha \varphi$ *iff* $N_Q \models \varphi, \varphi \in \mathcal{T}^c_\alpha$.

Proof*. This lemma is proved by induction on the length of φ in more or less the usual manner, and it suffices to deal with only two selected cases.

(a) $N_Q \models \sigma \in \tau$ implies $\mathrm{val}_Q(\sigma) \in \mathrm{val}_Q(\tau)$ implies, for some σ' such that $\mathrm{rnk}(\sigma') < \mathrm{rnk}(\tau)$, and $\mathrm{ord}_\alpha(\sigma') < \mathrm{ord}_\alpha(\tau)$, that

$$Q \Vdash_\alpha \sigma' \in \tau, \qquad \mathrm{val}_Q(\sigma') = \mathrm{val}_Q(\sigma).$$

By Lemma 2.908, $Q \Vdash_\alpha \sigma \approx \sigma'$; hence by Lemma 2.811, $Q \Vdash_\alpha \sigma \in \tau$. To see the reverse implication, assume $Q \Vdash_\alpha \sigma \in \tau$; then by Lemma 2.81, for some σ' such that

$$\mathrm{rnk}(\sigma') < \mathrm{rnk}(\tau), \qquad \mathrm{ord}_\alpha(\sigma') < \mathrm{ord}_\alpha(\tau),$$

we have $Q \Vdash_\alpha \sigma' \in \tau \wedge \sigma' \approx \sigma$; which implies by Lemma 2.908 that

$$\mathrm{val}_Q(\sigma') = \mathrm{val}_Q(\sigma), \qquad \mathrm{val}_Q(\sigma') \in \mathrm{val}_Q(\tau);$$

therefore $N_Q \models \sigma \in \tau$.

(b) $N_Q \models N_\beta(\sigma)$ implies that, for some τ with $\mathrm{stg}_\alpha(\tau) \leqslant \beta$ and $\mathrm{ord}_\alpha(\tau) \leqslant^* \mathrm{ord}_\alpha(\sigma)$, we have $\mathrm{val}_Q(\tau) = \mathrm{val}_Q(\sigma)$. Now, if $\mathrm{stg}_\alpha(\sigma) \leqslant \beta$, then immediately $Q \Vdash_\alpha N_\beta(\sigma)$; if $\mathrm{stg}_\alpha(\sigma) > \beta$, then by Lemma 2.908 $Q \Vdash_\alpha \tau \approx \sigma$, therefore $Q \Vdash_\alpha \mathrm{V}_\beta(\sigma)$. To see the reverse implication, assume $Q \Vdash_\alpha N_\beta(\sigma)$; hence for some τ such that $\mathrm{stg}_\alpha(\tau) \leqslant \beta$, $\mathrm{ord}_\alpha(\tau) \leqslant^* \mathrm{ord}_\alpha(\sigma)$, we have $Q \Vdash_\alpha \tau \lesseqgtr \sigma$; and by Lemma 2.908, $\mathrm{val}_Q(\sigma) = \mathrm{val}_Q(\tau)$ implies $N_Q \vDash N_\beta(\sigma)$.

By the absoluteness of the forcing relation we have that

$$\{p \mid p \in P_\alpha, p \Vdash_\alpha \varphi\}, \quad \varphi \in \mathcal{T}^c_\alpha,$$

is a set in M. This fact together with the last lemma gives, by the standard arguments [2], the following:

2.91. Lemma. *N_Q is a model of* ZF.

The following lemma and its corollaries are very useful.

2.911. Lemma. *$p \Vdash_\alpha \varphi$ iff φ is true in every model N_Q for which $p \in Q$.*

Proof. "If $p \Vdash_\alpha \varphi$, then φ" is obvious. To see the other direction, assume φ is true in every N_Q for which $p \in Q$; but $p \nVdash_\alpha \varphi$, then by Lemma 2.79 there is a $q \supseteq p$ such that $q \Vdash_\alpha \neg\varphi$. Let Q be a generic set of conditions such that $q \in Q$; then by Lemma 2.909, $N_Q \vDash \neg\varphi$; this is a contradiction since also $p \in Q$.

2.912. Corollary. *If $p \Vdash_\alpha \varphi$ and* ZF $\vdash \varphi \to \psi$, *then $p \Vdash_\alpha \psi$.*

2.913. Corollary. *If φ is a theorem of* ZF, *then* $0 \Vdash_\alpha \varphi$.

3. The iterated forcing lemma

Let $Q \subseteq P_\alpha$ be generic for $\mathcal{L}_\alpha$ and denote $M[I_\beta] = N_\beta$, $\beta \leqslant \alpha$. Then $M \subseteq N_\gamma \subseteq N_\beta$, $\gamma \leqslant \beta \leqslant \alpha$, since obviously $I_\beta \supseteq I_\gamma$, $\gamma \subseteq \beta$. Consider $Q_\beta = Q \cap P_\beta$, $\beta \leqslant \alpha$. Q_β is clearly generic since all basic statements of $\mathcal{L}_\beta$ are decided by Q_β, (this is shown in Lemma 3.20). We want to show that the equalities

(3.0) $$N_{Q_\beta} = N_\beta = \langle \{\mathrm{val}_Q(\sigma) \mid \sigma \in \mathcal{T}^c_\alpha, \mathrm{stg}_\alpha(\sigma) \leqslant \beta\}, \in \rangle$$

hold.

The main tool used in dealing with the iteration here is the iterated forcing lemma (3.10), from which the connection between Q_γ, $\Vdash_\gamma$, sat_{Q_γ} and Q_β, $\Vdash_\beta$, sat_{Q_β}, $\gamma \leqslant \beta \leqslant \alpha \leqslant \aleph_1^{(M)}$ will be established. To be more precise, we use the following:

3.01. Notation. Let $p \in P_\alpha \supseteq P_\beta$, then $p \Vdash_\beta \varphi$ if $\hat{p}^\beta \Vdash_\beta \varphi$. Then Lemma 3.10 says that:

$$p \Vdash_\alpha (\varphi)^{(N_\beta)} \quad \text{iff} \quad p \Vdash_\beta \varphi, \qquad \beta \leqslant \alpha,\ \ p \in P_\alpha,\ \ \varphi \in \mathcal{T}^c_\beta.$$

In order to prove this we need some quite obvious auxiliary lemmas concerning the predicate symbols $N_\gamma(\cdot)$ and relativization, the proofs of which are technical, essentially amounting to direct verifications.

3.02. Lemma. *Assume* $\rho_i \in \mathcal{T}^c_\gamma$, $1 \leqslant i \leqslant n$, $\gamma \leqslant \alpha$, *and*

$$p \Vdash_\alpha \rho_1 = (\rho_1)^{(N_\gamma)} \wedge \ldots \wedge \rho_n = (\rho_n)^{(N_\gamma)};$$

then

(a) (i) $p \Vdash_\alpha \dot{\{}\rho_1, \ldots, \rho_n\dot{\}} = (\dot{\{}\rho_1, \ldots, \rho_n\dot{\}})^{(N_\gamma)}$,

(ii) $p \Vdash_\alpha \dot{\langle}\rho_1, \ldots, \rho_n\dot{\rangle}^{(N_\gamma)} = \dot{\langle}\rho_1, \ldots, \rho_n\dot{\rangle}$;

(b) (i) $p \Vdash_\alpha \dot{\{}(\rho_1)^{(N_\gamma)}, \ldots, (\rho_n)^{(N_\gamma)}\dot{\}} = (\dot{\{}\rho_1, \ldots, \rho_n\dot{\}})^{(N_\gamma)}$,

(ii) $p \Vdash_\alpha \dot{\langle}(\rho_1)^{(N_\gamma)}, \ldots, (\rho_n)^{(N_\gamma)}\dot{\rangle} = (\dot{\langle}\rho_1, \ldots, \rho_n\dot{\rangle})^{(N_\gamma)}$;

(c) *if* $\gamma, \beta \leqslant \alpha$ *and*

$$p \Vdash_\alpha (\rho_1)^{(N_\gamma)} = (((\rho_1))^{(N_\gamma)})^{(N_\beta)} \wedge \ldots \wedge (\rho_n)^{(N_\gamma)}$$

$$= (((\rho_n))^{(N_\gamma)})^{(N_\beta)} ,$$

$\rho_i \in \mathcal{T}_\gamma, \rho_i \in \mathcal{T}_\beta$, $1 \leqslant i \leqslant n$, then

(i) $p \Vdash_\alpha (\dot{\{}\rho_1, ..., \rho_n\dot{\}})^{(N_\gamma)} = ((\dot{\{}\rho_1, ..., \rho_n\dot{\}})^{(N_\gamma)})^{(N_\gamma)}$,

(ii) $p \Vdash_\alpha (\dot{\langle}\rho_1, ..., \rho_n\dot{\rangle})^{(N_\gamma)} = ((\dot{\langle}\rho_1, ..., \rho_n\dot{\rangle})^{(N_\gamma)})^{(N_\gamma)}$.

Proof*. (a)(i) $\dot{\{}\rho_1, ..., \rho_n\dot{\}} = (\gimel w)(w = \rho_1 \vee ... \vee w = \rho_n)$,

$$(\dot{\{}\rho_1, ..., \rho_n\dot{\}})^{(N_\gamma)} = (\gimel w)(N_\gamma(w) \wedge (w = (\rho_1)^{(N_\gamma)} \vee ... \vee w = (\rho_n)^{(N_\gamma)})).$$

Since $p \Vdash_\alpha \rho_i = (\rho_i)^{(N_\gamma)}$, and $\rho_i \in \mathcal{L}_\gamma$, implies $p \Vdash_\alpha N_\gamma(\rho_i)$, $1 \leqslant i \leqslant n$ (by Lemma 2.812 and Definition 2.64). Now, assume that for $q \supseteq p$,

$$q \Vdash_\alpha \tau \in \dot{\{}\rho_1, ..., \rho_n\dot{\}}, \qquad q \Vdash_\alpha \tau \in (\dot{\{}\rho_1, ..., \rho_n\dot{\}})^{(N_\gamma)},$$

$$q \Vdash_\alpha \tau = \rho_i, \qquad 1 \leqslant i \leqslant n.$$

Then for some j, $q \Vdash_\alpha \tau = \rho_j = (\rho_j)^{(N_\gamma)}$ and by Lemmas 2.812 and 2.811(d), also $q \Vdash_\alpha N_\gamma(\tau)$; which implies

$$q \Vdash_\alpha \tau \in (\dot{\{}\rho_1, ..., \rho_n\dot{\}})^{(N_\gamma)};$$

which implies

$$p \Vdash_\alpha \dot{\{}\rho_i, ..., \rho_n\dot{\}} \subseteq (\dot{\{}\rho_1, ..., \rho_n\dot{\}})^{(N_\gamma)}.$$

The reverse inclusion is even simpler.

(a)(ii) Consider first the case $n = 2$. Then

$$\langle\rho_1, \rho_2\rangle = \dot{\{}\dot{\{}\rho_1\dot{\}}, \dot{\{}\rho_1, \rho_2\dot{\}}\dot{\}}$$
$$= (\gimel w)(w = \dot{\{}\rho_1\dot{\}} \vee w = \dot{\{}\rho_1, \rho_2\dot{\}}),$$

$$(\langle\rho_1, \rho_2\rangle)^{(N_\gamma)} = (\gimel w)(N_\gamma(w) \wedge (w = (\dot{\{}\rho_1\dot{\}})^{(N_\gamma)} \vee w = (\dot{\{}\rho_1, \rho_2\dot{\}})^{(N_\gamma)}).$$

Assume $p \subseteq r \Vdash_\alpha \tau \in \dot{\langle}\rho_1, \rho_2\dot{\rangle}$; then for some $q \supseteq r$, either $q \Vdash_\alpha \tau = \dot{\{}\rho_1\dot{\}}$ or $q \Vdash_\alpha \tau = \dot{\{}\rho_1, \rho_2\dot{\}}$. Then by (a)(i), $q \Vdash_\alpha \tau = (\dot{\{}\rho_1, \rho_2\dot{\}})^{(N_\gamma)}$ or $q \Vdash_\alpha \tau = (\dot{\{}\rho_1\dot{\}})^{(N_\gamma)}$; in any case we also have $q \Vdash_\alpha N_\gamma(\tau)$, as above; hence

$$q \Vdash_\alpha \tau \in (\dot{\langle}\rho_1, \rho_2\dot{\rangle})^{(N_\gamma)}.$$

Therefore $p \Vdash_\alpha \dot{\langle}\rho_1, \rho_2\dot{\rangle} \dot{\subseteq} (\dot{\langle}\rho_1, \rho_2\dot{\rangle})^{(N_\gamma)}$. The reverse is even simpler; thus $p \Vdash_\alpha \dot{\langle}\rho_1, \rho_2\dot{\rangle} = (\dot{\langle}\rho_1, \rho_2\dot{\rangle})^{(N_\gamma)}$. The general case is proven by induction on $n \geqslant 2$. By assumption,

$$p \Vdash_\alpha \dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle} = (\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle})^{(N_\gamma)};$$

and from (a)(i), (a)(ii), $n = 2$ we get:

$$p \Vdash_\alpha \dot{\{}\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle}\dot{\}} =$$

$$= (\dot{\{}\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle}\dot{\}})^{(N_\gamma)} \wedge (\dot{\{}\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle}, \rho_n \dot{\}})^{(N_\gamma)}$$

$$= \dot{\{}\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle}, \rho_n \dot{\}}.$$

By applying (a)(i) for $n = 2$, we get

$$p \Vdash_\alpha \dot{\{}\dot{\{}\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle}\dot{\}}, \dot{\{}\dot{\langle} \rho_n \dot{\}}, \dot{\{}\dot{\langle} \rho_1, \ldots, \rho_{n-1} \dot{\rangle}\dot{\}}\dot{\}}\dot{\}} =$$

$$= \dot{\langle} \rho_1, \ldots, \rho_n \dot{\rangle} = (\dot{\langle} \rho_1, \ldots, \rho_n \dot{\rangle})^{(N_\gamma)}.$$

(b) is an immediate result of (a) and Lemma 2.811(b).

(c) is proven in a manner completely analogous to (a), relying on the fact that if

$$q \Vdash_\alpha \tau = (\rho_i)^{(N_\gamma)} = ((\rho_i)^{(N_\gamma)})^{(N_\beta)},$$

then

$$q \Vdash_\alpha N_\gamma(\tau) \wedge N_\beta(\tau).$$

We omit further details.

3.03. Lemma. *For all* $\sigma \in \mathcal{T}_\alpha^c$, $\varphi \in \mathcal{F}_\alpha^c$ *and* $\gamma \leqslant \beta \leqslant \alpha$,

(i) $p \Vdash_\alpha ((\varphi)^{(N_\gamma)})^{(N_\beta)}$ *iff* $p \Vdash_\alpha (\varphi)^{(N_\gamma)}$,

$p \Vdash_\alpha ((\varphi)^{(N_\gamma)})^{(N_\beta)}$ *iff* $p \Vdash_\alpha ((\varphi)^{(N_\beta)})^{(N_\gamma)}$;

(ii) $0 \Vdash_\alpha (\sigma)^{(N_\gamma)} \approxeq ((\sigma)^{(N_\gamma)})^{(N_\beta)} \wedge ((\sigma)^{(N_\gamma)})^{(N_\beta)} \approxeq ((\sigma)^{(N_\beta)})^{(N_\gamma)}$.

Proof*. First, observe that for $\gamma \leqslant \beta$,

$$\mathrm{ord}_\alpha(((\varphi)^{(N_\gamma)})^{(N_\beta)}) = \mathrm{ord}_\alpha((\varphi)^{(N_\gamma)}) = \mathrm{ord}_\alpha(((\varphi)^{(N_\beta)})^{(N_\gamma)}),$$

$$\mathrm{ord}_\alpha(((\sigma))^{(N_\gamma)})^{(N_\beta)}) = \mathrm{ord}_\alpha((\sigma)^{(N_\gamma)}) = \mathrm{ord}_\alpha(((\sigma)^{(N_\beta)})^{(N_\gamma)}).$$

(i) will first be proven for ranked sentences by induction on

$$\mathrm{rnk}_\alpha(((\varphi)^{(N_\gamma)})^{(N_\beta)}), \qquad \mathrm{rnk}_\alpha((\varphi)^{(N_\gamma)}),$$

with use of (ii) for σ such that

$$\text{ord}_\alpha(((\sigma)^{(N_\gamma)})^{(N_\beta)}) < \text{ord}_\alpha(((\varphi)^{(N_\gamma)})^{(N_\beta)}) = (\eta, \xi) ,$$

$$\text{ord}_\alpha((\sigma)^{(N_\gamma)}) < \text{ord}_\alpha((\varphi)^{(N_\gamma)}) = (\eta, \xi) .$$

Then (ii) will be shown for terms of $\text{order}_\alpha = (\eta, \xi)$.

To prove (i), consider the different cases of Definition 2.64. (a)–(f) and (l), (m) are immediate since they are all of the form

$$p \Vdash_\alpha ((\sigma \in \tau)^{(N_\gamma)})^{(N_\beta)} = p \Vdash_\alpha ((\sigma)^{(N_\gamma)})^{(N_\beta)} \in ((\tau)^{(N_\gamma)})^{(N_\beta)}$$

and by the induction hypothesis concerning terms,

$$0 \Vdash_\alpha ((\sigma)^{(N_\gamma)})^{(N_\beta)} \overset{\alpha}{\approx} (\sigma)^{(N_\gamma)} \wedge ((\sigma)^{(N_\gamma)})^{(N_\beta)} \overset{\alpha}{\approx} ((\sigma)^{(N_\beta)})^{(N_\gamma)};$$

similarly for τ. Thus by substitution of equals using Lemmas 2.812 and 2.814, we obtain

$$p \Vdash_\alpha (\sigma)^{(N_\gamma)} \in (\tau)^{(N_\gamma)} \wedge ((\sigma)^{(N_\beta)})^{(N_\gamma)} \in ((\tau)^{(N_\beta)})^{(N_\gamma)}.$$

In a same manner we obtain the inverse implications.

Case (g). $p \Vdash_\alpha ((N_\zeta(\sigma))^{(N_\gamma)})^{(N_\beta)}$ iff $p \Vdash_\alpha N_\zeta(((\sigma)^{(N_\gamma)})^{(N_\beta)})$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \tau)\,[\text{stg}_\alpha(\tau) \leqslant \zeta$$
$$\text{and } \text{ord}_\alpha(\tau) \prec^* \text{ord}_\alpha(((\sigma)^{(N_\gamma)})^{(N_\beta)})$$
$$\text{and } q \Vdash_\alpha ((\sigma)^{(N_\gamma)})^{(N_\beta)} \overset{\alpha}{\approx} \tau] .$$

Since

$$\text{ord}_\alpha(((\sigma)^{(N_\gamma)})^{(N_\beta)}) < \text{ord}_\alpha(N_\zeta(((\sigma)^{(N_\gamma)})^{(N_\beta)}) ,$$

we can apply the induction hypothesis concerning (ii), giving

$$0 \Vdash_\alpha ((\sigma)^{(N_\gamma)})^{(N_\beta)} \overset{\alpha}{\approx} (\sigma)^{(N_\gamma)} \wedge ((\sigma)^{(N_\gamma)})^{(N_\beta)}$$
$$\overset{\alpha}{\approx} ((\sigma)^{(N_\beta)})^{(N_\gamma)};$$

hence by Lemmas 2.812 and 2.814,

$$q \Vdash_\alpha (\sigma)^{(N_\gamma)} \overset{\alpha}{\approx} \tau \wedge ((\sigma)^{(N_\beta)})^{(N_\gamma)} \overset{\alpha}{\approx} \tau .$$

Now, as observed above,

$$\text{ord}_\alpha((\sigma)^{(N_\gamma)}) = \text{ord}_\alpha(((\sigma)^{(N_\gamma)})^{(N_\beta)})$$
$$= \text{ord}_\alpha(((\sigma)^{(N_\beta)})^{(N_\gamma)}) \,{}^*\!\succ \text{ord}_\alpha(\tau) ,$$

implies $p \Vdash_\alpha N_\xi((\sigma)^{(N_\gamma)}$ and $p \Vdash_\alpha N_\xi(((\sigma)^{(N_\beta)})^{(N_\gamma)})$. The reverse implications are also seen to be immediate. Cases (h), (i) and (i) are trivially obtained by a reduction in length. Consider (j) for example:

$$p \Vdash_\alpha ((\neg\varphi)^{(N_\gamma)})^{(N_\beta)} = p \Vdash_\alpha \neg((\varphi)^{(N_\gamma)})^{(N_\beta)}$$

iff $\neg(\exists q \supseteq p)\,[q \Vdash ((\varphi)^{(N_\gamma)})^{(N_\beta)}]$ and by the induction hypothesis this implies

$$\neg(\exists q \supseteq p)\,[q \Vdash_\alpha (\varphi)^{(N_\gamma)} \quad \text{or} \quad q \Vdash_\alpha ((\varphi)^{(N_\beta)})^{(N_\gamma)}]\,,$$

which implies $p \Vdash_\alpha \neg(\varphi)^{(N_\gamma)}$ and $p \Vdash_\alpha \neg\,((\varphi)^{(N_\beta)})^{(N_\gamma)}$. Similarly for the reverse implication.

Case (k).

$$p \Vdash_\alpha ((\forall_\lambda v)\,(\varphi(v))^{(N_\gamma)})^{(N_\beta)} =$$

$$= p \Vdash_\alpha (\forall_\lambda v)\,(N_\beta(v) \to (N_\gamma(v) \to ((\varphi)^{(N_\gamma)})^{(N_\beta)}(v)))$$

iff

$$(\forall \sigma \in \mathcal{T}_\alpha^c)\,[\mathrm{rnk}_\alpha(\sigma) < \lambda,\ \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\forall_\lambda v, ((\forall_\lambda v)\,(\varphi)^{(N_\gamma)})^{(N_\beta)})$$

$$\text{and } p \Vdash_\alpha N_\beta(\sigma) \to (N_\gamma(\sigma) \to (\varphi^{(N_\gamma)(N_\beta)}(\sigma))]\,.$$

Since $\mathrm{ord}_\alpha(((\sigma)^{(N_\gamma)})^{(N_\beta)} < \mathrm{ord}_\alpha(((\forall_\lambda v)\,(\varphi)^{(N_\gamma)})^{(N_\beta)}$, we have by the induction hypothesis concerning (ii) that

$$0 \Vdash_\alpha ((\sigma)^{(N_\gamma)})^{(N_\beta)} \approx (\sigma)^{(N_\gamma)} \wedge ((\sigma)^{(N_\gamma)})^{(N_\beta)} \approx ((\sigma)^{(N_\beta)})^{(N_\gamma)}.$$

Moreover, since $\mathrm{stg}_\alpha(\sigma) \leqslant \gamma$, we shall have by Corollary 3.04 that $0 \Vdash_\alpha (\sigma)^{(N_\gamma)} \approx \sigma$. Therefore

$$p \Vdash_\alpha ((\varphi)^{(N_\gamma)})^{(N_\beta)}(((\sigma)^{(N_\gamma)})^{(N_\beta)})\,;$$

and since there is a reduction in formula rnk_α,

$$p \Vdash_\alpha (\varphi)^{(N_\gamma)}((\sigma)^{(N_\gamma)}) \wedge ((\varphi)^{(N_\beta)})^{(N_\gamma)}(((\sigma)^{(N_\beta)})^{(N_\gamma)}).$$

Again by substitution we get

$$p \Vdash_\alpha (\varphi)^{(N_\gamma)}(\sigma) \wedge ((\varphi)^{(N_\beta)})^{(N_\gamma)}(\sigma)\,.$$

Now, as observed above,

$$\begin{aligned}\mathrm{ord}_\alpha(\forall_\lambda v, (\forall_\lambda v)\,(N_\gamma(v) \to (\varphi)^{(N_\gamma)}(v))) &= \\ &= \mathrm{ord}_\alpha(\forall_\lambda v, (\forall_\lambda v)\,(N_\beta(v) \to (N_\gamma(v) \to ((\varphi)^{(N_\gamma)})^{(N_\beta)}))) \\ &= \mathrm{ord}_\alpha(\forall_\lambda v, (\forall_\lambda v)\,(N_\gamma(v) \to (N_\beta(v) \to ((\varphi(v))^{(N_\beta)})^{(N_\gamma)}))) \\ &= (\mu, \lambda) \leqslant (\gamma, \lambda);\end{aligned}$$

$$\begin{aligned}\mathrm{ord}_\alpha(\sigma) = \mathrm{ord}_\alpha((\sigma)^{(N_\gamma)}) &= \mathrm{ord}_\alpha(((\sigma)^{(N_\gamma)})^{(N_\beta)}) \\ &= \mathrm{ord}_\alpha(((\sigma)^{(N_\beta)})^{(N_\gamma)}) < (\mu, \lambda)\,.\end{aligned}$$

Therefore we have shown that $p \Vdash_\alpha ((\forall_\lambda v)\,(\varphi)^{(N_\gamma)})^{(N_\beta)}$ implies $p \Vdash_\alpha (\forall_\lambda v)\,(\varphi)^{(N_\gamma)} \wedge ((\forall_\lambda v)\,(\varphi)^{(N_\beta)})^{(N_\gamma)}$. The reverse implications are analogously shown.

We now turn to (ii). It has to be shown that for terms $\sigma \in \mathcal{T}_\alpha^c$ such that $\mathrm{ord}_\alpha(\sigma^{(N_\gamma)}) = (\eta, \xi)$ we have

$$0 \Vdash_\alpha (\sigma^{(N_\gamma)})^{(N_\beta)} \approx (\sigma)^{(N_\gamma)} \wedge ((\sigma)^{(N_\gamma)})^{(N_\beta)} \approx ((\sigma)^{(N_\beta)})^{(N_\gamma)}.$$

If σ is an individual constant, then

$$(\sigma)^{(N_\gamma)} = ((\sigma)^{(N_\gamma)})^{(N_\beta)} = ((\sigma)^{(N_\beta)})^{(N_\gamma)} = \sigma$$

and there is nothing to prove. If σ is of the form $K(\tau)$ or $\theta(\tau)$, $\theta \in \Gamma$, then $(K(\tau))^{(N_\delta)} = K((\tau)^{(N_\delta)})$ and similarly for $\theta(\tau)$, $\delta \leqslant \aleph_1^{(M)}$. Therefore by Lemma 2.8111, if

$$0 \Vdash_\alpha (\tau)^{(N_\gamma)} \approx ((\tau)^{(N_\gamma)})^{(N_\beta)} \wedge ((\tau)^{(N_\gamma)})^{(N_\beta)} \approx ((\tau)^{(N_\beta)})^{(N_\gamma)},$$

then 0 forces_α the corresponding equalities for $K(\tau)$ and $\theta(\tau)$. Hence we consider σ which is neither an individual constant nor the result of an elementary operation; i.e. σ is an abstraction term. Assume $\sigma = (\gimel_\lambda u)\,(\psi(u))$; thus

$$(\sigma)^{(N_\gamma)} = (\gimel_\lambda u)\,(N_\gamma(u) \wedge (\psi)^{(N_\gamma)})\,,$$

$$((\sigma)^{(N_\gamma)})^{(N_\beta)} = (\gimel_\lambda u)\,(N_\beta(u) \wedge N_\gamma(u) \wedge ((\psi)^{(N_\gamma)})^{(N_\beta)})\,,$$

$$((\sigma)^{(N_\beta)})^{(N_\gamma)} = (\gimel_\lambda u)\,(N_\gamma(u) \wedge N_\beta(u) \wedge ((\psi)^{(N_\beta)})^{(N_\gamma)})\,.$$

Now,

$$0 \Vdash_\alpha (\sigma)^{(N_\gamma)} \approx ((\sigma)^{(N_\gamma)})^{(N_\beta)} =$$

$$= 0 \Vdash_\alpha (\forall_\xi u)\,(N_\eta(u) \to (u \in (\sigma)^{(N_\gamma)} \leftrightarrow u \in ((\sigma)^{(N_\gamma)})^{(N_\beta)}))$$

if and only if for all τ, with $\mathrm{rnk}(\tau) = \zeta < \xi$ and $\mathrm{ord}_\alpha(\tau) < (\eta, \xi)$,

$$0 \Vdash_\alpha \tau \in (\sigma)^{(N_\gamma)} \leftrightarrow \tau \in ((\sigma)^{(N_\gamma)})^{(N_\beta)} .$$

Note that since $\mathrm{ord}_\alpha(\tau) \leqslant (\eta, \zeta) < (\eta, \xi)$, we have $0 \Vdash_\alpha N_\eta(\tau)$; and by the induction hypothesis,

$$0 \Vdash_\alpha \tau \approx (\tau)^{(N_\gamma)} \approx ((\tau)^{(N_\gamma)})^{(N_\beta)} \approx ((\tau)^{(N_\beta)})^{(N_\gamma)} .$$

Using the induction hypothesis and substitution of equals (Lemmas 2.812, 2.814), we have

$$0 \subseteq r \Vdash_\alpha \tau \in (\sigma)^{(N_\gamma)} \quad \text{iff} \quad r \Vdash_\alpha (\psi)^{(N_\gamma)}(\tau)$$

$$\text{iff} \quad r \Vdash_\alpha (\psi)^{(N_\gamma)}((\tau)^{(N_\gamma)}) \quad \text{iff} \quad r \Vdash_\alpha ((\psi)^{(N_\gamma)})^{(N_\beta)}(((\tau)^{(N_\gamma)})^{(N_\beta)})$$

$$\text{iff} \quad r \Vdash_\alpha ((\psi)^{(N_\gamma)})^{(N_\beta)}(\tau) \wedge N_\gamma(\tau) \wedge N_\beta(\tau)$$

$$\text{iff} \quad r \Vdash_\lambda \tau \in ((\sigma)^{(N_\gamma)})^{(N_\beta)}$$

which implies $0 \Vdash_\alpha (\sigma)^{(N_\gamma)} \approx ((\sigma)^{(N_\gamma)})^{(N_\beta)}$. A similar argument shows also that

$$0 \Vdash_\alpha ((\sigma)^{(N_\gamma)})^{(N_\beta)} \approx ((\sigma)^{(N_\beta)})^{(N_\gamma)} .$$

Finally the lemma is proved for global formulas by a direct induction on length. We omit the details.

3.04. Corollary. *If* $\mathrm{stg}_\alpha(\sigma) = \eta$, *then for all* $\gamma \geqslant \eta$,

$$0 \Vdash_\alpha (\sigma)^{(N_\gamma)} \approx \sigma .$$

Proof*. If $\eta < \alpha$, assume $\sigma = (\ldots (\rho)^{(N_{\eta_1})}) \ldots)^{(N_{\eta_m})}$, where $\tau_i = \eta$; then the result is easily proven by induction on m using Lemma 2.801. If $\eta \geqslant \alpha$, then the result follows by Lemma 2.8112.

The following lemma will be proven in Section 6, but as this seems to be a natural place for it, we inserted it here. Although seemingly trivial, it cannot be proven directly from Definition 2.64 (see also the discussion in 3.53). We will not make use of this lemma.

3.05. Lemma. *$p \Vdash_\alpha N_\eta(\tau) \wedge \sigma \in \tau$ implies $p \Vdash_\alpha N_\eta(\sigma)$ for $\eta < \alpha$, $\sigma, \tau \in \mathcal{T}_\alpha^c$.*

3.06. Lemma. *For all $\sigma, \tau \in \mathcal{T}_\eta^c$, $\eta \leqslant \alpha$,*

$$p \Vdash_\alpha (\sigma)^{(N_\eta)} \approx (\tau)^{(N_\eta)} \quad \textit{iff} \quad p \Vdash_\alpha (\sigma \approx \tau)^{(N_\eta)}.$$

Proof*. $p \Vdash_\alpha (\sigma)^{(N_\eta)} \approx (\tau)^{(N_\eta)} = p \Vdash_\alpha (\forall_\xi u)\,(N_\mu(u) \to (u \in (\sigma)^{(N_\eta)}$ iff $u \in (\tau)^{(N_\eta)}))$, where $\xi = \max(\mathrm{rnk}((\sigma)^{(N_\eta)}), \mathrm{rnk}((\tau)^{(N_\eta)}))$, and $\mu = \max(\mathrm{stg}_\alpha((\sigma)^{(N_\eta)}), \mathrm{stg}_\alpha((\tau)^{(N_\eta)}))$. Clearly $\mu \leqslant \eta$. If $\mu < \eta$, then also $\max(\mathrm{stg}_\alpha(\sigma), \mathrm{stg}_\alpha(\tau)) = \mu$; thus by Lemma 2.801(c), for all ρ, $p \Vdash_\alpha N_\mu(\rho)$ implies $p \Vdash_\alpha N_\eta(\rho)$. Hence

$$p \Vdash_\alpha (\forall_\xi u)\,(N_\mu(u) \to (u \in (\sigma)^{(N_\eta)} \leftrightarrow u \in (\tau)^{(N_\eta)}))$$

iff

$$p \Vdash_\alpha (\forall_\xi u)\,(N_\eta(u) \to (N_\mu(u) \to (u \in (\sigma)^{(N_\eta)} \leftrightarrow u \in (\sigma)^{(N_\eta)}))),$$

i.e., $p \Vdash_\alpha (\sigma)^{(N_\eta)} \approx (\tau)^{(N_\eta)}$ iff $p \Vdash_\alpha (\sigma \approx \tau)^{(N_\eta)}$. If $\mu = \eta$, then

$$\max(\mathrm{stg}_\alpha(\sigma), \mathrm{stg}_\alpha(\tau)) = \gamma \geqslant \eta,$$

$$\sigma \approx \tau = (\forall_\xi u)\,(N_\gamma(u) \to (u \in \sigma \leftrightarrow u \in \tau)).$$

Hence

$$(\sigma \approx \tau)^{(N_\eta)} = (\forall_\xi u)\,(N_\eta(u) \to (N_\gamma(u) \to (u \in (\sigma)^{(N_\eta)} \leftrightarrow \omega \in (\tau)^{(N_\eta)}))).$$

Now $\gamma \geqslant \eta$; therefore by applying Lemma 2.801(c), we again get

$$p \Vdash_\alpha (\sigma)^{(N_\eta)} \approx (\tau)^{(N_\eta)} \quad \text{iff} \quad p \Vdash_\alpha (\sigma \approx \tau)^{(N_\eta)}.$$

3.10. Iterated forcing lemma. *If φ is a sentence of $\mathcal{L}_\beta$, $\beta < \alpha$ and $p \in P_\alpha$, then $p \Vdash_\alpha (\varphi)^{(N_\beta)}$ iff $p \Vdash_\beta \varphi$*

Proof. The lemma will first be proven for local sentences by induction on $\mathrm{rnk}_\alpha((\varphi)^{(N_\beta)})$, and $\mathrm{rnk}_\beta(\varphi)$ depending on the direction of implication that is being shown. We check the different cases of Definition 2.64.

(a) $p \Vdash_\alpha (\sigma \in \dot{s})^{(N_\beta)} = p \Vdash_\alpha (\sigma)^{(N_\beta)} \in \dot{s}$, $\dot{s}$ a set constant, iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists s' \in s)\,[q \Vdash_\alpha (\sigma)^{(N_\beta)} \approxeq \dot{s}'] \;.$$

By Lemma 3.06,

$$q \Vdash_\alpha (\sigma)^{(N_\beta)} \approxeq \dot{s}') \;\text{ iff }\; q \Vdash_\alpha (\sigma \approxeq \dot{s}')^{(N_\beta)}$$

(by the induction hypothesis) iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists s' \in s)\,[q \Vdash_\beta \sigma \approxeq \dot{s}']$$

iff $p \Vdash_\beta \sigma \in s$. (Note that $\approxeq$ is identical with $\approxeq$. For if $\mathrm{stg}_\alpha((\sigma)^{(N_\beta)}) = \eta < \beta$, then $\mathrm{stg}_\beta(\sigma) = \eta < \beta$; and if $\mathrm{stg}_\alpha((\sigma)^{(N_\beta)}) = \beta$, then also $\mathrm{stg}_\beta(\sigma) = \beta$. This is the case in all such instances throughout the proof, and will generally be tacitly assumed.)

(b) $p \Vdash_\alpha (\sigma \in a_i)^{(N_\beta)} = p \Vdash_\alpha (\sigma)^{(N_\beta)} \in a_i$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\exists \delta < 2)\,[q \Vdash_\alpha (\sigma)^{(N_\beta)} \approxeq (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)},\; \langle i, j, \delta\rangle \in q] \;.$$

By Lemma 3.03,

$$0 \Vdash_\alpha (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \approxeq ((\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)})^{(N_\beta)};$$

thus by Lemma 2.811(a) and 3.06,

$$q \Vdash_\alpha (\sigma)^{(N_\beta)} \approxeq (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)}$$

iff

$$q \Vdash_\alpha (\sigma \approxeq (\langle j, \delta\rangle)^{(N_0)})^{(N_\beta)}$$

iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\exists \delta < 2)\,[q \Vdash_\beta \sigma \approxeq (\langle \dot{i}, j, \dot{\delta}\rangle)^{(N_0)}\; \langle i, j, \delta\rangle \in q]$$

(by the induction hypothesis) iff

$$p \Vdash_\beta \sigma \in a_i \;.$$

(c) $p \Vdash_\alpha (\sigma \in {}^{(t,l)}\theta(\tau))^{(N_\beta)} = p \Vdash_\alpha (\sigma)^{(N_\beta)} \in {}^{(t,l)}\theta((\tau)^{(N_\beta)})$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \kappa < \omega)\,(\exists \delta < 2)$$

$$[q \Vdash_\alpha (\sigma)^{(N_\beta)} \supsetneq (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)}$$

$$\text{where either } k < m \text{ and } t(k) = \delta,$$
$$\text{or } k \geqslant m \text{ and } q \Vdash_\alpha (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \in (\tau)^{(N_\beta)},$$
$$\text{where } j = k - m + l\,].$$

Again, by Lemmas 3.03, 2.811(a) and 3.06,

$$q \Vdash_\alpha (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \in (\tau)^{(N_\beta)}$$

iff

$$q \Vdash_\alpha (\langle \dot{j}, \dot{\delta}\rangle \in \tau)^{(N_\beta)}, \qquad q \Vdash_\alpha (\sigma)^{(N_\beta)} \supsetneq (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)}$$

iff

$$q \Vdash_\alpha ((\sigma \in (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)})^{(N_\beta)}$$

iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists k < \omega)\,(\exists \delta < 2)$$

$$[q \Vdash_\beta \sigma \supsetneq (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)},$$

$$\text{where either } \kappa < m \text{ and } t(k) = \delta,$$

$$\text{or } k \geqslant m \text{ and } q \Vdash_\alpha (\langle \dot{j}, \dot{\delta}\rangle)^{(N_0)} \in \tau];$$

(by the induction hypothesis) iff

$$p \Vdash_\beta \sigma \in {}^{(t,l)}\theta(\tau)\,.$$

(d) $p \Vdash_\alpha (\sigma \in K(\tau))^{(N_\beta)} = p \Vdash_\alpha (\sigma)^{(N_\beta)} \in K((\tau)^{(N_\beta)})$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \theta \in \Gamma)\,[q \Vdash_\alpha (\sigma)^{(N_\beta)} \in \theta((\tau)^{(N_\beta)})]$$

iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \theta \in \Gamma)\,[q \Vdash_\alpha (\sigma \in \theta(\tau))^{(N_\beta)}];$$

iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \theta \in \Gamma)\,[q \Vdash_\beta \sigma \in \theta(\tau)]$$

(by the induction hypothesis) iff

$$p \Vdash_\beta \sigma \in K(\tau)\,.$$

(e) $p \Vdash_\alpha (\sigma \in \chi_e)^{(N_\beta)}$ iff

$$(\forall r \supseteq p(\exists q \supseteq r)\,(\exists \bar{\theta}^n \in \bar{\Gamma}^n)\,(\exists l < \omega)\,(\exists \delta < 2)$$
$$[q \Vdash_\alpha (\sigma)^{(N_\beta)} \approx (\dot{\langle}\dot{\langle}\bar{\theta}\cdot \bar{a}_h\dot{\rangle}, \dot{\langle} l, \delta\dot{\rangle}\dot{\rangle})^{(N_\alpha)}$$
$$\text{and } \langle \epsilon, \bar{\theta}^n, l, \delta\rangle \in q\},$$

where $e = (h, g, s)$. The result is easily seen to follow using Lemmas 3.03, 2.811(a) and 3.06 as in the previous cases.

(f) The case $p \Vdash_{\alpha} (\sigma \in I_\gamma)^{(N_\beta)}$, $\gamma \leqslant \beta$, is similarly shown using the fact that $\mathrm{stg}_\alpha((\dot{\langle} e, \chi_e\dot{\rangle})^{(N_\gamma)}) = \gamma$; thus Lemma 3.03 is applicable, i.e.,

$$0 \Vdash_\alpha (\dot{\langle}\dot{e}, \chi_e\dot{\rangle})^{(N_\gamma)} \approx ((\dot{\langle}\dot{e}, \chi_e\dot{\rangle})^{(N_\gamma)})^{(N_\beta)}\,.$$

(g) $p \Vdash_\alpha (N_\gamma(\sigma))^{(N_\beta)} = p \Vdash_\alpha N_\gamma((\sigma)^{(N_\beta)})$ iff

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \tau \in \mathcal{T}^c_\alpha)$$
$$[\mathrm{stg}_\alpha(\tau) \leqslant \gamma,\ \mathrm{ord}_\alpha(\tau) \prec^* \mathrm{ord}_\alpha((\sigma)^{(N_\beta)}),$$
$$\text{and } q \Vdash_\alpha (\sigma)^{(N_\beta)} \approx \tau]\,.$$

Since $\mathrm{stg}_\alpha(\tau) \leqslant \gamma$, we have by Lemma 3.04 that $0 \Vdash_\alpha \tau \approx (\tau)^{(N_\beta)}$. Hence $q \Vdash_\alpha (\sigma)^{(N_\beta)} \approx (\tau)^{(N_\beta)}$. By Lemma 3.06 and the induction hypothesis, this implies $q \Vdash_\beta \sigma \approx \tau$. If $\mathrm{stg}_\alpha(\sigma) = \eta < \beta$, then $\mathrm{stg}_\beta(\sigma) = \eta$; hence $\mathrm{ord}_\beta(\tau) \prec^* \mathrm{ord}_\beta(\sigma)$; if $\mathrm{stg}_\alpha((\sigma)^{(N_\beta)}) = \beta$, then σ is not of the form $(\rho)^{(N_\eta)}$, $\eta < \beta$; therefore $\mathrm{ord}_\beta(\sigma) = (\beta, \mathrm{rnk}(\sigma)) \ {}^*\!\succ \mathrm{ord}(\tau)$. In any case we obtain $p \Vdash_\beta N_\gamma(\sigma)$. The other direction is even simpler.

Cases (h), (i), (j) are trivial, resulting from the induction hypothesis via an immediate reduction in length.

(k) Assume $p \Vdash_\alpha (\forall_\xi v)\,(\varphi(v))^{(N_\beta)} = p \Vdash_\alpha (\forall_\xi v)\,(N_\beta(v) \to (\varphi)^{(N_\beta)}(v))$. To show that $p \Vdash_\alpha (\forall_\xi v)\,\varphi(v)$, consider $\sigma \in \mathcal{T}^c_\beta$ such that $\mathrm{rnk}(\sigma) < \xi$ and

$$\mathrm{ord}_\beta(\sigma) \leqslant \mathrm{ord}_\beta(\forall_\xi v, (\forall_\xi v)\,(\varphi)) = (\eta, \xi)$$
$$= \mathrm{ord}_\alpha(\forall_\xi v, (\forall_\xi v)\,(\varphi))^{(N_\beta)}.$$

Consider two cases:

(i) $\eta < \beta$; then by Lemma 3.04, $0 \Vdash_\alpha \sigma \approx (\sigma)^{(N_\beta)}$. Now by assumption we have $p \Vdash_\alpha (\varphi)^{(N_\beta)}(\sigma)$; thus by substitution of equals (Lemmas 2.814, 2.812), this implies $p \Vdash_\alpha (\varphi)^{(N_\beta)}((\sigma)^{(N_\beta)})$ and by the induction hypothesis we get $p \Vdash_\beta \varphi(\sigma)$, as required.

(ii) If $\eta = \beta$, then

$$\mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) = (\beta, \lambda) < \mathrm{ord}_\alpha(\forall_\xi v, (\forall_\xi v)(\varphi)^{(N_\beta)})$$
$$= (\beta, \xi), \quad \lambda < \xi .$$

Hence immediately $p \Vdash_\alpha (\varphi)^{(N_\beta)}((\sigma)^{(N_\beta)})$, which implies by the induction hypothesis that $p \Vdash_\beta \varphi(\sigma)$. Thus $p \Vdash_\beta (\forall_\xi v)(\varphi)$.

To see the other direction assume $p \Vdash_\beta (\forall_\xi v)(\varphi(v))$, and let σ be such that $\mathrm{rnk}_\alpha(\sigma) < \xi$ and $\mathrm{ord}_\alpha(\sigma) \leqslant \mathrm{ord}_\alpha(\forall_\xi v, (\forall_\xi v)(\varphi)^{(N_\beta)}) = (\eta, \xi)$.

(i) If $\eta < \beta$, then $\mathrm{ord}_\beta(\forall_\xi v, (\forall_\beta v)(\varphi)) = (\eta, \xi)$. Thus by assumption $p \Vdash_\beta \varphi(\sigma)$, and this implies by the induction hypothesis that $p \Vdash_\alpha (\varphi(\sigma))^{(N_\beta)}$. Since $0 \Vdash_\alpha \sigma \approx (\sigma)^{(N_\beta)}$, $p \Vdash_\alpha (\varphi)^{(N_\beta)}(\sigma)$ by substitution of equals, as required.

(ii) If $\eta = \beta$, then also $\mathrm{ord}_\beta(\forall_\xi v, (\forall_\xi v)(\varphi)) = (\beta, \xi)$ which implies $p \Vdash_\alpha \varphi(\sigma)$; and, by the induction hypothesis,

$$p \Vdash_\alpha (\varphi(\sigma))^{(N_\beta)} = p \Vdash_\alpha (\varphi)^{(N_\beta)}((\sigma)^{(N_\beta)}) .$$

Since $\sigma \in \mathcal{T}^c_\beta$, $\mathrm{stg}_\alpha(\sigma) \leqslant \beta$, and $0 \Vdash_\alpha \sigma \approx (\sigma)^{(N_\beta)}$, we again get $p \Vdash_\alpha (\varphi)^{(N_\beta)}(\sigma)$ as required.

(l) $p \Vdash_\alpha (\sigma \in \tau)^{(N_\beta)}$, $\tau = (\mathsf{X}_\xi u)(\varphi(u))$, $\mathrm{rnk}((\sigma)^{(N_\beta)}) = \mathrm{rnk}(\sigma) < \xi$, and $\mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) < \mathrm{ord}_\alpha((\tau)^{(N_\beta)})$, iff

$$p \Vdash_\alpha N_\beta((\sigma)^{(N_\beta)}) \wedge (\varphi)^{(N_\beta)}((\sigma)^{(N_\beta)}) ,$$

which by the induction hypothesis, implies $p \Vdash_\beta \varphi(\sigma)$. If $\mathrm{ord}_\beta(\tau) = (\beta, \xi)$, then $\mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\tau)$; hence $p \Vdash_\alpha \sigma \in \tau$. If $\mathrm{stg}_\beta(\tau) < \beta$, then

$$\mathrm{stg}_\alpha((\sigma)^{(N_\beta)}) = \mathrm{stg}_\beta(\sigma) < \mathrm{stg}_\beta(\tau) = \mathrm{stg}_\alpha((\tau)^{(N_\beta)}) .$$

Therefore $\mathrm{ord}_\beta(\sigma) < \mathrm{ord}_\beta(\tau)$ which implies $p \Vdash_\alpha \sigma \in \tau$. To see the other direction assume $p \Vdash_\beta \sigma \in \tau$ and $\mathrm{ord}_\beta(\sigma) < \mathrm{ord}_\beta(\tau)$; then $p \Vdash_\beta \varphi(\sigma)$, and by the induction hypothesis

$$p \Vdash_\alpha (\varphi)^{(N_\beta)}((\sigma)^{(N_\beta)}) .$$

(i) If $\mathrm{stg}_\alpha(\tau) = \beta$, then also

$$\mathrm{stg}_\alpha((\tau)^{(N_\beta)}) = \beta , \qquad \mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) < \mathrm{ord}_\alpha((\tau)^{(N_\beta)}) ;$$

thus $p \Vdash_\alpha (\sigma \in \tau)^{(N_\beta)}$.

(ii) If $\mathrm{stg}_\beta(\tau) = \eta < \beta$, then also $\mathrm{stg}_\beta(\sigma) \leqslant \eta < \beta$; hence by Lemma 3.04, $0 \Vdash_\alpha (\sigma)^{(N_\beta)} \approxeq \sigma$, which implies $p \Vdash_\alpha (\varphi)^{(N_\beta)}(\sigma)$; and since

$$\mathrm{ord}_\alpha(\sigma) = \mathrm{ord}_\beta(\sigma) < \mathrm{ord}_\beta(\tau) = \mathrm{ord}_\alpha((\tau)^{(N_\beta)}) = (\eta, \xi), \quad \eta < \beta,$$

we get $p \Vdash_\alpha \sigma \in (\tau)^{(N_\beta)}$; and again by substitution of equals. $p \Vdash_\alpha (\sigma)^{(N_\beta)} \in (\tau)^{(N_\beta)}$ as required.

(m) $p \Vdash_\alpha (\sigma)^{(N_\beta)} \in (\tau)^{(N_\beta)}$, τ is an abstraction term and $\mathrm{rnk}((\sigma)^{(N_\beta)}) \geqslant \mathrm{rnk}((\tau)^{(N_\beta)})$ or $\mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) \geqslant \mathrm{ord}_\alpha((\tau)^{(N_\beta)})$ iff

$$\begin{aligned}(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \sigma' \in \mathcal{T}^c_\alpha)\,[&\mathrm{rnk}(\sigma') < \mathrm{rnk}((\tau)^{(N_\beta)}),\\ &\mathrm{ord}_\alpha(\sigma') < \mathrm{ord}_\alpha((\tau)^{(N_\beta)}),\\ &q \Vdash_\alpha \sigma \approxeq \sigma' \wedge \sigma' \in (\tau)^{(N_\beta)}].\end{aligned}$$

Since $\mathrm{stg}_\alpha(\sigma') \leqslant \beta$ implies by Lemma 3.04 that $0 \Vdash_\alpha \sigma' \approxeq (\sigma')^{(N_\beta)}$; hence

$$q \Vdash_\alpha (\sigma)^{(N_\beta)} \approxeq (\sigma)^{(N_\beta)} \wedge (\sigma')^{(N_\beta)} \in (\tau)^{(N_\beta)}.$$

By Lemma 3.06,

$$q \Vdash_\alpha (\sigma \approxeq \sigma' \wedge \sigma' \in \tau)^{(N_\beta)}$$

iff

$$q \Vdash_\beta \sigma \approxeq \sigma' \wedge \sigma' \in \tau,$$

by the induction hypothesis. Now,

$$\mathrm{rnk}(\sigma') < \mathrm{rnk}((\tau)^{(N_\beta)}) = \mathrm{rnk}(\tau)\,;$$

thus if $\mathrm{stg}_\alpha((\tau)^{(N_\beta)}) = \eta < \beta$, then

$$\mathrm{ord}_\alpha(\sigma') = \mathrm{ord}_\beta(\sigma') < \mathrm{ord}_\beta(\tau) = \mathrm{ord}_\alpha((\tau)^{(N_\beta)})\,.$$

If $\mathrm{stg}_\alpha((\tau)^{(N_\beta)}) = \beta$, then $\mathrm{stg}_\beta(\tau) = \beta$. In any case $\mathrm{ord}_\beta(\sigma') < \mathrm{ord}_\beta(\tau)$, therefore $p \Vdash_\beta \sigma \in \tau$. To see the other direction assume $p \Vdash_\beta \sigma \in \tau$. Then

$$\begin{aligned}(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \sigma' \in \mathcal{T}^c_\beta)\,[&\mathrm{rnk}(\sigma') < \mathrm{rnk}(\tau),\\ &\mathrm{ord}_\beta(\sigma') < \mathrm{ord}_\beta(\tau),\\ &q \Vdash_\beta \sigma \approxeq \sigma' \wedge \sigma' \in \tau].\end{aligned}$$

By the induction hypothesis and Lemma 3.06 as in previous cases,

$$q \Vdash_\alpha (\sigma)^{(N_\beta)} \not\approx (\sigma')^{(N_\beta)} \wedge (\sigma')^{(N_\beta)} \in (\tau)^{(N_\beta)};$$

and since by previous considerations $\mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) < \mathrm{ord}_\alpha((\tau)^{(N_\beta)})$, we have

$$p \Vdash_\alpha (\sigma \in \tau)^{(N_\beta)}.$$

For global φ the lemma is now proven by direct induction on the length of φ, relying on the fact that the lemma is true for local sentences. We omit the details which are trivial.

Let Q be a generic set of conditions for $\mathcal{L}_\alpha$. In N_Q the languages $\mathcal{L}_\beta$, $\beta \leqslant \aleph_1^{(M)}$ can be defined together with sat_β-val_β functions for $\beta \leqslant \alpha$. Hence the models $N_\beta = M[I_\beta]$ can also be defined in N_Q, $\beta \leqslant \alpha$. Now Q not only contains all the basic information determining the truth of statements in N_Q, but also all necessary information determining the truth of statements of $\mathcal{L}_\beta$ in N_β. Moreover all information responsible for the satisfaction of $\mathcal{L}_\beta$ in N_β is already contained in $Q_\beta = Q \cap P_\beta$. Since we will have to refer to the intermediate stage models N_β, $\beta < \alpha$, early in this work, the constants $N_\gamma(\cdot)$ were added, enabling us to circumvent the entire process of defining the $\mathcal{L}_\beta, P_\beta, \Vdash_\beta$ and N_β in N_Q at the outset. This, however, will eventually have to be done in order to prove one of our main lemmas (9.30).

The connection between $\mathcal{L}_\beta, P_\beta, \Vdash_\beta$, sat_β and satisfaction in N_Q is now established via the iterated forcing lemma. We first prove:

3.20. Lemma. *If $Q \subseteq P_\alpha$ is a generic set of conditions for $\mathcal{L}_\alpha$, then*

$$Q_\beta = Q \cap P_\beta, \quad \beta < \alpha,$$

is a generic set of conditions for $\mathcal{L}_\beta$, and

$$Q_\beta = \{\hat{p}^\beta \mid p \in Q\}.$$

Proof. (a) Let φ be a sentence of $\mathcal{L}_\beta$; then there exists a $p \in Q$ such that $p \parallel_\alpha (\varphi)^{(N_\beta)}$. By the Lemma 3.10 $\hat{p}^\beta \parallel_\beta \varphi$; and since $\hat{p}^\beta \in Q_\beta$, we have Q_β complete for $\mathcal{L}_\beta$.

(b) If $p, q \in Q_\beta$, let $p \subseteq p' \in Q$ and $q \subset q' \in Q$; then there exists $r' \in Q$ such that $r' \supseteq p' \cup q'$; thus $\hat{r}^\beta \in Q_\beta$, $\hat{r}^\beta \supseteq p \cup q$.

(c) If $p \in Q_\beta$ and $q \in P_\beta$, $q \subseteq P$, then $q \in Q$ implies $q = \hat{q}^\beta \in Q_\beta$.

$Q_\beta = \{\hat{p}^\beta \mid p \in Q\}$ also follows directly from property (c) for Q.

3.30. Lemma. *If Q is generic for $\mathcal{L}_\alpha$ and $\mathrm{stg}_\alpha(\tau) \leq \beta$, then*

$$\mathrm{val}_Q(\tau) = \mathrm{val}_{Q_\beta}(\tau).$$

Proof. This is proven by induction on $\mathrm{ord}_\alpha(\tau)$. If $x \in \mathrm{val}_Q(\tau)$, then for some $\rho \in \mathcal{T}^c_\alpha$, $\mathrm{val}_Q(\rho) = x$ and $Q \Vdash_\alpha \rho \in \tau$. By Lemma 2.81 we can assume that $\mathrm{rnk}(\rho) < \mathrm{rnk}(\tau)$ and $\mathrm{ord}_\alpha(\rho) < \mathrm{ord}_\alpha(\tau)$. Hence also $\rho \in \mathcal{T}^c_\beta$. By Lemma 3.04,

$$Q \Vdash_\alpha \rho \approx (\tau)^{(N_\beta)} \wedge \tau \approx (\tau)^{(N_\beta)};$$

thus by substitution of equals,

$$Q \Vdash_\alpha (\rho)^{(N_\beta)} \in (\tau)^{(N_\beta)};$$

which implies, by Lemma 3.10, that $Q_\beta \Vdash_\beta \rho \in \tau$. By assumption $\mathrm{val}_{Q_\beta}(\rho) = \mathrm{val}_Q(\rho)$; thence

$$\mathrm{val}_Q(\tau) \subseteq \mathrm{val}_{Q_\beta}(\tau)\ .$$

To see the reverse inclusion, assume $p \in Q_\beta$, $p \Vdash_\beta \rho \in \tau$ where $\mathrm{rnk}(\rho) < \mathrm{rnk}(\tau)$, and $\mathrm{ord}_\alpha(\rho) < \mathrm{ord}_\alpha(\tau)$. Then

$$p \Vdash_\alpha (\rho)^{(N_\beta)} \in (\tau)^{(N_\beta)}$$

by Lemma 3.10. Since

$$0 \Vdash_\alpha \rho \approx (\rho)^{(N_\beta)} \wedge \tau \approx (\rho)^{(N_\beta)},$$

(by Lemma 3.04), $p \Vdash_\alpha \rho \in \tau$. By assumption $\mathrm{val}_Q(\rho) = \mathrm{val}_{Q_\beta}(\rho)$; thus $\mathrm{val}_{Q_\beta}(\tau) \subseteq \mathrm{val}_Q(\tau)$. Therefore $\mathrm{val}_Q(\tau) = \mathrm{val}_{Q_\beta}(\tau)$, as was to be shown.

3.31. Lemma. *For all $\sigma \in \mathcal{T}^c_\beta$, $\beta < \alpha$, and Q generic for $\mathcal{L}_\alpha$,*

$$\mathrm{val}_Q((\sigma)^{(N_\beta)}) = \mathrm{val}_{Q_\beta}(\sigma)\ .$$

Proof. By induction on $\mathrm{ord}_\alpha((\sigma)^{(N_\beta)})$. If $x \in (\sigma)^{(N_\beta)}$, then for some $\rho \in \mathcal{L}_\alpha$ such that $\mathrm{ord}_\alpha(\rho) < \mathrm{ord}_\alpha((\sigma)^{(N_\beta)})$ and $\mathrm{rnk}(\rho) < \mathrm{rnk}((\sigma)^{(N_\beta)})$, there exists a $p \in Q$ such that $p \Vdash_\alpha \rho \in (\sigma)^{(N_\beta)}$. Since $\mathrm{stg}_\alpha(\rho) \leq \beta$, $0 \Vdash_\alpha \rho \approx (\rho)^{(N_\beta)}$; hence

$$p \Vdash_\alpha (\rho)^{(N_\beta)} \in (\sigma)^{(N_\beta)},$$

and by Lemma 3.10, $\hat{p}^\beta \Vdash_\beta \mu \in \sigma$. By Lemma 3.30, $\mathrm{val}_Q(\rho) = \mathrm{val}_{Q_\beta}(\rho)$; hence

$$\mathrm{val}_Q((\sigma)^{(N_\beta)}) \subseteq \mathrm{val}_{Q_\beta}(\sigma) .$$

To see the reverse inclusion assume $p \in Q_\beta$, $p \Vdash_\beta \rho \in \sigma$, where $\mathrm{rnk}(\rho) < \mathrm{rnk}(\sigma)$ and $\mathrm{ord}_\beta(\rho) < \mathrm{ord}_\beta(\sigma)$. Then by Lemma 3.10,

$$p \Vdash_\alpha (\rho)^{(N_\beta)} \in (\sigma)^{(N_\beta)} ;$$

and since

$$\mathrm{ord}_\alpha((\rho)^{(N_\beta)}) < \mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) ,$$

$$\mathrm{rnk}_\alpha((\rho)^{(N_\beta)}) = \mathrm{rnk}_\beta(\rho) < \mathrm{rnk}_\beta(\sigma) = \mathrm{rnk}_\alpha((\sigma)^{(N_\beta)}) ,$$

we have by the induction hypothesis that

$$\mathrm{val}_Q((\rho)^{(N_\beta)}) = \mathrm{val}_{Q_\beta}(\rho) ;$$

which implies

$$\mathrm{val}_{Q_\beta}(\sigma) \subseteq \mathrm{val}_Q((\sigma)^{(N_\beta)}) ,$$

as required.

3.40. Corollary. *If Q is generic for $\mathcal{L}_\alpha$, then for all $\sigma \in \mathcal{T}^c_\beta$, $\varphi \in \mathcal{F}^c_\beta$, $\beta < \alpha$, we have*

$$\mathrm{val}_Q((\sigma)^{(N_\beta)}) = \mathrm{val}_{Q_\beta}(\sigma),$$

$$N_{Q_\beta} \models \varphi \quad \textit{iff} \quad (\exists p \in Q_\beta)\, [p \Vdash_\alpha (\varphi)^{(N_\beta)}] .$$

The content of Lemma 3.31 is that although the interpretation of φ, σ varies in the model extensions, the interpretation of $(\varphi)^{(N_\beta)}$, $(\sigma)^{(N_\beta)}$ is the same in all further extensions and pertains to the model N_{Q_β}.

It is clear that,

3.50. Lemma. $N_{Q_\beta} = N_\beta = M[I_\beta]$.

In this connection the following facts are worth stressing.

3.51. Definition. Let $U_\beta = \{\mathrm{val}_Q(\sigma) \mid N_Q = N_\beta(\sigma)\}$, where Q is generic for $\mathcal{L}_\alpha$ and $\beta < \alpha$.

Denote $N_\beta^* = \langle U_\beta, \in \restriction U_\beta\rangle$.

3.52. Lemma. $N_\beta^* = N_\beta = N_{Q\beta}$.

Proof. Since $N_Q = N_\beta(\sigma)$ iff there is a $\tau \in \mathcal{T}_\alpha^c$ such that $\mathrm{stg}_\alpha(\tau) \leqslant \beta$, $\mathrm{ord}_\alpha(\tau) \prec^* \mathrm{ord}_\alpha(\sigma)$ and $N_Q \models \tau = \sigma$, it suffices to prove that

$$N_\beta^* = \langle\{\mathrm{val}_Q(\sigma) \mid \mathrm{stg}_\alpha(\sigma) \leqslant \beta\}, \in \restriction U_\beta\rangle = N_\beta .$$

By Lemma 3.30, we have $\mathrm{val}_Q(\sigma) = \mathrm{val}_{Q\beta}(\sigma)$ for $\mathrm{stg}_\alpha(\sigma) \leqslant \beta$; hence $N_\beta^* \subseteq N_\beta$. On the other hand, if $\sigma \in \mathcal{T}_\beta^c$, then by Lemma 2.8112

$$\mathrm{val}_{Q\beta}(\sigma) = \mathrm{val}_{Q\beta}((\sigma)^{(N_\beta)}) ;$$

and since

$$\mathrm{val}_Q((\sigma)^{(N_\beta)}) = \mathrm{val}_{Q\beta}((\sigma)^{(N_\beta)}) ,$$

(again by Lemma 3.30), $N_\beta \subseteq N_\beta^*$.

3.53. In order to show that the predicate $N_\beta(\cdot)$ actually describes N_β in N_α we must still prove that if $\sigma \in \mathcal{T}_\alpha^c$ is such that if $\mathrm{val}_\alpha(\sigma) \in N_\beta$, then $N_\alpha = N_\beta(\sigma)$. Therefore we will have to find a term $\tau \in \mathcal{T}_\beta^c$ such that

$$\mathrm{rnk}(\tau) < \mathrm{rnk}(\sigma) + \aleph_1^{(M)} \cdot 2 ,$$

$$N_\alpha \models \sigma = (\tau)^{(N_\beta)} .$$

The difficulty here is the bound required on $\mathrm{rnk}(\tau)$. For this reason the proof of the following lemma will have to be delayed until we show the countable chain condition and define $\mathrm{sat}_\beta^\alpha$, $\mathrm{val}_\beta^\alpha$ functions in N_α, $\beta \leqslant \alpha < \aleph_1^{(M)}$. We will not actually make use of this lemma.

3.54. Lemma. $\mathrm{val}_\alpha(\sigma) \in N_\beta$ *iff* $N_\alpha \models N_\beta(\sigma)$. (See Lemmas 8.40 and 8.42.)

We end this section with a few observations concerning the levels of constructibility of the model elements and their *orders*.

3.60. Definition. Let Q be a generic set of conditions for $\mathcal{L} = \mathcal{L}_{\aleph_1^{(M)}}$, and

$$x \in N_\alpha \subseteq N_Q\ , \quad \alpha \leqslant \aleph_1^{(M)}\ ;$$

then

(i) $\mathrm{RNK}_\alpha^Q(x)$, (or simply $\mathrm{RNK}_\alpha(x)$), is the least ordinal ξ for which there is a term $\sigma \in \mathcal{T}_\alpha^c$ such that $\mathrm{rnk}_\alpha(\sigma) = \xi$ and $\mathrm{val}_{Q_\alpha}(\sigma) = x$. $\mathrm{RNK}_\alpha(x)$ is said to be the *α-rank of x*.

(ii) $\mathrm{ORD}_\alpha^Q(x)$ (or simply $\mathrm{ORD}_\alpha(x)$), is the least ordinal pair, (η, ξ) for which there is a $\sigma \in \mathcal{T}_\alpha^c$ such that $\mathrm{ord}_\alpha(\sigma) = (\eta, \xi)$ and $\mathrm{val}_{Q_\alpha}(\sigma) = x$. $\mathrm{ORD}_\alpha(x)$ is said to be the *α-order of x*.

3.70. Lemma. (a) *For all* $s \in M$ *and* $\alpha \leqslant \aleph_1^{(M)}$,

$$\mathrm{RNK}_\alpha(s) = \mathrm{Rank}(s) = \mathrm{rnk}_\alpha(\dot{s})\ ;$$

(b) *for all* $0 \leqslant \beta < \alpha \leqslant \aleph_1^{(M)}$ *and* $x \in N_\beta$, $\mathrm{RNK}_\alpha(x) \leqslant \mathrm{RNK}_\beta(x)$;
(c) $\mathrm{ORD}_\alpha(x) = \mathrm{ORD}_\beta(x)$.

Proof. (a) We show by induction on $\mathrm{Rank}(s)$, that if $\mathrm{val}_{Q_\alpha}(\sigma) = s$, then $\mathrm{rnk}_\alpha(\sigma) \geqslant \mathrm{Rank}(s)$. From this it is immediate that $\mathrm{RNK}_\alpha(s) = \mathrm{Rank}(s)$, since $\mathrm{rnk}_\alpha(\dot{s}) = \mathrm{Rank}(s)$. For all $s' \in s$, $\mathrm{rnk}(\dot{s}') < \mathrm{rnk}(\dot{s})$, and $0 \Vdash_\alpha \dot{s}' \in s$. If $\mathrm{val}_{Q_\alpha}(\sigma) = s$, then there is a $p \in Q_\alpha$ such that

$$p \Vdash_\alpha \sigma = \dot{s} \wedge \dot{s}' \in \dot{s}\ .$$

By Lemmas 2.81, 2.812, there is a $\tau \in \mathcal{L}_\alpha$ for which $\mathrm{rnk}(\tau) < \mathrm{rnk}(\sigma)$ and $p \Vdash_\alpha \dot{s}' = \tau$. By the induction hypothesis $\mathrm{rnk}(\tau) \geqslant \mathrm{rnk}(\dot{s}') = \mathrm{Rank}(s')$; therefore

$$\mathrm{rnk}(\sigma) \geqslant \bigcup_{s' \in s} \mathrm{rnk}(\dot{s}') = \bigcup_{s' \in s} \mathrm{Rank}(s') = \mathrm{Rank}(s) = \mathrm{rnk}(\dot{s})\ .$$

(b) This is immediate from Lemma 3.31; for if $x = \mathrm{val}_{Q_\beta}(\sigma)$, then

$$x = \mathrm{val}_{Q_\alpha}((\sigma)^{(N_\beta)})\ , \qquad \mathrm{rnk}(\sigma) = \mathrm{rnk}((\sigma)^{(N_\beta)})\ .$$

(c) If $x = \mathrm{val}_{Q_\beta}(\sigma) = \mathrm{val}_{Q_\alpha}((\sigma)^{(N_\beta)})$ and $\mathrm{rnk}(\sigma) = \xi$, then

$$\mathrm{ord}_\beta(\sigma) = (\eta, \xi) \leqslant (\beta, \xi)\ , \qquad \mathrm{ord}_\alpha((\sigma)^{(N_\beta)}) \leqslant (\beta, \xi)\ .$$

If $\eta < \beta$, then also $\mathrm{stg}_\alpha((\sigma)^{(N_\beta)}) = \eta$; and if $\eta = \beta$, then also $\mathrm{stg}_\alpha((\sigma)^{(N_\beta)}) = \beta$; hence in any case $\mathrm{ord}_\beta(\sigma) = \mathrm{ord}_\alpha((\sigma)^{(N_\beta)})$. Therefore $\mathrm{ORD}_\beta(x) \geqslant \mathrm{ORD}_\alpha(x)$.

Now if $\mathrm{ORD}_\alpha(x) < \mathrm{ORD}_\beta(x)$, then for some σ such that $\mathrm{stg}_\alpha(\sigma) \leqslant \beta$ and $\mathrm{val}_{Q_\alpha}(\sigma) = x$, we have $\mathrm{ord}_\alpha(\sigma) < \mathrm{ORD}_\beta(x)$; but then by Lemma 3.30, $\mathrm{val}_{Q_\alpha}(\sigma) = \mathrm{val}_{Q_\beta}(\sigma)$. Since in this case $\mathrm{ord}_\beta(\sigma) = \mathrm{ord}_\alpha(\sigma)$, we obtain a contradiction; thus $\mathrm{ORD}_\alpha(x) = \mathrm{ORD}_\beta(x)$ as required.

3.71. Corollary. *If α is an ordinal in M, then* $\mathrm{RNK}_\beta(\alpha) = \alpha$.

3.72. Remark. If we had ensured that $\mathrm{RNK}_\alpha(x) = \mathrm{RNK}_\beta(x)$ for all $\beta < \alpha \leqslant \aleph_1^{(M)}$, $x \in N_\beta$, then we would have avoided dealing with the orders of terms and formulas. This appears to involve considerable difficulties in general, while the method adopted here is readily applicable in any iterated forcing procedure.

3.73. Corollary. $\mathrm{RNK}_\alpha(a_i) = \mathrm{RNK}_\alpha(\boldsymbol{\theta}(a_i)) = \omega$, $\mathrm{RNK}_\alpha(K(a_i)) = \omega + 1$, $i \in \omega$, $\theta \in \Gamma$.

Proof. Clearly the ranks are at least those quoted. To see that they are not less, assume that for some p and σ rank less than ω, $p \Vdash_\alpha \sigma = a_i$. If $\mathrm{rank}(\sigma) = m < \omega$, set $q = p \cup \{\langle i, n, \delta\rangle\}$ for sufficiently large $n > m$, and proper $\delta < 2$. Then

$$q \Vdash_\alpha \dot{\langle} \dot{n}, \dot{\delta} \dot{\rangle} \in a_i \wedge \sigma = a_i \,,$$

hence $q \Vdash_\alpha \dot{\langle} \dot{n}, \dot{\delta} \dot{\rangle} \in \sigma$. From Lemma 3.70 we obviously have

$$\mathrm{RNK}_\alpha(\dot{\langle} \dot{n}, \dot{\delta} \dot{\rangle}) \geqslant \max(n, \delta) \,.$$

Therefore, by Lemma 2.81 we must have $\mathrm{rnk}(\sigma) \geqslant \max(n, \delta) > m$, which is a contradiction. Similarly we can prove $\mathrm{RNK}_\alpha(\boldsymbol{\theta}(a_i)) = \omega$, and $\mathrm{RNK}_\alpha(K(a_i)) = \omega + 1$.

3.74. Lemma. (a) $\mathrm{RNK}_\alpha(\dot{\{} \rho_1, \ldots, \rho_n \dot{\}}) = \max_i (\mathrm{RNK}_\alpha(\rho_i) + 1)$
(b) $\mathrm{RNK}_\alpha(\dot{\langle} \rho_1, \ldots, \rho_n \dot{\rangle}) = \max_i (\mathrm{RNK}_\alpha(\rho_i) + 2n)$.

Proof. (a) follows directly from the definitions and (b) follows from (a) by finite induction.

4. The primitive generic elements

4.0. We now turn to the primitive elements adjoined to the model M, obtained by a realization of $\mathcal{L}$ with respect to a generic set of conditions $Q \subseteq P = P_{\aleph_1}(M)$, which will be fixed throughout the rest of this paper, and val_{Q_α}, sat_{Q_α} will be denoted val_α, sat_α, respectively. We want to show that these elements have the intended properties. To this end we first prove some lemmas concerning the amount of freedom there is in extending conditions. Also Lemma 2.72 concerning the induction hypothesis is proven here; the proof is first preceded by some lemmas that are directly or indirectly involved, particularly regarding the properties of the H_σ.

4.01. Lemma. (a) $0 \Vdash_\alpha$ "$\theta(a_i) \equiv 2^\omega$", $i < \omega$, $\theta \in \Gamma$.
(b) $0 \Vdash_\alpha \theta(a_i) \neq \theta^*(a_j)$, $i, j < \omega$, $i \neq j$, $\theta, \theta^* \in \Gamma$.

Proof. (a) Let $\theta = {}^{(t,l)}\theta$. By the forcing definition, (and Lemmas 2.812, 3.02), for all p, σ, τ such that

$$p \Vdash_\alpha \sigma \in \theta(a_i) \wedge \tau \in \theta(a_i)\,,$$

there are $m, n < \omega$, $\epsilon, \delta < 2$ and $q \supseteq p$ such that

$$q \Vdash_\alpha \sigma = \langle \dot{m}, \dot{\delta} \rangle \wedge \tau = \langle \dot{n}, \dot{\epsilon} \rangle\,.$$

It remains to be shown that if $m = n$, then $\delta = \epsilon$. This is obvious if $n < |t|$. If $n \geqslant |t|$, then by the forcing definition

$$q \Vdash_\alpha \langle \dot{n} - |\dot{t}| + \dot{l}, \dot{\delta} \rangle \in a_i \wedge \langle \dot{n} - |\dot{t}| + \dot{l}, \dot{\epsilon} \rangle \in a_i\,;$$

hence

$$\langle i, n - |t| + l, \delta \rangle \in q, \qquad \langle i, n - t + l, \epsilon \rangle \in q,$$

and by the requirements of a condition, $\delta = \epsilon$, as required.

(b) If $0 \not\Vdash_\alpha \theta(a_i) \neq \theta^*(a_j)$, then for some $p \in P_\alpha$,

$$p \Vdash_\alpha \theta(a_i) = \theta^*(a_j)\,.$$

Let $\theta = {}^{(t,l)}\theta$, $\theta^* = {}^{(t^*,l^*)}\theta$. Choose an integer k such that

$$k - l - l^* - |t| - |t^*| > 0$$

is larger than all integers mentioned in p, and set

$$q = p \cup \{\langle i, k + l - |t|, 0 \rangle, \langle j, k + l^* - |t^*|, 1 \rangle\}\,.$$

q is a condition since by our assumption no information can clash. From the forcing definition and the usual elementary lemmas,

$$q \Vdash_\alpha \langle \dot{k}, \dot{0} \rangle \in \theta(a_j) \wedge \langle \dot{k}, \dot{1} \rangle \in \theta^*(a_j)\,.$$

By assumption,

$$p \subseteq q \Vdash_\alpha \theta(a_j) = \theta^*(a_j)\,,$$

hence

$$q \Vdash_\alpha \langle \dot{k}, \dot{0} \rangle \in \theta^*(a_j) \wedge \langle \dot{k}, \dot{1} \rangle \in \theta^*(a_j)\,;$$

this yields a contradiction as in (a) of this lemma.

4.02. Lemma. *If $p \Vdash_\alpha \dot{\langle \bar{\sigma}^n \rangle} \in K(\bar{b}^n)$, then for all $r \supseteq p$, there is a $q \supseteq r$ and a $\bar{\theta}^n \in \bar{\Gamma}$ such that*

$$q \Vdash_\alpha \bigwedge_{i=1}^{n} (\sigma_i = \theta_i(b_i))\,.$$

Proof. $p \Vdash_\alpha \dot{\langle \bar{\sigma} \rangle} \in K(\bar{b})$; hence by Definition 2.30 and Lemma 2.81, for all $r \supseteq p$ there are σ_i' with rnk $\sigma_i' < \omega + 1$ and τ_i with rnk$(\tau_i) < \omega + 2$ and $q \supseteq r$ such that

$$q \Vdash_\alpha \bigwedge_{i=1}^{n} (\sigma_i = \sigma_i') \wedge \bigwedge_{i=1}^{n} (\sigma_i' \in \tau_i \wedge \dot{\langle i, \tau_i \rangle} \in J_0)\,.$$

Using the Definition 2.64 and Lemmas 2.812, 2.816 and 3.02, there is a $q^* \supseteq q$ and $\bar{\theta} \in \Gamma$ such that

$$q^* \Vdash_\alpha \bigwedge_{i=1}^{n} \tau_i = K(a_i) \wedge \sigma_i' = \sigma_i = \theta_i(b_i)\,,$$

which is the required result.

4.03. Lemma. *Let $\sigma(\bar{b}), \tau(\bar{b}) \in \mathcal{T}^c_\alpha$, $\bar{b} = \bar{a}_n$, then:*

(a) $0 \Vdash_\alpha H_\sigma(\bar{\theta} \cdot \bar{b}) \dot{\subseteq} K(\bar{b})$,

(b) $0 \Vdash_\alpha \dot{\langle \bar{\theta} \cdot \bar{b} \rangle} \in H_\sigma(\bar{\theta} \cdot \bar{b})$,

(c) *for all $\bar{\theta}_1, \bar{\theta}_2 \in \bar{\Gamma}$,*

$$0 \Vdash_\alpha H_\sigma(\bar{\theta}_1 \cdot \bar{b}) = H_\sigma(\bar{\theta}_2 \cdot \bar{b}) \vee H_\sigma(\bar{\theta}_1 \cdot \bar{b}) \dot{\cap} H_\sigma(\bar{\theta}_2 \cdot \bar{b}) = \dot{\emptyset}\,,$$

(d) *if $p \Vdash_\alpha \dot{\langle \bar{\theta}^* \cdot \bar{b} \rangle} \in H_\sigma(\bar{\theta} \cdot \bar{b})$, then $p \Vdash_\alpha H_\sigma(\bar{\theta}^* \cdot \bar{b}) = H_\sigma(\bar{\theta} \cdot \bar{b})$,*

(e) $0 \Vdash_\alpha \dot{\bigcup} H_\sigma = K(\bar{b})$,

(f) $p \Vdash_\alpha H_\sigma \neq H_\tau$ *implies that for some* $\bar{\theta} \in \bar{\Gamma}$ *and* $q \supseteq p$,

$$q \Vdash_\alpha H_\sigma(\bar{\theta}\cdot\bar{b}) \neq H_\tau(\bar{\theta}\cdot\bar{b}).$$

Proof. (a) and (b) follow easily from the definition of $H_\sigma(\bar{\theta}\cdot\bar{b})$, (2.3) and Lemma 4.02. To see (c), assume

$$0 \subseteq p \Vdash_\alpha H_\sigma(\bar{\theta}\cdot\bar{b}) \mathbin{\dot{\cap}} H_\sigma(\bar{\theta}_2\cdot\bar{b}) \neq \dot{\emptyset};$$

then for some $\bar{\theta}\cdot\bar{b}$, and $q \supseteq p$,

$$q \Vdash_\alpha \sigma(\bar{\theta}_1\cdot\bar{b}) = \sigma(\bar{\theta}\cdot\bar{b}) = \sigma(\theta_2\cdot\bar{b}) ,$$

(see 2.26, Definition 2.30, and (a), (b) above). We wish to show that

$$q \Vdash_\alpha (\forall w)\,(w \in H_\sigma(\bar{\theta}_1\cdot\bar{b}) \leftrightarrow w \in H_\sigma(\bar{\theta}_2\cdot\bar{b})) .$$

By (a) we can consider only terms $\dot{\langle}\bar{\theta}^*\cdot\bar{b}\dot{\rangle}$, $\bar{\theta}^* \in \bar{\Gamma}$. Let $q' \supset q$ be such that

$$q' \Vert_\alpha \dot{\langle}\bar{\theta}^*\cdot\bar{b}\dot{\rangle} \in H_\sigma(\bar{\theta}_1\cdot\bar{b}), \qquad q' \Vert_\alpha \dot{\langle}\bar{\theta}^*\cdot\bar{b}\dot{\rangle} \in H_\sigma(\bar{\theta}_2\cdot\bar{b}) ,$$

and assume

$$q' \Vdash_\alpha \dot{\langle}\bar{\theta}^*\cdot\bar{b}\dot{\rangle} \in H_\sigma(\bar{\theta}_1\cdot\bar{b}) .$$

Then

$$q' \Vdash_\alpha \sigma(\bar{\theta}^*\cdot\bar{b}) = \sigma(\bar{\theta}_1\cdot\bar{b}) ,$$

$$q' \Vdash_\alpha \sigma(\bar{\theta}_1\cdot\bar{b}) = \sigma(\bar{\theta}^*\cdot\bar{b}) = \sigma(\bar{\theta}_2\cdot\bar{b});$$

hence by transitivity

$$q' \Vdash_\alpha \sigma(\bar{\theta}^*\cdot\bar{b}) = \sigma(\bar{\theta}_2\cdot\bar{b}) ;$$

therefore

$$q' \Vdash_\alpha \dot{\langle}\bar{\theta}^*\cdot\bar{b}\dot{\rangle} \in H_\sigma(\bar{\theta}_2\cdot\bar{b}) .$$

The other direction is shown by a symmetrical argument; thus

$$q \Vdash_\alpha H_\sigma(\bar{\theta}_1\cdot\bar{b}) = H_\sigma(\bar{\theta}_2\cdot\bar{b}) .$$

This implies that

$$0 \Vdash_\alpha H_\sigma(\bar{\theta}_1\cdot\bar{b}) = H_\sigma(\bar{\theta}_2\cdot\bar{b}) \vee H_\sigma(\bar{\theta}_1\cdot\bar{b}) \mathbin{\dot{\cap}} H_\sigma(\bar{\theta}_2\cdot\bar{b}) = \dot{\emptyset} .$$

(d) follows easily from (c). For if

$$p \not\Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) = H_\sigma(\bar\theta^*\cdot\bar b)\,,$$

then for some $q \supseteq p$,

$$q \Vdash_\alpha H_\sigma(\bar\theta^*\cdot\bar b) \neq H_\sigma(\bar\theta\cdot\bar b)\,.$$

By (c) and (b),

$$q \Vdash_\alpha H_\sigma(\bar\theta^*\cdot\bar b) \mathbin{\dot\cap} H_\sigma(\bar\theta\cdot\bar b) = \dot\emptyset \wedge \dot{\langle}\bar\theta^*\cdot\bar b\dot{\rangle} \in H_\sigma(\bar\theta^*\cdot\bar b)$$
$$\wedge\ \dot{\langle}\bar\theta^*\cdot\bar b\dot{\rangle} \in H_\sigma(\bar\theta\cdot\bar b)\,.$$

This is a contradiction, hence (d) follows.

(e) follows from the definitions of H_σ and $\dot\cup$ (see 2.26, 2.30) because by (a) above, for any $\bar\theta \in \bar\Gamma$,

$$0 \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \in H_\sigma \wedge \dot{\langle}\bar\theta\cdot\bar b\dot{\rangle} \in H_\sigma(\bar\theta\cdot\bar b)\,.$$

(f) $p \Vdash_\alpha H_\sigma \neq H_\tau$ implies that for some $\bar\theta \in \bar\Gamma$ and $q \supseteq p$, either

$$q \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \notin H_\tau\,,$$

or

$$q \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \notin H_\sigma\,.$$

Without loss of generality, assume $q \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \notin H_\tau$. If

$$q \not\Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \neq H_\tau(\bar\theta\cdot\bar b)\,,$$

then for some $r \supseteq q$, $r \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) = H_\tau(\bar\theta\cdot\bar b)$. By Lemmas 2.811, 2.812, we get

$$q \subseteq r \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \in H_\tau \wedge H_\sigma(\bar\theta\cdot\bar b) \notin H_\tau\,;$$

a contradiction.

4.04. (Proof of Lemma 2.72.) It must be shown that if $\sigma(\bar b), \tau(\bar b) \in \mathcal{L}^c_\alpha$, and $s^\alpha_\tau = s^\alpha_\sigma$, then $0 \Vdash_\alpha H_\sigma = H_\tau$. If $0 \not\Vdash_\alpha H_\sigma = H_\tau$, then for some p, $0 \subseteq p \Vdash_\alpha H_\sigma = H_\tau$. By Lemma 4.03(f), there are a $\bar\theta$ and a $q \supseteq p$ such that

$$q \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \neq H_\tau(\bar\theta\cdot\bar b)\,.$$

Therefore, by definition of $H_\sigma(\bar\theta\cdot\bar b)$, $H_\tau(\bar\theta\cdot\bar b)$ and Lemma 4.02, for some $\bar\theta' \in \bar\Gamma$ and $q' \supseteq q$,

$$q' \Vdash_\alpha \sigma(\bar\theta \cdot \bar b) = \sigma(\bar\theta' \cdot \bar b) \wedge \tau(\bar\theta \cdot \bar b) \neq \tau(\bar\theta' \cdot \bar b) .$$

Let $p^* \subseteq q'$ be minimal such that

$$p^* \Vdash_\alpha \sigma(\bar\theta' \cdot \bar b) = \sigma(\bar\theta \cdot \bar b) .$$

Since $s^\alpha_\tau = s^\alpha_\sigma$, we have $[\sigma, \bar\theta', \bar\theta] = [\tau, \bar\theta', \bar\theta]_\alpha$; thus

$$p^* \Vdash_\alpha \tau(\bar\theta \cdot \bar b) = \tau(\bar\theta' \cdot \bar b)$$

which is impossible, (by Lemmas 2.68 and 2.69). Therefore $0 \Vdash_\alpha H_\sigma = H_\tau$. The remaining facts in Lemma 2.72 now follow easily using the proper articles of Lemma 4.03. We omit the details which are trivial.

4.10. Terminology. On many occasions, conditions, or sets of precondicitons, are referred to as *information.* When we say "all information in p on χ_e" we mean the subset of p consisting of all preconditions in which e occurs in the first place. Similarly "all information in p on $\chi_e(\bar\theta \cdot b)$" is the set of preconditions of p in which e and $\bar\theta$ occur. Information of type $\langle e, \bar\theta, i, \delta \rangle$ is said to be "coordinate information on $\chi_e(\bar\theta \cdot \bar b)$", and information of type $\langle e, l_1, \bar\theta_1, l_2, \bar\theta_2 \rangle$ is referred to as a "connection between $\chi_e(\bar\theta_1 \cdot \bar b)$ and $\chi_e(\bar\theta_2 \cdot \bar b)$".

4.11. Definition. Let $p \in P$, $\chi_e \in \mathcal{X}_{\alpha+1} - \mathcal{X}_\alpha$, $e = (h, g, s^\alpha_\sigma)$, $\bar a_h = \bar b$.

(a) We say that *p is prepared for $\chi_e(\bar\theta \cdot \bar b)$* if,

$$\hat p^\alpha \Vdash_\alpha [H_\sigma(\bar\theta \cdot \bar b) \text{ is nearly } (-l)\text{-definable}] ,$$

when $l \in g$; and

$$\hat p^\alpha \Vdash_\alpha [H_\sigma(\bar\theta \cdot \bar b) \text{ is not nearly } (-l)\text{-definable}]$$

when $l \in h - g$; and for any $\chi_e(\bar\theta^* \cdot \bar b)$ mentioned by p,

$$\hat p^\alpha \Vert_\alpha H_\sigma(\bar\theta \cdot \bar b) = H_\sigma(\bar\theta^* \cdot \bar b), \qquad \hat p^\alpha \Vert_\alpha H_\sigma(\bar\theta \cdot \bar b) = H_\sigma(\bar\theta^* \cdot \bar b).$$

(b) For any $\mu \subseteq \mu' \in 2^l$, $l < \omega$, denote

$$p(e, \bar\theta, \mu) = \{\langle e, \bar\theta, i, \mu(i) \rangle \mid i \in \mathrm{dom}(\mu)\} .$$

4.20. Lemma. *Let p be prepared for $\chi_e(\bar\theta \cdot \bar b)$, $e = (h, g, s^\alpha_\sigma)$ and assume that for all $\chi_e(\bar\theta^* \cdot \bar b)$ mentioned by p,*

$$\hat p^\alpha \Vdash_\alpha H_\sigma(\bar\theta^* \cdot \bar b) \neq H_\sigma(\bar\theta \cdot \bar b) .$$

Then for any $l < \omega$, *and* $\mu \subseteq \mu' \in 2^l$,

$$p \cup p(e, \bar\theta, \mu) \cup \{\langle e, \bar\theta, 0, \bar\theta, 0\rangle\}$$

is extendable to a condition.

Proof. Obvious.

4.21. Lemma. *Assume that p is prepared for* $\chi_e(\bar\theta \cdot \bar b)$, $e = (h, g, s_\sigma^\alpha)$. *If* $\chi_e(\bar\theta' \cdot \bar b)$ *is mentioned by p and*

$$\hat p^\alpha \Vdash_\alpha H_\sigma(\bar\theta' \cdot \bar b) = H_\sigma(\bar\theta \cdot \bar b),$$

then

$$p \cup \{\langle e, \bar\theta, 0, \bar\theta', 0\rangle\}$$

can be extended to a condition.

Proof. This is done by making $\chi_e(\bar\theta' \cdot \bar b)$ and $\chi_e(\bar\theta \cdot \bar b)$ informationwise equal. Set

$$J = \{\langle e, \bar\theta, j, \delta\rangle \mid \langle e, \bar\theta', j, \delta\rangle \in P\},$$

and

$$V = \{\langle e, \bar\theta, l, \bar\theta^*, l^*\rangle, \langle e, \bar\theta^*, l^*, \bar\theta, l\rangle, \langle e, \bar\theta', l, \bar\theta^*, l^*\rangle \in P\}.$$

It is clearly seen that

$$p \cup \{\langle e, \bar\theta, 0, \bar\theta', 0\rangle, \langle e, \bar\theta', 0, \bar\theta, 0\rangle\} \cup J \cup V = q$$

is a condition.

4.22. Lemma. *Assume that* $\chi_e(\bar\theta \cdot \bar b)$ *is mentioned by p,* $e = (h, g, s_\sigma^\alpha)$. *Let* $\mu \subseteq \mu' \in 2^l$, $l < \omega$, *be such that for any* $i \in \operatorname{dom}(\mu)$ *and* $\langle e, \bar\theta, i, \delta\rangle \in P$ *we have* $\mu(i) = \delta$. *Then*

$$q = p \cup p(e, \bar\theta, \mu) \cup \{\langle e, \bar\theta^*, i - s + t, \mu(i)\rangle \mid \langle e, \bar\theta, s, \bar\theta^*, t\rangle \in p,\ i \geq s\}$$

is a condition.

Proof. Denote the last set of the union by J. We show by a direct check that q fulfills the requirements specified in Definition 2.50. First observe that q fulfills 2.50(e) by assumption; and that q fulfills 2.50(a) if $\hat{q}^{\alpha+1}$ fulfills 2.50(b), (c), (d), (e), since $\hat{q}^{\alpha} = \hat{P}^{\alpha}$ and $q - \hat{q}^{\alpha+1} = P - \hat{P}^{\alpha+1}$.

2.50(b). We must show that for no $\bar{\theta}^*$, i, are $z_1 = \langle e, \bar{\theta}^*, j, 0\rangle$ and $z_2 = \langle e, \bar{\theta}^*, j, 1\rangle$ in q. They cannot both be in p, nor both be in $p(e, \bar{\theta}, \mu)$. If they are both in J, say

$$\langle e, \bar{\theta}^*, i - s + t, 0\rangle \in J, \quad \langle e, \bar{\theta}^*, i' - s' + t', 1\rangle \in J,$$

which implies

$$\langle e, \bar{\theta}, s, \bar{\theta}^*, t\rangle \in P, \quad \langle e, \bar{\theta}, s', \bar{\theta}^*, t'\rangle \in p$$

which implies by 2.50(c)(2), that $t - s = t' - s'$; thus $i = i'$; and $0 = \mu(i) = 1$, which is impossible. If $z_1 \in P$ and $z_2 \in P(e, \bar{\theta}, \mu)$, then we have an immediate contradiction to our assumptions on μ. If $z_1 \in P$, $z_2 \in J$, then

$$z_2 = \langle e, \bar{\theta}^*, i - s + t, \mu(i)\rangle, \quad i \geq s,$$

where $\langle e, \bar{\theta}, s, \bar{\theta}^*, t\rangle \in p$. Then

$$z_1 = \langle e, \bar{\theta}^*, i - s + t, 1 - \mu(i)\rangle \in P;$$

and by 2.50(c)(1) we get $\langle e, \bar{\theta}, \cancel{t} + i - \cancel{t}, 1 - \mu(i)\rangle \in P$, again contradicting our assumptions on μ. If $z_1 \in P(e, \bar{\theta}, \mu)$, $z_2 \in J$, then $z_1 = \langle e, \bar{\theta}, i, \mu(i)\rangle$; hence

$$z_2 = \langle e, \bar{\theta}, i' - s + t, 1 - \mu(i)\rangle, \quad i' \geq s,$$

where $\langle e, \bar{\theta}, s, \bar{\theta}, t\rangle \in p$. Thus $s = t$, by 2.50(d)(3); so

$$z_2 = \langle e, \bar{\theta}, I, 1 - \mu(i)\rangle,$$

which is impossible.

2.50(c)(1). It must be shown that if

$$z_1 = \langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in q,$$

$$z_2 = \langle e, \bar{\theta}_1, l_1 + k, \delta\rangle \in q,$$

then $z_3 = \langle e, \bar{\theta}_2, l_2 + k, \delta\rangle \in q$. We consider the three cases. If $z_1, z_2 \in p$, the result follows from the fact that p is a condition. If $z_1 \in p$ and $z_2 \in P(e, \bar{\theta}, \mu)$, then $z_3 \in J$. If $z_1 \in p$ and $z_2 \in J$, then

$$z_2 = \langle e, \bar{\theta}_1, i - s + t, \mu(i)\rangle,$$

where

$$\langle e, \bar\theta, s, \bar\theta_1, t\rangle \in p, \quad i \geqslant s,$$
$$i - s + t = l_1 + k .$$

By applying 2.50(c)(3) to p we get

$$\langle e, \bar\theta, s + l_1 - n, \bar\theta_2, t + l_2 - n\rangle \in p$$

for some $n \geqslant \min(t, l)$. Thus

$$\langle e, \bar\theta_2, i - (s + l_1 - n) + (t + l_2 - n), \mu(i)\rangle \in J.$$

Cancelling terms and substituting $i - s + t = l_1 + k$, we obtain

$$\langle e, \bar\theta_2, l_1 + k + l_2 - l_1, \mu(i)\rangle \in J;$$

thus $\langle e, \bar\theta_2, l_2 + k, \mu(i)\rangle \in q$, as required.

2.50(c)(2), 2.50(c)(3) and 2.50(c)(4) are trivial, since no new connections have been added to p. 2.50(d) is trivial since any $\chi_e(\bar\theta^*\cdot\bar b)$ mentioned by q is mentioned by p.

4.3. The last condition-extension lemma needed is concerned with connecting a new element $\chi_e(\bar\theta\cdot\bar b)$ to one in P. This lemma is preceded by a few definitions of notions involved and their properties.

4.30. Definition. (a) If $\chi_e(\bar\theta^*\cdot\bar b)$ and $\chi_e(\bar\theta^{**}\cdot\bar b)$ are mentioned by p, and

$$\hat p^\alpha \Vdash_\alpha H_\sigma(\bar\theta^*\cdot\bar b) \neq H_\sigma(\bar\theta^{**}\cdot\bar b),$$

then $\chi_e(\bar\theta^*\cdot\bar b)$, $\chi_e(\bar\theta^{**}\cdot\bar b)$ are said to be *different in p*, or *p-different*; otherwise they are said to be *p-equal.*

(b) If $\langle e, \bar\theta^*, l, \bar\theta^{**}, l\rangle \in P$, then $\chi_e(\bar\theta^*\cdot\bar b)$ and $\chi_e(\bar\theta^{**}\cdot\bar b)$ are said to be *slash*$_r$*-equal* in P, $l \leqslant r$.

(c) A set of p-different, slash$_r$-equal elements in p, $\chi_e(\bar\theta_j\cdot\bar b)$, $1 \leqslant j \leqslant m$, is said to be *maximal* if for any

$$\langle e, \bar\theta, l, \bar\theta_j, l\rangle \in P, \qquad l \leqslant r, \quad 1 \leqslant j \leqslant m$$

there exists an i, $1 \leqslant i \leqslant m$ such that

$$\hat p^\alpha \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) = H_\sigma(\bar\theta_i\cdot\bar b).$$

(d) A maximal set of slash$_r$-equal, p-different elements is said to be *full* if $m = 2^r$; otherwise it is said to be *deficient*.

(e) Denote

$$^p\mu(e, \bar{\theta}, r) = \{\langle i, \delta\rangle \mid \langle e, \bar{\theta}, i, \delta\rangle \in p,\ i < r\}\ .$$

4.31. Lemma. *If $\{\chi_e(\bar{\theta}_j \cdot \bar{b}) \mid 1 \leqslant j \leqslant m\}$ is a set of p-different, slash$_r$-equal elements in p, then the functions $^p\mu(e, \bar{\theta}_j, r)$, $1 \leqslant j \leqslant m$ are compatible in pairs.*

Proof. This is an immediate consequence of Definition 2.50(d)(1).

4.32. Corollary. *The number of elements in a maximal set of p-different, slash$_r$-equal elements in p, is less or equal to 2^r.*

4.33. Lemma. *The number of elements in a maximal set of p-different slash$_r$-equal elements in p is independent of the representatives chosen amongst the various p-equal elements.*

Proof. This is an immediate consequence of Definition 2.50(d)(3).

4.34. Lemma. *Let $\chi_e(\bar{\theta}_j \cdot \bar{b})$ be a set of p-different slash$_r$-equal elements in p, and denote*

$$\mu_j = {}^p\mu(e, \bar{\theta}_j, r), \quad 1 \leqslant j \leqslant m.$$

If $\mu_j \subseteq \mu_j^* \subseteq \mu_j' \in 2^r$, then

$$p \cup \bigcup_{1 \leqslant j \leqslant m} p(e, \bar{\theta}_j, \mu_j^*)$$

$$\cup \{\langle e, \bar{\theta}^*, i - s + t, \mu_j^*(i)\rangle \mid i \geqslant s,\ \langle e, \bar{\theta}_j, s, \bar{\theta}^*, t\rangle \in p,\ 1 \leqslant j \leqslant m\}$$

is a condition.

Proof. This lemma results from a successive application of Lemma 4.22, relying on the fact that the $\chi_e(\bar{\theta}_j \cdot \bar{b})$ are p-different in pairs, and on Definition 2.50(d)(1).

4.35. Definition. Let $\mu_j \subseteq \mu_j' \in 2^n$, $1 \leqslant j \leqslant m$; then $\mu \subseteq \mu' \in 2^n$ is said to be *r-new with respect to* $\{\mu_j \mid 1 \leqslant j \leqslant m\}$ if for any μ_j there exists an $0 \leqslant i < r$ such that either $\mu(i) = 1 - \mu_j(i)$ or $i \notin \operatorname{dom}(\mu_j)$.

4.36. Lemma. *If* $\{\chi_e(\bar\theta_j\cdot\bar b) \mid 1 \leqslant j \leqslant m\}$ *is a maximal set of p-different slash*$_r$*-equal elements in p, which is deficient, then there exists a* $\mu \subseteq \mu' \in 2^r$ *which is new with respect to the functions*

$$\{{}^p\mu(e, \bar\theta_j, r) \mid 1 \leqslant j \leqslant m\}\ .$$

Finally,

4.40. Lemma. *Let p be prepared for* $\chi_e(\bar\theta\cdot\bar b)$, *where* $e = (h, g, s^\alpha_\sigma)$. *Assume that for all* $\chi_e(\bar\theta^*\cdot\bar b)$ *mentioned by p,*

$$\hat P^\alpha \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b) \neq H_\sigma(\bar\theta^*\cdot\bar b)\ .$$

Let $\{\chi_e(\bar\theta_j\cdot\bar b) \mid 1 \leqslant j \leqslant m\}$ *be a maximal set of p-different slash*$_r$*-equal elements in p, which is deficient. Assume that* $\langle e, \bar\theta', s_j', \bar\theta_j, s_j\rangle \in p$, *for*

$$s_j' \leqslant r', \quad s_j' - s_j = r - r', \quad 1 \leqslant j \leqslant m\ .$$

Then for any $\mu \in 2^r$ *which is r-new with respect to the functions*

$${}^p\mu(e, \bar\theta_j, r) = \mu_j, \quad 1 \leqslant j \leqslant m\ .$$

$$p \cup \{\langle e, \bar\theta, r, \bar\theta', r'\rangle\} \cup p(e, \bar\theta, \mu)$$

can be extended to a condition q which contains only the additional information on χ_e *required of a condition by Definition* 2.50. (See Fig. 5.)

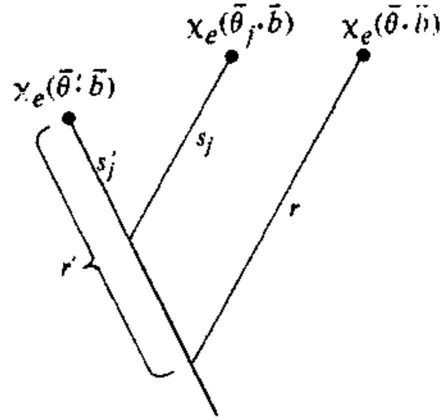

Fig. 5. $s_j' \leqslant r'$, $s_j - s_j' = r - r'$; $\chi_e(\bar\theta_j\cdot\bar b)$, and $\chi_e(\bar\theta\cdot\bar b)$ will be different elements in the same layer of the tail tree.

4.41. Remark. This lemma is needed in order to show that elements of dom($\dot{x}_e$), (see (2.45)), are mapped into the same tail. Moreover, if there are $\aleph_0$ $x \in \mathrm{dom}(\dot{x}_e)$, then they are mapped *generically onto a tail.*

Proof of Lemma 4.40. This proof is direct and tedious. The idea is to make the necessary connections between $\chi_e(\bar{\theta}\cdot\bar{b})$ and the $\chi_e(\bar{\theta}^*\cdot\bar{b})$ connected to $\chi_e(\bar{\theta}'\cdot\bar{b})$ via $\langle e, \bar{\theta}', r', \bar{\theta}, r\rangle$, then to properly transfer all the information in p on $\chi_e(\bar{\theta}'\cdot\bar{b})$ to $\chi_e(\bar{\theta}\cdot\bar{b})$ via $\langle e, \bar{\theta}', r', \bar{\theta}, r\rangle$. A certain amount of easy computation is needed to verify that this process actually suffices. Observe that by Lemma 4.34 we can assume that μ actually contradicts each of the μ_j.

Define:

$$w^1 = \{\langle e, \bar{\theta}, r, \bar{\theta}^*, t+r'-s\rangle \mid \langle e, \bar{\theta}', s, \bar{\theta}^*, t\rangle \in p,\ r' \geqslant s\},$$

$$w^2 = \{\langle e, \bar{\theta}, r+s-r', \bar{\theta}^*, t\rangle \mid \langle e, \bar{\theta}', s, \bar{\theta}^*, t\rangle \in p,\ r' < s\}.$$

Clearly $w^1 \cap w^2 = \emptyset$; set $w^* = w^1 \cup w^2$, and define

$$w^{**} = \{\langle e, \bar{\theta}^1, l_1, \bar{\theta}^2, l_2\rangle \mid \langle e, \bar{\theta}^2, l_2, \bar{\theta}^1, l_1\rangle \in w^*\}.$$

Then set $w = w^* \cup w^{**}$. Note that $\langle e, \bar{\theta}, r, \bar{\theta}', r'\rangle \in w$. (See Fig. 6.)

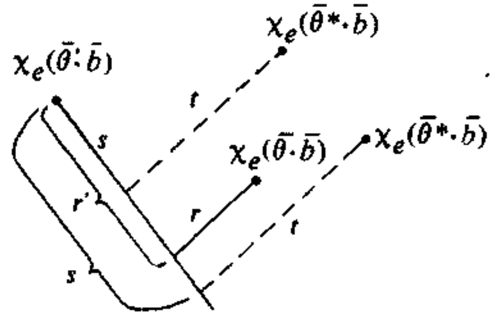

Fig. 6.

Note that it will not be necessary to transfer coordinate information from $p(e, \bar{\theta}, \mu)$ to any $\chi_e(\bar{\theta}^*\cdot\bar{b})$, since for all $\langle e, \bar{\theta}, l, \bar{\theta}^*, l^*\rangle \in w$, we have $l \geqslant r$.

Set

$$R = \{\langle e, \bar{\theta}, j, \delta\rangle \mid \langle e, \bar{\theta}', i, \delta\rangle \in p, j = i - r' + r, 1 \geqslant r'\},$$

and define

$$q = p \cup p(e, \bar{\theta}, \mu) \cup R \cup w \cup \{\langle e, \bar{\theta}, 0, \bar{\theta}, 0\rangle\}.$$

We wish to show that q is the required condition. This is shown by systematically checking the requirements of a condition, i.e., Definition 2.50. First observe that q satisfies 2.50(e) since p is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$ by assumption. Also, since $\hat{q}^\alpha = \hat{p}^\alpha$ and $q - \hat{q}^{\alpha+1} = p - p^{\alpha+1}$, q satisfies 2.50(a) if $\hat{q}^{\alpha+1}$ satisfies 2.50(b), (c), (d), (e); hence it remains to show that $\hat{q}^{\alpha+1}$ fulfills 2.50(b), (c), (d).

Consider 2.50(b). It must be shown that for no $\bar{\theta}^* \in \bar{\Gamma}$ does q contain $\langle e, \bar{\theta}^*, j, 0\rangle$ and $\langle e, \bar{\theta}^*, j, 1\rangle$. Suppose that for some $\bar{\theta}^*$, q contains two such elements z_1, z_2. They cannot both be in p nor both in $p(e, \bar{\theta}, \mu)$. If they are both in R, then $\langle e, \bar{\theta}', j-r+r', 0\rangle \in p$, and $\langle e, \bar{\theta}', j-r+r', 1\rangle \in p$, which is impossible. Obviously we cannot have $z_1 \in p$ and $z_2 \in p(e, \bar{\theta}, \mu)$ or $z_2 \in R$ since we are assuming that p does not mention $\chi_e(\bar{\theta}\cdot\bar{b})$. There remains the possibility that, say, $z_1 \in p(e, \bar{\theta}, \mu)$, $z_2 \in R$. But this is also impossible, since $\bar{\mu}$ contains information on coordinates $i < r$ and z_2 contains information on coordinates $i \geqslant r$.

2.50(c)(1). It must be shown that if

$$z_1 = \langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in q,$$

$$z_2 = \langle e, \bar{\theta}_1, l_1 + k, \delta\rangle \in q, \qquad k \geqslant 0;$$

then also $\langle e, \bar{\theta}_2, l_2 + k, \delta\rangle \in q$. If $z_1 \in p$, then also $z_2 \in p$, (since p does not mention $\chi_e(\bar{\theta}\cdot\bar{b})$), and the result follows from the fact that p is a condition. The case $z_1 = \langle e, \bar{\theta}, 0, \bar{\theta}, 0\rangle$ is also trivial. Thus assume $z_1 \in w$, and first assume $z_1 \in w^*$, i.e.

$$z_1 = \langle e, \bar{\theta}, l_1, \bar{\theta}^*, l_2\rangle, \qquad z_2 \in p(e, \bar{\theta}, \mu) \cup R.$$

The case $z_2 \in p(e, \bar{\theta}, \mu)$ is trivial since $l_1 \geqslant r$ and $p(e, \bar{\theta}, \mu)$ contains information only on coordinates $i < r$. Assume $z_2 \in R$, i.e. $z_2 = \langle e, \bar{\theta}, j, \delta\rangle$. Then

$$\langle e, \bar{\theta}', j - r + r', \delta\rangle \in p, \quad j \geqslant r.$$

If $z_1 \in w'$, then $l_1 = r$ and

$$z_1 = \langle e, \bar{\theta}, r, \bar{\theta}^*, t+r-s\rangle,$$

where

$$\langle e, \bar{\theta}', s, \bar{\theta}^*, t\rangle \in p, \quad r' \geqslant s;$$

hence since $j-r+r' \geqslant s$, and Definition 2.50(c)(1) holds for p,

$$\langle e, \bar{\theta}^*, t+(j-r+r'-s), \delta\rangle \in p;$$

therefore $\langle e, \bar{\theta}^*, j+l_1-l_2, \delta\rangle \in p$, as required. If $z_1 \in w^2$, then

$$z_1 = \langle e, \bar{\theta}, r+s-r', \bar{\theta}^*, t\rangle$$

where $\langle e, \bar{\theta}', s, \bar{\theta}^*, t\rangle \in p$, $r' < s$. If $j-r+r' \leqslant s$, there is nothing to prove; otherwise

$$\langle e, \bar{\theta}^*, t+(j-r+r')-s, \delta\rangle \in p, \quad t = l_2, \quad -r-s+r' = -l_1$$

therefore $\langle e, \bar{\theta}^*, l-l_1+j, \delta\rangle \in p$, as required. The case $z_2 \in w^{**}$ is shown by an entirely symmetrical argument.

2.50(c)(2). We must show that if q contains

$$z_1 = \langle e, \bar{\theta}^1, l^1, \bar{\theta}^2, l^2\rangle, \qquad z_2 = \langle e, \bar{\theta}^1, m^1, \bar{\theta}^2, m^2\rangle,$$

then $l_1 - l_2 = m_1 - m_2$. The cases where $z_1, z_2 \in p$ or $z_1 = \langle e, \bar{\theta}, 0, \bar{\theta}, 0\rangle$ are trivial. We cannot have $z_1 \in p$ and $z_2 \in w$ since p does not mention $\chi_e(\bar{\theta}\cdot\bar{b})$ by assumption. Hence we consider $z_1, z_2 \in w$, and without loss of generality we may assume $z_1, z_2 \in w^*$. Suppose $z_1, z_2 \in w$; then

$$z_1 = \langle e, \bar{\theta}, r, \bar{\theta}^*, t+r'-s\rangle, \qquad z_2 = \langle e, \bar{\theta}, r, t_2+r'-s_2\rangle$$

where $\langle e, \bar{\theta}', s, \bar{\theta}^*, t\rangle \in p$, $\langle e, \bar{\theta}', s_2, \bar{\theta}^*, t_2\rangle \in p$. By applying Definition 2.50(c)(2) to p, we get $t_1 - s_1 = t_2 - s_2$; hence

$$\begin{aligned} r-(t_1+r'-s_1) &= (r-r')+(s_1-t_1) \\ &= r-(t_2+r'-s_2) = (r-r')+(s_2-t_2), \end{aligned}$$

as required. If $z_1, z_2 \in w^2$, then

$$z_1 = \langle e, \bar{\theta}, r+s_1-r, \bar{\theta}^*, t\rangle, \quad z_2 = \langle e, \bar{\theta}, r+s_2-r', \bar{\theta}^*, t_2\rangle;$$

hence

$$\begin{aligned} (r+s_1-r')-t_1 &= (r-r')+(s_1-t_1) \\ &= (r+s_2-r')-t_2 = (r-r')+(s_2-t_2) \end{aligned}$$

as required, since $s_1 - t_1 = s_2 - t_2$. If $z_1 \in w^1$, $z_2 \in w^2$, then

$$z_1 = \langle e, \bar\theta, r, \bar\theta^*, t + r' - s_1\rangle, \quad z_2 = \langle e, \bar\theta, r + s_2 - r', \bar\theta^*, t_2\rangle .$$

Hence we again get

$$\begin{aligned}(r + s_2 - r') - t_2 &= r - r' + (s_2 - t_2)\\ &= r - (t_1 + r' - s_1) = r - r' + (s_1 - t_1),\end{aligned}$$

as required.

2.50(c)(3). It must be shown that if

$$z_1 = \langle e, \bar\theta^1, l^1, \bar\theta^2, l^2\rangle, \qquad z_2 = \langle e, \bar\theta^2, m^2, \bar\theta^3, m^3\rangle ,$$

then $\langle e, \bar\theta^1, l^1 + m^2 - k, \bar\theta^3, m^3 + l^2 - k\rangle \in q$ for some $k \geqslant \min(m^2, l^2)$. The case where one of z_1, z_2 is $\langle e, \bar\theta, 0, \bar\theta, 0\rangle$ is trivial; hence, as before, we can consider only the cases where $z_1, z_2 \in w$. Assume first that

$$z_1 \in (w')^* \subseteq w^{**}, \qquad z_2 \in w' \subseteq w^*.$$

Then

$$z_1 = \langle e, \bar\theta^1, t^1 + r' - s^1, \bar\theta, r\rangle, \quad r' \geqslant s^1 ,$$

where

$$\langle e, \bar\theta^1, t^1, \bar\theta', s^1\rangle \in p;$$

$$z_2 = \langle e, \bar\theta, r, \bar\theta^3, t^3 + r' - s^3\rangle, \quad r' \geqslant s^3 ,$$

where

$$\langle e, \bar\theta', s^3, \bar\theta^3, t^3\rangle \in p.$$

By applying Definition 2.50(c)(3) to p, we obtain

$$y = \langle e, \bar\theta^1, t^1 + s^3 - k, \bar\theta^3, s^1 + t^3 - k\rangle \in p, \tag{4.42}$$

for some $k \geqslant \min(s^1, s^3)$.

We need

$$x = \langle e, \bar\theta^1, t^1 + r' - s^1 + r - k^*, \bar\theta^3, r + t^3 + r' - s^3 - k^*\rangle \in q,$$

for some $k^* \geqslant \min(r, r) = r$. Define

$$k^* = k + r + r' - s^3 - s^1 .$$

By substituting for k in y, (4.44), we obtain $x \in p$. If $\min(s^1, s^3) = s^1$, then, since

$$r^1 \geqslant s^1, s^3, \quad k^* \geqslant s^1 + r + r' - s^3 - s^1 \geqslant r\ ;$$

if $\min(s^1, s^3) = s^3$, then

$$k^* \geqslant s^3 + r + r' - s^1 - s^3 \geqslant r.$$

Thus x is a required element. Assume $z_1 \in (w^2)^* \subseteq w^{**}$ and $z_2 \in w^2$. Thus

$$z_1 = \langle e, \bar\theta^1, t^1, \bar\theta, r + s^1 - r' \rangle, \quad r' < s^1,$$

$$\text{where } \langle e, \bar\theta^1, t^1, \bar\theta', s^1 \rangle \in p;$$

$$z_2 = \langle e, \bar\theta, r + s^3 - r', \bar\theta^3, t^3 \rangle, \quad r' < s^3,$$

$$\text{where } \langle e, \bar\theta', s^3, \bar\theta^3, t^3 \rangle \in p.$$

Since p satisfies Definition 2.50(c)(3) we have

$$y = \langle e, \bar\theta^1, t^1 + s^3 - k, \bar\theta^3, t^3 + s^1 - k \rangle \in p, \tag{4.43}$$

$$\text{for some } k \geqslant \min(s^1, s^3).$$

We require that

$$x = \langle e, \bar\theta^1, t^2 + r + s^3 - r' - k^*, \bar\theta^3, r + s^1 - r' + t^3 - k^* \rangle \in q$$

for some $k^* \geqslant \min(r + s^1 - r', r + s^3 - r') \geqslant 0$. Define $k^* = k + r - r'$. Then by substituting for k in (4.43), we obtain $x \in p$, and

$$k^* \geqslant \min(r + s^1 - r', r + s^3 - r'),$$

since $r' < s^1, s^3$. Hence x is the required element. Assume $z_1 \in (w')^* \subseteq w^{**}$, $z_2 \in w^2$. Thus

$$z_1 = \langle e, \bar\theta^1, t^1 + r' - s^1, \bar\theta, r \rangle, \quad r' \geqslant s',$$

$$\text{where } \langle e, \bar\theta^1, t^1, \bar\theta', s^1 \rangle \in p;$$

$$z_2 = \langle e, \bar\theta, r + s^3 - r', \bar\theta^3, t^3 \rangle, \quad r' < s^3,$$

$$\text{where } \langle e, \bar\theta', s^3, \bar\theta^3, t^3 \rangle \in p.$$

Since p satisfies Definition 2.50(c)(3), we have

$$\langle e, \bar\theta^1, t^1 + s^3 - k, \bar\theta^3, t^3 + s^1 - k \rangle \in p, \tag{4.44}$$

for some $k \geqslant \min(s^1, s^3)$.

It is required that

$$\langle e, \bar{\theta}^1, (t^1+r'-s^1)+(r+s^3-r')-k^*, \bar{\theta}^3, t^3+r-k^*\rangle \in q,$$

for some $k^* \geqslant \min(r, r+s^3-r') = r$, (since $s^3 > r' \geqslant s^1$). Taking $k = k^* - r + s^1$ in (4.44), we get

$$\langle e, \bar{\theta}', (t^1+s^3+r-s') - k^*, \bar{\theta}^3, t^3+r-k^*\rangle \in p.$$

Since $k^* = r - s' + k \geqslant r$, (because $k \geqslant s^1$), this is the required element. These are essentially all the cases.

2.50(c)(4). This requirement is satisfied directly from the definition of q.

2.50(d)(1). This requirement is fulfilled by q, since μ was assumed to contradict each μ_j, $1 \leqslant j \leqslant m$.

2.50(d)(2). This requirement is fulfilled by q, since $\langle e, \bar{\theta}, 0, \bar{\theta}, 0\rangle \in q$.

2.50(d)(3). That this requirement is fulfilled is an immediate consequence of the fact that $w \subseteq q$. Lemma 4.40 has thus been shown.

4.45. Lemma. *If $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in p$ and $m_1 - m_2 = l_1 - l_2$, and*

(a) *if $m_1 \geqslant l_1$, then*

$$q = p \cup \{\langle e, \bar{\theta}_1, m_1, \bar{\theta}_2, m_2\rangle, \langle e, \bar{\theta}_2, m_2, \bar{\theta}_1, m_1\rangle\}$$

is a condition;

(b) *if $l_1 > m_1$, and*

$${}^p\mu(e, \bar{\theta}_1, l_1) - {}^p\mu(e, \bar{\theta}_1, m_1) = {}^p\mu(e, \bar{\theta}_2, l_2) - {}^p\mu(e, \bar{\theta}_2, m_2),$$

(see Definition 4.30(3)), then

$$q = p \cup \{\langle e, \bar{\theta}_1, m_1, \bar{\theta}_2, m_2\rangle, \langle e, \bar{\theta}_2, m_2, \bar{\theta}_1, m_1\rangle\}$$

can be extended to a condition.

Proof. To see (a) observe that the transitivity requirement is satisfied. For assume $\langle e, \bar{\theta}_2, s, \bar{\theta}_3, t\rangle \in p$; then

$$\langle e, \bar{\theta}, l_1+s-k, \bar{\theta}_3, t+l_2-k\rangle \in p, \quad k \geqslant \min(l_2, s).$$

if $m_1 = l_1 + d$, $m_2 = l_2 + d$, then

$$\langle e, \bar{\theta}_1, m_1+s-(k+d), \bar{\theta}_3, t+m_2-(k+d)\rangle \in p,$$

$$(k+d) \geqslant \min(m_2, s),$$

as required. The other requirements are even more easily seen. In (b), all that has to be done to obtain a condition from q is to add all the necessary connections to comply with the transitivity requirement. Set

$$w_1^1 = \{\langle e, \bar\theta, t+m_2-s, \bar\theta_1, m_1\rangle \mid \langle e, \bar\theta_2, s, \bar\theta, t\rangle \in p, m_2 \geqslant s\},$$

$$w_2^1 = \{\langle e, \bar\theta, t, \bar\theta_1, m_1+s-m_2\rangle \mid \langle e, \bar\theta_2, s, \bar\theta, t\rangle \in p, m_2 < s\},$$

$$w_1^2 = \{\langle e, \bar\theta, t+m_2-s, \bar\theta_2, m_1\rangle \mid \langle e, \bar\theta_1, s, \bar\theta, t\rangle \in p, m_2 \geqslant s\},$$

$$w_2^2 = \{\langle e, \bar\theta, t, \bar\theta_2, m_1+s-m_2\rangle \mid \langle e, \bar\theta_1, s, \bar\theta, t\rangle \in p, m_2 < s\};$$

and take

$$w^* = w_1^1 \cup w_2^1 \cup w_1^2 \cup w_2^2\ ,$$

$$w^{**} = \{\langle e, \bar\theta', l', \bar\theta'', l''\rangle \mid \langle e, \bar\theta'', l'', \bar\theta', l'\rangle \in w^*\},$$

$$w = w^* \cup w^{**}.$$

As in the proof of Lemma 4.40, it is easily verified that $q \cup w$ is already a condition. Note that Definition 2.50(c)(1) and (b) are satisfied because of our assumptions with regard to ${}^p\mu(e, \bar\theta_i, m_i)$, ${}^p\mu(e, \bar\theta_i, l_i)$, $1 \leqslant i \leqslant 2$. We omit the details.

4.5. In order to extend a condition p by adding certain elements to $\hat p^{\alpha+1} - \hat p^\alpha$, it is generally necessary to first extend $\hat p^\alpha \subseteq q \in p_\alpha$ to prepare for these additions. When one wants to add information concerning $\chi_e(\bar\theta\cdot\bar b)$, $\chi_e(\bar\theta'\cdot\bar b)$ to $\hat p^{\alpha+1} - \hat p^\alpha$, it is convenient if $\hat p^{\alpha+1}$ already mentions $\chi_e(\bar\theta\cdot\bar b)$, $\chi_e(\bar\theta'\cdot\bar b)$. The next two lemmas are concerned with preparations of this kind.

4.50. Definition. $\chi_e(\bar\theta\cdot\bar b) = (\forall w)\,(\exists u)\,(w = \langle\langle\bar\theta\cdot\bar b\rangle, u\rangle \in \chi_e)$,

$$e = (h, g, s), \quad \bar a_h = \bar b.$$

4.51. Lemma. *Let* $e = (h, g, s_\sigma^\alpha)$, $\bar a_h = \bar b$, $l \in h$, $\bar\theta \in \Gamma$ *and assume that* p *is a condition fulfilling the following conditions:*

(a) *if* $l \in g$, *then some extension* q *of* $\hat p^\alpha$ *forces*$_\alpha$ $H_\sigma(\bar\theta\cdot\bar b)$ *to be nearly* $(-l)$*-definable, and*

(b) *if* $l \in h-g$, *then some extension* q' *of* $\hat p^\alpha$ *forces*$_\alpha$ $H_\sigma(\bar\theta\cdot\bar b)$ *to be not nearly* $(-l)$*-definable. Then there is an extension* q *of* $\hat p^\alpha$ *such that* $p \cup q$ *is a condition prepared for* $\chi_e(\bar\theta\cdot\bar b)$.

If p does not satisfy one of the conditions (a), (b), *then* $p \Vdash \chi_e(\bar{\theta}\cdot\bar{b}) = \dot{\emptyset}$. ($\dot{0} = \dot{\emptyset}$ *is the set constant for the empty set*).

Proof. If $l \in g$, then since p satisfies (a) there is a $q \supseteq \hat{p}^\alpha$, r, $q \in P_\alpha$ and a formula $\varphi(u_1, \ldots, u_m, v) \in \mathcal{F}_\alpha$ without generic-real constants, and a $\rho \in \mathcal{T}^c_\alpha$ such that $q \Vdash_\alpha \varphi(\bar{c}, \rho) \wedge$ "ρ is finite" $\wedge\, H_\alpha(\bar{\theta}\cdot\bar{b}) \in \rho \wedge (\exists! u)\,(\varphi(\bar{c}, u))$, where $\bar{c}$ are real constants, not including a_l. Let $q^* \in P_\alpha$ be an extension of q deciding$_\alpha$ all statements

4.52 $\qquad H_\alpha(\bar{\theta}\cdot\bar{b}) = H_\alpha(\bar{\theta}'\cdot\bar{b})$,

for all $\chi_e(\bar{\theta}'\cdot\bar{b})$ mentioned by p. Then $q^* \cup p$ is a condition prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$. If $l \in h-g$, let $q \supseteq \hat{p}^\alpha$ force$_\alpha$ $H_\alpha(\bar{\theta}\cdot\bar{b})$ to be not nearly $(-l)$-definable. Let $q^* \supseteq q$ decide all statements in (4.52) above. Then $p \cup q^*$ is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$. If p does not satisfy one of (a), (b), then no extension of p is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$; thus no extension of p can mention $\chi_e(\bar{\theta}\cdot\bar{b})$. Now, if

$$p \not\Vdash \chi_e(\bar{\theta}\cdot\bar{b}) = \dot{\emptyset},$$

then for some $q \supseteq p$,

$$q \Vdash \chi_e(\bar{\theta}\cdot\bar{b}) \neq \dot{\emptyset}.$$

Hence for some τ and $q' \supseteq q$,

$$q' \Vdash_\alpha \tau \in \chi_e(\bar{\theta}\cdot\bar{b}) \;;$$

by 4.5, the forcing definition and Lemmas 3.02, 2.812, for some $q'' \supseteq q'$ and $i < \omega$, $k < 2$,

$$q'' \Vdash_\alpha \tau = \ddot{\langle}\ddot{\langle}\bar{\theta}\cdot\vec{b}\ddot{\rangle}, \dot{\langle} i, \dot{\delta}\dot{\rangle}\ddot{\rangle} \qquad \langle e, \bar{\theta}, i, k\rangle \in q'' \; .$$

This has just been seen to be impossible.

4.521. Lemma. *If p is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$, then any extension q of p has an extension q' which is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$.*

Proof. Obvious, since $\hat{p}^\alpha$ decides$_\alpha$ "$H_\alpha(\bar{\theta}\cdot\bar{b})$ is nearly $(-l)$-definable".

4.53. Lemma. *For any condition p, and $e = (h, g, s^\alpha_\sigma)$, $\bar{a}_h = \vec{b}$ such that p is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$, there is a $q \supseteq p$ which mentions $\chi_e(\bar{\theta}\cdot\bar{b})$.*

Proof. If for some $\chi_e(\bar{\theta}'\cdot\bar{b})$ mentioned by p,

$$\hat{p}^\alpha \Vdash_\alpha H_a(\bar{\theta}\cdot\bar{b}) = H_a(\bar{\theta}'\cdot\bar{b}) \,,$$

then by Lemma 4.21, $p \cup \{\langle e, \bar{\theta}', 0, \bar{\theta}, 0\rangle\}$ can be extended to a condition. If for all $\chi_e(\bar{\theta}'\cdot\bar{b})$ mentioned by p,

$$\hat{p}^\alpha \Vdash_\alpha H_a(\bar{\theta}\cdot\bar{b}) \neq H_a(\bar{\theta}'\cdot\bar{b}) \,;$$

then by Lemma 4.40, for sufficiently large r,

$$p \cup \{\langle e, \bar{\theta}', 0, \bar{\theta}, r\rangle\}$$

can be extended to a condition.

4.60. Lemma. (a) $0 \Vdash$ "*a_i is a non-constructible real*", $i \in \omega$.

(b) *If p is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$, then $p \Vdash$ "$\chi_e(\bar{\theta}\cdot\bar{b})$ is a non-constructible real*".

Proof. By standard arguments,

$$0 \Vdash \text{“}a_i \text{ is a function”} \wedge \text{“dom}(a_i) = \omega\text{”} \wedge \text{“rng}(a_i) = 2\text{”}.$$

Moreover, for no set constant $\dot{s}$ can any p force $a_i = \dot{s}$; thus by the absoluteness of constructibility, $0 \Vdash$ "a_i is non-constructible". If p is prepared for $\chi_e(\bar{\theta}\cdot\bar{b})$, then by Lemmas 4.521, 4.53 for any $q \supseteq p$ there is an extension $q' \supseteq q$ mentioning $\chi_e(\bar{\theta}\cdot\bar{b})$. Then by Lemma 4.22, for any $j < \omega$ such that $\langle e, \bar{\theta}, j, \delta\rangle \notin q'$ for any $\delta < 2$, $q' \cup \{\langle e, \bar{\theta}, j, \epsilon\rangle\}$ can be extended to a condition, $\epsilon < 2$. Thus the usual arguments are applicable; therefore

$$p \Vdash \text{“}\chi_e(\bar{\theta}\cdot\bar{b}) \text{ is a function} \wedge \text{dom}(\chi_e(\bar{\theta}\cdot\bar{b})) = \omega$$
$$\wedge\ \text{rng}(\chi_e(\bar{\theta}\cdot\bar{b})) = 2\text{”} \,,$$

$$p \Vdash \text{“}\chi_e(\bar{\theta}\cdot\bar{b}) \text{ is non-constructible”}.$$

The next lemma shows that a connection yields the desired result.

4.61. Lemma. *If $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in p$, then*

$$p \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_1\cdot\bar{b})) = {}^{l_2}\theta(\chi_e(\bar{\theta}_2\cdot\bar{b})) \,.$$

Proof. By Lemma 4.60 we know that p forces $\chi_e(\bar{\theta}\cdot\bar{b})$ to be real. Therefore, we consider any $j \in \omega$, $\epsilon < 2$ and $q \supseteq p$, such that

$$q \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{b}))\,(j) = \epsilon\ .$$

By Definitions 2.64 and 4.50, we have

$$q \Vdash \chi_e(\theta_1 \cdot \bar{b})\,(l_1 + j) = \epsilon\ ;$$

therefore

$$\langle e, \bar{\theta}_1, l_1 + j, \epsilon\rangle \in q\ ,$$

(by Definition 2.64). Since

$$\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle \in q$$

implies

$$\langle e, \bar{\theta}_2, (l_1 + j) + (l_2 - l_1), \epsilon\rangle \in q\ ,$$

thus, again applying Definitions 2.64 and 4.50,

$$q \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b}))\,(j) = \epsilon\ .$$

The other direction is shown by a symmetric argument. This implies that

$$p \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{b})) = {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b}))\ ,$$

since if not, then by Corollary ?.79 for some $q' \supseteq p$,

$$q' \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b})) \neq {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b}))\ ;$$

and taking $q \supseteq q'$ as in the previous considerations, this would yield a contradiction.

The next lemma is almost the inverse of the previous lemma and will be needed later.

4.62. Lemma. *If* $p \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{b})) = {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b}))$, *then either*

$$p \Vdash \chi_e(\bar{\theta}_1 \cdot \bar{b}) = \check{\emptyset} = \chi_e(\bar{\theta}_2 \cdot \bar{b})\ ,$$

or

$$p \cup \{\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle\}$$

can be extended to a condition.

Proof. First we show that

$$p \Vdash \chi_e(\bar{\theta}_1 \cdot \bar{b}) = \dot{\emptyset} \leftrightarrow \chi_e(\bar{\theta}_2 \cdot \bar{b}) = \dot{\emptyset}\,.$$

Let $p' \supseteq p$ be such that

$$p' \parallel \chi_e(\bar{\theta}_1 \cdot \bar{b}) = \dot{\emptyset}, \qquad p' \parallel_\alpha \chi_e(\bar{\theta}_2 \cdot \bar{b}) = \dot{\emptyset}\,;$$

and assume that $p' \Vdash \chi_e(\bar{\theta}_1 \cdot \bar{b}) = \dot{\emptyset}$ but $p' \Vdash \chi_e(\bar{\theta}_2 \cdot \bar{b}) \neq \dot{\emptyset}$. Then by Lemma 4.51, p' must fulfill the conditions (a) and (b) of that lemma concerning $H_\sigma(\bar{\theta}_2 \cdot \bar{b})$; in which case there is an extension $q \supseteq p'$ such that q mentions $\chi_e(\bar{\theta}'_2 \cdot \bar{b})$.

$$q' \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{b})) = {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b})) \wedge \chi_e(\bar{\theta}_1 \cdot \bar{b}) = \dot{\emptyset}$$

gives, by Definitions 2.64 and 4.50

$$q' \Vdash {}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{b}) = \dot{\emptyset}\,;$$

therefore by Lemmas 2.811 and 2.812

$$q' \Vdash {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b})) = \dot{\emptyset}$$

and hence

$$q' \Vdash \neg(\exists u > l_2)\,(\exists v < 2)\,[\langle \dot{u}, \dot{v}\rangle \in \chi_e(\bar{\theta}_2 \cdot \bar{b})]\,.$$

But by Lemma 4.22, for $k > l_2$ sufficiently large, $q' \cup \{\langle e, \bar{\theta}_2, k, 0\rangle\}$ can be extended to a condition q''. This yields a contradiction, since $q'' \Vdash \chi_e(\bar{\theta}_2 \cdot \bar{b})\,(k) = 0$. Thus the first part of the lemma is proven. Let $p' \supseteq p$ be such that

$$p' \Vdash \chi_e(\bar{\theta}_1 \cdot \bar{b}) \neq \dot{\emptyset} \wedge \chi_e(\bar{\theta}_2 \cdot \bar{b}) \neq \dot{\emptyset}\,.$$

Then by Lemmas 4.51, 4.53, there is an extension q of p' such that q mentions both $\chi_e(\bar{\theta}_1 \cdot \bar{b})$ and $\chi_e(\bar{\theta}_2 \cdot \bar{b})$. By Definition 2.50(d)(3), for some m_1, m_2,

$$\langle e, \bar{\theta}_1, m_1, \bar{\theta}_2, m_2\rangle \in q\,.$$

If $m_1 = l_1$, $m_2 = l_2$, we are finished. Hence assume $\langle m_1, m_2\rangle \neq \langle l_1, l_2\rangle$. If $m_1 - m_2 = l_1 - l_2$, and $m_1 < l_1$, or $m_1 > l_1$ and

$${}^q\mu(e, \bar{\theta}_1, m_1) - {}^q\mu(e, \bar{\theta}_1, l_1) = {}^q\mu(e, \bar{\theta}_2, m_2) - {}^q\mu(e, \bar{\theta}_2, l_2)\,,$$

the result follows from Lemma 4.45. If $m_1 - m_2 = l_1 - l_2$, $m_1 > l_1$ but for some $k, m_1 > k \geqslant l_1$,

$$\langle e, \bar{\theta}_1, k, \epsilon\rangle \in q \quad \text{and} \quad \langle e, \bar{\theta}_2, k - l_1 + l_2, 1-\epsilon\rangle \in q \ ,$$

then

$$q \Vdash \chi_e(\bar{\theta}_1 \cdot \bar{b})\,(\dot{k}) = \dot{\epsilon} \neq \chi_e(\bar{\theta}_2 \cdot \bar{b})\,(k - \dot{l}_1 + l_2) = 1 \dot{-} \epsilon \ ;$$

therefore

$$q \Vdash \dot{\langle} \dot{k} - l_1, \dot{\epsilon} \dot{\rangle} \in {}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{b})) \wedge \dot{\langle} k \dot{-} l_1, 1 \dot{-} \epsilon \dot{\rangle} \in {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b})) \ .$$

Since $q \Vdash {}^{l_1}\theta(\chi_1(\bar{\theta}_1 \cdot \bar{b})) = {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{b}))$, we get also

$$\langle e, \bar{\theta}_2, k - l_1 + l_2, \epsilon\rangle \in q \ ,$$

which contradicts the definition of a condition. If $m_1 - m_2 \neq l_1 - l_2$, then for sufficiently large k,

$$q \cup \{\langle e, \bar{\theta}_1, l_1 + k, \epsilon\rangle, \langle e, \bar{\theta}_2, l_2 + k, 1 - \epsilon\rangle\}$$

can be extended to a condition, yielding a contradiction as in previous cases. Thus Lemma 4.62 is proven.

4.63. Lemma. *$p \Vdash \chi_e(\bar{\theta} \cdot \bar{b})\,(j) = \epsilon$ iff $\langle e, \bar{\theta}, j, \epsilon\rangle \in p$.*

Proof. This is immediate from Definitions 4.50 and 2.64.

4.70. Notation. In the following lemmas the basic properties of the primitive generic elements are proven. With regard to these elements the following notation is employed. If σ is a term of $\mathcal{T}^c$, we denote $\mathrm{val}(\sigma) = \boldsymbol{\sigma}$. This will also be used with regard to the elementary operation symbols. In particular we generally write:

$$\mathrm{val}({}^{(t,l)}\theta(\sigma)) = {}^{(t,l)}\boldsymbol{\theta}(\boldsymbol{\sigma})$$

instead of $\mathrm{append}_t(\mathrm{slash}_l(\sigma))$. Note that the functions, ${}^{(t,l)}\theta$ are more general than the functions append_t, slash_l. Hence also the function $\boldsymbol{K}$ is more general than the tail operation (Definition 1.20) which was defined only for reals.

4.701. Lemma. (a) $\mathrm{val}_\alpha(\dot{\{}\sigma_1, ..., \sigma_n\dot{\}}) = \{\mathrm{val}_\alpha(\sigma_1), ..., \mathrm{val}_\alpha(\sigma_n)\}$.
(b) $\mathrm{val}_\alpha(\dot{\langle}\sigma_1, ..., \sigma_n\dot{\rangle}) = \langle \mathrm{val}_\alpha(\sigma_1), ..., \mathrm{val}_\alpha(\sigma_n)\rangle$.

Proof. By (2.904),

$$\mathrm{val}_\alpha(\tau) = \{\mathrm{val}_\alpha(\sigma) \mid \mathrm{rnk}(\sigma) < \mathrm{rnk}(\tau), \mathrm{ord}_\alpha(\sigma) < \mathrm{ord}_\alpha(\tau),$$
$$Q_\alpha \Vdash_\alpha \sigma \in \tau\} ,$$
$$\dot{\{}\sigma_1, ..., \sigma_n\dot{\}} = (\exists w)\,(w = \sigma_1 \vee ... \vee w = \sigma_n) .$$

Clearly $Q_\alpha \Vdash_\alpha \rho \in \tau$ iff there is an i, $Q_\alpha \Vdash_\alpha \rho = \sigma_i$, $1 \leq i \leq n$; hence

$$\mathrm{val}_\alpha(\dot{\{}\sigma_1, ..., \sigma_n\dot{\}}) = \{\mathrm{val}_\alpha(\sigma_1), ..., \mathrm{val}_\alpha(\sigma_n)\}.$$

Similarly, $Q_\alpha \Vdash_\alpha \rho \in \dot{\langle}\sigma_1, \sigma_2\dot{\rangle}$ iff $Q_\alpha \Vdash_\alpha \rho = \dot{\{}\sigma_1\dot{\}} \vee \rho = \dot{\{}\sigma_1, \sigma_2\dot{\}}$; thus

$$\mathrm{val}_\alpha(\dot{\langle}\sigma_1, \sigma_2\dot{\rangle}) = \langle \mathrm{val}_\alpha(\sigma_1), \mathrm{val}_\alpha(\sigma_2)\rangle .$$

The general case is proved by induction on n.

4.71. Lemma. *Assume $\sigma \in \mathcal{T}^c_\alpha$; then*

(a) $\mathrm{val}_\alpha(\theta(\sigma)) = \boldsymbol{\theta}(\mathrm{val}_\alpha(\sigma))$, $\mathrm{val}_\alpha(K(\sigma)) = \boldsymbol{K}(\mathrm{val}_\alpha(\sigma))$.

(b) $\mathrm{val}_\alpha(a_i) = \boldsymbol{a}_i$ *is a generic real; moreover if $i \neq j$,*

$$\boldsymbol{K}(\boldsymbol{a}_i) \cap \boldsymbol{K}(\boldsymbol{a}_j) = \emptyset, \qquad 0 \leq i, j < \omega .$$

(c) $\mathrm{val}_\alpha(I_0) = \boldsymbol{I}_0\ \{\langle i, \boldsymbol{K}(\boldsymbol{a}_i)\rangle \mid i \in \omega\}$.

Proof. (a) is immediate from Definitions and Lemmas 4.701, 3.02, 2.908, 2.812, 2.906, 2.911.

(b) That $\boldsymbol{a}_i$ is a generic real follows from Lemmas 4.60(a) and 2.911. If

$$\boldsymbol{K}(\boldsymbol{a}_i) \cap \boldsymbol{K}(\boldsymbol{a}_i) \neq \emptyset, \quad i \neq j ,$$

then for some ${}^{(t,l)}\theta = \theta$, $t \in 2^{l'}$, $\theta(a_i) = a_j$. Hence there exists a $p \in Q$ such that $p \Vdash \theta(a_i) = a_j$. Let $k > l + l'$ be an integer larger than any mentioned in p. Then

$$p \cup \{\langle i, k, 0\rangle, \langle j, k - l + l', 1\rangle\} = q$$

is a condition. We have

$$q \Vdash \theta(a_i)\,(k - l + l') = 0 \wedge a_j(k - l + l') = 1 \wedge \theta(a_i) = a_j .$$

This is a contradiction.

(c) follows immediately from Definition 2.64 and Lemmas 2.812, 2.906, 2.908, 2.911, 3.02, 4.701, and (a) of this lemma.

4.72. Definition. ${}^{\alpha}H_{\sigma} = (H_{\sigma})^{(N_{\alpha})}$, ${}^{\alpha}H_{\sigma}(\bar{\theta}\cdot\bar{b}) = (H_{\sigma}(\bar{\theta}\cdot\bar{b}))^{(N_{\alpha})}$.

4.73. Lemma. *Assume* $\chi_e \in \mathfrak{X}_{\alpha+1}$, $e = (h, g, s^{\alpha}_{\sigma}) = e_g$, $\bar{a}_h = \bar{b}$.

(a) $\mathrm{val}_g(\chi_e) = \chi_e$, χ_e *is a binary relation; and*

$$\mathrm{dom}(\chi_e) \subseteq K(\bar{b}), \qquad \bigcup_{h \supseteq g} \mathrm{dom}(\chi_{e_g}) = K(\bar{b}),$$

$$\mathrm{dom}(\chi_{e_{g*}}) \cap \mathrm{dom}(\chi_{e_{g**}}) = \emptyset, \qquad g^* \neq g^{**}, \quad g^*, g^{**} \subseteq h.$$

Assume that $\bar{\theta}\cdot\bar{b}$ *and* $\bar{\theta}'\cdot\bar{b}$ *are in* $\mathrm{dom}(\chi_e)$ *then.*

(b) $\chi_e(\bar{\theta}\cdot\bar{b})$ *is a generic real;*

(c) *if* ${}^{\alpha}H_{\sigma}(\bar{\theta}\cdot\bar{b}) = {}^{\alpha}H_{\sigma}(\bar{\theta}'\cdot\bar{b})$, *then* $\chi_e(\bar{\theta}\cdot\bar{b}) = \chi_e(\bar{\theta}'\cdot\bar{b})$;

(d) *if* $\bar{\theta}\cdot\bar{b}$, $\bar{\theta}'\cdot\bar{b} \in \mathrm{dom}(\chi_e)$ *and* ${}^{\alpha}H_{\sigma}(\bar{\theta}\cdot\bar{b}) \neq {}^{\alpha}H_{\sigma}(\bar{\theta}'\cdot\bar{b})$, *then*

$$\chi_e(\bar{\theta}\cdot\bar{b}) \neq \chi_e(\bar{\theta}'\cdot\bar{b}), \quad K(\chi_e(\bar{\theta}\cdot\bar{b})) = K(\chi_e(\bar{\theta}'\cdot\bar{b}));$$

(e) *if* $\mathrm{dom}(\mathring{\chi}_e)$ *is infinite, then* $\mathrm{rng}(\chi_e)$ *is a tail.*

(f) $\mathrm{val}_{\beta}(I_{\alpha}) = I_{\alpha} = \{\langle e, \chi_e\rangle \mid e \text{ is a } \gamma+1\text{-index}, \gamma < \alpha\} \cup I_0$, $\beta \geq \alpha$.

Proof. (a) That $\mathrm{val}_{\beta}(\chi_e) = \chi_e$, $\beta < \alpha$, follows from Lemma 3.10, i.e., Lemma 3.40 and Definition 2.60. That χ_e is a binary relation with $\mathrm{dom}(\chi_e) \subseteq K(\bar{b})$, is immediate from the forcing relation by Lemmas 4.71, 4.701, 3.02, 2.911, 2.908, 2.906 and 2.812. To see that

$$\mathrm{dom}(\chi_{e_{g*}}) \cap \mathrm{dom}(\chi_{e_{g**}}) = \emptyset,$$

assume that for some $k_1, k_2 < \omega$, $i_1, i_2 < 2$, $\bar{\theta} \in \bar{\Gamma}$ and p,

$$p \Vdash \ddot{\langle}\langle\bar{\theta}\cdot\bar{b}\rangle, \dot{\langle} k_1, i_1\ddot{\rangle}\rangle \in \chi_{e_{g*}} \wedge \ddot{\langle}\langle\bar{\theta}\cdot\bar{b}\rangle, \dot{\langle} k_2, i_2\ddot{\rangle}\rangle \in \chi_{e_{g**}};$$

then by Definition 2.64, $\langle e, \bar{\theta}, k_1, i_1\rangle \in p$, $\langle e, \bar{\theta}, k_2, i_2\rangle \in p$. Hence by Definition 2.50, $\hat{p}^{\alpha}$ forces$_{\alpha}$ $H_{\sigma}(\bar{\theta}\cdot\bar{b})$ to be nearly $(-l)$-definable and $\hat{p}^{\alpha}$ forces$_{\alpha}$ $H_{\sigma}(\bar{\theta}\cdot\bar{b})$ to be not nearly $(-l)$-definable, for some $l \in g$, or vice versa. This is impossible. Hence

$$0 \Vdash \mathrm{dom}(\chi_{e_{g*}}) \cap \mathrm{dom}(\chi_{e_{g**}}) = \emptyset.$$

The result now follows by Lemma 2.911. To show that

$$\bigcup_{h \supseteq g} \operatorname{dom}(\chi_{e_g}) = K(\bar{b}),$$

assume that for some $\bar{\theta} \in \Gamma$ and p,

$$p \Vdash \bar{\theta} \cdot \bar{b} \notin \operatorname{dom}(\chi_{e_g});$$

then for no k, i is $p \cup \{\langle e, \bar{\theta}, k, i \rangle\}$ extendible to a condition. Then by Lemma 4.51, $\hat{p}^{\alpha}$ must force_{α} $H_{\sigma}(\bar{\theta} \cdot \bar{b})$ to be not nearly $(-l)$-definable for some $l \in g$. Therefore by Lemmas 4.51 and 4.53, for any $r \supseteq p$, there is a g^* and $q \supseteq r$ mentioning $\chi_{e_{g^*}}(\bar{\theta} \cdot \bar{b})$. By Lemma 4.22 there is a $q' \supseteq q$ and k, j such that $\langle e, \bar{\theta}, k, j \rangle \in q'$. Thus $p \Vdash \bar{\theta} \cdot \bar{b} \in \operatorname{dom}(\chi_{e_{g^*}})$. Therefore,

$$0 \Vdash \bigcup_{h \supseteq g} \bar{\theta} \cdot \bar{b} \in \operatorname{dom}(\chi_{e_g}), \quad \bar{\theta} \in \bar{\Gamma}.$$

This implies

$$0 \Vdash (\forall w)\,(w \in K(\bar{b}) \to \bigvee_{h \supseteq g} w \in \operatorname{dom}(\chi_{e_g}))\,.$$

The result now follows by Lemma 2.831.

(b) If $\bar{\theta} \cdot \bar{b} \in \operatorname{dom}(\chi_e)$, then for some $p \in Q$,

$$p \Vdash \bar{\theta} \cdot \bar{b} \in \operatorname{dom}(\chi_e)\,.$$

Thus, for some k, j, $\langle e, \bar{\theta}, k, j \rangle \in p$. The result now follows from Lemmas 4.60, 2.909.

(c) If (c) is not true, then for some $p \in Q$,

$$p \Vdash \bar{\theta} \cdot \bar{b} \in \operatorname{dom}(\chi_e) \wedge \bar{\theta}' \cdot \bar{b} \in \operatorname{dom}(\chi_e) \wedge \chi_e(\bar{\theta} \cdot \bar{b}) \neq \chi_e(\bar{\theta}' \cdot \bar{b})$$
$$\wedge\, {}^{\alpha}H_{\sigma}(\bar{\theta} \cdot \bar{b}) = {}^{\alpha}H_{\sigma}(\bar{\theta}' \cdot \bar{b})\,.$$

Therefore there is a $q \in Q$, $q \supseteq p$, and a $j < \omega$, $\epsilon < 1$, such that

$$q \Vdash \chi_e(\bar{\theta} \cdot \bar{b})\,(j) = \epsilon \wedge \chi_e(\bar{\theta}' \cdot \bar{b}) = 1 - \epsilon\,.$$

By Lemma 4.63, $\langle e, \bar{\theta}, j, \epsilon \rangle \in p$, and $\langle e, \bar{\theta}', j, 1 - \epsilon \rangle \in p$. By Lemma 3.10,

$$\hat{p}^{\alpha} \Vdash_{\alpha} H_{\sigma}(\bar{\theta} \cdot \bar{b}) = H_{\sigma}(\bar{\theta}' \cdot \bar{b})\,.$$

This is a contradiction to Definition 2.50(d)(3).

(d) Assume that (d) is not true, and for some $p \in Q$,

$$p \Vdash {}^{\alpha}H_{\sigma}(\bar{\theta} \cdot \bar{b}) \neq {}^{\alpha}H_{\sigma}(\bar{\theta}' \cdot \bar{b}) \wedge \chi_e(\bar{\theta} \cdot \bar{b}) = \chi_e(\bar{\theta} \cdot \bar{b}) \neq \dot{\emptyset}\,.$$

By Lemmas 4.51, 4.53, we can assume that p mentions $\chi_e(\bar\theta\cdot\bar b)$ and $\chi_e(\bar\theta'\cdot\bar b)$. Now, by using Lemma 4.22 twice, (using the fact that $\hat p^\alpha \Vdash_\alpha H_\sigma(\bar\theta\cdot\bar b)\neq H_\sigma(\bar\theta'\cdot\bar b)$),

$$p\cup\{\langle e,\bar\theta,k,0\rangle,\langle e,\bar\theta',k,1\rangle\}$$

can be extended to a condition q for sufficiently large k.

$$q\Vdash \chi_e(\bar\theta\cdot\bar b)=\chi_e(\bar\theta'\cdot\bar b)\wedge\chi_e(\bar\theta\cdot\bar b)(k)\neq\chi_e(\bar\theta'\cdot\bar b)(k)\,,$$

which gives a contradiction. To see the other part, assume $p\in Q$ and

$$p\Vdash {}^\alpha H_\sigma(\bar\theta\cdot\bar b)\neq{}^\alpha H_\sigma(\bar\theta'\cdot\bar b)\wedge K(\chi_e(\bar\theta\cdot\bar b))\cap K(\chi_e(\bar\theta'\cdot\bar b))=\emptyset$$
$$\wedge\,\chi_e(\bar\theta\cdot\bar b)\neq\dot\emptyset\neq\chi_e(\bar\theta'\cdot\bar b)\,.$$

By Lemmas 4.51, 4.53, there is an extension q of p mentioning $\chi_e(\bar\theta\cdot\bar b)$, $\chi_e(\bar\theta'\cdot\bar b)$. Now, by Definition 2.50(d)(4), for some $l,l'<\omega$, $\langle e,\bar\theta,l,\bar\theta',l'\rangle\in p$, and by Lemma 4.61,

$$q\Vdash {}^l\theta(\chi_e(\bar\theta\cdot\bar b))={}^{l'}\theta(\chi_e(\bar\theta'\cdot\bar b))\,.$$

Since $p\subset q$, this yields a contradiction.

(e) If (e) is not true, then for some $p\in Q$, $p\Vdash|\mathrm{dom}(\mathring\chi_e)|=\dot\aleph_0$, and for some $r,r'<\omega$, $\mu\in 2^r$,

$$p\Vdash {}^{\langle\mu,r'\rangle}\theta(\chi_e(\bar\theta'\cdot\bar b))\notin\mathrm{rng}(\chi_e)\wedge\bar\theta'\cdot\bar b\in\mathrm{dom}(\chi_e)\,.$$

Let $\{\chi_e(\bar\theta_j\cdot\bar b)\mid 1\leqslant j\leqslant m\}$ be a maximal set of p-different slash$_r$-equal elements in p, with

$$\langle e,\bar\theta',s_j',\bar\theta_j,s_j\rangle\in p$$

where $s_j-s_j'=r-r'$, and $s_j\leqslant r$. We can assume (by Lemma 4.22) that

$$\mu_j={}^p\mu(e,\bar\theta_j,r)\in 2^r.$$

This set must be deficient, i.e., $m<2^r$; for otherwise some $\mu_j=\mu$, which would already yield a contradiction. Since $|\mathrm{dom}(\mathring\chi_e)|=\aleph_0$ there is a $\bar\theta\in\bar\Gamma$, and a $q\supseteq p$, $q\in Q$, such that

$$q\Vdash\bigwedge_{j=1}^m {}^\alpha H_\sigma(\bar\theta\cdot\bar b)\neq{}^\alpha H_\sigma(\bar\theta_j\cdot\bar b)\,.$$

By Lemmas 3.10 and 4.40,

$$q \cup \{\langle e, \bar{\theta}, r, \bar{\theta}', r'\rangle \cup p(e, \bar{\theta}, \mu)$$

can be extended to a condition q^*. Clearly

$$p \subseteq q^* \Vdash^{(\mu, r')} \theta(\chi_e(\bar{\theta}' \cdot \bar{b})) = \chi_e(\bar{\theta} \cdot \bar{b}) \in \mathrm{rng}(\chi_e)\,,$$

which is a contradiction.

(f) This fact follows directly from Definition 2.64 by Lemmas 4.701, 3.02, 2.911, 2.908, 2.906 and 2.812.

4.80. Definition. Assume that $\tau \in N_\alpha$ and that there exists a formula $\varphi \in \mathcal{F}_\alpha$, with v as its only free variable, such that $N_\alpha \models (\exists! v)(\varphi(v) \wedge \varphi(\tau))$, then:

(a) if in φ there do not occur any generic real constants, then τ is said to be *definable*$_\alpha$, (definable in N_α), by φ. τ is said to be definable$_\alpha$ if for some $\varphi \in \mathcal{F}_\alpha$, τ is definable$_\alpha$ by φ.

(b) If $\varphi = \varphi(\bar{a}_h, v)$, where $\bar{a}_h$ are all generic real constants occurring in φ, then for any $f \supseteq h$, we may say that τ is f-definable$_\alpha$. τ is said to be f-definable$_\alpha$, if for some $\varphi \in \mathcal{F}_\alpha$, τ is f-definable$_\alpha$ by φ. (Note that "$\emptyset$-definable$_\alpha$" is equivalent to "definable$_\alpha$".)

(c) We shall also say that τ is $(-f)$-definable$_\alpha$ if there are $h \subseteq \omega$, such that $h \subseteq \omega$, such that $h \cap f = \emptyset$ and τ is h-definable$_\alpha$.

(d) τ is said to be nearly f-definable$_\alpha$ (by φ) if for some finite $\sigma \in N_\alpha$, which is f-definable$_\alpha$ (by φ), $\tau \in \sigma$. Similarly, τ is said to be nearly $(-f)$-definable$_\alpha$ if for some finite $\sigma \in N_\alpha$, which is nearly $(-f)$-definable$_\alpha$, $\tau \in \sigma$.

4.801. Remark. Note that the notion of nearly $(-f)$-definable$_\alpha$ complies with the parallel notion defined in 2.50(e), where $f = \{l\}$.

4.81. Definition. ${}^\alpha H^g_\sigma$, $g \subseteq h$, $\sigma = \sigma(\bar{b})$, are the following sets:

${}^\alpha H^g_\sigma$ is the subset of ${}^\alpha H_\sigma$ consisting of all elements which are nearly $(-\{l\})$-definable$_\alpha$, for every $l \in g$, but for every $l \in h - g$ they are not nearly $(-\{l\})$-definable$_\alpha$.

4.82. Remark. Observe that it has not yet been shown that these elements (in Definition 4.81) are actually values of terms. This will be shown below.

The next lemma follows immediately from Definition 4.81.

4.83. Lemma. ${}^{\alpha}H_{\sigma}^{g^*} \cap {}^{\alpha}H_{\sigma}^{g^{**}} = \emptyset$, $g^* \neq g^{**}$, $\bigcup_{h \supseteq g} {}^{\alpha}H_{\sigma}^{g} = {}^{\alpha}H_{\sigma}$.

4.84. Definition. The functions $\mathring{\chi}_e$, $\mathring{\chi}_d$, in which we shall actually be interested, are now derived from the χ_e. Let $e_g = (h, g, s_{\sigma}^{\alpha})$ and $|h| = n$.

(a) $\mathring{\chi}_e = (\text{Ɛ}w)\,(\exists u)\,(\exists v)\,(\overline{\exists \bar{u}^n})\,[\dot{\langle}\bar{u}^n\dot{\rangle} \in \operatorname{dom}(\chi_e) \wedge v = \chi_e(\bar{u}^n)$
$\wedge\, u = (\text{Ɛ}w')\,(\overline{\exists \bar{v}^n})\,(w' = \dot{\langle} v_1, \ldots, v_n \dot{\rangle} \wedge v = \chi_e(\bar{v}^n)$
$\wedge\, w = \dot{\langle} u, v \dot{\rangle}]$

(b) Denoting $(h, g, s_{\sigma}^{\alpha}) = e_g$, $d = (h, s_{\sigma}^{\alpha})$, set:

$$\chi_d = \dot{\bigcup_{h \supseteq g}} \chi_{e_g}, \qquad \mathring{\chi}_d = \dot{\bigcup_{h \supseteq g}} \mathring{\chi}_{e_g}.$$

4.85. Definition. If x is a set of reals, we define

$$K(x) = \bigcup_{y \in x} K(y).$$

If X, Y are sets of reals, then they are said to be K-disjoint if

$$K(X) \cap K(Y) = \emptyset.$$

4.86. Lemma. *Let* $e = (h, g, s_{\sigma}^{\alpha})$, *then*

(i) $\mathring{\chi}_e$ *is a univalent function,*

(ii) $\operatorname{dom}(\mathring{\chi}_e) = {}^{\alpha}H_{\sigma}^{g}$,

(iii) $\operatorname{rng}(\mathring{\chi}_e)$ *is included in a tail,*

(iv) *if* ${}^{\alpha}H_{\sigma}^{g}$ *is infinite, then* $\operatorname{rng}(\mathring{\chi}_e)$ *is a tail,*

(v) *if* $g^* \neq g^{**}$, $g^*, g^{**} \subseteq h$, *then* $\operatorname{rng}(\mathring{\chi}_{e_{g^*}})$, $\operatorname{rng}(\mathring{\chi}_{e_{g^{**}}})$ *are K-disjoint.*

Proof. (i) follows from the definition of $\mathring{\chi}_e$ and Lemma 4.73(d). (ii) follows from Definitions 2.50(e) and 4.81. (iii) follows from Lemma 4.73(d). (iv) follows from Lemma 4.73(e). (v) follows from Lemma 4.73(a).

Finally,

4.87. Lemma. *Let* $e = (h, g, s_{\sigma}^{\alpha})$, $d = (h, s_{\sigma}^{\alpha})$; *then:*

(i) $\mathring{\chi}_d$ *is a univalent function;*

(ii) $\operatorname{dom}(\mathring{\chi}_d) = {}^{\alpha}H_{\sigma}$;

(iii) $\operatorname{rng}(\mathring{\chi}_d \restriction {}^{\alpha}H^g_\sigma)$ *is included in a tail;*
(iv) *if* ${}^{\alpha}H^g_\sigma$ *is infinite, then* $\operatorname{rng}(\mathring{\chi}_d \restriction {}^{\alpha}H^g_\sigma)$ *is a tail;*
(v) $\operatorname{rng}(\mathring{\chi}_d \restriction {}^{\alpha}H^g_\sigma)$ *are K-disjoint,* $g \subseteq h$.

Proof. Immediate from Lemmas 4.86 and 4.73(a).

4.88. Definition. Let

$$\mathring{I}_\alpha = I_0 \cup \{\langle e, \mathring{\chi}_e\rangle \mid e \text{ is a } \gamma+1\text{-index}, \gamma < \alpha\},$$

$$J_\alpha = I_0 \cup \{\langle d, \mathring{\chi}_d\rangle \mid d = (h,s),\ (h,g,s) \text{ is a } \gamma+1\text{-index}, \\ g \subseteq h, \gamma < \alpha\}\ .$$

From J_α a well-ordering of the $\mathring{\chi}_d$ will be derived in N_α. Note that $\mathring{I}_\alpha$, and J_α are all defined in N_β, $\beta \geqslant \alpha$.

5. Symmetry properties of the models

In this section we study the symmetry properties of the models N_α, and the corresponding relations $\Vdash_\alpha$, $\alpha \leqslant \aleph_1^{(M)}$. To get the gist of things, consider a model $N' = M[x]$ obtained from M by adjoining a single generic real $x \in 2^\omega$. Let y be a generic real derived from x in an *infinite way*. For instance assume y is such that $y(i) = x(2i)$, $i < \omega$. In this case, y is, obviously, also a generic real and $M[y] \subseteq M[x]$. Since y was obtained from x by omitting an *infinite amount of information*, $M[y] \subsetneqq M[x]$. The situation is entirely different if y differs from x in a finite way, as in the case where $y = \text{append}_t(\text{slash}_l(x))$. Here, $M[y] = M[x]$, since y is defined from x and vice versa. Moreover, x, y are *equally generic*. This has the following result:

Assume $x = \text{val}_Q(a)$, $\varphi(a)$ is a sentence of the local language (a), and that for some $p \in Q$, $p \Vdash \varphi(a)$; in which case $M[x] = \varphi[x]$. *Now, if $y \in K(x)$ also satisfies p, (i.e., $\langle i, \delta\rangle \in p$ implies $y(i) = \delta$), then also $N' = M[x] = M[y] \models \varphi[y]$.*

For the moment, let us call this property of N', *elementary tail symmetry*. We would expect the same state of affairs to hold in a more intricate construction, provided that the exchange of x with $y \in K(x)$ does not affect the special sets added that are not mentioned in the condition. For example consider

$$N' = M[I_0] = N_0, \quad \text{where } I_0 = \{\langle i, K(a_i)\rangle \mid i \in \omega\}.$$

Then for any $\theta_i \in \Gamma$,

$$I_\Gamma = \{\langle i, K(\boldsymbol{\theta}(a_i))\rangle \mid i \in \omega\}$$

thus N' presumably has elementary tail symmetry (with respect to all the tails $K(a_i)$, $i \in \omega$). On the other hand, if

$$N' = M[J], \quad \text{where } J = \{\langle i, a_i\rangle \mid i \in \omega\},$$

then, in general,

$$J \neq \{\langle i, \boldsymbol{\theta}_i(a_i)\rangle \mid i \in \omega\};$$

hence N' obviously does not have elementary tail symmetry. We will show that the models N_α have elementary tail symmetry (with respect to the tails $K(a_i)$, $i \in \omega$). Moreover, in Section 10 it will be shown that for any $p(\bar{a}_h) \in Q$ there are $\aleph_0$ $\bar{x} \in K(\bar{a}_h)$ satisfying p. From this fact the negation of AC in N is easily shown, using the elementary symmetry property of N.

5.0. In order to study the effect of replacing a_i with $\theta(a_i)$, we define the following operations $T^{\theta,i}$ on conditions and expressions of the local language, $\theta \in \Gamma$, $i \in \omega$. First, the composition $\theta_2 \circ \theta_1$, $\theta_1, \theta_2 \in \Gamma$, will be defined.

5.01. Definition. ${}^{(t_2,l_2)}\theta \circ {}^{(t_1,l_1)}\theta = {}^{(t_2,l_1+l_2-|t_1|)}\theta$, if $l_2 \geqslant |t_2|$; and

$$ {}^{(t_2,l_2)}\theta \circ {}^{(t_1,l_1)}\theta = {}^{(t_2 \cup (t_1 - t_1 \upharpoonright l_2),l_1)}\theta \,, $$

if $l_2 < |t_1|$.

5.01. Corollary. ${}^{t}\theta \circ {}^{l}\theta = {}^{(t,l)}\theta$.

5.02. Remark. This coincides with our interpretation of ${}^{(t,l)}\theta$ as append$_t$(slash$_l$), see also Lemma 5.15.

5.03. Definition. (a) For every expression $\varphi \in \mathcal{F} \cup \mathcal{T}$, $T^{\theta,i}(\varphi)$ is the expression obtained from φ by exchanging a_i with $\theta(a_i)$ everywhere.

(b) For a condition p, $T^{\theta,i}(p)$ is a condition which is like p concerning all generic reals a_j other than a_i, and which for a_i contains the information so that $\theta(a_i)$ will satisfy p. This essentially amounts to a proper *shift* (translation) of coordinate information on a_i. To allow for this substitution, we will also have to make a proper adjustment of the $\bar{\theta}$'s occurring in preconditions of p mentioning the various $\chi_e(\bar{\theta}\cdot\bar{b})$. $T^{\theta,i}$ acts on preconditions as follows: Assume $\theta = {}^{(t,l)}\theta$ where $t \in 2^{l^*}$, $l, l^* < \omega$, then

$$ \text{(i)} \quad T^{\theta,i}(\langle j, m, \delta\rangle) = \begin{cases} \langle j, m, \delta\rangle, & i \neq j\,, \\ \langle j, m+l-l^*, \delta\rangle, & i = j,\ m+l \geqslant l^*, \\ \text{nothing}, & i = j,\ m+l < l^*. \end{cases} $$

(ii) Assume that $\chi_e(\bar{\theta}\cdot\bar{a}_h)$ is mentioned in p, where $\bar{\theta} = \langle\theta_1, \ldots, \theta_n\rangle \in \bar{\Gamma}^h$; then define $\bar{\theta}' = \langle\theta'_1, \ldots, \theta'_n\rangle$ as the sequence such that $\theta'_j = \theta_j$ if $h_j \neq i$, and $\theta'_j = \theta_j\cdot\theta$ if $h_j = i$, ($h = \{h_1, \ldots, h_n\}$). (Notice that $\bar{\theta}'$ depends on i, $\bar{\theta}$ and e; and that θ and $\bar{\theta}'$ are unrelated.) Now define:

$$T^{\theta,i}(\langle e, \bar\theta, m, \delta\rangle) = \langle e, \bar\theta', m, \delta\rangle,$$

$$T^{\theta,i}(\langle e, \bar\theta_1, l_1, \bar\theta_2, l_2\rangle) = \langle e, \bar\theta'_1, l_1, \bar\theta'_2, l_2\rangle .$$

(iii) $T^{\theta,i}(p) = \{T^{\theta,i}(p') \mid p' \in p\}$.

5.04. Remark. Note that $T^{\theta,i}$ may remove some information from a condition. It is not yet clear that $T(p)$ is a condition. This will soon be shown.

5.1. We precede the main lemma by a few auxiliary lemmas, the proofs of which follow easily from the definition of forcing by earlier elementary lemmas.

Lemma 5.11 expresses the fact that for any $x, y \in K(a_i)$ we can find t, l such that $\text{append}_t(\text{slash}_l(x)) = y$. We will need the forcing variation of this fact, which is as follows:

5.11. Lemma. (a) *For $l^* \geqslant l$,*

$$0 \Vdash_\alpha {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i)) = {}^{(t^*,l^*)}\theta(a_i) ,$$

provided that $l' = |t| + (l^* - l)$ and $t' = t^*$.

(b) *If $l^* < l$, then*

$$p_0 = \{\langle i, k, \delta\rangle \mid l^* \leqslant k < l,\ t'(k) = \delta\}$$

$$p_0 \Vdash_\alpha {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i)) = {}^{(t^*,l^*)}\theta(a_i) ,$$

provided that $l' = |t|, |t'| = |t^*| + l - l^*$ and $t' \restriction |t^*| = t^*$.

Proof*. (a) By Lemmas 2.812, 3.02 we may avoid the use of $\lessapprox$ and relativization to N_0 when considering the forcing definition. For any p, and $\sigma \in \mathcal{T}^c_\alpha$ we must show that if

5.111 $\quad p \parallel_\alpha \sigma \in {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i))$ and $p \parallel_\alpha \sigma \in {}^{(t^*,l^*)}\theta(a_i)$,

then

5.112 $\quad p \Vdash_\alpha \sigma \in {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i))$ iff $p \Vdash_\alpha \sigma \in {}^{(t^*,l^*)}\theta(a_i)$.

Assume that the left side of (5.112) holds. Then

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\exists \delta < 2)$$

$$[q \Vdash_\alpha \sigma = \langle \dot{j}, \dot{\delta} \rangle \text{ and either } t'(j) = \delta \text{ or } j > |t'|$$

$$\text{and } p \Vdash_\alpha \langle j + l' \dot{-} |t'|, \dot{\delta} \rangle \in {}^{t,l}\theta(a_i).$$

In the first case we are done, since $t^* = t'$. In the second case, either $\langle j + l' - |t'|, \delta \rangle \in t$, or $j + l' - |t'| > |t|$ and $\langle i, j+l' - |t'| + l - |t|, \delta \rangle \in q$. The first case is impossible since $l' \geqslant |t|, j > |t'|$. In the second case, since $|t| = l' + l - l^*$ we get

$$\langle i, j - |t'| + l^*, \delta \rangle \in q\ ,$$

and since $t' = t^*$ we get

$$q \Vdash_\alpha \langle \dot{j}, \dot{\delta} \rangle \in {}^{(t^*, l^*)}\theta(a_j)\ .$$

Thus $p \Vdash_\alpha \sigma \in {}^{(t^*, l^*)}\theta(a_i)$. To see the other direction, assume that the right side of (5.112) holds; then for all $r \supseteq p$, there is a $q \supseteq r$ such that, either $q \Vdash_\alpha \sigma \in \dot{t}^*$, in which case we are done since $t' = t^*$; or for some $\delta < 2, |t^*| \leqslant j < \omega$,

$$q \Vdash_\alpha \sigma = \langle \dot{j}, \dot{\delta} \rangle \qquad \langle i, j + l^* - |t^*|, \delta \rangle \in q\ .$$

Since $l' = |t| + l^* - l$ and $t' = t^*$, we get

$$\langle i, j + l' + l - |t| - |t'|, \delta \rangle \in q\ ;$$

therefore $q \Vdash_\alpha \langle \dot{j}, \dot{\delta} \rangle \in {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i))$. This implies that

$$p \Vdash_\alpha \sigma \in {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i))$$

as required.

(b) Reasoning along the previous lines, assume $p_0 \subsetneq p$, $\sigma \in \mathcal{T}^c_\alpha$ satisfy the left side of (5.112); then

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\exists \delta < 2)\,[q \Vdash_\alpha \sigma = \langle \dot{j}, \dot{\delta} \rangle]\ .$$

Either $j < |t'|$, in which case $t'(j) = \delta$; or $j \geqslant |t'|$ and

$$q \Vdash_\alpha \langle j + l' - |t'|, \dot{\delta} \rangle \in {}^{(t,l)}\theta(a_i)\ .$$

In the first case we are done since $t' \upharpoonright |t^*| = t^*$. In the second case, we must have, (as in (a)), that

$$\langle i, j+l'-|t'|+l-|t|, \delta\rangle \in q .$$

Substituting, $l' = |t|$ and $|t'| = |t^*| + l - l^*$ we get $\langle i, j+l^* - |t^*|, \delta\rangle \in q$; hence $\sigma \in \mathcal{T}^c_\alpha$, $p \Vdash_\alpha \sigma \in {}^{(t^*, l^*)}\theta(a_i)$ as required. To see the reverse direction, assume $p \supseteq p_0$, $\sigma \in \mathcal{T}^c_\alpha$ satisfy the right side of (5.112). Then

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists j < \omega)\,(\delta < 2)\;[q \Vdash_\alpha \sigma = \langle j, \delta\rangle]$$

and either $j < |t^*|$, and $t^*(j) = \delta$, or $j \geqslant t^*$ and $\langle i, j+l^* - |t^*|, \delta\rangle \in q$. Observe that we have $l^* \leqslant j + l^* - |t^*|$. Now if

$$l^* \leqslant j + l^* - |t^*| = k < l,$$

then $t'(k) = \delta$; hence

$$q \Vdash_\alpha \sigma \in {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i)) ,$$

as required. Consider the case where $k \geqslant l$. Since $|t^*| = |t'| + l^* - l$, we have

$$\langle i, j+l-|t'|, \delta\rangle \in q ;$$

and since $l' = |t|$ implies $\langle i, j+l+l'-|t|-|t'|, \delta\rangle \in q$, we have

$$p \Vdash_\alpha \sigma \in {}^{(t', l')}\theta({}^{(t,l)}\theta(a_i)) ,$$

as required.

5.12. Corollary. *For any $p \in P_\alpha$, $i \in \omega$ and $\theta, \theta^* \in \Gamma$ there exists some $q \supseteq p$ and $\theta' \in \Gamma$ such that*

$$q \Vdash_\alpha \theta'(\theta(a_i)) = \theta^*(a_i) .$$

5.13. Corollary. $0 \Vdash_\alpha {}^{(t,l)}\theta(a_i) = {}^t\theta({}^l\theta(a_i))$, $i \in \omega$, $\alpha \leqslant \aleph_1^{(M)}$.

5.14. Lemma. $0 \Vdash_\alpha {}^{(0,0)}\theta(a_i) = {}^0\theta(a_i) = a_i$.

Proof. Immediate from Definition 2.50.

The following lemma concerning composition of elements of Γ is proven in a manner very similar to Lemma 5.11.

5.15. Lemma.

$$0 \Vdash_\alpha {}^{(t_2,l_1)}\theta \circ {}^{(t_1,l_1)}\theta(a_i) = {}^{(t_2,l_2)}\theta({}^{(t_1,l_1)}\theta(a_i))$$

$$= {}^{(t_2,\, l_1+l_2-|t_1|)}\theta,$$

if $l_2 \geqslant |t_1|$;

$$0 \Vdash_\alpha {}^{(t_2,l_2)}\theta \circ {}^{(t_1,l_1)}\theta(a_i) = {}^{(t_2 \cup (t_1 - t_1 \restriction l_2),\, l_1)}\theta(a_1),$$

if $l_2 < |t_1|$, $i \in \omega$.

5.16. Definition. For $l \in \omega$, $t \in 2^l$, $p(i, t)$ is the condition;

$$\{\langle i, j, t(j)\rangle \mid j \in l\} = p(i, t) \in P_0.$$

5.17. Lemma. *Let* $t \in 2^l$, $l \in \omega$, *and* $i \in \omega$, *then*

(a) $0 \Vdash_\alpha {}^l\theta({}^t\theta(a_i)) = a_i$;
(b) $P(i, t) \Vdash_\alpha {}^t\theta({}^l\theta(a_i)) = a_i$;
(c) $0 \Vdash_\alpha t \subseteq {}^t\theta(\sigma)$, $\sigma \in \mathcal{T}_\alpha$.

Proof. (a) and (b) follow from Lemmas 5.11, 5.14. (c) follows easily from the forcing definition.

On certain occasions we will need the following lemma, which is a slight generalization of some of the previous ones which were proven for a_i, but the exact same argument gives the lemma with any $\rho \in \mathcal{T}_\alpha^c$ instead of a_i.

5.171. Lemma. (a) $0 \Vdash_\alpha {}^{(t,l)}\theta(\rho) = {}^t\theta({}^l\theta(\rho))$;
(b) *If* $t \in 2^l$, *then* $0 \Vdash_\alpha {}^l\theta({}^t\theta(\rho)) = \rho$.

We omit the proofs which can also be easily obtained directly from the forcing definition relying on the usual lemmas; or by seeing that the above statements must be true in every model.

5.18. Lemma. $0 \Vdash_\alpha K(a_i) = K(\theta(a_i))$, $i \in \omega$, $\theta \in \Gamma$.

Proof. It must be shown that for all p, σ such that

$$p \parallel_\alpha \sigma \in K(a_i), \qquad p \parallel_\alpha \sigma \in K(\partial(a_i)),$$

we have

$$p \Vdash_\alpha \sigma \in K(a_i) \quad \text{iff} \quad p \Vdash_\alpha \sigma \in K(\theta(a_i)) .$$

Assume that p, σ satisfy the left side of the bi-implication above. Then

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \theta' \in \Gamma)\,[q \Vdash_\alpha \sigma = \theta'(a_i)] .$$

By Lemma 5.12, for some $\theta'' \in \Gamma$, and $q' \supseteq q$,

$$q' \Vdash_\alpha \theta'(a_i) = \theta''(\theta(a_i)) .$$

Therefore $q' \Vdash_\alpha \sigma = \theta''(\theta(a_i))$; hence $p \Vdash_\alpha \sigma \in K(\theta(a_i))$. To see the reverse implication, assume $p \Vdash_\alpha \sigma \in K(\theta(a_i))$. Then

$$(\forall r \supseteq p)\,(\exists q \supseteq r)\,(\exists \theta' \in \Gamma)\,[q \Vdash_\alpha \sigma = \theta'(\theta(a_i))] .$$

By Lemma 5.10, there is a $q' \supseteq q$, and $\theta^* \in \Gamma$ such that

$$q' \Vdash_\alpha \theta'(\theta(a_i)) = \theta^*(a_i);$$

hence $p \Vdash_\alpha \sigma \in K(a_i)$.

5.2. The next lemma shows that any operation $T^{\theta,i}$, $\theta = {}^{(t,l)}\theta$, can be essentially viewed as a composition of operations namely

$$T^{\theta,i} = T^{l,i} \circ T^{t,i} ,$$

(note the order).

5.20. Lemma. (a) *For any* $\varphi \in \mathcal{F}_\alpha$, $0 \Vdash_\alpha T^{\theta,i}(\varphi) \leftrightarrow T^{l,i}(T^{t,i}(\varphi))$, $\theta = {}^{(t,l)}\theta$, $t \in 2^{l^*}$

(b) *For any* $p \in P_\alpha$,

$$T^{\theta,i}(p) = T^{l,i}(T^{t,i}(p)) \cup \{\langle i, j+l-l^*, \delta\rangle \mid \langle i, j, \delta\rangle \in p,\ l^* > j \geqslant l^* - l\} .$$

Proof. (a) follows from Lemmas 5.13, 2.812, and (b) is immediate from the definition.

The following lemma follows directly from Definitions 5.00, 5.03 and 5.01.

5.21. Lemma. *Assume $t \in 2^l$; then $T^{t,i}(T^{l,i}(p)) = p$, and if*

$$p \cap \{i\} \times l \times 2 \subseteq p(i, t),$$

then

$$T^{l,i}(T^{t,i}(p)) \cup p(i, t) \supseteq p.$$

Proof. The part concerned with the coordinate information on a_i is immediate from Definition 5.03(b)(i). Some simple computations, which we omit, show that for an element of the form $\langle e, \bar{\theta}, k, \delta\rangle$,

$$T^{t,i}(T^{l,i}(\langle e, \bar{\theta}, k, \delta\rangle)) = T^{l,i}(T^{t,i}(\langle e, \bar{\theta}, k, \delta\rangle)) = \langle e, \bar{\theta}, k, \delta\rangle\,,$$

(see Definition 5.00). Similarly for an element of the form $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2\rangle$.

5.22. Definition. Assume $\theta = {}^{(t,l)}\theta$; then $T^{\theta,i}$ is said to be *consistent* with a condition p if $p \cup p(i, t)$ is a condition. This is denoted by $\mathrm{Con}(T^{\theta,i}, p)$.

5.3. Let ${}^{(t,l)}\theta \in \Gamma$; then, since for any formula term or condition x, $T^{\theta,i}(x)$ can essentially be effected by the sequence of operations, $T^{l,i}(T^{t,i}(x))$, it suffices to formulate the transformation lemma, for each of these two types of operations separately.

5.30. Lemma. *Assume $p \in P_\alpha$, $l < \omega$, $t \in 2^l$ and $\varphi \in \mathcal{F}_\alpha^c$; then*

(a) *$T^{l,i}(p)$ is a condition.*

(b) *If $T^{t,i}$ is consistent with p, then $T^{t,i}(p)$ is a condition.*

(c) *$p \Vdash_\alpha \varphi$ iff $T^{l,i}(p) \Vdash_\alpha T^{l,i}(\varphi)$.*

(d) *$p \Vdash_\alpha \varphi$ implies $T^{t,i}(p) \Vdash_\alpha T^{t,i}(\varphi)$.*

(e) *If $T^{t,i}$ is consistent with p, and $T^{t,i}(p) \Vdash_\alpha T^{t,i}(\varphi)$, then $p \cup p(i, t) \Vdash_\alpha \varphi$.*

Proof. This lemma is proved by double induction, on α and $\mathrm{rnk}_\alpha(\varphi)$. It is first shown that the $T(p)$ are conditions.

$\alpha = 0$. In this case $T^{\theta,i}(p)$ is always a condition.

$\alpha > \beta \geqslant 0$.

(a) Clearly $\widehat{(T^{\theta,i}(p))}^\beta = T^{\theta,i}(\hat{p}^\beta) \in P$, by the induction hypothesis.

Let $e = (h, g, s)$ be a $\beta + 1$-index, where $\bar{a}_h = \bar{b}$ and $s = s_\sigma^\beta$. We will show that p satisfies the requirements of a condition concerning the information on χ_e in p. This is done by making a systematic check of the requirements listed in Definition 2.50.

(b) Since p satisfies (b), the only way in which $T^{\theta,i}(p)$ may violate (b) is if $\langle e, \bar{\theta}, k, 0\rangle$, $\langle e, \bar{\theta}^*, k, 1\rangle \in p$, with $\theta_j = \theta_j^*$ for $j \neq i$ and $\theta_i \neq \theta_i^*$, but $\theta_i \circ \theta = \theta_i^* \circ \theta$. By 2.50(d)

$$\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}\cdot\bar{b}) = H_\sigma(\bar{\theta}^*\cdot\bar{b}) .$$

If $\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}\cdot\bar{b}) \neq H_\sigma(\bar{\theta}^*\cdot\bar{b})$, then by the induction hypothesis

$$T^{\theta,i}(\hat{p}^\beta) \Vdash_\beta H_\sigma(\bar{\theta}'\cdot\bar{b}) \neq H_\sigma(\bar{\theta}^{*\prime}\cdot\bar{b}) ,$$

(see Definition 5.03(b)(ii)). This means in particular that we cannot have $\bar{\theta}' = \bar{\theta}^{*\prime}$; hence we must have $\theta_i \circ \theta \neq \theta_i^* \circ \theta$ in contradiction to our assumption. If

$$\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}\cdot\bar{b}) = H_\sigma(\bar{\theta}^*\cdot\bar{b}) ,$$

then by 2.50(d)(2). $\langle e, \bar{\theta}, 0, \bar{\theta}^*, 0\rangle \in p$. Hence by 2.50(c)(1), $\langle e, \bar{\theta}, k, 0\rangle \in p$ implies $\langle e, \bar{\theta}^*, k, 0\rangle \in p$; which is impossible, since we are assuming $\langle e, \bar{\theta}^*, k, 1\rangle \in p$. Therefore 2.50(b) is satisfied by $T(p)$.

(c)(1). Assume

$$\langle e, \bar{\theta}^{1\prime}, l_1, \bar{\theta}^{2\prime}, l_2\rangle \in T^{\theta,i}(p) ,$$

$$\langle e, \bar{\theta}^{*\prime}, l_1 + k, \delta\rangle \in T^{\theta,i}(p) ,$$

where $\bar{\theta}^{*\prime} = \bar{\theta}^{1\prime}$. Then $\langle e, \bar{\theta}^1, l_1, \bar{\theta}^2, l_1\rangle, \langle e, \bar{\theta}^*, l_1 + k, \delta\rangle \in p$. If $\bar{\theta}^* = \bar{\theta}^1$, then by (c)(1), $\langle e, \bar{\theta}^2, l_2 + k, \delta\rangle \in p$; hence $\langle e, \bar{\theta}^{2\prime}, l_2 + k, \delta\rangle \in T^{\theta,i}(p)$ as required. If $\bar{\theta}^* \neq \bar{\theta}^1$, (i.e. $\bar{\theta}_j^* = \theta_j^1$ for $j \neq i$, but $\theta_i^* \neq \theta_i^1$), and $\theta_i^* \circ \theta_1 = \theta_i^1 \circ \theta$, we consider

$$\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}'\cdot\bar{b}) = H_\sigma(\bar{\theta}^*\cdot\bar{b}) .$$

If $\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}^1\cdot\bar{b}) \neq H_\sigma(\bar{\theta}^*\cdot\bar{b})$, then by the induction hypothesis,

$$T^{\theta,i}(\hat{p}^\beta) \Vdash_\beta H_\sigma(\bar{\theta}^{1\prime}\cdot\bar{b}) \neq H_\sigma(\bar{\theta}^{*\prime}\cdot\bar{b}) .$$

Hence we must have $\bar{\theta}^{1\prime} \neq \bar{\theta}^{*\prime}$, in contradiction to our assumption. If $\hat{p}^\beta \Vdash_\beta H_\sigma(\bar{\theta}^1\cdot\bar{b}) = H_\sigma(\bar{\theta}^*\cdot\bar{b})$, then by (d)(2), $\langle e, \bar{\theta}', 0, \bar{\theta}^*, 0\rangle \in p$. Hence by (c)(3) and (c)(4), $\langle e, \bar{\theta}^*, l_1, \bar{\theta}^2, l_2\rangle \in p$, (see remark at end of 2.50(d)(2). Therefore, by (c)(1), $\langle e, \bar{\theta}^2, l_2 + k, \delta\rangle \in p$ which gives $\langle e, \bar{\theta}^{2\prime}, l_2 + k, \delta\rangle \in T^{\theta,i}(p)$, as required.

(c)(2). Assume $\langle e, \bar\theta_1^{1\prime}, l_1, \bar\theta_2^{1\prime}, l_2\rangle, \langle e, \bar\theta_1^{2\prime}, m_1, \bar\theta_2^{2\prime}, m_2\rangle \in T^{\theta,i}(p)$, where $\bar\theta_1^{1\prime} = \bar\theta_2^{1\prime}$ and $\bar\theta_1^{2\prime} = \bar\theta_2^{2\prime}$. If $\bar\theta_1^1 = \bar\theta_2^1$ and $\bar\theta_1^2 = \bar\theta_2^2$, then since $\langle e, \bar\theta_1^1, l_1, \bar\theta_2^2, l_2\rangle, \langle e, \bar\theta_1^1, m_1, \bar\theta_2^2, m_2\rangle \in p$ we get by (c)(2) that $l_1 - l_2 = m_1 - m_2$ as required. If $\bar\theta_1^1 \neq \theta_2^1$ or $\bar\theta_1^2 \neq \bar\theta_2^2$, we reason as before, considering $\hat p^\beta \Vdash_\beta H_\sigma(\bar\theta_1^1 \cdot \bar b) = H_\sigma(\bar\theta_2^1 \cdot \bar b)$ and $\hat p^\beta \Vdash_\beta H_\sigma(\bar\theta_1^2 \cdot \bar b) = H_\sigma(\bar\theta_2^2 \cdot \bar b)$. If p forces any of the inequalities, we can show as before using the induction hypothesis, that a contradiction is reached. Thus assume $\hat p^\beta$ forces$_\beta$ both equalities. Then by (d)(2), $l_1 = l_2$ and $m_1 = m_2$, yielding the desired result.

(c)(3). Assume $\langle e, \bar\theta^{1\prime}, l_1, \bar\theta^{2\prime}, l_2\rangle, \langle \bar\theta^{*\prime}, m_2, \bar\theta^{3\prime}, m_3\rangle \in T^{\theta,i}(p)$, where $\bar\theta^{2\prime} = \bar\theta^{*\prime}$. If $\bar\theta^* = \bar\theta^2$, the result follows immediately from the fact that p satisfies (c)(3). If $\bar\theta^* \neq \bar\theta^2$ we use the fact that $\hat p^\beta \Vdash_\beta H_\sigma(\bar\theta^* \cdot \bar b) = H_\sigma(\bar\theta^2 \cdot \bar b)$. If $\hat p^\beta$ forces$_\beta$ the inequality, we get by the induction hypo thesis that $T^{\theta,i}(\hat p^\beta) \Vdash_\beta H_\sigma(\bar\theta^{*\prime} \cdot \bar b) = H_\sigma(\bar\theta^{2\prime} \cdot \bar b)$; hence in particular we must have $\bar\theta^{*\prime} \neq \bar\theta^{2\prime}$, contradicting our hypothesis. If $\hat p^\beta$ forces the equality, then $\langle e, \bar\theta^*, 0, \bar\theta^2, 0\rangle \in p$. Hence by the remarks following 2.50(d)(2), we have $\langle e, \bar\theta^1, l_1, \bar\theta^2, l_2\rangle \in p$. Since also $\langle e, \bar\theta^2, m_2, \bar\theta^3, m_3\rangle \in p$, we get by (c)(3) that for some $k \geq \min(l_2, m_2)$,

$$\langle e, \bar\theta^1, l_1 + m_2 - k, \bar\theta^3, m_3 + l_2 - k\rangle \in p .$$

Therefore

$$\langle e, \bar\theta^{1\prime}, l_1 + m_2 - k, \bar\theta^{3\prime}, m_3 + l_2 - k\rangle \in p .$$

Hence

$$\langle e, \bar\theta^{1\prime}, l_1 + m_2 - k, \bar\theta^{3\prime}, m_3 + l_2 - k\rangle \in T^{\theta,i}(p),$$

as required.

(c)(4). $\langle e, \bar\theta_1', l_1, \bar\theta_2', l_2\rangle \in T^{\theta,i}(p)$ implies that $\langle e, \bar\theta_1, l_1, \bar\theta_2, l_2\rangle \in p$ implies that $\langle e, \bar\theta_2, l_2, \bar\theta_1, l_1\rangle \in p$, hence

$$\langle e, \bar\theta_2', l_2, \bar\theta_1', l_1\rangle \in T^{\theta,i}(p) ,$$

as required.

(d). If $\chi_e(\bar\theta_1' \cdot \bar b), \chi_e(\bar\theta_2' \cdot \bar b)$ are mentioned by $T^{\theta,i}(p)$, then $\chi_e(\bar\theta_1 \cdot \bar b)$, $\chi_e(\bar\theta_2 \cdot \bar b)$ are mentioned by p, and

$$\hat p^\beta \Vdash_\beta H_\sigma(\bar\theta_1 \cdot \bar b) = H_\sigma(\bar\theta_2 \cdot \bar b).$$

Therefore by the induction hypothesis

$$T^{\theta,i}(\hat p^\beta) \Vdash_\beta H_\sigma(\bar\theta_1' \cdot \bar b) = H_\sigma(\bar\theta_2' \cdot \bar b) .$$

Assume $\langle e, \bar\theta'_1, l_1, \bar\theta'_2, l_2\rangle \in T^{\theta,i}(p)$, then $\langle e, \bar\theta_1, l_1, \bar\theta_2, l_2\rangle \in p$; and:

(d)(1). If $T^{\theta,i}(\hat p^\beta) \Vdash_\beta H_\sigma(\bar\theta'_1 \cdot \bar b) \neq H_\sigma(\bar\theta'_2 \cdot \bar b)$, then

$$\hat p^\beta \Vdash_\beta H_\sigma(\bar\theta_1 \cdot \bar b) \neq H_\sigma(\bar\theta_2 \cdot \bar b)\,.$$

Hence either $l_1 \neq l_2$ or $l_1 = l_2$ and for some $j < l_1$, $\delta < 2$, $\langle e, \bar\theta_1, j, \delta\rangle \in p$ and $\langle e, \bar\theta_2, j, 1-\delta\rangle \in p$; therefore, either $l_1 \neq l_2$ or for some $j < l_1$, $\delta < 2$, $\langle e, \bar\theta'_1, j, \delta\rangle, \langle e, \bar\theta'_2, j, 1-\delta\rangle \in T^{\theta,i}(p)$ as required.

(d)(2). $T^{\theta,i}(\hat p^\beta) \Vdash_\beta H_\sigma(\bar\theta'_1 \cdot \bar b) = H_\sigma(\bar\theta'_2 \cdot \bar b)$ implies that

$$\hat p^\beta \Vdash_\beta H_\sigma(\bar\theta_1 \cdot \bar b) = H_\sigma(\bar\theta_2 \cdot \bar b)\,,$$

hence $l_1 = l_2$ and $\langle e, \bar\theta_1, 0, \bar\theta_2, 0\rangle \in p$; thus also $\langle e, \bar\theta'_1, 0, \bar\theta'_2, 0\rangle \in T^{\theta,i}(p)$.

(d)(3). If $\chi_e(\bar\theta'_1 \cdot \bar b)$, and $\chi_e(\bar\theta'_2 \cdot \bar b)$ are mentioned by $T^{\theta,i}(p)$, then $\chi_e(\bar\theta_1 \cdot \bar b)$ and $\chi_e(\bar\theta_2 \cdot \bar b)$ are mentioned by $\hat p^\beta$; hence $\langle e, \theta_1, l_1, \theta_2, l_2\rangle \in p$ for some $l_1, l_2 < \omega$; thus $\langle e, \bar\theta'_1, l_1, \bar\theta'_2, l_2\rangle \in T(\hat p^\beta)$ as required.

(e). Assume $e = (h, g, s^\beta_\sigma)$, and that $\chi_e(\bar\theta' \cdot \bar b)$ is mentioned in $T^{\theta,i}(p)$. It must be shown that $T^{\theta,i}(\hat p^\beta) = \widehat{T^{\theta,i}(p)}^\beta$ forces $H_\sigma(\bar\theta' \cdot \bar b)$ to be nearly $(-l)$-definable$_\beta$, if $l \in g$; and not nearly $(-l)$-definable$_\beta$, if $l \in h - g$. Assume $l \in g$. Since $\chi_e(\bar\theta \cdot \bar b)$ is mentioned in p, there is a formula $\varphi(u_1, ..., u_m, v) \in \mathcal{F}_\beta$ without generic real constants and some $h^* = \{h^*_1, ..., h^*_m\}$ such that $l \notin h^*$, and a constant term $\rho \in \mathcal{F}^c_\beta$, such that,

$$\hat p^\beta \Vdash_\beta (\exists! v)\,(\varphi(\bar a^m_{h^*}, v)) \wedge \varphi(\bar a^m_{h^*}, \rho) \wedge |\rho| < \dot\aleph_0 \wedge \rho \subseteq H_\sigma \wedge H_\sigma(\bar\theta \cdot \bar b) \in \rho. \tag{5.31}$$

By the induction hypothesis,

$$\begin{aligned} T(\hat p^\beta) \Vdash_\beta\ & (\exists! v)\,(\varphi(T(\bar a_{h^*}), v)) \\ & \wedge \varphi(T(\bar a_{h^*}), T(\rho)) \wedge |T(\rho)| < \dot\aleph_0 \\ & \wedge T(\rho) \subseteq T(H_\sigma) \wedge T(H_\sigma(\bar\theta \cdot \bar b) \in T(\rho)\,. \end{aligned}$$

Denote the formula $\varphi(T(\bar a_{h^*}), v)$ by $\varphi'(\bar a_{h^*}, v)$, and $T(\rho)$ by ρ': then since $T^{\theta,i}(H_\sigma(\bar\theta \cdot \bar b)) = H_\sigma(\bar\theta' \cdot \bar b)$ and $T^{\theta,i}(H_\sigma) = H_\sigma$ (see Remark 2.31), we get

$$\begin{aligned} T(\hat p^\beta) \Vdash_\beta (\exists! v)\,(\varphi'(\bar a_{h^*}, v)) \wedge \varphi'(\bar a_{h^*}, \rho') \wedge |\rho'| < \dot\aleph_0 \\ \wedge \rho' \subseteq H_\sigma \wedge H_\sigma(\bar\theta' \cdot \bar b) \in \rho', \end{aligned}$$

i.e. $T(\hat p^\beta)$ forces$_\beta$ $H_\sigma(\bar\theta' \cdot \bar b)$ to be $(-l)$-definable$_\beta$ as required. The other case is slightly more involved. Assume $l \in h - g$ and that for some $\varphi(\bar{u}^m, v) \in \mathcal{F}_3$ without generic real constants, and $\bar{a}_{h^*}$ such that $l \notin h^*$, and some term $\rho \in \mathcal{T}^c_\beta$, and some extension q of $T(\hat{p}^\beta)$,

$$q \Vdash_\beta (\exists! v)\,(\varphi(\bar{a}_{h^*}, v)) \wedge \varphi(\bar{a}_{h^*}, \rho) \wedge |\rho| < \dot{\aleph}_0 \wedge \rho \subseteq H_0 \wedge H_\sigma(\bar{\theta}' \cdot \bar{b}) \in \rho. \tag{5.32}$$

Assume first that $T = T^{l,i}$. Choose $t \in 2^l$ such that $T^{t,i}$ is consistent with q. Then by Lemma 5.21,

$$\hat{p}^\beta = T^{t,i}(T^{l,i}(\hat{p}^\beta))\,;$$

hence

$$T^{t,i}(T^{l,i}(q)) \supseteq \hat{p}^\beta\,.$$

Denoting the statement forced in (5.32) by ψ, we have by the induction hypothesis that $\hat{p}^\beta \subseteq T^{t,i}(q) \Vdash_\beta T^{t,i}(\psi)$. If $T^{l,i}$ does not operate on $\bar{b} = \bar{a}_h$ we have obviously arrived at a contradiction. If $T^{l,i}$ operates on $\bar{a}_h$, assume $i = h_k = b_k$. Then

$$\begin{aligned} T^{t,i}(\bar{\theta}' \cdot \bar{b}) &= T^{t,i}(T^{l,i}(\bar{\theta} \cdot \bar{b}) \\ &= \theta_1(b_1), \ldots, \theta_{k-1}(b_{k-1}), \theta_k \circ {}^l\theta \circ {}^t\theta(b_k), \\ &\quad \theta_{k+1}(b_{k+1}) \ldots \theta_n(b_n)\,; \end{aligned}$$

hence

$$T^{t,i}(T^{l,i}(H_\sigma(\bar{\theta} \cdot \bar{b}))) = H_\sigma(\theta_1(b_1) \ldots \theta_{k-1}(b_{k-1}), \theta_k({}^l\theta({}^t\theta(b_k))), \ldots, \theta_n(b_n)).$$

By Lemma 5.17, $0 \Vdash_\beta {}^l\theta({}^t\theta(b_k)) = b_k$; hence by the lemma on substitution of equals

$$0 \Vdash_\beta H_\sigma(\bar{\theta} \cdot \bar{b}) = T^{t,i}(T^{l,i} H_\sigma(\bar{\theta} \cdot \bar{b}))\,.$$

Thus from $T^{t,i}(\psi)$ we get,

$$\begin{aligned} \hat{p}^\beta \subseteq T^{t,i}(q) \Vdash_\beta (\exists! v)\,(\varphi(T^{t,i}(\bar{a}_{h^*}), v)) &\wedge \varphi(T^{t,i}(\bar{a}_{h^*}), T^{t,i}(\rho)) \\ &\wedge T^{t,i}(\rho) \subseteq H_\sigma \\ &\wedge |T^{t,i}(\rho)| < \dot{\aleph}_0 \\ &\wedge H_\sigma(\bar{\theta} \cdot \bar{b}) \in T^{t,i}(\rho)\,. \end{aligned}$$

This is a contradiction. Assume $T = T^{t,i}$, $t \in 2^l$, where $T^{t,i}$ is consistent with p. By Lemma 5.21, $T^{l,i}(T^{t,i}(p)) \cup p(i, t) \supseteq p$. By the induction hypothesis $T^{l,i}(q) \Vdash_\beta T^{l,i}(\psi)$. If $T^{t,i}$ does not act on $\bar{b}$ we obviously have a contradiction. Thus assume $i = h_k$; in which case,

$$T^{l,i}(\bar{\theta}'\cdot\bar{b}) = T^{l,i}(T^{t,i}(\bar{\theta}\cdot\bar{b}))$$
$$= \theta_1(b_1), ..., \theta_{k-1}(b_{k-1}), \theta_k \circ {}^t\theta \circ {}^l\theta(b_k), ..., \theta_n(b_n).$$

By Lemma 5.17, $p(i, t) \Vdash_\beta {}^t\theta({}^l\theta(b_k) = b_k)$, thus by substitution of equals,

$$p(i, t) \Vdash_\beta H_\sigma(\bar{\theta}\cdot\bar{b})$$
$$= H_\sigma(\theta_1(b_1), ..., \theta_{k-1}(b_{k-1}), \theta_k({}^t\theta({}^l\theta(b_k))), ..., \theta_n(b_n)).$$

Denoting $T^{l,i}(\varphi(\bar{a}_{h^*}, v)$ by $\varphi'(\bar{a}_{h^*}, v)$, and $T(\rho) = \rho'$ we get

$$p \subseteq T^{l,i}(q) \cup p(i, t) \Vdash_\beta (\exists! v)\, (\varphi'(\bar{a}_{h^*}, v))$$
$$\wedge\, \varphi'(\bar{a}_{h^*}, \rho') \wedge \rho' \subseteq H_\sigma \wedge H_\sigma(\bar{\theta}\cdot\bar{b}) \in \rho' \wedge |\rho'| < \check{\aleph}_0 .$$

This contradicts the fact that $T^{l,i}(q) \cup p(i, t)$ is a condition mentioning $\chi_e(\bar{\theta}\cdot\bar{b})$, ($T^{l,i}(q)$ is a condition by the induction hypothesis). Therefore it has been shown that $T^{t,i}(p)$ and $T^{l,i}(p)$ are conditions, $p \in P_\alpha$.

We now prove (c), (d) and (e) of Lemma 5.30 for α, by induction on $\mathrm{rnk}_\alpha(\varphi)$, assuming the lemma true for $\beta < \alpha$. We check the cases of the forcing definition for $T^{l,i}$, $T^{t,i}$ separately, in one direction only, the other direction will be shown to follow easily from the former. Although we show most cases, the only cases of any interest here are those of negation $\sigma \in a_k$, $\sigma \in \chi_e$, $\sigma \in I_0$.

We first make the following provision.

5.33. If $T = T^{l,i}$ and $r \supseteq T(p)$, let $r_T^* = T^{t,i}(r) \supseteq p$ for some choice of $t \in 2^l$.

If $T = T^{t,i}$, $t \in 2^l$ and $r \supseteq T^{t,i}(p)$, let $r_T^* = T^{l,i}(r)$. Hence by Lemma 5.21, $r^* \cup p(i, t) \supseteq p$.

5.331. Note that if $q \supseteq r^*$, then $T(q) \cup r$ is a condition. Because if $T = T^{t,i}$, then

$$T(r^*) = T^{t,i}(T^{l,i}(r)) = r\ ;$$

thus $T(q) \supseteq r$; and if $T = T^{l,i}$, then $T^{l,i}(q)$ does not contain information on coordinates $j < l$ of a_i, and since

$$r = T(r^*) \cup r \upharpoonright \{i\} \times l \times 2$$

with $T(r^*) \subseteq T(q)$, $T(q) \cup r$ is a condition.

(a) Assume $p \Vdash_\alpha \sigma \in \dot{s}$ and

(i)

$$r \supseteq T^{l,i}(p) = T(p) .$$

Let $q \supseteq r^* \supseteq p$ and $s' \in s$ be such that $q \Vdash_\alpha \sigma \approx \dot{s}'$; then by the induction hypothesis, $T(q) \Vdash_\alpha T(\sigma) \approx \dot{s}'$; and since by 5.331 $T(p) \subseteq r \subseteq T(q) \cup r$, it follows that for an arbitrary extension r of $T(p)$ there is one extension $T(q) \cup r$ of r, and an s' such that

$$T(q) \cup r \Vdash_\alpha T(\sigma) \approx \dot{s}' ,$$

hence by the forcing definition $T(p) \Vdash_\alpha T(\sigma \in \dot{s})$ is required.

(ii) For any $r \supseteq T^{l,i}(p) = T(p)$, let

$$q \supseteq r^* \cup p(i, t) \supseteq p,\ s' \in s$$

be such that $q \Vdash_\alpha \sigma \approx \dot{s}'$. Then by the induction hypothesis, $T(q) \Vdash_\alpha T(\sigma) \approx \dot{s}'$; and since $T(p) \subseteq r \subseteq T(q)$, it follows as above that $T(p) \Vdash_\alpha T(\sigma \in \dot{s})$.

(b) Assume $p \Vdash_\alpha \sigma \in a_k$ and

(i)

$$r \supseteq T^{l,i}(p) = T(p) .$$

Let $q \supseteq r^* \supseteq p$ and $j < \omega$, $\delta < 2$ be such that $q \Vdash_\alpha \sigma \approx (\langle \dot{j}, \dot{\delta} \rangle)^{(\aleph_0)}$ and $\langle k, j, \delta \rangle \in q$. If $i \neq k$, we have by the induction hypothesis

$$T(q) \Vdash T(\sigma) \approx (\langle \dot{j}, \dot{\delta} \rangle)^{(\aleph_0)}, \qquad \langle k, j, \delta \rangle \in T(q),$$

and since $T(p) \subseteq r \subseteq T(q) \cup r$ we get as above that

$$T^{l,i}(p) \Vdash_\alpha T^{l,i}(\sigma) \in a_k \ .$$

If $k = i$, then $\langle k, j + l, \delta \rangle \in T(q)$; hence by case (c) of the forcing definition $T(q) \Vdash_\alpha T(\sigma) \in {}^t\theta(a_k)$; whence $T(q) \Vdash_\alpha T(\sigma \in a_k)$. Since $T(q) \cup r \supseteq r \supseteq T(p)$, we have $T(p) \Vdash_\alpha T(\sigma \in a_k)$.

(ii) For any $r \supseteq T^{l,i}(p)$ let $q \supseteq r^* \cup p(i, t) \supseteq p$ and $j < \omega$, $\delta < 2$ be such that $q \Vdash_\alpha \sigma \approx (\langle j, \delta \rangle)^{(\aleph_0)}$ and $\langle k, j, \delta \rangle \in q$.

The case $i \neq k$ is immediate, as seen above, hence consider the case where $k = i$. If $j \geqslant l$, then $\langle k, j-l, \delta\rangle \in T(q)$, and by the forcing definition

$$T(p) \subseteq r \subseteq T(q) \Vdash_\alpha T(\sigma) \in {}^t\theta(a_k) :$$

therefore $T(p) \Vdash_\alpha T(\sigma \in a_k)$. If $j < l$, then, since we are assuming that $T^{t,i}$ is consistent with p, $\langle j, \delta\rangle \in t$; thus by the forcing definition $T(q) \Vdash_\alpha T(\sigma) \in {}^t\theta(a_k)$, which implies that $T(p) \Vdash_\alpha T(\sigma \in a_k)$.

(c) Assume $p \Vdash_\alpha \sigma \in {}^{\langle t', l'\rangle}\theta(\tau)$ and that for $k < \omega$, $\delta < 2$,

$$p \subseteq p(i, t) \cup r_T^* \subseteq q \Vdash_\alpha \sigma \approx (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)},$$

where either $t(k) = \delta$, or $k \geqslant m$, and for $j = k-m+l$, $q \Vdash_\alpha (\dot{\langle j, \delta\rangle})^{(N_0)} \in \tau$. Then by the induction hypothesis,

$$T(q) \Vdash_\alpha T(\sigma) \approx (\langle \dot{k}, \dot{\delta}\rangle)^{(N_0)}, \qquad T(q) \Vdash_\alpha (\dot{\langle j, \delta\rangle})^{(N_0)} \in T(\tau),$$

respectively. This implies that $T(p) \Vdash_\alpha T(\sigma) \in {}^{\langle t', l'\rangle}\theta(T(\tau))$, i.e. $T(p) \Vdash_\alpha T(\sigma \in {}^{\langle t', l'\rangle}\theta(\tau))$.

(d) Assume $p \Vdash_\alpha \sigma \in K(\tau)$ and that for $p \subseteq p(i, t) \cup r_T^* \subseteq q$, $\theta \in \Gamma$, $q \Vdash_\alpha \sigma \approx \theta(\tau)$. By the induction hypothesis $T(q) \Vdash_\alpha T(\sigma) \approx \theta(T(\tau))$. By previous arguments and the forcing definition, this implies that

$$T(p) \Vdash_\alpha T(\sigma) \in K(T(\sigma)) ,$$

i.e. $T(p) \Vdash_\alpha T(\sigma \in K(\tau))$.

(e) Assume $p \Vdash_\alpha \sigma \in \chi_e$ and that for some $k < \omega$, $\delta < 2$, and q

$$p \subseteq r_T^* \cup p(i, t) \subseteq q \Vdash_\alpha \sigma \approx (\dot{\langle}\dot{\langle}\bar{\theta}\cdot\bar{a}_h\dot{\rangle}, \langle\dot{k}, \dot{\delta}\rangle\dot{\rangle})^{(N_0)} ,$$

and $\langle e, \bar{\theta}, k, \delta\rangle \in q$, where h is the first component of e. Then by the induction hypothesis,

$$T(q) \Vdash_\alpha T(\sigma) \approx (\dot{\langle}\dot{\langle}\bar{\theta}'\cdot\bar{a}_h\dot{\rangle}, \langle\dot{k}, \dot{\delta}\rangle\dot{\rangle})^{(N_0)} ,$$

and $\langle e, \bar{\theta}', k, \delta\rangle \in q$, where $\bar{\theta}' = \bar{\theta}$ if T does not act on any elements of the sequence $\bar{a}_h$; and $\bar{\theta}'\cdot\bar{a}_h = T(\bar{\theta}\cdot\bar{a}_h)$ otherwise (see also Definition 5.03). Thus $T(q) \Vdash_\alpha T(\sigma) \in \chi_e$, and by previous arguments this implies that $T(p) \Vdash_\alpha T(\sigma \in \chi_e)$.

(f) Assume $p \Vdash_\alpha \sigma \in I_0$, and that for some $q \supseteq r \cup p(i, t) \supseteq p$, $j < \omega$,

$$q \Vdash_\alpha \sigma \approx (\langle \dot{j}, K(a_j)\dot{\rangle})^{(N_0)} .$$

Applying the induction hypothesis we get

$$T(q) \Vdash_\alpha T(\sigma) \lesssim (\langle \check{j}, K(\theta(a_j))\rangle\check{\ })^{(N_0)}, \qquad (T = T^{\theta, i}) .$$

By Lemma 5.18, $0 \Vdash_\alpha K(a_j) = K(\theta(a_j))$; hence by substitution of equals (Lemma 2.812),

$$T(q) \Vdash_\alpha T(\sigma) \lesssim (\langle \check{j}, K(a_j)\rangle\check{\ })^{(N_0)} ;$$

therefore $T(q) \Vdash_\alpha T(\sigma) \in I_0$; thus $T(p) \Vdash_\alpha T(\sigma \in I_0)$. If

$$p \Vdash_\alpha \sigma \in I_\beta , \quad \beta > 0,$$

the result follows immediately from the induction hypothesis and previous arguments.

(g) Assume $p \Vdash_\alpha N_\gamma(\sigma)$ and that for some $q \supseteq r^* \cup p(i, t) \supseteq p$, τ with

$$\mathrm{stg}_\alpha(\tau) \leqslant r, \qquad \mathrm{ord}_\alpha(\tau) \leqslant \mathrm{ord}_\alpha(\sigma), \qquad q \Vdash_\alpha \sigma \lesssim \tau .$$

Applying the induction hypothesis, we get $T(q) \Vdash_\alpha T(\sigma) \lesssim T(\tau)$. Since T does not alter the rank, stage$_\alpha$ or order$_\alpha$ of a term (see Definitions 2.21(f), 2.6103), $T(q) \Vdash_\alpha N_\gamma(T(\sigma))$; this implies that $T(p) \Vdash_\alpha T(N_\gamma(\sigma))$.

(h), (i) are trivial cases.

(j) This case demands a little care. Assume that $p \Vdash_\alpha \neg\varphi$.

(i) Let $T = T^{l,i}$. If $T^{l,i} \nVdash_\alpha \neg T^{l,i}(\varphi)$, then there is a $q \supseteq T^{l,i}(p)$ such that $q \Vdash_\alpha T^{l,i}(\varphi)$. Let $t \in 2^l$ be such that $T^{t,i}$ is consistent with q. By the induction hypothesis,

$$T^{t,i}(T^{l,i}(p)) \subseteq T^{t,i}(q) \Vdash T^{t,i}(T^{l,i}(\varphi)) .$$

But

$$T^{t,i}(T^{l,i}(p)) = p, \qquad T^{t,i}(T^{l,i}(\varphi(a_i)) = \varphi({}^l\theta({}^t\theta(a_i))),$$

and by Lemma 5.17 $0 \Vdash_\alpha {}^l\theta({}^t\theta(a_i)) = a_i$; thus we get by substitution of equals, $p \subseteq T^{t,i}(q) \Vdash_\alpha \varphi$, which is a contradiction.

(ii) Assume that $T^{t,i}$ is consistent with p and that

$$T^{t,i}(p) \nVdash_\alpha T^{t,i}(\varphi) .$$

Then there is a $q \supseteq T^{t,i}(p)$ such that $q \Vdash_\alpha T^{t,i}(\varphi)$. Assume $t \in 2^l$; then by the induction hypothesis,

$$T^{l,i}(T^{t,i}(p)) \subseteq T^{l,i}(q) \Vdash_\alpha T^{l,i}(\varphi(a_i))) ;$$

hence $T^{l,i}(q) \Vdash_\alpha \varphi({}^t\theta({}^l\theta(a_i)))$. $T^{l,i}(q)$ does not contain information of the type $\langle i, j, \delta\rangle$ for $0 \leqslant j < l$; therefore

$$\bar{q} = p(i, t) \cup T^{l,i}(q)$$

is a condition. Moreover, since $T^{t,i}$ is consistent with p, $p \subseteq p(i, t) \cup T^{l,i}(q) = \bar{q}$; and by Lemma 5.17, $\bar{q} \Vdash_\alpha {}^t\theta({}^l\theta(a_i)) = a_i$; therefore $p \subseteq \bar{q} \Vdash_\alpha \varphi(a_i)$. This is a contradiction.

(k) Assume $p \Vdash_\alpha (\forall_\xi v)(\varphi(v))$ and that $T(p) \nVdash_\alpha T((\forall_\xi v)(\varphi(v)))$. Then for some σ with

$$\operatorname{rnk}(\sigma) < \xi, \qquad \operatorname{ord}_\alpha(\sigma) < \operatorname{ord}_\alpha(\forall_\xi v, (\forall_\xi v)(T(\varphi))),$$

and $q \supseteq T(p)$, $q \Vdash_\alpha \neg T(\varphi)(\sigma)$. If $T = T^{l,i}$, let $T^{t,i}$ be consistent with q, $t \in 2^l$; then by case (j) just shown,

$$p \subseteq T^{t,i}(q) \Vdash \neg T^{t,i}(T^{l,i}(\varphi))(\sigma),$$

i.e.,

$$p \subseteq T^{t,i}(q) \Vdash_\alpha \neg\varphi({}^l\theta({}^t\theta(a_i)), T^{t,i}(\sigma)).$$

By Lemma 5.17,

$$T^{t,i}(q) \Vdash_\alpha \neg\varphi(a_i, T^{t,i}(\sigma));$$

and since the rank, stage_α and order_α of a term are not altered by an operation, we have obtained a contradiction. Similarly if $T = T^{t,i}$.

(l) Assume $p \Vdash_\alpha \sigma \in \tau$ where τ is an abstraction term,

$$\tau = (\lambda_\xi v)(\varphi(v)), \qquad \operatorname{rnk}(\sigma) < \xi, \quad \operatorname{ord}_\alpha(\sigma) < \operatorname{ord}_\alpha(\tau).$$

Then $p \Vdash_\alpha \varphi(\sigma)$, and by the induction hypothesis, $T(p) \Vdash_\alpha T(\varphi(\sigma))$; i.e. $T(p) \Vdash_\alpha \varphi^*(T(\sigma))$, where $T(\tau) = (\lambda_\xi v)(\varphi^*)$. Since the rank, stage_α and order_α of terms are not altered by an operation, we get

$$T(p) \Vdash_\alpha T(\sigma) \in T(\tau),$$

as required.

(m) This case follows directly from the induction hypothesis using the fact that the orders_α of terms are not altered by operations.

We have so far shown that $p \Vdash_\alpha \varphi$ implies $T^{l,i}(p) \Vdash_\alpha T^{l,i}(\varphi)$; and that if $T^{t,i}$ is consistent with p, then $T^{t,i}(p) \Vdash_\alpha T^{t,i}(\varphi)$. It must be shown that $T^{l,i}(p) \Vdash_\alpha T^{l,i}(\varphi)$ implies $p \Vdash_\alpha \varphi$, and $T^{t,i}(p) \Vdash_\alpha T^{t,i}(\varphi)$ implies $p \cup p(i, t) \Vdash_\alpha \varphi$, provided that $T^{t,i}$ is consistent with p. This follows easily from what we have previously shown. For assume

(i) $T^{l,i}(p) \Vdash_\alpha T^{l,i}(\varphi)$. For all $t \in 2^l$ we have $T^{t,i}$ consistent with $T^{l,i}(p)$, since for any bit of information $\langle i, j, \delta\rangle \in T^{l,i}(p)$, we have $i \geqslant l$; therefore by the direction of implication already shown, we get

$$p = T^{t,i}(T^{l,i}(p)) \Vdash_\alpha T^{t,i}(T^{l,i}(\varphi)) ;$$

therefore

$$p \Vdash_\alpha \varphi({}^l\theta({}^t\theta(a_i)) ;$$

hence $p \Vdash_\alpha \varphi$.

(ii) If $T^{t,i}(p) \Vdash_\alpha T^{t,i}(\varphi)$, and $T^{t,i}$ is consistent with p, $t \in 2^l$, then by applying $T^{l,i}$, we get from what has already been proven that

$$T^{l,i}(T^{t,i}(p)) \Vdash_\alpha T^{l,i}(T^{t,i}(\varphi)) ,$$

therefore

$$p \supseteq T^{l,i}(T^{t,i}(p)) \Vdash_\alpha \varphi({}^t\theta({}^l\theta(a_i)) ;$$

and since $T^{t,i}$ is consistent with p, $p \cup p(i, t)$ is a condition, hence by Lemma 5.17,

$$p \cup p(i, t) \Vdash_\alpha \varphi(a_i) ,$$

as required.

For global formulas the lemma is proven by induction on length in a manner similar to the corresponding local cases.

5.34. Lemma. *Assume $p \in P_\alpha$, ${}^{(t,l)}\theta \in \Gamma$, $t \in 2^{l^*}$ and $\varphi \in \mathcal{F}^c_\alpha$, then if $T^{\theta,i}$ is consistent with p,*

(a) *$T^{\theta,i}(p)$ is a condition;*

(b) *$p \Vdash_\alpha \varphi$ implies $T^{\theta,i}(p) \Vdash_\alpha T^{\theta,i}(\varphi)$;*

(c) *$T^{\theta,i}(p) \Vdash_\alpha T^{\theta,i}(\varphi)$ implies $p \cup p(i, t) \Vdash_\alpha \varphi$.*

Proof. (a) Since $T^{t,i}$ is consistent with p, $T^{t,i}(p)$ is a condition; hence also $T^{l,i}(T^{t,i}(p))$. Again by the consistency of $T^{t,i}$ with p we have that

$$q = T^{l,i}(T^{t,i}(p)) \cup \{\langle i, j+l-l^*, \delta\rangle \mid \langle i, j, \delta\rangle \in p,\ l^* > j \geqslant l^* - l\}$$

is a condition. By Lemma 5.20, $q = T^{\theta,i}(p)$.

(b) By (5.3) $p \Vdash_\alpha \varphi$ implies $T^{t,i}(p) \Vdash_\alpha T^{t,i}(\varphi)$, therefore

$$T^{l,i}(T^{t,i}(p)) \Vdash_\alpha T^{l,i}(T^{t,i}(\varphi)) .$$

Hence

$$T^{l,i}(T^{t,i}(p)) \subseteq T^{\theta,i}(p) \Vdash_\alpha T^{l,i}(T^{t,i}(\varphi))$$

and by Lemma 5.2(a) we finally get $T^{\theta,i}(p) \Vdash_\alpha T^{\theta,i}(\varphi)$.

(c) If $p \cup p(i, t) \not\Vdash_\alpha \varphi$, then by Corollary 2.79 there exists an $r \supseteq p \cup p(i, t)$ such that $r \Vdash_\alpha \neg\varphi$. Since r is consistent with $T^{t,i}$, we have by Lemma 5.3 that

$$T^{t,i}(r) \Vdash_\alpha \neg T^{t,i}(\varphi) ;$$

hence also

$$T^{l,i}(T^{t,i}(p)) \subseteq T^{l,i}(T^{t,i}(r)) \Vdash_\alpha \neg T^{l,i}(T^{t,i}(\varphi)) ;$$

i.e.,

$$T^{l,i}(T^{t,i}(r)) \Vdash_\alpha \neg T^{\theta,i}(\varphi) .$$

Since $T^{t,i}$ is consistent with $r \supseteq p \cup p(i, t)$ it follows that,

$$q = T^{l,i}(T^{t,i}(r)) \cup \{\langle i, j+l-l^*, \delta\rangle \mid \langle i, j, \delta\rangle \in p(i, t),\ l^* > j \geqslant l^* - l\}$$

is a condition. Thus

$$T^{\theta,i}(p) \subseteq q = T^{\theta,i}(r) \Vdash_\alpha \neg T^{\theta,i}(\varphi) .$$

Therefore $T^{\theta,i}(p) \not\Vdash_\alpha T^{\theta,i}(\varphi)$ in contradiction to our assumption.

5.341. Definition. (a) If $i_k \neq i_l$, $1 \leqslant k, l \leqslant n$, $k \neq l$, and x is either a condition or an expression of $\mathcal{L}_\alpha(x \in \mathcal{T}_\alpha \cup \mathcal{F}_\alpha)$, then

$$T^{\bar\theta,\bar i}(x) = T^{\theta_1,i_1}(T^{\theta_2,i_2}(\dots (T^{\theta_n,i_n}(x)) \dots)) .$$

The operation $T^{\bar\theta,\bar i}$ is also denoted $T^{\bar\theta,h}$ where $h = \{i_1, \dots, i_n\}$.

(b) If $t_k \in 2^{l_k}$, $l_k \in \omega$, $1 \leqslant k \leqslant n$, then denote

$$p(\bar i, \bar t) = \bigcup_{1 \leqslant k \leqslant n} p(i_k, t_k) , \qquad i_k \neq i_l, \quad k \neq l.$$

(c) We say that $T^{\bar\theta,\bar i}$ is consistent with p if T^{θ_k,i_k} is consistent with p, for all $1 \leqslant k \leqslant n$.

5.342. Remark. We limit ourselves to iterations of this form, since if $i \neq j$, $T^{\theta,i}$, $T^{\theta',j}$ commute; and if $i = j$, then for $\theta^* = \theta' \circ \theta$, $T^{\theta^*,i}$ is essentially the same as $T^{\theta',i} \circ T^{\theta,i}$

The transformation lemma is easily generalized by induction to:

5.35. General Transformation Lemma. *For any $p \in P_\alpha$ and sentence $\varphi \in \mathcal{F}^c_\alpha$, and $T^{\bar\theta,\bar l}$ such that $T^{\bar\theta,\bar l}$ is consistent with p, we have:*

(a) *$T^{\bar\theta,\bar l}(p)$ is a condition of P_α;*

(b) *$p \Vdash_\alpha \varphi$ implies $T^{\bar\theta,\bar l}(p) \Vdash_\alpha T^{\bar\theta,\bar l}(\varphi)$;*

(c) *$T^{\bar\theta,\bar l}(p) \Vdash_\alpha T^{\bar\theta,\bar l}(\varphi)$ implies $p \cup p(\bar i, \bar l) \Vdash_\alpha \varphi$.*

5.4. We are now in a position to give a rigorous justification of the fact that if $\bar x \in K(\bar a_h)$ satisfies $p(\bar a_h)$, and $p \Vdash_\alpha \varphi(\bar a_h)$, then $N_\alpha \models \varphi[x]$. This is intuitively clear, because the x_i are also generic; hence we may interpret the a_{h_i} as x_i instead of a_{h_i} while retaining the interpretation of χ_e as χ_e and I_β as I_β. This is a generic interpretation yielding the same model; and since $p \Vdash_\alpha \varphi$ means that φ is true in every model $N'_\alpha = N_Q$ realizing $\mathcal{L}_\alpha$ such that $p \in Q$, we must have $N_\alpha = N'_\alpha \models \varphi[x]$.

5.40. Definition. If p is a condition, we denote $p = p(\bar a_h)$, where $\bar a_h$ are the following constants:

(i) if for some j, δ, $\langle i, j, \delta\rangle \in p$, then $i \in h$;

(ii) if $\langle e, \bar\theta, k, \delta\rangle \in p$ or $\langle e, \bar\theta_1, l_1, \bar\theta_2, l_2\rangle \in p$ and h^* is the first component of e, then $h^* \subseteq h$.

In general, these will be the only constants occurring in $\bar a_h$, in which case we say that p mentions $\bar a_h$; but for notational convenience we may allow $\bar a_h$ to include constants not mentioned in p.

5.41. Definition. (a) $\bar x \in K(\bar a_h)$ is said to *satisfy* $p(\bar a_h)$ iff

(i) $\langle h_i, j, \delta\rangle \in p$ implies $x_{h_i}(j) = \delta$;

(ii) if $\langle e, \bar\theta, j, \delta\rangle \in p$, and h^* is the first component of e, and $\bar x^* = \bar x \restriction h$, then

$$\bar\theta \cdot \bar x \in \operatorname{dom}(\chi_e), \qquad \chi_e(\bar\theta \cdot \bar x^*)(j) = \delta$$

(see Definition 4.50);

(iii) if $\langle e, \bar\theta_1, l_1, \bar\theta_2, l_2\rangle \in p$, then $\bar\theta \cdot \bar x^*, \bar\theta_2 \cdot \bar x^* \in \operatorname{dom}(\chi_e)$, and

$$^{l_1}\theta(\chi_e(\bar\theta_1 \cdot \bar x^*)) = {}^{l_2}\theta(\chi_e(\bar\theta_2 \cdot \bar x^*)) .$$

(b) If $\varphi(\bar a_h)$ is a sentence of $\mathcal{L}_\alpha$, and $\bar x = \bar\theta \cdot \bar a_h$, we say $\bar x$ *satisfies*$_\alpha$ φ (in N_α) if $N_\alpha \models \varphi(\bar\theta \cdot \bar a_h)$. We also allow this notation if $\bar a_h$ includes all generic real constants in φ.

If $\bar{x}$ satisfies $p(\varphi)$ in N_α, we write $N_\alpha \models p[\bar{x}]$ $(N_\alpha \models \varphi[\bar{x}])$.

(c) $\bar{x} \in K(\bar{a}_h)$ is said to *$\exists$-satisfy* $p(\bar{a}_h)$ in N_α if for some $\bar{y} \in K(\bar{a}_h)$,

$$\bar{y} \restriction h \cap h^* = \bar{x} \restriction h \cap h^*, \qquad N_\alpha \models p[\bar{y}].$$

It follows easily from the definitions that:

5.42. Lemma. *If $\bar{x} = \bar{\theta}\cdot\bar{a}_h$ satisfies $p(\bar{a}_h) = p$, then $T^{\bar{\theta},h}$ is consistent with p, and $\bar{a}_h$ satisfies $T^{\bar{\theta},h}(p)$.*

Proof. To verify that $T^{\bar{\theta},h}$ is consistent with p, assume $\langle h_i, j, \delta\rangle \in p$; then ${}^{(t_i, l_i)}\theta_i(a_{h_i})(j) = \delta$ by assumption; and if $j < |t_i| = l_i^*$, we have $t_i(j) = \delta$, which is what was to be shown. Thus $T^{\bar{\theta},h}(p)$ is a condition. To show that $\bar{a}_h$ satisfies $T^{\bar{\theta},h}(p)$, assume first that $\langle h_i, j, \delta\rangle \in p$ and $j \geqslant l_i^* - l_i$; then

$$\langle h_i, j + l_i - l_i^*, \delta\rangle \in T(p).$$

If $a_{h_i}(j + l_i - l_i^*) = 1 - \delta$, then it follows from the forcing definition that

$$\theta_i(a_{h_i})(j + l_i - l_i^* - l_i + l_i^*) = 1 - \delta,$$

i.e.,

$$\theta_i(a_{h_i})(j) = 1 - \delta\ ,$$

which is a contradiction. If $j < l_i^* - l_i$, then for no ϵ is $\langle h_i, j + l_i - l_i^*, \epsilon\rangle$ in $T^{\bar{\theta},h}(p)$; hence there is nothing to prove. If $\langle e, \bar{\theta}^*, j, \delta\rangle \in p$, then $\langle e, \bar{\theta}^{*\prime}, j, \delta\rangle \in T^{\theta}(p)$ (see Definition 5.03). Now, $\bar{\theta}\cdot\bar{a}_h$ satisfies p, hence $\chi_e(\bar{\theta}^*\cdot\bar{\theta}\cdot\bar{a}_h)(j) = \delta$; but

$$\bar{\theta}^*\cdot\bar{\theta}\cdot\bar{a}_h = (\bar{\theta}^*\cdot\bar{\theta})\bar{a}_h = \bar{\theta}^{*\prime}\cdot\bar{a}_h\,,$$

thus $\chi_e(\bar{\theta}^{*\prime}\cdot\bar{a}_h)(j) = \delta$, i.e., $\bar{a}_h$ satisfies $\langle e, \bar{\theta}^{*\prime}, j, \delta\rangle$ as was to be shown. Similarly for $\langle e, \bar{\theta}^*, l^*, \bar{\theta}^{**}, l^{**}\rangle \in p$. Therefore $\bar{a}_h$ satisfies $T^{\bar{\theta},h}(p)$.

5.43. Symmetry Lemma. *If $p(\bar{a}_h) \Vdash_\alpha \varphi(\bar{a}_h)$, where $\bar{a}_h$ are all generic real constants occurring in p or φ, and $\bar{x} \in K(\bar{a}_h)$ satisfies p in N_α, then $N \models \varphi[x]$.*

Proof. Let $\bar{x} = \bar{\theta}\cdot\bar{a}_h$. Then by Lemma 5.42, $T^{\bar{\theta},h}$ is consistent with p, and $\bar{a}_h$ satisfies $T^{\bar{\theta},h}(p)$. Therefore $T^{\bar{\theta},h}(p) \in Q_\alpha$. By Lemma 3.35, $T^{\bar{\theta},h}(p) \Vdash T^{\bar{\theta},h}(\varphi)$, thus $N_\alpha \models T^{\bar{\theta},h}(\varphi)$. But $T^{\bar{\theta},h}(\varphi)$ is none other than $\varphi(\bar{\theta}\cdot\bar{a}_h)$, i.e., $N_\alpha \models \varphi[\bar{x}]$; as required.

The preceding lemma can also be given the following form:

5.44. Symmetry Lemma. *If $p(\bar{a}_{h^*}) \Vdash_\alpha \varphi(\bar{a}_h)$, where $\bar{a}_{h^*}$ are all generic real constants mentioned in p, and $\bar{a}_h$ are all generic real constants mentioned in φ, and $\bar{x} \in K(\bar{a}_h)$ $\exists$-satisfies p in N_α, then $N_\alpha \models \varphi[x]$.*

Proof. Let $h' = h \cup h^*$, and let $\bar{y} \in K(\bar{a}_{h'})$ be such that $\bar{y} \restriction h = \bar{x}$ and satisfies p in N_α, (such a $\bar{y}$ exists by assumption). The result now follows by Lemma 5.43.

This lemma has a kind of converse, namely:

5.45. Lemma. *If $\varphi(\bar{a}_h) \in \mathcal{F}^c_\alpha$, $\bar{x} \in K(\bar{a}_h)$ are such that $N_\alpha \models \varphi[\bar{x}]$, then for some $p(a_{h^*}) \in P_\alpha$ we have $p \Vdash_\alpha \varphi$ and $\bar{x}$ $\exists$-satisfies p in N_α.*

Proof. Let $\bar{x} = \bar{\boldsymbol{\theta}} \cdot \bar{a}_h$. Then for some $q(a_{h^*}) \in Q$, $q \Vdash_\alpha \varphi(\bar{\theta} \cdot \bar{a}_h)$, (note that $q \in Q$ iff q is satisfied by a_{h^*}). Let $\theta_i = {}^{t_i}\theta \cdot {}^{l_i}\theta$, where length $t_i = l_i^*$. Let $t_i^* = a_{h_i} \restriction l_i$, and set $\theta_i^* = {}^{t_i^*}\theta \cdot {}^{l_i^*}\theta$; whence $\bar{\boldsymbol{\theta}}^* \cdot \bar{\boldsymbol{\theta}} \cdot \bar{a}_h = \bar{a}_h$. Since $t_i^* = a_{h_i} \restriction l_i$, and $\bar{a}_h$ $\exists$-satisfies q, $T^{\bar{t}^*,h}$ is consistent with q; hence $T^{\bar{l}^*,h}(T^{\bar{t}^*,h}(q))$ is a condition p, and

$$T^{\bar{l}^*,h}(T^{\bar{t}^*,h}(q)) = p \Vdash_\alpha T^{\bar{l}^*,h}(T^{\bar{t}^*,h}(\varphi(\theta \cdot \bar{a}_h)) .$$

Then $T^{\bar{\theta},h}$ is consistent with p and

$$T^{\bar{\theta},h}(p) \Vdash_\alpha \varphi(\bar{\theta} \cdot \bar{\theta}^* \cdot \bar{\theta} \cdot \bar{a}_h) .$$

Now $\theta_i \cdot \theta_i^* \cdot \theta_i(a_{h_i})$ can be represented as

$${}^{t_i}\theta \cdot \underbrace{{}^{l_i}\theta \cdot {}^{t_i^*}\theta} \cdot \underbrace{{}^{l_i^*}\theta \cdot {}^{t_i}\theta} \cdot {}^{l_i}\theta(a_{h_i}) ,$$

(by Lemma 5.171). Consider the grouping marked above; then again by Lemma 5.171,

$$0 \Vdash_\alpha {}^{l_i}\theta \cdot {}^{t_i^*}\theta = {}^\circ\theta \wedge {}^{l_i^*}\theta = {}^\circ\theta .$$

Therefore we get

$$0 \Vdash_\alpha \bigwedge_i \theta_i \cdot \theta_i^* \cdot \theta_i(a_{h_i}) = \theta_i(a_{h_i}) = {}^{t_i}\theta({}^{l_i}\theta(a_{h_i})) ,$$

from which it follows that

$$T^{\bar{\theta},h}(p) \Vdash_\alpha \varphi(\bar{\theta} \cdot \bar{a}_h) .$$

By Lemma 3.35, $p(\bar{i}, \bar{t}) \cup p \Vdash_\alpha \varphi(\bar{a}_h)$. To see that $\bar{\theta}\cdot\bar{a}_h$ $\exists$-satisfies $p(\bar{i}, \bar{t}) \cup p$, first observe that $\bar{\theta}\cdot\bar{a}_h$ obviously satisfies $p(\bar{i}, \bar{t})$. To see that it $\exists$-satisfies p, consider

$$T^{\bar{\theta},h}(p) = T^{\bar{\theta},h}(T^{\bar{i}^*,h}(T^{\bar{i}^*,h}(q))) \subseteq q .$$

Thus, since $\bar{a}_h$ $\exists$-satisfies q, we get $\bar{\theta}\cdot\bar{a}_h$ $\exists$-satisfies p; as required.

Combining Lemmas 5.44 and 5.45 we have:

5.46. Generic Symmetry Lemma. *For any $\varphi(\bar{a}_h) \in \mathcal{F}^c_\alpha$, and $\bar{x} \in K(\bar{a}_h)$, $N_\alpha \models \varphi[\bar{x}]$ if and only if there is a condition $p \in P_\alpha$ such that $p \Vdash_\alpha \varphi(\bar{a}_h)$ and $\bar{x}$ $\exists$-satisfies p.*

The meaning of the last lemma is that all sequences of reals $a_i' \in K(a_i)$ are generically of the same status with respect to the model N. Moreover, the interpretation of the constants a_i as $a_i' = \theta_i(a_i)$ instead of a_i, while retaining the interpretation of the other constants, yields a corresponding interpretation switch of the entire local language complying with the following rule:

$$\langle N_\alpha, I_\beta, \chi_e, a_i\rangle \models \varphi(\bar{a}_h) \quad \text{iff} \quad \langle N_\alpha, I_\beta, \chi_e, a_i'\rangle \models \varphi(\bar{\theta}_h \cdot \bar{a}_h) ,$$

for any $\varphi \in \mathcal{F}^c_\alpha$.

5.5. In order to pay a debt owing from the introduction (Example 1.9), we shall prove lemmas concerning some properties of $N_0 = M[I_0]$, using the symmetry lemma.

Here, $p \in P_0$ are only finite bits of coordinate information on the a_i's. Thus, there are a denumerable number of $\bar{x} \in K(\bar{a}_h)$, from every layer, satisfying such a p. A condition $p(\bar{a}_h) \in P_0$ is a union of conditions $p(a_{h_i})$ speaking only about a_{h_i}.

5.50. Lemma. *There is no choice function for the set,*

$$\{I_0(i) \mid i \in \omega\} = K ,$$

in N_0.

Proof. Assume $\varphi(\sigma)$ states that σ is a choice function for K and that for

$p \in Q_0$, $p \Vdash_0 \varphi(\sigma)$. Let $l < \omega$ be an integer larger than any index of a generic real constant mentioned in p or φ. Then there is a $q \in Q_0$, $q \supseteq p$, such that

$$q \Vdash_0 \varphi(\sigma) \wedge \sigma(K(a_l)) = \theta(a_l) ,$$

for some $\theta \in \Gamma$. Let

$$q = q_1(a_{h_1}) \cup \ldots \cup q_m(a_{h_m}) \cup q_l(a_l) .$$

Then for any $x \in K(a_l)$ satisfying $q_l(a_l)$, we have $(a_{h_1}, \ldots, a_{h_m}, x)$ satisfying q. By Lemma 5.44,

$$N_0 = M[I_0] \models (\varphi(\sigma) \wedge \sigma(K(a_l)) = \theta(a_l)) \, [(a_{h_1}, \ldots, a_{h_m}, x)] .$$

If $x \neq a_l$, then $\boldsymbol{\theta}(x) \neq \boldsymbol{\theta}(a_l)$, but $\boldsymbol{\sigma}(K(a_l)) = \boldsymbol{\sigma}(K(x))$; this yields a contradiction, (because a_l is not mentioned in $\varphi(\sigma)$, and there are denumerably many such x).

Definition 5.501. For $x \in K(a_i)$, define

$$\mathrm{lr}(x) = \{y \mid (\exists l < \omega)\, {}^l\boldsymbol{\theta}(x) = {}^l\boldsymbol{\theta}(y)\} ,$$

$\mathrm{lr}(x)$ is called the *layer* of x. Denote by R the set of all layers of $\bigcup_{i \in \omega} I_0(i)$, i.e.,

$$R = \{\mathrm{lr}(x) \mid (\exists i)\, (x \in I_0(i))\} .$$

The corresponding terms in $\mathcal{L}_0$ are denoted by $\dot{R}$, $\dot{\mathrm{l}}\mathrm{r}(x)$.

5.51. Lemma. *There is no univalent function in* $M[I_0]$ *mapping* R *onto* $R \times 2$. (*Therefore* $(2\,|R| \neq |R|)^{N_0}$.)

Proof. Let $\varphi(\sigma)$ state that $\boldsymbol{\sigma}$ is such a map, and that for some $p \in Q_0$, $p \Vdash_0 \varphi(\sigma)$. Let l be an integer larger than any index of generic real constatns mentioned in p or φ. Now there exists $\theta, \theta' \in \Gamma$, $\kappa < \omega$, $\delta < 2$ and a $q \supseteq p$, $q \in Q_0$ such that

$$q \Vdash_0 \varphi(\sigma) \wedge \sigma(\dot{\mathrm{l}}\mathrm{r}(\theta(a_l))) = \langle \dot{\mathrm{l}}\mathrm{r}(\theta'(a_k), \dot{\delta} \rangle .$$

By choosing an $x \in K(a_l)$ satisfying $q(a_l) = q \upharpoonright a_l$ and not from $(\mathrm{lr}(\boldsymbol{\theta}(a)))$, we obtain a contradiction as in the previous lemma.

By similar arguments we can also show that the ordering principle does not hold in N_0.

5.52. Definition. Let $E_{m,l}$ be the following equivalence relation on $K(a_l)$: for $x, y \in K(a_l)$

$$xE_{m,i}y \quad \text{iff} \quad (\exists n)\,(n' < \omega)\,[\mathrm{slash}_{n\cdot m}(x) = \mathrm{slash}_{n'\cdot m}(y)]\ .$$

This is clearly an equivalence relation on $K(a_i)$, and we denote the set of equivalence classes by $H_{m,i}$. $H_{m,i}$ consists of m components each of which is a union of layers which are distant from each other by multiples of m, (see Definition 1.9(b)). Let

$$U_m = \bigcup\{H_{m,i} \mid i < \omega,\ 2 \leqslant m < \omega\}\ .$$

We have:

5.53. Lemma. *For any $2 \leqslant m < \omega$, U_m is not orderable in N_0.*

Proof. Assume $\prec_m$ orders U_m in N_0, and that $\varphi(a_h) \in \mathcal{L}_0$ states that $\prec_m$ orders U_m. Then for some $p(a_{h'}) \in Q$. $p \Vdash_0 \varphi(a_h)$. Let l be an integer not in $h \cup h'$. For some $p \subseteq q \in Q$ we must have,

$$\text{(5.54)} \qquad \text{either } q \Vdash_0 H_{m,l}({}^1\theta(a_l)) \prec_m H_{m,l}(a_l)$$
$$\text{or} \qquad q \Vdash_0 H_{m,l}(a_l) \prec_m H_{m,l}({}^1\theta(a_l))\ .$$

Clearly the following is true in N_0

$$\text{(5.55)} \qquad \bigwedge_{0 \leqslant j < k < m} (H_{m,l}({}^j\theta(a_l)) \neq H_{m,l}({}^k\theta(a_l))) \wedge$$
$$\wedge\, H_{m,l}(a_l) = H_{m,l}({}^m\theta(a_l))\ .$$

Let $q = q'(a_{h''}) \cup q(a_l)$ and let $\theta_j(a_l)$ be an element of $\mathrm{lr}({}^j\theta(a_l))$ satisfying $q''(a_l)$, $0 \leqslant j < m$. As stated above such elements obviously exist. Without loss of generality assume that the left disjunct of (5.54) holds. By the lemma we get:

$$N_0 \models \varphi \wedge \bigwedge_{0 \leqslant j < m} (H_{m,l}({}^1\theta(\theta_j(a_l))) \prec_m H_{m,l}(\theta_j(a_l)))\ .$$

Now,

$$H_{m,l}({}^1\theta(\theta_j(a_l))) = H_{m,l}(\theta_{j+1}(a_l)) = H_{m,l}({}^{j+1}\theta(a_l))\ ;$$

hence

$$N_0 \models \bigwedge_{0<j<m} (H_{m,l}({}^{j+1}\theta(a_l)) \prec_m H_{m,l}({}^{j}\theta(a_l))) ,$$

and by the transitivity of $\prec_m$ we get

$$N_0 \models H_{m,l}({}^{m}\theta(a_l)) \prec H_{m,l}(a_l) \wedge H_{m,l}({}^{m}\theta(a_l)) = H_{m,l}(a_l) .$$

This is clearly a contradiction. Thus,

5.56. Lemma. *The ordering principle does not hold in* N_0.

5.57. Remark. Lemma 5.44 shows that the amount of symmetry we have in N_α depends upon the amount of symmetry there is with respect to conditions. In N_0 there are denumerably many elements satisfying a condition; this accounts for the negation of choice in the model. The restrictions imposed by the χ_e reduce the number of those tail elements, possibly to a finite number. It will be shown that this is not the case; we shall later prove, in one of our main lemmas, that if a condition is satisfiable by elements of the corresponding tails, then it is satisfied by countably many such elements. This will enable us to negate choice also in N.

6. Combinatorial lemmas and preservation of alephs

6.0. In this section we prove two combinatorial lemmas that will later be seen to be responsible for the fact that in N no new elementary tail partitions arise. In one, it is shown that the amount of *essential information* involved in the truth of a statement is countable. In the other, it is shown that any set of mutually incompatible conditions is countable. From the latter it follows by a standard argument that alephs are preserved in the extensions.

6.00. Definition. $p, q \in P_\alpha$ are said to be *compatible*, if for some $r \in P_\alpha$, $p \cup q \subseteq r$; otherwise they are said to be *incompatible* or *contradictory*.

6.01. Notation. (a) If p is a condition, denote by $\mathrm{sym}(p)$ the set of generic real constants or function constants mentioned by p.

(b) Let s be a finite set of generic real constants, or function constants; then by a *batch of information* on s we mean a finite set of preconditions (possibly contradictory) mentioning only the symbols of s.

It easily follows from the fact that $\bigcup_{n\in\omega} \bar{\Gamma}^n$ is countable in M, that:

6.02. Lemma. *The number of different batches of information in M mentioning a finite set of symbols s is denumerable.*

6.1. We now show that the amount of essential information involved in the truth of a statement is countable in M.

6.10. Lemma. *For every $\varphi \in \mathcal{T}^c_\alpha$, the cardinality of $\|\varphi\|_\alpha$ is at most countable in M.*

Proof. Let $\|\varphi\|^n_\alpha$ be the set of n element conditions of $\|\varphi\|_\alpha$. If the cardinality of $\|\varphi\|_\alpha$ is larger than $\aleph_0$ in M, then using choice in M, it follows that for some n, $\|\varphi\|^n_\alpha$ is not denumerable in M. There is a condition q of maximal cardinality such that

$$|\{p \supseteq q \mid p \in \|\varphi\|^n_\alpha\}| = |\,\|\varphi\|^n_\alpha|$$

in M, (q may possibly be empty). Let

$$R = \{p \in \|\varphi\|^n_\alpha \mid q \subseteq p\}$$

for such a q. Since the set of all batches of information on sym(q) is countable, there must be a subset R_0 of R such that $|R_0|^{(M)} = \aleph_1^{(M)}$, and all $p \in R_0$ contain the same information on sym(q). Using choice, a sequence (p_ξ, R_ξ), $\xi > \aleph_1^{(M)}$, can be defined in M such that $p_\xi \in R_\xi$, $R_\xi \subseteq R_\eta$, $\xi > \eta$, $|R_\xi|^{(M)} = \aleph_1^{(M)}$, $|R_\xi - R_{\xi+1}|$ is denumerable in M, and

$$\mathrm{sym}(p_\xi) \cap \mathrm{sym}(p_\eta) = \mathrm{sym}(q)\,,$$

$\xi \neq \eta$. Let p_0 be any element of R_0. Assume that p_η, R_η have been defined for $\eta \leqslant \xi$ and that they have the aforementioned properties. There can only be countably many $p \in R_\xi$ such that

$$\mathrm{sym}(p) \cap \mathrm{sym}(p_\xi) \neq \mathrm{sym}(q)\,;$$

for if otherwise, then since there are only countably many batches of information on sym(p_ξ), we would obtain a contradiction to the maximality of $|q|^{(M)}$. Let $R_{\xi+1}$ be the elements of R_ξ other than those countably many for which $\mathrm{sym}(p) \cap \mathrm{sym}(p_\xi) \neq \mathrm{sym}(q)$. Let $p_{\xi+1}$ be any element of $R_{\xi+1}$. If $\xi < \aleph_1^{(M)}$ is a limit ordinal, set $R_\xi = \bigcup_{\eta<\xi} R_\eta$, and choose an arbitrary $p_\xi \in R_\xi$. Clearly p_ξ, R_ξ have the properties stated above, except perhaps $|R_\xi|^{(M)} = \aleph_1^{(M)}$. This is shown by transfinite induction. For successor ξ, the result simply follows from the fact that $R_{\xi+1} - R_\xi$ is countable; and for limit ξ, the result follows from the fact that $\aleph_1^{(M)}$ is regular in M.

We now claim that $q \Vdash_\alpha \varphi$, in which case $|R| = 1$, contradicting our assumption that R is a non-denumerable set in M. If $q \nVdash_\alpha \varphi$, then for some $p \supseteq q$, $p \Vdash_\alpha \neg\varphi$. Since the sequence p_ξ is uncountable, there are elements p_ξ such that

$$\mathrm{sym}(p) \cap \mathrm{sym}(p_\xi) = \mathrm{sym}(q)\,;$$

moreover p_ξ does not contain information on sym(q) which is not in $q \subseteq p_\xi$. Hence $p \cup p_\xi$ is a condition. This is impossible, since $p \cup p_\xi$ would force $\varphi \wedge \neg\varphi$. Therefore $\|\varphi\|_\alpha$ is countable in M, as required.

6.2. We now prove that P_α has the countable anti-chain condition, (c.a.c.).

6.20. Lemma. *Any set of mutually incompatible conditions $R \subseteq P_\alpha$ in M, is countable in M.*

Proof. Since M satisfies AC, we can assume that all elements of R are of cardinality n. Let q be a batch of information of maximal cardinality less than n for which

$$|\{p \in R \mid q \subseteq p\}|^{(M)} = \aleph_1^{(M)} ,$$

and let

$$R_1 = \{p \in R \mid q \subsetneq p\} ,$$

(note that q is not necessarily a condition and that q may be empty). Since there are only countably many batches of information concerning only sym(q), we can choose $R_2 \subseteq R_1$ in M, such that $|R_2|^{(M)} = \aleph_1^{(M)}$, and all $p \in R_2$ have the same information s on sym(q). Let $p \in R_2$. Since R_2 has $\aleph_1^{(M)}$ members, there must be a symbol x mentioned by p such that $\aleph_1^{(M)}$ members of R_2 mention x; otherwise, p would be compatible with elements of R_2 not mentioning any of the elements of sym(p)-sym(q). For such an x, let $R_3 \subseteq R_2$ be a non-countable set in M, all members of which mention x. Since there are only countably many batches of information on x, there must be $\aleph_1^{(M)}$ members of R_3 with identical information on x. This contradicts the maximality of q. Hence $|R|^{(M)} = \aleph_0$ as required.

This lemma has the following corollaries:

6.21. Corollary. *For all ordinals $\eta \in M$, the cofinality of η in N_α is equal to the cofinality of η in M, $\alpha \leqslant \aleph_1^{(M)}$.*

Proof. Assume that for some $\xi < \eta$, there exists a univalent function $\tau \in N_\alpha$ such that dom(τ) = ξ and rng(τ) $\subseteq \eta$. Let $\varphi(\tau, \xi, \eta)$ be the statement "τ is a univalent function with dom(τ) = ξ and rng(τ) $\subseteq \eta$". For any $\zeta < \xi$ and $\lambda < \eta$, define:

$$[\xi, \lambda] = \{p \in P_\alpha \mid p \Vdash_\alpha \tau(\zeta) = \lambda \wedge \varphi\} .$$

If $p_1 \in [\zeta, \lambda_1]$, $p_2 \in [\zeta, \lambda_2]$, where $\lambda_1 \neq \lambda_2$, then p_1, p_2 are incompatible because otherwise we would have a condition $q \supseteq p_1 \cup p_2$ forcing$_\alpha$

$$\tau(\zeta) = \lambda_1 \wedge \tau(\xi) = \lambda_2 \wedge \varphi ,$$

yielding a contradiction. Set

$$[\zeta] = \{\lambda \mid [\zeta, \lambda] \neq \emptyset\} .$$

Then $[\zeta]$ is countable in M. For if not, then using choice in M we could choose an element from each of the $[\zeta, \lambda] \neq \emptyset$, yielding an uncountable set of incompatible conditions in M, contradicting the previous lemma. Thus we see that for each $\zeta < \xi$, the number of possible values of $\tau(\zeta)$ calculated in M is countable. Moreover there is a function F in M with $\mathrm{dom}(F) = \xi$, and $\mathrm{rng}(F) \subsetneq \eta$, such that for all $\zeta < \xi$, $\tau(\zeta)$ in N_α is less or equal to $F(\zeta)$ in M, namely: $F(\zeta) = \bigcup[\zeta]$. If η is regular in M, then clearly $\mathrm{rng}(F)$ is bounded by an ordinal less than η, i.e. η is regular in N_α. From this it also follows that the cofinality of singular cardinals in N_α is not less than the cofinality in M. For if the cofinality of η is ξ in M where ξ is regular in M, and there is an increasing sequence of length μ in N_α cofinal with η, and the cardinality of μ in N_α is less than that of ξ, then we could define by induction in N_α, a sequence, G, of length μ cofinal with ξ. Namely, for $\zeta < \mu$, $G(\zeta + 1)$ = the least $\lambda \in \mu$ such that $\lambda > G(\zeta)$; and if ζ is a limit ordinal, then $G(\zeta)$ = the least $\lambda < \mu$ such that $\lambda > \bigcup_{\delta<\zeta} G(\delta)$. It is easily checked that G is the required function. This contradicts the fact that ξ is also regular in N_α.

6.22. Corollary. *$(\aleph_\xi)^{(M)} = (\aleph_\xi)^{N_\alpha}$, for all $\xi \in M$, $\alpha \leqslant \aleph_1^{(M)}$.*

6.3. The following simple fact is worthwhile knowing, although we do not make actual use of it.

6.30. Lemma. $|P_0|^{(M)} = \aleph_0$, $|P_\alpha|^{(M)} = |\mathcal{X}_\alpha|^{(M)} = \aleph_1^{(M)}$, $0 < \alpha \leqslant \aleph_1^{(M)}$.

Proof. This lemma is proven by transfinite induction. The case $\alpha = 0$ is trivial. Assume the lemma true for $\beta < \alpha$ and prove for α. First assume $\alpha = \beta + 1$. For all $\sigma \in \mathcal{T}_\alpha^c$, $\bar\theta, \bar\theta' \in \Gamma$, $[\pm\sigma, \bar\theta, \bar\theta']_\beta$ is countable in M; hence, using CH in M, the number of $[\pm\sigma, \bar\theta, \bar\theta']_\beta$ is at most $\aleph_1^{(M)}$. Again using CH and choice in M, we have $(\aleph_1^{\aleph_0})^{(M)} = \aleph_1^{(M)}$; therefore the number of s_σ in M is at most $\aleph_1^{(M)}$. Thus the number of $\beta + 1$-indices calculated in M is at most $\aleph_1^{(M)}$; hence $|\mathcal{X}_\alpha|^{(M)} = \aleph_1^{(M)}$. In order to see that $|\mathcal{X}_\alpha|^{(M)} = \aleph_1^{(M)}$, it is enough to observe that there are $\aleph_1^{(M)}$ elementary partitions of any tail in N_0. For any constructible real $a \in M$, an elementary tail partition H_a can be defined on $K(b)$ via an equivalence relation E_a as follows: if $x, y \in K(b)$, then $E_a(x,y)$ iff for all $i < \omega$,

$$x(i) = y(i) \quad \text{iff} \quad a(i) = 1 .$$

It is easily seen that E_a determines an elementary partition H_a of $K(b')$; and if $a \neq a'$, then $H_a \neq H_{a'}$. It is a simple matter to determine a term $\sigma = \sigma_{\dot{a}}$, ($\dot{a}$ the name of a in M), determining the partition H_a above. Thus for every real $a \in M$, there is a corresponding $s^c_\sigma = s_{\dot{a}}$ in M, (see Definition 2.41); and by Lemma 2.72, $\dot{a} \neq \dot{a}'$ iff $s_{\dot{a}} \neq s_{\dot{a}'}$. Therefore $|\mathcal{X}_1|^{(M)} \geqslant \aleph_1^{(M)}$. Since $\mathcal{X}_1 \subseteq \mathcal{X}_\alpha$, $1 \leqslant \alpha$, this implies that $|\mathcal{X}_\alpha|^{(M)} = \aleph_1^{(M)}$, as required. Since the number of preconditions concerning a symbol is countable, also $|P_\alpha|^{(M)} = \aleph_1^{(M)}$, as required. If α is a limit ordinal, then

$$\mathcal{X}_\alpha = \bigcup_{\beta<\alpha} \mathcal{X}_\beta, \qquad P_\alpha = \bigcup_{\beta<\alpha} P_\beta,$$

and the result follows from the fact that a countable union of sets of cardinality $\aleph_1$ is of cardinality $\aleph_1$ in M.

7. Syntax and semantics in N_α

We will need the definition of the sequence $(\mathcal{L}_\beta, P_\beta, \Vdash_\beta)$, $\beta \leq \aleph_1^{(M)}$, together with all auxiliary terms and notions in the models N_α. This is done simply by repeating the definitions given in the M-language $\mathcal{L}_M$. There is only a minor difficulty concerning the set constants since we have not supplied a predicate $M(\cdot)$ to $\mathcal{L}_\alpha$ for distinguishing the elements of M. However since the definition of the constructible hierarchy L in these models is absolute, and the realization of L in them is M, this can be easily overcome. We just alter the definition slightly by taking "for every set x in L a constant $\langle 4, x\rangle$" instead of "for every set x in M a constant $\langle 4, x\rangle$".

Hence we have:

7.00. Lemma. *In $\mathcal{L}_\alpha$, $\alpha \leq \aleph_1^{(M)}$, there is a formula defining in N_α the hierarchy $(\mathcal{L}_\beta, P_\beta, \Vdash_\beta)$, $\beta < \aleph_1^{(M)}$ together with all auxiliary terms and notions, which coincides with the hierarchy in M, defined in $\mathcal{L}_M$, and all the terms and notions involved are absolute.*

The extent which the semantics of $\mathcal{L}_\beta$ can be developed in N_α is determined by the fact that although the correspondences $I_\beta \to \boldsymbol{I}_\beta$, and $e \to \chi_e \to \boldsymbol{\chi}_e$, exist in N_α, for e a $\beta+1$ index, $\beta < \alpha$, the correspondence $a_i \to \boldsymbol{a}_i$ does not. Because, as will be shown in Section 10, $\{\langle i, \boldsymbol{a}_i\rangle \mid i < \omega\}$ is not an element of N_α. Nevertheless, we have in N_α a satisfaction for local formulas φ of $\mathcal{L}_\beta$, $\beta \leq \alpha$, relative to assignments s, which assigns to each of the finitely many a_i's occurring in φ a member $s(i)$ of $\boldsymbol{K}(\boldsymbol{a}_i)$. Similarly we shall have a value function in $\mathcal{L}_\alpha$ for the terms of $\mathcal{L}_\beta$, $\beta \leq \alpha$. In order to establish the satisfaction and value functions in the manner stated above, essential use will be made of Lemma 5.46. Roughly, this will be done by first defining satisfaction for conditions $p(\bar{a}_h) \in P_\beta$, relative to assignments s, where

$$\operatorname{dom}(s) = h, \qquad s(h_i) \in \boldsymbol{K}(\boldsymbol{a}_{h_i}) = \boldsymbol{I}_0(h_i)\ ;$$

and then defining

$$\operatorname{sat}_\beta(\varphi(\bar{a}_h), s) = 1 = \text{truth}$$

iff for some $p(\bar{a}_{h^*}) \in P_\beta$ such that $p \Vdash_\beta \varphi$, and some s^* such that

$$s^* \restriction h \cap h^* = s \restriction h \cap h^*,$$

we have

$$\mathrm{sat}_\beta(p(\bar{a}_{h^*}), s^*) = \text{truth} = 1.$$

7.10. Definition. Let p be a condition, and denote by R_p the following set of statements:

(1) $[\dot{\langle} \dot{j}, \dot{\delta} \dot{\rangle} \in a_i] \in R_p$ iff $\langle i, j, \delta \rangle \in p$;

(2) $[\dot{\langle} \ddot{\langle}\bar{\theta} \cdot \bar{a}_h \dot{\rangle}, \dot{j}, \dot{\delta} \dot{\rangle} \in \chi_e] \in R_p$ iff $\langle e, \bar{\theta}, j, \delta \rangle \in p$;

(3) $[{}^{l_1}\theta(\chi_e(\bar{\theta}_1 \cdot \bar{a}_h)) = {}^{l_2}\theta(\chi_e(\bar{\theta}_2 \cdot \bar{a}_h))] \in R_p$ iff $\langle e, \bar{\theta}_1, l_1, \bar{\theta}_2, l_2 \rangle \in p$;

(4) these are the only elements of R_p. $\tilde{p}$ is defined to be the conjunction of all the statements in R_p, i.e. $\tilde{p} = \bigwedge_{\varphi \in R_p} \varphi$.

7.20. Lemma. *In $\mathcal{L}_\beta$ there are formulas $\widetilde{\mathrm{sat}}_\beta(u, v, w, w')$, $\widetilde{\mathrm{val}}_\beta(u, v, w, w')$, such that for all $\alpha \geqslant \beta$, if $\varphi(a_{h_1}, \ldots, a_{h_n}, v_{g_1}, \ldots, v_{g_m}) \in M$ is a formula of $\mathcal{L}_\beta$, and $\sigma(a_{h_1}, \ldots, a_{h_n}, v_{g_1}, \ldots, v_{g_m}) \in M$ is a term of $\mathcal{L}_\beta$, where we have displayed all generic real constants and free variables occurring in φ, σ, and if $s \in N_\alpha$ is an assignment to the real constants, i.e.,* $\mathrm{dom}(s) = h$, *$s(h_i) \in I_0(h_i) = K(a_{h_i})$, $1 \leqslant i \leqslant n$, and $t \in N_\alpha$ is an assignment to the free variables, i.e.,* $\mathrm{dom}(t) = g$, *then*

$$7.21 \qquad N_\alpha \models \widetilde{\mathrm{sat}}_\beta(\varphi, s, t, 1) \quad \text{iff} \quad N_\beta \models \varphi[s(h_1), \ldots, s(h_n), t(g_1), \ldots, t(g_m)];$$

and for $x \in N_\alpha$,

$$7.22 \qquad N_\alpha \models \widetilde{\mathrm{val}}_\beta(\sigma, s, t, x) \quad \text{iff} \quad N_\beta \models \sigma[s(h_1), \ldots, s(h_n), t(g_1), \ldots, t(g_m)] = x.$$

(See Definitions 2.9071, 5.41 and Lemma 5.44.)

Proof. In order to employ the symmetry lemma in a convenient way, we first define $\widetilde{\mathrm{sat}}_\beta$, $\widetilde{\mathrm{val}}_\beta$ for $\tilde{p}$, for inclusive assignments, and we write in function form paralleling the ordinary definition.

Let $A(u)$ be a sentence in $\mathcal{L}_\beta$ stating that u is a function and $\mathrm{dom}(u) \subseteq \omega$.

$$\text{(a) } \widetilde{\mathrm{val}}_\beta(a_i, s, t) = \begin{cases} s(i) & \text{if } A(s) \text{ and } i \in \mathrm{dom}(s) \text{ and } s(i) \in I_0(i); \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\text{(b) } \widetilde{\mathrm{sat}}_\beta(\dot{\langle} \dot{j}, \dot{\delta} \dot{\rangle} \in a_i, s, t) = \begin{cases} 1 & \text{if } A(s),\ i \in \mathrm{dom}(s),\ s(i) \in I_0(i) \text{ and} \\ & \quad s(i)(j) = \delta; \\ 0 & \text{otherwise.} \end{cases}$$

(c) $\widetilde{\text{sat}}_\beta(\langle\langle\bar\theta\cdot\bar a_h\rangle, \langle j, \delta\rangle\rangle \in \chi_e, s, t) =$

$$= \begin{cases} 1 & \text{if } A(s), h \subseteq \operatorname{dom}(s), s(h_i) \in I_0(h_i) \text{ and} \\ & \langle\langle\theta_1(s(h_1)), \ldots, \theta_n(s(h_n))\rangle, \langle j, \delta\rangle\rangle \in I_\beta(e); \\ 0 & \text{otherwise.} \end{cases}$$

(d) $\widetilde{\text{val}}_\beta(\chi_e(\bar\theta\cdot\bar a_h), s, t) =$

$$= \begin{cases} \{\langle j, \delta\rangle \mid \langle\langle\theta_1(s(h_1)), \ldots, \theta_n(s(h_n))\rangle, \langle j, \delta\rangle\rangle \in I_\beta(e)\} \\ \quad \text{if } A(s), h \subseteq \operatorname{dom}(s), \text{and } s(h_i) \in I_0(h_i); \\ 0 \quad \text{otherwise.} \end{cases}$$

(e) $\widetilde{\text{sat}}_\beta({}^{l_1}\theta(\chi_e(\bar\theta_1\cdot\bar a_h)) = {}^{l_2}\theta(\chi_e(\bar\theta_2\cdot\bar a_h)), s, t) =$

$$= \begin{cases} 1 & \text{if } A(s), h \subseteq \operatorname{dom}(s), s(h_i) \in I_0(h_i) \text{ and} \\ & {}^{l_1}\theta(\widetilde{\text{val}}_\beta(\chi_e(\bar\theta_1\cdot\bar a_h), s, t)) = {}^{l_2}\theta(\widetilde{\text{val}}_\beta(\chi_e(\bar\theta_2\cdot\bar a_h), s, t)); \\ 0 & \text{otherwise.} \end{cases}$$

(f) $\widetilde{\text{sat}}_\beta(\varphi_1 \wedge \varphi_2, s, t) = \widetilde{\text{sat}}_\beta(\varphi_1, s, t) \cdot \widetilde{\text{sat}}_\beta(\varphi_2, s, t)$.

(g) $\widetilde{\text{sat}}_\beta((\varphi), s, t) = \widetilde{\text{sat}}_\beta(\varphi, s, t)$.

It follows that for every $p \in P_\beta$, $\widetilde{\text{sat}}_\beta(\tilde p, s, t)$ is defined.

We now define for general φ, σ not containing free variables. The mode of the rest of the definition is motivated by the symmetry lemma and proceeds by induction on the rank of terms.

(h) $\widetilde{\text{sat}}_\beta(\varphi(\bar a_h), s, t) =$

$$= \begin{cases} 1 & \text{if } A(s), h \subseteq \operatorname{dom}(s), s(h_i) \in I_0(h_i), \text{and for some} \\ & \tilde p(\bar a_{h^*}), p \in P_\beta, \text{such that } p \Vdash_\beta \varphi, \text{and some } s^* \text{ such that} \\ & s^* \restriction h \cap h^* = s \restriction h \cap h^*, \text{we have } \widetilde{\text{sat}}_\beta(\tilde p(\bar a_{h^*}), s^*, t) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

(i) $\widetilde{\text{val}}_\beta(\sigma(\bar a_h), s, t) =$

$$= \begin{cases} \{\widetilde{\text{val}}_\beta(\tau, s, t) \mid \tau \in \mathcal{T}^c_\beta, \operatorname{rnk}(\tau) < \operatorname{rnk}(\sigma), \operatorname{stg}(\tau) \leqslant \operatorname{stg}(\sigma) \\ \qquad \text{and there exist a } P(a_{h^*})\, P_\beta \text{ such that} \\ \qquad p \Vdash_\beta \tau \in \sigma \text{ and for some } s^* \text{ such that} \\ \qquad s^* \restriction h^* \cap h = s \restriction h \cap h^*, \text{we have} \\ \qquad \widetilde{\text{sat}}_\beta(\tilde p(\bar a_{h^*}), s^*, t) = 1]\}, \\ \quad \text{if } A(s), h \subseteq \operatorname{dom}(s) \text{ and } s(h_i) \in I_0(h_i), 1 \leqslant i \leqslant |h|; \\ \emptyset \quad \text{otherwise.} \end{cases}$$

For formulas and terms with free variables we define as follows:

(j) $\tilde{\text{sat}}_\beta(\varphi(\bar{a}_h, v_{g_1}, ..., v_{g_m}), s, t) =$

$$= \begin{cases} 1 & \text{if } A(s), h \subseteq \text{dom}(s), s(h_i) \in I_0(h_i), 1 \leq i \leq |h| \text{ and} \\ & A(t), g \subseteq \text{dom}(t) \text{ and there exist constant terms } \tau_j, \\ & 1 \leq j \leq m \text{ such that } \text{val}_\beta(\tau_j, s, t) = t(g_j), 1 \leq j \leq m \\ & \text{and } \tilde{\text{sat}}_\beta(\varphi(\bar{a}_h, \tau_1, ..., \tau_m), s, t) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

(k) $\tilde{\text{val}}_\beta(\sigma(\bar{a}_h, \bar{v}_g^m), s, t) =$

$$= \begin{cases} x & \text{if } A(s), h \subseteq \text{dom}(s), s(h_i) \in I_0(h_i), 1 \leq i \leq |h|, \text{and} \\ & \text{there exist constant terms } \tau_j, 1 \leq j \leq m \text{ of rank} < \omega+1 \\ & \text{such that } \tilde{\text{val}}_\beta(\tau_j, s, t) = t(g_j), 1 \leq j \leq m, \text{ and} \\ & x = \tilde{\text{val}}_\beta(\sigma(\bar{a}_h, \bar{\tau}^m), s, t); \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{\text{sat}}_\beta$, $\tilde{\text{val}}_\beta$ are well defined in $\mathcal{L}_\beta$. Moreover, it is an immediate consequence of Lemma 5.44 and the absoluteness properties of $\mathcal{L}_\beta$, P_β, $\Vdash_\beta$, and all associated notions in N_α, that (7.21) and (7.22) hold for all α, such that $\beta \leq \alpha \leq \aleph_1^{(M)}$.

7.23. Notation. If $\varphi \in \mathcal{F}_\beta^c$, then we write $\text{sat}_\beta(\varphi, s)$ instead of $\tilde{\text{sat}}(\varphi, s, t)$; similarly for $\tilde{\text{val}}_\beta$.

7.24. Corollary. *If* $\varphi(\bar{a}_h) \in \mathcal{F}_\beta^c$, $\sigma(\bar{a}_h) \in \mathcal{T}_\beta^c$ *and s is such that* $\text{dom}(s) = h$, $s(h_i) = a_{h_i}$, $1 \leq i \leq |h|$, *then* $N_\alpha \models \tilde{\text{sat}}_\beta(\varphi, s)$ *iff* $N_\beta \models \varphi$, *and* $N_\alpha \models \tilde{\text{val}}_\beta(\sigma, s, x)$ *iff* $\text{val}_\beta(\sigma) = x$, *where* $x \in N_\alpha$, $\beta \leq \alpha \leq \aleph_1^{(M)}$.

7.30. Lemma. (a) *The following notions are expressible in* $\mathcal{L}_\alpha$ *for* $\beta \leq \alpha \leq \aleph_1^{(M)}$: *definable*$_\beta$ *by* φ, *definable*$_\beta$, *f-definable*$_\beta$ *by* φ, *f-definable*$_\beta$, *(−f)-definable*$_\beta$ *by* φ, *(−f)-definable*$_\beta$, *nearly definable*$_\beta$ *by* φ, *nearly definable*$_\beta$, *nearly-(f)-definable*$_\beta$ *by* φ, *(f)-definable*$_\beta$ *by* φ, *nearly-(f)-definable*$_\beta$, *nearly-(−f)-definable*$_\beta$ *by* φ, *nearly-(−f)-definable*$_\beta$.

(b) *Let* $\sigma(\bar{a}_h) \in \mathcal{T}_\beta^c$, *and* $g \subseteq h$; *then there are terms* ${}^\beta H_\sigma^g$ *such that* $\text{val}_\alpha({}^\beta H_\sigma^g) = {}^\beta H_\sigma^g$. (See Definition 4.81.)

Proof. (a) This is a consequence of the fact that the syntax and semantics of N_β are expressible in $\mathcal{L}_\alpha$. Therefore Definition 4.80 can be repeated in $\mathcal{L}_\alpha$. We omit the technical details of this, which are obvious by routine methods concerning formalization of syntax and semantics in object languages.

(b) is a consequence of (a) and Definition 4.81. We omit, of course the actual description of the terms; but we will assume that the symbols above pertain to particular terms with the aforementioned properties.

8. Termination of tail partition formation and related facts.

We shall now show that every elementary tail partition in N can already be found in some N_α for $\alpha < \aleph_1^{(M)}$. It will be seen that the reason for this is the fact that $\|\varphi\|$ is countable in N, for any $\varphi \in \mathcal{T}^c$. It can also be shown that this is not true for any ordinal $\alpha < \aleph_1^{(M)}$; i.e., there are elementary tail partitions in N_α which do not exist in any earlier model. (In order to see this, just show that for any limit ordinal $\beta < \aleph_1^{(M)}$, there are reals in N_β which are not in any N_γ, $\gamma < \beta$. By example 1.9(c) new reals give new partitions.)

8.00. Lemma. *For any $\sigma(\bar{a}_h) \in \mathcal{T}^c_\alpha$, there exists an $\alpha < \aleph_1^{(M)}$ and a term $\tau(\bar{a}_h) \in \mathcal{T}^c_\alpha$ such that*

$$^{\alpha}H_\tau = {}^{\aleph_1^{(M)}}H_\sigma .$$

Proof. Since $\|\varphi\|$ is countable in N for any $\varphi \in \mathcal{T}^c$, there exists an $\alpha < (\aleph_1)^{(M)}$ such that for every $\bar{\theta}_1, \bar{\theta}_2 \in \bar{\Gamma}$,

$$\|\sigma(\bar{\theta}_1 \cdot \bar{a}_h) = \sigma(\bar{\theta}_2 \cdot \bar{a}_h)\| \subseteq P_\alpha ,$$

$$\|\sigma(\bar{\theta}_1 \cdot \bar{a}_h) \neq \sigma(\bar{\theta}_2 \cdot \bar{a}_h)\| \subseteq P_\alpha .$$

Now, observe that although sat_α is not necessarily defined on $\sigma(\bar{a}_h)$, it is defined for every $\bar{p}$, $p \in P_\alpha$. Hence by considering things in N_α, we can take $\tau(\bar{a}_h)$ as the following term:

$$\tau(\bar{a}_h) = (\lambda w)\,(\exists \bar{\theta} \in \bar{\Gamma})$$
$$[w = \bar{\theta} \cdot \bar{a}_h \wedge (\exists p \in P_\alpha)\,(p \Vdash \sigma(\bar{\theta} \cdot \bar{a}_h) = \sigma(\bar{a}_h) \wedge$$
$$\wedge (\exists s)\,(h \subseteq \mathrm{dom}(s) \wedge s(h_i) = a_{h_i},$$
$$1 \leqslant i \leqslant |h|, \wedge \, \tilde{\mathrm{sat}}_\alpha(\bar{p}, s) = 1 = \mathrm{truth}))] .$$

This is easily seen to be a local term of $\mathcal{T}^c_\alpha$, (by Lemmas 7.00, 7.20). It has to be shown that τ induces in N_α the partition $H_\sigma^{(\aleph_1)}$; i.e., we must show that

$$N_\alpha \models \tau(\bar{\theta}_1 \cdot \bar{a}_h) = \tau(\bar{\theta}_2 \cdot \bar{a}_h)$$

iff

$$N \models \sigma(\bar{\theta}_1 \cdot \bar{a}_h) = \sigma(\bar{\theta}_2 \cdot \bar{a}_h) .$$

Let $\bar{y} \in \tau^{(N_\alpha)}(\bar{\theta}_1 \cdot \bar{a}_h)$; then by definition of τ together with the absoluteness properties, we must have $\bar{y} = \bar{\lambda}_1 \cdot \bar{\theta}_1 \cdot \bar{a}_h$; and for some $p \in P_\alpha$ such that $p \Vdash \sigma(\bar{a}_h) = \sigma(\bar{\lambda}_1 \cdot \bar{a}_h)$, $\bar{\theta}_1 \cdot \bar{a}_h$ $\exists$-satisfies p. By Lemma 5.44, we have

$$N \models \sigma(\bar{\theta}_1 \cdot \bar{a}_h) = \sigma(\bar{\lambda}_1 \cdot \bar{\theta}_1 \cdot \bar{a}_h) .$$

If $N_\alpha \models \tau(\bar{\theta}_1 \cdot \bar{a}_h) = \tau(\bar{\theta}_2 \cdot \bar{a}_h)$, then also $y \in \tau^{(N_\alpha)}(\bar{\theta}_2 \cdot \bar{a}_h)$ and similarly we get

$$y = \bar{\lambda}_2 \cdot \bar{\theta}_2 \cdot \bar{a}_h \qquad N \models \sigma(\bar{\theta}_2 \cdot \bar{a}_h) = \sigma(\bar{\lambda}_2 \cdot \bar{\theta}_2 \cdot \bar{a}_h) .$$

Since $\bar{\lambda}_1 \cdot \bar{\theta}_1 \cdot \bar{a}_h = \bar{\lambda}_2 \cdot \bar{\theta}_2 \cdot \bar{a}_h$, we get $N \models \sigma(\bar{\theta}_1 \cdot \bar{a}_h) = \sigma(\bar{\theta}_2 \cdot \bar{a}_h)$. If on the other hand $\bar{y} \notin \tau^{(N_\alpha)}(\bar{\theta}_2 \cdot \bar{a}_h)$, then for no $p \in P_\alpha$ such that $p \Vdash \sigma(\bar{\lambda}_2 \cdot \bar{a}_h) = \sigma(\bar{a}_h)$ does $\bar{\theta}_2 \cdot \bar{a}_h$ $\exists$-satisfy p. Thus by Lemma 5.45 we must have

$$N \models \sigma(\bar{\theta}_2 \cdot \bar{a}_h) \neq \sigma(\bar{\lambda}_2 \cdot \bar{\theta}_2 \cdot \bar{a}_h) .$$

(For if $N \models \sigma(\bar{\theta}_2 \cdot \bar{a}_h) = \sigma(\bar{\lambda}_2 \cdot \bar{\theta}_2 \cdot \bar{a}_h)$, then by Lemma 5.45 for some $p \in P$, such that $p \Vdash \sigma(\bar{a}_h) = \sigma(\bar{\lambda}_2 \cdot \bar{a}_h)$, $\bar{\theta}_2 \cdot \bar{a}_h$ $\exists$-satisfies p. Let $q \supseteq p$ be minimal for $\sigma(\bar{a}_h) = \sigma(\bar{\lambda}_2 \cdot \bar{a}_h)$; then $\bar{\theta}_2 \cdot \bar{a}_h$ also $\exists$-satisfies q and $q \in P_\alpha$; a contradiction.) Hence if

$$\bar{y} \in \tau^{(N_\alpha)}(\bar{\theta}_1 \cdot \bar{a}_h) , \qquad \bar{y} \notin \tau^{(N_\alpha)}(\bar{\theta}_2 \cdot \bar{a}_h) ,$$

we get from the above that

$$N \models \sigma(\bar{\theta}_1 \cdot \bar{a}_h) = \sigma(\bar{\lambda}_1 \cdot \bar{\theta}_1 \cdot \bar{a}_h) \wedge \sigma(\bar{\theta}_2 \cdot \bar{a}_h) \neq \sigma(\lambda_2 \cdot \theta_2 \cdot a_h) .$$

Therefore, since $\bar{\lambda}_1 \cdot \bar{\theta}_1 \cdot \bar{a}_h = \bar{\lambda}_2 \cdot \bar{\theta}_2 \cdot \bar{a}_h$, we have

$$N \models \sigma(\bar{\theta}_1 \cdot \bar{a}_h) \neq \sigma(\bar{\theta}_2 \cdot \bar{a}_h).$$

It follows that

$$\tau^{(N_\alpha)}(\bar{\theta}_1 \cdot \bar{a}_h) = \tau^{(N_\alpha)}(\bar{\theta}_2 \cdot \bar{a}_h)$$

iff

$$\sigma^{(N)}(\bar{\theta}_1 \cdot \bar{a}_h) = \sigma^{(N)}(\theta_2 \cdot \bar{a}_h) ;$$

as required.

8.10. Corollary. *For every $\sigma(\bar{a}_h) \in \mathcal{T}^c$, there exists an $\alpha < \aleph_1^{(M)}$ and a term $\tau(\bar{a}_h) \in \mathcal{T}^c_\alpha$ such that for all $g \subseteq h$, we have ${}^\alpha H^g_\tau = H^g_\sigma$.*

The following lemma is useful in showing that certain sets in N already occur in an earlier stage model.

8.20. Lemma. *If $N \ni \boldsymbol{\sigma}(\bar{a}_h) \subseteq N_\alpha$, and there is a set of terms $A \in M$ such that every member of $\boldsymbol{\sigma}(a_h)$ is denoted by an element of A, and for every $\rho \in A$, $\|\rho \in \sigma\| \subseteq P_\alpha$, then $\boldsymbol{\sigma}(\bar{a}_h) \in N_\alpha$.*

Proof. Similarly to the proof of Lemma 8.00 we take:

$$\begin{aligned}\tau(\bar{a}_h) = (\lambda w)\,(\exists \rho \in A)\,(\exists p \in P_\alpha)\\ [p(\bar{a}_{h'}) \Vdash (\sigma(\bar{a}_{h''}) \in \sigma(\bar{a}_h) \wedge\\ \wedge (\exists s)\,\{\mathrm{dom}(s) \supseteq h \cup h' \cup h'' \wedge\\ \wedge s(j) = a_j,\, j \in h \cup h' \cup h'' \wedge\\ \wedge \tilde{\mathrm{sat}}_\alpha(\tilde{p}, s) = 1 \wedge\\ \wedge w = \mathrm{val}_\alpha(\rho, s)]\,\}.\end{aligned}$$

This is clearly a local term of $\mathcal{L}_\alpha$. It must be shown that $\boldsymbol{\tau}^{(N_\alpha)}(\bar{a}_h) = \boldsymbol{\sigma}^{(N)}(\bar{a}_h)$. If $x \in \tau^{(N_\alpha)}(\bar{a}_h)$, then for some $p(a'_h) \in P_\alpha$, $\rho(\bar{a}_{h''}) \in \mathcal{T}^c_\alpha$, and assignment s to $\bar{a}_{h \cup h' \cup h''}$,

$$p \Vdash (\rho)^{(N_\alpha)} \in \sigma, \qquad \tilde{\mathrm{sat}}_\alpha(p, s) = 1, \qquad \tilde{\mathrm{val}}_\alpha(\rho, s) = x.$$

By absoluteness, $x \in \boldsymbol{\sigma}$. If $x \in \boldsymbol{\sigma}$, then $x \in N_\alpha$ by assumption, and for some $p \in P$, and $\rho \in \mathcal{T}^c_\alpha$, $x = \mathrm{val}_\alpha(\rho) = \mathrm{val}((\rho)^{(N_\alpha)})$ and $p \Vdash (\rho)^{(N_\alpha)} \in \sigma$. Let $q \subseteq p$ be minimal for $(\rho)^{(N_\alpha)} \in \sigma$; then $q \in P_\alpha$; and again by absoluteness and the definition of τ, we clearly get $x = \mathcal{L}^{(N_\alpha)} \in \boldsymbol{\tau}^{(N_\alpha)}$.

8.30. Corollary. *If $\boldsymbol{\tau} \subseteq K(\bar{a}_h)$, then there is an $\alpha < \aleph_1^{(M)}$ such that $\boldsymbol{\tau} \in N_\alpha$.*

8.31. Corollary. *If $\boldsymbol{\tau} \subseteq H^g_\sigma$, then there is an $\alpha < \aleph_1^{(M)}$ such that $\boldsymbol{\tau} \in N_\alpha$.*

Proof. By Corollary 8.30 every element $\boldsymbol{\rho}$ of $\boldsymbol{\tau}$ is in some N_{α_ρ}, $\alpha_{\boldsymbol{\rho}} < \aleph_1^{(M)}$, and since $\boldsymbol{\tau}$ is countable, $\boldsymbol{\tau} \subseteq N_\alpha$, where $\alpha = \sup_{\boldsymbol{\rho} \in \boldsymbol{\tau}}(\alpha_{\boldsymbol{\rho}})$. Let A be the set of $\rho \in \mathcal{T}^c$ such that $\mathrm{rnk}(\rho) < \mathrm{rnk}(\tau)$. Since τ is countable, there exists an $\alpha^* \geqslant \alpha$ such that $\|\rho \in \tau\| \subseteq P_{\alpha^*}$ for every $\rho \in A$. The result now follows from Lemma 8.20.

Note that from Corollary 8.31 we can obtain an alternative proof of Lemma 8.00. We shall now prove Lemma 3.54, that $\mathrm{val}_\alpha(\sigma) \in N_\beta$ iff $N_\alpha \models N_\beta(\sigma)$, $\beta < \alpha$ (see Discussion in 3.53). We first prove:

8.40. Lemma. *If* $\mathrm{val}_\alpha(\sigma) \subseteq N_\beta$, $\beta < \alpha$, *then for some term* $\sigma^* \in \mathcal{T}^c_\beta$ *with* $\mathrm{rnk}(\sigma^*) \leqslant \mathrm{rnk}(\sigma) + \aleph_1^{(M)} \cdot 2$, *we have* $N_\alpha = \sigma = (\sigma^*)^{(N_\beta)}$.

Proof. The lemma will be proven by induction on $\mathrm{rnk}(\sigma)$. $\mathrm{val}_\alpha(\sigma) \in N_\beta$, thus for some $\tau \in \mathcal{T}^c_\beta$, $N_\alpha \models \sigma = (\tau)^{(N_\beta)}$. Hence for some $p^* \in Q_\alpha$, $p^* \Vdash_\alpha \sigma = (\tau)^{(N_\beta)}$. We claim that for any $\rho \in \mathcal{T}^c_\beta$ such that $\mathrm{stg}_\alpha(\rho) \subseteq \beta$,

(8.41) $\quad p^* \subseteq r \Vdash_\alpha \rho \in \sigma$ implies that $p^* \cup \tilde{r}^\beta \Vdash_\alpha \rho \in \sigma$.

Since $r \Vdash_\alpha \rho \in \sigma = (\tau)^{(N_\beta)}$, $\mathrm{stg}_\alpha(\rho) \leqslant \beta$, we get by the Lemmas 3.10 and 3.04, $\tilde{r}^\beta \Vdash_\beta \rho \in \tau$; but again, by these lemmas $\tilde{r}^\beta \Vdash_\alpha \rho \in (\tau)^{(N_\beta)}$. Hence $p^* \cup \tilde{r}^\beta \Vdash_\alpha \rho \in \sigma = (\tau)^{(N_\beta)}$, as required. Let $\mathrm{rnk}(\sigma) + \aleph_1^{(M)} \cdot 2 = \mu$, and note that for a term ρ, $\mathrm{rnk}(\rho) < \mu$, iff the rank of ρ as an element of $M \subseteq N_\beta$ is less than μ. Using this particular $p^*(a_{h^*}) \in Q_\alpha$, we take σ^* to be the following term:

$$\begin{aligned}
\sigma^* = (\lambda w)\, (\exists_{\aleph_1^{(M)}} p \in P_\beta)\, (\exists_\mu \rho \in \mathcal{T}^c_\beta)
&[\mathrm{stg}_\alpha(\rho) \leqslant \dot{\beta} \wedge \dot{p}^* \cup p(a_{h'}) \text{ is a condition } \wedge \\
&\quad \wedge\, \dot{p}^* \cup p \Vdash_\alpha \rho(a_{h''}) \in \sigma(a_h) \wedge \\
&\quad \wedge\, (\exists_{\omega \cdot 2} s)\, (\mathrm{dom}(s) = h \cup h^* \cup h' \cup h'' \cup \wedge \\
&\quad \wedge\, (\forall_\omega j \in h \cup h^* \cup h' \cup h'') \\
&\qquad \wedge\, (s(j) = a_j) \wedge \widetilde{\mathrm{sat}}_\beta(\hat{p}, s) = 1 = \text{truth} \wedge \\
&\qquad\qquad \wedge\, w = \widetilde{\mathrm{val}}_\beta(\rho, s))].
\end{aligned}$$

σ^* is clearly a local term of $\mathcal{L}_\beta$ and we wish to show that $N_\alpha \models (\sigma^*)^{(N_\beta)} = \sigma$. We first show that $N_\alpha \models (\sigma^*)^{(N_\beta)} \subseteq \sigma$; and since $N_\alpha \models \sigma = (\tau)^{(N_\beta)}$, it suffices to show that $N_\beta \models \sigma^* = \tau$. If $N_\beta \models \rho \in \sigma^*$, then by the definition of σ^*, and the absoluteness of the terms and notions involved, for some $\rho'(a_{h''}) \in \mathcal{T}^c_\beta$ with $\mathrm{stg}_\alpha(\rho) \leqslant \beta$ and $\mathrm{rnk}(\rho') < \mu$, and for some $p \in P_\beta$, $p^* \cup p \Vdash_\beta \rho' \in \sigma$ and $\mathrm{val}_\alpha(\rho) = \mathrm{val}_\alpha(\rho')$. Moreover, there is an assignment s to the generic real constants of $\sigma, \rho', p^*, \tilde{p}$, such that $s(j) = a_j$, $j \in h \cup h^* \cup h' \cup h''$, and $\mathrm{sat}_\beta(\tilde{p}, s) = 1$. Therefore $p \in Q_\beta$, whence

$$p^* \cup p \in Q_\alpha, \qquad p^* \cup p \Vdash_\alpha \rho' \in \sigma = (\tau)^{(N_\beta)}.$$

Thus $N_\alpha \models \rho' \in (\tau)^{(N_\beta)}$, which gives $N_\beta \models \rho' \in \tau$, hence also $N_\beta \models \rho \in \tau$; as required. In reverse, assume that $N_\alpha \models \rho \in \sigma$; then since $\mathrm{val}(\sigma) \in N_\beta$, also $\mathrm{val}(\rho) \in N_\beta$; moreover we may assume that $\mathrm{rnk}(\rho) < \mathrm{rnk}(\sigma)$. By the induction hypothesis there is a term $\rho^* \in \mathcal{T}^c_\beta$ such that $N_\alpha \models (\rho^*)^{(N_\beta)} = \rho$, and

$$\mathrm{rnk}(\rho^*) \leqslant \mathrm{rnk}(\rho) + \aleph_1^{(M)} \cdot 2 \leqslant \mathrm{rnk}(\sigma) + \aleph_1^{(M)} \cdot 2 = \mu .$$

Since we have $N_\alpha \models (\rho^*)^{(N_\beta)} \in \sigma$, there is by Definition 4.81 a $p \in Q_\beta$ such that

$$p^* \cup p \Vdash_\alpha (\rho^*)^{(N_\beta)} \in \sigma .$$

Hence by the definition of σ^*, $N_\beta \models (\rho^*)^{(N_\beta)} \in \sigma^*$, which implies that

$$N_\alpha \models (\rho^*)^{(N_\beta)} \in (\sigma^*)^{(N_\beta)} ;$$

hence also $N_\alpha \models \rho \in (\sigma^*)^{(N_\beta)}$; as required.

It remains to evaluate the rank of σ^*. Consider the rank of the individual terms. We have

$$\mathrm{rnk}(\dot{p}^*) < \aleph_1^{(M)}, \qquad \mathrm{rnk}(\dot{P}_\beta) < \aleph_1^{(M)},$$

(see Lemma 2.73); moreover, $u \Vdash_\alpha v \in \sigma$, $\mathrm{stg}_\alpha(u)$, $\widetilde{\mathrm{sat}}_\beta(u, v)$, $\widetilde{\mathrm{val}}_\beta(u, v)$, and $\mathcal{T}^c_\alpha$ are given by formulas and terms, of which the bounds of quantifiers, abstraction symbols, and terms do not exceed $\aleph_1^{(M)} + \omega$. This can be seen by reviewing the corresponding definitions that were given in the M-language and Lemma 2.73. Other quantifiers in σ^* necessary for saying that s is an assignment to the generic real constants occurring in ρ, p, p^*, σ such that $s(j) = a_j$, can obviously be bounded by $\omega \cdot 2$. Hence $\mathrm{rnk}(\sigma^*) < \mathrm{rnk}(\sigma) + \aleph_1^{(M)} \cdot 2$.

8.42. Lemmas 3.54 and 3.05 now easily follow from the preceding lemma and Definition 2.50. We omit the details.

9. In N, infinite sets are idemmultiple and orderable

We now come to the central theme of this construction, which may roughly be described as follows: using Lemma 5.44, every set $z \in N$ can be decomposed in N into a well-ordered sequence of disjoint sets, z_α, which are canonically equinumerous to subsets y_α of elementary tail partitions. With the help of the generic mappings, $\mathring{\chi}_e$, of the elementary tail partitions into generic tails, and I_0, we are able to construct in N an ordering of $\bigcup_\alpha y_\alpha \times \{\alpha\} = y$. If y is infinite, a bijection of y onto $y \times 2$ can also be constructed in N from

$$\mathring{I} = \{\langle e, \mathring{\chi}_e\rangle \mid e \text{ is a } \beta{+}1\text{-index } \beta < \aleph_1^{(M)}\}.$$

Our result will then follow from the fact that y is equinumerous to z, in N.

9.00. Definition. Let $g \subseteq h \subset \omega$ and define:

(a)(i) If $\bar{x} = \langle x_1, \ldots, x_n\rangle \in K(\bar{a}_h)$, then

$$\mathrm{proj}(\bar{x}, g) = \langle x_{i_1}, \ldots, x_{i_m}\rangle,$$

where $\{i_1, \ldots, i_m\} = g$;

(ii) If $Y \subseteq K(\bar{a}_h)$, then

$$\mathrm{proj}(Y, g) = \{x \mid (\exists y \in Y)\,(x = \mathrm{proj}(y, g))\};$$

(iii) If $Y \subseteq \mathcal{P}(K(\bar{a}_h))$, where $\mathcal{P}(\cdot)$ is the power set operation, then

$$\mathrm{proj}(Y, g) = \{z \mid (\exists \bar{z}' \in Y)\,(z = \mathrm{proj}(z, g))\};$$

$\mathrm{proj}\,(Y, g)$ is said to be the *g-projection of* Y.

(b) If $Y \subseteq \mathcal{P}(K(\bar{a}_h))$, then by the *g-section of Y based at* $x \subseteq K(\bar{a}_g)$, we mean the set

$$z = \{x' \in Y \mid \mathrm{proj}(x', g) = x\}.$$

z is said to be a *g-section* of Y.

It will be necessary to consider the following refinements of the equivalence relations ${}^\alpha E_\sigma$ and their induced partitions, ${}^\alpha H_\sigma$.

9.001. Definition. Let $\sigma(\bar{a}_h) \in \mathcal{T}^c_\alpha$, $\alpha \leqslant \aleph_1^{(M)}$ and $g \subseteq h$. $\bar{x}, \bar{y} \in K(\bar{a}_h)$ are said to be g-equal if $\mathrm{proj}(\bar{x}, g) = \mathrm{proj}(\bar{y}, g)$. This is clearly an equivalence relation and the conjunction of this relation with ${}^{\alpha}E_{\sigma}$ yields the refinement ${}^{\alpha}E^{g}_{\sigma}$ of ${}^{\alpha}E_{\sigma}$ defined by:

$$\bar{x}\,{}^{\alpha}E^{g}_{\sigma}\,\bar{y} \quad \text{iff} \quad \mathrm{proj}(\bar{x}, g) = \mathrm{proj}(\bar{y}, g) \text{ and } \sigma^{(N_\alpha)}[\bar{x}] = \sigma^{(N_\alpha)}[\bar{y}] .$$

The partition of $K(\bar{a}_h)$ induced by the equivalence relation ${}^{\alpha}E^{g}_{\sigma}$ is denoted by ${}^{\alpha}D^{g}_{\sigma}$. ${}^{\alpha}D^{g}_{\sigma}$ is said to be an *elementary g-partition.* Clearly if $g = \emptyset$ then ${}^{\alpha}D^{g}_{\sigma} = {}^{\alpha}H_{\sigma}$. If $\bar{y} \in K(\bar{a}_h)$, then ${}^{\alpha}D^{g}_{\sigma}[\bar{y}]$ will denote the component $\{\bar{z} \in K(\bar{a}_h) \mid \bar{z}\,{}^{\alpha}E^{g}_{\sigma}\,\bar{y}\}$ of ${}^{\alpha}D^{g}_{\sigma}$. On the other hand if $\bar{x} \in K(\bar{a}_g)$ then

$${}^{\alpha}D^{g}_{\sigma}[\bar{x}] = \{y \in {}^{\alpha}D^{g}_{\sigma} \mid \mathrm{proj}(y, g) = \{\bar{x}\}\} .$$

(${}^{\alpha}D^{g}_{\sigma}[\bar{x}]$ is a g-section of ${}^{\alpha}D^{g}_{\sigma}$ based at $\{\bar{x}\}$.

9.002. Lemma. *An elementary g-partition is an elementary tail partition, i.e., for every $\sigma(\bar{a}_h) \in \mathcal{T}^{c}_{\alpha}$, $\alpha \leqslant \aleph_1^{(M)}$ and $g \subseteq h$ there exists a term $\sigma_g(\bar{a}_h)$ such that ${}^{\alpha}H_{\sigma_g} = {}^{\alpha}D^{g}_{\sigma}$.*

Proof. Set $\sigma_g(\bar{a}_h) = \dot{\langle} \sigma(\bar{a}_h), \bar{a}_g \dot{\rangle} = \dot{\langle} \sigma, a_{g_1}, ..., a_{g_m} \dot{\rangle}$, $g = \{g_1, ..., g_m\}$, (see 2.26 and 2.27).

Since $\sigma_g(\bar{a}_h) = \langle \sigma, \bar{a}_g \rangle$, σ_g obviously induces the partitions ${}^{\alpha}D^{g}_{\sigma}$.

9.003. Definition. For $\sigma(\bar{a}_h) \in \mathcal{T}^{c}_{\alpha}$, $g, f \subseteq h$, ${}^{\alpha}D^{g,f}_{\sigma}$ will denote ${}^{\alpha}H^{f}_{\sigma_g}$ and for $\bar{x} \in K(\bar{a}_g)$, $\bar{y} \in K(\bar{a}_h)$, we denote

$${}^{\alpha}D^{g,f}_{\sigma}[\bar{y}] = {}^{\alpha}H^{f}_{\sigma_g}[\bar{y}] ,$$

$${}^{\alpha}D^{g,f}_{\sigma}[\bar{x}] = \{z \mid z \in {}^{\alpha}H^{f}_{\sigma_g}, \mathrm{proj}(z, g) = \{\bar{x}\}\} .$$

9.01. Definition. If Y is a partition, then a *subpartition of Y* is a set of the form

$$W = \{x \cap z \mid x \in Y\} .$$

This is obviously a partition of the set $(\bigcup Y) \cap Z$. W is also said to be a *subpartition of Y on Z.*

9.02. Definition. For any condition $p \in P$ denote:

(a)(i) ${}_{p}K(\bar{a}_h) = \{\bar{y} \in K(\bar{a}_h) \mid \bar{y}\ \exists\text{-satisfies } p\}$;

(ii) if $\bar{x} \in K(\bar{a}_g)$, then

$$_pK_{\bar{x}}(\bar{a}_h) = \{\bar{y} \in {}_pK(\bar{a}_h) \mid \mathrm{proj}(\bar{y}, g \cap h) = \mathrm{proj}(\bar{x}, g \cap h)\}\ .$$

(b) Let $\sigma(\bar{a}_h) \in \mathcal{T}^c$, $f, g \subseteq h$; then the subpartition of $D^{g,f}_\sigma$ on $_pK(\bar{a}_h)$ is said to be an *elementary p-sub-g-partition*, and is denoted by $_pD^{g,f}_\sigma$. For $\bar{x}, \bar{y} \in K(a_h)$, we denote

$$^\alpha_pD^{g,f}_\sigma[\bar{y}] = {}^\alpha D^{g,f}_\sigma[\bar{y}] \cap {}_pK(a_k)\ ,$$

$$^\alpha_pD^{g,f}_\sigma[\bar{x}] = \{z \cap {}_pK(a_h) \mid z \in {}^\alpha D^{g,f}_\sigma[\bar{x}]\}\ .$$

If $f = \emptyset$ we omit it, and if $\alpha = \aleph^{(M)}_1$ we also omit it.

9.021. Notation. If $\bar{a}_h$ are all the generic real constants occurring in the terms $\bar{\sigma}$ or mentioned by the conditions $\bar{P}$, then $K(\bar{\sigma}, \bar{p})$ will denote $K(\bar{a}_h)$.

9.03. Definition. Let g be a finite subset of ω; then g is said to *support* the term τ if $\tau = \tau(\bar{a}_g)$. $x \in N$ is said to be *supported by* g if there exists a term $\tau(\bar{a}_h) \in \mathcal{T}^c$, such that $x = \mathrm{val}(\tau(\bar{a}_h))$ and $h \subseteq g$.

9.04. Remark. Note that this does not comply with common usage since here g is not unique.

9.05. Remark. Note also that by the standard Skolem–Löwenheim arguments, if $x \in N$ is definable in the global language from members supported by g, then x is supported by g.

9.10. Definition. $\Omega_{g,\bar{x}} = \{_pD^g_\sigma[\bar{x}] \mid \sigma(\bar{a}_h) \in \mathcal{T}^c, p \in P, g \subseteq h\}$, where $g = \{0, 1, ..., n\}, \bar{x} \in K(\bar{a}_g)$.

9.20. Fundamental representation lemma. *For every initial segment g of ω and $\bar{x} \in K(\bar{a}_g)$, there is a function Φ_g in N with domain $\Omega_{g,\bar{x}}$ such that for every $z \in \Omega_{g,\bar{x}}$,*

$$\Phi_g(z) = (z', z'', \Sigma_{z'}, \mathrm{WO}_{z''}, \mathrm{O}_z)\ ,$$

where $z = z' \cup z''$, $z' \cap z'' = \emptyset$, and $\Sigma_{z'}$ is a bijection of $z' \times 2$, $\mathrm{WO}_{z''}$ is a well-ordering of z'', and O_z is an ordering of z.

We delay the proof of this lemma and show how to derive from it a proof of the following main lemma.

9.30. Lemma. *Every $z \in N$ is orderable in N; and if z is infinite, then z is idemmultiple in N.*

Proof. $z \in N$; hence for some $\tau(\bar{a}_g) \in \mathcal{T}^c$, $z = \mathrm{val}(\tau)$. It can be assumed that $g = \{0, 1, ..., n\}$ is an initial segment of ω, (for we can always insert a_i, $0 \leqslant i \leqslant n$ in any term in such a way as to yield a term with the same value).

Our first objective will be to obtain a natural decomposition of z in N, into sets z_α where

$$\bigcup_\alpha z_\alpha = z, \qquad z_\alpha \cap z_\beta = \emptyset, \quad \alpha \neq \beta,$$

such that each element z_α has a natural bijection ψ_α of z_α into some element of $\Omega_{g,\bar{a}_g}$. The manner in which this is done is based on the following idea. Assume $y = \mathrm{val}(\sigma(\bar{a}_h)) \in z = \tau$ and, as mentioned above, we may assume that $g \subseteq h$; then for some $p \in Q$, $p \Vdash \sigma \in \tau$. Now by the symmetry lemma, if $\bar{x} \in {}_pK_{\bar{a}_g}(\sigma, \tau)$ $\exists$-satisfies p, then $\sigma[\bar{x}] \in \mathrm{val}(\tau) = \tau[\bar{x}]$. Set

$$z' = \{\sigma[\bar{x}] \mid \bar{x} \in {}_pK_{\bar{a}_g}(\sigma, \tau)\} \subseteq z\,.$$

z' has a natural bijection ψ' onto ${}_pD^g_\sigma(\bar{a}_g)$. The required decomposition is to be obtained by repeating the above process, considering at each state the remaining part of z, until z is completely exhausted. The entire argument is to take place in N. Since there is a definition in N of all terms and notions concerning the sequence $\mathcal{L}_\alpha, \mathcal{F}_\alpha, \Vdash_\alpha, \alpha \leqslant \aleph_1^{(M)}$, there is a standard well-ordering of all pairs (q, σ), $q \in P$, $\sigma \in \mathcal{T}^c$. Let $\mathrm{rnk}(\tau) = \lambda$. z is to be decomposed, in N, using this well ordering, by defining in N the following sequence:

$$B(\beta) = (q_\beta, \sigma_\beta, \rho_\beta, \tau_\beta), \qquad \beta < \alpha \leqslant (|\lambda|^+)^{(M)} + \aleph_1^{(M)},$$

where q_β is a condition, σ_β is a term with $\mathrm{rnk}(\sigma) < \lambda$, and ρ_β, τ_β are terms such that $\mathrm{val}(\rho_\beta) = {}_{q_\beta}K_{\bar{a}_g}(\sigma_\beta)$, $\mathrm{val}(\tau_\beta) \subseteq \tau$. Assume that $B(\beta)$ has been defined for all $\gamma < \beta$; we define $B(\beta)$, $\beta < \alpha$. If there are no pairs q, $\sigma(\bar{a}_h)$ such that $\mathrm{rnk}(\sigma) < \lambda$, $g \subseteq h$, and $q \Vdash \sigma \in \tau$, and for all $\gamma < \beta$, $q \Vdash \neg(\sigma \in \tau_\beta)$, then B is to be undefined for ordinals $\gamma \geqslant \beta$. If there are such pairs, let (q_β, σ_β) be the first according to the standard well-ordering. Let $\sigma_\beta = \sigma_\beta(\bar{a}_{h_\beta})$ and define:

$$\rho_\beta = (\lambda \bar{w})\,(\exists_{\omega\cdot 2} s)\,[\mathrm{dom}(s) \subset \omega \wedge \bigwedge_{i \in \mathrm{dom}(s)} s(i) \in K(a_i) \wedge$$

$$\wedge \text{ ("}s\text{ is an assignment to the generic real constants mentioned in } \sigma_\beta, \tau, q_\beta\text{")} \wedge$$

$$\wedge \bigwedge_{j \in h_\beta} s(j) = w_j \wedge \bigwedge_{j \in g} s(j) = a_j \wedge$$

$$\wedge\ \widetilde{\mathrm{sat}}(q_\beta, s) = 1 \text{ (truth)}].$$

ρ_β is clearly a local term of $\mathcal{L}$. Let $h_\beta = \{i_1, ..., i_m\}$, and set

$$\tau_\beta = (\lambda w)\,(\exists_{\omega\cdot 2} s)\,[\mathrm{dom}(s) = h_\beta \wedge \dot{\langle} s(i_1), ..., s(i_m)\dot{\rangle} \in \rho_\beta \wedge$$

$$\wedge\ \widetilde{\mathrm{val}}(\sigma_\beta, s) = w].$$

Define $B(\beta) = (q_\beta, \sigma_\beta, \rho_\beta, \tau_\beta)$. Clearly, B is a well defined sequence in N of length $\alpha < (|\lambda|^+)^{(M)} + \aleph_1^{(M)}$, because there are at most $|\lambda|^{(M)}$ terms of rnk less than λ, and at most $\aleph_1^{(M)}$ conditions in P; moreover no pair can occur twice in the sequence. It is noteworthy that B is even defined in M. Let

$$\mathrm{val}(\tau_\beta) = z_\beta, \qquad z^* = \bigcup_{\beta<\alpha} z_\beta \in N$$

($z^* = \mathrm{val}(\tau^*)$, where

$$\tau^* = (\lambda w)\,(\exists u)\,(\exists v_1 v_2 v_3 v_4)\,[u \in \dot{\alpha} \wedge B(u) = \langle v_1, v_2, v_3, v_4\rangle \wedge$$

$$\wedge\ w \in \widetilde{\mathrm{val}}(v_4, \bar{a}_g)].)$$

An examination of all the terms $\rho_\beta, \tau_\beta, \tau^*$ shows that they are all supported by g.

We first prove, $z_\beta \subseteq z$, hence also $z^* \subseteq z$. If $y \in \mathrm{val}(\tau_\beta)$, then by absoluteness, for some $\bar{x} \in K(q_\beta, \sigma_\beta, \tau)$ such that $\mathrm{proj}(\bar{x}, g) = \bar{a}_g$, and $\bar{x}$ satisfies q_β, we have $y = \sigma_\beta[\bar{x}]$. Since $q_\beta \Vdash \sigma_\beta \in \tau$, we get by the symmetry lemma that $y = \sigma_\beta[\bar{x}] \in \tau(\bar{a}_g)$, as required.

Next, we show that,

$$z_\gamma \cap z_\beta = \emptyset, \quad \gamma < \beta < \alpha.$$

If $y \in z_\beta$, then, as above, $y = \sigma_\beta[\bar{x}]$ for some $\bar{x} \in K(q_\beta, \sigma_\beta, \tau)$ such that

$\mathrm{proj}(\bar{x}, g) = \bar{a}_g$ and $\bar{x}$ satisfies q_β. $q_\beta \Vdash \sigma_\beta \notin \tau_\gamma$, by definition; hence by the symmetry lemma,

$$y = \sigma_\beta[\bar{x}] \notin \tau_\gamma[\bar{x}].$$

Since τ_β is supported by g, i.e. $\tau = \tau(\bar{a}_g)$, we get

$$y = \sigma_\beta[\bar{x}] \notin z_\gamma = \mathrm{val}(\tau_\gamma),$$

as required.

We claim that $\bigcup_{\beta<\alpha} z_\beta = z$. If not, let $y \in z - \bigcup_{\beta<\alpha} z_\beta$, and assume that $y = \mathrm{val}(\sigma)$, where $\mathrm{rnk}(\sigma) < \lambda$. As previously mentioned we may also assume that $\sigma = \sigma(\bar{a}_h)$, where $g \subseteq h$. Then for some $q \in Q$,

$$q \Vdash \sigma \in \tau, \qquad q \Vdash \sigma \notin \tau,$$

i.e., $q \Vdash \sigma \notin \tau_\beta$, for all $\beta < \alpha$. But by our assumptions B terminated at the α'th step; hence such a (q, σ) cannot exist.

It is obvious from the definitions that

$$\mathrm{val}(\rho_\beta) = {}_{q_\beta}K_{\bar{a}_g}(\sigma_\beta) = {}_{q_\beta}K_{\bar{a}_g}(\bar{a}_{h_\beta}),$$

and that the partition induced by σ_β on ${}_{q_\beta}K_{\bar{a}_g}(\sigma_\beta)$ is denoted by ${}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g)$. We shall define in N a sequence of functions ψ_β, $\beta < \alpha$ mapping z_β onto ${}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g)$. For any $y \in \tau_\beta$, there exists a member $\bar{x}$ of ρ_β such that $y = \sigma_\beta[\bar{x}]$. Define

$$\psi_\beta(y) = {}_{q_\beta}D^g_{\sigma_\beta}[\bar{x}] \in {}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g).$$

ψ_β is univalent because for any $\bar{x}, \bar{y} \in \rho_\beta$,

$$\sigma_\beta[\bar{x}] = \sigma_\beta[\bar{y}] \quad \text{iff} \quad {}_{q_\beta}D^g_{\sigma_\beta}[\bar{x}] = {}_{q_\beta}D^g_{\sigma_\beta}[\bar{y}].$$

ψ_β is obviously surjective. Note that the functions ψ_β and the mapping $\beta \to \psi_\beta$, $\beta < \alpha$ are all supported by g.

${}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g) \in \Omega_{g,\bar{a}_g}$, and by Lemma 9.20 there is a canonic representation of each ${}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g)$, as a disjoint union,

$${}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g) = w'_\beta \cup w''_\beta,$$

with canonic univalent maps Σ_β of $w'_\beta \times 2$ onto w'_β, and well-orderings WO_β of w''_β, and orderings O_β of ${}_{q_\beta}D^g_{\sigma_\beta}(\bar{a}_g)$. Then, via ψ_β, we obtain a canonic representation of the z_β, $\beta < \alpha$,

$$z_\beta = z'_\beta \cup z''_\beta, \quad z'_\beta \cap z''_\beta = \emptyset$$

with univalent maps $\Sigma_{z'_\beta}$ of $z'_\beta \times 2$ onto z'_β, and well-orderings $\mathrm{WO}_{z''_\beta}$ and canonic orderings O_{z_β} of z_β. We simply take

$$z'_\beta = \psi_\beta^{-1}(w'_\beta), \qquad z''_\beta = \psi_\beta^{-1}(w''_\beta)$$

and define:

$$\Sigma_{z'_\beta}(\langle x, \delta\rangle) = y \quad \text{iff} \quad \Sigma_\beta(\langle \psi(x), \delta\rangle) = \psi(y)\,,$$

$$x \leqslant_{\mathrm{WO}_{z''_\beta}} y \quad \text{iff} \quad \psi(x) \leqslant_{\mathrm{WO}_\beta} \psi(y)\,,$$

$$x \leqslant_{\mathrm{O}_{z_\beta}} y \quad \text{iff} \quad \psi(x) \leqslant_{\mathrm{O}_\beta} \psi(y)\,.$$

Define

$$z' = \bigcup_{\beta<\alpha} z'_\beta, \qquad z'' = \bigcup_{\beta<\alpha} z''_\beta$$

Then $z = z' \cup z''$ and $z' \cap z'' = \emptyset$; moreover,

$$z'_\gamma \cap z'_\beta = \emptyset, \qquad z''_\gamma \cap z''_\beta = \emptyset, \quad \gamma < \beta < \alpha.$$

We claim:

(a) $(|z' \times 2|)^{(N)} = (|z'|)^{(N)}$; this follows directly from the fact that the z'_β are disjoint; hence $\Sigma = \bigcup_{\beta<\alpha} \Sigma_{z'_\beta}$ is a univalent map in N of $z' \times 2$ onto z'.

(b) z'' has a well-ordering in N; this also follows easily from the fact that the z''_β are disjoint, because a well-ordering can be defined on z' as follows: if $x, y \in z''$, say $x \in z''_\beta, y \in z''_\gamma$, then

$$x \leqslant y \quad \text{iff} \quad [\beta < \gamma \vee (\beta = \gamma \wedge x \leqslant_{\mathrm{WO}_{z''_\beta}} y)]\,.$$

This is clearly a well-ordering since any subset of z'' obviously has a first element according to $\leqslant$. We distinguish the following cases:

(i) If $z' = \emptyset$, then z is well-ordered; hence if z is infinite, it is idemmultiple.

(ii) If $z' \neq \emptyset$ and z'' is infinite, then $z = z' \cup z''$ is idemmultiple, since in this case there also is a bijection of $z'' \times 2$ onto z''.

(iii) if $z' \neq \emptyset$ and z'' is finite, then z' is infinite (since z is); hence *by a single choice of an element of* z' we can map $z' \cup z'' = z$ univalently onto z', in which case z is idemmultiple.

To see this last point, choose an element $x \in z'$; we shall define by induction from Σ and x an infinite sequence of different elements of z', $x_1, x_2, \ldots$ as follows:

$$\text{if } \Sigma(\langle x,0\rangle) \neq x, \quad \text{let } x_1 = \Sigma(\langle x,0\rangle)\,;$$

$$\text{if } \Sigma(\langle x,0\rangle) = x, \quad \text{let } x_1 = \Sigma(\langle x,1\rangle)\,;$$

then $x_1 \neq x$ since Σ is univalent. Assume $x_1, \ldots, x_n$ have been defined and let

$$R_n = \{\Sigma(\langle x_i, \delta\rangle) \mid 1 \leqslant i \leqslant n,\ \delta < 2\}\,.$$

R_n has $2n$ elements because, by the induction hypothesis, the x_i, $1 \leqslant i \leqslant n$, are different and Σ is univalent. Choose the first $\langle i, \delta\rangle$, (according to any natural ordering of $\omega \times 2$), such that

$$\Sigma(\langle x_i, \delta\rangle) \notin \{x_1, \ldots, x_n\}\,,$$

and set $x_{n+1} = \Sigma(\langle x_i, \delta\rangle)$. That such an $\langle i, \delta\rangle$ exists follows from the above remark that R_n has $2n$ elements. Note that $x_1, \ldots, x_{n+1}$ are all different; thus the required sequence X has been defined. Let Y be the elements of the sequence. z'' is finite by assumption, hence there is a bijection of $z'' \cup Y$ onto Y; therefore there is a bijection of $z' \cup z''$ onto z'; and since z' is idemmultiple, so is z.

That z is orderable is also an easy consequence of the representation:

$$z = \bigcup_{\beta<\alpha} z_\beta\,, \qquad z_\gamma \cap z_\beta = \emptyset, \quad \gamma < \beta,$$

with O_{z_β} ordering z_β. We can order z as follows: for any $x, y \in z$, say $x \in z_\beta, y \in z_\gamma$:

$$x \leqslant_{\mathrm{O}_z} y \quad \text{iff} \quad (\beta < \gamma) \vee (\beta = \gamma \wedge x <_{\mathrm{O}_{z_\beta}} y)\,.$$

This is an ordering of z in N. This completes the proof of the main lemma, on the basis of Lemma 9.20.

9.4. The following notions and their elementary properties will be needed to prove one of our key lemmas.

9.40. Definition. Let K be a tail; then for any $x \in K$, *the branch of x in K*, denoted branch(x), is the set:

$$\{y \in K \mid (\exists l \in \omega)\,[\mathrm{slash}_l(y) = x]\} = \{y \in K \mid (\exists t \in 2^{\omega})\,[\mathrm{append}_t(x) = y]\}\,.$$

9.41. Lemma. branch(x) *has a natural well-ordering defined from* x.

Proof. Let $\prec^*$ be the natural well-ordering of $\bigcup_{l<\omega} 2^l$, (i.e., $t_1 \prec^* t_2$ iff t_1 is shorter than t_2, or they are of the same length; and if j is the first integer such that $t_1(j) \neq t_2(j)$, then $t_2(j) > t_1(j)$). Define:

$$\text{append}_{t_1}(x) \leqslant^* \text{append}_{t_2}(x) \quad \text{iff} \quad t_1 \prec^* t_2 .$$

This is clearly a well-ordering of branch(x).

9.42. Lemma. (a) $\text{append}_t(x) = y$ *implies* $\text{branch}(y) \subseteq \text{branch}(x)$.

(b) *If* x, y *are incomparable with respect to the natural partial order of* K, (i.e., $x \leqslant y$ *iff* $(\exists l < \omega)\,\{\text{slash}_l(x) = y \text{ or } \text{slash}_l(y) = x\}$), *see Definition* 1.4), *then*

$$\text{branch}(x) \cap \text{branch}(y) = \emptyset.$$

Proof. (a) If $\text{append}_t(x) = y$ and $z \in \text{branch}(y)$, then for some t', $\text{append}_{t'}(y) = z$; thus

$$z = \text{append}_{t'}(\text{append}_t(x)) = \text{append}_{t''}(x),$$

where $t \in 2^l$, $t' \in 2^{l'}$ and $t'' \in 2^{l+l'}$ are such that

$$t''(j) = \begin{cases} t(j), & j < l; \\ t'(j-l), & l \leqslant j < l+l'. \end{cases}$$

(b) If x, y are incomparable, then for no $l < \omega$ does $\text{slash}_l(x) = y$ or $\text{slash}_l(y) = x$.

$$z \in \text{branch}(x) \cap \text{branch}(y)$$

implies that for some $t_1 \in 2^{l_1}$, $t_2 \in 2^{l_2}$,

$$\text{append}_{t_1}(x) = z, \qquad \text{append}_{t_2}(y) = z.$$

Thus $\text{slash}_{l_1}(z) = x$ and $\text{slash}_{l_2}(z) = y$. If $l_2 \geqslant l_1$, then $\text{slash}_{l_2 - l_1}(x) = y$; and if $l_1 \geqslant l_2$, then $\text{slash}_{l_1 - l_2}(y) = x$. It follows that x, y are comparable; a contradiction. Hence,

$$\text{branch}(x) \cap \text{branch}(y) = \emptyset.$$

The set of all branch(x), $x \in K$ is commonly taken to be a family of basic open sets for a topology on K. In keeping with topological terminology, we say that:

9.43. Definition. A set s included in a tail K is *dense in* K if for all $x \in K$, $s \cap \operatorname{branch}(x) \neq \emptyset$.

The following simple but essential fact was recognized at an early stage of this research. It was precisely this point, in conjunction with the realization that a generic subset of a tail must be dense (see Lemma 9.45), that led the author to the present solution of this problem.

9.44. Lemma. *If s is dense in a tail K, then there exists a bijection of $s \times 2$ onto s definable from s.*

Proof. We first define a univalent map ψ from $s \times 2$ into s. Let $t_0 = \{\langle 0, 0\rangle\}$, and $t_1 = \{\langle 0, 1\rangle\}$. Since

$$s \cap \operatorname{branch}(\operatorname{append}_{t_\delta}(x)) \neq \emptyset$$

for any $x \in K$, $\delta < 2$, we can define $\psi(x, \delta)$ to be the first element of $s \cap \operatorname{branch}(\operatorname{append}_{t_\delta}(x))$ according to the natural well-ordering of K defined above, $x \in s$, $\delta < 2$. That ψ is univalent is seen as follows: if $x, y \in s$ are incomparable, then obviously $\psi(x, \delta) \neq \psi(y, \delta')$, (because

$$\operatorname{branch}(\operatorname{append}_{t_\delta}(x)) \subset \operatorname{branch}(x)$$

is disjoint from

$$\operatorname{branch}(\operatorname{append}_{t_{\delta'}}(y)) \subset \operatorname{branch}(y)\,,$$

by Lemma 9.42). If $x \neq y \in s$ are comparable, say $x = \operatorname{slash}_l(y)$, $l > 0$, then $\psi(x, \delta) \notin \operatorname{branch}(\operatorname{append}_{t_{\delta'}}(y))$; hence $\psi(x, \delta) \neq \psi(y, \delta')$. If $x = y$ and, $\delta \neq \delta'$, the result again follows from the fact that the corresponding branches are disjoint. There is a trivial univalent map ψ^* from s into $s \times 2$, namely,

$$\psi^*(x) = \langle x, 0\rangle, \quad x \in s.$$

ψ and ψ^* have been defined from s. Now, by the Cantor–Bernstein theorem and its proof, there exists a bijection of $s \times 2$ onto s defined from ψ and ψ^*, hence from s; as required.

9.45. Lemma. *Assume that*

$$\sigma(\bar a_h) \in \mathcal{T}^{\circ}_{\alpha}, \quad \alpha < \aleph_1^{(M)}, \quad g \subseteq f \subseteq h,$$

$$z \subseteq {}^{\alpha}D^{g,f}_{\sigma}(\bar{\theta}\cdot\bar a_h), \quad z \in N_\alpha$$

and that z is infinite. Let $e = (h, f, s^{\alpha}_{\sigma_g})$; *then* $\mathring{\chi}_e(z)$ *is dense in the tail* $K(\chi_e(\bar{\theta}\cdot\bar a_h))$.

Proof. That $\mathring{\chi}_e(z) \subseteq K(\chi_e(\bar{\theta}\cdot\bar a_h))$ follows from Lemmas 4.73, 4.86. We claim that for every $x \in K(\chi_e(\bar{\theta}\cdot\bar a_h))$,

$$\mathring{\chi}_e(z) \cap \text{branch}(x) \neq \emptyset.$$

If not, let $z = \text{val}_\alpha(\tau)$ and $x = \theta(\chi_e(\bar{\theta}\cdot\bar a_h))$ be such that for some $p \in Q_{\alpha+1} = Q \cap P_\alpha$,

$$\begin{aligned}(9.451)\quad p \Vdash_{\alpha+1} \mathring{\chi}_e((\tau)^{(N_\alpha)}) \cap \text{branch}(\theta(\chi_e(\bar\theta\cdot\bar a_h)) = \dot{\emptyset} \wedge \\ \wedge |(\tau)^{(N_\alpha)}| = \aleph_0 \wedge \\ \wedge (\tau)^{(N_\alpha)} \subseteq {}^{\alpha}D^{g,f}_{\sigma}(\bar\theta\cdot\bar a_h).\end{aligned}$$

By Lemmas 7.30 and 9.002, ${}^{\alpha}D^{g,f}_{\sigma}(\bar\theta\cdot\bar a_h)$ is a term in $\mathcal{L}_\alpha$, and the notions of nearly-(l)-definable$_\alpha$, nearly-$(-l)$-definable$_\alpha$ and their negations are definable in $\mathcal{L}_\alpha$, and are absolute.

By Lemma 3.10,

$$\hat{p}^{\alpha} \Vdash_\alpha |\tau| = \aleph_0 \wedge \tau \subseteq D^{g,f}_{\sigma}(\bar\theta\cdot\bar a_h).$$

Let $\bar{\theta}^*_1\cdot\bar a_h, \ldots, \bar{\theta}^*_m\cdot\bar a_h$ be all reals such that $\chi_e(\bar{\theta}^*_j\cdot\bar a_h)$, $1 \leqslant j \leqslant m$, are mentioned in p. Then there is a $\bar\theta^* \in \bar\Gamma$ such that for some $q^* \in Q_\alpha$, $q^* \supseteq \hat{p}^{\alpha}$,

$$(9.452)\quad q^* \Vdash_\alpha \bigwedge_{j=1}^{m} (H_{\sigma_g}(\bar\theta^*_j\cdot\bar a_h) \neq H_{\sigma_g}(\bar\theta^*\cdot\bar a_h)) \wedge H_{\sigma_g}(\bar\theta^*\cdot\bar a_h) \in \tau.$$

Since ${}^{\alpha}H_{\sigma_g}(\bar{\theta}^*\cdot\bar a_h) \in \tau \subseteq {}^{\alpha}D^{g}_{\sigma}(\bar{\theta}\cdot\bar a_h)$, we can assume that q^* forces$_\alpha$ $H_{\sigma_g}(\bar\theta^*\cdot\bar a_h)$ to be, or not to be $(-l)$-definable$_\alpha$ depending upon $l \in f$. Thus by absoluteness, $q^* \cup p$ is prepared for $\chi_e(\bar\theta^*\cdot\bar a_h)$. By Lemma 4.86 we know that $\mathring{\chi}_e$ maps onto $K(\chi_e(\bar{\theta}\cdot\bar a_h))$; hence for some $\bar\theta' \in \bar\Gamma$, let

$$x = \theta(\chi_e(\bar{\theta}\cdot\bar a_h)) = \chi_e(\bar{\theta}'\cdot\bar a_h);$$

we can also assume that $p \Vdash_{\alpha+1} \chi_e(\bar{\theta}'\cdot\bar{a}_h) = \theta(\chi_e(\bar{\theta}\cdot\bar{a}_h))$. By Lemma 4.40,

$$q^* \cup p \cup \{(e, \bar{\theta}^*, l, \bar{\theta}', 0)\}$$

can be extended to a condition $\bar{p} \in P_{\alpha+1}$ for sufficiently large l. This yields a contradiction to (9.451), because $p \subseteq \bar{p}$, and we must have, (for instance by Corollary 2.912), that

$$\begin{aligned} \bar{p} \Vdash_{\alpha+1} {}^{\alpha}H_{\sigma_g}(\bar{\theta}^*\cdot\bar{a}_h) &\in (\tau)^{(N_\alpha)} \wedge \\ &\wedge \mathring{\chi}_e({}^{\alpha}H_{\sigma_g}(\bar{\theta}^*\cdot\bar{a}_h)) \in \mathrm{branch}(\chi_e(\bar{\theta}'\cdot\bar{a}_h)) \wedge \\ &\wedge \mathrm{branch}(\chi_e(\bar{\theta}'\cdot\bar{a}_h)) = \mathrm{branch}(\theta(\chi_e(\bar{\theta}\cdot\bar{a}_h))). \end{aligned}$$

Thus $\mathring{\chi}_e(z)$ is dense in $K(\chi_e(\bar{\boldsymbol{\theta}}\cdot\bar{a}_h))$, as was to be shown.

This lemma has the following important corollary:

9.50. Corollary. *Assume that $\sigma(\bar{a}_h) \in \mathcal{T}^c_\alpha$, $\alpha < \aleph_1^{(M)}$, and that*

$$z \subseteq {}^{\alpha}D^{g,f}_{\sigma}(\bar{\boldsymbol{\theta}}\cdot\bar{a}_h)\,,$$

where $g, f \subseteq h$, $z \in N_\alpha$ and z is infinite; then there is a bijection $\psi = \psi_{z,\sigma,f,g,\alpha,\boldsymbol{I}}$ of $z \times 2$ onto z defined in N from z, σ, f, g, α and $\boldsymbol{I}$.

Proof. Let $e = (h, f, s^{\alpha}_{\sigma_g})$. Then by Lemma 9.45, $\mathring{\chi}_e(z)$ is dense in $K(\chi_e(\bar{\boldsymbol{\theta}}\cdot\bar{a}_h))$. By Lemma 9.44, there is a bijection φ of $s \times 2$ onto s, defined from $\mathring{\chi}_e(z)$. The required function can now be defined as follows: for every $x \in z$, $\epsilon < 2$,

$$\psi(x, \epsilon) = \mathring{\chi}_e^{-1}(\varphi(\mathring{\chi}_e(x), \epsilon))\,.$$

ψ is univalent since $\mathring{\chi}_e$ and φ are; and since φ maps onto $\mathring{\chi}_e(z)$, ψ maps onto z. φ is defined from $\mathring{\chi}_e(z)$; and since $\mathring{\chi}_e$ is defined from σ, g, f, α and $\boldsymbol{I}$, ψ is defined from z, σ, g, f, α and $\boldsymbol{I}$.

It has been shown that,

9.51. Corollary. *There is a class function Δ defined in N such that*

$$\begin{aligned} \mathrm{dom}(\Delta) = \{(z, \sigma, g, f, \alpha) \mid \alpha < \aleph_1^{(M)}, \sigma \in \mathcal{T}^c_\alpha, z \in N_\alpha,& \\ \sigma = \sigma(\bar{a}_h), g, f \subseteq h,& \\ z \textit{ is a subset of } {}^{\alpha}D^{g,f}_{\sigma}\};& \end{aligned}$$

$$\Delta(z, \sigma, g, f, \alpha) = (\Sigma_z, \leqslant_z),$$

where Σ_z is a bijection of $z \times 2$ onto z if z is infinite, and $\Sigma_z = \emptyset$ otherwise, and $\leqslant_z$ is an ordering of z.

Proof. If z is infinite, Σ_z can be taken as the ψ supplied by the previous lemma. If $e = (h, f, s^{\alpha}_{\sigma_g})$, then $\mathring{\chi}_e$ is defined on z; and we define $\leqslant_z$ by $x \leqslant_z y$ iff $\mathring{\chi}_e(x) \leqslant \mathring{\chi}_e(y)$, where $x, y \in z$ and $\leqslant$ is the natural ordering of the reals.

The following representation lemma is fundamental.

9.60. Lemma. *For every finite $g \subset \omega$, there exists a function $\Lambda_g \in N$ whose domain U consists of all the subsets of g-sections of elementary tail g-partitions in N, (i.e., $z \in$ dom Λ_g iff z is a subset of some $D^g_\sigma(\bar{\theta}\cdot\bar{a}_g)$, $\sigma(\bar{a}_h) \in \mathcal{T}^c$ (see Definition 9.02(b)); and*

$$\Lambda_g(z) = \langle z', z'', \Sigma_{z'}, \mathrm{WO}_{z''}, \mathrm{O}_z\rangle,$$

where $z' \cup z'' = z$, $z' \cap z'' = \emptyset$, $\Sigma_{z'}$ is a bijection of $z' \times 2$ onto z', $\mathrm{WO}_{z''}$ is a well-ordering of z'', and O_z is an ordering of z. (z' and $\Sigma_{z'}$ may be empty.)

Proof. Assume $z \subseteq D^g_\sigma(\bar{\theta}\cdot\bar{a}_g)$, $\sigma(\bar{a}_h) \in \mathcal{T}^c$. For each $f \subseteq h$ let $z_f = z \cap D^{g,f}_\sigma$. By Lemmas 8.00 and 9.002 there exists an $\alpha < \aleph_1^{(M)}$ and $\tau(\bar{a}_n) \in \mathcal{T}^c_\alpha$ such that $D^{g,f}_\sigma = H^f_\tau$, hence

$$z_f \subseteq D^{g,f}_\sigma(\bar{\theta}\cdot\bar{a}_g) \subseteq {}^{\alpha}H^f_\tau .$$

By Lemma 9.51 there is a pair $(\Sigma_f, \leqslant_f)$ such that if z_f is infinite then Σ_f maps $z \times 2$ onto z and $\Sigma_f = \emptyset$ otherwise, and $\leqslant_f$ orders z_f. Define

$$z' = \bigcup\{z_f \mid f \subseteq g,\ z_f \text{ is infinite}\},$$

$$z'' = z - z'.$$

Set

$$\Sigma_{z'} = \bigcup\{\Sigma_f \mid |z_f| = \aleph_0\}.$$

Clearly $\Sigma_{z'}$ is a bijection of $z' \times 2$ onto z'. Let $\prec$ be a natural ordering of $\mathcal{P}(h)$ and define $\mathrm{WO}_{z''}$ as follows: for $x, y \in z''$

$$x\,\mathrm{WO}_{z''}\,y \quad \text{iff} \quad (x, y \in z_f \text{ and } x \leqslant_f y) \text{ or } (x \in z_{f'} \text{ and } y \in z_{f''} \text{ and } f' \prec f'')$$

$\mathrm{WO}_{z''}$ is clearly a well-ordering of z''. To define O_z, let $e_f = (h, f, s_\tau^\alpha)$, then $\chi = \bigcup_{f \subseteq h} \mathring{\chi}_{e_f}$ is a univalent function defined on $D_\sigma^g = {}^\alpha H_\tau$. Therefore we can define for $x, y \in z$,

$$x \,\mathrm{O}_z\, y \quad \text{iff} \quad \chi(x) \leqslant \chi(y),$$

where $\leqslant$ is the natural ordering of the reals. We have thus defined

$$\Lambda_g(z) = \langle z', z'', \Sigma_{z'}, \mathrm{WO}_{z''}, \mathrm{O}_z \rangle$$

for every $z \in U$ as required.

9.70. Corollary. *For every finite* $g \subseteq \omega$, *there exists a function* $\Xi_g \in N$ *whose domain* U *consists of all the g-sections of elementary tail sub-g-partitions,* (*see Definition* 9.02) *and for* $z \in U$,

$$\Xi_g(z) = \langle z', z'', \Sigma_{z'}, \mathrm{WO}_{z''}, \mathrm{O}_z \rangle\,,$$

where $z' \cup z'' = z$, $z' \cup z'' = \emptyset$, $\Sigma_{z'}$ *is a bijection of* $z' \times 2$ *onto* z', $\mathrm{WO}_{z''}$ *is a well-ordering of* z'', *and* O_z *is an ordering of* z. (z' *and* $\Sigma_{z'}$ *may be empty.*)

Proof. If $z \in U$, then for some $\sigma(\bar{a}_h) \in \mathcal{G}^c$, $g, f \subseteq h$ and $\bar{\theta} \in \bar{\Gamma}$,

$$z = \{(\textstyle\bigcup z) \cap y \mid y \in D_\sigma^g(\bar{\theta}\cdot\bar{a}_g)\}\,,$$

where $D_\sigma^g(\bar{\theta}\cdot\bar{a}_g) = \{y \in D_\sigma^g \mid \mathrm{proj}(y, g) = \bar{\theta}\cdot\bar{a}_g\}$. Let

$$z^* = \{y \in D_\sigma^g(\bar{\theta}\cdot\bar{a}_g) \mid y \cap (\textstyle\bigcup z) \neq \emptyset\}\ .$$

Hence $z^* \in \mathrm{dom}(\Lambda_g)$. Let

$$\Lambda_g(z^*) = \langle z^{*\prime}, z^{*\prime\prime}, \Sigma_{z^{*\prime}}, \mathrm{WO}_{z^{*\prime\prime}}, \mathrm{O}_z \rangle\,,$$

and define:

$$z' = \{y \cap (\textstyle\bigcup z) \mid y \in z^{*\prime}\}, \qquad z'' = \{y \cap (\textstyle\bigcup z) \mid y \in z^{*\prime\prime}\}\,,$$

$$\Sigma_{z'} = \{\langle\langle y_1, \epsilon\rangle, y_2\rangle \mid y_i = x_i \cap (\textstyle\bigcup z),\ i = 1, 2, \epsilon < 2 \text{ and } \langle\langle x_1, \epsilon\rangle, x_2\rangle \in \Sigma_{z^{*\prime}}\}\,.$$

$$\mathrm{WO}_{z''} = \{\langle y_1, y_2\rangle \mid y_i = x_i \cap (\textstyle\bigcup z),\ i = 1, 2 \text{ and } \langle x_1, x_2\rangle \in \mathrm{WO}_{z^{*\prime\prime}}\}$$

$$O_z = \{\langle y_1, y_2\rangle \mid y_i = x_i \cap (\mathbf{U}z),\ i = 1, 2 \text{ and } \langle x_1, x_2\rangle \in O_{z*}\}.$$

Clearly $\Xi_g(z) = \langle z', z'', \Sigma_{z'}, \mathrm{WO}_{z''}, O_z\rangle$ is the required representation.

9.80. Corollary. *Lemma* 9.20 *is obtained as a specialization of Corollary* 9.70, *because any element of the form* ${}_pD^g_\sigma(\bar{\theta}\cdot\bar{a}_g)$ *is a g-section of an elementary tail sub-(g)-partition.*

9.90. Definition. Let κ be any cardinal. By the κ-multiple choice axiom we shall mean the following statement (denoted $Z(\kappa)$):

> "for every set x of disjoint sets there is a multiple choice function F such that for all $y \in x$, $F(y) \subseteq y$ and $|F(y)| \leqslant \kappa$".

By the ${}^{<}\kappa$-multiple choice axiom (denoted $Z({}^{<}\kappa)$) we mean the statement obtained from the above statement by taking $|F(y)| < \kappa$ instead of $\leqslant \kappa$.

It was shown by Lévy [9] that in set theory permitting urelements (where $\mathfrak{S}$ is a Bernay–Gödel type set theory as used in Mostowski [13]) the axiom of choice is equivalent to $Z(n)$ for any finite n; but that $Z({}^{<}\aleph_0)$ does not imply the axiom of choice in $\mathfrak{S}$.

On the other hand, as Lévy has pointed out to me:

9.92. Lemma. *In* ZF, $Z({}^{<}\aleph_0)$ *implies the axiom of choice, i.e.,* $\mathrm{ZF} \vdash Z({}^{<}\aleph_0) \to \mathrm{AC}$.

Proof. First note that:

9.93. $Z({}^{<}\aleph_0)$ implies that every ordered set can be well-ordered. Let $\prec_X$ be an ordering of X. We shall obtain a well-ordering of X from $\prec_X$. By the proof of the well-ordering theorem, it suffices to show that there exists a choice function for $\mathcal{P}(X)$ (the power set of X). Let

$$Y = \{y \times \{y\} \mid y \in \mathcal{P}(X)\}\ .$$

Then Y is a set of disjoint sets, and by $Z({}^{<}\aleph_0)$ there exists a function F such that for all $y \in Y$, $F(y) \subseteq y$ and $|F(y)| < \aleph_0$. Set

$$G = \{\langle x, y\rangle \mid (\exists u)(\exists x')(\exists y')\,[x' = x \times \{u\} \wedge y' = y \times \{u\} \wedge \langle x', y'\rangle \in F]\}.$$

For all $x \subseteq X$, $G(x) \subseteq x$ is finite; hence using $\prec_X$ we can choose an element of $G(x), x \subseteq X$. This yields the required choice function.

We now use this fact to show that each $R(\alpha)$ can be well ordered (for definition of $R(\alpha)$ see 2.21(f)(iv)). This is proven by transfinite induction on α.

Assume that $\alpha = \beta + 1$ and that $\prec_\beta$ is a well-ordering of $R(\beta)$. Let $\prec'_{\beta+1}$ be the lexicographic ordering of $\mathcal{P}(R(\beta)) - R(\beta)$. Using 9.93 we can obtain a well-ordering $\prec^*_{\beta+1}$ of $\mathcal{P}(R(\beta)) - R(\beta)$, and hence a well-ordering $\prec_{\beta+1}$ extending $\prec_\beta$, by defining for $x, y \in R(\beta+1)$:

9.94 $$x \prec_{\beta+1} y \quad \text{iff} \quad (x, y \in R(\beta) \wedge x \prec_\beta y) \vee$$
$$\vee\, (x, y \in \mathcal{P}(R(\beta)) - R(\beta) \wedge x \prec^*_\beta y) \vee$$
$$\vee\, (x \in R(\beta) \wedge y \in \mathcal{P}(R(\beta)) - R(\beta)).$$

This is clearly the desired well-ordering.

Assume that α is a limit number, and let κ be a cardinal larger than $\bigcup_{\beta<\alpha} |R(\beta)|$, (by the induction hypothesis $|R(\beta)|$ is an initial ordinal, $\beta < \alpha$). By 9.93 there is a well-ordering $\prec^*$ of $\mathcal{P}(\kappa)$. Using $\prec^*$ we show that there is a sequence of orderings $\prec'_\beta$, $\beta < \alpha$ such that $\prec'_\gamma \subseteq \prec'_\beta$, $\gamma \leqslant \beta < \alpha$ and $\prec'_\beta$ well-orders $R(\beta)$, $\beta < \alpha$.

The sequence is defined by transfinite recursion.

(i) Let $\gamma + 1 = \beta < \alpha$, and assume $\prec'_\gamma$ defined for $R(\gamma)$. Define $\prec_{\gamma+1}$ as follows: let f_γ be the unique function mapping $\langle R(\gamma), \prec_\gamma \rangle$ isomorphically onto an initial segment $\langle \beta^*, \leqslant \rangle$ of κ. Using $\prec^*$ we have a choice function for $\mathcal{P}(\mathcal{P}(\beta^*))$, and via f_γ^{-1} we obtain a canonic choice function for $\mathcal{P}(\mathcal{P}(R(\gamma)))$. Hence by the proof of the well-ordering theorem, we obtain a canonic well-ordering $\preccurlyeq^*_{\gamma+1}$ of $\mathcal{P}(R(\gamma)) - R(\gamma)$. The required well-ordering $\prec'_{\gamma+1}$ is now defined from $\preccurlyeq^*_{\gamma+1}$ as in (9.94).

(ii) Assume $\beta < \alpha$ is a limit ordinal, and that $\prec'_\gamma$ are defined for $\gamma < \beta$. Then $\bigcup_{\gamma<\beta} \prec'_\gamma = \prec'_\beta$ is easily seen to be the required well-ordering of $R(\beta) = \bigcup_{\gamma<\beta} R(\gamma)$.

Now set $\preccurlyeq_\alpha = \bigcup_{\beta<\alpha} \preccurlyeq'_\beta$. $\prec_\alpha$ is clearly a well-ordering of $R(\alpha) = \bigcup_{\beta<\alpha} R(\beta)$, as required.

The question whether in ZF, $Z(\aleph_0)$ implies the axiom of choice had been raised by Lévy, and remained open till now. In the next lemma we show that $Z(\aleph_0)$ holds in N. This fact in conjunction with the fact proven in Section 10, that AC does not hold in N, settles the problem in the negative.

9.95. Lemma. $N \models Z(\aleph_0)$.

Proof. Let z be a set of disjoint sets in N. Assume $z = \mathrm{val}(\tau(\bar{a}_g))$ where $g = \{0, 1, ..., n\}$ is an initial segment of ω. Let $p \in Q$ be such that $p \Vdash$ "τ is a set of disjoint sets", and consider the sequence B defined in the proof of Lemma 9.30, with the following refinements: B' is to be a sequence defined for

$$\beta < \alpha \leqslant (|\mathrm{rnk}(\tau)|^+)^{(M)} + \aleph_1^{(M)}$$

such that $B'(\beta) = (q_\beta, \sigma_\beta, \sigma'_\beta, \rho'_\beta, \tau_\beta), \beta < \alpha$, where $p \subseteq q_\beta$ is a condition, $\sigma_\beta, \sigma'_\beta, \rho_\beta$ and τ_β are constant terms such that

$$q_\beta \Vdash \sigma'_\beta \in \sigma_\beta \in \tau_\beta, \qquad \mathrm{val}(\rho_\beta) = {}_{q\beta}K_{\bar{a}_g}(\sigma_\beta, \sigma'_\beta),$$

$$\mathrm{val}(\tau_\beta) \subseteq \tau.$$

B' is defined by induction as follows. Assume $B'(\gamma)$ defined for $\gamma < \beta$; we define $B'(\beta), \beta < \alpha$. If there are no triples $q, \sigma(a_h), \sigma'(a'_h)$ such that

$$\mathrm{rnk}(\sigma') < \mathrm{rnk}(\sigma) < \mathrm{rnk}(\tau), \quad g \subseteq h,$$

$$p \subseteq q \Vdash \sigma \in \tau \wedge \sigma' \in \sigma$$

and for all $\gamma < \beta$,

$$q \Vdash \neg(\sigma \in \tau_\beta),$$

then B' is to be undefined for ordinals $\gamma \geqslant \beta$. If there are such triples, let $(q_\beta, \sigma_\beta, \sigma'_\beta)$ be a minimal triple according to some standard well-ordering of $\mathcal{Q} \times \mathcal{Q} \times P$ such that $\mathrm{rnk}(\sigma'_\beta) < \mathrm{rnk}(\sigma_\beta) < \mathrm{rnk}(\tau)$, and $p \subseteq q_\beta \Vdash \sigma'_\beta \in \sigma_\beta \wedge \sigma_\beta \in \tau_\beta$. Let $\sigma_\beta = \sigma_\beta(\bar{a}_{h_\beta}), \sigma'_\beta = \sigma'(\bar{a}_{h'_\beta})$, and define ρ'_β similarly to the ρ_β of $B(\beta)$ (see Lemma 9.30),

$$\begin{aligned}\rho'_\beta = (\curlyvee \bar{w})(\exists_{\omega,2} s)\,[\mathrm{dom}(s) \subseteq \omega \wedge \bigwedge_{i \in \mathrm{dom}(s)} s(i) \in K(a_i) \wedge \\ \wedge (\text{"}s \text{ is an assignment to the generic} \\ \text{real constants in } \sigma_\beta, \sigma'_\beta, \tau, q''_\beta\text{"}) \wedge \\ \wedge \bigwedge_{j \in h_\beta \cup h'_\beta} (s(j) = w_j) \wedge \bigwedge_{j \in g} (s(j) = a_j) \wedge \\ \wedge\, \widetilde{\mathrm{sat}}(q_\beta, s) = 1 \text{ (truth)}].\end{aligned}$$

τ_β is defined exactly as in the proof of (9.3):

$$\tau_\beta = (\lambda w)\,(\exists_{\omega\cdot 2} s)\,[\mathrm{dom}(s) = h_\beta \wedge \langle s(i_1), \ldots, s(i_m)\rangle \in \rho_\beta \wedge \wedge\ \mathrm{v\hat{a}l}(\sigma_\beta, s) = w]\,.$$

As seen in Lemma 9.30, B' is a well defined sequence in N (and even in M) of length

$$\alpha < (|\mathrm{rnk}(\tau)|^+)^{(M)} + \aleph_1^{(M)}.$$

Setting $z_\beta = \mathrm{val}(\tau_\beta)$, we have, as in Lemma 9.30, that

$$z = \bigcup_{\beta<\alpha} z_\beta, \qquad z_\beta \cap z_\gamma = \emptyset, \quad \gamma < \beta < \alpha;$$

also

$$\mathrm{val}(\rho'_\beta) = {}_{q\beta}K_{\bar{a}_g}(\sigma_\beta, \sigma'_\beta) = {}_{q\beta}K_{\bar{a}_g}(\bar{a}_{h_\beta \cup h'_\beta})\,.$$

In N, define the equivalence relation $\sim_\beta$ on $\boldsymbol{\rho}'_\beta$ as follows: let $\bar{x}, \bar{y} \in \boldsymbol{\rho}'_\beta$, and set

$$\bar{x} \sim_\beta \bar{y} \quad \text{iff} \quad \sigma_\beta[\mathrm{proj}(\bar{x}, h_\beta)] = \sigma_\beta[\mathrm{proj}\,\bar{y}, h_\beta]\,,$$
$$\sigma'_\beta[\mathrm{proj}(\bar{x}, h'_\beta)] = \sigma'_\beta[\mathrm{proj}(\bar{y}, h'_\beta)]\,.$$

This is clearly an equivalence relation and the corresponding partition of $\boldsymbol{\rho}'_\beta$ is denoted by ${}_{q\beta}\boldsymbol{D}_{\sigma_\beta, \sigma'_\beta}(\bar{a}_g)$. Moreover for every $Y \in {}_{q\beta}\boldsymbol{D}_{\sigma_\beta, \sigma'_\beta}(\bar{a}_g)$, there is an $X \in {}_{q\beta}\boldsymbol{D}_{\sigma_\beta}(\bar{a}_g)$ such that

$$\mathrm{proj}(Y, h_\beta) \subseteq X\,.$$

Let X be an element of ${}_{q\beta}\boldsymbol{D}_{\sigma_\beta}(\bar{a}_g)$, and define:

$$C_X = \{Y \in {}_{q\beta}D_{\sigma_\beta, \sigma'_\beta}(\bar{a}_g) \mid \mathrm{proj}(Y, h_\beta) \subseteq X\}\,.$$

Set $A_X = \{\sigma'_\beta[\bar{y}] \mid y \in \bigcup C_X\}$. Then by Lemma 5.44

$$A_X \subseteq \sigma_\beta[\bar{x}], \qquad \bar{x} \in X,$$

and obviously $|A_X| \leqslant \aleph_0$. Moreover, if $X_1, X_2 \in {}_{q\beta}\boldsymbol{D}_{\sigma_\beta}(\bar{a}_g)$, $X_1 \neq X_2$, then $A_{X_1} \cap A_{X_2} = \emptyset$, since

$$p \subseteq q_\beta \Vdash \text{“}\tau \text{ is a set of disjoint sets”}.$$

Set

$$A^\beta = \{A_X \mid X \in {}_{q\beta}D_{\sigma_\beta}(\bar{a}_g)\}.$$

The multiple choice function F for τ can now be defined in N as follows: if $y \in \tau$, then for some unique $\beta < \alpha$, and

$$\bar{x} \in X \in {}_{q_\beta}D_{a_\beta}(\bar{a}_g)\ ,$$

we have $y = \sigma_\beta[\bar{x}]$; hence define $F(y) = A_X \in A^\beta$.

F has clearly been defined in N and is the required $\aleph_0$-multiple choice function.

The above argument can be carried out in any N_α, $0 \leqslant \alpha \leqslant \aleph_1^{(M)}$; hence

9.96. Corollary. $N_\alpha \models \mathrm{Z}(\aleph_0)$, $0 \leqslant \alpha \leqslant \aleph_1^{(M)}$.

10. Negation of the Axiom of Choice in N.

This section is devoted to proving that the axiom of choice does not hold in N.

10.0. The *amount of choice* in a model is determined by the *amount of symmetry* that the model has. In Cohen extensions, this is expressed by the symmetry of the forcing relation. Generally, symmetry is obtained by constructing the forcing relation so that it is invariant under certain transformations which pertain to permutations of atom-like *independent* generic elements with the right information. In our case, on the contrary, the invariance of the forcing relation pertains to the permutation of elements of the same tail with the right information, as expressed by Lemma 5.44. By the *amount of symmetry* in a model, we mean: the assured number of elementary tail reals which satisfy an arbitrary condition. In Lemmas 5.50 and 5.51, the fact that any condition of P_0 is satisfied by *countably many tail members* was responsible for the negation of choice in N_0. Since the structure of N is so much richer than that of N_0, this conceivably might not be true for arbitrary conditions of P, in which case choice would hold in N. Most of this section deals with a proof of the fact that satisfiable conditions of P are satisfied by countably many reals of the corresponding tails. This will readily account for the negation of choice in N, (Note that not all conditions of P_α, $\alpha > 0$ are satisfiable. To see this consider $\mathring{\chi}_e$ defined on nearly definable elements.)

10.00. Notation. (a) $I = I_{\aleph_1^{(M)}}$ is defined on ω as well as for $\beta+1$-indices, $\beta < \aleph_1^{(M)}$, and $I(i) = K(a_i)$. If $h \subset \omega$, $I(h)$ will denote $K(\bar{a}_h)$.

(b) We remind the reader that if $\bar{x} \in I(h)$, $\bar{x} = \bar{\theta}\cdot\bar{a}_h$ and $\sigma(\bar{a}_h) \in \mathcal{T}^c$, then $\sigma(\bar{a}_h)[\bar{x}]$ stands for the element $\sigma(\bar{\theta}\cdot\bar{a}_h) \in N$, where ${}^{(t,l)}\bar{\theta}$ is chosen in a unique manner by taking $x = {}^{(t,l)}\theta(a_i)$ for minimal l. Thus for instance ${}^{\alpha}H_\sigma(\bar{\theta}^*\cdot\bar{a}_h)[\bar{x}] = {}^{\alpha}H_\sigma(\bar{\theta}^*\cdot\bar{\theta}\cdot\bar{a}_h)$.

Also recall that if p is a condition, then $p[\bar{x}]$ stands for the condition $T^{\bar{\theta},\bar{h}}(p)$. This notation is also employed when p is a precondition or an arbitrary set of preconditions.

10.01. Definition. The tail $I(i)$ is said to be *dependent on the tails* $I(h_1), \ldots, I(h_n)$ *in* N_α if some member of $I(i)$ is supported by $h = \{h_1, \ldots, h_n\}$, in N_α; i.e. there exists a term $\tau(\bar{a}_g) \in \mathcal{T}^c_\alpha$ with $g \subseteq h$ such that $\mathrm{val}_\alpha(\tau(\bar{a}_g)) \in I(i)$; equivalently $I(i)$ has a $(-i)$-definable$_\alpha$ member. If $I(i)$ is not dependent on the tails $I(i_1), ..., I(i_n)$ in N_α, then $I(i)$ is said to be *independent of the tails* $I(i_1), ..., I(i_n)$ *in* N_α.

10.02. Definition. *Let* $g_1, g_2, h \subset \omega$; then $\bar{x}^1 \in I(g_1)$, $\bar{x}^2 \in I(g_2)$ are said to be *h-congruent* if $\mathrm{proj}(\bar{x}^1, h \cap g_1) = \mathrm{proj}(\bar{x}^2, h \cap g_2)$.

10.1. The following lemma is the principal lemma of this section.

10.10. Lemma. (a) *Assume that* $p(\bar{a}_g) \in P_\alpha$ *is satisfied by* $\bar{z}^* \in I(g)$ *in* N_α, *and let* $h = g - \{l\}$, *where* $l \in g$; *then there are countably many h-congruent* $\bar{z} \in I(g)$ *which satisfy p in* N_α.

(b) *For no i, h, where* $i \in \omega$, $h \subset \omega$, *and* $i \notin h$ *is* $I(i)$ *dependent on* $I(h_1), ..., I(h_n)$ *in* N_α, *where* $h = \{h_1, ..., h_n\}$; (*i.e., all are independent from each other in* N_α, $\alpha \leqslant \aleph_1^{(M)}$).

Proof. This lemma is proved by induction on α.

If $p \in P_0$, then

$$p(\bar{a}_h) = p(a_{h_1}) \cup ... \cup p(a_{h_n})$$

where $p(a_{h_i})$ are the subconditions of p mentioning only a_{h_i}. Each $p(a_{h_i})$ contains only coordinate information on a_{h_i}. Thus it is clear that any $p(a_{h_i})$ is satisfied by an infinite number of $x \in I(h_i)$ from every layer of $I(h_i)$, (see Definition 1.90(a)); therefore (a) is true for $\alpha = 0$. To prove (b) from (a), assume

$$\theta(a_l) = \mathrm{val}_\alpha(\sigma(\bar{a}_f)) ,$$

where $l \notin f$; then for some $p(\bar{a}_g) \in Q_\alpha$,

$$p(\bar{a}_g) \Vdash_\alpha \sigma(\bar{a}_f) = \theta(a_l) .$$

Let $h = g \cup f - \{l\}$, $h^* = h \cup \{l\}$; then by (a) there are denumerably many $\bar{z} \in I(h^*)$ which are h-congruent to $\bar{a}_{h^*}$ and satisfy $p(\bar{a}_g)$. Hence for some such $\bar{z}$ we have

$$\mathrm{proj}(\bar{z}, \{l\}) = \theta^*(a_l) ,$$

where $\theta(\theta^*(a_l)) \neq \theta(a_l)$, and $p[\bar{z}] \Vdash_\alpha \sigma(\bar{a}_f) = \theta(\theta^*(a_l))$. Therefore by

Lemma 5.44,

$$N_\alpha \models \sigma(\bar{a}_f) = \theta(\theta^*(a_l)) \wedge \theta(\theta^*(a_l)) \neq \theta(a_l) \wedge \sigma(\bar{a}_f) = \theta(a_l).$$

This is a contradiction; thus (b) is true for $\alpha \geq 0$. Before proceeding to the general case we first consider an example worthy of study:

10.11. Example. Define the partition, H, on $I(7)$ as follows: for all $x, y \in I(7)$, define the relation E by

$$x\,E\,y \quad \text{iff} \quad [x(0) = y(0) = 1 \text{ or } x(0) = y(0) = 0 \wedge$$
$$\wedge\ (\exists l \in \omega)\ (\text{slash}_l(x) = \text{slash}_l(y))].$$

This is clearly seen to be an equivalence relation. Let H consist of the equivalence classes of E. Denote

$$R_i = \{x \mid x \in I(7),\ x(0) = i\}, \quad i < 2.$$

Then H consists of R_1 and the layers of R_0, (see Example 1.90). This partition is induced by the following term $\sigma \in \mathcal{T}_0^c$:

$$\sigma(a_7) = (\text{Э}v)\ (\exists u \in \dot{\omega})\ (\forall w \in \dot{\omega})$$
$$[v(0) = 1 \wedge a_7(0) = 1 \vee (v(0) = 0 \wedge a_7(0) = 0 \wedge$$
$$\wedge\ v \in I_0(7) = K(a_7) \wedge a_7(u+w) = v(u+w))].$$

It is easily seen that for $x, y \in I(7)$, $N_0 \models \sigma[x] = \sigma[y]$ iff either $x, y \in R_1$, or x, y are in the same layer of R_0. Set $f = \{7\}$ and $e_0 = (\{7\}, \{1\}, s_\sigma^0)$, $e_1 = (\{7\}, \emptyset, s_\sigma^0)$. Choose the operations: ${}^t\theta, {}^{l_0}\theta, {}^{l_1}\theta, {}^{m_0}\theta, {}^{m_1}\theta$, such that $t = \langle 0, 0\rangle \in 2^1$, $l_1 \neq l_0$, $m_1 \neq m_0$; and take p to be the following condition:

$$\{\langle 7, l_1, 1\rangle, \langle 7, l_0, 0\rangle, \langle e_0, {}^t\theta, 0, 0\rangle, \langle e_1, {}^{l_1}\theta, 0, 1\rangle,$$
$$\langle e_0, {}^t\theta, m_0, {}^{l_0}\theta, m_1\rangle\}.$$

p says that

$$[a_7(l_1) = 1 \wedge a_7(l_0) = 0 \wedge \chi_{e_0}({}^t\theta(a_7))(0) = 0$$
$$\wedge\ \chi_{e_1}({}^{l_1}\theta(a_7))\,(0) = 1\ \wedge$$
$$\wedge\ {}^{m_0}\theta(\chi_{e_0}({}^t\theta(a_7))) = {}^{m_1}\theta(\chi_{e_0}({}^{l_0}\theta(a_7)))]$$

(see Fig. 7).

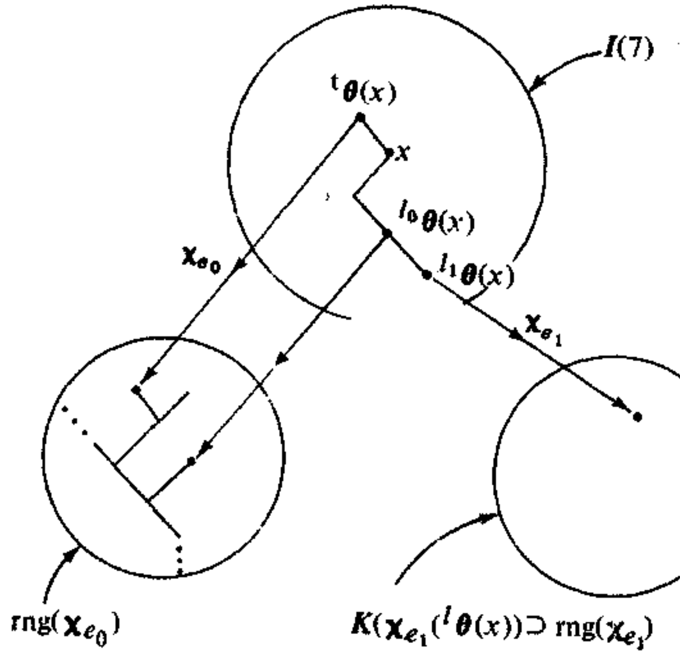

Fig. 7.

p is a condition because:

(1) for all x: ${}^{t}\theta(x)$, ${}^{l_0}\theta(x)$ are in different layers and $m_0 \neq m_1$;

(2) $0 \Vdash$ "$H_\sigma({}^{l_0}\theta(a_7))$ and $H_\sigma({}^{t}\theta(a_7))$ are in different layers of R_0, and $H_\sigma({}^{l_1}\theta(a_7)) = R_1$";

(3) $R_1 \in H_\sigma$ is definable and forced$_0$ so by 0, and no layer of R_0 is nearly (-7)-definable$_0$. We give a brief proof of this last statement. If some layer of R_0 belongs to a finite (-7)-definable$_0$ subset of layers of R_0, then, since we have a definable natural ordering of layers (see Example 1.90(a)), we would have a (-7)-definable layer of $I(7)$. Assume that for some formula $\varphi(v) \in \mathcal{F}_0$, with v as its only free variable, and not containing the generic real-constant a_7, and for some $x = \theta(a_7)$, there is a $q \in Q_0$ such that $q \Vdash_0 (\exists! v)(\varphi(v)) \wedge \varphi(H_\sigma(\theta(a_7)))$. Let $\bar{a}_g$ be all the generic real constants occurring in q or φ other than 7, and $h = g \cup \{7\}$. There are countably many $y \in I(7)$ from every layer satisfying $q(a_7)$. Choose a $y \in I(7)$ such that $y \in \bigcup R_0$, $H_\sigma(y) \neq H_\sigma(x)$, $y \neq a_7$ and y satisfies q. If $y = \theta^*(a_7)$, then, by Lemma 5.44,

$$N_0 \models (\exists ! v)(\varphi(v)) \wedge \varphi(H_\sigma(\theta(\theta^*(a_7)))).$$

Since also

$$N_0 \models H_\sigma(\theta(a_7)) \neq H_\sigma(\theta(\theta^*(a_7))) \wedge \varphi(H_\sigma(\theta(a_7))),$$

this is a contradiction.

The other requirements on conditions are trivially fulfilled in this case (see Definition 2.50).

We must show that there are $\aleph_0$ $z \in I(7)$ satisfying p, if it is satisfied by some $x \in I(7)$. Assuming $x_1, \ldots, x_n \in I(7)$ satisfy p, we show that there is an $x_{n+1} \in I(7)$, $x_{n+1} \neq x_i$, $1 \leqslant i \leqslant n$, such that x_{n+1} satisfies p. There are $\aleph_0$ layers of R_0; and there are $\aleph_0$ pairs of layers disjoint from those pairs determined by $x_1, \ldots, x_n$, for which there is an $x^* \in I(7)$ such that $x^*(l_0) = 0$, $x^*(l_1) = 1$, and append$_l(x^*)$ is in one layer and slash$_{l_0}(x^*)$ is in another layer. It is impossible to force$_1$, by a finite condition of P_1, all those pairs to be mapped by $\mathring{\chi}_{e_0}$ to elements of rng$(\mathring{\chi}_{e_0})$ so as not to satisfy p, i.e., to elements y_1, y_2 such that $y_1(0) \neq 0$, or slash$_{m_1}(y_1) \neq$ slash$_{m_2}(y_2)$. (Here is where the genericity of the functions χ_e comes in.) Hence we can find x_{n+1} as required, and by induction (in the universe), there exists an infinite such sequence, as required.

Note that if the nearly (-7)-definable components and the non-nearly (-7)-definable components of H_σ were mapped into the same tail, this argument would not hold!

The proof in the general case bears out the simple idea presented here, but turns out to be somewhat tricky.

To continue with the inductive proof of Lemma 10.10, we assume the lemma true for β, $0 \leqslant \beta \leqslant \alpha < \aleph_1^{(M)}$, and we prove for $\alpha + 1$. (Note that this is sufficient, for if $p \in P_\gamma$, then for some $\alpha < \gamma$, $p \in P_{\alpha+1}$.)

It will be shown later that:

10.12. If, for every $p(\bar{a}_h) \in Q_{\alpha+1}$ with $l \in h$, and arbitrary m, $1 \leqslant m < \omega$, there are $\bar{z}^1, \ldots, \bar{z}^m \in I(h)$ such that $\bar{z}^j$ are g-congruent to $\bar{a}_g$, where $g = h - \{l\}$, $1 \leqslant j \leqslant m$, and $p[\bar{z}^1] \cup \ldots \cup p[\bar{z}^m]$ can be extended to a condition, then:

10.121. For any $p^*(\bar{a}_{h^*}) \in P_{\alpha+1}$ which is satisfied by some $\bar{z} \in I(h^*)$ where $l \in h^*$, there are countably many $\bar{z} \in I(h^*)$ that satisfy p^* which are g^*-congruent to $\bar{z}^*$, where $g^* = h^* - \{l\}$.

Therefore we set out to establish (10.12) for arbitrary $p(\bar{a}_h) \in Q_{\alpha+1}$ such that $l \in h$, for any fixed $l < \omega$.

$\bar{a}_h$ satisfies $p(\bar{a}_h)$ in $N_{\alpha+1}$ (since $p \in Q_{\alpha+1}$); hence it also satisfies $\hat{p}^\alpha$ in N_α. By the induction hypothesis there are $\aleph_0$ $\bar{z} \in I(h)$ which satisfy $\hat{p}^\alpha$. We shall show that we can find an arbitrary large number, m, of these $\bar{z}^j$, $1 \leqslant j \leqslant m$ such that $\bigcup_{j=1}^m p[\bar{z}^j]$ can be extended to a condition. To this end we first enumerate $p - \hat{p}^\alpha$ as follows:

10.13. Notation. (a) $q_1, \ldots, q_k = \bar{q}^k$ is an enumeration of $p - \hat{p}^\alpha$ such that if $\langle e_q, \theta'_q, l'_2, \theta''_q, l''_q\rangle = q \in p - \hat{p}^\alpha$, then q occurs twice in succession in the sequence $\bar{q}^k$.

(b) To each q_i in the sequence we associate a term $G_{q_i} = G_i$ as follows: If $q_i = \langle e_i, \bar{\theta}_i, j_i, \delta_i\rangle$, where

$$e_i = e_q = (h_i, g_i, s_i^\alpha), \qquad \bar{a}_{h_i} = \bar{b}_i \subseteq \bar{b} = \bar{a}_g,$$

then

$$G_{q_i} = G_i = {}^\alpha H_{\sigma_i}^{g_i}(\bar{\theta}_i \cdot \bar{b}_i)\ ;$$

if $q_i = \langle e_i, \bar{\theta}'_i, l'_i, \bar{\theta}''_i, l''_i\rangle$ occurs for the first time in the sequence, then

$$G_i = {}^\alpha H_{\sigma_i}^{g_i}(\bar{\theta}'_i \cdot \bar{b}_i); \quad \text{otherwise} \quad G_i = {}^\alpha H_{\sigma_i}^{g_i}(\bar{\theta}''_i \cdot \bar{b}_i)\,.$$

(c) For any $\bar{z} \in I(h) = K(\bar{a}_h)$, we denote

$$G_i(z) = {}^\alpha H_{\sigma_i}^{g_i}(\bar{\theta}_i \cdot \bar{a}_{h_i})\,[\bar{z}_{h_i}]\,,$$

where $\bar{z}_{h_i} = \mathrm{proj}(\bar{z}, h_i)$, $\bar{a}_{h_i} = \bar{b}_i$, $1 \leqslant i \leqslant k$.

10.14. Note that if there are countably many $\bar{z} \in I(h)$ which are g-congruent to $\bar{a}_g$, satisfy $\hat{p}^\alpha$, and $G_i(\bar{z}) = G_i(\bar{a}_h)$, $1 \leqslant i \leqslant k$, then the result is immediate. Since we would have

$$\chi_{e_i}(\bar{\theta}_i \cdot \bar{z}) = \chi_{e_i}(\bar{\theta}_i \cdot \bar{a}_h),$$

hence each one of these $\bar{z}$ would satisfy p, (i.e., $\bigcup_{\bar{z}} p[\bar{z}] \subseteq Q_{\alpha+1}$), in which case any finite union of the $p[\bar{z}]$ would be extendable to a condition of $Q_{\alpha+1}$.

10.15. Definition. An element $\bar{z} \in I(h)$ which is g-congruent to $\bar{a}_g$ and satisfies $\hat{p}^\alpha$ is said to be *proper*.

Note that for any proper $\bar{z}$, and $1 \leqslant i, j \leqslant k$, if $e_i = e_j$ we have

$$(10.16)\quad {}^{\alpha}H^{g_i}_{\sigma_i}(\bar{\theta}_i \cdot \bar{b}_i)\,[\bar{z}] = {}^{\alpha}H^{g_j}_{\sigma_j}(\bar{\theta}_j \cdot \bar{b}_j)\,[\bar{z}]$$

iff

$${}^{\alpha}H^{g_i}_{\sigma_i}(\bar{\theta}_i \cdot \bar{b}_i) = {}^{\alpha}H^{g_j}_{\sigma_j}(\bar{\theta}_j \cdot \bar{b}_j)\,.$$

This follows from the definition of a condition, Lemma 5.44, and the absoluteness of the notions nearly $(-n)$-definable$_\alpha$, and not nearly $(-n)$-definable$_\alpha$, $n < \omega$.

10.161. By the induction hypothesis, there are countably many proper $\bar{z} \in I(h)$.

10.17. Notation. Let h^+ be the set of all j, $1 \leqslant j \leqslant k$, for which $G_j(\bar{a}_h)$ is nearly $(-l)$-definable$_\alpha$; and let h^- be those for which $G_j(\bar{a}_h)$ is not nearly $(-l)$-definable$_\alpha$. (Obviously, $\{j \mid 1 \leqslant j \leqslant k\} = h^+ \cup h^-$, and $h^+ \cap h^- = \emptyset$.)

If $f \subseteq \{j \mid 1 \leqslant j \leqslant k\}$, then f^+ will denote $f \cap h^+$, and f^- will denote $f \cap h^-$.

10.18. Definition. (a) Let f be a subset of $\{j \mid 1 \leqslant j \leqslant k\}$. $\bar{z}, \bar{z}' \in I(h)$ are said to be $\bar{G}_{f^+}$*-coherent* if, for all $j \in f^+$,

$$G_j(\bar{z}) = G_j(\bar{z}').$$

(b) $\bar{z}, \bar{z}' \in I(h)$ are said to be $\bar{G}_{f^-}$*-free* if, for all $i, j \in f^-$ for which $e_i = e_j$, we have

$$G_i(\bar{z}) \neq G_j(\bar{z}')\,.$$

10.2. The following obvious facts are worthy of note here.

10.20. If $l \notin h_j$, (where $e_j = (h_j, g_j, s_j^\alpha)$), $1 \leqslant j \leqslant k$, then any $\bar{z} \in I(h)$ which is g-congruent to $\bar{a}_g$ is G_j-coherent with $\bar{a}_h$.

10.201. If all $G_j(\bar{a}_h)$ are nearly $(-l)$-definable$_\alpha$, $1 \leqslant j \leqslant k$, then it is possible that any $\bar{z} \in I(h)$ which is g-congruent to $\bar{a}_g$ and satisfies p, is $\bar{G}^k$-coherent with $\bar{a}_h$.

10.21. Definition. A set $W \subseteq I(h)$ is said to be $(\bar{G}_f, \bar{C})$-*good* if
(a) all $\bar{z} \in W$ are proper,
(b) all $\bar{z}, \bar{z}' \in W$ such that $\bar{z} \neq \bar{z}'$ are $\bar{G}_{f^-}$-free,
(c) for all $\bar{z} \in W$, $\bar{G}_{f^+}(\bar{z}) = \langle G_{f_1^+}(\bar{z}), ..., G_{f_{m^+}^+}(\bar{z})\rangle = \bar{C}$, where $f^+ = \{f_1^+, ..., f_{m^+}^+\}$.

Note that if $f^+ = \emptyset$, then W is $(\bar{G}_f, \bar{C})$-good if and only if all distinct $\bar{z}, \bar{z}' \in W$ are proper and $\bar{G}_f$-free; in which case we say that W is $\bar{G}_f$-good.

Our immediate goal will be to show that there are arbitrary large $(\bar{G}^k, \bar{G}_{h^+}(\bar{a}_h))$-good sets. First, we state a few simple facts concerning the definability of the previous notions that will be needed.

10.22. Sub-lemma. (a) *The notion of "proper" is g-definable$_\alpha$, (see Definition 4.80(d)), i.e., there is a formula $\psi_1(\bar{u}, v) \in \mathcal{F}_\alpha$ not containing generic real constants such that $\bar{z}$ is proper iff $N_\alpha \models \psi_1(\bar{a}_g, \bar{z})$. (In particular the notion of "proper" is $(-l)$-definable$_\alpha$.)*

(b) *The notion of "$\bar{G}_{f^-}$-free" is $(-l)$-definable$_\alpha$; i.e., there is a formula $\psi_2(\bar{u}, v, v')$, not containing generic real constants and h^*, such that $l \notin h^*$; and for all $\bar{z}, \bar{z}' \in I(h)$, $\bar{z}, \bar{z}'$ are $\bar{G}_{f^-}$-free if and only if $N_\alpha \models \psi_2(\bar{a}_{h^*}, \bar{z}, \bar{z}')$.*

(c) *For every proper $\bar{x}$ there is a finite $(-l)$-definable$_\alpha$ set $V_{\bar{x}} \in N_\alpha$ such that if $\bar{z} \in I(h)$ is $\bar{G}_{f^+}$-coherent with $\bar{x}$, then $\bar{G}_{f^+}(\bar{z}) \in V_{\bar{x}}$.*

Proof. (a) and (b) are easily seen to be true by inspecting Definitions 10.15, 10.18. To see (c), let V_j^f be finite $(-l)$-definable$_\alpha$ sets of minimal cardinality such that $G_j(\bar{x}) \in V_j$ for $j \in f^+$; then set $V_{\bar{x}}^f$ to be the Cartesian produce of the V_j according to increasing index $j \in f^+$. $V_{\bar{x}} = \bigcup_f V_{\bar{x}}^f$ is the required set. If W is a $(\bar{G}_f, \bar{C})$-good set, then for all $\bar{z}, \bar{z}' \in W$, $V_{\bar{z}}^f = V_{\bar{z}'}^f$; hence we may denote this set by V_W.

10.23. Note that if $\bar{z}$ is proper, then $\{\bar{z}\}$ is a $(\bar{G}_f, \bar{G}_{f^+}(\bar{z}))$-good set, $f \subseteq \{j \mid 1 \leqslant j \leqslant k\}$.

10.24. Sub-lemma. *For any $f \subseteq \{j \mid 1 \leqslant j \leqslant k\}$, and any finite non-empty $(\bar{G}_f, \bar{C})$-good set W, there is a $\bar{z} \notin W$ such that $W \cup \{\bar{z}\}$ is a $(\bar{G}_f, \bar{C})$-good set.*

Proof. This lemma is proven by induction on $|f^-|$.

Assume $f^- = \emptyset$; then either $f = \emptyset$, in which case the lemma follows by 10.161; or for all members $\bar{z}$ of W, $G_j(\bar{z})$ is nearly $(-I)$-definable$_\alpha$, $j \in f = f^+$. If there are $\aleph_0$ proper $\bar{z}$ such that $G_j(\bar{z}) = C_j$ (i.e., $G_f(\bar{z}) = \bar{C}$), then any one of them not in W will do. We shall show that the other alternative (that there are only finitely many proper $\bar{z}$ such that $\bar{G}_f(\bar{z}) = \bar{C}$) is impossible. Let V_W be the set given by Sub-lemma 10.22(c); and define V^* as the set of all $y \in V_W$ for which there exists a proper $\bar{z}$ such that $y = \bar{G}_f(\bar{z})$, and there are only finitely many proper $\bar{z}'$ such that $\bar{G}_f(\bar{z}') = \bar{G}_f(\bar{z})$. V^* is clearly a finite non-empty $(-I)$-definable$_\alpha$ set. Let

$$A = \{\bar{z} \mid \bar{z} \text{ is proper and } \bar{G}_f(\bar{z}) \in V^*\}.$$

A is obviously $(-I)$-definable$_\alpha$; moreover, by our assumption it follows that *A is finite and non-empty.* Therefore, $\mathrm{proj}(A, \{I\})$ is finite and $(-I)$-definable$_\alpha$. Hence, using the natural ordering of the reals, we obtain a $(-I)$-definable element of $I(I)$, in contradiction to the induction hypothesis, (see Lemma 10.10(b)).

Assume $f^- \neq \emptyset$, and that for some finite non-empty $(\bar{G}_f, \bar{C})$-good set W, there does not exist a $\bar{z} \notin W$ such that $W \cup \{\bar{z}\}$ is $(\bar{G}_f, \bar{C})$-good. We shall show that this leads to a contradiction by constructing a finite $(-I)$-definable$_\alpha$ set B containing elements of the form $G_j(\bar{z}), j \in f^-$, $z \in W$, which is obviously impossible. For every $j \in f^-$, let $\bar{G}_f^{(-j)}$ be the sequence obtained from $\bar{G}_f$ by omitting the element G_j. By the induction hypothesis, for every $j \in f^-$ there is an infinite $(\bar{G}_f^{(-j)}, \bar{C})$-good set W_j extending W. Let B_i^W be the sets

$$\{G_i(\bar{z}) \mid \bar{z} \in W\}, \quad i \in f^-.$$

For every $\bar{z} \in W_j - W$ there exists an $i \in f^-$ such that $e_i = e_j$ and $G_j(\bar{z}) \in B_i^W$; otherwise $\bar{z}$ would be $\bar{G}_f$-free from all $\bar{z}' \in W$, hence $W \cup \{\bar{z}\}$ would be $(\bar{G}_{f^*}, \bar{C})$-good, contradicting the assumption. Define $B_{i,j}^{W,\bar{C}}$ to be the set of all elements $y \in B_i^W$ for which there exists an infinite $(\bar{G}_f^{(-j)}, \bar{C})$-good $W_j \supset W$ such that $G_j(\bar{z}) = y$ for all $\bar{z} \in W_j - W$.

10.25. Set $B^{W,\bar{C}} = \bigcup_{i,j} B_{i,j}^{W,\bar{C}}$. Then $B^{W,\bar{C}}$ is a *non-empty finite set.* We shall show that $B^{W,\bar{C}}$ can be defined from $\bar{C}$, *independent of* W. First assume that there exists another finite $(\bar{G}_f, \bar{C})$-good W', which is not extendible to a larger such set. We can then define as above the sets $B_i^{W'}$, $B_{i,j}^{W',\bar{C}}$, and $B^{W',\bar{C}}$, $i, j \in f^-$. *We shall show that* $B^{W,\bar{C}} = B^{W',\bar{C}}$.

Assume $y \in B^{W,\bar{C}}$; then for some i, j, $y \in B^{W,\bar{C}}_{i,j}$. Hence there is an infinite $(\bar{G}_f^{(-l)}, \bar{C})$-good set W^* such that for all $\bar{z} \in W^*$, $G_j(\bar{z}) = y \in B_i^W$. Since W^* is $(\bar{G}_f^{(-l)}, \bar{C})$-good, there must be an infinite subset W^{**} such that for all $\bar{z} \in W^{**}$, $W' \cup \{\bar{z}\}$ is $(\bar{G}_f^{(-l)}, \bar{C})$-good But since W' is not extendible to a larger set which is $(\bar{G}_f, \bar{C})$-good, we must have for all $\bar{z} \in W^{**}$,

$$y = G_j(\bar{z}) \in \bigcup_{i \in f^-} B_i^{W'} ;$$

hence $y \in B^{W',\bar{C}}_{i',j}$ for some $i' \in f^-$. Therefore $B^{W,\bar{C}} \subseteq B^{W',\bar{C}}$. A completely symmetric argument gives $B^{W',\bar{C}} \subseteq B^{W,\bar{C}}$. Thus $B^{W,\bar{C}} = B^{\bar{C}}$ can be defined from $\bar{C}$ only, i.e., $B^{\bar{C}}$ is independent of the particular non-extendible good set W which was chosen. $B^{\bar{C}}$ is non-empty, finite and definable from

$$\bar{C} \in V_W = V_{\bar{z}}, \quad \bar{z} \in W.$$

(see 10.22(c)). V_W is a finite $(-l)$-definable set. Let V^{**} be the set of all $y \in V_W$ such that for some proper $\bar{z}$, $\bar{G}_{f^*}(\bar{z}) = y$, and for which there exists a finite non-empty $(\bar{G}_f, y)$-good set which is not extendible to a larger $(\bar{G}_f, y)$-good set. V^{**} is clearly a finite, non-empty, $(\bar{C} \in V^{**})$, and $(-l)$-definable$_\alpha$ set. Define

$$B = \bigcup_{\bar{C} \in V^*} B^{\bar{C}} .$$

Then B is non-empty, finite, and $(-l)$-definable$_\alpha$. This is a contradiction, because on the other hand, B contains only sets $G_j(\bar{z})$, which are not $(-l)$-definable$_\alpha$.

10.26. Corollary. There is an infinite $(\bar{G}^k, \bar{G}_{h^*}(\bar{a}_h))$-good set W such that $\bar{a}_h \in W$.

This follows from the fact that $K(a_l)$ is well ordered in N.

10.3. The crucial step of Lemma 10.10 has been completed; however the separation requirement Definition 2.5(e) on conditions is essential for the success of the next step, in which we show:

10.30. Sub-lemma. *If $W = \{\bar{z}^j\} \mid 1 \leqslant j \leqslant m\}$ is a finite $(\bar{G}^k, \bar{C})$-good set, then $p[\bar{z}^1], ..., p[\bar{z}^m]$ can be extended to a condition.*

Proof. $\bar{z} \in W$ are proper, hence $N_\alpha \models \hat{p}^\alpha[\bar{z}^j]$, $1 \leqslant j \leqslant m$; thus

10.31 $\quad p' \Vdash_\alpha \hat{p}^\alpha[\bar{z}^1] \wedge \dots \wedge \hat{p}^\alpha[\bar{z}^m]$, for some $p' \in Q_\alpha$,

(where p has been identified with $\tilde{p}$, see Definition 7.10). We can also assume that p' forces$_\alpha$ everything required of the $(\bar{G}^k, \bar{C})$-good set W, as listed in Definition 10.21. Moreover, we assume that $p' \in Q_\alpha$ decides$_\alpha$ all the statements

$$H^{g_i}_{\sigma_i}(\bar{\theta}_i \cdot \bar{b}_i)\,[\bar{z}] = H^{g_j}_{\sigma_j}(\bar{\theta}_j \cdot \bar{b}_j)\,[\bar{z}'], \qquad 1 \leqslant i, j \leqslant k,\ \ \bar{z}, \bar{z}' \in W.$$

10.311. By Lemmas 4.62, 4.63, we can extend p' to a condition $\bar{p} \in P_\alpha$ such that

$$\bigcup_{j=1}^{m} \hat{p}^\alpha[\bar{z}^j] \subseteq \bar{p}\,.$$

(Note that $\bar{p}$ can even be assumed to be in Q_α.)

We now intend to turn $\bar{p} \cup \bigcup_{\bar{z}\in W}(p - \hat{p}^\alpha)[\bar{z}]$ into a condition. This will be done by adding connections to $\bigcup_{\bar{z}\in W}(p - \hat{p}^\alpha)[\bar{z}]$ complying with the requirements (d), (e), (c)(2), (c)(3) and (c)(4) of Definition 2.50, and then by shifting coordinate information in accordance with requirement 2.51(c)(1). This will automatically take care of requirement 2.51(b); 2.51(a) will trivially follow. (The reader will recall that by a connection we mean a pre-condition of the form $\langle e, \bar{\theta}, l, \bar{\theta}', l' \rangle$, in which case we say that it is a connection between the elements $\chi_e(\bar{\theta}\cdot\bar{b})$ and $\chi_e(\bar{\theta}'\cdot\bar{b})$, see also (4.10).)

In order to deal with the requirements of 2.51(d) and 2.51(c), we will have to differentiate between the elements of

$$\bigcup_{\bar{z}\in W}(p - \hat{p}^\alpha)\,[\bar{z}]$$

according to the

$$G_j(\bar{z}) = {}^\alpha H^{g_j}_{\sigma_j}(\bar{\theta}_j \cdot \bar{b}_j)\,[\bar{z}]\ .$$

It will be seen that since W was chosen to be $(\bar{G}^k, \bar{C})$-good, the $G_j(\bar{z})$ are separated in a manner enabling us to connect up the elements of $\bigcup_{\bar{z}\in W}(p - \hat{p}^\alpha)[\bar{z}]$ as desired.

First $\{j \mid 1 \leqslant j \leqslant k\}$ is divided into the following disjoint sets:

10.32. Definition.

$${}^0_aR = \{j \mid 1 \leqslant j \leqslant k,\ l \in g_j\}, \quad {}^0_bR = \{j \mid 1 \leqslant j \leqslant k,\ l \notin h_j\}$$

${}^0_aR$, ${}^0_bR$, are obviously disjoint; set ${}^0R = {}^0_aR \cup {}^0_bR$.

$${}^1R = \{j \mid 1 \leqslant j \leqslant k,\ l \notin g_j\}$$

$${}^1_eR = \{j \in {}^1R \mid \text{the first component of } g_j \text{ is } e\},$$

e is an $\alpha+1$-index. ${}^0R$, ${}^1_eR$ are disjoint and

$$\{j \mid 1 \leqslant j \leqslant k\} = {}^0R \cup {}^1R.$$

Note that according to the previous division of

$$p - \hat{p}^\alpha = \{q_j \mid 1 \leqslant j \leqslant k\}$$

into

$$\{q_j \mid j \in h^+\} \cup \{q_j \mid j \in h^-\},$$

we have

$${}^0R = \{j \mid j \in h^+\}, \qquad {}^1R = \{j \mid j \in h^-\}.$$

10.33. Definition. For any $\bar{z} \in I(h)$, define the following sets:

$${}^0R(\bar{z}) = \{G_j(\bar{z}) \mid j \in {}^0R\},$$

$${}^0_aR(\bar{z}) = \{G_j(\bar{z}) \mid j \in {}^0_aR\},$$

$${}^0_bR(\bar{z}) = \{G_j(\bar{z}) \mid j \in {}^0_bR\},$$

$${}^1R(\bar{z}) = \{G_j(\bar{z}) \mid j \in {}^1R\}, \quad {}^1_eR(\bar{z}) = \{G_j(\bar{z}) \mid j \in {}^1_eR\},$$

e an $\alpha + 1$-index.

Since W is $(\bar{G}^k, \bar{C})$-good, we obviously have that:

$$(10.34) \quad {}^0R(\bar{z}) = {}^0R(\bar{a}_h), \qquad {}^0_aR(\bar{z}) = {}^0_aR(\bar{a}_h),$$
$$\qquad {}^0_bR(\bar{z}) = {}^0_bR(\bar{a}_h), \qquad \text{for all } \bar{z} \in W.$$

Note also that,

$$(10.35) \quad {}^1_eR(\bar{z}) \cap {}^1_eR(\bar{z}') = \emptyset \quad \text{for } \bar{z}, \bar{z}' \in W, \quad \bar{z} \neq \bar{z}'.$$

We now divide $\mathbf{U}_{\bar{z} \in W}(p - \hat{p}^\alpha)[\bar{z}]$ into sets corresponding to the divisions we have made of

$$\{G_q(\bar{z}) \mid q \in p - \hat{p}^\alpha, \bar{z} \in W\},$$

from which it will become clear how to add the necessary connections.

10.40. Definition.

$$^0V = \{q_j[\bar{z}] \mid j \in {}^0R, \bar{z} \in W\},$$

$$^0_aV = \{q_j[\bar{z}] \mid j \in {}^0_aR, \bar{z} \in W\},$$

$$^0_bV = \{q_j[\bar{z}] \mid j \in {}^0_bR, \bar{z} \in W\},$$

$$^1V = \{q_j[\bar{z}] \mid j \in {}^1R, \bar{z} \in W\},$$

$$^1_eV = \{q_j[\bar{z}] \mid j \in {}^1_eR, \bar{z} \in W\},$$

$$^1_eV(\bar{z}) = \{q_j[\bar{z}] \mid j \in {}^1_eR\}, \quad \bar{z} \in W.$$

We want to show that $\bar{p} \cup {}^0V \cup {}^1V$ can be extended to a condition, in which case we are done, since

$$\bar{p} \cup {}^0V \cup {}^1V \supseteq \bigcup_{\bar{z} \in W} p[\bar{z}].$$

We shall successively extend $\bar{p} \in P_\alpha$ to a condition in $P_{\alpha+1}$, the final result of which will contain

$$\bar{p} \cup {}^0V \cup {}^1V \cup {}^2V \supseteq \bigcup_{\bar{z} \in W} p[\bar{z}].$$

First recall that the $p[\bar{z}]$ are conditions and that

$$\bar{p} \supseteq \bigcup_{\bar{z} \in W} \hat{p}^\alpha[\bar{z}]$$

(see 10.31, 10.311). Since for all $j \in {}^0_bR$, $\bar{z} \in W$, $q_j[\bar{z}] = q_j$, it obviously follows that $\bar{p} \cup {}^0_bV$ is a condition. Moreover, $\bar{p} \cup {}^0_bV \cup {}^0_aV$ is easily extended to a condition by adding the following connections: let $j, j' \in {}^0R$ be such that

$$e_j = e_{j'} = e, \qquad \bar{b}_j = \bar{b}_{j'} = \bar{b},$$

and that they mention $\chi_e(\bar{\theta}_j \cdot \bar{b})$, $\chi_e(\bar{\theta}_{j'} \cdot \bar{b})$ respectively. Suppose also, that

$$\langle e, \bar{\theta}_j, l, \bar{\theta}_{j'}, l' \rangle \in p - \hat{p}^\alpha;$$

then if for $\bar{z}, \bar{z}' \in W$,

$$G_j(\bar{z}) = {}^{\alpha}H^{gj}_{\alpha j}(\bar{\theta}_j \cdot \bar{b})\,[\bar{z}] = {}^{\alpha}H^{gj}_{\alpha j}(\bar{\theta}^* \cdot \bar{b})$$
$$= G_j(\bar{a}_h) = {}^{\alpha}H^{gj}_{\alpha j}(\bar{\theta}_j \cdot \bar{b})\,,$$

$$G_{j'}(\bar{z}') = {}^{\alpha}H^{gj'}_{\alpha j'}(\bar{\theta}_{j'} \cdot \bar{b})\,[\bar{z}] = {}^{\alpha}H^{gj}_{\alpha j}(\bar{\theta}_{j'} \cdot \bar{b})\,[\bar{z}]$$
$$= {}^{\alpha}H^{gj}_{\alpha j}(\bar{\theta}^{**} \cdot \bar{b}) = G_{j'}(\bar{a}_h) = {}^{\alpha}H^{gj}_{\alpha j}(\bar{\theta}_{j'} \cdot \bar{b})\,,$$

then the connections $\langle e, \bar{\theta}^*, l, \bar{\theta}^{**}, l'\rangle$, $\langle e, \bar{\theta}^{**}, l', \bar{\theta}^*, l\rangle$, $\langle e, \bar{\theta}^*, 0, \bar{\theta}_j, 0\rangle$, $\langle e, \bar{\theta}^{**}, 0, \bar{\theta}_{j'}, 0\rangle$, $\langle e, \bar{\theta}^*, l, \bar{\theta}_{j'}, l'\rangle$, $\langle e, \bar{\theta}_j, l, \bar{\theta}^{**}, l'\rangle$, are to be added to p_0. Clearly if the result of this operation is $p_0 \supseteq \bar{p} \cup {}^0V$, then p_0 is a condition. Note that the coordinate information on $\chi_e(\bar{\theta}^* \cdot \bar{b})$, $\chi_e(\bar{\theta}^{**} \cdot \bar{b})$ will automatically correspond, and all other necessary connections, demanded by the transitivity requirement and (c)(4) will be in p_0. Note also that since the $\bar{z}$ are proper, requirements 2.51(d), and 2.51(e) are fulfilled; see (10.16).

The first thing that has to be done in order to obtain a condition from $p_1 \cup {}^1V$ is to properly add connections between all elements $\chi_e(\bar{\theta} \cdot \bar{b})$ mentioned in ${}^1_eV(\bar{z}')$, $\bar{z} \neq \bar{z}'$. This is done as follows:

for some fixed e, let $V_1, \ldots, V_n$ be all the different ${}^1_eV(\bar{z})$ included in ${}^1_eV$. *We remark that since W is $(\bar{G}^k, \bar{C})$-good, they are disjoint.* Because if $\chi_{e_m}(\bar{\theta} \cdot \bar{b})$ is mentioned in V_i and $\chi_{e_{m'}}(\bar{\theta}' \cdot \bar{b})$ is mentioned in V_j, $e_m = e_{m'}$, $i \neq j$, then

$${}^{\alpha}H^{gm}_{\alpha m}(\bar{\theta} \cdot \bar{b}) \neq {}^{\alpha}H^{gm}_{\alpha m}(\bar{\theta}' \cdot \bar{b})\,,$$

and in particular $\bar{\theta} \neq \bar{\theta}'$. If χ_e is mentioned in V_1, then χ_e is mentioned in each of the V_i, $1 \leq i \leq n$. A representative element $\chi_e(\bar{\theta}^{(i)} \cdot \bar{b}^{(e)})$ mentioned in V_i is chosen for each i, $1 \leq i \leq n$. A connection will be properly chosen between $\chi_e(\bar{\theta}^{(i)} \cdot \bar{b}^{(e)})$ and $\chi_e(\bar{\theta}^{(i+1)} \cdot \bar{b}^{(e)})$, $1 \leq i \leq n-1$, as required by (d)(4) and then ${}^1_eV$ will be closed by adding all connections that necessarily follow from 2.51(c)(3), 2.51(c)(2), and 2.51(c)(4). The idea is to make $\chi_e(\bar{\theta}^{(i)} \cdot \bar{b}^{(e)})$ and $\chi_e(\bar{\theta}^{(i+1)} \cdot \bar{b}^{(e)})$ so far apart in the natural tail order that there will be no conflicting information resulting from the transfer of initial information as specified in 2.51(c)(1). Note that no connections exist in $p_1 \cup {}^1V$ between $\chi_e(\bar{\theta}^{(i)} \cdot \bar{b}^{(e)})$ and

$\chi_e(\bar{\theta}^{(j)} \cdot \bar{b}^{(e)})$, $i \neq j$, because $\chi_e(\bar{\theta}^{(i)} \cdot \bar{b}^{(e)})$ is mentioned in some $q[\bar{z}] \in V_i$, and $\chi_e(\bar{\theta}^{(j)} \cdot \bar{b}^{(e)})$ is mentioned in some $q'[\bar{z}'] \in V_j$ with $e_q = e_{q'}$ and $\bar{z} \neq \bar{z}'$; (see definition of ${}_eV$, and the previous remark).

Let s_i be an integer larger than an integer mentioned in V_i, $1 \leqslant i \leqslant n$ and define the following set Y_e of connections:

$$Y_e = \{\langle e, \bar{\theta}^{(1)}, 0, \bar{\theta}^{(2)}, 2(s_1 + s_2)\rangle, \langle e, \bar{\theta}^{(2)}, 0, \bar{\theta}^{(3)}, 2(s_2 + s_3)\rangle, \ldots,$$

$$\langle e, \bar{\theta}^{(n-1)}, 0, \bar{\theta}^{(n)}, 2(s_{n-1} + s_n)\rangle\} ,$$

(see Fig. 8).

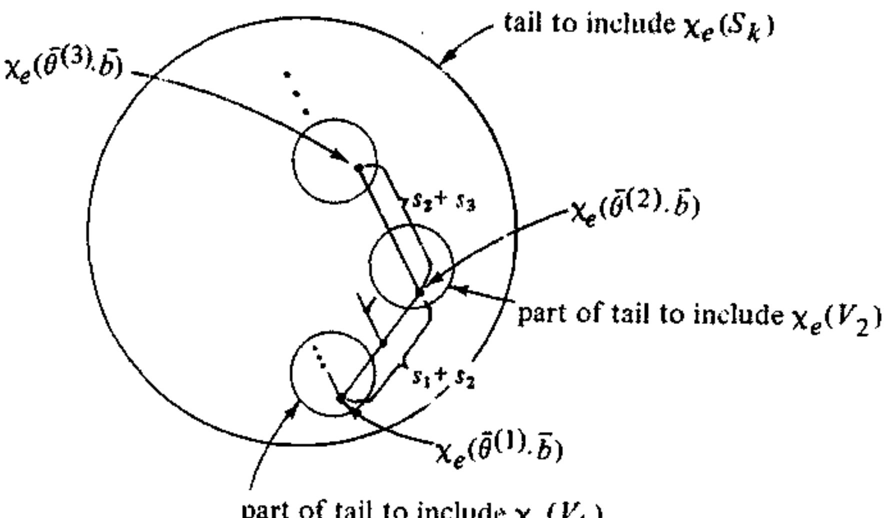

Fig. 8.

Let $Y = \bigcup_e Y_e$ and consider $p_1 \cup {}^1V \cup Y = p_1'$. Close p_1' by adding all the connections implied by Definition 2.5(c)(2), (3), (4), and transfer coordinate information according to (c)(1). Denote the result by p_2. It is readily shown that p_2 is a condition, since s_j was chosen so large that coordinate information of $V_j \subseteq {}_e^1V$, $i \neq j$ in p_2. (Note that if ${}_eU$ is the result of the closure operations on ${}_e^1V \cup Y_e$, then

$$p_2 = p_1 \cup \bigcup_e {}_e\overline{U} \ .)$$

We omit further details which amount to a direct systematic simple check of the requirements on a condition.

Sub-lemma 10.30 has thus been proven.

From 10.30 and Corollary 10.26 we obtain 10.12 which can be stated as follows:

10.41. Sub-lemma. *For every $p(a_l) \in Q_{\alpha+1}$ and arbitrary $m < \omega$, there are distinct*

$$z_1, \ldots, z_m \in I(l)$$

such that

$$p(a_l)\ [z_1] \cup \ldots \cup p(a_l)\ [z_m] \cup p(a_l)$$

can be extended to a condition.

(We have displayed only the generic real constant a_l mentioned by p.)

We can now complete the induction step concerning Sub-lemma 10.10(a):

10.50. *For all $p(\bar{a}_h) \in P_{\alpha+1}$ which is satisfied by some $\bar{z}^* \in I(h)$, and for any $l \in h$, there are $\aleph_0$ $\bar{z} \in I(h)$ which are g-congruent to $\bar{z}^*$ that satisfy p in $N_{\alpha+1}$, where $g = h - \{l\}$.*

Proof. In what follows we only display the generic real constant a_l which possibly occurs in the conditions or formulas. If the lemma is not true, then for some $z_j = \theta_j(\bar{a}_l)$, $1 \leq j \leq m$, and $p'(a_l) \in Q_{\alpha+1}$,

$$(10.51) \quad p'(a_l) \Vdash_{\alpha+1} \bigwedge_{j=1}^{m} \tilde{p}(\theta_j(a_l)) \wedge \\ \wedge (\forall_{\omega+1} u)\,(u \in I(l) \wedge \tilde{p}(u) \to \bigvee_{j=1}^{m} u = \theta_j(a_l)) \, .$$

By Sub-lemma 10.41, for any $n < \omega$ there are distinct $x_i \in I(l)$ for which

$$p'(a_l) \cup \bigcup_{1 \leq i \leq n} p'[x_i]$$

is extendible to a condition. Since the tail members are generic, hence non-periodic, there must be such an $x = \theta(a_l)$ for which

$$\theta_j(\theta(a_l)) \neq \theta_k(a_l), \qquad 1 \leq j, k \leq m.$$

We may assume that

$$Q_{\alpha+1} \ni p' \Vdash_{\alpha+1} \bigwedge_{1 \leq j, k \leq m} \theta_j(\theta(a_l)) \neq \theta_k(a_l).$$

Let p'' be a condition extending $p'(a_l) \cup p'(a_l)\,[x]$. By Lemma 5.44,

$$p'(a_l)\,[x] \Vdash_\alpha \bigwedge_{j=1}^{m} \tilde{p}(\theta_j(\theta(a_l))).$$

Hence by (10.51),

$$\begin{aligned} p' \cup p'[x] \subseteq p'' \Vdash_{\alpha+1} & \bigwedge_{j=1}^{m} \tilde{p}(\theta_j(a_l)) \wedge \bigwedge_{j=1}^{m} \tilde{p}(\theta_j(\theta(a_l))) \wedge \\ & \wedge \bigwedge_{k,j=1}^{m} (\theta_j(\theta(a_l)) \neq \theta_k(a_l)) \wedge \\ & \wedge (\forall_{\omega+1} u)\,(u \in I(l) \wedge \tilde{p}(u) \to \bigvee_{j=1}^{m} u = \theta_j(a_l)). \end{aligned}$$

This obviously yields a contradiction.

To complete the induction step concerning part (b) of Lemma 10.10, we assume this part of the lemma true for all $\alpha < \beta \leq \aleph_1^{(M)}$ and we prove for β. If the claim is not true, then for some $l \in \omega$, $h \subseteq \omega$, $l \notin h$, a_l is supported by h, i.e., there is a term $\sigma(\bar{a}_h) \in \mathcal{T}^c_\beta$ such that

$$N_\beta \models \sigma(\bar{a}_h) = a_l.$$

Thus for some $p(a_l) \in Q_\beta$,

$$p(a_l) \Vdash_\beta \sigma(\bar{a}_h) = a_l.$$

For some $\alpha < \beta$, $p \in P_{\alpha+1}$, thus we can apply 10.50, and as in that proof, there is an $x = \theta(a_l) \neq a_l$ such that

$$p(a_l) \cup p(a_l)\,[x]$$

is extendible to a condition $p'(a_l) \in Q_\beta$; we may also assume that

$$Q_\beta \quad p \Vdash_\beta \theta(a_l) \neq a_l.$$

By Lemma 5.44,

$$p(a_l)\,[x] \Vdash_\beta \sigma(\bar{a}_h) = \theta(a_l)\,.$$

Hence

$$p' \Vdash_\beta \sigma(\bar{a}_h) = a_l \wedge \sigma(\bar{a}_h) = \theta(a_l) \wedge a_l \neq \theta(a_l)\,.$$

This is a contradiction (by Lemmas 2.811(a), 2.812).

Lemma 10.10 has thus been proved.

10.52. Corollary. *For any $p(a_l) \in Q$, and $n < \omega$, there are n distinct $z_i \in I(l)$ such that*

$$N \models \bigwedge_{i=1}^{n} p(a_l)\,[z_i]\,,$$

and such that

$$p(a_l) \cup \bigcup_{i=1}^{n} p(a_l)\,[z_i]$$

is extendible to a condition.

Proof. Assume $n < \omega$; then by (10.50) there are n distinct $z_i \in I(l)$, $1 \leqslant i \leqslant n$, which satisfy p. Thus if $z_i = \theta_i(a_l)$, then

$$N \models \bigwedge_{i=1}^{n} \tilde{p}(\theta_i(a_l)) \wedge \tilde{p}(a_l)\,.$$

Hence for some $p' \in Q$,

$$p' \Vdash \tilde{p}(a_l) \wedge \bigwedge_{i=1}^{n} \tilde{p}(\theta_i(a_l))\,.$$

By Lemmas 4.62, 4.63, p' can be extended to a condition p'' which includes

$$p(a_l) \cup \bigcup_{i=1}^{n} p(a_l)\,[z_i]\,,$$

as required. Finally,

10.60. Lemma. *There is no choice function in N for the set* $\{I(i) \mid i < \omega\}$.

Proof. Assume that $\sigma(\bar{a}_h)$ is a choice function for $\{I(i) \mid i < \omega\}$ in N. Then for some $p \in Q$,

$$p \Vdash (\forall v)\,(v \in \dot{\omega} \to \text{“}|\sigma(\bar{a}_h) \cap I(v)| = i\text{”}).$$

Let l be an integer not in h; then for some $p' \in Q$, $p \subseteq p'$, and $\theta \in \Gamma$,

$$(10.61) \quad p'(a_l) \Vdash \sigma(\bar{a}_h) \cap I(l) = \theta(a_l)\,.$$

By Lemma 10.52 there is an unlimited number of $y \in I(l)$ for which $N \models p'(a_l)\,[y]$, and $p'(a_l) \cup p'(a_l)\,[y]$ is included in a condition. Hence, there is an $x = \boldsymbol{\theta}^*(a_l)$ such that

$$\boldsymbol{\theta}(\boldsymbol{\theta}^*(a_l)) = \boldsymbol{\theta}(a_l), \qquad N \models p'(a_l)\,[x]\,,$$

and for some $p'' \in P$, $p' \cup p'[x] \subseteq p''$. We may assume that

$$Q \ni p \Vdash \theta(\theta^*(a_l) \neq \theta(a_l)\,.$$

By Lemma 5.44, we get from (10.61) that,

$$(10.62) \quad p'(a_l)\,[x] \Vdash \sigma(\bar{a}_h) \cap I(l) = \theta(\theta^*(a_l))\,.$$

From (10.61) and (10.62) we get

$$\begin{aligned} p' \cup p'[x] \subseteq p'' \Vdash \sigma(\bar{a}_h) \cap I(l) &= \theta(a_l) \wedge \\ &\wedge\ \sigma(\bar{a}_h) \cap I(l) = \theta(\theta^*(a_l)) \wedge \\ &\wedge\ \theta(\theta^*(a_l)) \neq \theta(a_l)\,. \end{aligned}$$

This is a contradiction, (by Lemmas 2.811, 2.812).

All told we have shown that,

10.70. Theorem. Con(ZF) *implies* Con(ZF + *the ordering theorem + the* $\aleph_0$*-multiple choice axiom + every infinite cardinal is idemmultiple + the continuum is not well-orderable*).

Proof. 10.70 follows from Lemmas 2.91, 9.30, 9.95 and 10.60.

By Lemmas 9.96, 9.30 and Definition 5.52, it follows that

10.80. Theorem. *In* ZF, Z($\aleph_0$) *does not imply the ordering theorem, nor the idemmultiple hypothesis, nor that the continuum is well-orderable.*

Let Idm denote "the idemmultiple hypothesis". It would be interesting to know if:

$$\mathrm{ZF} \vdash \mathrm{Idm} \to \text{"the ordering theorem"};$$

$\mathrm{ZF} \vdash \mathrm{Idm} \to Z(\aleph_0)$; or even if Idm in conjunction with one of these two statements implies the other in ZF.

Pincus has shown (private communication), that

$$\mathrm{ZF} \nvdash (Z(\aleph_0) + \text{"the ordering theorem"}) \to \mathrm{Idm}.$$

We know that $(\forall \mathfrak{m} \nless \aleph_0)(\mathfrak{m}^2 = \mathfrak{m})$ is equivalent to AC; it would be interesting to know if there are any weak choice axioms equivalent to Idm. As mentioned in the introduction, Truss [24] has shown that

$$\mathrm{ZF} \nvdash \text{"every cardinal has a 3-successor"} \to \mathrm{Idm}.$$

On the other hand, the author has shown that

$$\mathrm{ZF} \nvdash \mathrm{Idm} \to \text{"every cardinal has a three successor"}.$$

It would be of interest to know whether

$$\mathrm{ZF} \vdash \mathrm{Idm} \wedge \text{"every cardinal has a 3-successor"} \to \mathrm{AC}.$$

It is not known to the author whether the Boolean prime ideal theorem, or the order extension principle holds in N. It is natural to consider the following generalization of Idm. Is it consistent to assume in ZF without choice that for all cardinals $\mathfrak{m}, \mathfrak{n}$, $\mathfrak{n} < \mathfrak{m}$ implies $\mathfrak{n}\mathfrak{m} = \mathfrak{m}$? A simple argument considering $\mathfrak{m} = \mathfrak{n} + \aleph(n)$, where $\aleph(n)$ is Hartog's aleph shows that this is impossible. On the other hand a straightforward generalization of our construction shows that it is consistent to assume in $\mathrm{ZF} \wedge \neg\mathrm{AC}$ that

$$(\forall \mathfrak{m})(\mathfrak{m} < \aleph_\alpha \vee \aleph_\alpha \cdot \mathfrak{m} = \mathfrak{m}),$$

for any fixed regular cardinal $\aleph_\alpha$. In this connection we have:

$$\mathrm{ZF} \vdash (\forall \mathfrak{m} \nless \aleph_0)(2\mathfrak{m} = \mathfrak{m}) \to (\forall \mathfrak{m} \nless \aleph_0)(\aleph_0 \mathfrak{m} = \mathfrak{m});$$

though is it probably not provable in ZF that

$$(\forall \mathfrak{m} \nless \aleph_0)\,[\forall \beta < \alpha)(\aleph_\beta \cdot \mathfrak{m} = \mathfrak{m} \to \aleph_\alpha \cdot \mathfrak{m} = \mathfrak{m})],$$

for any aleph, $\aleph_\alpha$, $\alpha > 0$.

References

[1] F. Bernstein, Untersuchungen aus der Mengenlehre, Math. Ann. 61 (1905) 117–155.
[2] P.J. Cohen, Set Theory and the Continuum Hypothesis (Benjamin, New York, 1966).
[3] W.B. Easton, Powers of regular cardinals, Ann. Math. Logic 1 (1970) 139–178.
[4] E. Ellentuck, Generalized idempotence in cardinal arithmetic, Fund. Math. 58 (1966) 241–258.
[5] J.D. Halpern and P.E. Howard, Cardinals m such that 2m = m, Proc. Am. Math. Soc. 26 (1970) 487–490.
[6] J.D. Halpern and P.E. Howard, Cardinal addition and the Axiom of Choice, Bull. Am. Math. Soc. 80 (1974) 584–586.
[7] J.D. Halpern and A. Lévy, The Boolean Prime Ideal Theorem does not imply the Axiom of Choice, in: Proc. Symp. Pure Math. 13 (I) (Am. Math. Soc., Providence, R.I.)
[8] T.J. Jech, The Axiom of Choice (North-Holland, Amsterdam, 1973).
[9] A. Lévy, Axioms of multiple choice, Fund. Math. 50 (1961–62) 475–483.
[10] A. Lévy, The Fraenkel–Mostowski method for independence proofs in set theory, in: J.W. Addison et al., eds., The Theory of Models (North-Holland, Amsterdam, 1965) 225–248.
[11] A. Lévy, Definability in Axiomatic Set Theory I, in: Y. Bar-Hillel, ed., Logic, Methodology and Philosophy of Science (North-Holland, Amsterdam, 1966) 127–151.
[12] A.R.D. Mathias, A Survey of Recent Results in Set Theory, to appear.
[13] A. Mostowski, Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip, Fund. Math. 32 (1939) 201–252.
[14] H. Rubin and J.E. Rubin, Equivalents of the Axioms of Choice (North-Holland, Amsterdam, 1963).
[15] J.E. Rubin, Non-constructive properties of cardinal numbers, Israel J. Math. 10 (1971) 504 et seq.
[16] W. Sierpiński, Sur l'égalite 2 m = 2 n pour les nombres cardinaux, Fund. Math. 3 (1922) 1–16.
[17] W. Sierpiński, Leçons sur les Nombres Transfinis (Warsaw, 1928).
[18] W. Sierpiński, Sur l'implication $(2m \leqslant 2n) \to (m \leqslant n)$ pour les nombres cardinaux, Fund. Math. (1947) 148–154.
[19] W. Sierpiński, Cardinal and Ordinal Numbers (Warsaw, 1955) 14–53.
[20] R. Solovay and S. Tannenbaum, Iterated Cohen extensions and Souslin's problem, Ann. Math. 94 (1971).
[21] A. Tarski, Sur quelques theoremes qui equivalent a l'Axiome du Choix, Fund. Math. 5 (1924) 147–154.
[22] A. Tarski, Cancellation laws in the arithmetic of cardinals, Fund. Math. 36 (1949) 77–92.
[23] A. Tarski, Theorems on the existence of successors of cardinals, Indag. Math. 16 (1954) 26–32.
[24] J. Truss, On successors in cardinal arithmetic, Fund. Math. 78 (1973) 7–21.