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Realism, nonstandard set theory, and large cardinals

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Abstract

Mathematicians justify axioms of set theory "intrinsically", by reference to the universe of sets of their intuition, and "extrinsically", for example, by considerations of simplicity or usefullness for mathematical practice. Here we apply the same kind of justifications to Nonstandard Analysis and argue for acceptance of BNST$^+$ (Basic Nonstandard Set Theory plus additional Idealization axioms). BNST$^+$ has nontrivial consequences for standard set theory; for example, it implies existence of inner models with measurable cardinals. We also consider how to practice Nonstandard Analysis in BNST$^+$, and compare it with other existing nonstandard set theories.

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1. Introduction

Since their introduction by Abraham Robinson [41], methods of Nonstandard Analysis have been found fruitful in a number of areas of pure and applied mathematics. Most practitioners of these methods work in the setting of superstructures devised by Robinson and Zakon [42]. This raises no foundational issues: superstructures, even with additional useful properties, such as various levels of saturation, can be proved to exist in ZFC. However, there are reasons to be less than satisfied with the superstructure framework. The definition of superstructures explicitly invokes the cumulative hierarchy, a concept not used in mathematics outside of logic. The cumulative hierarchy has height ω, so the standard sets in superstructures fail to satisfy ZFC. Superstructures thus do not allow the full use of existing mathematical techniques. Besides these technical shortcomings there is a more fundamental issue. Some of the objects admitted

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by Nonstandard Analysis into rigorous mathematics, most notably infinite natural numbers, seem so basic, and their presence effects the universe of mathematical objects so profoundly, that one would like to see them treated as an integral part of it, as, for instance, natural numbers are.

These concerns, together with a desire to facilitate the practice of Nonstandard Analysis, led to the formulation of axiomatic nonstandard set theories by Hrbacek [22], Nelson [37] and others; Di Nasso's recent paper [10] provides an up-to-date survey.

In addition to the above-mentioned common concerns and motivations, all of the existing nonstandard set theories are formulated with a view that it should be possible to regard nonstandard methods as purely formal and, in principle, eliminable from mathematical practice. This takes the concrete form of requiring that the nonstandard set theory (NST) be a conservative extension of ZFC, and all axiomatizations of NST surveyed in [10] are. Nevertheless, these axiomatizations differ widely, and often are mutually incompatible. Few theoretical grounds have been offered for preferring one to another; Kanovei and Reeken [27] do discuss some criteria for doing so, Reduction Property and Model Enlargement Property in particular, but the rationale for these requirements is certainly debatable (see Section 7 for more details). In particular, it seems to be tied to the above-mentioned formalist view of the proper role of Nonstandard Analysis.

Here I shall follow a different route. I propose to look for axioms for NST which are "*true*", without (*a priori*) regard for needs of practitioners or the formalist requirement that the resulting theory be conservative over ZFC. Obviously, this entails some kind of realist position vis-à-vis existence of nonstandard mathematical objects. In this paper, I take it for granted that nonstandard objects exist in the same sense as the standard ones, and examine the *mathematical* consequences of this view. I shall not attempt to address the manifold *philosophical* issues such a position raises, beyond a few comments and historical remarks.

According to Robinson [41], Leibniz thought of infinitesimals as ideal elements, a useful fiction one can always dispense with. For example, in a letter to P. Varignon of February 2, 1702, Leibniz writes

> d'où il s'ensuit, que si quelque'un n'admet point des lignes infinies et infiniment petites à la rigueur métaphysique et comme des choses réelles, il peut s'en servir sûrement comme des notions idéales qui abrègent le raissonnement... (quoted in [41, p. 262]).

This seems to have been the prevalent view at the time. Nevertheless, there were contemporaries of Leibniz, for example l'Hospital, who did admit infinite and infinitely small "lines" as "real things". All this was of course well before actual infinity became entrenched in rigorous mathematics by work of Cantor, so perhaps his views have more relevance for modern mathematics. It is well known that Cantor was strongly opposed to infinitesimals. Dauben [7, pp. 129–132, 233–236] discusses Cantor's views on infinitesimals in detail. Cantor extended the counting process into the transfinite to produce "enumerals of well-ordered sets" [21, p. 51] with ω being the smallest transfinite enumeral (ordinal number). In this picture there seems to be no place for infinitely large natural numbers. Another reason to reject infinitesimals was Cantor's commitment to his (and Dedekind's) theory of the continuum (as the complete linearly ordered set $(\mathbf{R}, <)$), which does not allow for infinitely small and large real numbers.

Of course, both Leibniz and Cantor were concerned with defending their own theories (Infinitesimal Calculus and Set Theory, resp.) against criticism by sceptics of their time. Much has been learned about logic and set theory since then, and we can feel on safer grounds handling nonstandard concepts. Practitioners of Nonstandard Analysis nowadays claim intuition about nonstandard objects that differs little from the intuition about standard ones. For example Robinson [41, p. 282] acknowledges:

> "Whatever our outlook and in spite of Leibniz' position, it appears to us today that the infinitely small and infinitely large numbers of a non-standard model of analysis are neither more nor less real than, for example, the standard irrational numbers."

Nelson [37, p. 1197] states "...infinitesimals and other idealized elements were there all along in the sets with which we are familiar..."

I think that there is a *prima facie* case for the realist approach.

Another objection to this project can be raised on the grounds of set-theoretic reductionism. It may be helpful to distinguish here between *strong reductionism*, i.e. the position that all objects of mathematical interest *are* (standard) sets, and *weak reductionism* (all mathematical structures are *representable* as, or *isomorphic* to, sets). Weak reductionism has served modern mathematics well and should not be abandoned lightly; in fact, it will be one of our guiding principles. On the other hand, I think there are epistemic reasons for abandoning strong reductionism in this situation (and in others): considering some mathematical objects as primitive (rather than as pure sets) may give us access to intuition about them that is not, or at least not readily, available from their (isomorphic) set-theoretic representation. (In Section 4 we shall see that this is helpful even for the understanding of the Power Set Axiom.)

Finally, it should be noted that one does not have to embrace the realist position for this paper to retain at least some meaning. Even realists tend to buttress their axioms by reasons that are partially or fully extrinsic, i.e., based on considerations of simplicity, utility, elegance, etc. (Maddy [35] catalogued the kinds of reasons set theorists use for this purpose.) Such arguments may carry some conviction even in the absence of realist grounding.

In summary, I shall attempt to justify acceptance of the proposed axioms for NST by arguments that are of the same kind as the arguments usually offered in support of various axioms of standard set theory. As it turns out, these "intuitively obvious" NST axioms have consequences for standard set theory that go beyond ZFC; in particular they imply consistency of measurable cardinals. We can regard this entire enterprise as an "experiment" with realism in mathematics, and look for confirmation of the "predictions" of NST regarding the standard universe by purely standard arguments.

The plan of the rest of the paper is as follows. In Section 2 I briefly describe the intuitive universe of nonstandard set theory, state the basic axioms (BNST), and derive some easy consequences from them. In Section 3 I discuss the reasons for adopting these axioms in more detail. The axiom of Power Set is crucial, and receives special attention in Section 4. There I show how it results from the need to provide a (standard) set-theoretic model of the continuum and its generalizations, and how similarly, the desire to provide nonstandard models of the continuum and its generalizations leads to axiom A_1. In Section 5 axiom A_1 and its variants (A_0, A_2, B_1, B_2) are arrived at from more general considerations of existence of "ideal" elements. Section 6 is technical; it is devoted to showing that the nonstandard universe has many subuniverses. The results are used in Section 7 to prove that BNST $+$ A_0 implies existence of highly nonregular ultrafilters in the standard universe. It then follows, via well-known results in set theory, that there exist inner models with measurable cardinals. Hence the "intuitively true" axiomatic system for NST proposed here is not a conservative extension of ZFC; rather, it can be viewed as a large cardinal assumption of, at present, unknown consistency strength. Finally, in Section 8 I critique other nonstandard set theories from the realist position and compare them with BNST. The last part of Section 8 takes up the issue of adequacy of BNST for the actual practice of Nonstandard Analysis.

The realist perspective has a bearing on virtually every issue that has been raised by attempts to axiomatize Nonstandard Analysis; as a result, this paper constitutes a survey of this work, albeit from a novel vantage point. I hope that it will help bring nonstandard set theory into the mainstream. (In fact, one of its main points is that nonstandard set theory *is* nothing but the usual set theory, in which the axiom of foundation has been replaced by a (rather complicated) Anti-Foundation Axiom.) Almost all proof techniques used in the paper have been employed before (perhaps in different contexts), either in nonstandard set theory or in model theory. Thus the principal innovation, such as it is, lies in the point of view adopted, and in the axiomatic system proposed.

I have tried to give credit for the first appearance of the ideas discussed here to the best of my knowledge, and I apologize to people whose contributions may have been inadvertently slighted. I am grateful to David Ballard, Mauro Di Nasso and Vladimir Kanovei for pointing out errors in an earlier version of the paper, and for their inspiring comments on its general conception.

This paper was presented at the 1999–2000 ASL Annual Meeting at Urbana (June 3–7, 2000), where Harvey Friedman brought to my attention his unpublished work [16–18]. Although he is not motivated by Nonstandard Analysis, there seems to be a number of points of contact between his theories and BNST; for example, axioms akin to Standardization, Transfer, and "All sets are of standard size", (a crucial consequence of BNST axioms) all play a role in them. On the other hand, there seem to be also many substantial differences; in particular, his "standard" universe W is usually a *set*, and

"Reflection" appears to be the agent responsible for the emergence of large cardinals; this is *not* the case in BNST (on the contrary; see the remark at the end of Section 5). Further study will be needed for the full understanding of the connections.

2. The axioms

We begin with an informal description of the nonstandard universe. First, there are the sets of the conventional set theory; from now on, we shall refer to them as the *standard sets*. We assume that the universe $\mathbb{S}$ of all standard sets satisfies the axioms of ZFC. For a "standard mathematician", these are the only objects of interest. He may agree that $\mathbb{S}$ can be consistently extended by various ideal objects (such as infinitely large natural numbers), but does not consider them real, or at least, does not admit them into mathematics (strong reductionism).

In contrast, our (idealized) "nonstandard mathematician" does admit reality of (at least some) such "ideal" objects. Thus a standard set, although determined by its standard elements, may nevertheless contain also some nonstandard elements. We shall refer to (standard and nonstandard) elements of standard sets that are of interest to a "nonstandard mathematician" as *internal sets*, and postulate that elementary set operations (such as $\cup$, $\times$ etc.) can be extended from the standard universe $\mathbb{S}$ to the *internal universe* $\mathbb{I}$ so as to retain their basic properties. This is merely a generalization to set theory of the fundamental insight of Leibniz that one can perform arithmetic operations on infinitely small and infinitely large numbers according to the usual rules, a *sine qua non* of nonstandard analysis:

> "...et il se trouve que les règles du fini réussissent dans l'infini... et que vice versa les règles de l'infini réussissent dans le fini"... (quoted in [41, p. 262]).

The next assumption concerning our nonstandard set theorist is that, except for his awareness of existence of more objects, he is as much as possible a *classical* set theorist. For example, as each internal set is either standard or nonstandard, "standardnes" is a "definite" property (see [15]) and the nonstandard mathematician recognizes the set A of all nonstandard internal natural numbers. (This is one of the points where our position differs from Nelson's; see his [38].) We shall see that A does not have an $\in$-least element, and is not internal. Thus there is a third kind of sets, noninternal or *external* sets. The usual arguments for believing axioms of ZFC valid in the (standard) set-theoretic universe can now be applied by the nonstandard set theorist to the universe of all *sets* (internal and external) [1] to conclude that all of ZFC except Foundation holds in it. At least initially, we can assume that sets are organized in a von Neumann-like cumulative hierarchy starting with (the class $\mathbb{I}$ of) internal sets instead of $\emptyset$.

[1] For emphasis, we shall sometimes refer to the universe $\mathbb{U}$ of all sets as the *external universe*.

Finally, the nonstandard mathematician can use his extra machinery to describe a collection of standard elements of some standard set. Such description constitutes a "definite" property; as the standard universe is supposed to contain *all* standard sets, it is "obvious" that such a collection forms a standard set. This is Standardization, actually a strong nonstandard version of Separation. (For example, the set of all people who have seen a ghost is a standard set, even though its description refers to "nonstandard" objects.)

We shall elaborate on some aspects of this picture in Section 3; next however we present a formal system that captures the basic properties of the just-described nonstandard universe of sets.

The primitive concepts of BNST (Basic Nonstandard Set Theory) are: variables $x, y, \ldots$ (ranging over *sets*), a binary predicate $\in$ (*membership*), and unary predicates $\mathbb{S}$ (*standard*) and $\mathbb{I}$ (*internal*).

We shall informally use (definable) *classes*, in particular:

$$\mathbb{S} := \{x \mid \mathbb{S}(x)\}, \quad \mathbb{I} := \{x \mid \mathbb{I}(x)\}, \quad \mathbb{U} := \{x \mid x = x\}.$$

If ϕ is an $\in$-formula, we let $\phi^{\mathbb{S}}$ be the formula obtained from ϕ by replacing each subformula of the form $(\exists x)\psi$ [$(\forall x)\psi$, resp.] by $(\exists x)(\mathbb{S}(x) \wedge \psi) \equiv (\exists x \in \mathbb{S})\psi$ [$(\forall x)(\mathbb{S}(x) \Rightarrow \psi) \equiv (\forall x \in \mathbb{S})\psi$, resp.]. Similar conventions will be used for $\mathbb{I}$ and other classes. Also, if R (c, F, resp.) is a relation (constant, operation, resp.) defined by ϕ, then $R^{\mathbb{S}}$ ($c^{\mathbb{S}}$, $F^{\mathbb{S}}$, resp.) denotes the relation (constant, operation, resp.) defined by $\phi^{\mathbb{S}}$ (provided that the appropriate existence and uniqueness statements hold), and similarly for other classes. Occasionally it will be convenient to write $\mathbb{S} \vDash \phi$ in place of $\phi^{\mathbb{S}}$ and say that $\mathbb{S}$ *satisfies* ϕ, etc. (we shall never use $\vDash$ in its model-theoretic sense).

Axioms of BNST:

0. (i) $x \in \mathbb{S} \Rightarrow x \in \mathbb{I}$.
   (ii) $x \in \mathbb{I} \wedge y \in x \Rightarrow y \in \mathbb{I}$.

I. $\phi^{\mathbb{S}}$ where ϕ is any axiom of ZFC.

II. (*Transfer.*) $x_1 \in \mathbb{S} \wedge \cdots \wedge x_n \in \mathbb{S} \Rightarrow (\phi^{\mathbb{S}}(x_1, \ldots, x_n) \equiv \phi^{\mathbb{I}}(x_1, \ldots, x_n))$
   where ϕ is any $\in$-formula whose free variables are among $x_1, \ldots, x_n$.
   In particular, $\phi^{\mathbb{I}}$ holds for all axioms ϕ of ZFC.

III. (*Standardization.*) $(\exists y \in \mathbb{S})\ (\forall z \in \mathbb{S})\ (z \in y \equiv z \in x)$.

IV. ϕ where ϕ is any axiom of ZFC except the axiom of Foundation, with the understanding that the axiom schemata of Separation and Replacement allow arbitrary $\in$-$\mathbb{S}$-$\mathbb{I}$-formulas.

V. $(\exists x)\ (x \in \mathbb{I} \wedge x \notin \mathbb{S})$.

This concludes the list of axioms of BNST. The list is highly redundant (see below). We note that, in view of (IV.), BNST can be regarded as just an ordinary Zermelo–Fraenkel set theory with choice, in which the axiom of foundation has been replaced by a complicated Anti-Foundation Axiom (namely, the "conjunction" of (0.) + (I.) + (II.) + (III.) + (V.); we shall see below that this is equivalent to a finite conjunction), and Separation and Replacement have been appropriately strengthened.

We proceed to derive some consequences of the BNST axioms.

We first note that the transitivity of $\mathbb{I}$, postulated in (0.)(ii), implies that many basic set-theoretic formulas and operations are *absolute* for $\mathbb{I}$; for example, $x_1 \in \mathbb{I} \wedge \cdots \wedge x_n \in \mathbb{I} \Rightarrow (\phi^{\mathbb{I}}(x_1, \dots, x_n) \equiv \phi(x_1, \dots, x_n))$ holds for all restricted formulas, and $(\forall x, y \in \mathbb{I})\ (\mathscr{F}_i^{\mathbb{I}}(x, y) = \mathscr{F}_i(x, y))$ holds for all Gödel operations (see [25]).

Standardization postulates that for any set x there is a standard set y having the same standard elements as x; by Extensionality in the standard universe, this y is unique. We shall denote it by x°.

Our universe of sets satisfies ZFC except for Foundation, and thus allows the construction of the usual von Neumann cumulative hierarchy (see [25] for the definition of ordinal numbers in ZFC without Foundation):

$$V_0 := \emptyset;$$

$$V_{\alpha+1} := \mathscr{P}(V_\alpha);$$

$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta \text{ if } \alpha \text{ is a limit ordinal.}$$

Let $\mathbb{V} := \bigcup_{\alpha \in \mathbb{O}n} V_\alpha$ be the class of all *well-founded sets*.

We define a mapping $* : \mathbb{V} \to \mathbb{S}$ by transfinite recursion (justified by (IV.), Replacement for $\in$-$\mathbb{S}$-formulas):

$$^*x := \{^*y \mid y \in x\}^{\circ}.$$

Proposition 1 (BNST). *$*$ is an isomorphism between $(\mathbb{V}, \in)$ and $(\mathbb{S}, \in)$.*

Proof. $*$ is one–one:

Assume $* \restriction V_\alpha$ is one–one and $x_1, x_2 \subseteq V_\alpha$, $x_1 \neq x_2$. If say $y \in x_1 \setminus x_2$ then $^*y \in {^*x_1}$ but $^*y \notin {^*x_2}$ (otherwise, $^*y = {^*z}$ for some $z \in x_2$ so $y = z \in x_2$).

$*$ is onto:

Let $^*\mathbb{V} := \{^*x \mid x \in \mathbb{V}\} \subseteq \mathbb{S}$. Assume $a \in \mathbb{S} \setminus {^*\mathbb{V}}$. Construct the transitive closure A of $\{a\}$ (externally): $A := \bigcup_{n=0}^{\infty} A_n$ where $A_0 := \{a\}$, $A_{n+1} := \bigcup A_n$. Let $b := (A \cap \mathbb{S} \setminus {^*\mathbb{V}})^{\circ}$; we note that $b \in \mathbb{S}$ and $a \in b$. The axiom of Foundation in $\mathbb{S}$ implies that there is $c \in b \cap \mathbb{S}$ such that $d \in c \cap \mathbb{S} \Rightarrow d \notin b$. Then $c \in \mathbb{S} \setminus {^*\mathbb{V}}$ and $c \cap \mathbb{S} \subseteq {^*\mathbb{V}}$. [$d \in c \cap \mathbb{S} \setminus {^*\mathbb{V}}$ implies $d \in b$ because of transitivity of A]. Let $x := \{y \in \mathbb{V} \mid {^*y} \in c\}$ (use (IV.), replacement). Then $x \in \mathbb{V}$ and $^*x = c$, a contradiction.

Trivially now $y \in x \equiv {^*y} \in {^*x}$ for $x, y \in \mathbb{V}$. $\square$

Let (I$'$.) be $\Phi_1^{\mathbb{S}} \wedge \Phi_2^{\mathbb{S}}$ where Φ_1 is the Axiom of Extensionality and Φ_2 is the Axiom of Foundation.

Corollary 1. (I.) + (III.) + (IV.) *is equivalent to* (I$'$.) + (III.) + (IV.).

Proof. Extensionality and Foundation in $\mathbb{S}$ are the only axioms of group (I.) used in the proof of Proposition 1, in addition to (III.) and (IV.). As $\mathbb{V}$ satisfies ZFC, it follows that $\mathbb{S}$ does, too. $\square$

Proposition 2 (BNST). $(\forall y \in \mathbb{I})\ (\exists x \in \mathbb{S})\ (y \in x)$.

Proof. As $\mathbb{I}$ satisfies ZFC, $y \in V_\beta^{\mathbb{I}}$ for some internal ordinal β. There is a standard ordinal α such that $\alpha \notin \beta$; otherwise β° would be a standard set containing all standard ordinals. But then $\beta \leqslant \alpha$, so $y \in V_\alpha^{\mathbb{I}}$ and $V_\alpha^{\mathbb{I}} = V_\alpha^{\mathbb{S}}$ is standard. $\square$

Corollary 2 (BNST). $*$ *is a nontrivial elementary embedding of* $(\mathbb{V}, \in)$ *into* $(\mathbb{I}, \in)$.

More precisely, $* \neq \mathbb{I}d$ *and for any* $\in$*-formula* ϕ *whose free variables are among* $x_1, \dots, x_n$, $x_1 \in \mathbb{V} \wedge \cdots \wedge x_n \in \mathbb{V} \Rightarrow (\phi^{\mathbb{V}}(x_1, \dots, x_n) \equiv \phi^{\mathbb{I}}({}^*x_1, \dots, {}^*x_n))$.

Proof. $* \neq \mathbb{I}d$: Otherwise $\mathbb{S} = \mathbb{V}$ and $\mathbb{S}$ is transitive. By Proposition 2 then $\mathbb{I} \subseteq \mathbb{S}$. This contradicts (V.). $\square$

In our axiomatization of BNST we view the standard sets as (special cases of) internal sets. We shall refer to this as the *internal picture*; it was introduced into nonstandard analysis independently by Hrbacek [22] and Nelson [37]. It seems to be more in the spirit of Leibniz, and to provide easier access to intuition about the nonstandard universe, than the alternative *external picture* established by Corollary 2. In the latter, the universe $\mathbb{V}$ of well-founded sets is viewed as the primary, "standard" universe. Well-founded sets are generally not internal, but the elementary embedding $*$ provides each $x \in \mathbb{V}$ with an internal "copy" ${}^*x \in \mathbb{I}$. This is the original view of Robinson [41], and is used by the practitioners who work with superstructures. As pointed out already in [23], the two pictures are equivalent and interchangeable. [2] In the context of nonstandard set theory the external picture has been used by Kawai [31], Ballard and Hrbacek [4], Di Nasso [11] and Kanovei and Reeken [28, 30]. A practitioner who prefers the internal picture can regard ${}^*x, {}^*y \dots$ (read: *st*and*ar*d $x, y \dots$) as variables ranging over the standard universe $\mathbb{S}$. A formula of the form $(\forall x \in \mathbb{V})\ (\exists y \in \mathbb{V})\phi({}^*x, {}^*y) \equiv (\forall^* x)\ (\exists^* y)\phi({}^*x, {}^*y)$ is then merely an equivalent of $(\forall x \in \mathbb{S})\ (\exists y \in \mathbb{S})\ \phi(x, y)$, etc. In the rest of the paper we shall use whichever formulation seems most convenient. We shall also write ${}^*R, {}^*c, {}^*F$ in place of $R^{\mathbb{S}}, c^{\mathbb{S}}, F^{\mathbb{S}}$ explained above.

Corollary 2 also permits further elaboration of the remark following the list of BNST axioms. Let us temporarily write j in place of $*$. In this notation, the axioms of BNST amount to this:

There is an elementary embedding $j : \mathbb{V} \to \mathbb{I} \subseteq \mathbb{U}$, $j \neq \mathbb{I}d$, where $\mathbb{V}$ is the universe of well-founded sets, $\mathbb{I}$ is transitive, and $\mathbb{U}$ satisfies ZFC except for Foundation, and with Separation and Replacement for $\in$-j-formulas.

This has a flavor of a large cardinal axiom. In fact, we have:

Corollary 3. "BNST+Foundation" *and* "ZFC+there exists a measurable cardinal" *are mutually interpretable.*

[2] For NST's whose external universe satisfies at least standard-size Replacement, such as BNST, NS_1 or HST; the existence of the external picture is not provable in NS_2, but it is consistent with it, and provable in its extension NS_3.

Proof. In BNST + Foundation, $\mathbb{U} = \mathbb{V}$ and the above statement about j becomes the well-known equivalent form of the assertion that measurable cardinals exist (set theorists usually write $\mathbb{M}$ in place of $\mathbb{I}$). Conversely, given $j : \mathbb{V} \to \mathbb{M} \subseteq \mathbb{V}$ where $\mathbb{M}$ is a transitive inner model and $j \neq \mathbb{I}d$, we let $\mathbb{S} := \{j(x) \mid x \in \mathbb{V}\}$ to obtain an interpretation for BNST. □

So far, BNST postulates only existence of *some* nonstandard internal sets, to avoid triviality. Of course, from our point of view, Foundation is false in the external universe: infinitely large natural numbers exist. The key issue facing a nonstandard realist is, which "ideal" objects exist (as internal nonstandard sets). Users of superstructures and past proposers of nonstandard set theories deal with this issue by postulating some form of *saturation* of the internal universe.

Definition. A collection of sets $\mathscr{F}$ has the *finite intersection property* (FIP) if $\bigcap \mathscr{F}_0 \neq \emptyset$ for every finite $\mathscr{F}_0 \subseteq \mathscr{F}$.

Definition. *κ-enlargement property* (*κ-saturation property*, resp.) asserts: If $\mathscr{F}$ is a collection of standard sets (internal sets, resp.) with FIP and $|\mathscr{F}| < \kappa$ then $\bigcap \mathscr{F} \neq \emptyset$.

"Naively", we might like to postulate that κ-saturation holds for all κ. This amounts to saying that *every* ideal object (that is, one that by itself can be consistently added to the internal universe) actually exists in $\mathbb{I}$. However, such assumptions lead to a contradiction.

Proposition 3 (BNST). *The κ-enlargement property and the κ-saturation property fail for $\kappa > |{}^*\omega|$.*

Proof.[3] Let $\mathscr{F} := \{{}^*\omega \backslash v \mid v \in {}^*\omega\}$; then $\mathscr{F}$ has FIP, $|\mathscr{F}| = |{}^*\omega|$ and $\bigcap \mathscr{F} = \emptyset$. Hence $|{}^*\omega|^+$-saturation fails.

Given any κ for which the κ-enlargement property holds, let $A := \alpha$ where $\alpha < \kappa$. For every finite $F \subseteq A$ let $A_F := \{X \in \mathbb{S} \mid {}^*F \subseteq X \subseteq {}^*A \wedge X \text{ is finite in } \mathbb{S}\}^\circ$; A_F is standard. Let $\mathscr{F} := \{A_F \mid F \subseteq A,\ F \text{ finite}\}$. Then $\mathscr{F}$ has FIP, and $|\mathscr{F}| = |A| < \kappa$, so by κ-enlargement property there is $X \in \bigcap \mathscr{F}$. This X is internal, finite in $\mathbb{I}$, and ${}^*A \cap \mathbb{S} \subseteq X \subseteq {}^*A$. Thus there is an internal one–one mapping $\phi : X \to {}^*\omega$. $Y := \phi[{}^*A \cap \mathbb{S}]$ is an (external) subset of ${}^*\omega$ with $|Y| = |A|$. Hence $\kappa \leqslant |{}^*\omega|^+$. This shows that κ-enlargement property fails for $\kappa > |{}^*\omega|^+$. Its failure for $\kappa = |{}^*\omega|^+$ follows from Corollary 7 in Section 7. □

We shall consider the crucial question of existence of "ideal" objects (*Idealization*) further in Sections 4 and 5.

[3] The incompatibility of the full enlargement property with other axioms of NST was established in [22].

3. Discussion of the axioms

We now return to the issue of justification of the axioms of BNST.

3.1. Axiom groups (I.) and (IV.)

We do not intend to question the validity of the axioms of ZFC in the Standard Universe. The intuitive reasons for accepting them are well known. We hold that the nonstandard mathematician can advance essentially the same reasons to justify acceptance of ZFC (except Foundation) as a description of the Nonstandard Universe. We shall pass over the axioms of Extensionality, Pair, Union and Infinity; however, the remaining axioms call for some comments.

Separation. Even in standard set theory, reality of the set-theoretic universe needs to be appealed to in order to justify separation for properties with unbounded quantifiers. As this is also our position vis-à-vis the Nonstandard Universe, we can make the same appeal.

It is our position (borne out by the actual practice of nonstandard analysis) that a nonstandard mathematician can clearly distinguish between standard vs. nonstandard, and internal vs. noninternal objects. Hence references to $\mathbb{S}(x)$ and $\mathbb{I}(x)$ in properties used in Separation (and Replacement) are justified.

Replacement. The most compelling argument for replacement in the Standard Universe seems to be *limitation of size*. It has equal force in the Nonstandard Universe.

Choice. For a realist, the axiom of choice is one of the least problematic axioms (see [35, 21]), and this applies equally well to the Nonstandard Universe. However, what we really need to establish is the

Well-ordering principle (WO): Every set can be well ordered.

Besides Choice and Replacement, Zermelo's proof of WO essentially uses the axiom of Power Set, [4] which we regard as the most problematic, both in the Standard and in the Nonstandard Universe, and which we shall discuss last. However, WO can also be proved without reference to power set if one assumes *global axiom of choice* (GAC), i.e. a class function C such that $(\forall x)\ (x \neq \emptyset \to C(x) \in x)$ and Replacement allowing formulas with C (see [19]). If one is willing to accept GAC (e.g., on grounds similar to those used to justify Separation, which seem to require the ability of surveying the entire universe, in some sense), the previous sentence provides an argument for validity of WO which does not depend on the Power Set Axiom.

Foundation, or the lack of it. We assume the axiom of Foundation in the Standard Universe. This simplifies the presentation and does not seem to effect any issues of concern to us here, but it is not necessary.

If v is an infinitely large internal integer, the external set $\{v, v-1, v-2, \ldots\}$ has no $\in$-least element, so Foundation in the external universe fails. One could postulate (see [31])

[4] Zarach [45] constructed a model for ZF – Power Set + AC where WO fails.

Foundation over $\mathbb{I}$: $(\forall x \neq \emptyset)\ (\exists y \in x)\ (y \cap x \subseteq \mathbb{I}\ )$.

An inner model for BNST + Foundation over $\mathbb{I}$ can be constructed in BNST along the lines of von Neumann hierarchy: let

$W_{\alpha,0} := {}^*V_\alpha$;
$W_{\alpha,\beta+1} := \mathscr{P}(W_{\alpha,\beta})$;
$W_{\alpha,\beta} := \bigcup_{\gamma<\beta} W_{\alpha,\gamma}$ for β limit.

$\mathbb{W} := \bigcup_{\alpha,\beta \in \mathbb{O}n} W_{\alpha,\beta}$ is the class of *sets well-founded over* $\mathbb{I}$. The axiom of Foundation over $\mathbb{I}$ is equivalent to $\mathbb{U} = \mathbb{W}$.

For many considerations, this assumption is convenient and intuitive. However, there are reasons to reject it; we shall consider them in Section 6.

3.2. Axiom (III.): Standardization

Axiom III. involves two distinct ideas. The first one was mentioned in the introduction to Section 2: we are allowed to use nonstandard properties in the descriptions of standard sets, as long as they are "definite", like $\mathbb{S}(x)$ or $\mathbb{I}(x)$. This justifies

Bounded Standardization:

$$a \in \mathbb{S} \Rightarrow (\exists y \in \mathbb{S})\ (\forall z \in \mathbb{S})\ (z \in y \equiv z \in a \wedge \phi(z, a, x_1, \ldots, x_n))$$

where ϕ is any $\in$-$\mathbb{S}$-$\mathbb{I}$-formula.

We note that (IV.) implies existence of the set $x := \{z \in a \mid \phi(z, a, x_1, \ldots, x_n)\}$, and hence one can formulate bounded Standardization as a single axiom:

$$a \in \mathbb{S} \Rightarrow (\exists y \in \mathbb{S})\ (\forall z \in \mathbb{S})\ (z \in y \equiv z \in a \wedge z \in x).$$

A second idea is needed to obtain full Standardization: we have to exclude the possibility that

$$(\bigtriangledown) \quad (\exists x)\ \neg\ (\exists a \in \mathbb{S})\ (\forall z \in \mathbb{S})\ (z \in x \Rightarrow z \in a).$$

Let us suppose that bounded Standardization and $(\bigtriangledown)$ hold (in addition to $(0.) + (\mathrm{I}.) + (\mathrm{IV}.)$). Let $\rho^{\mathbb{S}}$ be the standard rank function. For any set x as in $(\bigtriangledown)$ the set $A := \bigcup \rho^{\mathbb{S}}[x \cap \mathbb{S}]$ contains all standard ordinals. A modification of the proof of Proposition 1 then establishes an isomorphism $* : (V_\alpha, \in) \to (\mathbb{S}, \in)$ where α is an (external) ordinal. Hence $(\bigtriangledown)$ is equivalent to the statement that $\mathbb{S}$ is a set.

We offer three reasons for rejection of $(\bigtriangledown)$, and hence for acceptance of full Standardization, as a "true" axiom.

The extended realism adopted here allows us to view the standard universe either in its internal picture $(\mathbb{S}, \in)$, or its external picture $(\mathbb{V}, \in)$, but the usual set-theoretic realism posits a *unique* (standard) universe. This seems to require $(\mathbb{S}, \in)$ and $(\mathbb{V}, \in)$ to be isomorphic, and hence $(\bigtriangledown)$ to fail. (It is easy to see directly that existence of such isomorphism is equivalent to full Standardization.)

Another line of reasoning goes roughly as follows (technical details will appear in [24]): Assume that $*$ is an isomorphism of $(V_\alpha, \in)$ and $(\mathbb{S}, \in)$. In (0.) + (I.) + (IV.) + (iii) (see below) one can show that $(\mathbb{I}, \in)$ is isomorphic to a "bounded" limit ultrapower of $(V_\alpha, \in)$ (see the proof of Proposition 11). Taking *the same* limit ultrapower, but of $(\mathbb{V}, \in)$, produces an end extension $(\mathbb{I}', \mathbb{E}, \mathbb{S}')$ of $(\mathbb{I}, \in, \mathbb{S})$ where $(\mathbb{S}', \mathbb{E} \upharpoonright \mathbb{S}')$ is isomorphic to $(\mathbb{V}, \in)$. Under further reasonable assumptions on the external universe (some weak form of Superuniversality), there are classes $\hat{\mathbb{I}} \supseteq \mathbb{I}$, $\hat{\mathbb{S}} \supseteq \mathbb{S}$ such that $\hat{\mathbb{I}}$ is transitive, $(\hat{\mathbb{I}}, \in, \hat{\mathbb{S}})$ is isomorphic to $(\mathbb{I}', \mathbb{E}, \mathbb{S}')$, and $*$ extends to an isomorphism of $(\mathbb{V}, \in)$ and $(\hat{\mathbb{S}}, \in)$. In other words, there is an extension of the "original" standard and internal universe for which full Standardization (as well as the rest of BNST) holds.

Finally, (∇) of course egregiously violates weak reductionism.

3.3. Axiom group (0.): Transitivity of $\mathbb{I}$

We do not claim that axiom (0.)(ii) is a necessary consequence of the realist viewpoint advocated here, and we adopt it only as a *simplifying assumption* (as do superstructures and all nonstandard set theories that postulate a unique internal universe; see [22, 37, 31, 27, 9, 2]). Intuitively, although internal sets might have external elements, we are not interested in them, and leave them out of the universe of objects under consideration here. (Proposition 14 in Section 7 demonstrates formally that 0.(ii) is consistent with the other axioms of BNST.) We make two further observations.

(1) The proofs of the key results of this paper, in particular Propositions 15–17 and Corollary 6 in Section 7, go through *without* the assumption of transitivity of $\mathbb{I}$.

(2) Transitivity of $\mathbb{I}$ is related to the question of *uniqueness* of the internal universe. For the purposes of this paper it is only necessary to posit *some* "nonstandard mathematician" whose internal universe $\mathbb{I}$ (together with its appurtenant external universe) is sufficiently comprehensive to satisfy BNST and the additional axioms of type A or B discussed below. We could allow that *another* "nonstandard mathematician" works with a different internal universe $\mathbb{I}'$. In that case 0.(ii) would fail either for $\mathbb{I}$ or $\mathbb{I}'$ (for example, let $x \in \mathbb{I} \setminus \mathbb{I}'$; taking $a \in \mathbb{S}$ such that $x \in a$ we have $a \in \mathbb{I}'$, $x \in a$, $x \notin \mathbb{I}'$). I find the hypothesis that there is a unique (largest) internal universe intuitively attractive, but we do not have to commit ourselves to it here. In any case, it still does not justify 0.(ii), as internal sets might have external elements for reasons not connected with Nonstandard Analysis.

There is yet another related issue. In Proposition 2 we proved that

(iii) $(\forall y \in \mathbb{I})\ (\exists x \in \mathbb{S})\ (y \in x)$.

Combined with 0.(i), (ii) this gives

$\mathbb{I}(y) \equiv (\exists x \in \mathbb{S})\ (y \in x)$.

One could use this statement as a *definition*, and thus eliminate $\mathbb{I}$ from the list of primitive concepts. We prefer not to do so, both because of the contingent nature of 0.(ii) and because the equivalence may fail when one allows standard non-well-founded sets.

3.4. Axiom group (II.): Transfer

As stated in the introduction to Section 2, we assume that the elementary set-theoretic operations can be extended from standard to internal sets. Specifically, we consider the 10 Gödel operations $\mathscr{F}_i$ $(i=1,\dots,10)$ (see [25]). Each of these operations has a defining formula $\phi_i(x,y,z)$ such that (in ZFC) one can prove $(\forall x,y)$ $(\exists! z)$ $\phi_i(x,y,z)$. We postulate

(II′.) $(\forall x,y\in\mathbb{S})$ $(\mathscr{F}_i^{\mathbb{S}}(x,y)=\mathscr{F}_i^{\mathbb{I}}(x,y))$ $(i=1,\dots,10)$.

More precisely, (II′.) is the conjunction of

$(\forall x,y\in\mathbb{I})$ $(\exists! z\in\mathbb{I})$ $\phi_i^{\mathbb{I}}(x,y,z)$ and

$(\forall x,y\in\mathbb{S})$ $(\forall z\in\mathbb{I})$ $(\phi_i^{\mathbb{I}}(x,y,z)\Rightarrow z\in\mathbb{S}\wedge\phi_i^{\mathbb{S}}(x,y,z))$ $(i=1,\dots,10)$.

It is an idea of Robinson and Zakon [42] (see also [3, 11]) that these 10 special cases imply the full Transfer. This is also true in the present setting, if we strengthen (0.) to $(0^+.)\equiv(0.)+(\text{iii})$.

Proposition 4. $(0^+.)+(\text{I}'.)+(\text{II}'.)+(\text{III}.)+(\text{IV}.)$ *is equivalent to* $(0.)+(\text{I}.)+(\text{II}.)+(\text{III}.)+(\text{IV}.)$.

Proof (*Outline*)**.** Corollary 1 in Section 2 already showed that $\Phi_1^{\mathbb{S}}\wedge\Phi_2^{\mathbb{S}}$ implies the remaining axioms of group (I.). The Normal Form Theorem (Theorem 30) in [25] holds in both $\mathbb{S}$ (it satisfies ZFC) and $\mathbb{I}$ (it is a Gödel closed transitive class); hence for every restricted formula $\phi(x_1,\dots,x_n)$ there is a composition of Gödel operations $\mathscr{F}(x_1,\dots,x_n)$ such that for all standard $X_1,\dots,X_n$

$$(x_1,\dots,x_n)^{\mathbb{S}}\in\mathscr{F}^{\mathbb{S}}(X_1,\dots,X_n)\equiv x_1\in X_1\wedge\cdots\wedge x_n\in X_n\wedge\phi^{\mathbb{S}}(x_1,\dots,x_n)$$

and

$$(x_1,\dots,x_n)^{\mathbb{I}}\in\mathscr{F}^{\mathbb{I}}(X_1,\dots,X_n)\equiv x_1\in X_1\wedge\cdots\wedge x_n\in X_n\wedge\phi^{\mathbb{I}}(x_1,\dots,x_n)$$

hold for all standard $x_1\in X_1,\dots,x_n\in X_n$.

Applying (II′.) to $\mathscr{F}_1(x,y)=\{x,y\}$ we see that $(x_1,\dots,x_n)^{\mathbb{S}}=(x_1,\dots,x_n)^{\mathbb{I}}$. (II′.) and the above equivalences imply that Transfer holds for all restricted formulas.

We next proceed by induction on the complexity of ϕ. Let us assume that Transfer holds for Π_m formulas and that $\mathbb{I}\models(\exists x)$ $\phi(x,x_1,\dots,x_n)$ where ϕ is Π_m and $x_1,\dots,x_n\in\mathbb{S}$. From (iii) it follows that there is a standard set A such that $\mathbb{I}\models(\exists x\in A)$ $\phi(x,x_1,\dots,x_n)$. It suffices to show that $\mathbb{S}\models(\exists x\in A)$ $\phi(x,x_1,\dots,x_n)$. If $m=0$, this follows immediately from Transfer for restricted formulas. If $m>0$ we can assume $\phi(x,x_1,\dots,x_n)\equiv(\forall y)$ $\psi(x,y,x_1,\dots,x_n)$ where ψ is Σ_{m-1}. Assume that $\mathbb{S}\models(\forall x\in A)$ $(\exists y)\neg\psi(x,y,x_1,\dots,x_n)$. As $\mathbb{S}\models\text{ZFC}$, there is a standard function F such that $\mathbb{S}\models(\forall x\in A)\neg\psi(x,F(x),x_1,\dots,x_n)$. The latter is a Π_m formula, so, by inductive assumption, $\mathbb{I}\models(\forall x\in A)\neg\psi(x,F(x),x_1,\dots,x_n)$, in contradiction with $\mathbb{I}\models(\exists x\in A)$ $(\forall y)$ $\psi(x,y,x_1,\dots,x_n)$. $\square$

Proposition 4 shows that the infinite list of axioms $(0.)+(\text{I}.)+(\text{II}.)+(\text{III}.)$ can be replaced by the finite list $(0^+.)$ + $(\text{I}'.)$ + $(\text{II}'.)$ + $(\text{III}.)$ (in the presence of (IV.)).

It also provides a justification of Transfer for restricted formulas. In order to be able to regard it as a justification of full Transfer, we have to give an independent argument for the validity of (iii); without stronger assumptions on $\mathbb{I}$ than those in (II$'$.) it does not seem possible to construct the internal von Neumann hierarchy used to prove (iii) in Proposition 2. However, we continue in the spirit of the introduction to Section 2 and postulate that the power set operation (exponentiation) can also be extended from standard to internal sets: $(\forall x \in \mathbb{S})\ (\mathscr{P}^{\mathbb{S}}(x) = \mathscr{P}^{\mathbb{I}}(x))$. Then it is seen easily (using restricted Transfer) that $(\forall y \in \mathbb{I})\ [(\exists x \in \mathbb{S})\ y \in x \equiv (\exists x \in \mathbb{S})\ y \subseteq x]$ and hence that $\mathbb{I}' := \{y \in \mathbb{I} \mid (\exists x \in \mathbb{S})\ y \in x\}$ is closed under $\mathscr{F}_i$, $i = 1, \ldots, 10$ and also satisfies (iii). We can thus disregard any objects that violate (iii) and take $\mathbb{I}'$ in place of $\mathbb{I}$ as the internal universe.

4. The continuum

This section is devoted to some considerations related to the Power Set axiom. Reputable intrinsic justifications for the Power Set axiom, even in standard set theory, seem to be in short supply [35]. We consider an obvious attempt for an extrinsic justification: the need to give a mathematical account of the continuum and its generalizations. We first review the standard model of the continuum and its interplay with the power set axiom. Analogous considerations for Nonstandard Models of the continuum lead to a crucial (partial) answer to the question of existence of "ideal" elements in nonstandard set theory.

There seems to be a general agreement that our spatio-temporal intuitions about the linear continuum (the straight line) are not sufficiently acute to ground a unique account of it. Nevertheless, there are some basic intuitions about it that go back to the prehistory of geometry, and should carry some conviction:

(i) There are points on the line.

Greek mathematicians viewed the line as *potentially* infinite only; a contemporary set-theoretic realist who accepts the axiom of Infinity should have no difficulty regarding it as a "completed totality", and thus conclude that the set P of all points on the line exists. (In fact, a finite line segment suffices for the forthcoming argument.)

(ii) There is a natural linear ordering $\prec$ of the points. (In fact, two of them; we fix one.)

(iii) One can establish a one-one order-preserving correspondence f between the rationals $\mathbf{Q}$ and a certain subset Q of P.

Finally: The line has no holes.

The precise meaning of this is of course the issue. But, with hindsight, it should imply *at least* the following:

(iv) For every Dedekind cut (A,B) in $(\mathbf{Q},<)$ (i.e., sets $A,B \neq \emptyset$ such that $A \cup B = \mathbf{Q}$, $a \in A,\ b \in B \Rightarrow a < b$ and B has no $<$-least element) there is a point $p \in P$ such that $a \in A,\ b \in B \Rightarrow f(a) \preccurlyeq p \preccurlyeq f(b)$.

We hasten to note that conditions (i)–(iv) do *not* imply that $(P, \prec)$ is isomorphic to $(\mathbf{R}, <)$; in fact, they are consistent both with the standard model of the continuum, i.e. $(\mathbf{R}, <)$, and various nonstandard models, such as hyperreals ${}^*\mathbf{R}$, or hyperfinite grids in [33]. The informal argument that follows can be formalized either in ZFC–Power Set, or BNST–Power Set.

Namely, let f be the order-isomorphism between $(\mathbf{Q}, <)$ and $(Q, \prec)$. For each $p \in P$ let $A_p := \{r \in \mathbf{Q} \mid r = f^{-1}(q) \text{ for some } q \preccurlyeq p,\ q \in Q\}$; $B_p := \mathbf{Q} \backslash A_p$. The mapping $p \mapsto (A_p, B_p)$ contains all Dedekind cuts in $(\mathbf{Q}, <)$ in its range, and its range is a set, by the axiom of Replacement.

We use this argument in two ways. Informally, we take it as a justification of the existence of $(\mathbf{R}, <)$, the order-completion of the rationals, based on the intuitive properties of the line. (For that purpose, it would be appropriate to regard P as a set of individuals.) Our intuition about other dense linearly ordered sets arises from this example and is of course much weaker, but it, such as it is, together with needs of modern mathematical practice to construct completions of all sorts, and the maxim to 'generalize' (see [35]), make a case for the

Axiom of Completion. Every dense linearly ordered set has an order-completion.

The second way to use this argument is to formalize it in BNST–Power Set to prove the following

Proposition 5 (BNST–Power Set).[5] *If* ${}^*\omega \backslash \omega \neq \emptyset$ *then* $\mathbf{R}$ *exists.*

Proof. The assumption implies that $({}^*\mathbf{Q}, {}^*{<})$ has infinitesimals. It then follows that letting $P := {}^*\mathbf{Q}$, $Q := {}^*\mathbf{Q} \cap \mathbb{S}, \prec := {}^*{<}$ and $f := * \upharpoonright \mathbf{Q}$ gives a set P with properties (i)–(iv). We verify only (iv): Let (A, B) be a Dedekind cut in $\mathbf{Q}$. Let v be a positive infinitesimal in ${}^*\mathbf{Q}$. Consider $\{kv \mid k \in {}^*\mathbf{Z}\}$; there is a $k \in {}^*\mathbf{Z}$ such that $kv \in {}^*A$, $(k+1)v \in {}^*B$. The interval $[kv, (k+1)v]$ contains at most one standard rational. From this, existence of $p \in (kv, (k+1)v)$ such that ${}^*a \prec p \prec {}^*b$ holds for all $a \in A$, $b \in B$ follows easily.

The construction of $\mathbf{R}$ outlined above can now be carried out formally. $\square$

Proposition 6. *In* ZF–Power Set + WO, *the Axiom of Completion implies the Axiom of Power Set. The same is true in* BNST–Power Set + WO.

Proof. We give the argument in ZF–Power Set + WO. We prove that $\mathscr{P}(\kappa)$ exists by transfinite induction on κ. So assume $\mathscr{P}(\lambda)$ exists for all $\lambda < \kappa$. Using Replacement we obtain $Z_\kappa := \bigcup_{\xi < \kappa} {}^\xi\mathbf{Z} = \{f \mid f \text{ is a function, dom } f = \xi < \kappa, \text{ ran } f \subseteq \mathbf{Z}\}$. The relation $\prec$ defined on Z_κ by "$f_1 \prec f_2 \equiv f_1 \subseteq f_2 \vee f_1(\eta) < f_2(\eta)$ where η is the least ordinal for

[5] The construction of $\mathbf{R}$ using infinitesimals is due to Martin Davis [8].

which $f_1(\eta)\neq f_2(\eta)$" is a dense linear ordering. Let $(\hat{Z}_\kappa,\prec)$ be an order-completion of $(Z_\kappa,\prec)$.

For each $x\in\hat{F}_\kappa$ define $f_x:\kappa\to\mathbf{Z}$ recursively as follows: $f_x(\xi):=$ the largest $n\in\mathbf{Z}$ such that $f_x\restriction\xi\cup\{\langle\xi,n\rangle\}\preccurlyeq x$ if such n exists; 0 otherwise. Let $P:=\{f_x\mid x\in\hat{Z}_\kappa\}$ (by Replacement). We claim that $P={}^\kappa\mathbf{Z}$; this implies that $\mathscr{P}(\kappa)$ exists.

Given $f:\kappa\to\mathbf{Z}$ let $A:=\{f\restriction\xi\mid\xi<\kappa\}\subseteq Z_\kappa$. A is bounded above, so $x:=\sup A$ exists in $\hat{Z}_\kappa$. It is easy to check that $f=f_x$.

This proves that every well-orderable set has a power set. As we assume WO, we are done.

In the case of BNST the same argument proves Power Set in $\mathbb{V}$. Hence Power Set holds both in the standard universe (via the isomorphism $*$) and the external universe (on account of WO and Replacement, every set is in a one–one correspondence with some set in $\mathbb{V}$). $\square$

We now return to the consideration of the nonstandard universe. Under (i)–(iv) we listed the intuitive properties that any mathematical model of the continuum should satisfy. One such mathematical model is the standard one. In this model the property (iv) is implemented in a strong sense: in every standard cut of the standard rationals there is a standard point. We now consider an apparently weaker form of (iv); we require merely that in every standard cut of ${}^*\mathbf{Q}$ there is an internal point. This is how (iv) is implemented in various nonstandard models of the continuum, such as the *hyperreal line* or the *hyperfinite grid*. Such models "appear" continuous to the standard observer, but may appear discrete to the internal one. It is the *raison d'être* of nonstandard analysis to formalize rigorously existence of such nonstandard models of the continuum. For this it suffices to postulate existence of points infinitesimally close to the origin. We again generalize, to arbitrary ordered fields. This leads to:

Axiom A_1. Every standard ordered field has infinitesimals.

Definition. If $\langle F,<,+,\times,0,1\rangle$ is a standard ordered field, $\xi\in F$ is an *infinitesimal* if $\xi\neq 0$ and $-r<\xi<r$ holds for all standard $r\in F$, $r>0$.

Proposition 7. *In* BNST–Power Set + WO, A_1 *implies the* Power Set *axiom.*

Proof. We need a result that is proved in the next section (without using the Power Set axiom; see the proofs of Proposition 10, $(\mathrm{A}_1)\Rightarrow(B_0)$ and Proposition 9, $(1_\kappa)\Rightarrow(2_\kappa)$): Axiom A_1 implies that for any infinite regular κ there exists $\xi<{}^*\kappa$ such that ${}^*\alpha<\xi$ for all $\alpha<\kappa$.

Let now $(L,<)$ be a dense linearly ordered set; letting $P:={}^*L, Q:={}^*L\cap\mathbb{S}$, $\prec:={}^*<$ and $f:=*\restriction L$ gives a set P with properties (i)–(iv), except that L takes the place of $\mathbf{Q}$. We verify (iv): Let (A,B) be a Dedekind cut in L. Let $\langle c(\alpha)\mid\alpha<\kappa\rangle$, κ regular, be a decreasing sequence of elements of B coinitial in B. If $\xi\in{}^*\kappa$ is such that $\xi>{}^*\alpha$ for all $\alpha<\kappa$, then ${}^*a\prec{}^*c(\xi)\prec{}^*b$ holds for all $a\in A$ and $b\in B$. As in the construction of **R** outlined above, it now follows that the set of all Dedekind cuts in $(L, <)$ exists. Thus A_1 implies the axiom of Completion. From this, the Axiom of Power Set follows by Proposition 6. $\square$

The main importance of axiom A_1 is that it provides a natural and interesting (partial) answer to the question of existence of "ideal" elements. We approach this question from a different direction in Section 5.

5. Existence of ideal elements

In this and the next section we work in BNST. We begin with a classification of internal sets. Let κ be a cardinal.

Definition. An internal set x is *κ-constrained* if $x \in {}^*a$ for some $a \in \mathbb{V}$ with $|a| \leqslant \kappa$.

(This is an "external" formulation; an equivalent "internal" formulation is: $x \in a$ for some $a \in \mathbb{S}$ with ${}^*|a| \leqslant {}^*\kappa$.) [6]

Proposition 8 (BNST). *If κ is finite and x is κ-constrained then x is standard.*

Proof. This amounts to proving that all elements of a standard finite set are standard. It suffices to do it for standard natural numbers, i.e., sets of the form ${}^*n$ where $n \in \omega$. We proceed by induction. ${}^*0 = 0^{\mathbb{I}}$ by Transfer, and $0^{\mathbb{I}} = 0$ by absoluteness ($\mathbb{I}$ is transitive). Assuming ${}^*n = n$ has been proved, we have ${}^*(n+1) = {}^*(n \cup \{n\}) = n^* \cup {}^*\{n\} = n \cup^{\mathbb{I}} \{n\}^{\mathbb{I}} = n \cup \{n\}$, using again first Transfer and then absoluteness of union and pair. $\square$

Our axioms must imply the existence of infinitely large integers, a *sine qua non* of nonstandard analysis. Such integers are examples of nonstandard, ω-constrained sets. It seems intuitively highly implausible that *all* internal sets are ω-constrained. This would also violate a number of maxims that have been successfully used in standard set theory to justify its axioms (see [35]), in particular *maximize*, the admonition against *whimsical identities* and the desire to *uniformize*. We thus propose:

Axiom $\mathbf{B}_1$. For every infinite κ there exist internal sets that are κ-constrained, but not λ-constrained for any $\lambda < \kappa$.

We now investigate this axiom.

Definition. An internal set is strictly κ-constrained if it is κ-constrained, but not λ-constrained for any $\lambda < \kappa$.

[6] This notion has been discovered independently several times. The earliest appearance is in Luxemburg [34]; his σ-quasistandard sets are exactly the ω-constrained sets in the terminology of this paper. The general concept was introduced in a circulated preprint of Hrbacek [23] (it has been omitted from the published version) and in Kanovei–Reeken [28], under various names.

Proposition 9 (BNST). *Let κ be a cardinal. Consider the statements*

(1_κ) *There exists an internal set x which is strictly κ-constrained.*

(2_κ) *There exists $\xi \in {}^*\kappa$ such that ${}^*\alpha < \xi$ for all $\alpha < \kappa$.*

(3_κ) *There exists an internal set X such that ${}^*\alpha \in X$ holds for all $\alpha < \kappa$, and $|X|^{\mathbb{I}} < {}^*\kappa$.*

Then $(1_\kappa) \Rightarrow (2_\kappa) \Rightarrow (3_\kappa)$. If κ is regular then $(1_\kappa) \equiv (2_\kappa) \equiv (3_\kappa)$.

Proof. $(1_\kappa) \Rightarrow (2_\kappa)$:

Let x be as in (1_κ). Then $x \in {}^*A$ where $|A| \leqslant \kappa$. Let $f : A \to \kappa$ be one–one and $\xi := {}^*f(x)$. Then $\xi \in {}^*\kappa$ and for any $\alpha < \kappa$, $B := f^{-1}[\alpha]$ has $|B| < \kappa$, so $x \notin {}^*B$ and $\xi \notin {}^*\alpha$.

$(2_\kappa) \Rightarrow (3_\kappa)$:

If $\xi \in {}^*\kappa$ is as in (2_κ), we have $|\xi|^{\mathbb{I}} < {}^*\kappa$. Let $X = \xi$.

$(3_\kappa) \Rightarrow (1_\kappa)$ if κ is regular:

Let X be as in (3_κ). From $|X|^{\mathbb{I}} < {}^*\kappa$ and internal regularity of ${}^*\kappa$ it follows that there is $\xi \in {}^*\kappa$ such that $X \cap {}^*\kappa \subseteq \xi$; in particular ${}^*\alpha < \xi$ for all $\alpha < \kappa$. Clearly ξ is κ-constrained. If $\xi \in {}^*B$ for some B with $|B| = \lambda < \kappa$ then, by regularity of κ, there is $\alpha < \kappa$ with $B \cap \kappa \subseteq \alpha$. So $\xi \in {}^*\alpha$, contradicting ${}^*\alpha < \xi$. $\square$

Proposition 9 suggests a weaker version of B_1:

Axiom $\mathbf{B}_0$. For every infinite regular cardinal κ there exist internal sets that are strictly κ-constrained.

An even weaker assumption is

Axiom $\mathbf{A}_0$. For every infinite cardinal λ there exist internal sets that are not λ-constrained.

By Proposition 9(2_κ), B_0 is equivalent to the statement that every standard regular ${}^*\kappa$ contains internal ordinals larger than all standard ordinals below ${}^*\kappa$, a claim with additional intuitive plausibility. By (3_κ), B_0 can be interpreted as saying that each regular ${}^*\kappa$ has "many more" nonstandard elements than standard ones. We also note that B_0 implies that (2_κ) and (3_κ) hold for all κ. [If κ is singular, and $X \subseteq \kappa$ of order type $\lambda = \mathrm{cf}(\kappa)$ is cofinal in κ, then λ is regular and (2_λ) implies that there is $\xi \in {}^*X$ such that ${}^*\alpha < \xi$ for all $\alpha \in X$, hence for all $\alpha < \kappa$. Thus (2_κ) holds.]

Proposition 10 (BNST). *The following statements are equivalent*:

(A_1) *Every standard ordered field has infinitesimals.*

(B_0) *For every infinite regular κ there exist internal sets that are strictly κ-constrained, but not λ-constrained for any $\lambda < \kappa$.*

Proof. $(\mathrm{B}_0) \Rightarrow (\mathrm{A}_1)$:

Let $\langle F, <, +, \times, 0, 1 \rangle$ be an ordered field in $\mathbb{V}$. Let $\langle a(\alpha) \mid \alpha < \kappa \rangle$ be a decreasing sequence of positive elements of F with $\inf\{a(\alpha) \mid \alpha < \kappa\} = 0$, where κ is regular.

B_0 guarantees some $\xi \in {}^*\kappa$ such that $\xi > {}^*\alpha$ for all $\alpha < \kappa$. Then ${}^*0 \, {}^*\!< {}^*a(\xi) \, {}^*\!< {}^*a({}^*\alpha)$ holds for all $\alpha < \kappa$, hence ${}^*a(\xi) \, {}^*\!< {}^*a$ for all $a \in F$, $a > 0$, and ${}^*a(\xi)$ is an infinitesimal in $\langle {}^*F, {}^*\!<, \ldots \rangle$.

$(A_1) \Rightarrow (B_0)$:

Let κ be regular. Let $\mathbf{F}_\kappa = \langle F_\kappa, <, +, \times, 0, 1 \rangle$ be an ordered field of cofinality κ with $|F_\kappa| = \kappa$. [One can easily construct such fields as follows. Let T_0 be the Skolemization of the theory of the ordered field of rationals. For $\xi < \kappa$ let $T_{\xi+1}$ be obtained from T_ξ by adding a new constant c_ξ and postulating that $\tau < c_\xi$ where τ is any closed term of T_ξ. Then $T := \bigcup_{\xi < \kappa} T_\xi$ is consistent and yields a desired model.] Let $A := \{a_\xi \mid \xi < \kappa\}$ be a sequence of elements of F_κ decreasing to 0. By A_1, the standard ordered field ${}^*\mathbf{F}_\kappa$ has an infinitesimal x. This $x \in {}^*F_\kappa$ where $|F_\kappa| = \kappa$, but for any set B with $|B| < \kappa$ there exists $\xi < \kappa$ such that ${}^*(-a_\xi, a_\xi) \cap {}^*B \subseteq \{0\}$, so $x \notin {}^*B$. Hence x is strictly κ-constrained. $\square$

It is instructive to compare the axiom A_1 with the "naive" assumption that the κ-enlargement property holds for all κ, which, as we have seen in Section 2, Proposition 3, leads to a contradiction. We of course do believe (and postulate) that ${}^*\omega \cap \mathbb{S}$ is a subset of an internal set of internal cardinality less than ${}^*\omega$. (This is equivalent to existence of infinitely large integers.) The consequence of the κ-enlargement property that yielded a contradiction is a generalization of the above to arbitrary κ as follows:

For every κ, ${}^*\kappa \cap \mathbb{S}$ is a subset of some internal set of internal cardinality less than ${}^*\omega$.

But this is an "improper" generalization: one occurrence of ω has been replaced by κ, but the other has not. The "proper" generalization is:

For every κ, ${}^*\kappa \cap \mathbb{S}$ is a subset of some internal set of internal cardinality less than ${}^*\kappa$.

This last statement is equivalent to A_1 (as pointed out in the remark preceding Proposition 10, $A_1 \equiv (\forall \kappa)(3_\kappa)$).

These considerations suggest the extent to which the enlargement property might hold in the nonstandard universe. We need a "proper" generalization of the situation at ${}^*\omega$. "Ideal" elements of ω are described by collections $\mathscr{F}$ of subsets of ω with the finite intersection property. To generalize this, we replace ω by an arbitrary regular κ.

Definition. A collection of sets $\mathscr{F}$ has the *κ-intersection property* (κ-IP) if $\bigcap \mathscr{F}' \neq \emptyset$ for every $\mathscr{F}' \subseteq \mathscr{F}$ with $|\mathscr{F}'| < \kappa$.

Axiom A_2. For every regular κ and every collection $\mathscr{F}$ of standard subsets of ${}^*\kappa$, if $\mathscr{F}$ has the κ-IP then $\bigcap \mathscr{F} \neq \emptyset$. [Equivalently, if $\mathscr{F}$ is any collection of subsets of κ with the κ-IP then $\bigcap_{A \in \mathscr{F}} {}^*A \neq \emptyset$.]

We note that A_2 for a fixed κ is a consequence of the $(2^\kappa)^+$-enlargement property; but of course we cannot assume that the latter holds for all regular κ (Proposition 3, Section 2). Clearly $A_2 \Rightarrow A_1$, but it seems unlikely that $A_2 \Rightarrow B_1$. Examination of the relationship between A_1 and B_1 suggests the following further generalization:

Axiom B_2. For every κ and every collection $\mathscr{F}$ of standard subsets of ${}^*\kappa$, if there are sequences $\langle \mathscr{F}_\tau \mid \tau < \mathrm{cf}(\kappa)\rangle$ and $\langle \kappa_\tau \mid \tau < \mathrm{cf}(\kappa)\rangle$ such that $\mathscr{F}_\tau \subseteq \mathscr{F}_{\tau'}$ and $\kappa_\tau \leqslant \kappa_{\tau'}$ for $\tau < \tau'$, each $\mathscr{F}_\tau$ has the κ_τ-IP, $\sup \kappa_\tau = \kappa$ and $\mathscr{F} = \bigcup_{\tau < \mathrm{cf}(\kappa)} \mathscr{F}_\tau$, then $\bigcap \mathscr{F} \neq \emptyset$.

To see that $B_2 \Rightarrow B_1$, let $\mathscr{F}_\tau := \{{}^*X \mid X \subseteq \kappa \wedge |\kappa \setminus X| < \kappa_\tau\}^\circ$ where $\langle \kappa_\tau \mid \tau < \mathrm{cf}(\kappa)\rangle$ is an increasing sequence of regular cardinals with limit κ.

One cannot fail noticing that the evidence for the axioms is getting weaker as the axioms grow stronger. For B_2 it amounts only to a mix of "maximize", "generalize" and "one step back from disaster" maxims. It is easy to propose (apparently) even stronger axioms, but I shall mention only two possible directions to generalize A_2.

(1) One could require that $\bigcap \mathscr{F} \neq \emptyset$ for any collection of standard subsets of ${}^*F(\kappa)$ with the κ-IP, where $F(\kappa)$ is a fixed nondecreasing function of κ.

(2) One could look for a principle that bears the same relationship to A_2 as the saturation property does to the enlargement property. A note of caution: simply replacing the word "standard" by "internal" in A_2 leads to a contradiction. [Let $\langle n_\xi \mid \xi < \kappa\rangle$ be an increasing sequence cofinal in $({}^*\omega, <)$, κ regular. Put $\mathscr{F} := \{\{v \mid n_\xi \leqslant v < {}^*\omega\} \mid \xi < \kappa\}$; then $\mathscr{F}$ has the κ-IP, but $\bigcap F = \emptyset$.] A more promising idea seems to be to consider a subuniverse $\hat{\mathbb{S}}$ of $\mathbb{I}$ such that $\mathbb{S} \prec \hat{\mathbb{S}} \prec \mathbb{I}$, which can be regarded as a "new" standard universe, and to which all of the previous considerations, including the formulations of the A and B-like axioms, could be relativized. Some results about subuniverses of $\mathbb{I}$ are presented in Section 6, but we shall not pursue their study here far enough to allow us to suggest relativizations of A and B with any confidence (for more details on this, see [24]).

There is one vestige of full saturation that has been found useful in applications, and would be a desirable addition to BNST. Under the label 'extension principle' it has been included (as an axiom or a theorem) in some existing nonstandard set theories (e.g., see [27]), but its first appearance in nonstandard analysis is in *comprehensive superstructures* of Robinson and Zakon [42]. I shall therefore call it the *Comprehensiveness Property* (see also [6, Theorem 4.4.23]). In BNST it can be formulated as follows:

(*Comp*) For every $f : {}^*X \cap \mathbb{S} \to \mathbb{I}$ there exists an internal $F : {}^*X \to \mathbb{I}$ such that $f \subseteq F$.

We note that (Comp) holds in any model of BNST whose internal universe is constructed as an ultraproduct of the standard universe (Section 7, Proposition 14). I expect the "correct" generalization of A_2 to internal sets to imply (Comp).

Reflection. It has to be noted that the axiom B_1 apparently contradicts the *reflection* maxim. By Proposition 2 in Section 2 (or, by axiom (iii)), every internal set belongs to some standard set. This can be reformulated as: "Every $\mathbb{O}n$-constrained internal set is λ-constrained by some $\lambda < \mathbb{O}n$" and, when reflected to a cardinal κ, gives a counterexample to B_1. However, I believe that this is not an appropriate application of reflection. It is well known that not all properties of $\mathbb{O}n$ can be reflected; unfortunately, even in the standard case there is no theory that would determine the properties for which reflection is legitimate. According to Reinhardt [40], Gödel proposed that "structural" properties can be reflected (but did not elaborate on the meaning of "structural"). I shall offer here only two sketchy remarks in support of the claim that use of reflection in the present situation is inappropriate. (A more ambitious attempt at a coherent theory will be made elsewhere.)

First, in "legitimate" applications of reflection, the cardinal(s) whose existence is being justified by it are "large". In the present case it is B_1 that asserts that (each) κ is "large"; reflection produces κ which are "small". E.g., in the trivial situation $\mathbb{I} = \mathbb{S}$, all infinite cardinals are counterexamples to B_1.

Second, the fact that all internal sets are λ-constrained for some λ seems to be merely a consequence of the Cantorian view of the universe as an "unfinished totality". In order to reflect on it, though, we have to regard $\mathbb{O}n$, $\mathbb{S}$ and $\mathbb{I}$ as *finished*, and perhaps even proceed one or several levels in the cumulative hierarchy beyond $\mathbb{O}n$, in some sense. (This is also necessary for "standard" reflection; for example, to obtain (strongly) inaccessible κ from $\mathbb{O}n$ by reflection, we need to consider *all* subclasses of $\mathbb{V}$, i.e., "$\mathbb{O}n+1$"-st level.) But, if $\mathbb{I}$ is regarded as finished, i.e., as a "set", there are no longer any reasons why $\xi \in \mathbb{I}$, $\xi \notin x$ for any $x \in \mathbb{S}$ could not exist, and the reflection argument loses its force.

6. Relatively standard sets

For any class $\mathscr{X} \subseteq \mathbb{I}$ we define the class of *sets standard relative to* $\mathscr{X}$:
$\mathbb{S}[\mathscr{X}] := \{ f(x_1,\dots,x_n) \mid f$ is a standard function, $x_1,\dots,x_n \in \mathscr{X},\ n \in \omega\}$.

Proposition 11. $\mathbb{S} \prec \mathbb{S}[\mathscr{X}] \prec \mathbb{I}$. [7]

Proof. [8] We first assume $\mathscr{X} = \{x_1,\dots,x_n\}$ is finite. We fix $A \in \mathbb{V}$ such that $x_1,\dots,x_n \in {}^*A$, let $I := A^n$, $\bar{x} := \langle x_1,\dots,x_n\rangle$, and define an ultrafilter $\mathscr{U} = \mathscr{U}_{\bar{x}}$ on $\mathscr{P}(I)$ by $P \in \mathscr{U} \equiv \bar{x} \in {}^*P$. Let $\mathbb{V}^I/\mathscr{U}$ be the ultraproduct of $\mathbb{V}$ over $\mathscr{U}$, and $k : \mathbb{V} \to \mathbb{V}^I/\mathscr{U}$ the canonical embedding $v \mapsto k_v$ where k_v is the constant function on I with value v. For $f : I \to \mathbb{V}$ let $\hat{f} := {}^*f(\bar{x}) \in \mathbb{S}[\mathscr{X}]$. We note that $\hat{k}_v = {}^*v$ (so $* = \hat{}\circ k$) and $\hat{\pi}_i = x_i$ (where $\pi_i(u_1,\dots,u_n) = u_i$) so $\mathbb{S} \subseteq \mathbb{S}[\mathscr{X}]$, $\mathscr{X} \subseteq \mathbb{S}[\mathscr{X}]$.

[7] We use statements and notation of the form " $\mathbb{S}$ is an elementary submodel of $\mathbb{I}$ ", $\mathbb{S} \prec \mathbb{I}$, as a convenient shorthand for Transfer; existence of the satisfaction relation in the model-theoretic sense is not implied.

[8] The argument is based on Theorem 6.4.4. in [6]. This idea was introduced into nonstandard set theory by Gordon [20]. In the full generality used here, it can be found in [29].

Claim. *The mapping* $\hat{}: \mathbb{V}^I/\mathscr{U} \to \mathbb{S}[\mathscr{X}]$ *is an* $\in$*-isomorphism. Furthermore*, $\mathbb{S}[\mathscr{X}] \models \phi({}^*f_1(\bar{x}), \ldots, {}^*f_k(\bar{x})) \equiv \{\bar{a} \in I \mid \mathbb{V} \models \phi(f_1(\bar{a}), \ldots, f_k(\bar{a}))\} \in \mathscr{U}$ *holds for all* $\in$*-formulas* ϕ.

Proof. We have $f =_{\mathscr{U}} g \equiv \{\bar{a} \in I \mid f(\bar{a}) = g(\bar{a})\} \in \mathscr{U} \equiv \bar{x} \in {}^*\{\bar{a} \in I \mid f(\bar{a}) = g(\bar{a})\} \equiv \bar{x} \in \{\bar{a} \in {}^*I \mid {}^*f(\bar{a}) = {}^*g(\bar{a})\} \equiv {}^*f(\bar{x}) = {}^*g(\bar{x})$. This shows $\hat{}$ is well-defined and one–one; it is obviously onto. Similarly, $f \in_{\mathscr{U}} g \equiv {}^*f(\bar{x}) \in {}^*g(\bar{x})$. Hence $\hat{}$ is an $\in$-isomorphism, and $\mathbb{S}[\mathscr{X}] \models \phi(\hat{f}_1, \ldots, \hat{f}_n) \equiv \mathbb{V}^I/\mathscr{U} \models \phi(f_1, \ldots, f_n) \equiv \{\bar{a} \in I \mid \mathbb{V} \models \phi(f_1(\bar{a}), \ldots, f_n(\bar{a}))\} \in \mathscr{U}$ (the last equivalence is Loś theorem). $\square$

We have $\mathbb{S} \subseteq \mathbb{S}[\mathscr{X}] \subseteq \mathbb{I}$; as $\mathbb{S} \prec \mathbb{I}$, it suffices to show $\mathbb{S}[\mathscr{X}] \prec \mathbb{I}$. We proceed by induction on complexity of formulas; the only nontrivial step is when $\mathbb{S}[\mathscr{X}] \models (\forall v)\phi(v, {}^*f_1(\bar{x}), \ldots, {}^*f_k(\bar{x}))$. Let $A := \{\bar{a} \in I \mid \mathbb{V} \models (\forall v)\ \phi(v, f_1(\bar{a}), \ldots, f_k(\bar{a}))\}$. Then $A \in \mathbb{V}$ and by Claim, $A \in \mathscr{U}$. We can conclude that $\bar{x} \in {}^*A = \{\bar{a} \in {}^*I \mid \mathbb{S} \models (\forall v)\phi(v, {}^*f_1(\bar{a}), \ldots, {}^*f_k(\bar{a}))\} = \{\bar{a} \in {}^*I \mid \mathbb{I} \models (\forall v)\phi(v, {}^*f_1(\bar{a}), \ldots, {}^*f_k(\bar{a}))\}$. (The last step is by Transfer.) Hence $\mathbb{I} \models (\forall v)\ \phi(v, {}^*f_1(\bar{x}), \ldots, {}^*f_k(\bar{x}))$.

If $\mathscr{X}_1 \subseteq \mathscr{X}_2$ are finite, we have $\mathbb{S}[\mathscr{X}_1] \prec \mathbb{S}[\mathscr{X}_2]$. Consider now an arbitrary $\mathscr{X}$. The classes $\mathbb{S}[\mathscr{Y}]$ for $\mathscr{Y} \subseteq \mathscr{X}$, $\mathscr{Y}$ finite, form a directed system of elementary submodels of $\mathbb{I}$, so we have $\mathbb{S} \prec \mathbb{S}[\mathscr{Y}] \prec \mathbb{S}[\mathscr{X}] = \bigcup\{\mathbb{S}[\mathscr{Y}] \mid \mathscr{Y} \subseteq \mathscr{X}, \mathscr{Y} \text{ finite}\} \prec \mathbb{I}$. $\square$

The proof shows that $\mathbb{S}[\mathscr{X}]$ is the least elementary submodel of $\mathbb{I}$ containing $\mathbb{S}$ and $\mathscr{X}$, and that every elementary submodel $\mathbb{N}$ of $\mathbb{I}$ containing $\mathbb{S}$ is of this form, that is, isomorphic to a limit ultraproduct of $(\mathbb{V}, \in)$ (because $\mathbb{N} = \mathbb{S}[\mathbb{N}]$).

Corollary 4. $\mathbb{S}[\mathscr{X}, \mathscr{Y}] = \mathbb{S}[\mathscr{X}][\mathscr{Y}] := \{g(y_1, \ldots, y_n) \mid g \in \mathbb{S}[\mathscr{X}], y_1, \ldots, y_n \in \mathscr{Y}\}$.

Let κ be an infinite cardinal. Set $\mathbb{C}_\kappa := \mathbb{S}[{}^*\kappa]$.

Proposition 12. $z \in \mathbb{C}_\kappa$ *if and only if* z *is* κ*-constrained.*

Proof. If $z \in \mathbb{C}_\kappa$ then $z = {}^*f(\bar{x})$ for some $\bar{x} = \langle x_1, \ldots, x_n \rangle \in ({}^*\kappa)^n$. We can assume dom $f = \kappa^n$; hence $z \in \operatorname{ran}({}^*f) = {}^*(\operatorname{ran} f)$ where $|\operatorname{ran} f| \leqslant \kappa$. Conversely, z κ-constrained means $z \in {}^*a$ where $a \in \mathbb{V}$, $|a| \leqslant \kappa$. Let $f \in \mathbb{V}$ be a mapping of κ onto a; then $z = {}^*f(x)$ for some $x \in {}^*\kappa$, and so $z \in \mathbb{S}[{}^*\kappa] = \mathbb{C}_\kappa$. $\square$

Corollary 5. $\mathbb{S} \prec \mathbb{C}_\kappa \prec \mathbb{I}$.

Proof. This can also be easily proved directly. $\square$

We let BNST^- denote BNST without the axiom 0.(ii).

Proposition 13. *Assume* $\mathscr{X} \cap (\mathbb{I} \setminus \mathbb{S}) \neq \emptyset$. *Then* $(\mathbb{U}, \in, \mathbb{S}, \mathbb{S}[\mathscr{X}])$ *satisfies* BNST^- (*here 'sets' are interpreted as elements of* $\mathbb{U}$, *'standard sets' are elements of* $\mathbb{S}$, *and 'internal sets' are elements of* $\mathbb{S}[\mathscr{X}]$).

Proof. Straightforward verification. $\square$

We shall use these results in Section 7 to deduce some consequences of our axioms for standard set theory, and in Section 8 to show how to practice Nonstandard Analysis in BNST. We conclude this section with some remarks on "relativization".

(0) The observation that the internal universe of nonstandard set theory has many definable proper subuniverses seems to occur first in the preprint of [23] mentioned in footnote 6, where the universes $\mathbb{C}_\kappa$ of κ-constrained (there, κ-quasistandard) sets were introduced and Corollary 5 was stated (including additional saturation properties of the elementary embeddings).

(1) Proposition 13 shows that letting $\hat{\mathbb{S}} := \mathbb{S}$, $\hat{\mathbb{I}} := \mathbb{S}[\mathscr{X}]$ and $\hat{\mathbb{W}} := \mathbb{U}$ yields an interpretation of BNST$^-$, but unlike $\mathbb{I}$, $\hat{\mathbb{I}}$ is *not* transitive. It seems natural and can be proved consistent with BNST [24] to assume that there is a transitive universe isomorphic to $\hat{\mathbb{I}}$; but this implies that $\mathbb{U}$ is not well-founded over $\mathbb{I}$. This is one reason why we do not include Foundation over $\mathbb{I}$ among the axioms of BNST.

(2) Another, related question is whether there are interpretations of BNST in which some $\hat{\mathbb{S}} \neq \mathbb{S}$ is the new *standard* universe. This question ("relativization of standardness") was considered first by Gordon [20] and Péraire [39] (in the context of IST). Gordon showed that, for example, $(\mathbb{I}, \in, \hat{\mathbb{S}})$ where $\hat{\mathbb{S}} = \mathbb{S}[\nu]$ for $\nu \in {}^*\omega \setminus \omega$ satisfies Transfer and (Bounded) Idealization, but Standardization fails, even for $x \in \mathbb{I}$. More recent results of Andreyev [1] and of [24] show that Standardization must fail for any (definable) $\hat{\mathbb{S}} \neq \mathbb{S}$ with ${}^*\omega \setminus \hat{\mathbb{S}} \neq \emptyset$.

In the opposite direction, Péraire *axiomatized* the notion of *relative standardness* and showed consistency of a theory (RIST) extending IST, where the universe of sets standard relative to any fixed x does satisfy versions of Standardization, Transfer and Idealization.

Returning to the setting of BNST, it can be proved that existence of many universes $\hat{\mathbb{S}} \neq \mathbb{S}$, $\hat{\mathbb{I}}$ and $\hat{\mathbb{W}}$ such that $(\hat{\mathbb{W}}, \in, \hat{\mathbb{S}}, \hat{\mathbb{I}})$ satisfies BNST is consistent with BNST; however, $\hat{\mathbb{W}}$ may not then be well-founded over $\mathbb{I}$. (The details will appear in [24].) This is our second reason for not accepting Foundation over $\mathbb{I}$ as a true axiom. (The third reason is the desire to have the ability to "layer" nonstandard arguments, discussed briefly in Section 8.)

It now seems reasonable that, for example, given any set $X \subseteq {}^*\kappa$ with $|X| \leqslant \kappa$ there exists $\hat{\mathbb{S}} \supseteq X$ and $\hat{\mathbb{I}}, \hat{\mathbb{W}}$ so that $(\hat{\mathbb{W}}, \in, \hat{\mathbb{S}}, \hat{\mathbb{I}})$ satisfies BNST and some or all of the axioms of type A and B. This appears to be a "proper" way to relativize these axioms, but we do not pursue it here.

7. BNST and standard set theory

Up to this point, we have eschewed any considerations of relative consistency of our proposed axioms, in order to let the arguments for their intuitive validity stand on their own. It would be presumptuous to claim that our intuition about nonstandard objects is so well developed as to guarantee that the proposed axioms cannot lead to contradictions. On the other hand, any consequences for standard set theory that these axioms may have are justified to the extent one finds the axioms themselves intuitively justified. In this section we study these and related issues. In particular, we shall show that BNST^+ is a large cardinal assumption of as yet unknown consistency strength.

Proposition 14. *Let κ be a definable cardinal* [*i.e., there is an $\in$-formula $\phi(x)$ such that* $\mathrm{ZFC} \vdash (\exists! x)\ \phi(x) \wedge (\forall x)\ (\phi(x) \Rightarrow x$ *is a cardinal*); *κ is defined by $\phi(x)$*]. *Then* $\mathrm{Con(ZFC)}$ *implies* $\mathrm{Con(BNST} + \kappa\text{-saturation} + \mathrm{Comp})$.

Proof. For details see [11]. It simplifies matters if one works in ZFBC, a theory obtained from ZFC by replacing the axiom of Foundation by the axiom of Superuniversality, and adding Global Choice (see [4][9]). It is known that ZFBC is a conservative extension of ZFC, in the sense that $\mathrm{ZFC} \vdash \phi$ if and only if $\mathrm{ZFBC} \vdash \phi^{\mathbb{V}}$ holds for any $\in$-formula ϕ.

Let $\mathscr{U}$ be a κ-good ultrafilter on $I := \kappa$ (see [6]). We consider the ultraproduct $\mathbb{V}^I/\mathscr{U}$ and the canonical embedding $k : \mathbb{V} \to \mathbb{V}^I/\mathscr{U}$. In ZFBC there is an isomorphism $l : \mathbb{V}^I/\mathscr{U} \to \mathbb{I}_\kappa$ where $\mathbb{I}_\kappa$ is a transitive class. Let $* : \mathbb{V} \to \mathbb{I}_\kappa$ be the composition $l \circ k$ and let $\mathbb{S}_\kappa$ be the range of $*$. Finally, obtain $\mathbb{W}_\kappa$ by constructing a cumulative hierarchy over $\mathbb{I}_\kappa$:
$W_{\alpha,0} := {}^*V_\alpha$;
$W_{\alpha,\beta+1} := \mathscr{P}(W_{\alpha,\beta})$;
$W_{\alpha,\beta} := \bigcup_{\gamma<\beta} W_{\alpha,\gamma}$ for β limit, and
$\mathbb{W}_\kappa := \bigcup_{\alpha,\beta \in \mathbb{O}n} W_{\alpha,\beta}$.

It is not difficult to verify that BNST holds in $(\mathbb{W}_\kappa, \in, \mathbb{S}_\kappa, \mathbb{I}_\kappa)$. κ-saturation is by [6, Theorem 6.1.8] and (Comp) by [6, Theorem 4.4.23]. □

Well-known "tricks" allow modifications of the above argument to show that BNST+ κ-saturation + (Comp) is in fact a conservative extension of ZFC; see [11].

The situation changes dramatically when A or B type axioms are added to BNST.

We recall that an ultrafilter $\mathscr{U}$ on κ is (τ,λ)-*regular* if there is some sequence $\langle A_\alpha \mid \alpha < \lambda \rangle$ where $A_\alpha \in \mathscr{U}$ for all $\alpha < \lambda$, and $\bigcap_{\alpha \in J} A_\alpha = \emptyset$ for all $J \subseteq \lambda$, $|J| \geqslant \tau$. $\mathscr{U}$ is *regular* if and only if it is (ω,κ)-regular.

In the proof of Proposition 11 in Section 6 we showed that for each $\xi \in {}^*\kappa$ there is an ultrafilter $\mathscr{U}_\xi$ on κ defined by $A \in \mathscr{U}_\xi \equiv \xi \in {}^*A$ (let $\mathscr{X} := \{\xi\}$). $\mathscr{U}_\xi$ is uniform if and only if ξ is not λ-constrained for any $\lambda < \kappa$; i.e., if and only if $\xi \notin {}^*A$ for any $A \subseteq \kappa$, $|A| < \kappa$. We also defined an isomorphism $\hat{\ }: \mathbb{V}^\kappa/\mathscr{U}_\xi \to \mathbb{S}[\{\xi\}] \prec \mathbb{I}$. The key observation now is

Proposition 15 (BNST). *If $\mathscr{U}_\xi$ on κ is (τ^+,λ)-regular then $\lambda \leqslant |{}^*\tau|$.*

[9] It is argued in [4] that ZFBC is a convenient vehicle for the practice of nonstandard analysis; I do not discuss it here because I consider it to be a *standard* set theory.

Proof. Let $\langle A_\alpha \mid \alpha < \lambda \rangle$ be a witness to (τ^+, λ)-regularity of $\mathscr{U}_\xi$. We define $f : \kappa \to \mathscr{P}(\lambda)$ by $f(x) := \{\alpha < \lambda \mid x \in A_\alpha\}$, and notice that $|f(x)| \leqslant \tau$. Hence $\mathbb{V}^\kappa/\mathscr{U}_\xi \vDash |f| \leqslant k_\tau \wedge k_\alpha \in f$, for all $\alpha < \lambda$. The isomorphism $\hat{}$ then gives $\mathbb{S}[\{\xi\}] \vDash |\hat{f}| \leqslant {}^*\tau \wedge {}^*\alpha \in \hat{f}$, for all $\alpha < \lambda$. As $\mathbb{S}[\{\xi\}] \prec \mathbb{I}$, these statements hold in $\mathbb{I}$ as well. Hence $\lambda \leqslant |\hat{f}| \leqslant |{}^*\tau|$. $\square$

Proposition 16 (BNST). *If* $\kappa > |{}^*\omega|$ *and*

(S_κ): *there exists a strictly* κ*-constrained set,*

then there exists a uniform ultrafilter on κ *which is not* (ω, λ)*-regular for any* $\kappa \geqslant \lambda > |{}^*\omega|$.

(*In particular, it is not regular.*)

Proof. Take $\xi \in {}^*\kappa$, ξ not μ-constrained for any $\mu < \kappa$. The corresponding $\mathscr{U}_\xi$ is a uniform ultrafilter on κ. If it were (ω, λ)-regular, hence (ω^+, λ)-regular, we would have $\lambda \leqslant |{}^*\omega|$. $\square$

Corollary 6. $\mathrm{Con}(\mathrm{BNST} + \mathrm{A}_0)$ *implies* $\mathrm{Con}(\mathrm{ZFC} + \text{there exists a measurable cardinal})$.

Proof. This follows from a theorem of Donder [12, Theorem 4.5] strengthening a previous result of Jensen: Unless there is an inner model with a measurable cardinal, every uniform ultrafilter on κ is (ω, λ)-regular for all $\lambda < \kappa$. $\square$

The assumption $\lambda > |{}^*\omega|$ can be weakened.

We say that an ultrafilter $\mathscr{U}$ on κ is *realized* if $\mathscr{U} = \mathscr{U}_\xi$ for some internal ξ. Statement (S_κ) is equivalent to: *Some uniform ultrafilter on* κ *is realized.*

Proposition 17 (BNST). *Let* $\tau \leqslant \lambda \leqslant \kappa$. *If* $2^\lambda > |{}^*(2^{<\tau})|$ *then every realized uniform ultrafilter on* κ *is* (τ, λ)*-nonregular.*

Proof (See [25], 38.6 for $\tau = \omega$, $\lambda = \kappa$). Assume that $\mathscr{U}$ is (τ, λ)-regular and realized. Let $\langle A_\alpha \mid \alpha < \lambda \rangle$ be a sequence of subsets of κ that witnesses (τ, λ)-regularity of $\mathscr{U}$; i.e., $A_\alpha \in \mathscr{U}$ for all $\alpha < \lambda$ and for any $x \in \kappa$, $C_x := \{\alpha \in \lambda \mid x \in A_\alpha\}$ has $|C_x| < \tau$. Fix an enumeration $\langle \alpha_i^x \mid i \in \sigma^x \rangle$ $(\sigma^x < \tau)$ of each C_x. For any $g : \lambda \to 2$ let $f_g(x) := \langle g(\alpha_i^x) \mid i \in \sigma^x \rangle \in 2^{<\tau}$. If $g \neq g'$ then $g(\alpha) \neq g'(\alpha)$ for some $\alpha < \lambda$ and one sees easily that, for all $x \in A_\alpha \in \mathscr{U}$, $f_g(x) \neq f_{g'}(x)$. Therefore $f_g/\mathscr{U} \neq f_{g'}/\mathscr{U}$ and $|{}^*(2^{<\tau})| \geqslant 2^\lambda$. $\square$

We note that existence of λ for which (S_{λ^+}) holds and $2^{\lambda^+} > |{}^*(2^{<\lambda})|$ implies existence of a fully nonregular ultrafilter on λ^+ (i.e., (λ, λ^+)-nonregular). However, it seems more likely that $2^{\lambda^+} \leqslant |{}^*(2^{<\lambda})|$ holds in the nonstandard universe (this of course does not preclude existence of such ultrafilters).

Corollary 7. *If* κ^+*-enlargement property holds then* $2^\kappa \leqslant |{}^*\omega|$.

Proof. Let $I := \mathscr{P}^{<\omega}(\kappa)$; κ^+-enlargement property implies existence of $x \in {}^*I$ such that ${}^*\alpha \in x$ for all $\alpha < \kappa$. The ultrafilter $\mathscr{U}_{\{x\}}$ on I is (ω, κ)-regular. By Proposition 17, $2^\kappa \leqslant |{}^*(2^{<\omega})| = |{}^*\omega|$. $\square$

We now consider the opposite direction.

Proposition 18. Con(ZFC + there exists a proper class of measurable cardinals) *implies* Con(BNST + A_0).

Proof. Again we work in ZFBC (+ existence of an increasing sequence $\langle \kappa_\nu \mid \nu \in \mathbb{O}n \rangle$ of measurable cardinals). We choose a κ_ν-complete ultrafilter $\mathscr{U}_\nu$ on each κ_ν and construct a transfinite sequence $(\mathbb{M}_\nu, \mathbb{E}_\nu, \mathbb{K}_{\mu,\nu} \mid \mu \leqslant \nu,\ \mu, \nu \in \mathbb{O}n)$ where $\mathbb{M}_0 := \mathbb{V}$, $\mathbb{E}_0 := \{(x, y) \in \mathbb{V} \mid x \in y\}$, $\mathbb{K}_{0,0} := \mathbb{I}\,d$; $(\mathbb{M}_{\nu+1}, \mathbb{E}_{\nu+1})$ is the ultraproduct of $(\mathbb{M}_\nu, \mathbb{E}_\nu)$ via the ultrafilter $\mathscr{U}_\nu$ on κ_ν, $\mathbb{K}_{\nu,\nu+1}$ is the canonical embedding of $\mathbb{M}_\nu$ into $\mathbb{M}_{\nu+1}$; $(\mathbb{M}_\nu, \mathbb{E}_\nu, \mathbb{K}_{\mu,\nu})$ for limit ν is the direct limit of the previously constructed system. Finally we let $(\mathbb{M}_\infty, \mathbb{E}_\infty, \mathbb{K}_{\mu,\infty})$ be the direct limit of $(\mathbb{M}_\nu, \mathbb{E}_\nu, \mathbb{K}_{\mu,\nu} \mid \mu, \nu \in \mathbb{O}n)$. The key point is that, for each $x \in \mathbb{M}_\infty$, $\{y \in \mathbb{M}_\infty \mid y\, \mathbb{E}_\infty\, x\}$ is a set. By Superuniversality and global choice, there is an isomorphism $l : (\mathbb{M}_\infty, \mathbb{E}_\infty) \to (\mathbb{I}, \in)$ where $\mathbb{I}$ is a transitive class. We let $* := l \circ \mathbb{K}_{0,\infty}$ and finish the proof as in Proposition 14. $\square$

One can easily modify this proof to show Con(BNST + A_0 + κ-saturation), for any definable cardinal κ. Also, let A_0^+ be the statement:

"There exist arbitrarily large κ such that every collection $\mathscr{F}$ of standard subsets of ${}^*\kappa$ with κ-IP has $\bigcap \mathscr{F} \neq \emptyset$". A similar argument proves Con(BNST + A_0^+) assuming Con(ZFC + there exists a proper class of strongly measurable cardinals), where κ is *strongly measurable* if every κ-complete filter on κ extends to a κ-complete ultrafilter.

Relative consistency of the stronger versions of A and B is at present unknown. Work of Foreman et al. [14] shows that nonregular ultrafilters are consistent relative to "ZFC + huge cardinals exist". As shown by Proposition 17, axiom A_1 requires existence of many very highly nonregular ultrafilters. We next describe more precisely what is needed to construct models for BNST + A_1. For simplicity we shall limit the discussion to models whose standard universe is isomorphic to V_θ, where θ is strongly inaccessible.

Definition. Let θ be a strongly inaccessible cardinal. $(\mathscr{U}, \mathscr{F})$ is an A_1-*suitable pair* (B_1-*suitable pair*, resp.) for θ if $\mathscr{U}$ is an ultrafilter over θ, $\mathscr{F}$ is a filter over $\theta \times \theta$ generated by (some) equivalence relations on θ having less than θ equivalence classes, and for all regular $\kappa < \theta$ (all $\kappa < \theta$, resp.)

(1) there exists $f : \theta \to \kappa$ such that $\mathrm{eq}(f) \in \mathscr{F}$ and $f_*[\mathscr{U}]$ is a uniform ultrafilter over κ;

(2) $|\prod_{\mathscr{U}|\mathscr{F}} \kappa| < \theta$.

Proposition 19 (ZFC). *Let θ be strongly inaccessible. Existence of an* A_1*-suitable pair on θ is equivalent to the existence of a model for* $\mathrm{BNST}+\mathrm{A}_1$ *whose standard universe is isomorphic to V_θ. Similarly for* B_1.

Proof. Given $(\mathscr{U},\mathscr{F})$ we let $\mathbb{I}$ be isomorphic to $\prod_{\mathscr{U}|\mathscr{F}} V_\theta$; the rest of the model is constructed in a way similar to the one used in the proof of Proposition 14. Condition (1) guarantees that A_1 holds. Condition (2) shows that ${}^*\kappa$ in the model has $|{}^*\kappa|<\theta$, which in turn suffices to show that Power Set and Replacement hold in the model.

Conversely, any model as described is isomorphic to $\prod_{\mathscr{U}|\mathscr{F}} V_\theta$ for some ultrafilter $\mathscr{U}$ on θ and a filter $\mathscr{F}$ on $\theta\times\theta$ (see [6, Theorem 6.4.10]). It is easy to check that $(\mathscr{U},\mathscr{F})$ has to have the properties from the definition. □

One can similarly characterize A_2-suitable pairs and prove a result akin to Proposition 19. The only change needed is replacing (1) with (1^+): for every collection $\mathscr{G}$ of subsets of κ with κ-IP there exists $f:\theta\to\kappa$ such that $\mathrm{eq}(f)\in\mathscr{F}$ and $\mathscr{G}\subseteq f_*[\mathscr{U}]$. There is an analogous formulation for B_2.

We use BNST^+ to refer generically to $\mathrm{BNST}+\mathrm{A}_1$ or any of the stronger axioms of type A or B. In contrast to the usual large cardinal axioms, it is not clear whether BNST^+ can be preserved by small forcing (in the standard universe). Although $\mathrm{BNST}+\mathrm{A}_0$ is easily seen consistent with any statement whose consistency with ZFC can be established by small forcing, such as the Continuum Hypothesis or its negation (see the proof of Proposition 18), the situation with respect to the stronger axioms might well be different.

We conclude this section with a brief discussion of two properties of Nonstandard Set Theories that Kanovei and Reeken [28] consider desirable.

A Nonstandard Set Theory T has the *Model Enlargement Property* with respect to ZFC if every countable model M of ZFC can be enlarged to a model M' of T containing M as the class of all standard sets. In view of Corollary 6, BNST^+ cannot have this property with respect to ZFC, but it should be possible to formulate a theory ZFC^+ (along the lines of Proposition 19) so that BNST^+ has the Model Enlargement Property *with respect to* ZFC^+. As long as the latter is a *true* theory, this would more than satisfy a realist.

A Nonstandard Set Theory T is *reducible* to ZFC if $T\vdash \mathrm{ZFC}^{\mathbb{S}}$ and for any formula $\phi(x_1,\dots,x_n)$ in the language of T there exists an $\in$-formula $\hat{\phi}(x_1,\dots,x_n)$ such that $T\vdash(\forall x_1\in\mathbb{S})\dots(\forall x_n\in\mathbb{S})\ [\phi(x_1,\dots,x_n)\equiv\hat{\phi}^{\mathbb{S}}(x_1,\dots,x_n)]$.

According to Kanovei and Reeken [28], this property is "important especially for those who are inclined to treat nonstandard methods as shortcuts for 'standard' reasoning". This is not our position here, and I do not see why a realist needs to insist on this property, any more than he needs to insist that the Nonstandard Set Theory be a conservative extension of ZFC. Nevertheless, this *Reduction Property* is certainly useful in practice (see [37, 38]). I do not know whether BNST^+ has the Reduction Property with respect to ZFC, but I suspect that it can have it with respect to some suitable ZFC^+.

8. Realism, nonstandard set theories, and practice of nonstandard analysis

In the preceding sections I have presented arguments to support the claim that the nonstandard universe satisfies (at the minimum) the axioms of BNST and some other axioms postulating existence of various "ideal" objects. Here I briefly survey the panorama of existing nonstandard set theories, and consider how they differ from BNST. I conclude with some remarks on the adequacy of BNST for nonstandard practice.

The earliest nonstandard set theories were proposed in [22, 37]. I shall first discuss the theories NS_1 and NS_2 from [22]. Their common part NS_0 agrees with BNST, except for

(1) NS_0 does not postulate Power Set, Choice and Replacement in the external universe;

(2) NS_0 does postulate the κ-enlargement property and the κ-saturation property for all κ (axioms (B4) and (B4$^+$) in [22]).

As Proposition 3 in Section 2 shows, κ-enlargement property for all κ is incompatible with full BNST. My original primary concern when axiomatizing nonstandard set theory was to accomodate the nonstandard practice of the time as much as possible; it appeared that κ-saturation for all κ was an essential tool for such practice, while a relatively weak external set theory would suffice. Once this crucial decision is made, one can choose between incompatible consistent theories $NS_1 := NS_0 +$ replacement and $NS_2 := NS_0 +$ Power Set + Well-Ordering Principle.

I believe that the considerations of Sections 2 and 3 show that BNST is better justified than the full κ-enlargement property. Also, both NS_1 and NS_2 violate weak reductionism, as it is provable in both of them that $({}^*\omega, <)$ is *not* isomorphic to any standard structure. Both NS_1 and NS_2 are conservative extensions of ZFC and hence can be used in practice without fear of contradiction, but from our realist point of view they should be regarded as "false".

Nelson's IST includes an axiom schema postulating existence of "ideal" elements (Idealization) that is even stronger than κ-enlargement for all κ. In particular, Idealization implies existence of an internal ordinal that is bigger than all standard ordinals; i.e., the property (∇) discussed in Section 3. (Accordingly, IST postulates only bounded Standardization.)

IST is a theory of standard and internal sets only, so the issues of validity of Power Set, Replacement and choice for external sets do not arise in it. An attempt to extend IST to a theory of external sets was made by Kawai [31, 32]. The full extent to which weak reductionism is violated then becomes manifest: in addition to $({}^*\omega, <)$ not being isomorphic to any standard set, the entire standard universe in Kawai's NST is a set, in accord with our discussion of (∇) in Section 3.

Another extension of IST was proposed by Péraire [39]; as such, RIST inherits the questionable features of the former. However, this paper introduced the important idea of relativizing the standard universe (see Section 6).

Kanovei and Reeken re-examined foundational issues connected with IST in a series of papers beginning with [26]; in particular see [27]. First, metamathematical properties of IST improve, and none of its practical usefulness is lost, if one assumes that every internal set is an element of some standard set, and weakens Idealization to bounded Idealization. In the resulting theory BST, one no longer has ordinals larger than all the standard ones. BST can be extended in a natural, definable way, to a theory of external sets HST. This HST is essentially NS_1, with the addition of Foundation over $\mathbb{I}$ and some weak forms of the Axiom of Choice. Thus the comments and objections to NS_1 raised earlier apply to HST as well.

Andreyev and Gordon [2] formulated another modification of IST. The sets of NCT satisfy the axioms of BST; in addition, NCT allows proper classes (in the style of Gödel–Bernays), some of which ("semisets") can be used in lieu of external subsets of $\mathbb{I}$. I see no intrinsic reasons why external "sets" should not be genuine sets. NCT does not allow "sets of external sets", etc., and it still violates weak reductionism.

All of the axiomatic systems discussed so far share a commitment to a unique standard and internal universe and (except for IST, RIST and BST, which do not allow external sets at all) also a unique external universe, although a rich structure of subuniverses exists in many of them. Fletcher [13] takes a fundamentally different *philosophical* position. For him, there is no privileged unique internal or external universe; a practitioner can choose them according to the desired level of κ-saturation. In Fletcher's SNST, for each fixed κ, the structure $(\mathbb{E}_\kappa, \in, \mathbb{S}, \mathbb{I}_\kappa)$ satisfies BNST + κ-saturation. The "catch-all" superuniverses $\mathbb{I} := \bigcup_\kappa \mathbb{I}_\kappa$ and $\mathbb{E} := \bigcup_\kappa \mathbb{E}_\kappa$ are of no interest, and no particular properties are postulated for them. Ballard [3] takes this relativistic worldview to its logical conclusion by developing a system (EST) in which there is even no fixed standard universe. Both SNST and EST are conservative extensions of ZFC, and so formally unexceptionable. They are not directly relevant to the project of this paper, which is predicated on the assumption that there is a fixed comprehensive nonstandard universe. However, it is of interest to consider the extent to which ideas of SNST and EST could be "true" in it. Various subuniverses allowing relativization of the internal, and even the standard universe, are available in (extensions of) BNST, as discussed in Section 5 and below. The additional facility of "layering" provided by EST (see also [4]) is the possibility of using a (relativized) external universe as the "new" standard universe for another "nonstandard" extension. Again, this feature is consistent with BNST to some extent (see below), and might lead to further insight into the structure of the nonstandard universe.

Yet another radically different philosophical position underlies the Alternative Set Theory (AST) developed extensively by Petr Vopěnka and his Prague school (see [44, 43]): the assumption that all sets are formally finite (hyperfinite, in the terminology of Nonstandard Analysis). For all its interest, AST does not seem to bear directly on this project, and cannot be discussed here. The same applies to nonstandard theories of finite sets proposed by Andreyev and Gordon [2] and Baratella and Ferro [5].

It is convenient at this point to insert a few remarks about the popular superstructure framework of Robinson and Zakon [42]. For the purposes of this paper, I do not consider it to be a *Nonstandard Set Theory*, because its users work with models rather than axiomatically, and because full ZFC does not hold in its standard universe. But, as the proposers were clearly aware and Keisler [33] makes explicit, superstructures do satisfy axioms of a weak set theory RZ. If formulated in the internal picture used here, RZ can be obtained from BNST roughly as follows. The list of primitive concepts of BNST is extended by two constants, X and Y; they are intended to denote the sets of standard and internal individuals, respectively. The axioms are modified as appropriate for a set theory with individuals. Infinity and Replacement are omitted from groups (I.) and (IV.). Transfer holds for restricted formulas only; more precisely, $x_1 \in \mathbb{S} \wedge \cdots \wedge x_n \in \mathbb{S} \Rightarrow \phi^{\mathbb{S}}(x_1,\ldots,x_n,X) \equiv \phi^{\mathbb{I}}(x_1,\ldots,x_n,Y)$ where $\phi(x_1,\ldots,x_n,z)$ is a restricted $\in$-formula. It is assumed that X is not finite in $\mathbb{S}$. Saturation can be added as desired.

Finally, yet another interesting proposal for an axiomatic nonstandard set theory was made recently by Di Nasso in [9, 11]. *ZFC takes $\in$ and $*$, rather than $\in$ and $\mathbb{S}$, as primitive concepts; in other words, it uses the external picture rather than the internal one. As we saw in Section 2, this is only a matter of preference. In the internal reformulation, axioms (1)–(3) of *ZFC are equivalent to BNST, (4) is Foundation over $\mathbb{I}$, and (5) (definable saturation schema) will be discussed below. Otherwise, there are only terminological differences; e.g. Di Nasso calls "standard sets" what we term "well-founded sets", etc.

Let us now consider the axiom schema (5) of *ZFC; in the language of BNST it asserts:

Definable saturation. Let $\phi(x)$ be an $\in$-formula with a single free variable. If $(\forall x, y \in \mathbb{V})\ (\phi^{\mathbb{V}}(x) \wedge \phi^{\mathbb{V}}(y) \Rightarrow x = y \wedge x$ is a cardinal$)$ then $(\forall \kappa)[\phi^{\mathbb{V}}(\kappa) \Rightarrow (\forall \mathscr{F})(\mathscr{F} \subseteq \mathbb{I} \wedge \mathscr{F}$ has FIP $\wedge |\mathscr{F}| < \kappa \Rightarrow \cap \mathscr{F} \neq \emptyset)]$.

This complicated statement merely says that κ-saturation holds for every cardinal κ that is $\in$-definable. At present I see no convincing reasons either for or against accepting definable Saturation as a true axiom (schema). But certainly

Proposition 20. Con(ZFC) *implies* Con(BNST + Definable Saturation).

Proof. By compactness, it suffices to prove consistency of BNST with a finite number of instances of Definable Saturation. This follows immediately from Proposition 14 in Section 7. □

A slightly more careful argument shows that BNST* := BNST + Definable Saturation is a conservative extension of BNST, in the sense that the same statements $\phi^{\mathbb{S}}$ (where ϕ is an $\in$-formula) are provable in both. One can similarly show that definable saturation can be conservatively added to BNST + Additional Axioms of type A or B, assuming the latter are consistent.

The interest in Definable Saturation comes from the fact, first pointed out by Di Nasso, that it is adequate for an important part of nonstandard practice. In fact, it was this observation of his that made me seriously contemplate the abandonment of the requirement of κ-saturation for all κ, and the defensibility of the realist stance in nonstandard set theory even from a practitioner's perspective. The following paraphrase of his result (see [11, Theorem 2.2]) shows that any standard statement about a standard set x that is provable in BNST* under the assumption of κ-saturation is provable in BNST* (hence, in ZFC) alone, provided only that the level of saturation that is needed for the proof is $\in$-definable in terms of x (as is always the case in practice).

Proposition 21. *Let $\phi(x)$, $\chi(x,y)$ be $\in$-formulas with the indicated free variables only. If* BNST$^* \vdash (\forall x \in \mathbb{V})\ (\exists \kappa)\ (\kappa$ *is a cardinal* $\bigwedge \chi^{\mathbb{V}}(x,\kappa)) \wedge (\forall x \in \mathbb{V})[(\exists \kappa)\ (\kappa$ *is a cardinal* $\bigwedge \chi^{\mathbb{V}}(x,\kappa) \wedge \kappa$*-saturation*$)\ \Rightarrow \phi^{\mathbb{S}}({}^*x)]$ *then* ZFC$\vdash (\forall x)\ \phi(x)$*.*

Proof. It suffices to prove that BNST$^* \vdash (\forall x \in \mathbb{V})\ \phi^{\mathbb{S}}({}^*x)$ (by conservativity of BNST* over ZFC). Working in BNST*, assume that, for some $x \in \mathbb{V}, \neg\phi^{\mathbb{S}}({}^*x)$, or equivalently, $\neg\phi^{\mathbb{V}}(x)$. Let κ be the smallest cardinal such that $(\exists x \in \mathbb{V})\ (\chi^{\mathbb{V}}(x,\kappa) \wedge \neg\phi^{\mathbb{V}}(x))$. Then κ is $\in$-definable in $\mathbb{V}$, so we have κ-saturation and $\phi^{\mathbb{S}}({}^*x)$ for any $x \in \mathbb{V}$ such that $\chi^{\mathbb{V}}(x,\kappa) \wedge \neg\phi^{\mathbb{V}}(x)$, a contradiction. $\square$

Proposition 21 allows *de facto* unlimited saturation in arguments that use nonstandard analysis to prove standard theorems. Typical example is the nonstandard proof of Tychonov's theorem; i.e., compactness of $X := \prod_{i\in I} X_i$ assuming all X_i are compact. We need $\kappa = |X|^+$-saturation, so let $\chi(x,y)$ be "$y = |x|^+$" and $\phi(x)$ be "if $x = \prod X_i$ where X_i are compact topological spaces, then x is compact".

Of course, methods of Nonstandard Analysis are used at least equally often to construct nonstandard objects, such as e.g. Loeb measures. Usually a particular, definable level of saturation is needed for such constructions; e.g. ω_1-saturation suffices for Loeb measures. Such arguments can be carried out in BNST* with no difficulties.

Practitioners have occasionally expressed a desire to be able to *iterate* nonstandard methods, that is, to consider internal, or even external, objects as "new standard objects." Examples of such techniques seem rare so far, but Gordon [20] and Péraire [39] use relatively standard sets to give some applications of "second-order" nonstandard methods to internal objects. Molchanov [36] obtained results in topology by applying "second-order" nonstandard techniques to external objects ("layering"). We have already discussed relativization in BNST; we conclude by proving a result for BNST* that sets the stage for applications of "second-order" nonstandard methods to external objects.

Proposition 22 (BNST*)**.** *Let κ be an $\in$-definable cardinal. There exists $\mathbb{I}_\kappa$ such that $(\mathbb{U}, \in, \mathbb{S}, \mathbb{I}_\kappa)$ satisfies* BNST$^-$+κ^+*-saturation and $|{}^*A \cap \mathbb{I}_\kappa| \leqslant 2^{2^\kappa}$ for all $A \in \mathbb{V}$ with $|A| \leqslant \kappa$.*

Proof. [10] In BNST*, $\mathscr{A} := \langle {}^*\kappa, \in, {}^*A\rangle_{A\subseteq\kappa}$ is a κ^+-saturated structure in a language $\mathscr{L}$ with $|\mathscr{L}| = 2^\kappa$. Let $\mathscr{X} := \langle X, \in, {}^*A \cap X\rangle_{A\subseteq\kappa}$ be an elementary submodel of $\mathscr{A}$ which is

[10] Kanovei and Reeken [29] prove a much stronger result in this direction.

κ^+-saturated and $|X| \leqslant 2^{2^\kappa}$ (Chang and Keisler [6, Lemma 5.1.4]). We let $\mathbb{I}_\kappa := \mathbb{S}[X]$; Proposition 13 shows that BNST$^-$ is satisfied in $(\mathbb{U}, \in, \mathbb{S}, \mathbb{I}_\kappa)$.

Every $\xi \in {}^*\kappa \cap \mathbb{I}_\kappa$ is of the form $\xi = {}^*f(x_1, \ldots, x_n)$ where $\bar{x} \subseteq X$ and $f \in \mathbb{V}$ is a function. As $X \subseteq {}^*\kappa$ and $\xi \in {}^*\kappa$, we can assume $f : \kappa^n \to \kappa$. Hence $|{}^*\kappa \cap \mathbb{I}_\kappa| \leqslant 2^{2^\kappa}$.

It remains to show that κ^+-saturation holds. Let $\{A_i \mid i \in \kappa\} \subseteq \mathbb{I}_\kappa$ have FIP, in the sense that for each finite $F \subseteq \kappa$ there is $\xi_F \in \mathbb{I}_\kappa \cap \bigcap_{i \in F} A_i$. Again, $\xi_F = {}^*f_F(\bar{x})$ for some $\bar{x} \subseteq X$ and some function $f_F : \kappa^n \to \mathbb{V}$. Let $B := \bigcup \{\operatorname{ran} f_F \mid F \subseteq \kappa,\ F \text{ finite}\} \in \mathbb{V}$. Then $|B| \leqslant \kappa$ and $\{{}^*B \cap A_i \mid i \in \kappa\}$ has FIP. Hence we can assume that $A_i \subseteq {}^*\kappa$ for all $i < \kappa$. Again we pick $G_i \in \mathbb{V}$, $G_i : \kappa^{n_i} \to \mathscr{P}(\kappa)$ and $\bar{\xi}_i \subseteq X$ so that $A_i = {}^*G_i(\bar{\xi}_i)$. Letting $H_i := \{(\bar{x}, y) \mid \bar{x} \in \kappa^{n_i}, y \in G_i(\bar{x})\} \in \mathbb{V}$, we have $H_i \subseteq \kappa$ ($\cong \kappa^{n_i} \times \kappa$) and $A_i := \{y \in {}^*\kappa \mid (\bar{\xi}_i, y) \in {}^*H_i\}$ is definable from ${}^*H_i$ and $\bar{\xi}_i$ in the structure $\mathscr{A}$, hence also in $\mathscr{X}$. As $\mathscr{X}$ is κ^+-saturated and $\{A_i \mid i \in \kappa\}$ has FIP, it follows that there is $y \in X \subseteq \mathbb{I}_\kappa$ such that $(\bar{\xi}_i, y) \in {}^*H_i$ holds for all $i < \kappa$, i.e., $y \in \bigcap_{i<\kappa} A_i$. $\square$

As a hypothetical example, let us consider applying "second-order" nonstandard techniques to a Loeb measure. So let H be a hyperfinite set, Σ the Loeb σ-algebra, and m the Loeb measure, constructed on H so that $m(B) - |B \cap H|^{\mathbb{I}} / |H|^{\mathbb{I}}$ is infinitesimal for all *internal* $B \subseteq H$. BNST* (in its present form) does not allow "layering", so we first need to "copy" the Loeb measure space into the standard universe. Axiom of Choice gives a set $\tilde{H} \in \mathbb{V}$ in one–one correspondence with H, and hence a measure space $(\tilde{H}, \tilde{\Sigma}, \tilde{m})$ isomorphic to (H, Σ, m), and an application of $*$ produces the sought-for "standard copy" of the Loeb measure space, $({}^*\tilde{H}, {}^*\tilde{\Sigma}, {}^*\tilde{m})$. The problem is that the cardinality of $\tilde{H}$ is greater than all $\in$-definable cardinals, while many nonstandard techniques customarily applied to measures [for example, the (pre-Loeb) construction of a hyperfinite set K such that ${}^*\tilde{H} \cap \mathbb{S} \subseteq K$ and ${}^*\tilde{m}({}^*B) - |{}^*B \cap K|^{\mathbb{I}} / |K|^{\mathbb{I}}$ is infinitesimal for all $B \in \tilde{\Sigma}$] require $|\tilde{\Sigma}|^+$-saturation.

The solution is to work with $\mathbb{I}_\omega$ rather than $\mathbb{I}$. As $\mathbb{I}_\omega$ satisfies ω_1-saturation, one can construct the Loeb σ-algebra Σ and the Loeb measure m on $H \cap \mathbb{I}_\omega$, and have $|H \cap \mathbb{I}_\omega| \leqslant |{}^*\omega \cap \mathbb{I}_\omega| \leqslant 2^{2^\omega}$. Now the measure algebra $(\tilde{H}, \tilde{\Sigma}, \tilde{m})$ isomorphic to $(H \cap \mathbb{I}_\omega, \Sigma, m)$ has $|\tilde{\Sigma}| \leqslant 2^\lambda$ where $\lambda := 2^{2^\omega}$, and 2^λ is $\in$-definable in $\mathbb{V}$. Hence $|\tilde{\Sigma}|^+$-saturation (and more) becomes available.

I think we are justified to conclude that BNST$^+$:= BNST + the additional axioms of type A and B and, possibly, others, while grounded on realist principles and developed largely without regard for the needs of practitioners, is capable of supporting arguments typically used by nonstandard analysts. In addition, the availability of full ZFC in the external universe makes BNST$^+$ an interesting set theory in the classical tradition.

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