Annals of Pure and Applied Logic 109 (2001) 1–8
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Petr Vopěnka (*16.5.1935)

P. Vopěnka had been engaged in work in at least five different areas during his stunningly rich and productive life: (a) topology, (b) classical axiomatic set theory, (c) non-classical alternatives to classical set theory, (d) philosophy and history of mathematics, and (e) work for the Czech society. In every one of his pursuits he left a significant mark. His mathematical results are well known and appreciated around the

0168-0072/01/$ - see front matter
PII: S0168-0072(01)00037-9

world as ground-breaking contributions to human knowledge. Rather than following the road well-traveled, P. Vopěnka sought and often found novel, original approaches both in mathematics and in interpretations of its history. His mathematical endeavors can be followed chronologically, of course, partially overlapping with his other activities.

P. Vopěnka was born in Prague on 16 May 1935. Both of his parents taught mathematics at high school. After completing his secondary education at the high school (gymnázium) in Ledeč nad Sázavou (1946–1953), he studied at the Faculty of Mathematics and Physics of Charles University during the period 1953–1958. He has been a member of the faculty ever since.

P. Vopěnka embarked on his studies in topology already during his student years. He was the last student to complete his masters thesis under the supervision of one of the most influential Czech mathematicians: Eduard Čech. For this reason it is hardly surprising that the first two of his papers deal with topology, in particular, dimension theory. The best known topological result of P. Vopěnka is his construction of compact Hausdorff spaces $X_{m,n}$, $Y_{m,n}$ for $0<m \leqslant n \leqslant \infty$ such that $\dim X_{m,n}=m$, $\operatorname{ind} X_{m,n}=n$, $\dim Y_{m,n}=m$ and $\operatorname{Ind} Y_{m,n}=n$, providing the ultimate answer as to the behavior of dimension functions on compact spaces [1]. Although it was many years until his next published paper on topology, P. Vopěnka's topological upbringing clearly influenced his approach to the method of forcing.

After he completed his studies, P. Vopěnka gradually directed his attention toward set theory. He regularly attended the seminars of L. S. Rieger, a true pioneer of Czech (and Czechoslovak) mathematical logic. Rieger (1916–1962) published works on the theory of groups and lattices, Boolean and Heyting algebras, intuitionistic and classical logic, and set theory. In his paper [R] from 1956, noting that Czechoslovak mathematical logic lagged behind in both absolute and relative terms, Rieger formulates the subject of logic as the study of the properties of the notion of consequence, and he discusses the role of set theory and the infinite in mathematical logic. Besides publishing, Rieger also started the aforementioned seminar and inspired a number of colleagues. His personal connections also influenced later close cooperation of Czechoslovak and Polish logicians. During Rieger's terminal illness in the early sixties Vopěnka filled the void and became a leader of Czech mathematical logic. After Rieger's death P. Vopěnka assumed command of the set theoretic seminar, though only two members of the original seminar remained. The new seminar consisted of an entirely new group of young researchers. P. Vopěnka's ability to attract young and talented mathematicians, and to share his fascination with set theory, made possible the emergence of a group which, by Czech standards, extraordinarily and strongly participated in the progress of mathematics. Among the members of this group were B. Balcar, L. Bukovský, P. Hájek, K. Hrbáček, T. Jech, K. Příkrý, A. Sochor and P. Štěpánek.

The beginning of this period of P. Vopěnka's mathematical life is marked by his work on non-standard interpretations of Gödel–Bernays set theory (GB) based primarily on the method of ultraproduct ([5,6,7], 1962!). Interestingly, at the same time A. Robinson's investigations of non-standard methods gave birth to non-standard analysis. P. Vopěnka's preference for GB was influenced by Gödel's work [G], while finiteness of the axiomatization, accomplished by introducing the concept of a class, also played a role. The difference between sets and classes is treated as a mere technicality in P. Vopěnka's early work, whereas the relationship between sets and classes became a central theme of his considerations later, in the early seventies.

In his habilitation [15], defended in 1964, P. Vopěnka investigated the satisfiability relation inside set theory and showed the relative consistency of the assumption that there are, in GB, integers for which there is no reasonable definition of the satisfiability relation (in other words, integers whose non-standardness can be exhibited internally). This work has never been published.

The most important impulse for the blooming of the Prague school of set theory came, however, with Cohen's method of forcing [C]—a method which P. J. Cohen developed in order to prove (using tools of model theory) the relative independence of the Continuum Hypothesis, and of the Axiom of Choice, from the Zermelo–Fraenkel set theory (ZF) (i.e. if ZF is consistent it remains consistent after adding this axiom, or its negation). Based on Cohen's method, P. Vopěnka constructed a relative (syntactical) interpretation of GB with added negation of the Continuum Hypothesis as well as an interpretation of GB with the negation of the Axiom of Choice. This was a substantial contribution as, while GB and ZF prove the same theorems about sets and Cohen's method provides an interpretation of ZF with added negation of the Continuum Hypothesis (or the Axiom of Choice) in ZF, it is known that interpretability of $\mathrm{ZF} \cup \{\phi\}$ in ZF (where ϕ is an additional axiom) does not in general, imply the interpretability of $\mathrm{GB} \cup \{\phi\}$ in GB (see [H,HH]).

This construction was, of course, only the beginning of the exploration of Cohen's method by the Prague set theory seminar. Once again, the crucial idea came from P. Vopěnka.

The topological approach chosen by P. Vopěnka in his work on ∇-models[1] [21, 30] simplified, and clarified, Cohen's method. P. Vopěnka employed and developed the theory of sheafs over topological spaces and their ultrafilter limits in order to construct non-standard syntactical models of GB. In [30] he showed that the kernel is always an extremally disconnected space; hence, the important parameters are: a Boolean algebra, and an ultrafilter on it. In [33] he developed the theory of Boolean valued models and generic extensions, completely independent of a virtually identical work of Solovay and Scott. The fact that P. Vopěnka managed to keep up with the leading researchers in set theory, together with the myriad possible applications of Cohen's method in proving the independence of various statements from standard axioms of set theory, made the Prague school of set theory famous. Among the results of that time are the following: The consistency of the existence of a Σ^1_2-set of cardinality $\aleph_1$ with the negation of the Continuum Hypothesis (Vopěnka–Bukovský), results on powers of alephs (Bukovský), results about the Continuum Hypothesis for the first measurable

[1] Why ∇-model: K. Gödel proved the consistency of $V = L$ by constructing a Δ-model, i.e. by shrinking the universe, the letter ∇ indicates that the direction is opposite, i.e. the universe is extended.

cardinal (Příkrý), the consistency of the so-called Church alternatives for the continuum (Hájek), the independence of the Suslin Hypothesis (Jech), results about partitions of metric spaces into nowhere dense sets (Vopěnka–Štěpánek [36]), results on systems of almost disjoint sets in connection with the forcing $([\kappa]^{\kappa}, \subseteq)$ (Vopěnka–Balcar [44]), and a general method of construction of interpretations of set theory with the Axiom of Foundation in case that an analogous interpretation, without the Axiom of Foundation, has been constructed using a permutation model (Jech–Sochor). The theorem of P. Vopěnka and K. Hrbáček [29] extends Scott's result about incompatibility of the Axiom of Constructibility ($V = L$) with the existence of a measurable cardinal—the existence of a strongly compact cardinal is incompatible with $V = L[a]$, i.e. with the assumption that the universe is a result of adjoining a set to the universe of constructible sets. A complete bibliography of P. Vopěnka's set theory seminar can be found in [H1] and [H2].

The history of *Vopěnka's principle*, i.e. the assertion that: "If $\mathscr{C}$ is a proper class of models of a given language then there are distinct $A, B \in \mathscr{C}$ such that A is elementarily embeddable into B" (see [J] or [K]) is different. Attempts to strengthen the important result that there is a rigid relation on every infinite set to a construction of a proper class of mutually rigid relations kept failing. Originally as a joke, P. Vopěnka postulated as an axiom that such a strengthening does not exist, and from this axiom he inferred some consequences which he himself did not consider very plausible. He hoped that his principle would be disproved. However, the future development took the joke seriously: T. Jech and A. Kanamori popularized the principle abroad, and the principle eventually found its place in the hierarchy of large cardinal assumptions, in between supercompact and huge cardinals (see [J]). The principle also has many applications in category theory [AR].

A. Tarski, one of the fathers of modern mathematical logic, commented on the results of P. Vopěnka's seminar in a letter: "I do not know if there is at this point another place in the world, having as numerous and cooperative a group of young and talented researchers in foundations of mathematics".

Over time, P. Vopěnka's authority among his colleagues and students progressively rose with, naturally, an attendant influence on faculty and administrative matters. In 1966 P. Vopěnka was appointed as an associate dean and as such used his new position to help found a Department of Mathematical Logic and also to create new curricula. Because of these changes, students could major in areas such as algebra, geometry, topology and most notably in theoretical cybernetics under the supervision of members of the newly founded department. Today, some of the most important figures in Czech computer science are former students of the department and theoretical cybernetics majors (e.g., M. Chytil, P. Kalášek, R. Kryl, A. Kučera, P. Pudlák, O. Štěpánková, and F. Franěk and P. Vojtáš abroad).

At the end of the 1960's, the political situation in Czechoslovakia substantially interfered with P. Vopěnka's life for the first time. Feeling unacceptably constrained by the Czechoslovak communist regime, a number of participants in P. Vopěnka's seminar decided to leave the country. After the invasion of Czechoslovakia in 1968, the exodus accelerated. Though other members of the seminar remained in the republic even in the era of the so-called "normalization", after the original enthusiastic group had been decimated, many decided to seek their own independent mathematical futures. In 1971, the Department of Mathematical Logic was abolished and P. Vopěnka lost his influence on the theoretical cybernetics major.

The turning point between the second and third periods of P. Vopěnka's mathematical work was the book "Theory of Semisets", which he coauthored with P. Hájek [a]. Besides being undoubtedly a turning point, this book can be understood both as a culmination of the period of interpretations of classical set theory and as the first stage of P. Vopěnka's non-classical approach to set theory. An obvious relation to Boolean valued models exemplifies the first, while the introduction of the notion of a *semiset*, i.e. a subclass of a set which itself does not have to be a set, indicates the second. At this juncture, the concept of a semiset became central to P. Vopěnka's treatment of set theory in a new way.

During the first half of the 1970's P. Vopěnka's non-classical approach to set theory crystallized in the form of his Alternative Set Theory (AST). This theory newly formulated the basic principles of set theory, regardless of the classical set theory. Underlying and driving this new approach was an attempt to describe the human way of understanding. Philosophically, the influence of Husserl's phenomenology is quite apparent in P. Vopěnka's ideas.

P. Vopěnka sought a compromise between accepting and rejecting actual infinity. On the one hand, he regarded accepting actual infinity as assuming a God-like position, which is something that clearly and completely transcends human possibilities. In this light classical set theory appears as a theory which does not adequately reflect *human* understanding of the world. On the other hand, rejecting actual infinity completely—that is confining to finite objects only—is viewed by P. Vopěnka as unacceptable, since all of infinitary mathematics would disappear. He felt that infinitary mathematics cannot be completely abandoned, if for nothing else, for its indisputable applicability. P. Vopěnka found a way out of this vicious circle by developing the idea of a class first introduced by J. von Neumann.

AST is based on the fundamental difference between the finite and the infinite. Sets are only allowed to be (formally) finite, while infinity is accepted exclusively in the form of proper classes. AST goes quite knowingly against the usual interpretations of Cantor's set theory and incorporates the theory of semiset which, as mentioned above, allows for the existence of parts of sets which are not sets themselves. P. Vopěnka's aim of devising a theory closely corresponding to human understanding led to severe restrictions on the number of cardinalities. Indeed, P. Vopěnka questioned how many infinite cardinals are really necessary. It is well known that the assumption that the collection of all subsets of a given set is a set, was an integral part of Cantor's intentions. This assumption (together with the existence of at least one infinite set) forces the existence of an unlimited scale of distinct infinite cardinalities as a consequence of Cantor's theorem. P. Vopěnka's approach represents countability by one type of class. As with classical set theory, one shows that there has to be another kind of infinity corresponding to the continuum, i.e. the cardinality of the real line. It turns out that, using this approach, no other kind of infinity is necessary; hence, the desire to limit the number of infinite cardinalities leads to the acceptance of the assumption that there are exactly two types of infinity, namely, the countable and the continuum.

The most important principle of AST is, however, the principle of prolongation. Formally, it constitutes a certain saturatedness requirement, but more important is the philosophical motivation behind the principle. Looking towards the horizon we fancy that a phenomenon approaching the horizon has to continue for a while in a similar fashion. The desire is to mathematize this human vision of the possibility of crossing the horizon, and the mathematization of this phenomenon is responsible for the existence of parts of sets which themselves are not sets in AST.

Once again, P. Vopěnka succeeded in forming a group of mathematicians devoted to developing these ideas of his. Among these mathematicians were, K. Brázdová-Trlifajová, K. Čuda, B. Kussová-Vojtášková, J. Mlček, J. Sgall, A. Sochor, A. Vencovská-Paris and J. Witzany in Prague, and J. Guričan, M. Kalina and P. Zlatoš in Slovakia. This time, however, the group was joined by researchers from abroad who would frequently visit Prague, including N. Prati and C. Markini from Italy, A. Tzouvaras from Greece, and A. Rašković from Yugoslavia.

In the book [c] P. Vopěnka published a comprehensive study of his new conception of set theory. The book was followed by a series of articles by other members of the seminar (often coauthored by Vopěnka) appearing since 1979, primarily in the journal Comment. Math. Univ. Carolinae. An overview of the results of the whole seminar is presented in [ČSZ].

The work on AST has covered several areas. It includes both alternative approaches to topics treated in the classical set theory and topics internal to AST arising naturally from the development of the theory. The first group of topics includes, most notably, constructions of topological spaces using relations of *indiscernibility equivalence* (P. Vopěnka [c,45]), investigations of their metrizability (Mlček) and dimension (Sgall-Witzany), alternative approaches to measure theory (Čuda, Kalina-Zlatoš, Tzouvaras) and reformulation of non-standard methods (Vopěnka-Sochor [44]). The other group of results deals for instance with the description of a motion in a topological space, including its discretization (P. Vopěnka [c]), shifting the horizon (Vopěnka-Sochor [50]), investigations of inner models of AST (Vencovská, Tzouvaras), study of "basic" equivalences (Čuda-Vojtášková, Mlček). Metamathematical problems of AST were also considered (Sochor, Vencovská).

In 1980, the Logic Colloquium was supposed to take place in Prague. Included in the program was a series of lectures intended to present AST and its methods to a broader mathematical audience. At this point, politics interfered with Vopěnka's life for the second time. Dissident, philosopher and mathematician V. Benda was put under arrest, which triggered a reaction among mathematicians, who wrote a petition and collected signatures. Fearing of public protests, the communist government intervened and three weeks before its supposed beginning, the conference was called off. The intended massive presentation of AST thus never came to fruition.

Besides his direct involvement with mathematical research, P. Vopěnka also devoted a lot of time and energy to the history of mathematics. P. Vopěnka's history is not history in the usual sense, but rather a thorough study of the leading ideas of a particular era with particular emphasis on its significance for the development of modern mathematics. P. Vopěnka focused primarily on geometry and the beginnings of set theory, especially on the heritage of B. Bolzano. His insightful analysis of the differences between Cantor's and Bolzano's views of infinity points to the fact that the intellectual heritage of Bolzano has yet to be fully appreciated and reflected. This idea of Vopěnka still awaits further elaboration.

It was in 1983 that P. Vopěnka began reading his thoughts on the history of mathematics at the so-called philosophical seminar he organized at the Faculty of Mathematics and Physics. Soon after 1983, the seminar became a platform for other personalities to present their ideas. Often these were thinkers not allowed to teach at the university (S. Sousedík, J. Polívka, Z. Neubauer, R. Palouš, P. Rezek), and so, because of its non-conformity, P. Vopěnka's philosophical seminar constantly balanced on the edge of legality. Closely watched by secret police through student-agents, the seminar was just barely tolerated and often severely restricted by the political representation.

In addition to the history and philosophy of mathematics, P. Vopěnka devoted much of his time to mathematical education. He gave numerous lectures to mathematics teachers, in the firm belief that furthering the horizons of teachers will have a positive impact on the level of mathematical knowledge of students.

In his *Analysis*, published in 1996, P. Vopěnka tried to follow as closely as possible the original approach of the infinitesimal calculus of Leibniz and Newton. This work is, of course, influenced by non-standard analysis and also his experience with AST.

Political circumstances interfered with Vopěnka's life for the third time in 1989. Only after the fall of communism in 1990 did he become a full professor. In June 1990, P. Vopěnka joined the government as the minister of education of Czechoslovakia. He tried to help the educational system to rise from an obedient tool of the communist power to a force helping the formation of new Czech intelligence. For this he has been praised by some and condemned by others.

Since the end of his term as a member of the government, P. Vopěnka devoted his energy almost exclusively to the history of mathematics. He published another volume of *Rozpravy s Geometrií* (Debates with Geometry) [f, g, h, i] and offered illuminating observations about the influence of the Czech baroque on mathematics and, in particular, on set theory [k]. These books are, unfortunately, so far available in Czech only. In 1998, the president of the Czech republic awarded P. Vopěnka with the medal "Za zásluhy".

A. Sochor
Mathematical Institute,
Czech Academy of Sciences,
Žitná 25, 115 67 Praha 1,
Czech Republic
E-mail address: sochor@math.cas.cz

References

[AR] J. Adámek, J. Rosický, Locally Presentable and Accessible Categories, London Mathematical Society, Lecture Notes Series 189, Cambridge University Press, 1994, 316pp.

[C] P.J. Cohen, The independence of the continuum hypothesis, Proc. Nat. Acad. Sci. U.S.A. 50 (1963) 1143–1148, 52 (1964) 105–110.

[ČSZ] K. Čuda, A. Sochor, P. Zlatoš, in: J. Mlček, M. Benešová, B. Vojtášková (Eds.), Guide to alternative set theory, Proceedings of the First Symposium Mathematics in the alternative set theory, JSMF, Bratislava, 1989, pp. 44–131.

[G] K. Gödel, The consistency of the axiom of choice and the generalized continuum hypothesis, Ann. Math. Studies, Vol. 3, Princeton University Press, Princeton, NJ, 1940.

[H] P. Hájek, On interpretability in set theories, Comm. Math. Univ. Carolinae 12 (1971) 73–79.

[H1] P. Hájek, Sets, Semisets, Models, in: D. Scott, (Ed.), Axiomatic set theory, AMS, 1971, pp. 67–81.

[H2] P. Hájek, Bibliography of the Prague seminar on foundations of set theory Part II, Czechoslovak Math. J. 23 (1973) 521–523.

[HH] P. Hájek, M. Hájková, On interpretability in theories concerning arithmetics, Fund. Math. 76 (1972) 131–137.

[J] T. Jech, Set Theory, Springer, Berlin, Heidelberg, New York, 1997, 634pp.

[K] A. Kanamori, The Higher Infinite, Perspectives in Mathematical Logic, Springer, Berlin, Heidelberg, New York, 1994, 536pp.

[R] L.S. Rieger, O některý ch základních otázkách matematické logiky, Časopis pro pě stování matematiky 81 (1956) 342–351.