Dedekind-finite Structures

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The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the works of others.

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Dla mojego Taty

Abstract

In this thesis we investigate the possible structures admitted by certain Dedekind-finite sets – infinite sets that do not have an infinite countable subset. Such sets only exist in models of set theory where the Axiom of Choice is false.

In Chapter 1, we give an overview of the history of questions surrounding Dedekind-finite sets, as well as an indication of the motivations for our particular approach.

In Chapter 2, we begin with a brief list of the model-theoretic notions we will use in later chapters. Then, we introduce the notion of Dedekind-finiteness and various other, stronger, definitions of finiteness. Some of these definitions are newly introduced, so we put the new notions in the context of the old.

In Chapter 3, we begin by reviewing the Fraenkel-Mostowski method for building models of set theory with atoms in which the Axiom of Choice may fail. We then sketch a more modern approach to a forcing model construction given by Plotkin, and relate the construction to the Fraenkel-Mostowski method. The ground model of the Plotkin construction is assumed to contain a countable model $\mathscr{A}$ of a first-order theory, and the resulting forcing model is one in which choice fails, and which contains a structure $\mathfrak{A}$ satisfying the same first-order formulas as $\mathscr{A}$. Certain model theoretic assumptions on $\mathscr{A}$ are necessary for $\mathfrak{A}$ to be finite according to specific definitions of finiteness stronger than that of Dedekind-finiteness: we give our refinements to the results of Plotkin.

In Chapter 4, we analyse the model theoretic properties of certain Dedekind-finite sets that carry a first-order structure. We find that various subclasses of the class of Dedekind-finite sets admit only theories with certain model-theoretic properties. In this way, we find that many of the model theoretic assumptions made in the Plotkin construction are not only necessary, but also sufficient to determine the finiteness properties of the resulting non-well-orderable sets.

In Chapter 5, we generalise the results of Chapters 3 and 4 to models axiomatised in an infinitary language $L_{\omega_1\omega}$.

Contents

Chapter 1

Introduction and Motivation

The structural similarities between certain classes of Dedekind-finite sets and classes of well-behaved theories have been noted in the past. Hitherto, the structures of Dedekind-finite sets have been explored in a direct fashion, by testing the properties of the sets by methods of *ad hoc* model construction or by direct proofs that carefully avoid the use of the Axiom of Choice (AC). Examples of such approaches can be found in, for example, [Tru95, MT03, CT00].

For some time, many investigations in the realm of Dedekind-finite sets were felt to be analogous to model theory. This was inspired by the similarities between the notion of a strongly minimal theory and that of an amorphous set. These similarities were extensively explored in [Tru95], where it was discovered that amorphous sets and strongly minimal sets share many structural features. The studies in [Tru95] inspired the introduction of further finiteness notions that were similar to existing model theoretic notions. An example of such a notion is (MT-)rank, first defined in [MT03]. MT-rank was intended to be analogous to the notion of Morley rank, insomuch as it generalizes the notion of an amorphous set. However, as we will show, MT-rank is not the same as Morley rank. Our overall strategy for studying various classes of Dedekind finite sets and the structures they admit is to relate the classes to model theoretic notions.

In Chapter 4, we find that model theoretic methods can be applied more or less directly to determine the limitations placed on a first-order structure that can have as a domain an infinite (weakly) Dedekind-finite set. As a "bonus," by these methods we find that classical subclasses of the class of Dedekind-finite sets correspond quite neatly with certain classical model theoretic classifications. Thus, we can in some cases extend and apply results from the classification of first-order theories to the structural theory of Dedekind-finite sets.

We note that in both the cases of amorphous and MT-ranked sets, differences between the model theoretic and choiceless set theoretic notions were found. In the past, all of these differences were blamed on second order properties of the Dedekind-finite sets. In cases where a wider variety of behaviours was found than shown by analogous model theoretic structures, this reasoning was perhaps accurate. However, where the possible structures were found to be more limited than the behaviours of the apparently analogous model theoretic structures, we find that the limiting factor was not caused by second-order properties, but rather by the fact that the theories carried by such sets must be $\aleph_0$-categorical.

The main result of Chapter 4 is that weakly Dedekind-finite sets can only carry $\aleph_0$-categorical first-order theories. We then examine more restricted classes of sets, and determine their possible first-order structures.

On the other hand, in Chapter 3, we take the more traditional approach by beginning with countable structures to which we apply a forcing technique. The technique that we have chosen is an established one. The choice of technique is appropriate in that the models produced are in a sense minimal. We thus find many of the correlations of Chapter 4 to be exact.

Figure 1.1 illustrates the correspondences that will be established in Chapters 4 and 3. These results will appear shortly in [WTar].

weakly Dedekind finite	$\longleftrightarrow$	$\aleph_0$-categorical
strictly Mostowski finite	$\longleftrightarrow$	$\aleph_0$-categorical and $\neg$SOP
MT-ranked	$\longleftrightarrow$	$\aleph_0$-categorical and $\aleph_0$-stable
weakly o-amorphous	$\longleftrightarrow$	$\aleph_0$-categorical and weakly o-minimal
o-amorphous	$\longleftrightarrow$	$\aleph_0$-categorical and o-minimal
amorphous	$\longleftrightarrow$	$\aleph_0$-categorical and strongly minimal

Figure 1.1: *Correlations between different finiteness notions and notions from model theory.*

In Chapter 5, we generalise the results of Chapters 3 and 4. The results of this chapter allow us to examine the structures on certain Dedekind-finite, not weakly Dedekind-finite structures. There are a few questions left open in this area, and we have suggested possible strategies for attempting solutions.

We emphasise that for the entirety of this thesis, we limit ourselves to structures having theories expressible in a finite or countably infinite first-order language L or with a language $L_{\omega_1\omega}$ with a countable vocabulary. The possibility of a Dedekind-finite language falls outside the scope of this thesis for two reasons. First of all, and most importantly, the techniques described are not applicable in such a case. The second, more minor, reason is habit: we were originally motivated by questions as to the possible nature of (weakly) Dedekind-finite groups; then rings, orders, lattices, etc., which in any case are naturally axiomatised in finite languages.

As mentioned, this thesis started out as a study of locally finite groups in models of set theory without choice. Over the years, it morphed into something else, as theses are wont to do. A shadow of the thesis's genesis remains in the prevalence of group theoretical examples contained herein. In particular, this shadow is most vivid in the Appendix, where we present early results using direct methods not involving metamathematical considerations.

Before the Axiom of Choice was shown to be consistent with and independent of the rest of set theory, questions surrounding models not admitting choice had strong connections with the philosophy of mathematics. Now that the problem of Consistency has been resolved, and a consensus has largely been reached about the acceptance of the Axiom, many (non-logician) mathematicians may ask why the study of Dedekind-finite sets is worthy of the effort. I have heard doing mathematics without the Axiom of Choice referred to as practice of poor taste by some. Perhaps it is. I do gently point out that the Liberace Museum is still a major draw in Vegas, and you never know when sequins and candelabras will come back into fashion.

More seriously: In the classification of theories, one often looks at the behaviours of models of a theory across models of various cardinalities, with the assumption of the Generalised Continuum Hypothesis, and without. Perhaps an examination of non-well-orderable models in such a vein may be worthwhile.

There is still probably a lot more to be said on this topic. But, doing mathematics in general, and writing a thesis in particular, is very much an exercise in rainbow-chasing.

A final word on notation: For reference, we give the numbers of models as in [HR98] whenever possible. We have also attempted to employ a suggestive system of notation: usually, well-ordered structures will be denoted with a script font, non-well-orderable (i.e. Dedekind-finite) sets and structures be denoted by upper-case Gothic letters, and upper-case calligraphic letters will be used when the well-orderablity is not specified. Instances where a Fraenkel-Mostowski construction is being carried out are an exception to these tendencies– there we will use the traditional notation. Otherwise, the notation we use is consistent with our understanding of "standard".

Chapter 2

Notation and Basic Definitions

2.1 Model Theoretic Notions

We will make reference to the following classical model-theoretic notions.

An *n-type* of a theory T is a set $\Phi(\bar{x})$ of formulas, with $\bar{x} = (x_0, \dots, x_{n-1})$, such that for some model $\mathcal{M}$ of T and some n-tuple $\bar{a}$ of elements of the domain M of $\mathcal{M}$, $\mathcal{M} \models \phi(\bar{a})$ for all ϕ in Φ. We call Φ a *type* if it is an n-type for some $n < \omega$.

A *complete (n-)type* is a maximal (n-)type.

Let $\Phi(\bar{x})$ be a set of formulas of a language L. We say that the type Φ is *realised in* an L-structure $\mathcal{M}$ if there is a tuple $\bar{a}$ of elements of M such that $\mathcal{M} \models \bigwedge \Phi(\bar{a})$; we say that $\mathcal{M}$ *omits* Φ if Φ is not realised in $\mathcal{M}$. By $\mathrm{tp}_M(\bar{a})$ we denote the complete type realised by the tuple $\bar{a}$ in $\mathcal{M}$.

Let T be the complete theory of a structure $\mathcal{M}$. We say that Φ is a *principal type* over T if there is a formula $\psi \in \Phi$ such that $T \cup \{\exists x \psi\}$ has a model, and for every formula $\phi(\bar{x})$ in Φ, $T \vdash \forall \bar{x}(\psi \to \phi)$.

We say that a structure $\mathcal{M}$ is *atomic* if for every tuple $\bar{a}$ of elements of M, the complete type of $\bar{a}$ in $\mathcal{M}$ is principal.

Definition 2.1.1. A structure $\mathcal{M}$ is said to be *locally finite* if any substructure generated by finitely many elements is finite.

The structure $\mathcal{M}$ is said to be *uniformly locally finite* if there is a function $f : \omega \to \omega$ such that for every substructure $\mathcal{B}$ of $\mathcal{M}$, if $\mathcal{B}$ has a generator set of cardinality $\leqslant n$, then B itself has cardinality $\leqslant f(n)$.

Uniform local finiteness is a first-order property, while local finiteness is not. However, one can state the property of being locally finite in terms of omitting each type in a particular set of quantifier-free first-order types. Namely, this set of types consists of types $p(\bar{x})$, where $\bar{x}$ is an n-tuple and p contains formulas of the form $\bigwedge_{i<j<n} t_i(\bar{x}) \neq t_j(\bar{x})$ for t_i, t_j terms in the language L appropriate for the structure, and arbitrarily large n.

A class of theories that will be of primary importance in this thesis is that of $\aleph_0$-categorical theories:

Definition 2.1.2. A theory T is said to be $\aleph_0$-categorical if it has, up to isomorphism, exactly one model of cardinality $\aleph_0$.

We note that a structure whose theory is $\aleph_0$-categorical is uniformly locally finite.

One major theorem that we will refer to later concerning $\aleph_0$-categorical theories is Theorem 2.1.3. We include the statement of the theorem here for reference:

Theorem 2.1.3 (Ryll-Nardzewski, Engeler, Svenonius)**.** *Let L be a countable first-order language and T a complete theory in L which has infinite models. Then the following are equivalent.*

1. *T is $\aleph_0$-categorical*

2. *If $\mathcal{M}$ is any countable model of T, then the automorphism group of $\mathcal{M}$,* $\mathrm{Aut}(M)$ *is* oligomorphic *(i.e. for every $n < \omega$,* $\mathrm{Aut}(M)$ *has only finitely many orbits in its action on n-tuples of elements of M).*

3. *T has a countable model $\mathcal{M}$ such that* $\mathrm{Aut}(M)$ *is oligomorphic.*

4. *Some countable model of T realises only finitely many complete n-types for each $n < \omega$.*

5. *For each $n < \omega$, the set of complete n-types for T, $S_n(T)$, is finite.*

6. *For each $\bar{x} = (x_0, \ldots, x_{n-1})$, there are only finitely many pairwise non-equivalent formulas $\phi(\bar{x})$ of L modulo T.*

7. *For each $n < \omega$, every type in $S_n(T)$ is principal.*

Let $S_1(A)$ denote the set of all complete types of formulas in one variable with parameters from a set A.

Definition 2.1.4. A model $\mathcal{M}$ is *κ-saturated* if for each set $A \subset M$ where $|A| < \kappa$, $\mathcal{M}$ realizes every type in $S_1(A)$. A model $\mathcal{M}$ is *saturated* if it is $|\mathcal{M}|$-saturated.

A countable model of an $\aleph_0$-categorical theory is $(\aleph_0)$-saturated.

Definition 2.1.5. Let L be a first order language, and let $\mathcal{M}$ be an L-structure. Let κ be an infinite cardinal. We say that $\mathcal{M}$ is *κ-homogeneous* if for all $A \subset M$ with $|A| < \kappa$, $a \in M$, and elementary maps $f : A \longrightarrow M$, there is an elementary map $g : A \cup \{a\} \longrightarrow M$ extending f.

If κ is a finite cardinal or $\aleph_0$, then $\mathcal{M}$ is κ-homogeneous if whenever $\lambda < \kappa$ and $\bar{a}$ and $\bar{b}$ are λ-tuples from M with $\mathrm{tp}_M(\bar{a}) = \mathrm{tp}_M(\bar{b})$, and c is an element from M, then there is an element $d \in M$ such that $\mathrm{tp}_M(\bar{a}, c) = \mathrm{tp}_M(\bar{b}, d)$.

We say a structure $\mathcal{M}$ is *homogeneous* if it is $|M|$-homogeneous.

Note that if $\mathcal{M}$ is a countable $(\aleph_0\text{-})$homogeneous structure, and $\mathrm{tp}_M(\bar{a}) = \mathrm{tp}_M(\bar{b})$, then there is an automorphism f of M such that $f(\bar{a}) = \bar{b}$.

It should be noted that $\aleph_0$-saturation implies $\aleph_0$-homogeneity.

Definition 2.1.6. Let L be a first order language, and let $\mathcal{M}$ be an L-structure. We say that $\mathcal{M}$ is *ultrahomogeneous* if whenever $\bar{a}$ and $\bar{b}$ are n-tuples from M, $n < \omega$, with the same quantifier-free types (i.e. they generate isomorphic structures), then there is an automorphism f of $\mathcal{M}$ such that $f(\bar{a}) = \bar{b}$.

An $\aleph_0$-categorical structure that has quantifier-elimination is uniformly locally finite and ultrahomogeneous.

Stability and related notions

We now give some basic notions from stability theory, as well as a notion that applies to certain unstable theories that allows the use of certain stability-theoretic techniques. These notions are fundamental to the classification of first-order theories.

Definition 2.1.7. A formula $\phi(\bar{x}, \bar{y})$ is said to have the *strict order property* (for a complete theory T) if in every model $\mathcal{M}$ of T, ϕ defines a partial ordering relation on the set of all n-tuples, which contains arbitrarily long finite chains. The theory T has the strict order property if some formula has the strict order property for T.

Definition 2.1.8. A formula $\phi(\bar{x}, \bar{y})$ is said to have the *independence property* (for a complete theory T) if in every model $\mathcal{M}$ of T there is, for each $n < \omega$, a family of tuples $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that for every subset X of n there is a tuple $\bar{a}$ in M for which $\mathcal{M} \models \phi(\bar{a}, \bar{b}_i) \leftrightarrow i \in X$. The theory T has the independence property if some formula has the independence property for T.

Definition 2.1.9. Let T be a complete theory in a first order language L. Let $\phi(\bar{x}, \bar{y})$ be a formula of L with free variables divided into two groups $\bar{x}$, $\bar{y}$. An *n-ladder* is a sequence $(\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{b}_0, \ldots, \bar{b}_{n-1})$ of tuples in some model $\mathcal{M}$ of T, such that for all $i, j < n$, $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_j) \leftrightarrow i \leqslant j$. We say ϕ is a *stable formula* if there is some $n < \omega$ such that no n-ladder exists for ϕ. We say T is a *stable theory* if all its formulas are stable.

It can be shown that a theory T is stable if it has neither the strict order property nor the independence property.

Definition 2.1.10. Let T be a complete theory (of any cardinality) with an infinite model. T is said to be $\aleph_0$-*stable* if for all $\mathcal{M} \models T$ and $A \subset M$ of cardinality at most $\aleph_0$, $|S_1(A)| \leqslant \aleph_0$.

Definition 2.1.11. Let T be a complete theory in a language L, and let $\mathfrak{C}$ denote a model of this theory which is saturated and of sufficiently large cardinality. The relation $\mathrm{MR}(\varphi) = \alpha$, for φ a formula in n variables and α an ordinal, -1, or ∞ is defined by the following recursion.

(i) $\mathrm{MR}(\varphi) = -1$ if φ is inconsistent;

(ii) $\mathrm{MR}(\varphi) \geqslant 0$ iff $\varphi(\mathfrak{C})$, i.e. the set of elements of $\mathfrak{C}$ that satisfies φ, is not empty;

(iii) $\mathrm{MR}(\varphi) \geqslant \alpha + 1$ iff there are L-formulas $\varphi_i(\bar{x})$ (where $i < \omega$ and $\bar{x}$ is an n-tuple of variables) with parameters from $\mathfrak{C}$, such that the sets $\varphi(\mathfrak{C}^n) \cap \varphi_i(\mathfrak{C}^n)$, with $i < \omega$, are pairwise disjoint and $\mathrm{MR}(\varphi \wedge \varphi_i) \geqslant \alpha$ for each $i < \omega$;

(iv) $\mathrm{MR}(\varphi) \geqslant \delta$, with $\delta \in \mathit{Lim}$ iff for all $\alpha < \delta$, $\mathrm{MR}(\varphi) \geqslant \alpha$.

This rank is called *Morley rank*.

If T is a theory, we take $\mathrm{MR}(T)$ to be the Morley rank of the formula $x = x$.

Definition 2.1.12. Let T, L, $\mathfrak{C}$, and φ all be as in Definition 2.1.11, and let $\mathrm{MR}(\varphi) = \alpha$. Then, the greatest integer d such that $\varphi(\mathfrak{C}^n)$ can be written as the union of disjoint sets $\varphi_0(\mathfrak{C}), \ldots \varphi_{d-1}(\mathfrak{C})$, all of Morley rank α is called the *Morley degree* of φ.
We will often write $\mathrm{MR}(\varphi) = (\alpha, d)$ to indicate that φ has Morley rank α and Morley degree d.

We note that in the case of $\aleph_0$-categorical theories, it suffices to calculate Morley rank in the (unique) countable model of the theory.

If a structure is axiomatised in a countable language and its theory admits Morley rank, then the theory is $\aleph_0$-stable. Note that the assumption about the countability of the language is essential.

Definition 2.1.13. A complete theory T is said to be *strongly minimal* if every parametrically definable subset of each of its models is either finite or cofinite.

Note that a theory with Morley rank 1 and Morley degree 1 will be strongly minimal. Thus, Morley rank can be viewed as a dimension-like generalisation of the notion of strong minimality.

Finally, the following notion applies to certain theories that are unstable. It has been found that certain techniques from stability theory can be applied to those theories that satisfy this condition.

Definition 2.1.14. A subset of a linearly ordered set $(X, \leqslant)$ is called an *interval* if it has the form $(a, b) = \{x : a < x < b\}$, $[a, b]$, $[a, b)$, or $(a, b]$, where $a, b \in X \cup \{\pm\infty\}$.
A subset is $Y \subseteq X$ is *convex* if $a < x < b \wedge a, b \in Y \Rightarrow x \in Y$.
Let T be a theory in a language L containing a symbol $\leqslant$ which, for all models $\mathcal{M}$ of T, linearly orders M. Then, T is said to be *o-minimal* if for all models $\mathcal{M}$, $\leqslant$ orders M in such a way that every subset of M which is first-order definable with parameters is a union of finitely many intervals and points. The theory T is said to be *weakly o-minimal* if every definable subset of M is a finite union of convex sets.
Clearly, o-minimal structures are weakly o-minimal, however, the converse does not hold.

2.2 Finiteness Notions

2.2.1 Definitions and historical notes

Definition 2.2.1. A set X is *Dedekind-finite* (or *IV-finite*) if X has no proper subset Y that can be mapped bijectively onto X. Equivalently, X is Dedekind-finite if there does not exist a bijection from ω onto a subset of X. If X is Dedekind-finite, then we say (taking notation as in [Tru74]) that $|X| \in \Delta$.

Definition 2.2.1 was the first formal definition of finiteness or infinity, and was given by Richard Dedekind in the 1870's[1].

Definition 2.2.2. A set X is *finite* if there exists a one-to-one and onto mapping between X and some $n \in \omega$. A set X is *infinite* if it fails the definition of finite.

This definition is not dependent on the axiom of choice, and is absolute for models of set theory. There are other definitions of finiteness that are equivalent to finite only if one assumes AC. In this thesis, we will focus on the following notions, using terminology and notation as found in [Tru74], [Deg94], [Gol97], [CT00], [MT03], and [Lev58]:

Definition 2.2.3. A set X is *weakly Dedekind-finite* (or *III-finite*) if there does not exist a surjection from X onto ω. Equivalently (see [Tru74]), X is weakly Dedekind-finite if $P(X)$ is Dedekind-finite. We say that $|X| \in \Delta_4$.

Perhaps a more appropriate name for the finiteness notion defined in 2.2.3 would be "strong Dedekind-finiteness". However, we use the terminology for this notion as established in [Deg94]. The notion first appeared in [Tar24] under the name *III-finite*, and was further studied in [Die74] under the name "almost finite".

We write $X \leqslant^* Y$ if Y can be mapped surjectively onto X.

The class of weakly Dedekind-finite sets is but a small subclass of the Dedekind-finite sets. In fact, the author of [Mon75] showed that for an arbitrary ordinal α, one can construct a model containing a Dedekind-finite set X that does not map onto α. With this in mind, we define the following generalisation of weakly Dedekind-finite sets:

[1]See §5 of [Ded88], where Dedekind gives his definition of finiteness. In a footnote, he mentions that he referred to this notion in September 1882 in a letter to Cantor, and also in earlier correspondence with Schwartz and Weber.

Definition 2.2.4. Let α be an ordinal. By $\Delta_{*\alpha}$ we denote the class of Dedekind-finite cardinals that cannot be mapped onto α, that is

$$\Delta_{*\alpha} = \{|X| : (\alpha \not\leq^* X) \wedge (X \in \Delta)\}.$$

Note that $\Delta_{*\omega} = \Delta_4$.

We will abuse terminology and call a set X a $\Delta_{*\alpha}$-set (or structure) if its cardinality $|X| \in \Delta_{*\alpha}$.

Definition 2.2.5. A set X is *Mostowski finite* if every linearly ordered subset of X is finite. If X is Mostowski finite, then $|X| \in \Delta_3$.

The class of Mostowski finite sets was first discussed in [Tru74] (under the name Δ_3), but arises naturally in models where the ordering principle fails. The name "Mostowski finite" is newly introduced for the purposes of this thesis, and is in honour of A. Mostowski's studies of the Ordering Principle in models of set theory without AC.

The following notion of strictly Mostowski finite is newly introduced specifically for the purposes of the work presented in this thesis, and as such, is inspired by a model-theoretic notion.

Definition 2.2.6. A set X is *strictly Mostowski finite* if any partial ordering on X^n, $n \in \omega$ only has chains of finite bounded length. We say that $|X| \in \Delta_3^\dagger$.

Definition 2.2.7. A set X is *II-finite* if every family of non-empty subsets of X which can be linearly ordered by inclusion has a maximal element.In other terms, X is II-finite if any linearly ordered partition of X is finite. If X is II-finite, then we say that $|X| \in \Delta_2$.

II-finiteness was first mentioned in [Tar24].

Definition 2.2.8. A set X is *amorphous* (or Ia-finite) if it cannot be expressed as the union of two disjoint infinite sets. Then, we say $|X| \in \Delta_1$.

Note that other authors use the term "amorphous" for *infinite* sets that cannot be expressed as the union of two disjoint infinite sets. This notion was introduced as *Ia-finiteness* in [Lev58].

We are interested in non-choice structures that admit the following rank, and corresponding degree, functions:

Definition 2.2.9. The relation $\mathrm{MT}(X) = \alpha$, for α an ordinal or -1, is defined by the following recursion.

(i) $\mathrm{MT}(X) = -1$ if $X = \emptyset$.

(ii)$\mathrm{MT}(X) = \alpha$ if $\mathrm{MT}(X) \nless \alpha$, and there is some $n \in \omega$ such that if $\{X_i : 0 \leqslant i \leqslant n\}$ are any $n+1$ pairwise disjoint subsets of X, then for some i, $\mathrm{MT}(X_i) < \alpha$.

We refer to this rank as *MT-rank.*

Definition 2.2.10. A set X of rank α has *MT-degree* k if X has k pairwise disjoint subsets of MT-rank α, but for any $k+1$ pairwise disjoint subsets, at least one of them has smaller rank.

MT-rank and degree were first introduced in [MT03], and were referred to as simply "rank" and "degree". It follows from Definition 2.2.9, that if X has MT-rank α, and k is the least n which will serve in the definition, then X has MT-degree k. We will write $\mathrm{MT}(X) = (\alpha, k)$ to indicate that X has MT-rank α and MT-degree k. The definition of MT-rank is meant to generalise the notion of an amorphous set in much the same way that Morley rank generalises strong minimality. In addition, as discussed below, the class of amorphous sets, Δ_1, is not closed under unions and products, as one would expect of a class of sets satisfying a finiteness notion. On the other hand, the MT-ranked sets are closed under unions and products, and thus

MT-rank can be considered a somewhat better-behaved notion of finiteness than amorphousness.

The notions in Definitions 2.2.11 and 2.2.12 were first given in [CT00], and are inspired by model theoretic notions.

Definition 2.2.11. A linearly ordered set $(X, \leqslant)$ is said to be *o-amorphous* if it is infinite and its only subsets (definable or not) are finite unions of intervals (see Definition 2.1.14) and points. We say that a set X is o-amorphous if there exists an ordering $\leqslant$ such that $(X, \leqslant)$ is o-amorphous.

Definition 2.2.12. If in Definition 2.2.11 we replace the word "intervals" with "convex sets", X is a *weakly o-amorphous* set.

As noted, many of these definitions of finiteness are classic notions. For example, Definitions 2.2.7, and 2.2.3 were defined, or at least referred to in [Tar24]. Others were inspired by model-theoretic notions.

2.2.2 Properties of finiteness notions

We now mention some basic properties of the various classes of sets satisfying the finiteness notions defined above. We concentrate on the relative strengths of the finiteness notions, and on their closure properties.

Relative strengths of finiteness notions

The authors of [Lev58, Tar24, Tru74] demonstrated the relative strengths of finiteness definitions as shown in figure 2.1.

We place the new finiteness notions, strict Mostowski finiteness and the class $\Delta_{*\omega_1}$, into this context, and improve certain older results.

In [MT03], the authors showed that sets admitting MT-rank have cardinality in Δ_4, as well as a few other basic properties. We note that from Lemma 4.1 of [MT03]

$$\begin{tikzcd}
& \text{finite} \arrow[d] & \\
& \text{amorphous} \arrow[d] & \\
& \text{II-finite} \arrow[dl] \arrow[dr] & \\
\text{Mostowski finite} \arrow[dr] & & \text{weakly Dedekind-finite} \arrow[dl] \\
& \text{Dedekind-finite} &
\end{tikzcd}$$

Figure 2.1: *Dependencies among definitions of finiteness, as established in [Lev58, Tar24, Tru74].*

it can be easily inferred that the class of MT-ranked sets is contained within the class of strictly Mostowski finite sets, and thus have cardinality in $\Delta_3^\dagger$:

Remark 2.2.13. *Let X have MT-rank α and MT-degree n. Then X is strictly Mostowski finite.*

Proof. Assume that X has MT-rank α and MT-degree n, and let $<$ be a partial order on X. Then, by Corollary 4.2 of [MT03], X can be decomposed into finitely many antichains. Then clearly, the chains on X must be finite, and of bounded size. Because X^n also admits MT-rank, again any partial ordering on X^n must only have finite chains of bounded length. $\square_{2.2.13}$

We place strict Mostowski finiteness in context:

Remark 2.2.14. $\Delta_3^\dagger \subset \Delta_2$

Proof. Let X be a strictly Mostowski finite set, and assume that $|X| \notin \Delta_2$. Then, there exists an infinite linearly ordered partition $(\pi, <) = (X_i : i \in I)$ of X. We can define a partial order $\prec$ on the elements of X utilizing the ordering of the partition $<$: Let $x_i, x_j \in X$, where $x_i \in X_i$ and $x_j \in X_j$. Then,

$$x_i \prec x_j \Leftrightarrow X_i < X_j.$$

Because the ordering $<$ has arbitrarily long finite chains, so does the partial ordering $\prec$. This contradicts the assumption that X is strictly Mostowski finite.

Model Construction 3.4.13 yields a set that is in Δ_2 but not $\Delta_3^\dagger$. $\square_{2.2.14}$

Finally, we have the following result, which we give without proof as it is clear from the definitions and Remark 2.2.14:

Remark 2.2.15.

$$\Delta_3^\dagger \subset \Delta_2 \subseteq \Delta_3 \cap \Delta_4$$

Figure 2.2 summarises the dependencies between the finiteness notions that are discussed in the rest of this thesis.

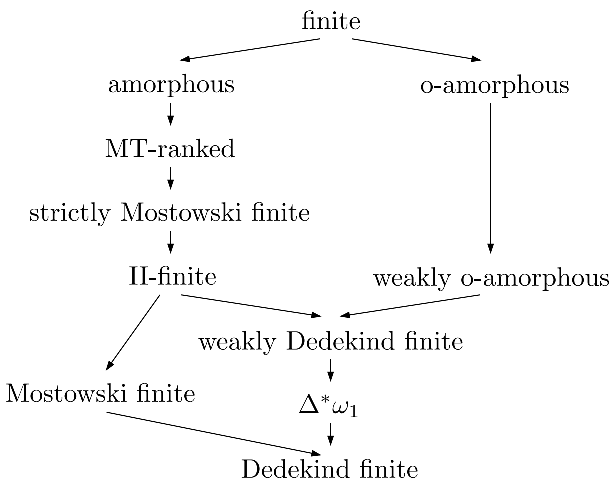

Figure 2.2: *Dependencies between different finiteness notions. For the purposes of the figure, by "o-amorphous," and "weakly o-amorphous," we mean that the set has an ordering under which it fulfils that definition. See also note 94 of [HR98] and updates on the associated website.*

Closure properties of finiteness notions

We build upon the results of [Tru74]. Let $\Delta_?$ be a class of Dedekind finite sets. We say that $\Delta_?$ is *closed under (disjoint) unions* if when $I \in \Delta_?$ and each $X_i \in \Delta_?$ (and the X_i are pairwise disjoint), then the union $\bigcup_{i\in I} X_i \in \Delta_?$. The various definitions of finiteness have the following closure properties:

Remark 2.2.16. *1. All classes of sets defined in this section are closed under subsets.*

2. Δ_2, Δ_4, $\Delta_{*\omega_1}$ (assuming ω_1 is regular), and the class of sets admitting MT-rank are closed under unions.

3. Δ_3 and Δ are closed under disjoint unions.

4. The class of sets having MT-rank α, for α an ordinal, the class of o-amorphous sets, and the class of weakly o-amorphous sets are closed under finite unions.

5. The class of sets admitting MT-rank, Δ_2, Δ_3, Δ_4, $\Delta_{*\omega_1}$, Δ, when understood to be classes of cardinals, are closed under $+$, and $\times$.

6. Basic closure properties of MT-ranked sets:

(a) The image of any set having MT-rank α under a function has MT-rank $\leqslant \alpha$;

(b) The union of finitely many sets of MT-rank $< \alpha$ has MT-rank $< \alpha$;

(c) If X has MT-rank α, and Y has MT-rank $< \alpha$, then $X \cup Y$ has MT-rank α, and the same MT-degree as X;

(d) A set X has MT-rank α if and only if it is a finite union of MT-rank α, MT-degree 1 sets.

(e) If X has finite MT-rank α, MT-degree m, and Y has finite MT-rank β, MT-degree n, then $X \times Y$ has MT-rank exactly $\alpha + \beta$ and MT-degree mn. (For the case of infinite MT-ranks, see Theorem 1.9 of [MT03].)

Proof. Results concerning all classes aside from $\Delta_3^\dagger$ can be found in [Tru74], [MT03], and [CT00]. In particular, the proofs of the properties of MT-ranked sets can be found in [MT03]. That $\Delta_3^\dagger$ is closed under subsets is evident.

We prove that o-amorphous sets are closed under finite unions. It suffices to demonstrate this for two sets. Let X and Y be o-amorphous, and choose relations $\leqslant_X$ and $\leqslant_Y$ on each of them witnessing this. The set $X \cup Y$ can be linearly ordered in the following manner: We take X first in the ordering $\leqslant_X$, followed by a point z, followed by $Y \setminus X$ in the ordering $\leqslant_Y$. The point z is added to ensure that X and Y are intervals in the resulting order. Since $Y \setminus X$ is a subset of Y, it is a finite union of intervals, thus $X \cup Y$ is the union of a finite sequence of o-amorphous intervals in order, and as such is o-amorphous. The proof for weakly o-amorphous sets is similar.

We show that $\Delta_{*\omega_1}$ is closed under $\times$: Suppose $X, Y \in \Delta_{*\omega_1}$. Because Δ is closed under $\times$, $X \times Y$ is a Dedekind-finite set. Suppose for a contradiction that there is a function f that maps $X \times Y$ onto ω_1. We examine the sets $f^{-1}(\alpha)$ for $\alpha \in \omega_1$.

Case I. Assume that for some $x \in X$,

$$X_x = \{\alpha : (\{x\} \times Y) \cap f^{-1}(\alpha) \neq \emptyset\}$$

is uncountable. In this case, Y can be mapped onto ω_1, giving a contradiction.

Case II. Assume that each X_x is countable. Let $\alpha_x = \sup X_x$. Then, we can map X into ω_1 by $x \mapsto \alpha_x$. This maps X to a cofinal subset of ω_1. By the assumption that ω_1 is regular, we have a contradiction. $\square_{2.2.16}$

Chapter 3

Permutation Model and Forcing Constructions

3.1 Introduction

As outlined earlier, our overall goal is to find correspondences, as exact as possible, between model-theoretic notions on the one hand, and classes of Dedekind-finite sets on the other.

In the present chapter, we will examine what sorts of Dedekind finite structures can be constructed from (countable) structures using the methods of Fraenkel-Mostowski permutation models and by a forcing construction.

We begin with a brief overview and explanation of notation for Fraenkel-Mostowski permutation model constructions, which give models of Set Theory with Atoms. Then we present and discuss a forcing construction, due to Plotkin ([Plo69]) that gives ZF models that are analogous to Fraenkel-Mostowski models built using finite supports. These are, in a sense, the minimal non-choice models for a given structure, and as such give an idea of correspondences which will be shown in most cases to be exact in later chapters. In particular, in Section 3.3, we will outline the Plotkin construction and show that this is the model construction that arises from the application of the Jech-Sochor embedding theorem on Fraenkel-Mostowski models built using finite supports. In Section 3.4, we discuss structures built from $\aleph_0$-categorical theories using this method, and show our refinements for determining the Dedekind-finite cardinalities of these structures.

3.2 Fraenkel-Mostowski Permutation Models

Set Theory with Atoms (or Urelemente) (ZFA or ZFU) is a modified version of set theory which admits objects other than sets, atoms (sometimes also referred to as Urelemente). Atoms are objects which do not have any elements. The language of ZFA consists of $=$ and $\in$. and of two constant symbols $\emptyset$ and U (the empty set and the set of all atoms). The axioms of ZFA are like the axioms of ZF, except for the following changes that make allowances for the existence of atoms:

0. *Empty set* $\neg\exists x[x \in \emptyset]$,

A. *Atoms* $\forall z[z \in U \leftrightarrow z \neq \emptyset \wedge \neg\exists x(x \in z)]$.

Atoms are the elements of U while *sets* are all objects which are not atoms. The following axioms are altered to take into account the existence of atoms:

A1. *Extensionality* $(\forall \text{ set } X)(\forall \text{ set } Y)(\forall x(x \in X \leftrightarrow x \in Y) \leftrightarrow X = Y)$.

A8. *Regularity* $(\forall \text{ nonempty } S)(\exists x \in S)(x \cap S = \emptyset)$.

Note that "X is nonempty" is not the same as "$X \neq \emptyset$" – it is the same only if X is a set. The set theoretic universe in this case is built analogously[1] to that of ZF set

[1]We should point out here that in the definition of an ordinal, we must insert a clause that an ordinal does not have any atoms among its elements.

theory, where here we define $P^\alpha(S)$ for any set S as follows:

$$\begin{aligned} P^0(S) &= S \\ P^{\alpha+1}(S) &= P^\alpha \cup P(P^\alpha(S)) \\ P^\alpha &= \bigcup_{\beta<\alpha} P^\beta(S) \qquad (\alpha \text{ a limit ordinal}) \\ P^\infty(S) &= \bigcup_{\alpha\in On} P^\alpha(S). \end{aligned}$$

Then we have

$$V = P^\infty(U).$$

The class $V = P^\infty(\emptyset)$, called the *kernel*, is a model of ZFC and contains all the ordinals.

The construction of permutation models is a method for showing the independence of AC from ZFA. The construction method was developed by A. Fraenkel in a series of papers from the 1920's and 30's, and further refined by A. Mostowski in the late 1930's and E. Specker in the 1950's.

We begin with a ground model of ZFA (with choice), $\mathscr{M}$. If π is a permutation of the set of atoms U, then using $\in$-recursion, we can define πx, x a set by:

$$\pi\emptyset = \emptyset, \qquad \pi(x) = \pi''x = \{\pi(y) : y \in x\}.$$

Thus π becomes an $\in$-automorphism of the universe.

Let $\mathscr{G}$ be a group of permutations of the set of atoms U. A set $\mathscr{F}$ of subgroups of $\mathscr{G}$ is a *normal filter* on $\mathscr{G}$ if it is a filter, and additionally,

$$\text{if } \pi \in \mathscr{G} \text{ and } H \in \mathscr{F}, \text{ then } \pi H \pi^{-1} \in \mathscr{F}$$

and

$$\text{for each } u \in U,\ \{\pi \in \mathscr{G} : \pi u = u\} \in \mathscr{F}.$$

We say that x is *symmetric* if the setwise stabiliser of x,

$$\mathscr{G}_{\{x\}} = \{g \in \mathscr{G} : g(y) \in x \text{ for all } y \in x\}$$

is such that $\mathscr{G}_{\{x\}} \in \mathscr{F}$. The class

$$\mathscr{N} = \{x : x \text{ is symmetric and } x \subseteq \mathscr{N}\}$$

consists of all *hereditarily symmetric* objects. We call $\mathscr{N}$ a *Fraenkel-Mostowski (FM) permutation model*. For verification that this is indeed a model, see, for example, [Jec73]. We also note that the kernel of $\mathscr{M}$ is contained in $\mathscr{N}$.

We call a family of subsets I of U a *normal ideal* if it is an ideal, and

$$\text{if } \pi \in \mathscr{G} \text{ and } E \in I, \text{ then } \pi'' E \in I;$$

and

$$\text{for each } u \in U, \{u\} \in I.$$

We can then define $\mathscr{F}$ to be the filter on $\mathscr{G}$ defined by the pointwise stabilisers of sets in I, $\mathscr{G}_E$, $E \in I$. This will be a normal filter. Note that a set x is symmetric if and only if there is $E \in I$ such that

$$\mathscr{G}_E \subseteq \mathscr{G}_{\{x\}}.$$

We then say that E is a *support* of x.

We point out that the ideal generated by all finite sets of atoms always produces a normal filter. For verification, see, for example, [Jec73]. We will not repeat this verification in the permutation model constructions carried out in this thesis.

Permutation models provide a method of showing independence in ZFA, but do not themselves show independence in ZF. However, there is a strong similarity between Fraenkel-Mostowski permutation models and symmetric models of ZF built using a forcing construction. This is made clear by the Jech-Sochor Theorem:

Theorem 3.2.1 (Jech, Sochor)**.** *Let $\mathscr{U}$ be a model of ZFA+AC (set theory with atoms with the axiom of choice), and let U be its set of atoms. Let $\mathscr{M}$ be the kernel of $\mathscr{U}$ (i.e. the class $P^\infty(\emptyset)$ in $\mathscr{U}$: a model of ZFC). Let α be an ordinal in $\mathscr{U}$.*

For every permutation model $\mathscr{V} \subseteq \mathscr{U}$ (a model of ZFA), there exists a symmetric extension $\mathscr{N} \supseteq \mathscr{M}$ (a model of ZF) and a set $\mathfrak{U} \in \mathscr{N}$ such that $(P^\alpha(U))^{\mathscr{V}}$ is $\in$-isomorphic to $(P^\alpha(\mathfrak{U}))^{\mathscr{N}}$.

A detailed proof can be found in [Jec73], but I include a brief sketch of the idea behind the theorem and proof.

Using forcing, we add to the kernel $\mathscr{M}$ (which is a model of ZFC) one generic set $\hat{u}$ of generic subsets of an ordinal κ for each $u \in U$. The sets $\hat{u}$ will have cardinality at most $|U| \times \kappa \times \kappa \times \kappa$. The set $\{\hat{u} : u \in U\}$ will play the role of the atoms in the new symmetric model. The ordinal κ is chosen to be a regular cardinal $\kappa > |(P^\alpha(U))^{\mathscr{U}}|$, and the generic extension $\mathscr{M}[G]$ is obtained using forcing conditions of cardinality less than κ. This guarantees that the sets $\mathfrak{u}$ do not appear in the first α levels of $(P^\alpha(\mathfrak{U}))^{\mathscr{M}[G]}$. Then, we construct $\mathscr{N}$ using a group and normal filter suitably adapted from those used to define $\mathscr{V}$.

Sets of sets of ordinals are used instead of merely sets of ordinals because there are non-trivial definable relations between sets of ordinals which would give the set that is meant to imitate atoms a structure.

The Jech-Sochor Theorem is usually used via the following:

Remark 3.2.2. *Let ϕ be a formula of the form $\exists X\, \psi(X, \beta)$, where the only quantifiers we allow in ψ are $\exists u \in P^\beta(X)$ and $\forall u \in P^\beta(X)$. Let $\mathscr{V}$ be a permutation model such* *that $\mathscr{V} \models \exists X\, \psi(X, \beta)$. Then there exists a symmetric model $\mathscr{N}$ of ZF such that $\mathscr{N} \models \exists X\, \psi(X, \beta)$.*

A sentence like $\exists X\, \psi(X, \beta)$ is called *boundable*. Remark 3.2.2 states that boundable statements are *transferable*: if such a statement is satisfied in a permutation model, there exists a Jech-Sochor symmetric model that also satisfies it. Examples of transferable statements include $\exists X$ (X is Dedekind finite) and $\exists X$ (X is amorphous), and similar.

Figure 3.1 illustrates the embedding of a permutation model in a symmetric model.

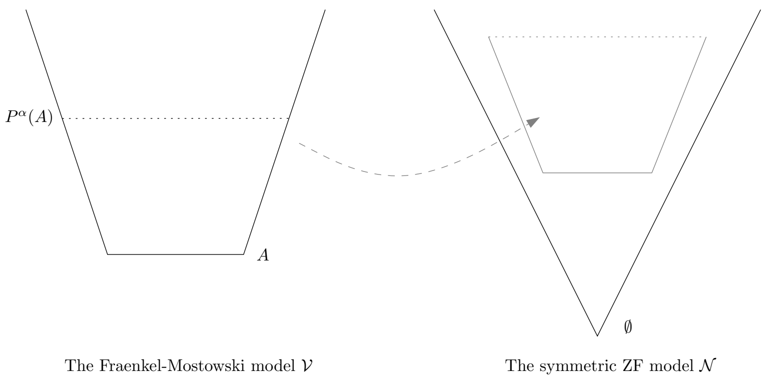

Figure 3.1: *An illustration of Jech-Sochor*

3.3 Plotkin's Construction

We now turn our attention to a forcing model construction. The construction in [Plo69] is carried out using ramified forcing. We find this obscures and complicates the forcing construction. Hence, we state the construction in terms of unramified forcing. We adopt terminology for forcing as in [Sho71].

Let $\mathscr{M}$ be a countable transitive model of $ZF + (V = L)$ – this model will serve as our base model in the forcing construction. Let $\mathscr{A}$ be a relational system in $\mathscr{M}$ such that $\mathscr{A}$ has an infinite domain $A \in \mathscr{M}$ and finitely many relations. For notational simplicity we assume $\mathscr{A}$ has only a binary relation R. We write $\mathscr{A}$ as $\langle A, R\rangle$.

In the following, $\mathscr{A}$ is the model of a complete first order theory T and equality is modelled in $\mathscr{A}$ by the identity relation on $\mathscr{A}$. The signature of T is denoted by L, $F(L)$ denotes the set of all formulas of L, while $F_n(L)$ denotes the set of L-formulas with at most n free variables. For $\mathscr{B}$ a model of T with domain B, $D_n(B)$ is the set of all n-ary relations on $\mathscr{B}$ which are definable with parameters from the model by $F(L)$. i.e. $s \in D_n(B)$ if

$$s = \{\langle b_1, \ldots, b_n\rangle : \mathscr{B} \models \varphi(b_1, \ldots, b_n, a_1, \ldots, a_m)\}$$

for some $\varphi(y_1, \ldots, y_n, x_1, \ldots, x_m) \in F(L)$ and $a_i \in B$ where $1 \leqslant i \leqslant m$.

All model theory will be done inside $\mathscr{M}$ – so in particular, when we say $\mathscr{A}$ is countable, we mean it is countable in $\mathscr{M}$.

Model Construction 3.3.1 ($\mathcal{M}22$ Plotkin's Model I)**.** Let C be the set of functions from a finite subset of $A \times \omega \times \omega$ into $\{0, 1\}$ partially ordered by reverse extension. We regard C as a notion of forcing. Let G be an $\mathscr{M}$-generic set over C, and $\mathscr{M}[G]$ the resulting extension. Then, since $G \in \mathscr{M}[G]$, $F = \bigcup G \in \mathscr{M}[G]$, and F is a function from $A \times \omega \times \omega$ into $\{0, 1\}$.

Let

$$\begin{aligned} S_{aj} &= \{n : F(a, j, n) = 1\} && \subseteq \omega \\ \mathfrak{A}_a &= \{S_{aj} : j \in \omega\} && \subseteq P(\omega) \\ \mathfrak{A} &= \{\mathfrak{A}_a : a \in A\} && \subseteq PP(\omega) \end{aligned}$$

and

$$\mathfrak{R} = \{\langle \mathfrak{A}_a, \mathfrak{A}_b\rangle : \langle a, b\rangle \in R\}.$$

We use bold symbols for the constants (names) in the forcing language $\mathscr{L}$.

Remark 3.3.2. *$\mathscr{M}[G] \models \mathbf{S_{aj}} = \mathbf{S_{bk}} \leftrightarrow (a = b \wedge j = k)$. Consequently, $\mathscr{M}[G] \models \boldsymbol{\mathfrak{A}}_{\mathbf{a}} = \boldsymbol{\mathfrak{A}}_{\mathbf{b}} \leftrightarrow a = b$.*

Let $\theta(a) = \mathfrak{A}_a$. Note that $\theta \in \mathscr{M}[G]$. Let $\mathscr{N}$ be the submodel of $\mathscr{M}[G]$ consisting of those of its elements which are hereditarily ordinal definable (HOD) over

$$\{S_{aj} : a \in A, j \in \omega\} \cup \{\mathfrak{A}_a : a \in A\} \cup \{\mathfrak{A}\} \cup \{\mathfrak{R}\}.$$

This completes the construction.

Before we continue, we make a remark about the symmetry of the model $\mathscr{N}$: Let $\mathscr{H}_1$ be the group of automorphisms of $\langle A, R\rangle$ which are in the base model $\mathscr{M}$. Let $\mathscr{H}_2$ be the group of permutations on ω whose elements move only finitely many $i \in \omega$ (all of which are in $\mathscr{M}$).

Let $p \in C$. For $\gamma \in \mathscr{H}_1$, define

$$\gamma(p) = \{\langle \gamma(a), j, n, i\rangle : \langle a, j, n, i\rangle \in p\}.$$

For $\pi \in \mathscr{H}_2$, define

$$\pi_a(p) = \{\langle a, \pi j, n, i\rangle : \langle a, j, n, i\rangle \in p\} \cup \{\langle b, j, n, i\rangle : \langle b, j, n, i\rangle \in p \text{ and } b \neq a\}.$$

For formulas $\phi_{\mathfrak{L}}$ of the forcing language $\mathfrak{L}$, we can define $\gamma(\phi_{\mathfrak{L}})$ to be the formula obtained by replacing each occurrence of S_{aj} and $\mathfrak{A}_a$ by $\gamma(S_{aj})$ and $\gamma(\mathfrak{A}_a)$ respectively. The formula $\pi_a(\phi_{\mathfrak{L}})$ is defined analogously. As shown in Lemma 2 of [Plo69], we then have that

$$p \Vdash \phi_{\mathfrak{L}} \Leftrightarrow \gamma(p) \Vdash \gamma(\phi_{\mathfrak{L}}) \text{ for all } \gamma \in \mathscr{H}_1,$$

and

$$p \Vdash \phi_{\mathfrak{L}} \Leftrightarrow \pi_a(p) \Vdash \pi_a(\phi_{\mathfrak{L}}) \text{ for all } \pi \in \mathscr{H}_2, a \in A.$$

The model $\mathscr{N}$ is thus an (hereditarily) symmetric extension of $\mathscr{M}$ with respect to the group generated by $\mathscr{H}_1$ and $\mathscr{H}_2$.

Finally, we close this section about Plotkin's construction with a remark relating it to models built using the Fraenkel-Mostowski method.

Remark 3.3.3. *Let $\mathscr{U}$ be a model of ZFA+AC (set theory with atoms with the axiom of choice), and let U be its set of atoms, indexed by the domain of a model A of a first-order theory T (as above). Let $\mathscr{M}$ be the kernel of $\mathscr{U}$ (i.e. $P^\infty(\emptyset) \in \mathscr{U}$: a model of ZFC). Let α be an ordinal in $\mathscr{U}$.*

Let $\mathscr{V} \subseteq \mathscr{U}$ be the Fraenkel-Mostowski permutation model (a model of ZFA) built using automorphisms of A as permutations, and using the filter of finite supports. Then, there exists a model $\mathscr{N} \supseteq \mathscr{M}$ (a model of ZF) and a set $\mathfrak{A} \in \mathscr{N}$ such that $(P^\alpha(U))^{\mathscr{V}}$ is $\in$-isomorphic to $(P^\alpha(\mathfrak{A}))^{\mathscr{N}}$. Furthermore, the model $\mathscr{N}$ and the set $\mathfrak{A}$ are as defined in the Plotkin model construction.

Proof. The result follows by direct application of the proofs of the Jech-Sochor Embedding Theorem and the Pincus Support Theorem (see Theorems 6.1 and 6.6 of [Jec73]). $\square_{3.3.3}$

3.4 The first-order $\aleph_0$-categorical case

In this section, we concentrate on assumptions about the theory T and its model $\mathscr{A}$ in $\mathscr{M}$, and their effect on the structure $\mathfrak{A}$ resulting from the Plotkin construction. We begin with a remark we state without proof.

Remark 3.4.1. *The correspondence $\theta : a \longrightarrow \mathfrak{A}_a$ is an isomorphism between the relational systems $\langle A, R\rangle$ and $\langle \mathfrak{A}, \mathfrak{R}\rangle$. As noted above, $\theta \in \mathscr{M}[G]$ but $\theta \notin \mathscr{N}$.*

Clearly, if $\mathfrak{r}$ is an n-ary relation on $\mathfrak{A}$ in $\mathscr{N}$, then it is hereditarily ordinal definable over a finite subset of $\mathfrak{A}$ (and also sets of the form S_{aj}, $\mathfrak{A}$, and $\mathfrak{R}$), by the definition of the model $\mathscr{N}$. We wish to show that under certain circumstances, all the n-ary relations on $\mathfrak{A}$ are also L-definable over $\mathfrak{A}$.

We will write formulas in $F(L)$ as small Greek letters, such as φ_L, and the corresponding formulas in the forcing language $\mathfrak{L}$ with quantifiers restricted to $\boldsymbol{\mathfrak{A}}$ as $\varphi_{\mathfrak{L}}$. Clearly, for $\varphi_L \in F(L)$, we have

$$\mathscr{A} \models \varphi_L(a_1, \dots, a_n) \leftrightarrow \mathscr{N} \models \varphi_{\mathfrak{L}}(\boldsymbol{\mathfrak{A}}_{\mathbf{a_1}}, \dots, \boldsymbol{\mathfrak{A}}_{\mathbf{a_n}}).$$

It remains to be shown that the relations in $\mathfrak{A}$ can be defined (with appropriate parameters) in L.

Now we define notation that will be in effect for Lemmas 3.4.2 and 3.4.3 and Corollary 3.4.4: Let $\varphi_{\mathfrak{L}}(\boldsymbol{\mathfrak{A}}_{\mathbf{b_1}}, \dots, \boldsymbol{\mathfrak{A}}_{\mathbf{b_k}})$ be a formula in the forcing language that defines a relation $\mathfrak{r}$ on $\mathfrak{A}$ in $\mathscr{N}$. That is,

$$\mathscr{N} \models (\mathfrak{A}_{a_1}, \dots, \mathfrak{A}_{a_n}) \in \mathfrak{r} \leftrightarrow \varphi_{\mathfrak{L}}(\mathfrak{A}_{a_1}, \dots, \mathfrak{A}_{a_n}, \boldsymbol{\mathfrak{A}}_{\mathbf{b_1}}, \dots, \boldsymbol{\mathfrak{A}}_{\mathbf{b_k}})$$

where $\boldsymbol{\mathfrak{A}}_{\mathbf{b_1}}, \dots, \boldsymbol{\mathfrak{A}}_{\mathbf{b_k}}$ indicates that $\varphi_{\mathfrak{L}}$ includes mention of $\boldsymbol{\mathfrak{A}}_{\mathbf{b_i}}$ or some $\mathbf{S_{b_ij}}$. We write $\varphi_{\mathfrak{L}}^{\mathscr{N}}(\mathfrak{A}_{b_1}, \dots, \mathfrak{A}_{b_k})$ to indicate the interpretation of $\varphi_{\mathfrak{L}}$ in the model $\mathscr{N}$.

In Lemmas Lemmas 3.4.2 and 3.4.3, we fix the $(n+k)$-ary formula $\phi_{\mathfrak{L}}$, hence n and k are both fixed. In the following, $(n+k)$-homogeneous means that tuples of length $n+k$ satisfying the same first-order formulas can be mapped to each other via an automorphism of the structure.

Lemma 3.4.2. *Assume $\mathscr{A}$ is an $(n+k)$-homogeneous model of a theory T.*

Let $\psi(u_1, \dots, u_n, v_1, \dots, v_k)$ be an atom of the boolean algebra $F_{n+k}(L)$. Let $\mathfrak{r}_{\psi_{\mathfrak{L}}}(\mathfrak{A}_{b_1}, \dots, \mathfrak{A}_{b_k})$ be the n-ary relation defined by (here, treating the relation as a *set of ordered n-tuples):*

$$\mathfrak{r}_{\psi_{\mathfrak{L}}}(\mathfrak{A}_{b_1},\dots,\mathfrak{A}_{b_k}) = \{\langle \mathfrak{A}_{a_1},\dots,\mathfrak{A}_{a_n}\rangle \in \mathfrak{A}^n : \psi_{\mathfrak{L}}(\mathfrak{A}_{a_1},\dots,\mathfrak{A}_{a_n},\mathfrak{A}_{b_1},\dots,\mathfrak{A}_{b_k})\}.$$

If $\mathfrak{r}_{\psi_{\mathfrak{L}}}(\mathfrak{A}_{b_1},\dots,\mathfrak{A}_{b_k}) \cap \varphi_{\mathfrak{L}}^{\mathscr{N}}(\mathfrak{A}_{b_1},\dots,\mathfrak{A}_{b_k}) \neq \emptyset$, then $\mathfrak{r}_{\psi_{\mathfrak{L}}} \subseteq \varphi_{\mathfrak{L}}^{\mathscr{N}}$.

Proof. By hypothesis, there is some n-tuple

$$\langle \mathfrak{A}_{d_1},\dots,\mathfrak{A}_{d_n}\rangle \in \mathfrak{r}_{\psi_{\mathfrak{L}}} \cap \varphi_{\mathfrak{L}}^{\mathscr{N}}.$$

Let $\langle \mathfrak{A}_{c_1},\dots,\mathfrak{A}_{c_n}\rangle$ be any other tuple in $\mathfrak{r}_{\psi_{\mathfrak{L}}}$. We assume that $\langle \mathfrak{A}_{c_1},\dots,\mathfrak{A}_{c_n}\rangle \notin \varphi_{\mathfrak{L}}^{\mathscr{N}}$ and derive a contradiction.

Since both $\langle \mathfrak{A}_{d_1},\dots,\mathfrak{A}_{d_n}\rangle, \langle \mathfrak{A}_{c_1},\dots,\mathfrak{A}_{c_n}\rangle \in \mathfrak{r}_{\psi_{\mathfrak{L}}}$, by definition we have

$$\mathscr{N} \models \psi_{\mathfrak{L}}(\mathfrak{A}_{d_1},\dots,\mathfrak{A}_{d_n},\mathfrak{A}_{b_1},\dots,\mathfrak{A}_{b_k})$$

and

$$\mathscr{N} \models \psi_{\mathfrak{L}}(\mathfrak{A}_{c_1},\dots,\mathfrak{A}_{c_n},\mathfrak{A}_{b_1},\dots,\mathfrak{A}_{b_k}).$$

Because $\theta : a \longrightarrow \mathfrak{A}_a$ is an isomorphism, we have

$$\mathscr{A} \models \psi_L(d_1,\dots,d_n,b_1,\dots,b_k)$$

and

$$\mathscr{A} \models \psi_L(c_1,\dots,c_n,b_1,\dots,b_k).$$

By assumption, ψ_L is an atom in $F_{n+k}(L)$, thus both $\langle d_1,\dots,d_n,b_1,\dots,b_k\rangle$ and $\langle c_1,\dots,c_n,b_1,\dots,b_k\rangle$ satisfy the same formulas of L. Because we have assumed that $\mathscr{A}$ is $(n+k)$-homogeneous, there is an automorphism γ of $\mathscr{A}$ such that $\gamma(b_i) = (b_i)$, $1 \leqslant i \leqslant k$, and $\gamma(d_j) = (c_j)$, $1 \leqslant j \leqslant n$. However, the model $\mathscr{N}$ is symmetric with regards to the automorphism γ, and so we come to a contradiction. $\square_{3.4.2}$

Lemma 3.4.3. *Assume $\mathscr{A}$ is an atomic, $(n+k)$-homogeneous model of T.*

Let $\varphi^{\mathscr{N}}_{\mathfrak{L}}$ be the $(n+k)$-ary formula defining an n-ary relation on $\mathfrak{A}$, as above. Then, $\varphi^{\mathscr{N}}_{\mathfrak{L}}$ is a disjoint union of the satisfaction sets of certain formulas which correspond to atoms in $F_{n+k}(L)$.

Proof. We continue with the notation as in the proof of Lemma 3.4.2.

Let $\langle \mathfrak{A}_{d_1}, \ldots, \mathfrak{A}_{d_n} \rangle \in \varphi^{\mathscr{N}}_{\mathfrak{L}}(\mathfrak{A}_{b_1}, \ldots, \mathfrak{A}_{b_k})$. Since $\mathscr{A}$ is atomic, there is an atom $\psi_L(u_1, \ldots, u_n, v_1, \ldots, v_k)$ in $F_{n+k}(L)$ such that

$$\mathscr{A} \models \psi_L(d_1, \ldots, d_n, b_1, \ldots, b_k).$$

But then, $\mathscr{N} \models \psi_{\mathfrak{L}}(\mathfrak{A}_{d_1}, \ldots, \mathfrak{A}_{d_n}, \mathfrak{A}_{b_1}, \ldots, \mathfrak{A}_{b_k})$. Therefore,

$$\langle \mathfrak{A}_{d_1}, \ldots, \mathfrak{A}_{d_n} \rangle \in \mathfrak{r}_{\psi_{\mathfrak{L}}}(\mathfrak{A}_{b_1}, \ldots, \mathfrak{A}_{b_k})$$

. By Lemma 3.4.2, $\mathfrak{r}_{\psi_{\mathfrak{L}}} \subseteq \varphi^{\mathscr{N}}_{\mathfrak{L}}$. Therefore we have

$$\varphi^{\mathscr{N}}_{\mathfrak{L}} = \bigcup_{i \in I} \mathfrak{r}_{\psi_{i\,\mathfrak{L}}},$$

where I indexes a set of atoms in $F_{n+k}(L)$. Furthermore, $\mathfrak{r}_{\psi_{i\,\mathfrak{L}}}(\mathfrak{A}_{b_1}, \ldots, \mathfrak{A}_{b_k}) \cap \mathfrak{r}_{\psi_{j\,\mathfrak{L}}}(\mathfrak{A}_{b_1}, \ldots, \mathfrak{A}_{b_k}) = \emptyset$ because ψ_i and ψ_j are atoms of $F_{n+k}(L)$. $\square_{3.4.3}$

Corollary 3.4.4.

1. *If $\mathscr{A}$ is an atomic $\aleph_0$-homogeneous model of T, every n-ary relation on $\mathfrak{A}$ has the form of a disjoint union $\bigsqcup_{i \in I} \mathfrak{r}_{\psi_{i\,\mathfrak{L}}}$, where $\psi_{i\,L}$ is an atom of $F_{n+k}(L)$, and I indexes the atoms of $F_{n+k}(L)$.*

2. *If $\mathscr{A}$ is a prime model of T, every n-ary relation on $\mathfrak{A}$ has the form of a disjoint union $\bigsqcup_{i \in I} \mathfrak{r}_{\psi_{i\,\mathfrak{L}}}$, where $\psi_{i\,L}$ is as in statement (1).*

3. *If $\mathscr{A}$ is a countable model of an $\aleph_0$-categorical theory T, then every n-ary relation on $\mathfrak{A}$ has the form of a finite disjoint union $\bigsqcup_{i=1}^{m} \mathfrak{r}_{\psi_i\,\mathfrak{L}}$. Thus, any n-ary relation on $\mathfrak{A}$ is definable with a single first-order L-formula. Thus, in this case, the complete first order theory of $\mathfrak{A}$,* $\mathrm{Th}(\mathfrak{A}) = T$.

We defer discussion of non-first order axiomatisable structures (i.e. those axiomatisable using a language $L_{\omega_1\omega}$) in this context to Section 5.4.

Proof. The first statement of the corollary is clear from Lemmas 3.4.3 and 3.4.2.

The second statement follows from the fact that a model of a countable first order theory is prime if and only if it is countable and atomic, as well as the fact that a countable and atomic model is $\aleph_0$-homogeneous.

The third statement follows easily from the fact that if T is $\aleph_0$-categorical, then for all $n \in \omega$, the set of atoms of $F_n(L)$ is finite. $\square_{3.4.4}$

In [Plo69], Plotkin concludes that for an $\aleph_0$-categorical theory T, $\mathfrak{A}$ (as defined above) is Dedekind-finite. We now show our refinements of this result.

Theorem 3.4.5. *Let T be $\aleph_0$-categorical. Then, $\mathfrak{A}$ (as defined above) is weakly Dedekind-finite.*

Proof. Assume that $|\mathfrak{A}| \notin \Delta_4$. Then, there are Z_i, $i \in \omega$, pairwise disjoint and non-empty, such that $\mathfrak{A} = \bigcup_{i\in\omega} Z_i$. Since each Z_i is a subset of $\mathfrak{A}$, each can be regarded as a unary relation on $\mathfrak{A}$. Since any relation can be expressed as the finite sum of atoms, each Z_i can be defined using a single formula. Thus we can find $\aleph_0$ many pairwise disjoint formulas, contrary to the $\aleph_0$-categoricity of T. Hence, $|\mathfrak{A}| \in \Delta_4$. $\square_{3.4.5}$

The following Theorem 3.4.6 gives the exactness of the correlation described in Theorem 4.3.1.

Theorem 3.4.6. *Let T be an $\aleph_0$-categorical theory that does not have the Strict Order Property. Then* $|\mathfrak{A}| \in \Delta_3^{\dagger}$.

Proof. Let T be a theory that is $\aleph_0$-categorical and does not have the strict order property. Then $|\mathfrak{A}| \in \Delta_4$. Assume that $|\mathfrak{A}| \notin \Delta_3^\dagger$. Then, for some $n \in \omega$, $\mathfrak{A}^n$ admits a partial order having chains of unbounded length. Using the same arguments as in the previous theorems, this would indicate that T has the strict order property, a contradiction. $\square_{3.4.6}$

Theorem 3.4.7 gives the exactness of the correlation established in Corollary 4.5.3.

Theorem 3.4.7. *If T is $\aleph_0$-categorical, $\aleph_0$-stable, then $\mathfrak{A}$ admits MT-rank. Furthermore, the Morley rank of T and the MT-rank of $\mathfrak{A}$ are of equal value.*

Proof. Assume that T is $\aleph_0$-categorical $\aleph_0$-stable. Then, T has finite Morley rank n, say, and Morley degree d. Thus, the formula $\phi =$ "$x = x$" has Morley rank n, degree d. Thus, there are L-formulas $\phi_i(\bar{x})$, $i \in \omega$ such that the sets $\phi(\mathscr{A}) \cap \phi_i(\mathscr{A})$ are pairwise disjoint and $MR(\phi \wedge \phi_i) \geqslant n - 1$ for each $i \in \omega$. Assume further that $\mathfrak{A}$ does not admit MT-rank. Then for all ordinals α, for each $m \in \omega$, there exist m pairwise disjoint subsets $X_i : 0 \leqslant i \leqslant m$ of $\mathfrak{A}$ such that $MT(X_i) \geqslant \alpha$, for all $i \in m$. However, as subsets of $\mathfrak{A}$ can be regarded as unary relations, each X_i corresponds to a single formula. This clearly contradicts the value of the Morley rank of the theory T.

The values of the ranks must be equal: Assume first that T has Morley rank n and Morley degree d, $\mathfrak{A}$ has MT-rank m, MT-degree c, and that $n < m$. Thus, if we divide $\mathfrak{A}$ into $c + 1$ pairwise disjoint sets, one piece must have rank $< m$. However, since each subset of $\mathfrak{A}$ can be defined by a single formula, this would give a contradiction with the the Morley rank of T. Now, let us assume that $m < n$. Then, there are first-order formulas ϕ_i in one variable whose satisfactions sets in the unique countable model of T satisfy the terms of the definition of Morley rank. By the construction of the model containing $\mathfrak{A}$, these formulas are then also satisfied in $\mathfrak{A}$, and thus there are subsets of $\mathfrak{A}$ that contradict the assumed MT-rank of $\mathfrak{A}$. $\square_{3.4.7}$

Note that the rank values are the same in this case. This is because the model $\mathfrak{A}$ only exhibits the first-order structure of the theory. Later, we will give an example where the ranks are not equal, showing that MT-rank measures more than just first-order structure.

The following Theorems 3.4.8, 3.4.9, and 3.4.10, are proved analogously to those results proved thus far in this section. We state them without proof.

Theorem 3.4.8. *If T is an $\aleph_0$-categorical o-minimal theory, then $\mathfrak{A}$ is o-amorphous.*

Theorem 3.4.9. *If T is an $\aleph_0$-categorical weakly o-minimal theory, then $\mathfrak{A}$ is weakly o-amorphous.*

Theorem 3.4.10. *If T is an $\aleph_0$-categorical strongly minimal theory, then $\mathfrak{A}$ is an amorphous set.*

Later, we will show that all of the correspondences mentioned in Theorems 3.4.5, 3.4.8, 3.4.6, 3.4.7, 3.4.9, and 3.4.10 are exact.

We now examine a final classical finiteness notion, II-finiteness, to determine some of the behaviours exhibited by sets of this type. We have not been able to find a neat and tidy correlation for sets that are II-finite. For example, a model constructed using the method of [Plo69] based on the random graph yields a set which is II-finite and has the Independence Property (see Definition 2.1.8). Similarly, it is also possible to construct a II-finite set whose theory has the Strict Order Property (see Definition 2.1.7) using the generic partial order. Because strictly Mostowski finite sets correspond to those sets that do not have the Strict Order Property, a model built using the method of [Plo69] based on Hrushovski's stable (not $\aleph_0$-stable) $\aleph_0$-categorical pseudoplane (see [Hru88]) is II-finite (because it is strictly Mostowski finite) but has neither the Strict Order Property nor the Independence Property. The same can be said about models built from $\aleph_0$-stable, $\aleph_0$-categorical theories.

Thus it seems that II-finite sets can exhibit a wide variety of first-order behaviours from a stability theoretic point of view.

We will construct the examples mentioned based on the random graph and the generic partial order. For clarity, we use the Fraenkel-Mostowski method, and implicitly assume application of the Jech-Sochor Theorem. Conclusions about a model based on the Hrushovski pseudoplane can be reached through theorems proved thus far. We omit that construction since little would be gained from a detailed exposition compared to the effort involved in describing the pseudoplane's construction.

Model Construction 3.4.11 (A II-finite set with the Independence Property)**.** Let Γ be the random graph. This is a countable, universal ultrahomogeneous, $\aleph_0$-categorical graph whose theory has the Independence Property. We will build a Fraenkel-Mostowski model using a finite support structure.

Indexing the set of atoms: Let U be a countable set of atoms. We index using the elements of Γ: Let $U = \{u_\gamma : \gamma \in \Gamma\}$.

Defining a permutation group: Let $\mathscr{G}$ be the group of permutations of U induced by $Aut(\Gamma)$.

Defining a normal filter: Let $\mathscr{F}$ be the filter generated by $\{\mathscr{G}_A : A \subset \Gamma, |A| \in \omega\}$. This is a filter based on the ideal of finite sets, and is thus normal.

Let $\mathscr{N}_\Gamma$ be the resulting Fraenkel-Mostowski model.

Theorem 3.4.12. *Let $\mathscr{N}_\Gamma$ and $U \in \mathscr{N}_\Gamma$ be as defined in Model Construction 3.4.11. Then,* $\mathrm{Th}(U) = \mathrm{Th}(\Gamma)$ *and so has the Independence Property, and* $|U| \in \Delta_2$.

Proof. That $\mathrm{Th}(U)$ has the Independence Property is clear by part 3 of Corollary 3.4.4.

Assume for a contradiction that U is not II-finite. Let $\{X_i : i \in I\}$ be a partition of U, where I is an infinite ordered set. Since we assume this ordered partition is in the model $\mathscr{N}_\Gamma$, it must be supported by a finite set $S \subset \Gamma$, where $|S| = s$, We write $S = \{\gamma_1, \ldots, \gamma_s\}$. Because Γ is $\aleph_0$-categorical, there are finitely many 1-types over S in $\mathrm{Th}(\Gamma)$. In other words, $Aut(\Gamma)$ is oligomorphic, and thus has finitely many orbits of $(s+1)$-tuples. Choose distinct members a, b of an infinite orbit over S that lie in different partition pieces, say $a \in X_i, b \in X_j$, $i \neq j$. Two such elements can be found because the partition is infinite, while the number of orbits is finite. Thanks to the symmetry of the graph relation, $(\gamma_1, \ldots, \gamma_s, a, b)$ and $(\gamma_1, \ldots, \gamma_s, b, a)$ have the same quantifier-free type, so g given by $ga = b$, $gb = a$, and $g\gamma_i = \gamma_i$, for all $\gamma_i \in S$, is a partial automorphism. Because the random graph is ultra-homogeneous, this automorphism extends to an automorphism on the whole of Γ. This automorphism is clearly in the subgroup supported by S, but interchanges elements of the partition, contrary to the antisymmetry of the ordering. Hence we come to a contradiction. $\square_{3.4.12}$

Model Construction 3.4.13 (A II-finite set with SOP)**.** Let $(\Psi, <)$ be the generic partial order. This is a countable, universal homogeneous, $\aleph_0$-categorical partial order whose theory has the Strict Order Property. We will build a Fraenkel-Mostowski model using a finite support structure.

Indexing the set of atoms: Let U be a countable set of atoms. We index using the elements of Ψ: Let $U = \{u_x : x \in \Psi\}$.

Defining a permutation group: Let $\mathscr{G}$ be the group of permutations of U induced by $Aut(\Psi)$.

Defining a normal filter: Let $\mathscr{F}$ be the filter generated by $\{\mathscr{G}_A : A \subset \Psi, |A| \in \omega\}$. This is a normal filter because it is generated by the ideal of finite sets.

Let $\mathscr{N}_\Psi$ be the resulting Fraenkel-Mostowski model.

Theorem 3.4.14. *Let $\mathscr{N}_\Psi$ and $U \in \mathscr{N}_\Psi$ be as defined in Model Construction 3.4.13. Then,* $\mathrm{Th}(U, <) = \mathrm{Th}(\Psi, <)$ *and so has the Strict Order Property, and* $|U| \in \Delta_2$.

Proof. That Th(U) has the Strict Order Property is clear by part 3 of Corollary 3.4.4. We note that $(U,<)$ has arbitrarily long finite chains in $\mathscr{N}$, however, in this model, $(U,<)$ does not have *infinite* chains, as we will show.

We note that the generic partial order is characterised by the following property: If A, B, and C are pairwise disjoint finite subsets of Ψ, such that for all $a \in A$, $b \in B$, $c \in C$,

$$b < c$$
$$c \not< a$$
$$a \not< b,$$

then there exists $z \in \Psi$ such that

$$b < z < c$$
$$a \not< z \wedge a \not> z,$$

for all $a \in A$, $b \in B$, $c \in C$.

Assume for a contradiction that U is not II-finite. Let $\{X_i : i \in I\}$ be a partition of U, where I is an infinite ordered set. Since we assume this ordered partition is in the model $\mathscr{N}_\Psi$, it must be supported by a finite set $S \subset \Psi$, where $|S| = s$. We write $S = \{a_1, \ldots, a_s\}$. Because Ψ is $\aleph_0$-categorical, there are finitely many 1-types over S in Th(Ψ). In other words, $Aut(\Psi)$ is oligomorphic, and thus has finitely many orbits of $(s+1)$-tuples. Thus, there must be two sets from the partition X_i, X_j, $i \neq j$, that each contain an element that has the same 1-type over S. Call these elements $x \in X_i$ and $y \in X_j$. Thus, there is a permutation g such that $ga_i = a_i$ for all $a_i \in S$, and $gx = y$.

We have two possible cases for the relation between x and y:

First, assume $x \not< y \wedge y \not< x$. Then the relation between x and y is symmetric, and g is a partial automorphism that can be extended to an automorphism of the structure. We can argue for the contradiction as in the proof of Theorem 3.4.12.

Second, assume that either $x < y$ or $y < x$. Because

$$\{z : tp(z|S) = tp(x|S)\} \cong \Psi,$$

we can choose an element $z \notin S$ such that $z \not< x \wedge z \not> x$ and $z \not< y \wedge y \not> x$. Since x and y are in different partition pieces, z must be in a different partition piece from at least one of them. Thus we have reduced this second case to the first, and we again can use the same argument as in the proof of Theorem 3.4.12, giving the desired contradiction. $\square_{3.4.14}$

Chapter 4

The use of model theoretic methods on weakly Dedekind-finite structures

4.1 Introduction

In this chapter, we will apply model theoretic methods in order to explore the possible first-order structures admitted by various classes of infinite weakly Dedekind-finite sets. From now on, we will assume that all sets under consideration are infinite (according to Definition 2.2.2).

Our strategy is to determine what kinds of first-order structures can be carried by each class of weakly Dedekind finite set. Because first-order statements are expressible in arithmetic, and arithmetic is absolute for models of set theory, we can apply any knowledge about the possible first-order definable structures of the theories to the corresponding classes of non-choice sets. In this way, we can apply theorems whose proofs ostensibly rely on the Axiom of Choice to sets in a non-choice setting.

4.2 Main Result: Weakly Dedekind-finite Structures

We begin with an observation that perhaps will suggest our main result.

Remark 4.2.1. *Let $\mathfrak{A}$ be a weakly Dedekind-finite structure that is axiomatisable in a language $L = \{f_i, r_j, c_k : i \in I, j \in J, k \in K\}$, where $I \cup J \cup K$ is well-ordered by $<$. Then $\mathfrak{A}$ is uniformly locally finite.*

Proof. Let $\mathfrak{A}$ be a structure such that the domain $|\mathfrak{A}| \in \Delta_4$. Then $\mathfrak{A}$ is locally finite (see the proof of Remark 5.3.1).

Assume that $\mathfrak{A}$ is not uniformly locally finite. Then, for some n, the cardinalities of the substructures generated by n-tuples are unbounded, hence the set of n-element subsets of $\mathfrak{A}$ can be mapped onto ω. However, as Δ_4 is closed under multiplication (see Remark 2.2.16), the set of n-element subsets of $\mathfrak{A}$ also belongs to Δ_4, a contradiction.

$\square_{4.2.1}$

As mentioned earlier, $\aleph_0$-categorical structures are uniformly locally finite. This perhaps gives a clue that leads to our following, main, result:

Theorem 4.2.2. *Let $\mathfrak{A}$ be a weakly Dedekind-finite set admitting a structure axiomatisable in a countable (finite or infinite) language. Let $T = \mathrm{Th}(\mathfrak{A})$ be the (complete) set of sentences in the (first-order) language appropriate for the structure true in $\mathfrak{A}$. Then, the set F_nT of formulas in n free variables with respect to T is finite.*

Thus T is $\aleph_0$-categorical.

We use the usual proof of the Engeler-Ryll-Nardzewski-Svenonius Theorem 2.1.3 to show that countable models of T are atomic, and that countable models are isomorphic. The key to our proof is to find the one out of the long list of equivalent statements of Theorem 2.1.3 that is easiest applied to a weakly Dedekind-finite structure. This is demonstrated in our first claim. We then show that the rest of the proof can be done without the use of the Axiom of Choice:

Proof. Claim: F_nT is finite.

Let $n \in \omega$. Because the language is countable, the set of formulas with n free variables is at most countable. For each formula $\varphi(x)$ in n free variables (i.e. in F_nT, in the usual notation), consider $\{\bar{a} \in \mathfrak{A}^n : \mathfrak{A} \models \varphi(\bar{a})\}$. This is a countable family of subsets of $\mathfrak{A}^n$. However, since the class of weakly Dedekind-finite sets is closed under products, $\mathfrak{A}^n$ is weakly Dedekind-finite. Hence, this family of subsets must be finite. Thus, we map F_nT onto a finite family of sets. Suppose the formulas φ and ψ determine the same set. Then, $\mathfrak{A} \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. Hence, the formula $\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})) \in T$, and so, $T \vdash \varphi \leftrightarrow \psi$, and ϕ and ψ are identified in F_nT. This means that F_nT is finite.

Claim: Every model of T is atomic.

Let $\mathscr{A}$ be a countable model of T, $\bar{a}$ a finite tuple from the set underlying $\mathscr{A}$, and let $\varphi_1, \ldots, \varphi_n$ be a list of the finitely many formulae, up to equivalence in T, which are satisfied by $\bar{a}$ in $\mathscr{A}$. Then, $\varphi_1 \wedge \ldots \wedge \varphi_n = \psi$ is consistent with T, and $T \models \psi \to \varphi_i$ for each $i \leqslant m$. Thus, $\varphi_1, \ldots, \varphi_n$ are isolated by ψ, and thus $\mathscr{A}$ is atomic.

Claim: Countable atomic models of T are isomorphic.

Let $\mathscr{A}$ and $\mathscr{B}$ be countable atomic models of T. Enumerate both of them with ω. Let a_0 be the first element of $\mathscr{A}$ in the enumeration, and $\varphi(a_0)$ the formula that isolates the finite set of formulae $\psi^1_{a_0} \ldots \psi^n_{a_0}$ which are satisfied by a_0 in $\mathscr{A}$. Since $\mathscr{B}$ is also a model of T, there is a first element in the enumeration of $\mathscr{B}$ satisfying $\varphi(x)$ in $\mathscr{B}$. Call this element b_0. Let b_1 be the first element of $\mathscr{B} \setminus \{b_0\}$, and find the first $a_1 \in \mathscr{A}$ such that the finite set of formulae satisfied by (a_0, a_1) is the same as the finite set of formulae satisfied by (b_0, b_1). Note that the existence of these is guaranteed by the completeness of T. We continue back and forth ω times. This gives enumerations $\{a_i : i < \omega\}$ and $\{b_i : i < \omega\}$ if $\mathscr{A}$ and $\mathscr{B}$ respectively, and the map taking a_i to b_i defines an isomorphism from $\mathscr{A}$ to $\mathscr{B}$. $\square_{4.2.2}$

The results of the rest of this chapter all depend heavily on Theorem 4.2.2.

4.3 Strictly Mostowski Finite Structures

Because many of the subclasses of the class of weakly Dedekind-finite sets are defined according to the relations that they admit, it is natural to examine these sets with the usual model-theoretic notions about definable relations in mind.

We have the following result:

Theorem 4.3.1. *Let $\mathfrak{A}$ be a strictly Mostowski finite set admitting a structure axiomatizable in a countable (finite or infinite) language. Then $T = \mathrm{Th}(\mathfrak{A})$ is $\aleph_0$-categorical and does not have the strict order property.*

Proof. Because strict Mostowski finiteness implies weak Dedekind-finiteness, T is $\aleph_0$-categorical. Let $\mathscr{A}$ be the unique countable model of T.

Assume T has the strict order property. Then T has a formula ϕ with the strict order property. This formula defines a partial ordering on the set of all n-tuples in $\mathscr{A}$ which contains arbitrarily long finite chains. As $\mathfrak{A}$ and $\mathscr{A}$ satisfy the same first-order sentences, $\mathfrak{A}$ admits such a first-order definable partial ordering, contrary to the strict Mostowski finiteness of $\mathfrak{A}$. $\square_{4.3.1}$

4.4 o-amorphous versus o-minimal

The similarity between o-amorphous sets and $\aleph_0$-categorical o-minimal theories was studied extensively in [CT00]. While we cannot add much to those results, we will show that the methods developed above can be used to simplify some of the proofs. We begin by noting that the complete theory of an o-amorphous set is $\aleph_0$-categorical.

Theorem 4.4.1. *Let $(\mathfrak{A}, \leqslant)$ be an o-amorphous set, and let $T = \mathrm{Th}(\mathfrak{A})$. Then T is $\aleph_0$-categorical, o-minimal.*

The proof is similar to that of Theorem 4.3.1.

Proof. Assume that $(\mathfrak{A}, \leqslant)$ is an o-amorphous set. By Lemma 2.5 of [CT00], $\mathfrak{A}$ is weakly Dedekind-finite. Thus the $\aleph_0$-categoricity of T is immediate by Theorem 4.2.2.

If T were not o-minimal, then there would be a formula ϕ that would define a subset of $\mathfrak{A}$ that is not a finite union of intervals and points. Then $\{a \in \mathfrak{A} : (\mathfrak{A}, \leqslant) \models \phi(a)\}$ would counter our assumption that $\mathfrak{A}$ is o-amorphous. $\square_{4.4.1}$

Thus, we can immediately deduce from [SP86] that, for example, no o-amorphous set can carry a group structure, and we can obtain the complete classification of o-amorphous sets in [CT00] from that for $\aleph_0$-categorical o-minimal sets in [SP86].

There is a corresponding relation between weakly o-amorphous sets (as defined in [CT00]) and weakly o-minimal theories (a discussion of which can be found in [HMM$^+$00]), which may be given by the methods above. Treating the easiest case for instance, $\aleph_0$-categorical weakly o-minimal sets may be given by means of a finite chain of refining equivalence relations on a densely linearly ordered set (which may be formed using a suitable amalgamation class for instance by use of Fraïssé's method), and these correspond to one of the classes of weakly o-amorphous sets described in [CT00]. There are more complicated examples also given, but as for the case of amorphous sets, not all can be captured in a first-order way, and one has to choose which language to use to describe them.

4.5 Structures Admitting MT-rank

We begin this section with a discussion of the special, and more well known, case of amorphous sets.

Remark 4.5.1. *Let $\mathfrak{A}$ be an amorphous set that carries a structure axiomatizable in a countable (finite or infinite) language. Then $T = \mathrm{Th}(\mathfrak{A})$ is $\aleph_0$-categorical strongly minimal.*

Remark 4.5.1 is simply a restricted case of Theorem 4.5.2 below, since amorphous sets have MT-rank $(1, 1)$ and strongly minimal theories have Morley rank $(1, 1)$.

The similarities between amorphous sets and strongly minimal theories were extensively studied in [Tru95]. In that paper, the author provided a "classification" of amorphous sets into four cases: bounded, unbounded not of projective type, projective type with a bound on the cardinality of the underlying field, and of projective type with no bound on the underlying field. (For definitions and explanation of these cases, the reader is referred to [Tru95]). $\aleph_0$-categorical strongly minimal sets, or more specifically, the first-order structures of $\aleph_0$-categorical strongly minimal theories, have been "classified" into three cases, based on their underlying geometry: They have either a trivial, affine, or projective geometry. We note that the amorphous sets of bounded type correspond to theories having trivial geometries, while the unbounded amorphous sets of projective type correspond to the affine and projective geometries. The case of unbounded amorphous sets not of projective type does not coincide with any $\aleph_0$-categorical strongly minimal theory, because such a set cannot carry a structure axiomatizable in a countable first-order language.

We now turn to the more general notion of MT-rank.

Theorem 4.5.2. *Let $\mathfrak{A}$ be a set such that $\mathrm{MT}(\mathfrak{A}) = (\alpha, k)$, and assume $\mathfrak{A}$ admits a first-order structure axiomatizable in a countable (finite or countably infinite) language. Let $\mathscr{A}$ be the unique countable model of $T = \mathrm{Th}(\mathfrak{A})$ guaranteed by Theorem 4.2.2. Then, $\mathscr{A}$ has Morley rank, and $\mathrm{MR}(\mathscr{A}) \leqslant \mathrm{MT}(\mathfrak{A}) = (\alpha, k)$, where $\leqslant$ denotes the lexicographic order.*

Before we prove Theorem 4.5.2, we state an immediate corollary:

Corollary 4.5.3. *Let $\mathfrak{A}$ be a set admitting MT rank that carries a structure axiomatizable in a countable (finite or infinite) language. Then $T = T(\mathfrak{A})$ is $\aleph_0$-stable.*

We turn to the proof of Theorem 4.5.2.

Proof. The theory T is $\aleph_0$-categorical by Theorem 4.2.2. Thus, it suffices to calculate Morley rank within $\mathscr{A}$.

We proceed with the proof by induction on the MT-rank of $\mathfrak{A}$.

Assume $\mathfrak{A}$ is amorphous, and thus $(n,k) = (1,1)$. If T is not strongly minimal, then there is a formula φ that definably divides $\mathscr{A}$ into two infinite subsets. Then, both $\exists^{\geqslant n} x\varphi(x) \in T$ and $\exists^{\geqslant n} x\neg\varphi(x) \in T$ for each n. But then, $\{a \in \mathfrak{A} : \mathfrak{A} \models \varphi(a)\}$ violates the amorphousness of $\mathfrak{A}$.

Now, assume that $\mathscr{A}$ does not have Morley rank less than or equal to $(\alpha, k) = \mathrm{MT}(\mathfrak{A})$. Then, there are formulas φ_i that divide $\mathscr{A}$ into $k+1$ subsets not having rank less than α. That is, $\varphi_0[\mathscr{A}], \varphi_1[\mathscr{A}], \ldots, \varphi_k[\mathscr{A}]$ are pairwise disjoint and not of Morley rank $< \alpha$. But then, $\varphi_0[\mathfrak{A}], \varphi_1[\mathfrak{A}], \ldots, \varphi_k[\mathfrak{A}]$ are pairwise disjoint, but by the definition of MT-rank, there is some $\varphi_i[\mathfrak{A}]$ having MT-rank $< \alpha$. By the inductive hypothesis, $\varphi_i[\mathscr{A}]$ has Morley rank $< \alpha$, a contradiction. $\square_{4.5.2}$

Remark 4.5.4. *Let $\mathfrak{A}$, T be as in Theorem 4.5.2. Then,* $\mathrm{MR}(\mathscr{A}) < \omega$.

Proof. $\mathscr{A}$ is $\aleph_0$-categorical and admits Morley rank. From [Mor65], $\mathscr{A}$ is $\aleph_0$-stable. By one of the main results of [CHL85], $\mathrm{MR}(\mathscr{A}) < \omega$. $\square_{4.5.4}$

We remark that all of the results cited in the proof of Remark 4.5.4 use AC, and so we must justify why we can appeal to these results: $\mathscr{A}$ is itself a (well-ordered) countable structure. So, even if the universe we are working in does not satisfy AC, we can nonetheless carry out all the stability theoretic arguments in a subuniverse containing $\mathscr{A}$ that *does* satisfy AC, for example, $L[T]$.

All theorems concerning first-order definable substructures of $\aleph_0$-categorical structures with Morley rank apply to structures admitting MT-rank. For example, there can be no infinite MT-ranked fields or boolean algebras. Likewise, through the work in [Fel78] and [BCM79], we can deduce the following:

Theorem 4.5.5. *A group $\mathfrak{A}$ whose domain admits MT-rank is abelian-by-finite.*

Proof. The statement of Theorem 4.5.5 follows from the fact that important group theoretic subgroups of an $\aleph_0$-categorical group are all first-order definable. Thus, these structures are still present in non-choice models, since first-order properties are absolute in this context. In particular, if a group $\mathfrak{A}$ has MT-rank, then $\mathrm{Th}(\mathfrak{A})$ is $\aleph_0$-stable. The unique countable model $\mathscr{A}$ of $\mathrm{Th}(\mathfrak{A})$ is thus abelian-by-finite. An $\aleph_0$-categorical abelian-by-finite group has a 0-definable abelian subgroup of finite index ([BCM79]). Because the sentence defining the subgroup has no parameters and is first order, it is also satisfied by $\mathfrak{A}$. $\square_{4.5.5}$

There are cases in which the two ranks are not equal: Let $\mathfrak{A} = \mathfrak{B} \times \mathfrak{B}$ be the direct product of two isomorphic amorphous elementary abelian p-groups. Then, $\mathrm{MT}(\mathfrak{A}) = (2,1)$, but $\mathrm{MR}(\mathscr{A}) = (1,1)$. Thus, the inequality in the statement of the above theorem cannot be improved upon. We can even build a model that contains a group of arbitrarily large MT-rank, but whose theory is strongly minimal. An example of such a construction is demonstrated in Model Construction 4.5.6 below.

Questions about the possible algebraic structures on sets that admit MT-rank initiated the work presented in this chapter. Some work had been done previously to this end: partial orders, and hence boolean algebras were considered in [MT03], and amorphous groups were briefly studied in [Tru95]. The remark in the latter work, that any amorphous group must be elementary abelian of prime exponent, was the primary inspiration for these considerations. Although that result was discovered independently, its proof was very similar to that of the corresponding model-theoretic result about strongly minimal $\aleph_0$-categorical groups (a proof can be found in, for example, [Poi01]). This, in a sense, hinted at a parallel development.

We outline an example of such parallel development: Given a group $\mathfrak{G}$ that has an MT-rank, one can develop much of the standard machinery for analyzing $\aleph_0$-categorical groups of finite Morley rank quite independently. For example, one can define the *connected component* $\mathfrak{G}^\circ$ of $\mathfrak{G}$ to be the intersection of all subgroups of finite index, and prove that $|\mathfrak{G} : \mathfrak{G}^\circ|$ is finite. Furthermore, $\mathfrak{G}^\circ$ has MT-degree 1 (and of course, the same MT-rank as $\mathfrak{G}$). More details further to this can be found in the Appendix, and in particular, A.4.1.

Considerations such as those outlined above give a possible route to an analysis of groups admitting MT-rank. However, ultimately the more efficient procedure described in this chapter is to be preferred. We do note that there are certain distinctions which apply in the non-choice, MT-ranked case which are obscured by these techniques. These distinctions arise from second-order features of the structures, some of which may have first-order consequences. An example of this is the fact that a group whose MT-rank is greater that its Morley rank can be constructed. Thus, MT-rank and Morley rank are not the same, as Morley rank only measures first-order properties, while MT-rank takes into account some vestigial second order information from the model construction. Another example of this lies in the possible structures of MT-rank 1 groups. Here, the structures largely depend on the automorphisms of the (amorphous) connected component of the group that are admissible in the model, and a wide variety of behaviours is possible. These need not be the same as in models where the Axiom of Choice is assumed. We now turn to the promised model construction. We use the Fraenkel-Mostowski construction method, as outlined in Section 3.2.

We construct a model containing a group of MT-rank α, for α an ordinal.

Model Construction 4.5.6 (A group having infinite MT-rank)**.** Let $V = \{v_k : k \in \omega\}$ be an $\aleph_0$-dimensional vector space over a finite field of prime order p with an additive group structure (i.e. V is an $\aleph_0$-dimensional elementary abelian p-group). We intend to construct a Fraenkel-Mostowski model that utilizes V in defining an indexing set for the set of atoms.

Indexing the set of atoms: Let V_i, $i \in \alpha$, be isomorphic copies of V, and let $\mathbf{b_i} = (b_{0i}, b_{1i}, \ldots)$ be a fixed basis of V_i. For each $\beta \leqslant \alpha$, let

$$\Upsilon_\beta = \{0_V\} \times \operatorname*{Dr}_{i\in\beta} V_i,$$

where $\mathrm{Dr}_{i\in\beta} V_i$ denotes the restricted external direct product of the V_i. Note that each Υ_β has cardinality $\max(\aleph_0, |\beta|)$. Denote by b_{ji}^β the vector

$$b_{ji}^\beta = (0_V, 0_V, \ldots, 0_V, b_{ji}, 0_V, \ldots) \in \Upsilon_\beta,$$

where the b_{ji} is on the i-th coordinate. Clearly, each Υ_β has a locally finite structure.

Let U be a set of atoms of cardinality $\max(\aleph_0, |\alpha|)$ indexed by $\bigcup_{\beta\leqslant\alpha} \Upsilon_\beta$. Let

$$U_\beta = \{u_\upsilon \in U : \upsilon \in \Upsilon_\beta\}.$$

For notational ease, we will refer to atoms by their index when this will not result in confusion.

Defining a permutation group: Let $\mathscr{G}$ be the group of permutations of U with the following properties:

(i) The sets U_β, $\beta \leqslant \alpha$ are fixed.

(ii) For $g \in \mathscr{G}$, the cardinality of the *basic support* S of g is finite:

$$|S| = |\{b_{ji}^\beta : g(b_{ji}^\beta) \neq b_{ji}^\beta, \beta \in \alpha\}| < \omega$$

(iii) The group acts as a linear automorphism on each U_β.

Defining a normal filter: For the purposes of defining a filter, we define the following functions: Let

$$F_\beta : U_\beta \longrightarrow U_{\beta+1}$$

be given by

$$F_\beta(u_\upsilon) = u_{\upsilon\smile 0},$$

where $\upsilon \in \Upsilon_\beta$ and $\upsilon\smile 0 \in \Upsilon_\beta \times \{0\} \subset \Upsilon_{\beta+1}$. For limit ordinals γ, let

$$F_{\beta\gamma} : U_\beta \longrightarrow U_\gamma$$

be given by

$$F_{\beta\gamma}(u_\upsilon) = u_{(\upsilon,\underbrace{0_V, 0_V, 0_V, \ldots}_{\gamma-\beta})}.$$

In the following, for $g \in \mathscr{G}$ and F a function F_β or $F_{\beta\gamma}$, gF denotes the image of F under g, i.e. $\{(gx, gy) : (x, y) \in F\}$.

Let $\mathscr{F}$ be the filter of subgroups generated by

$$\{\mathscr{G}_A : A \subset U, |A| < \omega\} \cup \{\mathscr{G}_{\{g^{-1}F_\beta\}} : \beta \in \alpha, g \in \mathscr{G}\} \cup \\ \{\mathscr{G}_{\{g^{-1}F_{\beta\gamma}\}} : \beta \in \alpha, \gamma \in Lim \text{ and } \gamma \leqslant \alpha, g \in \mathscr{G}\}. \quad (4.1)$$

This filter is clearly normal, as $\mathscr{G}_{\{g^{-1}F\}} = g^{-1}\mathscr{G}_F g$, where F is one of F_β or $F_{\beta\gamma}$. Let $\mathscr{N}_{MT_\alpha}$ be the resulting Fraenkel-Mostowski model.

In $\mathscr{N}_{MT_\alpha}$, each U_β is still a vector space over the field of order p, and F_β, $F_{\beta\gamma}$ are linear transformations from U_β to $U_{\beta+1}$, and from U_β to U_γ, respectively. There is, however, no set in the model which gives all these embeddings 'simultaneously'.

Theorem 4.5.7. *In $\mathscr{N}_{MT_\alpha}$, $\mathrm{MT}(U_\alpha) = (\alpha, 1)$.*

Proof. We prove that $\mathrm{MT}(U_\alpha) = (\alpha, 1)$, where U_α is as in the preceding model construction, by induction on $\beta \leqslant \alpha$.

For $\beta = 0$. $|U_0| = 1$.

We show the inductive step in two parts. We first show that $\beta \leqslant \mathrm{MT}(U_\beta)$, and then we show equality.

Assume $\beta \leqslant \alpha$ and that $\mathrm{MT}(U_\delta) = (\delta, 1)$ for all $\delta < \beta$. If $\beta = \delta + 1$ is a successor, then for any n, U_β has n pairwise disjoint subsets of MT-rank δ, namely, $F_\delta[U_\delta] + b^{\beta+1}_{j(\beta+1)}$ under for $j < n$. Hence, U_β does not have MT-rank less than β. If β is a limit ordinal, then for any $\delta < \beta$, U_β has a subset, namely $F_{(\delta+1)\beta}U_{(\delta+1)}$ of MT-rank greater than δ.

We still must show that U_β has MT-rank exactly β and MT-degree 1: Let $X \subseteq U_\beta$ be any subset in $\mathscr{N}_{MT_\alpha}$. Then, by the definition of the model, $\mathscr{G}_{\{X\}} \in \mathscr{F}$, and thus

$$\mathscr{G}_{\{X\}} \geqslant \mathscr{G}_A \cap \bigcap_{k<l} \mathscr{G}_{\{g_k^{-1}F_{\delta_k}\}} \cap \bigcap_{m<n} \mathscr{G}_{\{g_m^{-1}F_{\delta_m\gamma_m}\}}, \tag{4.2}$$

with $l, n \in \omega$, and $g_k, g_m \in \mathscr{G}$, and A a finite set. We assume without loss of generality that the set A is closed under the functions $g_k^{-1}F_{\delta_k}$ and $g_k F_{\delta_k}^{-1}$ for $k < l$ and $g_m^{-1}F_{\delta_m\gamma_m}$ and $g_m F^{-1}_{\delta_m\gamma_m}$ for $m < n$.

If $\beta = \delta + 1$ is a successor, let

$$Y = \langle (A \cap U_\beta) \cup \bigcup \{ g_k^{-1} F_{\delta_k} U_\delta : \delta_k = \delta \} \rangle,$$

while if $\beta \in Lim$, let

$$Y = \langle (A \cap U_\beta) \cup \bigcup \{ g_m^{-1} F_{\delta_m\gamma_m} U_{\delta_m} : \gamma_m = \beta \} \rangle.$$

Since each g_k has only finite support, Y is generated by the image of U_δ under at most one of the maps together with a finite set. Thus, because the field is finite, Y can be written as a finite union of sets of MT rank at most δ_k, whence $\mathrm{MT}(Y) < \beta$. We show that either $X \subseteq Y$ or $X \supseteq U_\beta \setminus Y$. Thus, we will show that U_β has MT-rank exactly β and degree 1 by showing that for any two disjoint subsets of U_β, one has MT-rank $< \beta$.

Let $u_\upsilon, u_\tau \in U_\beta \setminus Y$. Let $g \in \mathscr{G}$ be such that $g(u_\upsilon) = u_\tau$ and $g(u_\tau) = u_\upsilon$, and

$g(F(u_\upsilon)) = F(u_\tau)$ and $F(u_\tau) = F(u_\upsilon)$, where F ranges over all the (finitely many) possible compositions of the $g_k^{-1}F_{\delta_k}$ and $g_m^{-1}F_{\delta_m\gamma_m}$; g fixes all points outside the finitely-generated and hence finite set $\langle u_\upsilon, u_\tau, F(u_\upsilon), F(u_\tau)\rangle$. Then,

$$g \in \mathscr{G}_A \cap \bigcap_{k<l} \mathscr{G}_{\{g_k^{-1}F_{\delta_k}\}} \cap \bigcap_{m<n} \mathscr{G}_{\{g_m^{-1}F_{\delta_m\gamma_m}\}},$$

so, by 4.2, g preserves X, and hence $u_\upsilon \in X \Leftrightarrow u_\tau \in X$, giving the result. $\square_{4.5.7}$

We can apply the Jech-Sochor Embedding theorem to get an analogous result for ZF.

Chapter 5

Methods from Infinitary Logic

5.1 Introduction

We now turn our attention to methods beyond those of first-order logic for analysing the structures possible on Dedekind-finite sets. We will primarily have in mind certain Dedekind-finite sets that are not weakly Dedekind-finite. However, the results herein directly generalise those results on weakly Dedekind-finite sets presented elsewhere in this thesis.

For much of the following discussion, we assume that ω_1 is regular, and will restrict ourselves to sets in $\Delta_{*\omega_1}$. However, we suspect that many of the results might be extended to $\Delta_{*\alpha}$, where α is an arbitrary, or at least regular, ordinal.

Let $\mathfrak{A} \in \Delta_{*\omega_1}$ be a Dedekind-finite structure axiomatisable using a language with a vocabulary that is at most countable. We aim to present an analysis of this structure using methods from infinitary logic. We will show that structures with domains in $\Delta_{*\omega_1}$ can be uniquely correlated with countable structures with certain properties. In the literature, one can find numerous examples of comparison of various structures across set theoretic universes in this vein, however, the author has not seen these methods used for comparisons of non-choice universes with universes satisfying choice.

5.2 Setting the stage

In this section, we will define the infinitary notions to which we refer later, and develop the infinitary tools necessary for our considerations.

5.2.1 Potential isomorphisms and $L_{\omega_1\omega}$

We begin by recalling a few notions. Our exposition here follows that of [KK04, Kei71].

By L we denote a first-order language having at most countably many symbols. We will refer to the collection of symbols as the *vocabulary* of L, and in an abuse of notation, denote it as L. By $L_{\omega_1\omega}$ we denote the language with the same vocabulary as L, but allowing the conjunction and disjunction of a set of fewer than ω_1 formulas, with $< \omega_1$ individual variables, and allowing universal and existential quantification on a finite set of variables. The language $L_{\omega_1\omega}$ can thus appropriately be built from the atoms of L. By *sentence* we will mean a formula having no free variables. Finally, we will assume that $2^{\aleph_0}$ and hence the language $L_{\omega_1\omega}$ is well-orderable.

Like first-order logic, $L_{\omega_1\omega}$ is complete (see Chapter 4 of [Kei71]). The model theory of $L_{\omega_1\omega}$ is most notably different from that of classical first-order logic in that the Compactness Theorem fails. Thus, we will have to perform some gymnastics to make up for this loss.

Let $\mathcal{A}, \mathcal{B}$ be structures with universes A; B, respectively. We let $\mathcal{A} \equiv \mathcal{B}$ and $\mathcal{A} \equiv_{L_{\omega_1\omega}} \mathcal{B}$ mean that the structures satisfy exactly the same sentences of L and $L_{\omega_1\omega}$ respectively. A *partial isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair of tuples $(\bar{a}, \bar{b})$, of the same finite length, such that $\bar{a} \subset A$, $\bar{b} \subset B$, and $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free $L_{\omega_1\omega}$-formulas. We note that $(\bar{a}, \bar{b})$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ if and only if the empty pair $(\emptyset, \emptyset)$ is a partial isomorphism from $(\mathcal{A}, \bar{a})$ to $(\mathcal{B}, \bar{b})$. A *back-*

and-forth family for $\mathcal{A}$ and $\mathcal{B}$ is a set $\mathcal{P}$ of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ such that:

- $\mathcal{P} \neq \emptyset$;
- for each $(\bar{a}, \bar{b}) \in \mathcal{P}$ and $c \in A$, there exists $d \in B$ such that $(\bar{a}c, \bar{b}d) \in \mathcal{P}$;
- for each $(\bar{a}, \bar{b}) \in \mathcal{P}$ and $d \in B$, there exists $c \in A$ such that $(\bar{a}c, \bar{b}d) \in \mathcal{P}$.

We say that two structures $\mathcal{A}$ and $\mathcal{B}$, of arbitrary cardinality, are *potentially isomorphic* if there is a back-and-forth family for $\mathcal{A}$ and $\mathcal{B}$.

Theorem 5.2.1 (Karp's Theorem)**.** *Let $L_{\infty\omega}$ denote the language that allows disjunction and conjunction over arbitrarily large sets of formulas and quantification over finite sets, and assume that this language is well-orderable.*

A structure $\mathcal{A}$ is potentially isomorphic to another structure $\mathcal{B}$ if and only if they are $L_{\infty\omega}$-elementarily equivalent.

We present this proof much as given in in [KK04].

Proof. Assume first that $\mathcal{A}$ and $\mathcal{B}$ are potentially isomorphic. Note that if $\mathcal{P}$ is a back-and-forth family witnessing the potential isomorphism of $\mathcal{A}$ and $\mathcal{B}$ and $(\bar{a}, \bar{b}) \in \mathcal{P}$, then $(\mathcal{A}, \bar{a})$ and $(\mathcal{B}, \bar{b})$ are potentially isomorphic. By induction on complexity of formulas, it is clear that for all $(\bar{a}, \bar{b}) \in \mathcal{P}$, $\mathcal{A} \models \phi(\bar{a})$ iff $\mathcal{B} \models \phi(\bar{b})$.

Now, let as assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of $L_{\infty\omega}$. We define a back-and-forth family $\mathcal{P}$ consisting of the pairs $(\bar{a}, \bar{b})$ such that the $L_{\infty\omega}$-formulas satisfied by $\bar{a}$ in $\mathcal{A}$ are the same as those satisfied by $\bar{b}$ in $\mathcal{B}$. Let $(\bar{a}, \bar{b}) \in \mathcal{P}$, and let $c \in A$. We need d such that $(\bar{a}c, \bar{b}d) \in \mathcal{P}$. For each $d \in B$ such that $(\bar{a}c, \bar{b}d) \notin \mathcal{P}$, choose a formula $\phi_d(\bar{u}, x)$ satisfied by $\bar{a}c$ in $\mathcal{A}$ but not by $\bar{b}d$ in $\mathcal{B}$. (We have assumed that the language is well-orderable, and can thus take the first suitable formula.) Let $\psi(\bar{u}, x)$ be the conjunction of the formulas $\phi_d(\bar{u}, x)$. Since $\exists x \psi(\bar{u}, x)$ is true of $\bar{a}$ in $\mathcal{A}$, it is true of $\bar{b}$ in $\mathcal{B}$. Taking d such that $\mathcal{B} \models \psi(\bar{b}d)$, we get $(\bar{a}c, \bar{b}d) \in \mathcal{P}$. $\square_{5.2.1}$

By following the proof of Karp's Theorem 5.2.1, we immediately arrive at Corollary 5.2.2. The assumption below that we are comparing a structure that is not mappable onto ω_1 to a countable structure allows us to limit ourselves to $L_{\omega_1\omega}$-equivalence.

Corollary 5.2.2. *Let $\mathfrak{A}$ be a structure whose domain A is such that $|A| \in \Delta_{*\omega_1}$, and let $\mathscr{B}$ be a countable structure. Then $\mathfrak{A}$ and $\mathscr{B}$ are potentially isomorphic if and only if $\mathfrak{A} \equiv_{L_{\omega_1\omega}} \mathscr{B}$.*

Proof. Assume the structures $\mathfrak{A} \in \Delta_{*\omega_1}$ and $\mathscr{B}$ countable are potentially isomorphic and let $\mathcal{P}$ be a back-and-forth family witnessing this potential isomorphism. We proceed to show by a standard argument by induction on complexity of formulas ϕ of $L_{\omega_1\omega}$ that if $(\bar{a}, \bar{b}) \in \mathcal{P}$, then $\mathcal{A} \models \phi(\bar{a})$ iff $\mathcal{B} \models \phi(\bar{b})$. For ϕ atomic we have the implication by assumption that $\mathcal{P}$ is a back-and-forth family. Again, the case is clear for negations, and countably many conjunctions and disjunctions of formulas already shown to satisfy the implication. Suppose we have shown the implication for $\phi(x_1, \ldots, x_n)$. Let $(a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}) \in \mathcal{P}$. By the properties of the back-and-forth property, and the inductive hypothesis, the following are equivalent:

$\mathfrak{A} \models \exists x_n \phi[a_1, \ldots, a_{n-1}]$.

$\mathfrak{A} \models \phi[a_1, \ldots, a_{n-1}, a]$ for some $a \in A$

$\mathfrak{A} \models \phi[a_1, \ldots, a_{n-1}, a]$ for some a where $(a_1, \ldots, a_{n-1}, a, \bar{b}) \in \mathcal{P}$.

$\mathscr{B} \models \phi[b_1, \ldots, b_{n-1}, b]$ for some b where $(a_1, \ldots, a_{n-1}, a, b_1, \ldots, b_{n-1}, b) \in \mathcal{P}$.

$\mathscr{B} \models \phi[b_1, \ldots, b_{n-1}, b]$ for some $b \in B$

$\mathscr{B} \models \exists x_n \phi[b_1, \ldots, b_{n-1}]$.

Note that $\forall x_n \phi$ is equivalent to $\neg\exists x_n \neg\phi$. Thus we have shown that the two potentially isomorphic structures satisfy the same $L_{\omega_1\omega}$ sentences.

Note that because neither A nor B can be mapped onto ω_1, a game that corresponds to the back-and-forth family can have length that is at most countable. Thus, the formulas ϕ in the proof above cannot have quantifier rank $\geqslant \omega_1$, and so must be formulas of $L_{\omega_1\omega}$.

Now, let us assume that $\mathfrak{A}$ and $\mathscr{B}$ satisfy the same sentences of $L_{\omega_1\omega}$. We define a back-and-forth family $\mathcal{P}$ consisting of the pairs $(\bar{a}, \bar{b})$ such that the $L_{\omega_1\omega}$-formulas satisfied by $\bar{a}$ in $\mathfrak{A}$ are the same as those satisfied by $\bar{b}$ in $\mathscr{B}$. First of all, we allow $(\bar{a}, \bar{b}) \in \mathcal{P}$ if the pair extends to an isomorphism of a finitely generated structure $\mathfrak{A}_0$ onto a finitely generated structure $\mathscr{B}_0$. (Note that $\mathfrak{A}_0$ and hence $\mathscr{B}_0$ will both be finite by 5.3.1 below.) Now we can show that $\mathcal{P}$ is a back-and-forth family just as in the proof of Theorem 5.2.1. $\square_{5.2.2}$

Thus, a countable structure and a $\Delta_{*\omega_1}$ structure that are potentially isomorphic are very similar on a deeper level, despite their apparent differences. This is well illustrated by Theorem 5.2.3.

Theorem 5.2.3. *Let $\mathfrak{A}$ be a structure whose domain is in $\Delta_{*\omega_1}$, and let $\mathscr{A}$ be a countable structure that is potentially isomorphic to $\mathfrak{A}$. Then, there exists a generic extension of the set theoretical universe $\mathscr{M}$ in which $\mathfrak{A}$ is countable and isomorphic to $\mathscr{A}$.*

Proof. If $\mathfrak{A}$ and $\mathscr{A}$ are potentially isomorphic, then there exists a back-and-forth family $\mathcal{P}$ for the two structures. We can then take $\mathcal{P}$ to be a notion of forcing ordered by extension. Let G be an $\mathscr{M}$-generic set, and let $\mathscr{M}[G]$ be the generic extension of the model $\mathscr{M}$. Then $\bigcup G = f$ is a one-to-one function from a subset of $\mathfrak{A}$ into $\mathscr{A}$ and is in the model $\mathscr{M}[G]$. However, for all $a \in \mathfrak{A}$, $\{p \in \mathcal{P} : a \in \mathrm{dom}(p)\}$ is a dense open set. Hence, the domain of f is the whole of $\mathfrak{A}$, and similarly its range is the whole of $\mathscr{A}$, so f is a bijection. Thus, f provides an enumeration of the domain of $\mathfrak{A}$ showing that both $\mathfrak{A}$ and $\mathscr{A}$ are countable in the universe $\mathscr{M}[G]$. Because pairs of tuples in the back-and-forth family satisfy the same quantifier-free formulas, one can show by induction on the the complexity of formulas that f is also an isomorphism.

Notice that the proof does not depend on the choice of G. $\square_{5.2.3}$

We have shown that a countable structure and a $\Delta_{{}^*\omega_1}$ structure that are potentially isomorphic are in some sense equivalent, and that the differences depend largely on the set theoretical universe. Our next goal is to find a canonical way of identifying a countable structure that is potentially isomorphic to a given $\Delta_{{}^*\omega_1}$ structure.

5.2.2 Language fragments and Scott sentences

We will show that it is possible to define a Scott sentence and Scott rank for the structure $\mathfrak{A}$. We begin by defining the language, language fragments, and notions we will refer to later. Here, our exposition mirrors that of [Sac83].

First, we give a few words on language fragments. The language $L_{\omega_1\omega}$ can be problematic in that it may contain uncountably many formulas. To alleviate this problem, we introduce the notion of a fragment, which gives us a nicely manageable countable subset of $L_{\omega_1\omega}$ within which we will work.

Assume that the vocabulary L in which we work is at most countable. We can assume that each of the symbols of L is a natural number, and that each variable v_α, $\alpha \in \omega_1$ is the pair $\langle v, \alpha \rangle$. The atomic formulas of $L_{\omega_1\omega}$ are finite sequences of symbols, and so we consider them as sets containing the natural numbers associated with the symbols of which they are composed. In this manner, we can assume that each formula of $L_{\omega_1\omega}$ is a set. In an a minor abuse of notation, we let $L_{\omega_1\omega}$ denote the set of all codes of formulas of $L_{\omega_1\omega}$. For any set $\mathscr{C}$ of natural numbers, let $L_\mathscr{C} = L_{\omega_1\omega} \cap \mathscr{C}$. We call $L_\mathscr{C}$ a *fragment* of $L_{\omega_1\omega}$ iff

- $\mathscr{C}$ is a non-empty transitive set.

- If $c, d \in \mathscr{C}$ then $\{c, d\} \in \mathscr{C}, c \cup d \in \mathscr{C}, c \times d \in \mathscr{C}$.
- If $c \in \mathscr{C}$ and α is the least ordinal which is not in the transitive closure of c, then $\alpha \in \mathscr{C}$.
- If $\phi(x, \ldots) \in L_{\mathscr{C}}$ and $t \in \mathscr{C}$ is a term of $L_{\omega_1\omega}$, then $\phi(t, \ldots) \in L_{\mathscr{C}}$.

We have the following standard facts about fragments of $L_{\omega_1\omega}$: For any fragment $L_{\mathscr{C}}$,

1. $\omega \subset \mathscr{C}$.
2. $L_{\mathscr{C}}$ is closed under negation, finite conjunctions and disjunctions.
3. If $\varphi \in L_{\mathscr{C}}$ and $x \in \mathscr{C}$, then $\forall x\, \varphi \in L_{\mathscr{C}}$ and $\exists x\, \varphi \in L_{\mathscr{C}}$.
4. If $\varphi \in L_{\mathscr{C}}$, then every subformula of φ is in $L_{\mathscr{C}}$.
5. If $\psi(x_1, \ldots, x_n)$ is an atomic formula of L and the finite tuple $x_1, \ldots, x_n \in \mathscr{C}$, then $\psi(x_1, \ldots, x_n) \in L_{\mathscr{C}}$.
6. If Φ is a countable set of formulas and $\Phi \in \mathscr{C}$, then $\bigwedge \Phi \in L_{\mathscr{C}}$, $\bigvee \Phi \in L_{\mathscr{C}}$.
7. For every countable set $\Phi \in L_{\omega_1\omega}$, there is a least countable fragment $L_{\mathscr{C}}$ such that $\Phi \subset L_{\mathscr{C}}$.
8. Countable fragments of $L_{\omega_1\omega}$ are complete: if ϕ is a sentence of a countable fragment $L_{\mathscr{C}}$ of $L_{\omega_1\omega}$, then ϕ is provable with statements from $L_{\mathscr{C}}$ if and only if $\models \phi$.

Proofs for the above can be found in any standard text on infinitary logic, for example [Kei71].

We assume that the vocabulary, of the language is at most countable. Let $L(0, \mathfrak{A})$ be the $L_{\omega\omega}$ (first-order) language whose primitive symbols correspond to the relations, function, and distinguished elements of $\mathfrak{A}$. Note that this is a countable set of formulas. For each limit ordinal λ, let $L(\lambda, \mathfrak{A})$ be

$$\bigcup\{L(\delta, \mathfrak{A}) : \delta < \lambda\}.$$

For each δ, a successor ordinal, let $T(\delta, \mathfrak{A})$ be the complete theory of $\mathfrak{A}$ in the language $L(\delta, \mathfrak{A})$. Let $L(\delta+1, \mathfrak{A})$ be the least (countable) fragment of $L_{\omega_1\omega}$ that includes $L(\delta, \mathfrak{A})$ and satisfies the following closure conditions:

if $n < \omega$ and $p(x_1, \ldots, x_n)$ is a non-principal n-type of $T(\delta, \mathfrak{A})$ realised in $\mathfrak{A}$, then $\bigwedge\{F(\bar{x}) : F(\bar{x}) \in p\}$ belongs to the fragment.

Please note that all of the fragments defined above are countable.

Lemma 5.2.4. *Let $\mathfrak{A}$ be such that its domain $|A| \in \Delta_{*\omega_1}$, and assume that ω_1 is regular. Then, there exists a countable ordinal δ such that $L(\gamma, \mathfrak{A}) = L(\delta, \mathfrak{A})$ for all $\gamma > \delta$.*

Proof. Let

$$X = \{(\bar{x}, \bar{y}) : \bar{x}, \bar{y} \text{ finite one-to-one sequences of elements of } A, |\bar{x}| = |\bar{y}|\}.$$

Note that $X \in \Delta_{*\omega_1}$ since this class is closed under $\times$ and subsets.

If $L(\delta + 1, \mathfrak{A}) \neq L(\delta + 2, \mathfrak{A})$, then there exists a pair of n-tuples of A that are equivalent with respect to all n-ary formulas of $L(\delta, \mathfrak{A})$, but inequivalent with respect to some formula of $L(\delta + 1, \mathfrak{A})$. If a pair of tuples is inequivalent at some ordinal δ, it is clearly inequivalent for all $\gamma > \delta$.

Now, we show that the first ordinal at which equality of the language fragments occurs is countable. For each $(\bar{x}, \bar{y}) \in X$ such that $\bar{x}$ and $\bar{y}$ are inequivalent at some stage, we map this pair to the first stage where the inequivalence is exhibited. Thus, we map a subset of X into the class of ordinal numbers. The range of this mapping must be an ordinal, say δ. Since X cannot be mapped onto ω_1, and we have assumed ω_1 to be regular, δ must be countable. $\square_{5.2.4}$

Definition 5.2.5. Let $\mathfrak{A}$ be such that its domain $|A| \in \Delta_{*\omega_1}$. Let $r(\mathfrak{A})$ be the least ordinal δ such that $L(\delta, \mathfrak{A}) = L(\delta + 1, \mathfrak{A})$. We call $r(\mathfrak{A})$ the *Scott rank* of $\mathfrak{A}$.

The canonical *Scott sentence* $F_{\mathfrak{A}}$ for $\mathfrak{A}$ is a sentence of $L_{\omega_1\omega}$ that asserts "I am an atomic model of $T(r(\mathfrak{A}), \mathfrak{A})$".

Thus, it is clear from Corollary 5.2.2 that the Scott sentence of a $\Delta_{*\omega_1}$ structure $\mathfrak{A}$ is the same as the Scott sentence of any countable structure $\mathscr{A}$ to which $\mathfrak{A}$ is potentially isomorphic. Furthermore, the Scott ranks of $\mathfrak{A}$ and $\mathscr{A}$ are equal, and thus do not depend on the choice of the generic set G in the proof of Theorem 5.2.3. It should be noted that not all non-countable structures have a Scott sentence. It is our assumption that $\mathfrak{A}$ cannot be mapped onto ω_1 that guarantees this.

5.2.3 The existence of a companion

We are, by the above discussion, able to determine the Scott sentence $F_{\mathfrak{A}}$ for a structure $\mathfrak{A}$. Now, we show that there exists a countable model that satisfies $F_{\mathfrak{A}}$. First, we introduce some needed notation and notions.

Let C be a countable set of new constant symbols, and let M be the vocabulary formed by adding each $c \in C$ to L. We can then form the languages M and $M_{\omega_1\omega}$ as before. We do this so that in the following considerations, we can deal only with sentences, and not with formulas that have free variables.

If ϕ is a formula of $M_{\omega_1\omega}$, then by $\phi\neg$ we mean a formula that is obtained by moving the negation inside. This is done on increasing complexity of formulas: if ϕ is atomic, then $\phi\neg = \neg\phi$. Also, $(\neg\phi)\neg$ is ϕ, $(\bigwedge_{\phi\in\Phi}\phi)\neg$ is $\bigvee_{\phi\in\Phi}\neg\phi$, $(\forall x\phi)\neg$ is $\exists x\neg\phi$, disjunction and existential quantification formulas are defined analogously.

Let S be a set of countable sets of sentences of $M_{\omega_1\omega}$. S is said to be a *consistency property* iff for each $s \in S$, all of the following hold:

1. (Consistency Rule) Either $\phi \notin s$ or $(\neg\phi) \notin s$.

2. ($\neg$-rule) If $(\neg\phi) \in s$ then $s \cup \{\phi\neg\} \in S$.

3. ($\bigwedge$-rule) If $(\bigwedge \Phi) \in s$ then for all $\phi \in \Phi$, $s \cup \{\phi\} \in S$.

4. ($\forall$-rule) If $(\forall x\, \phi(x)) \in S$, then for all $c \in C$, $s \cup \{\phi(c)\} \in S$.

5. ($\bigvee$-rule) If $(\bigvee \Phi) \in s$, then for some $\phi \in \Phi$, $s \cup \{\phi\} \in S$.

6. ($\exists$-rule) If $(\exists x\, \phi(x)) \in s$, then for some $c \in C$, $s \cup \{\phi(c)\} \in S$.

7. (Equality Rules) Let t be either a constant symbol or a term of the form $F(c_1, \ldots, c_n)$ where $c_1, \ldots, c_n \in C$ and F is a function symbol of L. Let $c, d \in C$.

   - If $(c = d) \in s$, then $s \cup \{d = c\} \in S$.
   - If $c = t$, $\phi(t) \in s$, then $s \cup \{\phi(c)\} \in S$.
   - For some $e \in C$, $s \cup \{e = t\} \in S$.

The following Makkai Model Existence Theorem 5.2.6 is needed for a construction of the countable model we are looking for. We give this theorem without proof, however one can be found in Chapter 3 of [Kei71].

Theorem 5.2.6 (Makkai Model Existence Theorem)**.** *If S is a consistency property and $s_0 \in S$, then s_0 has a countable (though possibly finite) model.*

Theorem 5.2.7 (Companion Existence)**.** *Let $\mathfrak{A}$ be a $\Delta_{*\omega_1}$-structure, and let $F_{\mathfrak{A}}$ be its Scott sentence. Then there exists a countably infinite model $\mathscr{A}$, unique up to isomorphism, that shares the Scott sentence $F = F_{\mathscr{A}} = F_{\mathfrak{A}}$, has the same Scott rank $r = r(\mathscr{A}) = r(\mathfrak{A})$, and is potentially isomorphic to $\mathfrak{A}$.*

Proof. Denote by $L_{\mathscr{C}}$ the countable fragment $L(r(\mathfrak{A}), \mathfrak{A})$ as defined in the construction of the Scott sentence $F_{\mathfrak{A}}$. Let C be a countable set of constant symbols which are not in the vocabulary L. Let $M_{\mathscr{C}}$ be the set of all formulas $\phi(x_1, \ldots, x_n, c_1, \ldots, c_m)$

of $M_{\omega_1\omega}$ obtained from formulas $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m) \in L_{\mathscr{C}}$ by replacing each free occurrence of y_i by $c_i \in C$, $1 \leqslant i \leqslant m$. We denote by $S(\delta, \mathfrak{A})$ the set of $M_{\omega_1\omega}$ sentences that correspond to the $L_{\omega_1\omega}$ sentences of $T(\delta, \mathfrak{A})$.

The set

$$S = \{s_0 : s_0 \subset S(\delta, \mathfrak{A}) \text{ for some } \delta \leqslant r(\mathfrak{A}) \text{ and } s_0 \text{ is finite or countable } \}$$

is a consistency property. The conditions for a consistency property can be easily checked, and are satisfied thanks to the definition of $T(\delta, \mathfrak{A})$ as a set of formulas in a countable fragment that are true in the model $\mathfrak{A}$.

We note that the Scott sentence $F_{\mathfrak{A}} \in S$. Denote by Y the least set of sentences such that $F_{\mathfrak{A}} \in Y$; Y is closed under subformulas; if t is a term, $c \in C$, and $\phi(t) \in Y$, then $\phi(c) \in Y$; if $(\neg\phi) \in Y$ then $(\phi\neg) \in Y$; if $c, d \in C$ then $c = d \in Y$. The set Y is countable. Let X be the set of all sentences in Y, and enumerate them so that $X = \{\phi_0, \phi_1, \phi_2, \ldots\}$. Let $Z = \{t_0, t_1, t_2, \ldots\}$ be the set of all constant symbols and terms of the form $F(c_1, \ldots, c_n)$ where $c_1, \ldots, c_n$ are constants and F is a function symbol of L.

We construct a sequence $s_0 \subset s_1 \subset s_2 \subset \ldots$ of elements of S. Let $F_{\mathfrak{A}} = s_0$. We define the rest by induction, such that

- $s_n \subset s_{n+1} \in S$.
- If $s_n \cup \{\phi_n\} \in S$ then $\phi_n \in s_{n+1}$.
- If $s_n \cup \{\phi_n\} \in S$ and $\phi_n = \bigvee \Phi$, then for some $\psi \in \Phi$, $\psi \in s_{n+1}$.
- If $s_n \cup \{\phi_n\} \in S$ and $\phi_n = \exists x\, \phi(x)$, then for some $c \in C$, $\phi(c) \in s_{n+1}$.
- For some $c \in C$, $(c = t_n) \in s_{s+1}$.

We now define the countable model $\mathscr{A}$ of $s_0 = F_{\mathfrak{A}}$. Let $s_\omega = \bigcup_{n<\omega} s_n$. For $c, d \in C$, let $c \sim d$ iff $(c = d) \in s_\omega$. Because the consistency property satisfies the equality rules, $\sim$ is an equivalence relation on C.

Let $\mathscr{A}$ have the universe set $A = \{c/\sim : c \in C\}$. By the equality rules, if $\phi(c_1, \ldots, c_n) \in s_\omega$ and $c_1 \sim d_1, \ldots, c_n \sim d_n$, then $\phi(d_1, \ldots, d_n) \in s_\omega$. Thus, we can interpret the relation, constant, and function symbols of M in $\mathscr{A}$ in the following way. If t is a constant symbol or terms of the form $F(c_1, \ldots, c_n)$ where $c_1, \ldots, c_n$ are constants and F is a function symbol of L, and $c \in C$, then $\models (c = t)$ iff $(c = t) \in s_\omega$. Additionally, if R is an n-placed relation symbol and $c_1, \ldots, c_n \in C$, then $\mathscr{A} \models R(c_1, \ldots, c_n)$ iff $R(c_1, \ldots c_n) \in s_\omega$. These two conditions determine the model $\mathscr{A}$. It can be shown, using the definition of a consistency property, that any atomic or negated atomic sentence in s_ω holds in $\mathscr{A}$ and furthermore, every sentence in s_ω holds in $\mathscr{A}$. Thus, $\mathscr{A}$ is a model of s_ω and hence of s_0.

Note that here the model $\mathscr{A}$ cannot be finite. If it were, then the model $\mathfrak{A}$ itself would be finite, contrary to our assumptions.

It can be easily seen that the sentence $F_{\mathfrak{A}}$ is also a Scott sentence for $\mathscr{A}$. The usual version of Scott's Isomorphism Theorem asserts that if ϕ is a Scott sentence of a countable model $\mathscr{A}$, and there is another countable model $\mathscr{B}$ such that $\mathscr{B} \models \phi$, then $\mathscr{A} \cong \mathscr{B}$. Thus, the model $\mathscr{A}$ is the unique countable model, up to isomorphism, that shares the Scott sentence $F_{\mathfrak{A}}$ of $\mathfrak{A}$. $\square_{5.2.7}$

We can now define the following important notions:

Definition 5.2.8. Let $\mathfrak{A}$ be a $\Delta_{*\omega_1}$ structure and let $\mathscr{A}$ be the countable model as constructed in the proof of the Companion Existence Theorem 5.2.7. We will refer to the countable model $\mathscr{A}$ as the *countable companion model* (or simply, *companion model*) of the model $\mathfrak{A}$.

Let Aut($\mathscr{A}$) be the automorphism group of $\mathscr{A}$ in a "nice" set theoretical universe that contains $\mathscr{A}$ (say, $L[\mathscr{A}]$). We refer to Aut($\mathscr{A}$) as the *companion automorphism group* ComAut($\mathfrak{A}$) of $\mathfrak{A}$.

5.3 Restrictions on the companions

Now that we have identified some tools that can be used in our Dedekind-finite context, we now examine the restrictions on the companion model and companion automorphism group of an $\Delta_{*\omega_1}$ structure.

Remark 5.3.1. *Let $\mathfrak{A}$ be a structure whose domain is Dedekind finite, axiomatisable in a language $L = \{f_i, r_j, c_k : i \in I, j \in J, k \in K\}$, where $I \cup J \cup K$ is well-ordered by $<$. Then, K, the number of constants of the language is at most finite, and $\mathfrak{A}$ is locally finite, i.e. every finitely generated subset is finite.*

Proof. We recall that the substructure generated by some $X \subseteq \mathfrak{A}$, $\langle X \rangle$, is the smallest substructure of $\mathfrak{A}$ containing X. Then $\{c_k\}_{k \in K} \subset \langle X \rangle$. If there were countably infinitely many constant symbols, we could define a countable subset of $\mathfrak{A}$. Thus, the set of constants must be finite. Also, $\langle X \rangle$ is closed under each function of $\{f_i\}_{i \in I}$. However, $\langle X \rangle$ can be constructed internally: Let $X_0 = X$, and let $X_{n+1} = X_n \cup \{f_i(t) : t \in \mathfrak{A}^{[|f_i|]}, i \in I\}$, where $|f_i|$ is the arity of the function f_i. Each of the sets X_n is finite, and together form an ascending chain of sets such that $\langle X \rangle = \bigcup_{n \in \omega} X_n$. However, as $\mathfrak{A}$ is Dedekind-finite, it must be that $\langle X \rangle = X_n$ for some $n \in \omega$ because otherwise one could define a set that can be mapped bijectively onto ω. Hence, $\langle X \rangle$ is finite. $\Box_{5.3.1}$

We will strengthen this result below.

5.3.1 Automorphisms

Remark 5.3.2. *If $\mathfrak{A}$ is an $\Delta_{*\omega_1}$ structure, then $\mathfrak{A}$ has at most finitely many elements that can be $L_{\omega_1\omega}$-defined with a fixed finite set of parameters. In other terms, if $X \subset \mathscr{A}$ is a finite set, the $L_{\omega_1\omega}$-definable closure of X, $\mathrm{dcl}_{L_{\omega_1\omega}}(X)$, is at most finite.*

Proof. Assume otherwise. Then, because the language is assumed to be well-orderable, there is a list of countably many formulas that define elements of $\mathfrak{A}$. Because this list is well orderable, one could then order the set of definable elements of $\mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind-finite, the set must be finite. $\square_{5.3.2}$

There is a close correlation between $L_{\omega_1\omega}$-definable sets of a countable model $\mathscr{A}$, and the orbits of the its automorphism group $\mathrm{Aut}(\mathscr{A})$. For more on this, see Section 4.1 of [Hod93]. This correlation can be extended to the $L_{\omega_1\omega}$-definable sets of a $\Delta_{*\omega_1}$ structure via the companion model and the companion automorphism group.

In [Kue68], Kueker showed the following theorem:

Theorem 5.3.3. *Let $\mathscr{M}$ be a countable structure. The following are equivalent:*

1. *$\mathscr{M}$ has countably many automorphisms.*
2. *$\mathscr{M}$ has fewer than continuum many automorphisms.*
3. *There is some tuple of elements $n_1, \ldots, n_j \in M$ such that $(\mathcal{M}, n_1, \ldots, n_j)$ is rigid.*
4. *There is some tuple of elements $n_1, \ldots, n_j \in M$ such that for each $m \in M$, there is a formula $\phi(x_1, \ldots, x_j, y)$ of $L_{\omega_1\omega}$ such that*

$$M \models \exists! y \phi(n_1, \ldots, n_j, y) \wedge \phi(n_1, \ldots, n_j, m),$$

that is, m is definable from $n_1, \ldots, n_j$ in $\mathscr{M}$ by a formula of $L_{\omega_1\omega}$.

By Remark 5.3.2 and Theorem 5.3.3, we can immediately deduce the following:

Corollary 5.3.4. *Let $\mathfrak{A}$ be a $\Delta_{*\omega_1}$ structure. Then the cardinality of its companion automorphism group* $\mathrm{ComAut}(\mathfrak{A}) = 2^{\aleph_0}$.

It is a well-known fact that if a (countable) structure is $\aleph_0$-homogeneous, then its automorphism group is uncountable. However, the converse is in general not true. We suspect that the converse does hold for companion models.

Conjecture 5.3.5 (Homogeneity)**.** *Let $\mathfrak{A}$ be a $\Delta_{*\omega_1}$ structure, and let $\mathscr{A}$ be its countable companion model. If $\bar{a}$ and $\bar{b}$ are tuples from $\mathscr{A}$ satisfying the same (first order) L-formulas, then there is an automorphism of $\mathscr{A}$ taking $\bar{a}$ to $\bar{b}$. In other terms, $\mathscr{A}$ is $\aleph_0$-homogeneous.*

Conjecture 5.3.5 at first glance looks quite easy to prove, however, we do run into some difficulties. This is evident in the case that $\bar{a}$ and $\bar{b}$ satisfy the same first-order formulas, but the number of elements $L_{\omega_1\omega}$-definable over $\bar{a}$ does not equal the number of elements $L_{\omega_1\omega}$-definable over $\bar{b}$. If c is an element definable over $\bar{a}$, one may find an element d in $\mathscr{A} \setminus \bar{b}$ that satisfies the same first-order formulas, but there is no automatic guarantee that it will satisfy the same $L_{\omega_1\omega}$-formulas. To overcome this obstacle, one would likely need some form of a compactness argument. One way to show this would be to check if the fragments in which we are working are admissible (in the sense of Kripke-Platek set theory). Then, we could apply the Barwise Compactness Theorem to gain the result. One could also attempt a more bare-handed approach.

If the conjecture is false, then an example would be very interesting in that it would give an example of a model $\mathfrak{A}$ that cannot arise from the Plotkin Construction. A possible candidate would be something like a generalised Prüfer group of certain length, for example $\omega(\omega + 1) + 1$. By results contained in [Nad94], we would then expect such an example to have Scott rank $> \omega+1$. Since $\aleph_0$-homogeneous structures necessarily have Scott rank $\leqslant \omega$, the example would then not be homogeneous.

5.4 Plotkin's construction revisited

Now that we have recalled many notions from infinitary logic, we can now return to the Plotkin construction in an infinitary context. Many of the results shown in Section 3.4 can be in some sense extended to structures axiomatised with formulas of $L_{\omega_1\omega}$.

To avoid repetition of the proofs in the earlier section, I redefine some of the notation given in Section 3.3. Many of the proofs as written in Section 3.4 can be carried through with this newly redefined notation in mind.

Let $\mathscr{M}$ be as before, and let $\mathscr{A}$ be a countable structure in $\mathscr{M}$ that is $L_{\omega_1\omega}$-axiomatisable with a countable signature L. By T we mean the Scott sentence for $\mathscr{A}$. Let $L_{\mathscr{C}}$ be the appropriate countable fragment of $L_{\omega_1\omega}$ for T (i.e. $L(r(\mathscr{A}), \mathscr{A})$). Let $F_\infty(L)$ be the set of all formulas of $L_{\mathscr{C}}$. By $F_n(L)$ we denote the set of formulas with at most n free variables. Let $F(L) = F_{<\omega}(L)$. The set $D_n(A)$ is the set of all n-ary relations definable with finitely many parameters from A by $F_n(L)$. All other notation of Section 3.3 remains as given, and we carry out the construction as given there.

Our goal is to identify models $\mathscr{A}$ so that all $D_n(\mathfrak{A})$ are $F(L)$-definable over $\mathfrak{A}$.

In this new context, analogues for Lemmas 3.4.2 and 3.4.3 have fundamentally identical proofs to those given in Section 3.4. To Corollary 3.4.4, we add the following statement:

Corollary 5.4.1. *If $\mathscr{A}$ is a countable $\aleph_0$-homogeneous model with Scott sentence T, then every n-ary relation on $\mathfrak{A}$ has the form of a countable disjoint union $\bigsqcup_{i\in I} \mathfrak{r}_{\psi_i\,\mathfrak{L}}$,* *where I is a countable set indexing the atoms of $F_{n+k}(L)$. Thus, every n-ary relation on $\mathfrak{A}$ is definable with a single formula from $F(L)$.*

We then have the following Theorem:

Theorem 5.4.2. *Let $\mathscr{A}$ be a countable structure such that for each finite subset $X \subset \mathscr{A}$, the cardinality of definable closure $|\operatorname{dcl}(X)| < \omega$, and assume further that $\mathscr{A}$ is $\aleph_0$-homogeneous (i.e. having Scott rank $r(\mathscr{A}) \leqslant \omega$).*

Then $\mathfrak{A}$ is a $\Delta_{*\omega_1}$-structure.

Proof. Assume $\mathscr{A}$ is as in the hypothesis of the theorem, and first assume for a contradiction that $\mathfrak{A}$ is not Dedekind-finite. Then there is an infinite well-ordered subset $\mathfrak{B} \subseteq \mathfrak{A}$. This subset can be represented with a unary relation on $\mathfrak{A}$, and by Corollary 5.4.1, can be defined with a single formula in the language $F(L)$. The well-ordering can likewise be expressed with a single $F(L)$-formula. However, if $b \in \mathfrak{B}$, then using the conjunction of the formulas defining the infinite subset and the well-ordering, one finds that the definable closure of b is infinite, a contradiction.

Now, assume that $\omega_1 \leqslant^* |\mathfrak{A}|$. Then, there are Z_i, $i \in \omega_1$, pairwise disjoint and non-empty, such that $\mathfrak{A} = \bigcup_{i \in \omega_1} Z_i$. Since each Z_i is a subset of $\mathfrak{A}$, each can be regarded as a unary relation on $\mathfrak{A}$. Since any relation can be expressed as the countable sum of atoms from the countable fragment $L_{\mathscr{C}}$, each Z_i can be defined using a single formula. Thus we can find uncountably many pairwise disjoint formulas, contrary to the countability of the fragment $L_{\mathscr{C}}$. $\square_{5.4.2}$

Appendix A

Appendix: Direct Methods

This thesis has presented methods with which structures whose domains are Dedekind finite can be analysed. However, much of those considerations were heavily metamathematical. Here we present some results that are obtained without resort to metamathematics.

A.1 Boolean Algebras

A.1.1 Boolean Algebras with MT-Rank

The authors of [MT03] gave a few requirements for partial orders to admit MT-rank. It is therefore natural to ask similar questions of Boolean algebras. We shall show that infinite Boolean algebras with rank do not exist. Firstly, we examine the type of structure such an algebra would have to have under these conditions.

Remark A.1.1. *Let A be a Boolean algebra, and let A have MT-rank m and degree $k = 2^l q$, where q is an odd natural number. Then A has at most l atoms.*

Proof. We first show that for k an odd number, there can be no atoms. Then, using induction on the number of times 2 divides k, we show the conclusion.

Let A be an MT-rank m, degree k Boolean algebra. Assume that it has an atom, a. Then, let

$$A^a = \{x \in A : a \leqslant x\}$$

and

$$B = \{x \in A : a \leqslant -x\}\,.$$

(Please note that these are not the usual relative algebras). Since a is an atom, A^a and B are disjoint, with $A^a \cup B = A$. Moreover, there exists a bijection ϕ between A^a and B, defined by $\phi(x) = -x$, thus A^a and B must have the same MT-rank and degree, i. By Lemma 1.6 of [MT03], the degree of A is the sum of the degrees of A^a and B, thus $k = i + i$ contrary to k odd. From this we deduce that A cannot be split into two parts in this manner – something that the existence of an atom would allow us to do.

Notice that A^a contains no atoms of A other than a. Also B can be considered a Boolean algebra with $0_B = 0_A$ and $1_B = -a_A$, and the set of atoms of B is contained in the set of atoms of A.

For the general case, let A be a Boolean algebra with MT-rank m and degree $k = 2^l q$, with $l > 0$ and q an odd number. We split A into two parts, A^a and B, as above. They must both have MT-rank m, and degrees of $2^{l-1}q$. Thus, A^a contains one atom of A, while B, as a Boolean algebra of rank m and degree $2^{l-1}q$ contains at most $l-1$ atoms of A. Therefore, A has at most l atoms. $\square_{A.1.1}$

Remark A.1.2. *If a Boolean algebra A has an MT-rank and (at least) l atoms, then 2^l divides k, where $k = \mathrm{MTdeg}(A)$.*

Proof. Let X be a subset of the set of atoms of the Boolean algebra A such that X has l elements.

Consider the map $\psi : A \longrightarrow P(X)$, defined by

$$\psi(a) = \{x \in X : x \leqslant a\}$$

Then, ψ is an epimorphism, and so partitions A into 2^l pieces of the same size. $\square_{A.1.2}$

Theorem A.1.3. *There does not exist an infinite Boolean algebra with MT-rank.*

Proof. Suppose to the contrary.

Firstly, let A be an (infinite) atomless Boolean algebra of minimal (subject to the algebra being atomless) MT-rank α (> 0) and MT-degree n. We can choose $n+1$ pairwise disjoint $a_0, a_1, \ldots, a_n \in A$, since A is atomless. Then, the relative algebras $A_i = \{x : x \leqslant a_i\}$, with $1_{A_i} = a_i$, are all atomless Boolean algebras. However, since $\mathrm{MTdeg}(A) = n$, some A_i must have an MT-rank smaller than α, which leads to a contradiction: either we have found an infinite Boolean algebra of smaller rank, or a finite atomless Boolean algebra.

Now, suppose A is an infinite Boolean algebra with atoms, where $\mathrm{MT}(A) = \alpha$ is minimal, and $\mathrm{MTdeg}(A) = 2^l q$, where l is minimal. Let a be an atom, and we split A as above:

$$A^a = \{x \in A : a \leqslant x\}$$

and

$$B = \{x \in A : a \leqslant -x\}.$$

As above, A^a and B are isomorphic, thus of MT-rank α and MT-degree $2^{l-1}q$, contrary to minimality of l. $\square_{A.1.3}$

A.1.2 Power Set Algebras of Sets with MT-Rank

We consider the structure of the power set algebra of a set admitting MT-rank.

Lemma A.1.4. *Assume X is a set such that*

$$\mathrm{MT}(X) = \alpha \geqslant 1$$

$$\mathrm{MTdeg}(X) = 1.$$

Let β and k be such that $0 \leqslant \beta < \alpha$ and $k \in \omega \setminus \{0\}$. Then there exist disjoint Y and Z such that $Y \cup Z = X$ and

$$\mathrm{MT}(Y) = \beta \qquad \mathrm{MTdeg}(Y) = k$$

$$\mathrm{MT}(Z) = \alpha \qquad \mathrm{MTdeg}(Z) = 1.$$

Proof. First we wish to show that if $\mathrm{MT}(X) = \alpha \geqslant 1$, $\mathrm{MTdeg}(X) = 1$, and $0 \leqslant \beta < \alpha$, then X has a subset Y of MT-rank β. We work by transfinite induction on the MT-rank of X, α.

Since $\mathrm{MT}(X) \nleqslant \beta$ there exist disjoint $A, B \subseteq X$, both of which have MT-rank greater or equal to β. Because they are subsets of X, A and B have MT-rank less than or equal to α. However, since $\mathrm{MTdeg}(X) = 1$, they do not both have MT-rank α. Let us assume that $\mathrm{MT}(A) < \alpha$. Thus, $\beta \leqslant \mathrm{MT}(A) < \alpha$. If $\mathrm{MT}(A) = \beta$, then we let $Y = A$. If $\beta < \mathrm{MT}(A) < \alpha$, A has a subset Y of MT-rank β by the induction hypothesis.

Now we wish to show that if $\mathrm{MT}(X) = \alpha$, $\mathrm{MTdeg}(X) = 1$, $0 \leqslant \beta < \alpha$, and $0 < k < \omega$, then there exist disjoint $Y, Z \subseteq X$ with $Y \cup Z = X$ that fulfil our requirements. We have just shown that there is $Y' \subseteq X$ of MT-rank β. Since Y' can be written as the disjoint union of MT-rank β, MT-degree 1 sets, we may assume $\mathrm{MTdeg}(Y') = 1$. Let $Z' = X \setminus Y$. Then $\mathrm{MT}(Z') = \alpha$ and $\mathrm{MTdeg}(Z') = 1$. We repeat the process k times to find the desired Y and Z. $\square_{A.1.4}$

Definition A.1.5. Let A be a boolean algebra, and $At(A)$ its set of atoms. Let

$I^{At(A)}$ be the ideal of A generated by $At(A)$. We define the *canonical ideals* $I_\alpha(A)$ of A: Let

$$I_0(A) = \{0^A\}.$$

Suppose I_α has been defined. Let $\psi_\alpha : A \longrightarrow A/I_\alpha(A) = A^\alpha$ be the canonical homomorphism. Then let

$$I_{\alpha+1}(A) = \psi_\alpha^{-1}(I^{At(A^\alpha)}).$$

For δ a limit ordinal, let

$$I_\delta(A) = \bigcup_{\gamma<\delta} I_\gamma(A).$$

Note that the canonical ideals form an ascending chain. We call $A^\alpha = A/I_\alpha(A)$ the α^{th} *Cantor-Bendixson derivative* of A.

A Boolean algebra A is *superatomic* if every homomorphic image of A has at least one atom. If A is superatomic, then $|A^\alpha|$ is finite for some ordinal α (see Lemma 17.9 of [MB89]). Then α and $n = |At(A^\alpha)|$ are called the *Cantor-Bendixson invariants* of A.

Remark A.1.6. *A set X has MT-rank if and only if $P(X)$ is a superatomic Boolean algebra.*

It should be noted that a Boolean algebra admits a rank function[1] if and only if it is superatomic. See exercise 2, chapter 6 of [MB89]. Furthermore, in ZFC set theory, no infinite set has a superatomic powerset algebra.

[1]Here, a rank function on a boolean algebra A is a mapping r from A into the ordinals such that if $b \leq a$ in A, then $r(b) \leq r(a)$; and if $a = b + c$ in A, where b and c are disjoint, non-zero, then either $r(b) < r(a)$ or $r9c) < r(a)$. In the context of this section, MT-rank is a rank function according to this definition.

Proof. First of all, we note that for any set X, and any algebra $A = P(X)$

$$I_\alpha(A) = \{Y \subseteq X : \mathrm{MT}(Y) < \alpha\}$$

and that

$$a \in At(A^\alpha) \Leftrightarrow \mathrm{MT}(\psi_\alpha^{-1}(a)) = (\alpha, 1).$$

We prove these statements by transfinite induction.

To verify the first equation, we note that the case of $\alpha = 0$ is trivial.
Let $Y \in I_{\alpha+1}$. Then $\psi_\alpha(Y)$ is a finite union of atoms of A^α. In other words, Y is a finite union of MT-rank α, MT-degree 1 sets. Thus, $\mathrm{MT}(Y) \leqslant \alpha$, and so $\mathrm{MT}(Y) < \alpha + 1$. For limit ordinals, the composition of the canonical ideal is clear.

Next, we prove the second equation. For any ordinal α let $a \in A^\alpha$ such that $\mathrm{MT}(\psi_\alpha^{-1}(a)) = (\alpha, 1)$. We wish to show that $a \in At(A^\alpha)$. Assume there exists $x \in A^\alpha$, such that $0 <_\alpha x <_\alpha a$, where the subscripts indicate relations in the appropriate quotient algebra. (A Venn diagram may be helpful for the following:) Then

$$\mathrm{MT}(\psi_\alpha^{-1}(x) \triangle \psi_\alpha^{-1}(a)) \geqslant \alpha$$

since $a \neq_\alpha x$. Furthermore,

$$\mathrm{MT}(\psi_\alpha^{-1}(x) \setminus \psi_\alpha^{-1}(a)) < \alpha$$

since $x <_\alpha a$. Then,

$$\mathrm{MT}(\psi_\alpha^{-1}(a) \setminus \psi_\alpha^{-1}(x)) = \alpha$$

because of the MT-rank of the symmetric difference combined with the fact that $\mathrm{MT}(\psi_\alpha^{-1}(a)) = \alpha$. But then

$$\mathrm{MT}(\psi_\alpha^{-1}(a) \cap \psi_\alpha^{-1}(x)) < \alpha$$

since $\mathrm{MTdeg}(\psi_\alpha^{-1}(a)) = 1$, and thus $\mathrm{MT}(\psi_\alpha^{-1}(x)) < \alpha$, and so $\psi_\alpha^{-1}(x) \in I_\alpha$, and $x =_\alpha 0$.

Conversely, let us assume that $a \in At(A^\alpha)$. Then, $\psi_\alpha^{-1}(a)$ must have rank since all its subsets have an MT-rank (bounded by α). Whatever the MT-rank of $\psi_\alpha^{-1}(a)$, its degree must be 1 because otherwise there would be a contradiction with a being an atom of A^α. Thus, we can apply Lemma A.1.4 to determine that the MT-rank of $\psi_\alpha^{-1}(a)$ is indeed α.

In conclusion, if $\mathrm{MT}(X) = (\alpha, n)$, clearly $I_{\alpha+1} = A$. Thus $X \in I_{\alpha+1}$ and so is the sum of finitely many atoms of A^α, and must also contain all atoms of A^α. Thus $|At(A^\alpha)| = n$.

Let us now assume that $P(X)$, for some set X, is a superatomic Boolean algebra. We wish to show that X has an MT-rank, and that the MT-rank and MT-degree correspond to the Cantor-Bendixson invariants of $P(X)$.

Assume that the Cantor-Bendixson invariants of A are α and n that is, that $|At(A^\alpha)| = n$. Then $A^{\alpha+1}$ is the trivial algebra with $I_{\alpha+1} = A$. Hence X is the union of n sets of MT-rank α, MT-degree 1. Thus $\mathrm{MT}(X) = (\alpha, n)$. $\square_{A.1.6}$

A.2 Fields

Structures whose domains are weakly Dedekind-finite are locally finite (as demonstrated in Remark 4.2.1). This leads us to the following conclusion:

Remark A.2.1. *There does not exist a weakly Dedekind-finite (skew) field.*

Proof. Let F be a (commutative) field with $|F| \in \Delta_4$ and as such, cannot be mapped onto ω. Thus the field has a finite (prime) characteristic p. Likewise, the multiplicative group F^* has no element of infinite order, and the finite orders of its elements are bounded. Thus, the group has a finite exponent N. Then there exist infinitely many elements of F that satisfy the equation $x^N - 1 = 0$, which is a contradiction.

Suppose F is a skew field. Then there exist elements x, y that do not commute. However, according to the previous Remark 4.2.1, the substructure generated by x and y is finite, hence commutative, a contradiction. $\square_{A.2.1}$

In [Hic78, Hic80, Hic82], Hickman showed that Dedekind-finite (but, of course not weakly Dedekind-finite) fields do exist, and can be represented as a strictly increasing countable sequence $(F_n)_{n<\omega}$ of finite subfields. Furthermore, he showed that a Dedekind-finite field is algebraically closed if and only if every proper field extension is Dedekind-infinite. Clearly, Dedekind-finite fields have certain structural properties in common with pseudo-finite fields.

A.3 Groups

The author of [Tru95] has shown that any group G such that $\mathrm{MT}(G) = (1, 1)$ is elementary abelian of prime exponent. This result was extended in [MT03] to show that an abelian group G with $\mathrm{MT}(G) = (1, n)$ has an elementary abelian subgroup of prime exponent, the index of which is finite and bounded by n. We will show in Corollary A.4.7 that every infinite group with rank has an infinite abelian subgroup. This is a special case of results shown in the main body of the thesis, but the method here is "bare-handed" and direct. This will involve a modified and truncated version of the Kargapolov and Hall-Kulatilaka Theorem restricted to nonabelian groups with cardinality in Δ_4. We note that the general version of the Kargapolov and Hall-Kulatilaka theorem requires some weakened form of the axiom of choice, as shown in [Plo81]. Lemmas A.4.1, A.4.3, and A.4.5 are needed for the proof of our case. Lemma A.4.1 concerns infinite groups, thus we include the proof here. Lemmas A.4.3 and A.4.5, both due to Burnside, concern finite groups. The proofs are elementary, and we shall state them without proof here as a reminder. We largely follow the proofs as found in [Rob72], with adaptations to compensate for the lack of the Axiom of Choice.

A.4 Weakly Dedekind-finite Groups

Groups having a cardinality that lies in Δ_4, are uniformly locally finite, as demonstrated in Remark 4.2.1. Thus, we can apply the following lemmas and results.

Lemma A.4.1. *Let G be an infinite periodic group containing an involution. Then either the centre of G contains an involution or G has a proper infinite subgroup with non-trivial centre.*

Proof. We can assume that every involution in G has a finite centraliser since otherwise the truth of the theorem is evident. Let i be an involution in G: a non-trivial element $g \in G$ such that $g^i = i^{-1}gi = g^{-1}$ will be called *i-involuted.*

Claim (i): There are infinitely many i-involuted elements in G.

Since, by our assumption, $C_G(i)$ is finite, $|G : C_G(i)|$ is infinite and therefore G contains infinitely many involutions, for example, the conjugates of i. Hence, there exist infinitely many cosets $C_G(i)a$, different from $C_G(i)$, where a is an involution. Let $z = ai^{-1}ai$; then $i^{-1}zi = i^{-1}aia = z^{-1}$, and $z \neq 1$ since $i^{-1}ai \neq a$. Also $ai^{-1}ai = bi^{-1}bi$ and $a^2 = 1 = b^2$ imply that $C_G(i)a = C_G(i)b$. Thus we prove the assertion.

Claim (ii): Of the infinitely many i-involuted elements in G, infinitely (in fact, all but finitely) many have odd order.

Assume, to the contrary, that there is an infinite set X of i-involuted elements of even order. For each $x \in X$, let x have order $2m_x$. For $i^{-1}xi = x^{-1}$, and so $i^{-1}x^{m_x}i = (i^{-1}xi)^{m_x} = (x^{-1})^{m_x} = x^{m_x}$. In other words, i centralises each x^{m_x}. Hence by our initial assumption, the set $Y = \{x^{m_x} : x \in X\}$ is a finite set of involutions. Consequently, there is an involution in the set Y which is centralised by infinitely many elements of the set X, contrary to our hypothesis.

Let S be the set of all i-involuted elements with odd order.

Claim (iii): We define a particular infinite subset T of S, on which we will later concentrate our attentions.

Let a be a fixed element of S: Define $k = ia$ and note that $k^2 = iaia = a^i a = i^2 iaia = a^{-1}a = 1$ and $a \neq i$; hence k is an involution. Let b be any element of S and put $u = ik^b$. Then $u^i = k^b i = (ik^b)^{-1} = u^{-1}$. By assumption, $C_G(k)$ is finite, so, by allowing b to take values in the infinite set S, we obtain an infinity of distinct elements u. Furthermore, those b in S for which u has odd order form an infinite subset T of S; otherwise, using the methods of (ii), we would be able to find an involution with infinite centraliser.

Claim (iv): For each $b \in T$, there is an $\bar{h} \in C_G(k)$ such that $\bar{h}b$ is j-involuted, where j is an involution of the form ia_1^{-1}, $a_1 \in \langle a \rangle$.

Let b be any element of T. As before, we write $u = ik^b$. Since $u^i = u^{-1}$ and u has odd order, $\langle u, i \rangle$ is a finite dihedral group with Sylow 2-subgroups of order 2. As $k^b \in \langle u, i \rangle$, Sylow's Theorem shows that there is an element u_1 in $\langle u \rangle$ such that $i = (k^b)^{u_1}$. Similar analysis of the group $\langle a, i \rangle$ yields $i = k^{a_1}$ for some $a_1 \in \langle a \rangle$. Thus we get $k^{bu_1a_1^{-1}} = k$. Therefore, $h = bu_1a_1^{-1} \in C_G(k)$ and $u_1 = b^{-1}ha_1$. Let $g = b^{-1}h = u_1a_1^{-1}$, so we have that $u_1 = ga_1$ and $u_1^i = g^i a_1^i$. However, since u_1 and a_1 are powers of u and a respectively, we have $u_1^i = u_1^{-1}$ and $a_1^i = a_1^{-1}$. Thus, combining equations, we have $a_1^{-1}g^{-1} = u_1^{-1} = u_1^i = g^i a_1^{-1}$, and therefore, $g^{ia_1^{-1}} = g^{-1}$. Define $j = ia_1^{-1}$. Because $a_1^i = a_1^{-1}$ as stated before, j is an involution, and $g^j = g^{-1}$. Thus $(g^{-1})^j = g$, and writing $\bar{h} = h^{-1}$, we get $g = (g^{-1})^j = (\bar{h}b)^j = b^{-1}\bar{h}^{-1} = (\bar{h}b)^{-1}$. Notice that i and a, and so k, a_1, and j are fixed throughout this calculation and do not depend on b.

Claim (v): We find an element which has an infinite centraliser.

Since $C_G(k)$ is finite by our assumption, there is an infinite subset $T_1 \subseteq T$ such that for some $c \in C_G(k)$, the elements of the form cb', where $b' \in T_1$, are j-involuted. Let $b_1 \in T_1$. If $b' \in T_1$ is distinct from b_1, we have $(cb_1)^j = b_1^{-1}c^{-1}$ and $(cb')^j = b'^{-1}c^{-1}$, and so $(b'^{-1}b_1)^j = (b'b_1^{-1})^{c^{-1}}$. Since b_1 and all the b' are i-involuted and $j = ia_1^{-1}$, we have $(b'^{-1}b_1)^j = (b'b_1^{-1})^{a_1^{-1}}$. Combining these expressions, we have $(b'b_1^{-1})^{a_1^{-1}c} = b'b_1^{-1}$. Since the elements $b'b_1^{-1}$ are distinct for all $b' \in T' \setminus b_1$, $d = a_1^{-1}c$ has an infinite centraliser. Note that d is not an involution.

Claim (vi): $C_G(d)$ is a proper infinite subgroup.

Suppose to the contrary, that $C_G(d) = G$. Then every element commutes with d and in particular $d^k = d$. Since $c \in C_G(k)$, we have $ka_1^{-1}kc = ka_1^{-1}ck = d^k = d = a_1^{-1}c$, hence $a_1^k = a_1$. Now $a_1^k = a_1^{ia} = a_1^{-1}$ since $a_1 \in \langle a \rangle$. Hence $a_1^2 = 1$ and since a_1 has odd order, $a_1 = 1$. Then we have (by the proof of (iv)) $i = k^{a_1} = k = ia$ and $a = 1$, which is impossible. Thus $C_G(d)$ is a proper infinite subgroup of G and $d \neq 1$. $\square_{A.4.1}$

Definition A.4.2. A *Frobenius group* is a finite group G containing a non-trivial normal subgroup M such that if $x \in M \setminus \{1\}$, then $C_G(x) \leqslant M$. This unique M is referred to as the *kernel*. A subgroup H of G such that $G = MH$ and $M \cap H = \{1\}$ is called a *complement* of M. Such a complement always exists.

Lemma A.4.3. *If G is a Frobenius group with kernel M and complement H, and $P \in Syl_p(H)$, then*

(i) if $p \neq 2$, then P is cyclic,

(ii) if $p = 2$, then P is cyclic or generalised quaternion.

Definition A.4.4. A group G is *metacyclic (metabelian)* iff $G/[G,G]$ and $[G,G]$ are cyclic (abelian).

Lemma A.4.5. *Let G be a finite group. If all Sylow subgroups of G are cyclic, then G is metacyclic.*

The proofs to both Lemmas A.4.3 and A.4.5 can be found in standard group theory texts, for example, in [Sco64] Theorems 12.6.15 and 12.6.17.

Theorem A.4.6. *Let G be an infinite non-abelian group such that $|G| \in \Delta_4$. Then there is a non-trivial element $x \in G$ with infinite centraliser $C_G(x)$.*

Proof. Let us assume the contrary: $C_G(x)$ is finite for all $1 \neq x \in G$. We look for a contradiction.

Claim (i): If F is a non-trivial finite subgroup of G, than the normaliser $N_G(F)$ is finite.

Let $1 \neq a \in F$. Then $C_G(F) \leqq C_G(a)$. As the the latter is finite by assumption, $C_G(F)$ is also. Furthermore, $N_G(F)/C_G(F)$ is finite (since it is isomorphic to a subgroup of $\mathrm{Aut}(F)$), and so $N_G(F)$ is finite.

Claim (ii): There is a finite subgroup F such that $C_G(F) = 1$.

Let $1 \neq a \in G$ and let $1, b_1, \ldots, b_n$ be the distinct elements of $C_G(a)$. By our assumption, each $C_G(b_i)$ is finite, so we can choose an element $c_i \notin C_G(b_i)$ for each i (note that this is a finite set of choices). Let $F = \langle a, b_1, \ldots, b_n, c_1, \ldots, c_n \rangle$, a finite subgroup. Then $C_G(F) \leqq C_G(a)$ and yet no element of $C_G(a)$ except 1 centralises F. Therefore, $C_G(F) = 1$.

Claim (iii): For each prime p, G has no infinite p-subgroups.

To the contrary, let P be an infinite p-subgroup of G. Then, P cannot be abelian, for if it were, then for $x \in P \setminus \{1\}$, $C_G(x)$ would be infinite, contrary to our assumptions. Thus, P satisfies the initial hypotheses of G. Since P is an infinite, locally finite non-abelian group with cardinality in Δ_4, there is a finite subgroup F of P such that $C_P(F) = 1$. But then F is a non-trivial finite p-group; hence $1 < Z(F) \leqq C_P(F)$ (the first inequality holds since the centre of a finite p-group is non-trivial). Thus we find a contradiction.

Claim (iv): For each prime p the Sylow p-subgroups of G exist, and are finite and conjugate.

First, we search for maximal p-subgroups containing a particular p-subgroup R: by *(i)*, each term in the sequence $N(R), N(N(R)) = N^2(R), \ldots$ is finite. By our assumption on the cardinality of G, this sequence must stabilise after a finite number of terms, say at $N^{n_1}(R)$. Since this group is finite, we may speak of Sylow p-subgroups within this group, and so let $R_1^1, \ldots, R_1^{k_1}$ be the Sylow p-subgroups of $N^{n_1}(R)$ that contain R as a subgroup. Let $S_1 = \langle R_1^1, \ldots, R_1^{k_1} \rangle$. Note that S_1 is finite. We then find the normaliser limit of each of the R_1^l. Each limit, for the reasons given earlier, will be a finite group with Sylow p-subgroups. We label the Sylow p-subgroups containing R of each of the normaliser limits of the R_1^l as $R_2^1, \ldots, R_2^{k_2}$ and the groups generated by them $S_2 = \langle R_2^1, \ldots, R_2^{k_1} \rangle$. We continue in this manner, and so find a finitely branching tree of finite groups, and a sequence of finite groups $S_1, S_2, \ldots$. This sequence must also stabilise after a finite number of terms, say at S_j. Then $R_j^{k_1}, \ldots, R_j^{k_j}$ are maximal p-subgroups of G containing R. Note that these groups are indeed maximal. For let us assume that there is some p-group $T \supset R_j^k$ for some k. Without loss of generality, we may assume that $o(T) = o(R_j^k)p$. By Sylow's first theorem, R_j^k is normal in T. This would mean that $T \subseteq N(R_j^k)$, a contradiction. Finally, any two Sylow p-subgroups of G generate a finite group of which they are also Sylow p-subgroups; by Sylow's Theorem, they are conjugate in this group.

Claim (v): If $M \lhd G$ and P is a Sylow p-subgroup of M, then $G = N_G(P)M$.
Let $x \in G$; now P and P^x are Sylow p-subgroups of M, and by *(iv)* they are conjugate in M, say $P^x = P^y$ for some $y \in M$. Then $xy^{-1} \in N_G(P)$ and $x \in N_G(P)M$.

Claim (vi): Every proper homomorphic image of G is finite.
Let $1 \neq M \lhd G$ and let P be a non-trivial Sylow subgroup of M. Then P is finite by *(iv)*, and by *(i)* $N_G(P)$ is finite. Furthermore, *(vi)* shows that $G = N_G(P)M$, so G/M is finite.

Claim (vii): G contains no involutions and is locally solvable.
If G contained an involution, it would have a non-trivial element with infinite cen-

traliser, by Lemma A.4.1. Hence finitely generated subgroups of G have odd order and are solvable by the Feit-Thompson Theorem.

Claim (viii): G is not residually finite.

Suppose, to the contrary, that G is residually finite. Let P be a non-trivial Sylow p-subgroup of G and define

$$T = \langle C_G(x) : 1 \neq x \in P \rangle.$$

Then $P \leqq T$ and T is finite since P is finite. From the residual finiteness of G it follows that there is a normal subgroup K with finite index in G, such that $K \cap T = 1$. Clearly, $K \cap P = 1$, so we see that K has no elements of order p: for assume otherwise, that there exists $r \in K$ of order p. Then r belongs to some Sylow p-group Q; by *(iv)*, $P^s = Q$ for some $s \in G$. Thus there exists an $x \in P$ such that $s^{-1}xs = r$. However, K is normal, so $(s^{-1})^{-1}Ks^{-1} \subset K$ and in particular $(s^{-1})^{-1}s^{-1}xss^{-1} = x \in K$. A contradiction. Let Q be a non-trivial Sylow q-subgroup of K and let $N = N_G(Q)$. Then $p \neq q$ and $G = NK$ by *(v)*. Now $|P|$ divides $|G : K| = |N : N \cap K|$ since $K \cap P = 1$. Hence Sylow p-subgroups of N have the same order as P since P is wholly contained in N, and so are conjugate to P. Replacing P by a suitable conjugate if necessary, we can assume that $P \leqq N$ and $Q^P = Q$. Suppose that $a^b \in P$ where $1 \neq a \in P$ and $1 \neq b \in Q$. Then $[a, b] \in Q \cap P = 1$ and $a^b = a$; this implies that $b \in T$, by definition of T, and hence that $b \in K \cap T = 1$. From this we deduce that $P \cap P^b = 1$, if $1 \neq b \in Q$. Thus P is the unique complement of the Frobenius group PQ with p odd, and we may infer from Lemma A.4.3 that P is cyclic.

Thus we have shown that every Sylow subgroup of G is cyclic. Now, a finite group with cyclic Sylow subgroups is metacyclic (of derived length $\leqslant 2$), by Lemma A.4.5. Consequently G is locally metacyclic and therefore metabelian: as always, $G/[G, G]$ is abelian. If $a, b \in [G, G]$, then they have the form $[a_{1_1}, a_{2_1}] \cdots [a_{1_i}, a_{2_i}]$ and

$[b_{1_1}, b_{2_1}] \cdots [b_{1_j}, b_{2_j}]$ respectively for some $a_{1_1}, a_{2_1}, \ldots, a_{1_i}, a_{2_i}, b_{1_1}, b_{2_1}, \ldots, b_{1_j}, b_{2_j} \in G$ with $i, j < \omega$. However, the commutator subgroup of

$$H = \langle a_{1_1}, a_{2_1}, \ldots, a_{1_i}, a_{2_i}, b_{1_1}, b_{2_1}, \ldots, b_{1_j}, b_{2_j} \rangle$$

is cyclic, hence abelian. Thus a and b commute. By hypothesis, G cannot be abelian, so $G/[G,G]$ is finite by *(vi)*; but [G,G] is abelian, so it is also finite and hence G is finite.

Claim (ix): G has a minimal normal subgroup R which is of finite index, simple, and locally solvable.

Let R be the finite residual of G (that is, the intersection of the normal subgroups F_i of G such that G/F_i is finite). By *(viii)* $R \neq 1$ and by *(vi)* G/R is finite. Thus R is infinite, and must satisfy the initial hypotheses on G. R is simple, for if we take $1 \neq N \lhd R$, then R/N is finite, and $\bigcap_{g \in G} N^g$ is a normal subgroup of G of finite index, and so by the minimality of R, $N = R$. Therefore R is simple. By *(vii)*, R is locally solvable. R is clearly minimal.

Claim (x): We come to a final contradiction.

If R is indeed not abelian, there exist elements $a, b \in R$ such that $c = [a,b] \neq 1$. Since R is minimal, we have $R = \langle a^G \rangle$. Hence there is a finite subset X of G such that $\langle a, b \rangle = \langle c^X \rangle$. This is because $\langle a, b \rangle$ is a finite subgroup of R, and generators of R have the form a^g, $g \in G$. Put $H = \langle c, X \rangle$, noting that H is finite. Then $[a,b] = c \in \langle c^H \rangle'$, and so $\langle c^H \rangle = \langle c^H \rangle'$. However, as R is locally soluble, the (finite) group $H \cap R \supseteq \langle c^H \rangle$ is soluble. This cannot be since $c \neq 1$. Thus R is abelian. From *(ix)* we know that R is also simple. This implies that R is finite (since if R were infinite and abelian, it could not be simple). Again from *(ix)* we know that R has finite index in G, and so G must be finite, a contradiction. $\square_{A.4.6}$

A.4.1 Groups with MT-rank

The cardinality of groups with MT-rank lies in Δ_4 by Lemma 1.3 of [MT03]. Hence, we may build upon the results of the previous section.

Corollary A.4.7. *Let G be a non-abelian group such that* $\mathrm{MT}(G) = (n, k)$*, with n an ordinal and $k < \omega$. Then G has an infinite abelian subgroup.*

Proof. We proceed by induction.
In [Tru95], the author showed that all amorphous groups, that is those of MT-rank 1 MT-degree 1, are elementary abelian.

Let G be a group such that $\mathrm{MT}(G) = (n, k)$, and assume that our assertion has been proved true for all groups of (lexicographically) smaller rank and degree. Theorem A.4.6 guarantees the existence of at least one element with an infinite centraliser. Of these elements, let us choose an element x with infinite centraliser $C_G(x)$ of minimal MT-rank, and among those, of minimal MT-degree. We now have two cases:

I. $\mathrm{MT}(C_G(x)) < (n, k)$

Then, $C_G(x)$, and by consequence G, has an infinite abelian subgroup.

II. $\mathrm{MT}(C_G(x)) = (n, k)$

In such a case, $C_G(x) = G$, for otherwise G contains $C_G(x)$ and a coset of $C_G(x)$, two disjoint subsets of the same rank and degree as G, a contradiction. Thus $\{g \in G : C_G(g) \text{ is infinite}\} = \{g \in G : \mathrm{MT}(C_G(g)) = (n, k)\} = Z(G)$.

We have two subcases:

(A.) $Z(G)$ is an infinite set.

Clearly, G contains an infinite abelian subgroup.

(B.) $Z(G)$ is a finite set.

Notice that in light of Theorem A.4.6, the centre in this case cannot be trivial. We analyse the upper central series:

$$1 \lhd Z_1(G) \lhd Z_2(G) \lhd \ldots \lhd Z_{j-1}(G) \lhd Z_j(G) \lhd \ldots \lhd G$$

where Z_{i+1}/Z_i is the centre of G/Z_i. There exists such j such that $Z_{j-1}(G)$ is finite while $Z_j(G)$ is infinite with MT-rank $m > 0$ and MT-degree l. For ease of notation, let $Z_j(G) = H$.

If $x \notin Z_{j-1}(G)$, then $x \notin Z(G)$, and so $C_G(x)$ is finite, and hence so is $C_H(x)$. Thus the set of cosets of $C_H(x)$ in H must be of MT-rank m (the same as H), and so the conjugacy class of x must likewise have MT-rank m. Therefore, there can be at most l infinite conjugacy classes within H.

Let $a_1, \ldots, a_{l+1} \in H \setminus Z_{j-1}(G)$, where each lies in a different coset of $Z_{j-1}(G)$ with respect to H. At least two of these must be conjugate to one another in H, say $x^{-1}a_s x = a_t$, for some $x \in H$. Therefore,

$$[Z_{j-1}(G)x]^{-1}[Z_{j-1}(G)a_s][Z_{j-1}(G)x] = [Z_{j-1}(G)a_t].$$

However, since H/Z_{j-1} is abelian (as $H/Z_{j-1} = Z(G/Z_{j-1})$, we have

$$Z_{j-1}(G)a_s = Z_{j-1}(G)a_t,$$

a contradiction. $\square_{A.4.7}$

From this point, until the end of this chapter, we will develop the theory of the structure of MT-ranked sets closely following the development as found in [Che79].

Definition A.4.8. A group H is *connected* iff it has no subgroup of finite index. Let G be a group, H a connected subgroup of G having finite index in G. Then, we call H the *connected component* of G.

Remark A.4.9. *A group has a unique connected component.*

Proof. Let G be a group such that $\mathrm{MT}(G) = (n, k)$. If $k = 1$, then G is connected. Thus, assume $k > 1$, and let $H_1, H_2 < G$ be two different connected components with $[G : H_1] = h_1$, $[G : H_2] = h_2$. If $[G : H_1 \cap H_2]$ is finite, we apply Poincare's Theorem: Let $K = H_1 \cap H_2$, and let R be the set of right cosets of K, R_i the set of right cosets of H_i. Define the relation

$$T = \{(Kx, U) : x \in G, U \in R_1 \times R_2 \text{ and } U = (H_1x, H_2x)\}.$$

Then, T is a one-to-one function from R into $R_1 \times R_2$, and so $[G : H_1 \cap H_2] \leqslant [G : H_1][G : H_2] = h_1h_2$. $\square_{A.4.9}$

We will now write the connected component of G as G°.

Remark A.4.10. *For G a group, $G^\circ \lhd G$.*

Proof. Assume otherwise. Then, for some $x \in G$, $x^{-1}G^\circ x \neq G^\circ$, which would give G another connected component. $\square_{A.4.10}$

Theorem A.4.11. *Let G be a group admitting MT-rank. Then, G is connected if and only if* $\mathrm{MTdeg}(G) = 1$.

To prove this, we first need a lemma. In this lemma, we will denote the MT-rank of a set X as $\mathrm{MTr}(X)$, to avoid confusion with our previous notation of $\mathrm{MT}(X)$ for the rank and degree combination of the set X.

Lemma A.4.12. *Let G be a group admitting MT-rank that acts on a set X. Let $Y \subseteq X$, and let*

$$\mathit{Stab}(Y) = \{g \in G : \mathrm{MTr}(gY \triangle Y) < \mathrm{MTr}(Y)\}.$$

Then,

(i) $Stab(Y)$ is a subgroup of G;

(ii) If $Y \subseteq X$ is such that $\mathrm{MTr}(Y) = \mathrm{MTr}(X)$, then $Stab(Y)$ is a subgroup of finite index in G.

Proof. Firstly, note that when Y has MT-degree 1,

$$Stab(Y) = \{g \in G : \mathrm{MTr}(gY \cap Y) = \mathrm{MTr}(Y)\}.$$

For (i): Clearly, $1 \in Stab(Y)$. Let $g^{-1}, h \in Stab(Y)$. Then $g^{-1}h \in Stab(Y)$, $\mathrm{MTr}(gY \triangle Y) < n$, and $\mathrm{MTr}(hY \triangle Y) < n$. Whence clearly $\mathrm{MTr}(gY \triangle hY) < n$, and so $\mathrm{MTr}(Y \triangle g^{-1}hY) < n$. Thus, $g^{-1}h \in Stab(Y)$ and so $Stab(Y)$ is a subgroup of G.

For (ii): By decomposing Y into finitely many subsets of MT-degree 1, we can assume that $\mathrm{MTdeg}(Y) = 1$. Let $X_1 = Y$ and find X_i such that $\mathrm{MTr}(X_i) = \mathrm{MTr}(X) = n$, $\mathrm{MTdeg}(X_i) = 1$ and $X = X_1 \sqcup \ldots \sqcup X_k$. Set

$$G_i = \{g \in G : \mathrm{MTr}(gX_1 \cap X_i) = n\} = \{g \in G : \mathrm{MTr}(gX_i \triangle X_i) < n\}.$$

Clearly, $G = G_1 \sqcup \ldots \sqcup G_k$ and $G_1 = Stab(Y)$. If $g, h \in G_i$, then

$$\mathrm{MTr}(gX_1 \triangle hX_1) < n,$$

and so $h^{-1}g \in G_1$. Thus $G_1 \leqslant G$ and each non-empty G_i is a coset of G_1. Thus G_1 has finite index in G. $\square_{A.4.12}$

Proof. If $\mathrm{MTdeg}(G) = 1$, then G is clearly connected.

Assume G is connected and $\mathrm{MTr}(G) = n$, but $\mathrm{MTdeg}(G) = k > 1$. Let G act upon itself by left-multiplication, and let A and B be disjoint MT-rank n, MT-degree 1 subsets of G. Then, $G = Stab(A) = Stab(B)$, by the previous lemma. Now, consider the subsets

$$U = \{(a,b) \in A \times B : ab \in A\}$$

and

$$V = \{(a,b) \in A \times B : ab \in B\}.$$

Then $V = \bigsqcup_{a \in A}[\{a\} \times (a^{-1}B \cap B)]$. Note that $\mathrm{MT}(a^{-1}B \cap B) = n$. Consider the projective map $\pi V \longrightarrow A$. Then, for $a \in A$, $\pi^{-1}(b) = \{a\} \times (a^{-1}B \cap B)$, the MT-rank of which is n. From this we infer that $\mathrm{MT}(V) = \mathrm{MT}(A) + n = 2n$. A similar deduction yields $\mathrm{MT}(U) = 2n$. However, this contradicts the fact that $\mathrm{MT}(A \times B) = \mathrm{MT}(A) + \mathrm{MT}(B) = 2n$ and $\mathrm{MTdeg}(A \times B) = 1$. $\square_{A.4.11}$

Definition A.4.13. A group G is called *centraliser connected* if it has no proper subgroup of finite index which is the centraliser of a non-trivial element. The *centraliser connected component* is the intersection of the centralisers of finite index in G.

Remark A.4.14. *In a centraliser connected group, every finite normal subset is central.*

Proof. If G normalises its finite subset A, then each element of A has only finitely many conjugates: for if $A \lhd G$, then for all $g \in G$, $g^{-1}ag \in A$. Therefore, A is centralised by the centraliser connected component of G, since $[G : C(a)] = |cl(a)|$, where $cl(a)$ is the conjugacy class of a. Since G is centraliser connected, the elements of A are all centralised by G itself, hence $A \subset Z(G)$. $\square_{A.4.14}$

Remark A.4.15. *A group with only finitely many commutators (alternately: a group in which every element has finitely many conjugates) is central-by-finite.*

Proof. For every $a \in G$, the set of commutators $a^{-1}x^{-1}ax$ is finite. Therefore, a has only finitely many conjugates: Let $a^{-1}x^{-1}ax = y_i$, $i \in I$ where I is finite. Then,

$x^{-1}ax = a^x = ay_i$. By Proposition A.4.14, since $\{ay_i\}_{i\in I}$ is a finite set normalised by G, it is centralised by the centraliser connected component of G. Thus, every element $a \in G$ is centralised by the centraliser connected component. Thus the centraliser connected component (a subgroup of G of finite index) is contained in $Z(G)$. $\square_{A.4.15}$

Remark A.4.16.

(i) If G is centraliser connected and if its centre is finite, it also is equal to its second centre.

(ii) An infinite centraliser connected nilpotent group has infinite centre.

Proof. For (i): Assume $Z(G)$ is finite and $aZ \in Z(G/Z)$. Then, for all $x \in G$, $a^{-1}x^{-1}ax \in Z(G)$ because $a^{-1}Zx^{-1}ZaZxZ = a^{-1}ZaZx^{-1}ZxZ = Z$. Thus, a has finitely many conjugates, so by Propositions A.4.14 and A.4.15, the centraliser of a (and its conjugates) has finite index, and $a \in Z(G)$.

For (ii): If the group is nilpotent, the centre must be infinite, since otherwise the upper central series would not increase. $\square_{A.4.16}$

Remark A.4.17. *In a nilpotent centraliser connected group, every infinite normal subgroup contains infinitely many elements central in the whole group.*

Proof. Let N be an infinite normal subgroup of G. If $N \subset Z(G)$ or $Z(G) \subseteq N$, then the result is trivial. Thus, we assume otherwise. Then, NZ/Z defines a proper normal subgroup of G/Z, which must contain a non-identity element of the second centre of G: G/Z is nilpotent of class $c-1$ for some finite integer c. Thus, $Z_{c-1}(G/Z) = G/Z$ and there is a least positive integer i such that $Z_{i+1} \cap NZ = (\bmod Z) Z_i(G/Z) \cap NZ/Z \neq 1$. Now, $[N \cap Z_i(G/Z), G/Z] \leqslant N \cap Z_{i-1}(G/Z) = 1$, and $N \cap Z_i(G/Z) \leqslant N \cap Z_1(G/Z)$. Thus, $N \cap Z_1(G/Z) = N \cap Z_i(G) \neq 1$. Thus $Z_1(G/Z) = Z_2(G)$, whence $NZ \cap Z_2(G) \neq 1$. Let $1 \neq a \in N \cap Z_2(G)$. For each x in G, $a^{-1}x^{-1}ax \in Z \cap N$, because N is normal. Since a is not central, by Proposition A.4.16 it has infinitely many conjugates. Thus, we have infinitely many elements of $N \cap Z$. $\square_{A.4.17}$

Remark A.4.18. *In an infinite connected nilpotent group G having finite MT-rank n, every subgroup of infinite index has infinite index in its normaliser.*

Proof. Let G be infinite, nilpotent, and of MT-rank n, MT-degree 1. Then, G is centraliser connected, so by A.4.16(ii), has infinite centre $Z(G)$.

We continue the proof by induction on the MT-rank of G.

Let H be a subgroup of infinite index in G. Then, obviously, H is a proper subgroup of G of strictly smaller MT-rank; and H is not self-normalising (if H does not contain the centre, then this is clear, otherwise, examine the upper central series).

If the connected component Z° of the centre $Z(G)$ is contained in H, then we examine the group G/Z°. As the homomorphic image of a nilpotent group, G/Z° is nilpotent. Because the centre $Z(G)$ is infinite, $\mathrm{MT}(G/Z^\circ) < \mathrm{MT}(G) = n$. Furthermore, this group is connected as it is the homomorphic image of a connected group, or alternatively, must have MT-degree 1 by Remark 2.2.16(6.e). By the inductive hypothesis, the image of H under the canonical homomorphism from G to G/Z° is properly contained, and has infinite index in its normaliser in G/Z°. Thus, the same must be true for H in G.

In the other case, when Z° is not contained in H, then $Z^\circ H$ is a subgroup of $N_G(H)$ which has strictly greater MT-rank than H. $\square_{A.4.18}$

Theorem A.4.19. *A connected group of MT-rank 2 is soluble.*

Proof. Let us assume otherwise: G is a non-soluble group of MT-rank 2, MT-degree 1. We aim for a contradiction.

Claim (i): $Z(G)$ is finite.

If $Z(G)$ is infinite, then $\mathrm{MT}(G/Z(G)) = (1,1)$, hence $G/Z(G)$ is abelian, contrary to the insolubility of G.

Claim (ii): G contains a connected abelian subgroup of MT-rank 1.

By Corollary A.4.7, G contains an infinite abelian subgroup A. Clearly, A must have MT-rank 1, or otherwise our proof is complete. So, A° is our desired subgroup.

Without loss of generality, we consider $H = G/Z(G)$ in the rest of the proof. Thus, H is a non-soluble connected MT-rank 2 group. Let $A \subseteq H$ denote an amorphous subgroup, the existence of which is guaranteed by *(ii)* above. Let $N = N(A)$.

Claim (iii): $Z(H) = 1$.

Let Z_1 be the preimage of $Z(H)$ in G. That is, $Z_1/Z(G) = Z(H) = Z(G/Z(G))$. Since $Z_1 \lhd G$, Z_1 must be finite by the reasoning of *(i)*. By Proposition A.4.14, it must be central, and so, $Z(H) = 1$.

Claim (iv): $[H : N] = \infty$ and $[N : A] < \infty$.

Since A is amorphous, it is abelian. Suppose $N = H$. Then, $H/A = N/A$ is a connected group of MT-rank 1, and thus is abelian. Then, the series $1 \lhd A \lhd H$ would contradict the non-solubility of H. Thus, N must have MT-rank 1.

Claim (v): For $g, h \in H$, $A^g \cap A^h = 1$, unless $gh^{-1} \in N$.

Assume otherwise. Let $1 \neq b \in A^g \cap A^h$. Then, A^g, $A^h \subseteq C(b)$ and $C(b) \neq H$ because we assumed b is non-trivial and H is centreless. Thus, $\mathrm{MT}(C(b)) \leqslant 1$, and since A^g and A^h are both amorphous, they both have finite index in $C(b)$. However, a group may have only one connected subgroup. Thus, $A^g = A^h$, and so $gh^{-1} \in N$.

Claim (vi): Let $w \in H \setminus N$. Then $H = AwA \cup N$.

It suffices to show that AwA has MT-rank 2. Since H is connected, there can be only one such coset, for it is well-known that double cosets are either disjoint or identical. We must show that the expression of an element of AwA in the form $a_1 w a_2$ is unique: If $a_1 w a_2 = b_1 w b_2$, then

$$b_2 a_2^{-1} = w^{-1} b_1^{-1} a_1 w \in A \cap A^w = 1$$

and so $b_2 = a_2$, $b_1 = a_1$.

Claim (vii): $H \setminus N$ contains an involution.

Let x be an arbitrary element of $H \setminus N$. By *(vi)*, we can write

$$x^{-1} = a_1 x a_2$$

with a_1, $a_2 \in A$. Let $w = xa_1$. Then,

$$w^2 = xa_1xa_1 = xa_1xa_2a_2^{-1}a_1 = xx^{-1}a_2^{-1}a_1 = a_2^{-1}a_1 \in A.$$

Let $a = w^2$:

$$a^w = w^{-1}w^2w = w^{-1}a_2^{-1}a_1w = a_1^{-1}x^{-1}a_2^{-1}a_1xa_1 = xa_1xa_1 = w^2 = a$$

so $w^2 = a^w = a \in A \cap A^w = 1$. Thus, $w^2 = 1$ and $w \neq 1$, and so w is an involution.

Fix w.

Claim (viii): Let $K = N \cap A^w$. Then $N = A \rtimes K$.

Clearly $K \leq N$, and thus K normalises A. Since $A \cap A^w = 1$, $K \cap A = 1$. Thus, we must show that $N = AK$.

Since $w \in G \setminus N$, for all $n \in N$, $nw \notin N$. Thus, by *(vi)* we can write

$$nw = a_1wa_2.$$

Then, $a_1^{-1}n = a_2^w \in A^w \cap N = K$. Thus, any $n \in N$ can be written as $a_1a_2^w$. Hence $N = AK$.

(ix) $K = 1$, and thus $N = A$.

If K is nontrivial, then for any $a \in K^w$, $a \in A$ and $a^w \in N$. We claim that for all $g \in H$, $a^g \in N$: If $g \in N$, then this is clear. If $g \in H \setminus N$, then by *(vi)*, $g = a_1wa_2$

for unique a_1, $a_2 \in A$. So,

$$a^g = a_2^{-1}w^{-1}a_1^{-1}aa_1wa_2 = a_2^{-1}w^{-1}awa_2 = (a^w)^{a_2} \in N$$

Thus $a^g \in N$.

Let $B = \langle a^g : g \in H \rangle$. Then $B \lhd H$, $B \subseteq N$. By Proposition A.4.14, B must be infinite. Since $[N : A] < \infty$, we have $[B : A \cap B] < \infty$. If we conjugate by w (note that $B^w = B$), we have $[B : A^w \cap B] < \infty$, and so $[B : A \cap A^w \cap B] < \infty$. But B is infinite, so this contradicts $A \cap A^w = 1$. Thus, $A = 1$, so $K = 1$, and $N = A$.

Claim (x): $H = \bigcup_{g \in H} A^g$.

Let $X = \bigcup_{g \in H} A^g$. Since $A^g \cap A^h = 1$ when $gh^{-1} \notin A$, the MT-rank of X is 2. If $b \in H \setminus X$, the conjugacy class of b has MT-rank at most 1. Thus, $C(b)$ is infinite. Let $B = [C(b)]^\circ$. Then, we may follow the same reasoning of sets *(vi)* - *(ix)* with B in place of A. In particular, $B = N(B)$, implying $B = C(b)$ and $b \in B$. Now, let $Y = \bigcup_{g \in H} B^g$. Then, for the same reasons as above, Y has MT-rank 2. Thus X meets Y on a large set. Thus, since $A^g \cap A^h = 1$, unless $gh^{-1} \in A$, $B^g \cap B^h = 1$, unless $gh^{-1} \in B$, and $X \cap Y$ is MT-rank 2, we can assume that $A \cap B \neq 1$, replacing A or B with a suitable conjugate if necessary. Since A and B are both connected, $A \cap B$ is either finite, in which case we again apply Proposition A.4.14 to find that $A \cap B$ is central and hence trivial, or else $A = B$, contradicting the existence of an element $b \in B \setminus A$. In either case we have a contradiction. Thus $H = X$ as desired.

Claim (xi): The final step.

If $a \in H \setminus \{1\}$, then by steps *(i)* through *(x)*, $C(a)$ is a connected abelian group equal to its own normaliser. Let $w \in H$ be an involution outside $C(a)$. Then, by *(x)*, w is conjugate to an involution i of A. Then $iw \neq wi$ and $(iw)^i = (iw)^{-1}$. Let $B = C(iw)$. Then $iw \in B$ also, and so $iw = wi$, a contradiction. $\square_{A.4.19}$

Theorem A.4.20. *Let G be a MT-rank n soluble connected group. Then G is nilpotent.*

Proof. Assume otherwise: G is a soluble, non-nilpotent group such that $\mathrm{MT}(G) = (n, 1)$. Firstly, G' is not finite, since if it were, by Proposition A.4.15, G would be central-by-finite. Since it is connected, it would be simply central, hence abelian. Thus, assume G' is infinite.

If G' is not connected, then $G/[G']^\circ$ has a finite derived group (i.e. $G'/[G']^\circ$, because the derived group is the smallest group such that the quotient group is abelian). By the same Proposition A.4.15, $G/[G']^\circ$ would be central-by-finite. However, since both G and $[G']^\circ$ are connected, $G/[G']^\circ$ is connected. Hence, $G/[G']^\circ$ is abelian, contrary to the definition of G' as the smallest subgroup of G that gives an abelian quotient.

Thus we have established that G' is connected. The same reasoning applies for the rest of the derived series $G', G'', \ldots$. Let $G^{(n)}$ be the last non-trivial term. Then, $G^{(n)}$ is an abelian group on which the connected abelian group $G^{(n-1)}/G^{(n)}$ acts by inner automorphisms.

Lemma A.4.21. *Let M be a connected abelian group acting as an automorphism group on an infinite abelian group A, M and A both admitting MT-rank. Then, M fixes infinitely many points of A.*

Proof. Let M be a connected abelian group with identity element e_M and group operation $+_M$. Let A be an abelian group admitting MT-rank, with identity symbol 0_M and operation $+_A$. M acts upon A as an automorphism group. We wish to show that M fixes infinitely many points of A.

Suppose the contrary, and let $A_\circ$ be the finite set of fixed points. $A_\circ$ is non-empty because 0_A in particular is fixed by M.

If $a \notin A_\circ$ is an element that is not fixed, then the orbit $a^M = \{ma : m \in M\}$ is infinite: For if a^M were finite, i.e. $a^M = \{a_1, a_2, \ldots, a_n\}$, then M permutes the elements of a^M. Consider the group homomorphism

$$\phi : M \longrightarrow Sym(a^M)$$

defined by $\phi(m)(a_i) = ma_i$. We consider the kernel of ϕ with help from the homomorphism theorem: :

$$M/ker(\phi) \simeq \phi(M) \leqslant Sym(a^M).$$

But, since $Sym(a^M)$ is a finite group, $Ker(\phi)$ has finite index in M, a contradiction. Hence, for each $a \notin A_\circ$, a^M is infinite.

Let us consider the family of subgroups $\langle a^M \rangle$, $a \notin A_\circ$, of A. Choose one $a \notin A_\circ$ such that $\langle a^M \rangle$ is of minimal MT-rank and MT-degree, and call that group B.

Every element $b \in B$ can be written

$$b = m_1 a + \ldots + m_n a$$

where $m_i \in M$, and n is bounded (by the usual Δ_4 argument). Note we are not choosing a particular representation of b, only giving the form of the representations. Let R be the ring of endomorphisms of B generated by A: Let 0 be the endomorphism that takes all of B to the identity 0_A. Define the operation:

$$(r_m \otimes r_n)b = m(n(b))$$

$$(r_m \oplus r_n)b = mb +_M nb$$

These operations, and the addition of 0, give us a ring. Clearly, R is a commutative ring. Likewise, it is clear that $Ra = B$, and so, two endomorphisms of B are equal if they have the same value on a. Thus, the underlying set of R is $(M^\wedge 0)^n$, quotiented by an equivalence relation. Hence, R has MT-rank because the underlying set has MT-rank as a subset of a Cartesian product of finitely many sets with rank.

Clearly, B is an R-module. A proper R-submodule of B must be a proper subgroup of B that satisfies the module axioms.

Claim: Every proper submodule of B is contained in $B \cap A_\circ$:
Assume there is a proper submodule that contains an element $b' \notin A_\circ$. Then, the entire orbit b'^M must be contained in the submodule, but we have shown that this orbit is necessarily infinite, and a proper subgroup of B. This contradicts the minimality of the MT-rank of B.

Thus, every element of R is either surjective or zero. Hence R is an infinite commutative integral domain of finite exponent (the latter because the cardinality of R lies in Δ_4). Thus we come to a contradiction, since such rings can have at most n n-th roots of unity. $\square_{A.4.21}$

The connected group $G^{(n-1)}/G^{(n)}$ acts upon $G^{(n)}$ by inner automorphisms. Thus, by Lemma A.4.21, the group $G^{(n-1)}/G^{(n)}$ centralises an infinite subgroup A_1. Likewise, $G^{(n-2)}/G^{(n-1)}$ centralises an infinite subgroup A_2 of A_1, and so on. After a finite number of steps, we see that G has an infinite centre Z_1.
If $G \neq Z_1$, then G/Z_1 has an infinite centre Z_2/Z_1, and so on. Because these are structures with cardinality in Δ_4, this process eventually stops, and so G is nilpotent.

$\square_{A.4.20}$

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