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SUITABLE EXTENDER MODELS I*

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We investigate both iteration hypotheses and extender models at the level of one supercompact cardinal. The HOD Conjecture is introduced and shown to be a key conjecture both for the Inner Model Program and for understanding the limits of the large cardinal hierarchy. We show that if the HOD Conjecture is true then this provides strong evidence for the existence of an ultimate version of Gödel's constructible universe L. Whether or not this "ultimate" L exists is now arguably the central issue for the Inner Model Program.

Keywords: Extender; iteration tree; inner models; supercompact cardinal.

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*In memoriam, Gregory Hjorth.

1. Introduction

The primary goal of this investigation is the attempt to understand inner model theory at the level of a supercompact cardinal and beyond. The surprising discovery is that the inner model problem for exactly one supercompact cardinal emerges as the key problem (assuming that the supercompact cardinal of the inner model inherits its supercompactness from V in a natural fashion). This paper is Part I of a two part series and Part II will be also discussed in this introduction as necessary.

The original motivation for investigating inner models for very strong large cardinal axioms, was to determine the status of the Ω Conjecture, which is a conjecture that arises naturally from the following remarkable absoluteness result. Suppose there is a proper class of Woodin cardinals. Then for each Σ_2-sentence, ϕ, the Σ_2-sentence, ϕ^*, which asserts that ϕ holds in some forcing extension of V, is *absolute* between V and all generic extensions of V. Thus it follows for example, that if there is a huge cardinal (and a proper class of Woodin cardinals) then over *any* generic extension of V, there is a further generic extension in which there is again a huge cardinal *even* if there is only one huge cardinal to begin with and in the first generic extension that huge cardinal is collapsed to be countable.

This surprising absoluteness result motivates the definition of Ω-logic, the formulation of the Ω Conjecture, and it isolates the set $\mathcal{V}_\Omega$ of Ω-valid sentences as a set of fundamental set theoretic interest: the set $\mathcal{V}_\Omega$ is recursively equivalent to the set of all Σ_2-sentences which can hold in some forcing extension of V. We shall give an overview of the relevant definitions in Sec. 2.

The Ω Conjecture, if true, has important meta-mathematical consequences for the foundations of Set Theory. For example, it implies that if there is a proper class of Woodin cardinals then the set $\mathcal{V}_\Omega$ is *definable* in the structure, $H(\delta^+)$, given by those sets of hereditary cardinality at most δ where δ is the least Woodin cardinal (so in particular the set $\mathcal{V}_\Omega$, whose natural definition is Σ_2, is actually Δ_2-definable). This in turn is the basis (assuming there is a proper class of Woodin cardinals and that the Ω Conjecture holds) for an argument that the independence of a sentence (from large cardinal axioms) obtained by forcing is *not* a proof that the sentence has no answer.

For example, augmenting the Ω Conjecture with the AD^+ Conjecture (which we define on p. 134), one obtains that $\mathcal{V}_\Omega$ is in fact definable in $H(c^+)$. As a consequence the independence of the Continuum Hypothesis as demonstrated by forcing is arguably not a proof that the Continuum Hypothesis has no answer; the challenge for the skeptic (who denies formalism) is to exhibit for $H(c^+)$, a sentence ϕ whose truth in $H(c^+)$ is not absolute to all generic extensions of V together with the reasons that ϕ is true in $H(c^+)$. By the absoluteness result cited above, assuming there is a proper class of Woodin cardinals, every such sentence is qualitatively just like CH; over *any* forcing extension of V one can further force to produce an extension for which the sentence is true (in the $H(c^+)$ of that extension), and one can further force to produce an extension for which the sentence is false (in the $H(c^+)$ of that extension). Independent of the status of the AD^+ Conjecture, the challenge is simply to identify as true *some* Π_2-sentence which is not absolute to all generic extensions of V. This basic argument is presented in [21] in an expanded form. The conclusion argued for in [21] is that if the Ω Conjecture is true then there is at present no plausible conception of truth for Set Theory other than one that deems statements like CH to have determinate truth-values. This raises another challenge: give an example of an axiom which together with ZFC gives a *robust* conception of V which is compatible with all large cardinal axioms. The only known examples of axioms which do provide a robust conception of V are based on generalizations of Gödel's Axiom of Constructibility and these axioms are *not* compatible with (all) large cardinal axioms.

The Ω Conjecture itself is absolute between V and $V^{\mathbb{P}}$ for all partial orders, $\mathbb{P}$. Thus it is not unreasonable to expect both that the Ω Conjecture has an answer and further if that answer is that it is false, then the Ω Conjecture be *refuted* from some large cardinal hypothesis. Many of the meta-mathematical consequences of the Ω Conjecture follow from the non-trivial Ω-satisfiability of the Ω Conjecture; this is the assertion that there exist a partial order, $\mathbb{P}_0$, and an ordinal, α_0, such that

$$V_{\alpha_0}^{\mathbb{P}_0} \vDash \mathrm{ZFC} + \text{“There is a proper class of Woodin cardinals”}$$

and such that

$$V_{\alpha_0}^{\mathbb{P}_0} \models \text{"The } \Omega \text{ Conjecture"}.$$

This assertion is obviously a Σ_2-assertion and so assuming there is a proper class of Woodin cardinals, this assertion must also be absolute between V and $V^{\mathbb{P}}$ for all partial orders, $\mathbb{P}$. While the claim, if the Ω Conjecture is false then the Ω Conjecture must be refuted from some large cardinal hypothesis, is debatable, the corresponding claim for the non-trivial Ω-satisfiability of the Ω Conjecture (in the sense just defined) is much harder to argue against. The point here is that while there are many examples of sentences which are provably absolute for set forcing and which cannot be decided by any (standard) large cardinal axiom, there are no known examples where the sentence is Σ_2. In fact if the Ω Conjecture is true then there really can be no such example. Finally it seems unlikely the Ω Conjecture be false and yet the non-trivial Ω-satisfiability of the Ω Conjecture be true.

An abstract analysis shows that if a specific large cardinal hypothesis does refute the Ω Conjecture then that large cardinal hypothesis must be *beyond* a certain kind of inner model theory. This suggests looking at the possibilities for inner models for various large cardinals in order to find clues as to whether or not the Ω Conjecture is true. More precisely if the Ω Conjecture is false then the failure should correspond to some critical stage in the large cardinal hierarchy and this stage should be evident from inner model theory. Moreover finding this precise stage could also be a significant aid in finding the actual proof that the Ω Conjecture is false. On the other hand if the Ω Conjecture holds in all the inner models, so that there is no critical stage, then this would be very strong evidence that the Ω Conjecture is true. This was the motivation for this work and the results offer evidence that no known large cardinal hypothesis refutes the Ω Conjecture. Of course there may yet be large cardinal hypotheses which are far beyond anything currently imagined (moreover evidence is not a proof) and a natural question is how these results could apply to them. The answer is that the results show that the inner models at the level of just one supercompact cardinal are surprisingly "close" to V. Further, any such inner model (constructed with anything like the current methodology for building inner models) necessarily inherits essentially all large cardinals from the universe of sets — these inner models are analogous to L in the context where $0^\#$ does not exist.

Inner Model Theory has developed into an elaborate study of inner models generalizing L which is the minimum inner model for Set Theory (that contains all of the ordinals). These inner models are *Extender Models* which are of the form $L[\tilde{E}]$ where $\tilde{E}$ is a sequence of *extenders*. Extender models arise in one of two basic forms, the coarse extender models and the fine-structural partial extender models. A coarse extender model is of the form $L[\tilde{E}]$ where $\tilde{E}$ is a sequence of total extenders, that is true extenders from the universe of sets. The second variety, fine-structural extender models, are of the form $L[\tilde{E}]$ where $\tilde{E}$ is a sequence of partial extenders, that is extenders whose associated ultrafilters measure only the sets previously constructed. The least non-trivial coarse extender models are the inner models $L[\mu]$ where μ is a normal measure on some measurable cardinal κ. In contrast, the least non-trivial fine-structural extender model is $L[0^{\#}]$. For *larger* inner models the distinction becomes even more important but identifying the coarse extender models has been a useful precursor to identifying the fine-structural extender models.

Martin and Steel, [7], defined coarse extender models up to the level of superstrong cardinals. These are the Martin–Steel extender models and they generalize earlier constructions of Mitchell. Sometime later, Mitchell and Steel, [8], defined fine-structural extender models again up to the level of superstrong cardinals. These are the Mitchell–Steel extender models. However the definition of a Mitchell–Steel extender model requires an *iteration hypothesis* to establish existence. Currently, the best result is due to Neeman, [11], who proved, by establishing the required iteration hypothesis, that if there exists a Woodin cardinal which is a limit of Woodin cardinals then there is a Mitchell–Steel inner model in which there is a Woodin cardinal which is a limit of Woodin cardinals.

Iteration hypotheses are necessary to prove *comparison* for the Martin–Steel extender models, and as a consequence they are necessary to analyze the reals of these extender models, but such hypotheses are not necessary to define the models (unless one attempts to use a *Doddage* which can yield a definable extender model).

Reaching beyond superstrong cardinals it is straightforward to generalize the definition of Martin–Steel extender models to define extender models in which much larger cardinals exist (if such cardinals exist in the parent universe), though a priori, it is not at all obvious if one can reach the level of the existence of a non-trivial elementary embedding,

$$j : V_{\lambda+1} \to V_{\lambda+1}.$$

But there are serious obstacles to establishing comparison for these models from any iteration hypothesis whatsoever. As a result no progress has been made on generalizing Mitchell–Steel extender models to stronger large cardinal hypotheses.

Our main theorem on the problem of comparison in this general setting is a negative one: comparison is false for generalized Martin–Steel extender models (if sufficiently large cardinals exist in V). Remarkably, the failure occurs just past the Martin–Steel extender models and this seems to strongly show that there is no useful theory of coherent extender models past the Martin–Steel extender models, which leaves partial extender models as the only option of the Inner Model Program. However assuming that partial extender models satisfy *strong acceptability* which seems an essential feature for fine-structural partial extender models, the extent to which there are long extenders on the sequence is *significantly constrained* by the failure of comparison indicated above. Unless this also indicates the failure of the Inner Model Program itself, a new phenomenon must appear: this is the closure of the inner model under extenders which cohere the inner model but which are not initial segments of any extender on the sequence from which the inner model is constructed. The extent of this phenomenon will be dictated by the kinds of partial extenders on the sequence. There are two kinds of partial extenders. An extender E on the sequence $\mathbb{E}$ from which the inner model is constructed might be partial because the associated ultrafilters do not measure all sets $L[\mathbb{E}]$, or more generally, the extender E might be partial because it is not directly given as the restriction to the inner model of the initial segment of a strong extender in V. The failure of comparison for coarse extender sequences severely constrains the extenders of $L[\mathbb{E}]$ which are not partial in latter sense and which can be obtained (by a uniform proceedure) as the initial segment of an extender on the sequence $\mathbb{E}$.

There are strong indications that this phenomenon is precisely what happens and for this the level of one supercompact cardinal turns out to be the critical level. Suppose N is an inner model of ZFC which the contains the ordinals and that δ is supercompact in N. Suppose that the supercompactness of δ in N is witnessed by the restriction to N of extenders which witness the supercompactness of δ in V (so for each $\gamma > \delta$ is there is a normal fine measure μ on $\mathcal{P}_\delta(\gamma)$ such that $\mu(\mathcal{P}_\delta(\gamma) \cap N) = 1$ and $\mu \cap N \in N$). Therefore every set $a \subseteq \mathrm{Ord}$ with $|a| < \delta$ can be covered by a set $b \in N$ such that $|b| < \delta$. This implies that no singular cardinal above δ of countable cofinality can be a regular cardinal in N — and we will prove in addition that for *all* singular cardinals $\gamma > \delta$, γ is a singular cardinal in N and $\gamma^+ = (\gamma^+)^N$. But if N belongs to a hierarchy of canonical inner models then N cannot be small and so N should satisfy all large cardinal axioms which hold for a proper class of cardinals in V (and which are within reach of this hierarchy of canonical inner models). In fact (and this is a remarkable coincidence), N *does* satisfy essentially all large cardinal axioms which hold in V for a proper class of cardinals. The reason is that if E is any extender in V of strongly inaccessible length κ and with critical point above δ such that $j_E(N) \cap V_\kappa = N \cap V_\kappa$ then *necessarily* for all $\alpha < \kappa$, $E|\alpha \cap N \in N$.

This is the subject of Sec. 5 and shows the necessary universality of N for large cardinals at least up to the level of the existence of a nontrivial elementary embedding

$$j : V_\lambda \to V_\lambda.$$

We show in Sec. 6 that if N is an partial extender model this extends to the axiom which asserts the existence of a non-trivial elementary embedding, $j : V_{\lambda+1} \to V_{\lambda+1}$, and (in Part II of this paper) far beyond, though this is much more subtle. Thus one only needs to verify that the inner model is closed under extenders sufficient to witness that there is a supercompact cardinal.

Part of the problem in trying to generalize partial extender models to large cardinals beyond the level of superstrong cardinals is that one must allow *long* extenders on the sequence. An extender of length η is a long extender if η is greater than the image of the critical point of the elementary embedding from which the extender is defined. This in turn requires generalizing the fundamental notion of an *iteration tree* to accommodate long extenders. Here again there are potential problems even in the coarse case. There are a variety of possible ways to generalize iteration trees to the context of long extenders but the most liberal generalization namely allowing extenders from one model to be applied to another model whenever the two models are in sufficient agreement (as certified by the iteration tree) so that the relevant ultrapowers are defined — leads to a failure of iterability. This is by a result of Neeman. We generalize the notion of an iteration tree to that of an iteration tree which does allow the (non-trivial) use of long extenders. We prove that the generalization of the Martin–Steel Theorem on the existence of maximal realizable branches does hold for this notion of iteration tree (and in particular this eliminates the family of counterexamples identified by Neeman). The special cases of both this definition and the theorem, for iteration trees of length ω, are due to Steel, [16]; but for a different purpose namely analyzing the Mitchell order for arbitrary extenders. All current theorems on the iterability of V or of structures derived from extenders in V, trace back either directly, or indirectly, to the Martin–Steel Theorem. Therefore generalizing this theorem to whatever notion of iteration tree one defines is an important test for the correctness of that notion.

The definition of an iteration tree which we use, allows long extenders which are much *longer* than those relevant to just supercompactness, we just require that the extenders not be ω-huge in the models from which they are selected and even this restriction can be relaxed. Further the (admittedly few) positive results on iterability that we can prove do not require any further restriction on the length of the extenders. This might seem to suggest that the (coarse) iterability problem is really a distinct problem from the inner model problem and we initially conjectured this to be the case. Whether or not this is in fact the case will depend on the emerging details of the attempts to develop fine-structural notions in the context of long extenders, [22]. It could turn out at one extreme that iterability even in the coarse sense is constrained by possible extent of premice in the context of long extenders and this is bounded by the failure of comparison in the coarse case. At the other extreme, it could be that there is a very general iteration hypothesis which can be established for V and from which the entire fine-structural development can be carried out.

Nevertheless the results presented in this paper suggest the intriguing possibility that one can reduce the inner model problem for *all* known large cardinal axioms to the inner model problem for exactly *one* supercompact cardinal. To explore and develop this claim more fully (and this is the ultimate purpose of this paper) we define an abstract notion of a partial extender model which we call a *suitable extender model*. Roughly, a suitable extender model is a transitive inner model N which is sufficiently closed under extenders from V such that there is a cardinal δ_N of N which is supercompact in N as witnessed by extenders of N which are of the form $E \cap N$ where E is an extender in V (δ_N simply denotes the least such supercompact cardinal of N). We shall require that for some $X \in V_{\delta_N+1}$, N is Σ_2-definable in V from X and more importantly that there exist a sequence, $\langle E_\alpha : \alpha < \delta_N \rangle$ of extenders which witnesses that δ_N is a Woodin cardinal and such that

$$\langle E_\alpha \cap N : \alpha < \delta_N \rangle \in N.$$

The latter requirement is a key genericity requirement and this seems necessary to tranfer the strongest large cardinal axioms (the existence of λ at which I0 holds and beyond) from V to N.

Of course our notion of a suitable extender model is far too general with regard to the purpose of isolating candidates for genuine inner models at the level of a supercompact cardinal — if there is a supercompact cardinal in V then V itself is a suitable extender model. Our assumption is that whatever the exact form of the fine-structural inner model at the level of a supercompact cardinal, it will be a suitable extender model. Therefore by establishing properties for *all* possible suitable extender models, we establish properties of the fine-structural model. Suitable extender models have remarkable closure properties within V. For example N computes successor cardinals correctly for singular cardinals above δ_N and for essentially all known large cardinal axioms, if the axiom holds in V for a proper class of cardinals (possibly in a slightly strengthened form), then the axiom holds in N for a proper class of cardinals. This closeness suggests that no large cardinal axiom can refute the Ω Conjecture (since under very general assumptions Ω Conjecture must hold in any fine-structural partial extender model) and so this provides arguably substantial evidence that the Ω Conjecture is true.

But a far more interesting possibility is suggested by these results: that the existence of a supercompact cardinal *implies* that there is an ultimate enlargement of L, in one of two possible forms L^Ω or L^Ω_S, which is ordinal definable and which is close to V (in the sense just described). Such an enlargement would be a faithful analog of L in the context of $0^\#$ does not exist, but with *no* limiting assumption on the large cardinals of V. The inner model L^Ω would be constructed as a fine-structural partial extender model and L^Ω_S would be the strategic variation. One notable corollary of such a development would be that the Kunen inconsistency on the nonexistence of a non-trivial elementary embedding,

$$j : V \to V,$$

would follow in *just* ZF provided one assumes in addition that there are at least two supercompact cardinals. This is examined in some detail in Sec. 7.2 where the formulation of the notion of a supercompact cardinal in the context of just ZF is given. But there is another consequence of such a development which seems much more profound:

(ZF) *Suppose δ is a supercompact cardinal and there is a supercompact cardinal below δ. Then there is a transitive class M such that*

(1) $M \vDash \mathrm{ZFC}$,
(2) *there exists $X \in V_\delta$ such that M is Σ_2-definable from X and moreover the definition defines $M \cap V_\alpha$ in V_α for all limit $\alpha > \delta$.*

(3) *there is a partial order $\mathbb{P} \in M \cap V_\delta$ such that for every set $Y \subset \mathrm{Ord}$, $Y \in M[G]$ for some M-generic filter $G \subset \mathbb{P}$.*

This result, which as indicated would hold in just ZF, argues for the Axiom of Choice based solely on the existence of large cardinals. Notice that as a corollary, all successor cardinals above δ must be regular and for every regular cardinal $\kappa > \delta$, the filter on κ generated by the closed unbounded subsets of κ *must* be κ-complete. As a precursor to such a result, we do prove in Sec. 7.2 that the existence of a proper class of supercompact cardinals does imply nontrivial instances of the Axiom of Choice showing (in ZF) that if λ is singular limit of supercompact cardinals then λ^+ is a regular cardinal and the filter on λ^+ generated by the closed, cofinal, subsets of λ^+, is λ^+-complete.

Of course a more skeptical assessment of these results is simply that the large cardinal hierarchy as presently conceived beyond the level of a single supercompact cardinal is *not* well justified. This assessment would be based on the claim that the true justification for large cardinal axioms comes from the understanding of their inner model theory. But the results discussed so far suggest that the inner model theory for large cardinal axioms beyond the level of one supercompact cardinal is simply the inner model theory for one supercompact cardinal.

In Part II, we examine the large cardinal axioms at the level of ω-huge and the primary motivation was initially to just understand how such axioms transfer down to suitable extender models. The results obtained seem to re-affirm the analogy with determinacy axioms. For example, the transference problem leads to an analog of $\mathrm{AD}_\mathbb{R}$ at the level of ω-huge and we show that this axiom does transfer from V to suitable extender models. Further the proof of transference gives additional information about this $\mathrm{AD}_\mathbb{R}$-axiom showing for example that the existence of a proper class of λ at which this axiom holds is invariant under set forcing. Motivated by this, we formulate two conjectures, the first arises naturally from the transference problem and the second is suggested by the analogy with determinacy axioms. Either conjecture, if true, shows that essentially all such axioms (as ranked by a natural ordinal parameter quantifying strength) transfer from V to a small generic extension of N and that a substantial initial segment transfer to N. Here N is any suitable extender model.

We note that Laver has proposed the following axiom as the axiomatic analog of $\mathrm{AD}_\mathbb{R}$ at the level of λ. The axiom asserts the existence of a set $V_{\lambda+1} \subset \Gamma \subset V_{\lambda+2}$ together with an elementary embedding,

$$j : L(\Gamma) \to L(\Gamma),$$

with critical point below λ such that $\Gamma = L(\Gamma) \cap V_{\lambda+2}$, such that $\Gamma^\lambda \subset L(\Gamma)$, and such that $V_{\lambda+1}$-choice holds in $L(\Gamma)$ for all functions $F : V_{\lambda+1} \to \Gamma$. Laver's Axiom is motivated by the theorem that in the context of ZF+AD+DC; $\mathrm{AD}_\mathbb{R}$ is equivalent to $\mathbb{R}$-choice for all functions, $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ (i.e. to uniformization). The relationship between the $\mathrm{AD}_\mathbb{R}$-like axiom at λ which we consider and Laver's Axiom is unclear.

In part this is because it is not even known if Laver's Axiom even implies the consistency of the existence of an elementary embedding, $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, with critical point below λ.

In the sequel to this paper, we shall explore the possibilities for fine-structural versions of suitable extender models in the context of various iterability hypotheses on V. There are two possible forms of such models, the pure (non-strategic) extender models, L^Ω, and the strategic variations L^Ω_S where L^Ω_S is defined so that L^Ω_S is closed under the (fine-structural) iteration strategy which acts on all the levels of L^Ω_S. The remainder of this introduction is concerned with the optimistic scenario that at least one of these variations exists. Focusing first on L^Ω (similar considerations apply to L^Ω_S but there are key differences), for those cardinals λ for which in L^Ω there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ there *must* be a canonical structure theory for the corresponding $L(V_{\lambda+1})$. Within the current hierarchy of canonical inner models, Steel has shown that the *true* theory of $L(\mathbb{R})$ is revealed at the first stage where $L(\mathbb{R})$ fails to contain a wellordering of the reals. However the results of Part II actually suggest the possibility that the correct setting for the analysis of the hierarchy of ω-huge cardinals is not L^Ω. The only possible alternative at this stage (other than a generic extension of L^Ω) is L^Ω_S. If L^Ω_S exists then one can explicitly state the axiom $V = L^\Omega_S$ without even yet knowing the exact definition of L^Ω. It is the conjunction of:

(1) There is a supercompact cardinal.
(2) There exist a universally Baire set $A \subset \mathbb{R}$ and $\gamma < \Theta^{L(A,\mathbb{R})}$ such that

$$V \equiv (\mathrm{HOD})^{L(A,\mathbb{R})} \cap V_\gamma$$

for all Π_2-sentences (equivalently, for all Σ_2-sentences).

If there is a proper class of Woodin cardinals then for every universally Baire set $A \subset \mathbb{R}$, $L(A,\mathbb{R}) \vDash \mathrm{AD}^+$ and so for the set A which witnesses that (2) holds, necessarily $L(A,\mathbb{R}) \vDash \mathrm{AD}^+$. Thus assuming there is a proper class of Woodin cardinals and defining

$$M^A_\gamma = (\mathrm{HOD})^{L(A,\mathbb{R})} \cap V_\gamma$$

where $A \subset \mathbb{R}$ is universally Baire and $\gamma < \Theta^{L(A,\mathbb{R})}$, the sets M^A_γ are uniquely determined by $(\Theta^{L(A,\mathbb{R})}, \gamma)$.

If L^Ω_S exists then a natural conjecture is that for all λ such that in L^Ω_S there exists an elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}),$$

the structure theory of $(L(V_{\lambda+1}))^{L_S^\Omega}$ is analogous in both form and detail to that of $L(\mathbb{R})$ in the context of AD. If there is a converse, for example that elements of this theory require $V = L_S^\Omega$, then one would have a compelling argument that in fact $V = L_S^\Omega$.

A key problem for L_S^Ω (or for L^Ω if L_S^Ω does not exist), will be determining for which (uncountable) cardinals λ does

$$(L(\mathcal{P}(\lambda)))^{L_S^\Omega} \not\models \text{“The Axiom of Choice”}.$$

It would be a striking affirmation of the existence of ω-huge cardinals if necessarily any such λ be ω-huge in L_S^Ω (and the proof does not show there are no such λ). On the other hand if this is not the case then by examining the class of all such λ one may again be led to the correct structure theory of $L(V_{\lambda+1})$ in the context of the existence of an elementary embedding, $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, with critical point below λ. By results of Shelah, if λ is a singular strong limit cardinal of uncountable cofinality then

$$L(\mathcal{P}(\lambda)) \models \text{ZFC}.$$

Therefore for the question above and for λ which are singular in L_S^Ω, one is only interested in the case where λ has cofinality ω in L_S^Ω.

To summarize the potential that L_S^Ω seems to represent, I close with the following comments. There is an ever changing list of questions in set theory the answers to which would greatly increase our understanding of the universe of sets. The difficulty of course is the ubiquity of independence: almost always the questions are independent. As a consequence, while there have been remarkable advances in understanding a variety of aspects of the universe of sets, the progress on resolving many of the fundamental questions such as that of CH has been limited.

If L_S^Ω exists then this list of questions will expand to include a variety of questions about L_S^Ω itself and questions about V which, because of the closeness of V to L_S^Ω, will be reducible to questions in L_S^Ω. But then independence is probably only possible as a consequence of a coupling to some large cardinal axiom and so these questions *will have answers*. For L_S^Ω there are questions which cannot have a positive answer in either L^Ω or any generic extension of L^Ω but which cannot presently be ruled out to have positive answers in L_S^Ω. Thus one could have a separate and compelling argument for $V = L_S^\Omega$ in favor of $V = L^\Omega$ should both exist. This distinction would hinge on the fact that L^Ω must be a (non-trivial) set-generic extension of a copy of L^Ω combined with the fact that L_S^Ω cannot be the set-generic extension of any (proper) inner model, [22].

One has only to consider the state of our understanding of V had $0^\#$ been quickly refuted in order to appreciate the potential here. In fact there is already a specific list of questions, very likely absolute between V and L_S^Ω, the answers to which could provide a compelling solution to the problem of CH. Now, either the answers discovered will suffice to advance our understanding of the universe of sets to the point where CH will be answered, or the pattern of answers will be such that the ambiguity of CH is preserved. If the latter case always prevails then one will ultimately be forced to reevaluate the issue of the meaning of CH. There is a genuine distinction between an investigation of CH which is stalled because of the persistence of independence as opposed to an investigation of CH which is stalled because answers never lead to progress.

This paper is Part I of a two part series. The combined paper is the second substantial revision of a version completed in May of 2009 and that version was based on notes prepared in support of a tutorial given as part of the program, *Computational Prospects of Infinity*, held in the summer of 2005 at the *Institute for Mathematical Sciences* (IMS) of the *National University of Singapore*. Because of this purpose there is more introductory material and more detail given (particularly in the proof of Theorem 66), than customary. Both versions contain a substantial amount of additional material beyond the tutorial notes (for example, essentially all of Part II) and quite a number of other changes.

The (2009 May) version was the end result of a series of refinements which began with the notes of 2005. During this development the scenario that V is somehow based on L^Ω became increasingly unlikely if the structure theory associated to the hierarchy of the ω-huge cardinals was to play any role in determining V and yet by the universality of L^Ω, a structure theory associated to the hierarchy of ω-huge cardinals (possibly in the correct "V") is arguably essential for the validation of these axioms. But there seemed to be no alternative to L^Ω granting that L^Ω_S could not exist. But the claim that L^Ω_S could not exist was in turn based on the claim that in HOD of $L(A, \mathbb{R})$ there could be no cardinal which is strong past a Woodin cardinal where $A \subset \mathbb{R}$ and $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$, and the proof of this latter claim was not correct. Here I would like to also acknowledge Steel's persistent determination to understand my proof which helped me find the error in my arguments. While this claim I was attempting to prove is actually not yet known to be false without assuming an iteration hypothesis which is not yet known to be consistent, the (potential) consequences of the existence of L^Ω_S are so compelling that the claim is surely false because L^Ω_S must exist.

The (2009 May) version was based on a failure of comparison for coherent extender models at the level of the moving spaces problem which was a stage identified by Steel at which the methodology of comparison through iterations seemed likely to break down. This is the level at which there exist $\delta < \kappa$ such that δ is κ-supercompact and κ is a measurable, and this witnessed by the extenders on the sequence.

The second major revision, which occurred in the summer of 2010, was the result of the discovery of the failure of comparison just past the Martin–Steel extender models and so far below the level of the moving spaces problem. This renders vacuous (at least from an inner model theoretic perspective) the notion of a suitable extender sequence at the level of supercompactness which was the primary topic of both the original (2009 May) version and the first revision. The refined threshold for the failure of comparison was the result of the fine-structural development of suitable premice, [22]. The new counterexample to comparison is extreme enough that for the second revision, the chapter of the previous versions which dealt with ramifications of comparison has been moved to [22] since there seems to be no useful abstraction of comparison to suitable extender models without focusing on a specific class of such models. Any such specification involves partial extender sequences and so arguably must involve fine-structural notions. We take up this point again in Sec. 8.

2. Preliminaries

2.1. *Weakly homogeneous trees, universally Baire sets, and scales*

We briefly review the basic definitions concerning weakly homogeneous trees, universally Baire sets, and scales.

For any set X, $X^{<\omega}$ is the set of finite sequences of elements of X. If $s \in X^{<\omega}$ then $\ell(s)$ denotes the length of s, which formally is simply the domain of s. A *tree* T on a set X is a set of finite sequences from X which is closed under initial segments. So $T \subseteq X^{<\omega}$.

We deviate from this convention slightly and say that T is a *tree on* $\omega \times \kappa$ where κ is an ordinal if T is a set of pairs (s, t) such that

(1) $s \in \omega^{<\omega}$ and $t \in \kappa^{<\omega}$,
(2) $\ell(s) = \ell(t)$,
(3) for all $i < \ell(s)$, $(s|i, t|i) \in T$.

Suppose that T is a tree on $\omega \times \kappa$. For $s \in \omega^{<\omega}$ we let

$$T_s = \{t \in \kappa^{<\omega} \mid (s, t) \in T\}$$

and for each $x \in \omega^\omega$,

$$T_x = \cup\{T_{x|k} \mid k \in \omega\}.$$

Thus for each $x \in \omega^\omega$, T_x is a tree on κ. We let

$$[T] = \{(x, f) \mid x \in \omega^\omega, f \in \kappa^\omega, \text{ and for all } k \in \omega, (x|k,\ f|k) \in T\}$$

denote the set of infinite branches of T and we let

$$p[T] = \{x \in \omega^\omega \mid (x, f) \in [T] \text{ for some } f \in \kappa^\omega\}.$$

Thus $p[T] \subseteq \omega^\omega$, it is the *projection* of T, and clearly

$$p[T] = \{x \in \omega^\omega \mid T_x \text{ is not wellfounded}\}.$$

Regarding $\mathbb{R}$ as ω^ω, which we will almost always do, a set $A \subseteq \mathbb{R}$ is κ-Suslin if there exists a tree T on $\omega \times \kappa$ such that $A = p[T]$. The set A is $(<\kappa)$-Suslin if A is δ-Suslin for some $\delta < \kappa$ and the set A is *Suslin* if A is κ-Suslin for some κ.

Assuming the Axiom of Choice every set $A \subseteq \mathbb{R}$ is Suslin. By restricting to the case that $A = p[T]$ where T is a tree on $\omega \times \kappa$ which satisfies additional combinatorial requirements, the notion becomes a deep structural property of the set A particularly in the context of large cardinals. The relevant requirements on the tree T involves measures.

Suppose that X is a nonempty set. We let $m(X)$ denote the set of countably complete ultrafilters on the boolean algebra $\mathcal{P}(X)$. Our convention is that μ is a *measure on* X if $\mu \in m(X)$. As usual for $\mu \in m(X)$ and $A \subseteq X$, we write $\mu(A) = 1$ to indicate that $A \in \mu$.

Suppose that $X = Y^{<\omega}$ and that $\mu \in m(X)$. Since μ is countably complete, there is a unique $k \in \omega$ such that $\mu(Y^k) = 1$. Suppose that μ_1 and μ_2 are measures on $Y^{<\omega}$. Let k_1 and k_2 be such that $\mu_1(Y^{k_1}) = 1$ and $\mu_2(Y^{k_2}) = 1$. Then μ_2 *projects to* μ_1 if $k_1 < k_2$ and, for all $A \subseteq Y^{k_1}$, $\mu_1(A) = 1$ if and only if $\mu_2(A^*) = 1$ where

$$A^* = \{s \in Y^{k_2} \mid s|k_1 \in A\}.$$

We write $\mu_1 < \mu_2$ to indicate that μ_2 projects to μ_1.

For each $\mu \in m(X)$ there is a canonical elementary embedding

$$j_\mu : V \to M_\mu$$

where M_μ is the transitive inner model obtained from taking the transitive collapse of V^X/μ. Suppose that $\mu_1 \in m(Y^{<\omega})$, $\mu_2 \in m(Y^{<\omega})$ and $\mu_1 < \mu_2$. Then there is a canonical elementary embedding

$$j_{\mu_1,\mu_2} : M_{\mu_1} \to M_{\mu_2}$$

such that $j_{\mu_2} = j_{\mu_1,\mu_2} \circ j_{\mu_1}$.

Suppose that $\langle \mu_k : k \in \omega \rangle$ is a sequence of measures on $Y^{<\omega}$ such that for all $k \in \omega$, $\mu_k(Y^k) = 1$. The sequence $\langle \mu_k : k \in \omega \rangle$ is a *tower* if for all $k_1 < k_2$, $\mu_{k_1} < \mu_{k_2}$. The tower, $\langle \mu_k : k \in \omega \rangle$, is *countably complete* if for any sequence $\langle A_k : k \in \omega \rangle$ such that for all $k < \omega$, $\mu_k(A_k) = 1$, there exists $f \in Y^\omega$ such that $f|k \in A_k$ for all $k \in \omega$. It is completely standard that if $\langle \mu_k : k \in \omega \rangle$ is a tower of measures on $Y^{<\omega}$ then the tower is countably complete if and only if the direct limit of the sequence $\langle M_{\mu_k} : k < \omega \rangle$ under the system of maps,

$$j_{\mu_{k_1},\mu_{k_2}} : M_{\mu_{k_1}} \to M_{\mu_{k_2}} \quad (k_1 < k_2 < \omega),$$

is wellfounded.

We come to the key notions of homogeneous trees and weakly homogeneous trees. These definitions are due independently to Kunen and Martin.

Definition 1. Suppose that κ is an ordinal and $\kappa \neq 0$. Suppose that T is a tree on $\omega \times \kappa$.

(1) The tree T is δ-*weakly homogeneous* if there is a partial function

$$\pi : \omega^{<\omega} \times \omega^{<\omega} \to m(\kappa^{<\omega})$$

such that for all $(s,t) \in \mathrm{dom}(\pi)$, $\pi(s,t)(T_s) = 1$ and $\pi(s,t)$ is a δ-complete measure, and such that for all $x \in \omega^\omega$, $x \in p[T]$ if and only if there exists $y \in \omega^\omega$ such that

(a) $\{(x|k, y|k) \mid k < \omega\} \subseteq \mathrm{dom}(\pi)$,
(b) $\langle \pi(x|k, y|k) : k \in \omega \rangle$ is a countably complete tower.

(2) The tree T is $(<\delta)$-*weakly homogeneous* if T is α-weakly homogeneous for all $\alpha < \delta$.

(3) The tree T is *weakly homogeneous* if T is δ-weakly homogeneous for some δ.

Definition 2. Suppose that κ is an ordinal and $\kappa \neq 0$. Suppose that T is a tree on $\omega \times \kappa$.

(1) The tree T is δ-*homogeneous* if there is a partial function

$$\pi : \omega^{<\omega} \to m(\kappa^{<\omega})$$

such that for all $s \in \mathrm{dom}(\pi)$ then $\pi(s)(T_s) = 1$ and $\pi(s)$ is a δ-complete measure, and such that for all $x \in \omega^\omega$, $x \in p[T]$ if and only if

(a) $\{x|k \mid k \in \omega\} \subseteq \mathrm{dom}(\pi)$,
(b) $\langle \pi(x|k) : k \in \omega \rangle$ is a countably complete tower.

(2) The tree T is $(<\delta)$-*homogeneous* if T is α-homogeneous for all $\alpha < \delta$.

(3) The tree T is *homogeneous* if T is δ-homogeneous for some δ.

Any tree on $\omega \times \omega$ is δ-weakly homogeneous for all δ and similarly any tree on $\omega \times 1$ is δ-homogeneous for all δ. In each case the associated measures are principal.

The following gives a reformulation of the notion that a tree be weakly homogeneous.

Lemma 3. *Suppose that T is a tree on $\omega \times \kappa$. Then T is δ-weakly homogeneous if and only if there exists a countable set $\sigma \subseteq m(\kappa^{<\omega})$ such that*

(1) *every measure in σ is δ-complete*;
(2) *for all $x \in \omega^\omega$, $x \in p[T]$ if and only if there is a countably complete tower $\langle \mu_k : k \in \omega \rangle$ of measures in σ such that for all $k \in \omega$, $\mu_k(T_{x|k}) = 1$.*

Homogeneity is a rather restrictive condition on a tree, weak homogeneity, however, is not.

Theorem 4. *Suppose that δ is a Woodin cardinal and T is a tree on $\omega \times \kappa$ for some κ. Then there exists an ordinal $\alpha < \delta$ such that if $G \subseteq \mathrm{Coll}(\omega, \alpha)$ is V-generic then in $V[G]$, T is $(<\delta)$-weakly homogeneous.*

Theorem 5. *Suppose that δ is a limit of Woodin cardinals and T is a tree on $\omega \times \kappa$ for some κ. Then the following are equivalent.*

(1) *T is $(<\delta)$-weakly homogeneous.*
(2) *There is a tree S on $\omega \times \delta$ such that for all $\mathbb{P} \in V_\delta$, if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$,*

$$p[T] = \mathbb{R}^{V[G]} \backslash p[S].$$

A set of reals $A \subseteq \mathbb{R}$ is *universally Baire* if for all compact Hausdorff spaces, Ω, and for all continuous functions,

$$\pi : \Omega \to \mathbb{R},$$

the preimage of A under π, $\pi^{-1}[A]$, has the property of Baire in Ω; [1]. The more natural definition (see [1]) uses the language of forcing and this is the definition we give below.

Definition 6. Suppose that $A \subseteq \omega^\omega$.

(1) The set A is δ-*universally Baire*, where δ is a cardinal, if for all partial orders $\mathbb{P}$ of cardinality δ there exist trees S and T in $\omega \times \kappa$ for some κ such that
 (a) $A = p[T]$,
 (b) if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$,

$$p[T] = \mathbb{R} \backslash p[S].$$

(2) The set A is $(<\delta)$-*universally Baire*, where δ is an uncountable limit cardinal, if A is κ-universally Baire for all $\kappa < \delta$.
(3) The set A is *universally Baire* if A is δ-universally Baire for all δ.

If A is δ-universally Baire for some cardinal δ and $G \subseteq \mathbb{P}$ is V-generic where $|\mathbb{P}|^V \leq \delta$, then in $V[G]$ the set A has a canonical interpretation as a set $A_G \subseteq \mathbb{R}^{V[G]}$:

$$A_G = \cup\{(p[T])^{V[G]} \,|\, T \in V \text{ and } A = (p[T])^V\}.$$

It is straightforward to show that

$$A_G = (p[T])^{V[G]}$$

where $T \in V$ is *any* tree such that there exists a tree S in V which complements T in $V[G]$; i.e. such that $(p[T])^{V[G]} = \mathbb{R}^{V[G]} \backslash (p[S])^{V[G]}$.

Using this fact one can easily show that if A is universally Baire then A has all the classical regularity properties (A is Lebesgue measurable, has the property of Baire etc.). Further one can show that the universally Baire sets are closed under countable unions, complements and preimages by any function

$$f : \mathbb{R} \to \mathbb{R}$$

which is $\underset{\sim}{\Sigma}^1_1$, or more generally by any function which is *universally Baire measurable*. A function

$$f : \mathbb{R} \to \mathbb{R}$$

is universally Baire measurable if for each open set $O \subset \mathbb{R}$, $f^{-1}[O]$ is universally Baire. This condition implies that the graph of f is universally Baire (as a subset of $\mathbb{R} \times \mathbb{R}$) but the latter is in general a weaker condition. To illustrate suppose that $f : \mathbb{R} \to \mathbb{R}$ is universally Baire measurable and $V[G]$ is a set generic extension of V. Then in $V[G]$, f_G is a total function,

$$f_G : \mathbb{R}^{V[G]} \to \mathbb{R}^{V[G]}.$$

If one only assumes the graph of f is universally Baire then all one can conclude is that f_G is a partial function, see Lemma 14. For example, if $V = L$ then there is a Π^1_1 function $f : \mathbb{R} \to \mathbb{R}$ such that $f_G = f$ in $V[G]$ for all generic extensions of V; define $f(x) = y$ if y is (the code of) the theory of L_α where α is least such that $x \in L_\alpha$ and $L_\alpha \vDash \mathrm{ZC}$. By the choice of α it follows that every element of L_α is definable in L_α and so y codes L_α.

If $V = L$, or simply if $\mathbb{R} \subseteq L$, then for *every* set $A \subseteq \mathbb{R}$ there is a universally Baire set $B \subseteq \mathbb{R}$ and a continuous function, $f : \mathbb{R} \to \mathbb{R}$, such that $A = f[B]$. More generally this conclusion holds assuming CH and that there is an ω_1 sequence of distinct reals, $\langle x_\alpha : \alpha < \omega_1 \rangle$, such that the set of reals which code initial segments of the sequence, is universally Baire. These assumptions actually hold in the current generation of inner models below the first Woodin cardinal of the model, provided the inner model satisfies a certain natural closure condition (the inner model is closed under the *iteration strategies* for all of its initial segments which are bounded by the least Woodin cardinal of the inner model), but this is a deep fact.

Definition 7. Suppose that $A \subseteq \omega^\omega$.

(1) The set A is δ-*weakly homogeneously Suslin* if $A = p[T]$ for some tree T which is δ-weakly homogeneous. The set A is δ-*homogeneously Suslin* if $A = p[T]$ for some tree T which is δ-homogeneous.
(2) The set A is $(<\delta)$-*weakly homogeneously Suslin* if A is α-weakly homogeneously Suslin for all $\alpha < \delta$. The set A is $(<\delta)$-*homogeneously Suslin* if A is α- homogeneously Suslin for all $\alpha < \delta$.
(3) The set A is *weakly homogeneously Suslin* if A is δ-weakly homogeneously Suslin for some δ. The set A is *homogeneously Suslin* if A is δ-homogeneously Suslin for some δ.

The next theorem is an immediate corollary of the definitions and Theorem 5.

Theorem 8. *Suppose that δ is a limit of Woodin cardinals and that $A \subseteq \mathbb{R}$. Then the following are equivalent.*

(1) *A is $(<\delta)$-universally Baire.*
(2) *A is $(<\delta)$-weakly homogeneously Suslin.*

The elementary connection between the notions of being weakly homogeneously Suslin and being homogeneously Suslin is given in the following lemma.

Lemma 9. *Suppose that $A \subseteq \mathbb{R}$. Then A is δ-weakly homogeneously Suslin if and only if A is the continuous image of a set B which is δ-homogeneously Suslin.*

A much deeper connection is given by the seminal theorem of Martin and Steel, [6].

Theorem 10 (Martin–Steel). *Suppose that δ is a Woodin cardinal, $A \subseteq \mathbb{R}$ and that A is (δ^+)-weakly homogeneously Suslin. Then $\mathbb{R}\backslash A$ is $(<\delta)$-homogeneously Suslin.*

Putting everything together we obtain:

Theorem 11. *Suppose that δ is a limit of Woodin cardinals and that $A \subseteq \mathbb{R}$. Then the following are equivalent.*

(1) *A is $(<\delta)$-universally Baire.*
(2) *A is $(<\delta)$-weakly homogeneously Suslin.*
(3) *A is $(<\delta)$-homogeneously Suslin.*

Closely related to Suslin representations are the notions of a *norm* and a *scale*, these are due to Moschovakis. We give the definitions only in a special case and so ignore what are perhaps the most important aspects of the definitions, [9] is a good reference for the general definitions.

Definition 12. Suppose $A \subseteq \mathbb{R}$. A *norm* on A is a function;

$$\rho : A \to \mathrm{Ord}$$

with range an ordinal.

If $\rho : A \to \mathrm{Ord}$ is a norm on A we often identify the norm ρ with the associated *prewellordering* which we denote by $\leq_\rho$ and define by:

$$\leq_\rho = \{(x, y) \in A \times A \mid \rho(x) \leq \rho(y)\}.$$

In considering a norm ρ, we will often write that $\rho \in \Gamma$ meaning that $\leq_\rho \in \Gamma$. Generally it will be clear from the context whether in discussing a norm ρ we are referring to the actual norm or to the associated prewellordering.

Definition 13. Suppose $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. A *scale* on A is a sequence, $\langle \rho_i : i < \omega \rangle$, of norms on A such that for all sequences, $\langle x_i : i < \omega \rangle$, from A which converge to x, if for all $i < \omega$, the sequence,

$$\langle \rho_i(x_k) : k < \omega \rangle$$

is eventually constant then $x \in A$ and for all $k < \omega$,

$$\rho_k(x) \leq \rho_k(x_i)$$

for all sufficiently large i.

Suppose that T is a tree on $(\omega \times \kappa)$ and $A = p[T]$ (with $A \neq \emptyset$). For each $x \in p[T]$ let $f_x \in \kappa^\omega$ be such that for all $0 < i < \omega$

$$f_x|i = \min\{g|i \mid (x, g) \in [T]\},$$

where the minimum is calculated relative to the lexicographical order on Ord^i. Then $(x, f_x) \in [T]$.

For each $0 < i < \omega$ and for all $x, y \in A$ define $x \leq_i y$ if $f_x|i \leq f_y|i$ (in the lexicographical order) and let

$$\rho_i : A \to \mathrm{Ord}$$

be the associated norm. Then $\langle \rho_i : i < \omega \rangle$ is a scale on A, it is the scale of the tree T. Conversely if $\langle \rho_i : i < \omega \rangle$ is a scale on A with the property that for all $i < k$ and for all $x, y \in A$ if $\rho_k(x) \leq \rho_k(y)$ then $\rho_i(x) \leq \rho_i(y)$, then $\langle \rho_i : i < \omega \rangle$ is the scale of a tree T such that $p[T] = A$.

Suppose $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, Γ is closed under finite unions and complements and for any Δ^1_1 function,

$$f : \mathbb{R} \to \mathbb{R},$$

Γ is closed under preimages by f. Then Γ has an unambiguous interpretation as a subset of $\mathcal{P}(\mathbb{R} \times \mathbb{R})$. Suppose that $A \subseteq \mathbb{R}$ then A has a scale each norm of which is in Γ if there is a scale, $\langle \rho_i : i < \omega \rangle$, on Γ such that for each $i < \omega$;

$$\{(x, y) \mid \rho_i(x) \leq \rho_i(y)\} \in \Gamma.$$

In particular if δ is an uncountable cardinal then the notion that A has a scale each norm of which is $({<}\delta)$-universally Baire is well-defined. It is not difficult to show that if ρ is a norm on A and ρ is $({<}\delta)$-universally Baire then A is $({<}\delta)$-universally Baire. Further if $|\mathbb{P}| < \delta$ and if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$, ρ_G is a norm on A_G, more precisely in $V[G]$, $(\leq_\rho)_G$ is the prewellordering given by a norm on A_G which we denote by ρ_G. Whereas, $(\leq_\rho)_G \cap V = \leq_\rho$, in general $\rho_G|V \neq \rho$.

Similarly if $\langle \rho_i : i < \omega \rangle$ is a scale on A each norm of which is $({<}\delta)$-universally Baire and if $V[G]$ is a generic extension of V for a partial order $\mathbb{P}$ with $|\mathbb{P}|^V < \delta$, then in $V[G]$, $\langle (\rho_i)_G : i < \omega \rangle$ is a scale on A_G.

To verify these claims, fix a partial order $\mathbb{P}$ and fix $\alpha \in \mathrm{Ord}$ such that $\mathbb{P} \in V_\alpha$ and such that $V_\alpha \prec_{\Sigma_1} V$. Let $X \prec V_\alpha$ be a countable elementary substructure with $\mathbb{P} \in X$. Let M be the transitive collapse of X and let $\mathbb{P}_M$ be the image of $\mathbb{P}$ under the collapsing map. With this notation we have the following lemma from which the claims above follow immediately. This lemma allows one to fairly easily understand the generic absoluteness properties of universally Baire sets.

Lemma 14. *Suppose $G \subseteq \mathbb{P}_M$ is M-generic. Then for each $|\mathbb{P}|$-universally Baire set $B \in X$,*

$$(B_G)^{M[G]} = B \cap M[G].$$

Proof. Since $X \prec V_\alpha \prec_{\Sigma_1} V$, there exists $\kappa \in X \cap \alpha$ and there exists a pair of trees, (S, T), on $\omega \times \kappa$ such that

(1.1) $(S, T) \in X$,
(1.2) $B = p[T]$,
(1.3) $V^{\mathbb{P}} \vDash$ "$p[T] = \mathbb{R} \backslash p[S]$".

Let (S_M, T_M) be the image of (S, T) under the transitive collapse of X. Thus

$$M[G] \vDash \text{“}p[T_M] = \mathbb{R} \backslash p[S_M]\text{”},$$

and $(B_G)^{M[G]} = (p[T_M])^{M[G]}$. By absoluteness,

(2.1) $(p[T_M])^{M[G]} = p[T_M] \cap M[G]$
(2.2) $(p[S_M])^{M[G]} = p[S_M] \cap M[G]$,

and so $p[T_M] \cap M[G] = (\mathbb{R} \backslash p[S_M]) \cap M[G]$. Finally $p[T_M] \subseteq p[T]$ and $p[S_M] \subseteq p[S]$ and so $B \cap M[G] = p[T] \cap M[G] = p[T_M] \cap M[G] = (B_G)^{M[G]}$. $\square$

A very useful consequence of the property that a set of reals is $(<\delta)$-weakly homogeneously Suslin where δ is a limit of Suslin cardinals is given by the following theorem of Steel. This theorem provides a key tool for producing scales on sets of reals whose norms are also $(<\delta)$-weakly homogeneously Suslin.

Theorem 15 (Steel; [4]). *Suppose δ is a limit of Woodin cardinals and that A is $(<\delta)$-weakly homogeneously Suslin. Then A has a scale each norm of which is $(<\delta)$-weakly homogeneously Suslin.*

As an immediate corollary we get yet another equivalence that a set $A \subseteq \mathbb{R}$ be $(<\delta)$-universally Baire in the case that δ is a limit of Woodin cardinals.

Theorem 16. *Suppose that δ is a limit of Woodin cardinals and that $A \subseteq \mathbb{R}$. Then the following are equivalent.*

(1) *A is $(<\delta)$-universally Baire.*
(2) *A has a scale each norm of which is $(<\delta)$-universally Baire.*
(3) *A and $\mathbb{R} \backslash A$ each have scales each norm of which is $(<\delta)$-universally Baire.*

Finally while the situation of generic absoluteness for universally Baire sets can be subtle, as illustrated by the example in L of a Π^1_1 function, f, such that $f_G = f$ in $L[G]$ for all generic extensions of L; with the closure properties implied by the existence of large cardinals one obtains very strong forms of generic absoluteness — as a preliminary remark note that if $f : \mathbb{R} \to \mathbb{R}$ is function whose graph is κ-weakly homogeneously Suslin, then for all $\mathbb{P} \in V_\kappa$ if $G \subset \mathbb{P}$ is V-generic then in $V[G]$, f_G is a total function.

Theorem 17. *Suppose that $\Gamma \subset \mathcal{P}(\mathbb{R})$ and*:

(i) *Every set in Γ is universally Baire.*
(ii) *Every set in Γ has a scale each norm of which is in Γ.*

(iii) Γ *is closed under images and preimages by borel functions.*
(iv) *For each* $A \in \Gamma$, $(A, \mathbb{R})^\# \in \Gamma$.

Suppose that $V[G]$ *is a set generic extension of* V. *Then for each* $A \in \Gamma$ *there is an elementary embedding,*

$$j : L(A, \mathbb{R}) \to L(A_G, \mathbb{R}^{V[G]}).$$

Proof. Since Γ is closed under preimages by borel functions, for each $k < \omega$, there is an unambiguous interpretation of Γ as a subset of $\mathbb{R}^k$.

Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is a set in Γ. Let $\langle \rho_i : i < \omega \rangle$ be a scale on A each norm of which is in Γ. By modifying the scale $\langle \rho_i : i < \omega \rangle$ if necessary we can suppose that for all $i < \omega$, for all $a, b \in A$,

(1.1) if $\rho_i(a) = \rho_i(b)$ then $a|i = b|i$,
(1.2) if $\rho_{i+1}(a) \leq \rho_{i+1}(b)$ then $\rho_i(a) \leq \rho_i(b)$,

where for all $(x, y) \in \mathbb{R} \times \mathbb{R}$, $(x, y)|i = (x|i, y|i)$.

Let B be the projection of A and let

$$f : B \to \mathbb{R}$$

be the uniformization function for A given by the scale, $\langle \rho_i : i < \omega \rangle$. Thus for all $x \in B$, $(x, f(x)) \in A$ and for all $(x, y) \in A$,

$$\rho_i(x, f(x)) \leq \rho_i(x, y)$$

for all $i < \omega$. This uniquely specifies f.

It follows that for each basic open set $O \subseteq \mathbb{R}$, $f^{-1}[O] \in \Gamma$. Therefore since every set in Γ is universally Baire, f is universally Baire measurable.

This shows that every set in Γ can be uniformized by a universally Baire measurable function. Since for every set $A \in \Gamma$, $(A, \mathbb{R})^\# \in \Gamma$ the conclusion of the theorem follows. Again this is fairly straightforward using Lemma 14. Suppose $\mathbb{P}$ is a partial order and fix $\gamma \in \mathrm{Ord}$ such that $\mathbb{P} \in V_\gamma$ and such that

$$V_\gamma \prec_{\Sigma_1} V.$$

Suppose

$$X \prec V_\gamma$$

is countable with $\mathbb{P} \in X$. Let M be the transitive collapse of X and let $\mathbb{P}_M$ be the image of $\mathbb{P}$ under the collapsing map.

Suppose $G \subseteq \mathbb{P}_M$ is M-generic with $G \in V$. By Lemma 14, for each $A \in \Gamma \cap X$,

$$(A_G)^{M[G]} = A \cap M[G],$$

and for each function,

$$f : \mathbb{R} \to \mathbb{R}$$

such that f is universally Baire measurable, if $f \in X$ then

$$(f_G)^{M[G]} = f \cap M[G].$$

Since f is universally Baire measurable,

$$\mathrm{dom}((f_G)^{M[G]}) = \mathbb{R} \cap M[G]$$

and so it follows that $M[G]$ is closed under f. Since every set in Γ can be uniformized by a universally Baire measurable function and since for each $A \in \Gamma \cap X$, $(A, \mathbb{R})^\# \in \Gamma \cap X$, it follows that for each $A \in \Gamma \cap X$,

$$\langle V_{\omega+1} \cap M[G], (A, \mathbb{R})^\# \cap M[G] \rangle \prec \langle V_{\omega+1}, (A, \mathbb{R})^\# \rangle,$$

noting that if $B = (A, \mathbb{R})^\#$ then $B \in X$ and

$$(B_G)^{M[G]} = B \cap M[G].$$

Therefore there is an elementary embedding,

$$j : (L(A, \mathbb{R}))^M \to (L(A_G, \mathbb{R} \cap M[G]))^{M[G]}. \qquad \square$$

2.2. AD$^+$

We begin with some definitions.

Definition 18. Suppose $A \subseteq \mathbb{R}$. The set A is $^\infty$*-borel* if there exist a set $S \subseteq \mathrm{Ord}$, $\alpha \in \mathrm{Ord}$, and a formula $\phi(x_0, x_1)$ such that

$$A = \{y \in \mathbb{R} \mid L_\alpha[S, y] \vDash \phi[S, y]\}.$$

There are many equivalent definitions of the $^\infty$borel sets, for example, given a formula ϕ, for each set $S \subseteq \mathrm{Ord}$, the set

$$A = \{y \in \mathbb{R} \mid L[S, y] \vDash \phi[S, y]\},$$

is easily seen to be $^\infty$borel.

One can also define a set $A \subseteq \mathbb{R}$ to be $^\infty$borel if A has a transfinite borel code: but it is important that the transfinite borel code be (coded by) a set of ordinals; i.e. that the wellordered unions be effective.

Assuming $\mathrm{AD} + \mathrm{DC}$ there is yet another equivalent definition.

Lemma 19 (AD + DC). *Suppose $A \subseteq \mathbb{R}$. The following are equivalent.*

(1) *A is $^\infty$borel.*
(2) *There exists $S \subseteq \mathrm{Ord}$ such that $A \in L(S, \mathbb{R})$.*

Suppose $\lambda \in \mathrm{Ord}$ and that $A \subseteq \lambda^\omega$. The set A is *determined* if there exists a winning strategy for Player I or for Player II in the game on λ corresponding to the set A.

In contrast to the following well known and standard lemma, assuming

$$\mathrm{ZFC} + \text{“There is a Woodin cardinal”}$$

is consistent then so is

$$\mathrm{ZFC} + \text{“Every OD set } A \subseteq \omega_1^\omega \text{ is determined”}.$$

Lemma 20 (ZF). *There exists a set $A \subseteq \omega_1^\omega$ such that A is not determined.*

A very interesting question is whether

$$\text{ZFC} + \text{“Every OD set } A \subseteq \text{Ord}^\omega \text{ is determined”}$$

is consistent. We conjecture the answer is no, see p. 299 for the relevant conjecture.

Suppose T is a tree on $\omega \times \lambda$. We use the notation from Sec. 2.1 and define a set $A_T \subseteq \omega^\omega$ as follows. $x \in A_T$ if Player I has a winning strategy in the game corresponding to $B_x \subseteq \lambda^\omega$ where

$$B_x = [T_x] = \{f \in \lambda^\omega \,|\, (x, f) \in [T]\}.$$

The set A_T is easily verified to be ${}^\infty$borel. If the set

$$[T] \subseteq \omega^\omega \times \lambda^\omega$$

is clopen in the product space, $\omega^\omega \times \lambda^\omega$, we shall say that T is an ${}^\infty$borel code of A_T. Note that in the case that $\lambda = \omega$, A_T is necessarily $\underset{\sim}{\Sigma}^1_1$ and if $[T]$ is clopen then A_T is borel. It is not difficult to show that every ${}^\infty$borel set has an ${}^\infty$borel code.

One important feature of the ${}^\infty$borel sets is that assuming AD the property of being ${}^\infty$borel is a *local* property. One manifestation of this is given in the following lemma where for a set $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, M_Γ is the set of all sets X such that X can be coded by sets in Γ; i.e. there is a surjection

$$\pi : \mathbb{R} \to Y,$$

where Y is the transitive closure of X, such that both $\{a \,|\, \pi(a) \in X\}$ and $\{(a, b) \,|\, \pi(a) \in \pi(b)\}$ belong to Γ. If Γ is closed under finite unions, complements, and preimages by all Δ^1_1 functions,

$$f : \mathbb{R} \to \mathbb{R},$$

then Γ has an unambiguous interpretation as a set $\Gamma^* \subseteq \mathcal{P}(\mathbb{R} \times \mathbb{R})$ and so M_Γ is both well-defined and contains Γ. If in addition Γ is closed under preimages by *all* borel functions,

$$f : \mathbb{R} \to \mathbb{R},$$

then M_Γ is transitive.

Lemma 21 ($\mathbf{ZF + AD + DC_{\mathbb{R}}}$). *Suppose $A \subseteq \mathbb{R}$ and that A is ${}^\infty$borel. Let Γ be the pointclass of sets which are projective in A. Then there exists*

$$T \in M_\Gamma$$

such that T is an ${}^\infty$borel code for A.

Assuming AD many ordinal games *are* determined and this is closely related to the existence of Suslin representations for sets of reals.

Recall that Θ is the least ordinal which is not the range of a function with domain $\mathbb{R}$. The Axiom of Choice implies $\Theta = c^+$. Using the notation, M_Γ, defined above, Θ is the least ordinal α such that $\alpha \notin M_\Gamma$ where $\Gamma = \mathcal{P}(\mathbb{R})$.

We now give the definition of AD^+.

Definition 22 ($\mathbf{ZF} + \mathbf{DC}_{\mathbb{R}}$). AD^+:

(1) Suppose $A \subseteq \mathbb{R}$. Then A is ${}^\infty$borel.
(2) Suppose $\lambda < \Theta$ and

$$\pi : \lambda^\omega \to \omega^\omega$$

is a continuous function. Then for each $A \subseteq \mathbb{R}$ the set $\pi^{-1}[A]$ is determined.

The motivation for isolating the ordinal determinacy hypothesis in the definition of AD^+ originates from the following theorem in [3].

Theorem 23 ($\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}_{\mathbb{R}}$). *Suppose that $\lambda < \Theta$,*

$$\pi : \lambda^\omega \to \omega^\omega$$

is a continuous function, and that $A \subseteq \mathbb{R}$ is a set which is both Suslin and co-Suslin. Then the set $\pi^{-1}[A]$ is determined.

One important feature of AD^+ is that it is downward absolute.

Theorem 24 ($\mathbf{ZF} + \mathbf{DC}_{\mathbb{R}}$). *Assume* AD^+ *and that M is a transitive inner model of* ZF *such that $\mathbb{R} \subseteq M$. Then $M \vDash \mathrm{AD}^+$.*

Proof. Suppose $\delta < \Theta^M$. Then by the Moschovakis Coding Lemma, [9],

$$\mathcal{P}(\delta) \subseteq M.$$

The theorem follows by Lemma 21. □

The next theorem illustrates the utility of AD^+, showing that assuming $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ the basic analysis of $L(\mathbb{R})$ generalizes. In fact the theorem gives consequences of AD^+ which taken together are equivalent to AD^+ over the base theory $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$.

For each $x \in \mathbb{R}$, $\Delta^2_1(x)$ is the set of all $A \subseteq \mathbb{R}$ such that both A and $\mathbb{R}\backslash A$ are $\Sigma^2_1(x)$; i.e. such that both A and $\mathbb{R}\backslash A$ are Σ_1-definable in $L(\mathcal{P}(\mathbb{R}))$ from $(\mathbb{R}, x)$.

Theorem 25 (($\mathbf{ZF}+\mathbf{DC}_{\mathbb{R}}$): $\boldsymbol{\Sigma^2_1}$-Basis Theorem). *Assume* $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Then*:

(1) *The pointclass Σ^2_1 has the scale property.*
(2) *Suppose $A \subseteq \mathbb{R}$ is Σ^2_1. Then*

$$A = p[T]$$

for some tree $T \in \mathrm{HOD}$.
(3) *For all $x \in \mathbb{R}$, $M_{\Delta^2_1(x)} \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$.*

Conclusion (3) of Theorem 25 implies the Δ^2_1 basis theorem for AD^+: if $X \subseteq \mathcal{P}(\mathbb{R})$ is Σ_1-definable in $L(\mathcal{P}(\mathbb{R}))$ from $\{\mathbb{R}\}$ then there exists $Z \in X$ such that

Z is Δ^2_1. So for example, assuming $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}^+$, every Δ^2_1 set has a Δ^2_1-scale. We do not know if assuming $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}^+$ one can prove

$$M_{\Delta^2_1} \prec_{\Sigma_1} V,$$

but assuming $\mathrm{ZF}+\mathrm{AD}^+$ and just that DC holds in $L(\mathcal{P}(\mathbb{R}))$, it is possible that

$$M_{\Delta^2_1} \not\prec_{\Sigma_1} V.$$

A central open question in the theory of AD^+ is whether $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}$ *implies* AD^+. For stronger versions of AD the answer is yes.

Let *Uniformization* abbreviate the assumption that for all

$$A \subseteq \mathbb{R}\times\mathbb{R}$$

there exists a function $F:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$, if $(x,y)\in A$ for some $y\in\mathbb{R}$ then

$$(x, F(x)) \in A.$$

Uniformization is a trivial consequence of $\mathrm{AD}_{\mathbb{R}}$, which is the strengthening of AD that asserts that every real game of length ω is determined.

Theorem 26 (ZF + DC). *The following are equivalent.*

(1) AD + Uniformization.
(2) $\mathrm{AD}_{\mathbb{R}}$.
(3) AD + *Every set of reals is Suslin.*

As a corollary one obtains the following theorem which using Theorem 23, trivially shows that $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}_{\mathbb{R}}$ implies AD^+.

Theorem 27 (ZF + DC). *Assume* AD + Uniformization. *Then* AD^+.

Another theorem relevant to the question of whether $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}$ implies AD^+ is the following.

Theorem 28 ($\mathbf{ZF+AD+DC_{\mathbb{R}}}$). *Define*

$$\Gamma = \{A\subseteq\mathbb{R} \mid L(A,\mathbb{R})\vDash \mathrm{AD}^+\}$$

Then:

(1) $L(\Gamma,\mathbb{R})\vDash\mathrm{AD}^+$;
(2) *Suppose that* $\Gamma\neq\mathcal{P}(\mathbb{R})$ (*i.e. that* AD^+ *fails*) *then* $L(\Gamma,\mathbb{R})\vDash\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}_{\mathbb{R}}$.

The next two theorems record some basic facts relating to AD^+ in the context of universally Baire sets and large cardinals, which we shall need. These are not stated in the strongest forms, but are simply stated in the form we shall need.

Theorem 29. *Suppose there is a proper class of Woodin cardinals and that* $A\subseteq\mathbb{R}$ *is universally Baire. Then*

(1) $(A,\mathbb{R})^\#$ *is universally Baire*,
(2) $L(A,\mathbb{R})\vDash\mathrm{AD}^+$.

Theorem 30. *Suppose that δ is strongly inaccessible and δ is a limit of Woodin cardinals and a limit of cardinals which are strong in V_δ. Suppose $A \subseteq \mathbb{R}$ and A is universally Baire in V_δ. Then there exists $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ such that*

(1) *every set in Γ is universally Baire in V_δ,*
(2) $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$,
(3) $A \in \Gamma$,
(4) $L(\Gamma, \mathbb{R}) \vDash \mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$.

We shall also need the Derived Model Theorem.

Theorem 31 (Derived Model Theorem). *Suppose that δ is a limit of Woodin cardinals. Suppose $g \subseteq \mathrm{Coll}(\omega, {<}\delta)$ is V-generic and let*

$$\mathbb{R}_g = \cup\{(\mathbb{R})^{V[g|\alpha]} \mid \alpha < \delta\}.$$

Let Γ be the set of all

$$A \in \mathcal{P}(\mathbb{R}_g) \cap V(\mathbb{R}_g)$$

such that $L(A, \mathbb{R}_g) \vDash \mathrm{AD}^+$. Then in $V(\mathbb{R}_g)$ the following hold.

(1) $L(\Gamma, \mathbb{R}_g) \vDash \mathrm{AD}^+$.
(2) *For each $A \in \mathcal{P}(\mathbb{R}_g) \cap V(\mathbb{R}_g)$ the following are equivalent.*
   (a) *A is Suslin, co-Suslin in $V(\mathbb{R}_g)$.*
   (b) *$A \in \Gamma$ and A is Suslin, co-Suslin in $L(\Gamma, \mathbb{R}_g)$.*

Suppose that there is a proper class of Woodin cardinals. If there is a supercompact cardinal then by passing to a generic extension of V, the theory of $L(\Gamma^\infty)$ can be "sealed" in a very strong sense, where Γ^∞ is the set of all universally Baire sets. A proof of the following theorem which illustrates one aspect of this can be found in [4]. The actual "sealing" which occurs is much stronger.

Theorem 32. *Suppose that δ is supercompact and that there is a proper class of Woodin cardinals. Suppose that $V[G_0]$ is a generic extension of V, $V[G_0][G_1]$ is a generic extension of $V[G_0]$, and that $V_{\delta+2}$ is countable in $V[G_0]$. Then in $V[G_0][G_1]$ there is an elementary embedding*

$$j : (L(\Gamma^\infty, \mathbb{R}))^{V[G_0]} \to (L(\Gamma^\infty, \mathbb{R}))^{V[G_0][G_1]}.$$

There is a corollary of this theorem.

Theorem 33. *Suppose that δ is supercompact and that there is a proper class of Woodin cardinals. Suppose that $V[G_0]$ is a generic extension of V and that $V_{\delta+2}$ is countable in $V[G_0]$. Then in $V[G_0]$,*

$$(L(\Gamma^\infty, \mathbb{R}))^{V[G_0]} \vDash \mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$$

and $(\Gamma^\infty)^{V[G_0]} = \mathcal{P}(\mathbb{R}^{V[G_0]}) \cap (L(\Gamma^\infty, \mathbb{R}))^{V[G_0]}$.

2.3. Ω-*logic and the* Ω *Conjecture*

We briefly review the definition of Ω-logic.

Definition 34. Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Then

$$T \models_\Omega \phi$$

if for all partial orders, $\mathbb{P}$, for all ordinals, α, if

$$V_\alpha^{\mathbb{P}} \models T$$

then $V_\alpha^{\mathbb{P}} \models \phi$.

If there is a proper class of Woodin cardinals then the relation $T \models_\Omega \phi$ is generically absolute. This fact makes Ω-logic interesting from a meta-mathematical point of view. For example, the set

$$\mathcal{V}_\Omega = \{\phi \,|\, \emptyset \models_\Omega \phi\}$$

is generically absolute so for a given sentence, ϕ, the question whether or not ϕ is logically Ω-valid, i.e. whether or not $\phi \in \mathcal{V}_\Omega$, is absolute between V and all of its generic extensions.

Theorem 35. *Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Suppose that there exists a proper class of Woodin cardinals.*

Then for all partial orders, $\mathbb{P}$,

$$V^{\mathbb{P}} \models \text{“}T \models_\Omega \phi\text{”}$$

if and only if $T \models_\Omega \phi$.

We next define the proof relation, $T \vdash_\Omega \phi$. This we do in ZFC with no additional large cardinal assumptions. In fact the proper context for Ω-logic is under assumption that there is a proper class of Woodin cardinals and in this context the definition of the proof relation for Ω-logic is much simpler to state, see Lemma 41.

Suppose that $A \subseteq \mathbb{R}$ and for every borel function,

$$f : \mathbb{R} \to \mathbb{R},$$

$f^{-1}[A]$ has the property of Baire in $\mathbb{R}$. Then for all countable partial orders, $\mathbb{P}$, if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$, the set A has a canonical interpretation A_G where

$$A_G = \cup\{(p[T])^{V[G]} \,|\, T \in V \text{ and } A = (p[T])^V\}.$$

The *term relation* for A, denoted τ_A, is the set of all triples,

$$(p, \sigma, \mathbb{P}) \in H(\omega_1)$$

such that $\mathbb{P}$ is a partial order, $p \in \mathbb{P}$, $\sigma \in V^{\mathbb{P}}$ and

$$p \Vdash \sigma \in A_G.$$

Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then for each partial order, $\mathbb{P}$, if $G \subseteq \mathbb{P}$ is V-generic then A has a canonical interpretation $A_G \subseteq \mathbb{R}^{V[G]}$. This defines an extension of τ_A, denoted τ_A^∞, to all partial orders. Thus τ_A^∞ is the class of all triples $(p, \sigma, \mathbb{P})$ such that $p \in \mathbb{P}$, $\sigma \in V^{\mathbb{P}}$ and

$$p \Vdash \sigma \in A_G.$$

Suppose M is a transitive model of ZFC and $A \subseteq \mathbb{R}$ is universally Baire. Then M is A-*closed* if for all $b \in M$,

$$\tau_A^\infty \cap b \in M.$$

Notice that this is equivalent to the condition that if $\mathbb{P} \in M$ and if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$,

$$A_G \cap M[G] \in M[G].$$

The transitive set M is A-*full* if M is A-closed and if $g \subseteq \mathrm{Coll}(\omega, M)$ is V-generic then in $V[g]$ for all $\mathbb{P} \in M$, for all $G \subset \mathbb{P}$, if G is M-generic then

$$A_g \cap M[G] = \{I_G(\sigma) \mid (p, \sigma, \mathbb{P}) \in \tau_A^\infty \cap M \text{ for some } p \in G\}.$$

Remark 36. Suppose that $A \subseteq \mathbb{R}$ is universally Baire. The definitions of A-closed and A-full are simpler to state in the case that M is a transitive model of ZFC which is *countable*:

(1) M is A-closed if and only if for all $\mathbb{P} \in M$, for comeager many filters $G \subset \mathbb{P}$, if G is M-generic then

$$A \cap M[G] \in M[G].$$

(2) M is A-full if and only if M is A-closed and for all $\mathbb{P} \in M$, for all $G \subset \mathbb{P}$, if G is M-generic then

$$A \cap M[G] = \{I_G(\sigma) \mid (p, \sigma, \mathbb{P}) \in \tau_A \cap M \text{ for some } p \in G\}.$$

(3) Suppose that $A \cap M[G] \in M[G]$ for all M-generic filters G. Then M is A-closed.
(4) M is $\{x\}$-closed for all $x \in \mathbb{R}$. By taking x to be a Cohen real over M, it follows that the requirement that $A \cap M[G] \in M[G]$ for all M-generic filters G does not imply that M is A-full.

Remark 37. Suppose $V_\delta \vDash \mathrm{ZFC}$, $A \subseteq \mathbb{R}$ and A is $({<}\delta)$-universally Baire. Suppose that $X \prec V_\delta$, $A \in X$ and let M be the transitive collapse of X. Then M is A-full.

Lemma 38. *Suppose that $\Gamma_0 \subseteq \mathcal{P}(\mathbb{R})$ is countable, each $A \in \Gamma_0$ is universally Baire, and that Γ_0 is closed under complements, finite unions, and preimages by Δ_1^1 functions*

$$f : \mathbb{R} \to \mathbb{R}.$$

Suppose that each $A \in \Gamma_0$ admits a scale each norm of which is in Γ_0. Then for each transitive set M, if M is A-closed for each $A \in \Gamma_0$ then M is A-full for each $A \in \Gamma_0$.

Proof. By reflection it suffices to prove the lemma for countable transitive models and in this case we can use the alternative formulations for A-closed and A-full indicated above in Remark 36.

Suppose M is a countable transitive set, $M \vDash \mathrm{ZFC}$, and M is A-closed for each $A \in \Gamma_0$. Fix $\lambda \in M \cap \mathrm{Ord}$.

Fix $A_0 \in \Gamma_0$. Suppose $\sigma \in M^{\mathrm{Coll}(\omega,\lambda)}$ is a term for a real, $p \in \mathrm{Coll}(\omega, \lambda)$. Suppose that for almost all M-generic filters, $G \subset \mathrm{Coll}(\omega, \lambda)$, if $p \in G$ then $I_G(\sigma) \in A_0$. We show that this implies that for *all* M-generic filters $G \subseteq \mathrm{Coll}(\omega, \lambda)$, if $p \in G$ then $I_G(p) \in A_0$. The lemma follows. Clearly we can suppose that $p = 1$.

The key point is the following. Suppose ρ is a norm on a set $A \in \Gamma_0$ and that $\leq_\rho \in \Gamma_0$ where $\leq_\rho$ is the associated prewellordering. Then for almost all M-generic filters, $G_0 \subseteq \mathrm{Coll}(\omega, \lambda)$, for almost all $M[G_0]$-generic filters $G_1 \subseteq \mathrm{Coll}(\omega, \lambda)$, the following holds:

(1.1) For all $x \in M[G_0] \cap A$ there exists $y \in M[G_1] \cap A$ such that $\rho(x) = \rho(y)$.

Suppose toward a contradiction that this fails. Then since $\leq_\rho$ is universally Baire and since M is $\leq_\rho$-closed, there exists $p_0 \in \mathrm{Coll}(\omega, \lambda)$ and there exists a term $\tau_0 \in M$ such that for almost all M-generic filters

$$G \subseteq \mathrm{Coll}(\omega, \lambda)$$

if $p_0 \in G$ then $I_G(\tau_0) = \leq_\rho \cap M[G]$, where $I_G(\tau_0)$ is the interpretation of τ_0 by G.

This implies, since $\mathrm{Coll}(\omega, \lambda)$ is homogeneous, there exists a term $\tau \in M$ such that

$$I_G(\tau) = \leq_\rho \cap M[G]$$

for almost all M-generic filters $G \subseteq \mathrm{Coll}(\omega, \lambda)$.

Therefore for almost all M-generic filters, $G_0 \subseteq \mathrm{Coll}(\omega, \lambda)$, for almost all $M[G_1]$-generic filters $G_1 \subseteq \mathrm{Coll}(\omega, \lambda)$,

(2.1) $I_{G_0}(\tau) = \leq_\rho \cap M[G_0]$,
(2.2) $I_{G_1}(\tau) = \leq_\rho \cap M[G_1]$,
(2.3) there exists $x \in A \cap M[G_0]$ such that for all $y \in M[G_1] \cap A$, $\rho(x) \neq \rho(y)$,

where as above, $I_{G_0}(\tau)$ is the interpretation of τ by G_0 and $I_{G_1}(\tau)$ is the interpretation of τ by G_1.

Let $\mathbb{B}$ be the complete Boolean given by $\mathrm{Coll}(\omega, \lambda)$, as defined in M and let

$$\pi : \mathbb{B} \to \mathbb{B}$$

be a Boolean automorphism with $\pi \in M$. Then for almost all M-generic filters $G \subseteq \mathbb{B}$,

$$I_G(\tau) = I_{\pi[G]}(\tau) = \leq_\rho \cap M[G].$$

By the definability of forcing, for all M-generic filters $G \subseteq \mathbb{B}$, for all Boolean automorphisms,

$$\pi : \mathbb{B} \to \mathbb{B}$$

with $\pi \in M$,

(3.1) $I_G(\tau) = I_{\pi[G]}(\tau)$,
(3.2) $M[G] \vDash$ "$I_G(\tau)$ is a prewellordering with on a set $X \subseteq \mathbb{R}$".

This implies that for all M-generic filters $G \subseteq \mathbb{B}$, there is a function

$$\rho_G : \mathbb{R}^{M[G]} \to \mathrm{Ord}$$

such that

(4.1) ρ_G is definable in $M[G]$ from parameters in M,
(4.2) $I_G(\tau)$ is the prewellordering induced by ρ_G.

By the homogeneity of $\mathrm{Coll}(\omega, \lambda)$, both the formula ϕ and the parameter $a \in M$ which witness that (4.1) holds can be chosen independent of G. Finally

$$\mathrm{Coll}(\omega, \lambda) \cong \mathrm{Coll}(\omega, \lambda) \times \mathrm{Coll}(\omega, \lambda)$$

and (1.1) now follows.

Let $\langle \rho_i : i < \omega \rangle$ be a scale on A_0 with the associated prewellorderings in Γ_0. By the key point above, for all $i < \omega$, for almost all M-generic filters, $G_0 \subseteq \mathrm{Coll}(\omega, \lambda)$, for almost all $M[G_0]$-generic filters, $G_1 \subseteq \mathrm{Coll}(\omega, \lambda)$, the following holds: For all $x \in M[G_0] \cap A_0$ there exists $y \in M[G_1] \cap \mathbb{R}$ such that $x|i = y|i$ and $\rho_i(x) = \rho_i(y)$.

Now by assumption, for almost all M-generic filters $G \subseteq \mathrm{Coll}(\omega, \lambda)$, $I_G(\sigma) \in A_0$. Therefore for each $i < \omega$ there exists a dense set $D_i \subseteq \mathrm{Coll}(\omega, \lambda)$, with $D_i \in M$, such that for all $q \in D_i$ there is a term $\tau_q \in M^{\mathrm{Coll}(\omega,\lambda)}$ and a condition $q^* \in \mathrm{Coll}(\omega, \lambda)$ such that for almost all filters $G_0 \times G_1 \subseteq \mathrm{Coll}(\omega, \lambda) \times \mathrm{Coll}(\omega, \lambda)$, if $(q, q^*) \in G_0 \times G_1$ then

$$\rho_i(x) = \rho_i(y) \quad \text{and} \quad x|i = y|i$$

where $(x, y) = (I_{G_0}(\sigma), I_{G_1}(\tau_q))$.

Now suppose G is M-generic for $\mathrm{Coll}(\omega, \lambda)$ and that $p \in G$. Let $t = I_G(\sigma)$ we must show that $t \in A_0$. Let $\langle p_i : i < \omega \rangle$ be such that for each $i < \omega$ let $p_i \in G \cap D_i$. Thus there exists a sequence $\langle q_i : i < \omega \rangle$ of conditions in $\mathrm{Coll}(\omega, \lambda)$ and a sequence $\langle \tau_i : i < \omega \rangle$ of terms in $M^{\mathrm{Coll}(\omega,\lambda)}$ such that for all $i < \omega$:

(5.1) For almost all filters $G_0 \times G_1 \subseteq \mathrm{Coll}(\omega, \lambda) \times \mathrm{Coll}(\omega, \lambda)$, if $(p_i, q_i) \in G_0 \times G_1$ then $\rho_i(x) = \rho_i(y)$ and $x|i = y|i$ where $(x, y) = (I_{G_0}(\sigma), I_{G_1}(\tau_i))$.

But this implies, since $p_i \in G$ for all $i < \omega$, that for all $i < k < \omega$:

(6.1) For almost all M-generic filters $G_1 \subset \mathrm{Coll}(\omega, \lambda)$, for almost all M-generic filters $G_2 \subset \mathrm{Coll}(\omega, \lambda)$, if $(q_i, q_k) \in G_1 \times G_2$, then $x|i = y|i = t|i$ and

$$\rho_i(x) = \rho_k(y)$$

where $(x, y) = (I_{G_1}(\tau_i), I_{G_2}(\tau_k))$.

Therefore there exists a sequence $\langle x_i : i < \omega\rangle$ of elements of of A_0 such that for all $i < \omega$, $x_i|i = t|i$ and such that for all $i < k < \omega$,

$$\rho_i(x_i) = \rho_i(x_k).$$

But then $\lim_i x_i = t$ and for each $i < \omega$, the sequence $\langle \rho_i(x_k) : k < \omega\rangle$ is eventually constant. Since $\langle \rho_i : i < \omega\rangle$ is a scale on A_0, this implies $t \in A_0$. □

Another version of Lemma 38 is the following where we use the notation from the statement of the lemma (with the same assumptions). Let $\langle A_k : k < \omega\rangle$ enumerate Γ_0 and let

$$B = \cup\{\{k\} \times A_k \mid k < \omega\},$$

so that $B \subseteq \omega \times \mathbb{R}$. Let

$$\pi : \mathbb{R} \to \omega \times \mathbb{R}$$

be a bijection which is Δ^1_1 and let B^* be the preimage of B under π. Then for each transitive set M, M is B^*-closed if and only if M is B^*-full.

The proof of Lemma 38 easily adapts to prove an interesting (and very useful) condensation property. For example, suppose that Γ_0 is as in the statement of Lemma 38 and that M is a transitive model of ZFC such that M is A-closed for each $A \in \Gamma_0$. Suppose

$$X \prec (M, \tau_A \cap M : A \in \Gamma_0)$$

where for each $A \in \Gamma_0$, τ_A is the term-relation for A. Let M_X be the transitive collapse of X and for each $A \in \Gamma_0$, let τ^X_A be the image of $\tau_A \cap X$ under the collapsing map. Then for each $A \in \Gamma_0$, M_X is A-closed and moreover

$$\tau_A \cap M_X = \tau^X_A.$$

Definition 39. Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Then $T \vdash_\Omega \phi$ if there exists a set $A \subseteq \mathbb{R}$ such that:

(1) $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$,
(2) $(A, \mathbb{R})^\#$ exists and is universally Baire,
(3) for all countable transitive models, M, if M is A-closed and $T \in M$, then

$$M \vDash \text{“}T \vDash_\Omega \phi\text{”}.$$

Remark 40. One can require in the definition of $T \vdash_\Omega \phi$ that M be both A-closed and A-full. One can also simply use the intermediate notion that for all transitive sets N if N is a set generic extension of M then $A \cap N \in N$.

Assuming there is a proper class of Woodin cardinals then for all universally Baire sets, $A \subseteq \mathbb{R}$,

$$L(A, \mathbb{R}) \vDash \mathrm{AD}^+$$

and $(A, \mathbb{R})^{\#}$ is universally Baire. Thus in the context of a proper class of Woodin cardinals the relation, $T \vdash_{\Omega} \phi$, has a much simpler definition. This we state in the form of a lemma.

Lemma 41. *Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Suppose there is a proper class of Woodin cardinals. Then $T \vdash_{\Omega} \phi$ if and only if there exists a universally Baire set A such that for all countable transitive models, M, if M is A-closed and $T \in M$, then*

$$M \vDash \text{“}T \vDash_{\Omega} \phi\text{”}.$$

The relation, $T \vdash_{\Omega}$, is generically absolute. Assuming there is a proper class of Woodin cardinals this is relatively easy to show. Working in just ZFC the proof requires the following theorem from the theory of AD^{+}.

Theorem 42. *Suppose that $A \subseteq \mathbb{R}$, $(A, \mathbb{R})^{\#}$ exists,*

$$L(A, \mathbb{R}) \vDash \mathrm{AD}^{+},$$

and that for some Σ^2_1-sentence ϕ,

(i) $L(A, \mathbb{R}) \vDash \phi$,
(ii) *for all $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$, if*

$$L(B, \mathbb{R}) \vDash \phi$$

then $L(A, \mathbb{R}) = L(B, \mathbb{R})$.

Let Γ_0 be the set of all $B \subseteq \mathbb{R}$ such that B is Σ_1-definable in $L(A, \mathbb{R})$ with parameters from $\{\mathbb{R}\} \cup I$ where I is the class of Silver indiscernibles for $L(A, \mathbb{R})$.

Then every set in Γ_0 has a scale each norm of which is in Γ_0.

Theorem 43. *For all (T, ϕ) and for all partial orders $\mathbb{P}$,*

$$T \vdash_{\Omega} \phi \quad \text{if and only if} \quad V^{\mathbb{P}} \vDash \text{“}T \vdash_{\Omega} \phi\text{”}.$$

One can prove that *Soundness* holds for the proof relation, $T \vdash_{\Omega}$. The proof is an easy corollary of the following lemma together with the Σ^2_1-Basis Theorem, Theorem 25.

Lemma 44. *Suppose that $A \subseteq \mathbb{R}$,*

$$L(A, \mathbb{R}) \vDash \mathrm{AD}^{+}$$

and that $(A, \mathbb{R})^{\#}$ exists and is universally Baire.

Then for each set $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$, such that B is both Suslin and co-Suslin in $L(A, \mathbb{R})$ and for each $\alpha \in \mathrm{Ord}$, there exists a transitive set M such that $V_\alpha \in M$,

$$M \vDash \mathrm{ZFC},$$

and such that M is B-closed.

Lemma 45. *Suppose that $A \subseteq \mathbb{R}$ is universally Baire, M is a transitive set, $M \vDash$ ZFC, and M is A-closed. Then there exists a countable elementary substructure,*

$$X \prec M$$

such that the transitive collapse of X is A-closed.

Proof. Fix λ such that $M \in V_\lambda$ and such that $\lambda = |V_\lambda|$. Suppose

$$Y \prec V_\lambda,$$

Y is countable and that $\{M, A\} \in Y$. Let $X = Y \cap M$ and let N be the transitive collapse of X. Then N is A-closed. □

Theorem 46 (Soundness). *If $T \vdash_\Omega \phi$ then $T \vDash_\Omega \phi$.*

Proof. This is trivial modulo one potential problem: given an ordinal β and a set $A \subseteq \mathbb{R}$ which witnesses that $T \vdash_\Omega \phi$, we need that there exists a transitive set M such that $V_\beta \in M$, M is A-closed and such that $M \vDash$ ZFC. This would be immediate if we had a proper class of strongly inaccessible cardinals. Lemma 44 supplies the set M without this additional assumption.

Let $A \subseteq \mathbb{R}$ witness that $T \vdash_\Omega \phi$. By Theorem 25 we can assume that A is Suslin and co-Suslin in $L(A, \mathbb{R})$.

Suppose toward a contradiction that $T \nvDash_\Omega \phi$. Then there exist a partial order, $\mathbb{P}$, and an ordinal α such that $V_\alpha^{\mathbb{P}} \vDash T$ but

$$V_\alpha^{\mathbb{P}} \vDash (\neg\phi).$$

Fix $\beta > \alpha$ such that $\mathbb{P} \in V_\beta$. Let M be a transitive set such that $V_\beta \in M$,

$$M \vDash \text{ZFC}$$

and such that M is A-closed. Let $X \prec M$ be a countable elementary substructure with $\{\mathbb{P}, \alpha\} \in X$ and such that N is A-closed where N is the transitive collapse of X, by Lemma 45, X exists. Clearly

$$N \nvDash \text{“}T \vDash_\Omega \phi\text{”},$$

which contradicts that A witnessed $T \vdash_\Omega \phi$. □

We now come to the Ω Conjecture which in essence is simply the conjecture that the Gödel Completeness Theorem holds for Ω-logic; see [20] for a more detailed discussion.

Definition 47 (Ω Conjecture). Suppose that there exists a proper class of Woodin cardinals. Then for all sentences ϕ, $\emptyset \vDash_\Omega \phi$ if and only if $\emptyset \vdash_\Omega \phi$.

Letting $0^\Omega = \{\phi \mid \emptyset \vdash_\Omega \phi\}$, the Ω Conjecture simply asserts that if there is a proper class of Woodin cardinals then $\mathcal{V}_\Omega = 0^\Omega$.

With just the definition of A-closed, the Ω Conjecture can be reformulated as follows.

> *Suppose there is a proper class of Woodin cardinals and that ϕ is a Σ_2-sentence. Suppose that for all universally Baire sets, A, there exists a countable transitive set M such that M is A-closed, $M \vDash$ ZFC and $M \vDash \phi$. Then there exists a partial order $\mathbb{P}$ such that $V^{\mathbb{P}} \vDash \phi$.*

The following theorem (which we shall prove in a slightly stronger form as Lemma 217 on p. 315) is one indication that a large cardinal hypothesis which refutes the Ω Conjecture is beyond the reach of an inner model theory which is anything like the inner model theory which has been developed to date.

Theorem 48. *Suppose that there is a proper class of Woodin cardinals and that for every set $A \subseteq \mathbb{R}$, if A is* OD *then A is universally Baire. Then*

$$\mathrm{HOD} \vDash \text{“}\Omega \textit{ Conjecture”}.$$

2.4. *The complexity of* Ω*-logic and the* AD$^+$ *Conjecture*

A key question about Ω-logic concerns the complexity of the set

$$\mathcal{V}_\Omega = \{\phi \mid \emptyset \vDash_\Omega \phi\}.$$

Assuming the Ω Conjecture and that there is a proper class of Woodin cardinals then $\mathcal{V}_\Omega = 0^\Omega$ where

$$0^\Omega = \{\phi \mid \emptyset \vdash_\Omega \phi\}.$$

Therefore assuming the Ω Conjecture the key question concerns the complexity of 0^Ω. A natural conjecture (which we thought we could prove) is that the set 0^Ω is necessarily definable in $H(c^+)$. Currently the best known calculation is that 0^Ω is definable in $H(\delta^+)$ where δ is the least Woodin cardinal. Thus assuming there is a proper class of Woodin cardinals and that the Ω Conjecture holds, the set $\mathcal{V}_\Omega$ is Δ_2-definable in V.

The definability of 0^Ω in $H(c^+)$ is a consequence of the following conjecture.

Definition 49 (AD$^+$ Conjecture). Suppose that $L(A,\mathbb{R})$ and $L(B,\mathbb{R})$ each satisfy AD$^+$. Suppose that every set

$$X \in (L(A,\mathbb{R}) \cup L(B,\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$$

is ω_1-universally Baire. Then either $(\underset{\sim}{\Delta}{}^2_1)^{L(A,\mathbb{R})} \subseteq (\underset{\sim}{\Delta}{}^2_1)^{L(B,\mathbb{R})}$ or $(\underset{\sim}{\Delta}{}^2_1)^{L(B,\mathbb{R})} \subseteq (\underset{\sim}{\Delta}{}^2_1)^{L(A,\mathbb{R})}$.

There is a stronger version of this conjecture.

Definition 50 (Strong AD$^+$ Conjecture). Suppose that $L(A,\mathbb{R})$ and $L(B,\mathbb{R})$ each satisfy AD$^+$. Suppose that every set

$$X \in (L(A,\mathbb{R}) \cup L(B,\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$$

is ω_1-universally Baire. Then either $A \in L(B,\mathbb{R})$ or $B \in L(A,\mathbb{R})$.

Theorem 51 (AD^+ Conjecture). *The set*

$$0^\Omega = \{\phi \mid \emptyset \vdash_\Omega \phi\}$$

is definable in $H(c^+)$.

Proof. By the Σ^2_1-Basis Theorem, Theorem 25, for each sentence ϕ, the following are equivalent.

(1.1) $\phi \in 0^\Omega$.
(1.2) There exists a universally Baire set $A \subset \mathbb{R}$ and there exists a set $B \subset \mathbb{R}$ such that

(a) B is Δ^2_1-definable in $L(A, \mathbb{R})$,
(b) B witnesses $\emptyset \vdash_\Omega \phi$.

Let Γ be the set of all $A \subset \mathbb{R}$ such that

(2.1) $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$,
(2.2) every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is ω_1-universally Baire.

Let Σ be the set of $B \subseteq \mathbb{R}$ such that B is ω_1-universally Baire. Then Σ is definable in $H(c^+)$ and this implies that Γ is definable in $H(c^+)$.

For each $A \in \Gamma$, let 0^Ω_A be the set of all sentences ϕ such that

(3.1) B is Δ^2_1-definable in $L(A, \mathbb{R})$,
(3.2) for all countable transitive sets M if $M \vDash \mathrm{ZFC}$ and if M is B-closed then $M \vDash$ "$\emptyset \vDash_\Omega \phi$".

By the AD^+ Conjecture, for all $A_0, A_1 \in \Gamma$ either $0^\Omega_{A_0} \subseteq 0^\Omega_{A_1}$ or $0^\Omega_{A_1} \subseteq 0^\Omega_{A_0}$.

There are two cases. The first case is that for all $A \in \Gamma$, $0^\Omega_A \subseteq 0^\Omega$. Then

$$0^\Omega = \cup\{0^\Omega_A \mid A \in \Gamma\}$$

in which case 0^Ω is definable in $H(c^+)$ since Γ is definable in $H(c^+)$.

The second case is that there exists $A \in \Gamma$ such that $0^\Omega_A \not\subseteq 0^\Omega$. Then by AD^+ Conjecture and by the Σ^2_1-Basis Theorem, Theorem 25, there exists $A_0 \in \Gamma$ and a sentence ϕ_0 such that

(4.1) $\phi_0 \in 0^\Omega_{A_0} \backslash 0^\Omega$,
(4.2) for all $A \in \Gamma$, if $\phi_0 \notin 0^\Omega_A$ then $0^\Omega_A \subseteq 0^\Omega$.

But then for all sentences ϕ, the following are equivalent:

(5.1) $\phi \in 0^\Omega$;
(5.2) there exists $A \in \Gamma$ such that $\phi_0 \notin 0^\Omega_A$ and such that $\phi \in 0^\Omega_A$,

and again since Γ is definable in $H(c^+)$ we have that 0^Ω is definable in $H(c^+)$. □

Remark 52. (1) The AD^+ Conjecture is expressible (uniformly) as a Π_2-sentence in $(H(c^+), \mathcal{I}_{\mathrm{NS}})$ where $\mathcal{I}_{\mathrm{NS}}$ is the nonstationary ideal on ω_1.

(2) If ZFC + CH implies AD^+ Conjecture then so does ZFC.
(3) Quite a number of statements are known to imply the AD^+ Conjecture. For example the existence of an ω_1-dense ideal on ω_1 implies the AD^+ Conjecture.
(4) A natural weakening of the AD^+ Conjecture is obtained by replacing ω_1 with c in Definition 49, call this the *weak* AD^+ *Conjecture*. This conjecture suffices for the application that 0^Ω is definable in $H(c^+)$ (by essentially the same proof). The weak AD^+ Conjecture and the AD^+ Conjecture are the same in the context of CH therefore as noted in (2), if ZFC proves the weak AD^+ Conjecture then ZFC proves the AD^+ Conjecture.

    Martin's Maximum implies the weak AD^+ Conjecture and for this Martin's Maximum for the class of partial orders $\mathbb{P}$ with $|\mathbb{P}| \leq c$ suffices.
(5) A counterexample to the AD^+ Conjecture gives a counterexample to the *Mouse Set Conjecture* and so likely would give an inner model with a superstrong cardinal.

2.5. *The Ω Conjecture and inner models*

The connection between the Ω Conjecture and inner models is suggested in part by the following theorem.

Theorem 53. *For all partial orders*, $\mathbb{P}$,

$$V^{\mathbb{P}} \models \text{“}\textit{The } \Omega \textit{ Conjecture}\text{”}$$

if and only if $V \models$ “*The* Ω *Conjecture*”.

Since the Ω Conjecture is generically absolute, if it is false one would expect to refute it from some large cardinal hypothesis (see p. 104). Thus a natural approach to the Ω Conjecture is to attempt to understand the possibilities for inner models for large cardinals, supercompact and beyond, to see if one can isolate candidates for large cardinals axioms which refute the Ω Conjecture.

The search for inner models for very strong large cardinal axioms is one of the central programs of modern set theory. Its origins lie in Gödel's discovery of the inner model, L, which is simply the smallest transitive inner model of ZF containing the ordinals. The *Inner Model Program* is the attempt to define and analyze analogs of L for specific large cardinals. The stronger the larger cardinal the more interesting and difficult the problem.

What are the criteria for success? It is important to have a test question and Ω-logic does give a natural test question for inner model theory.

Suppose that $(\exists x \phi)$ is a large cardinal axiom.

Is there a (*countable*) *transitive set* M *such that the following hold*?

(1) $M \models \mathrm{ZFC} + (\exists x \phi)$;
(2) $M \models \mathrm{ZFC} +$ “*There is a proper class of Woodin cardinals*”;

(3) *Suppose that* $x \in \mathbb{R} \cap M$. *Then*

$$(\mathrm{ZFC} \vdash_\Omega \text{“}x \in \mathrm{HOD}\text{”})^M.$$

At the level (if it exists) of large cardinal axioms where the Ω Conjecture *fails*, one modifies (3) replacing $\vdash_\Omega$ with the logical relation, $\models_\Omega$.

Thus the (challenge) problem is to produce a model in which the large cardinal axiom holds and in which every real is ordinal definable in the strongest possible sense.

This test question cannot be solved positively by forcing, this is essentially by the (forcing) absoluteness of the relation, $\vdash_\Omega$.

Theorem 54. *Suppose that* $x \in \mathbb{R}$ *and*

$$\mathrm{ZFC} \not\vdash_\Omega \text{“}x \in \mathrm{HOD}\text{”}.$$

Then for all $\mathbb{P}$, *in* $V^{\mathbb{P}}$,

$$\mathrm{ZFC} \not\vdash_\Omega \text{“}x \in \mathrm{HOD}\text{”}.$$

The same is true for $\not\models_\Omega$, assuming there is a proper class of Woodin cardinals.

Theorem 55. *Suppose there is a proper class of Woodin cardinals.*

Suppose that $x \in \mathbb{R}$ *and*

$$\mathrm{ZFC} \not\models_\Omega \text{“}x \in \mathrm{HOD}\text{”}.$$

Then for all $\mathbb{P}$, *in* $V^{\mathbb{P}}$,

$$\mathrm{ZFC} \not\models_\Omega \text{“}x \in \mathrm{HOD}\text{”}.$$

3. Generalized Iteration Trees

3.1. *Long extenders*

Suppose that

$$j : V \to M$$

is an elementary embedding with critical point κ. Suppose that η is an ordinal, $\eta > \kappa$, and let $\hat{\eta}$ be the least ordinal such that $\eta \le j(\hat{\eta})$. From j one can define the *extender* of length η. This is, in essence, simply the function

$$F : \mathcal{P}(\hat{\eta}) \to V$$

given by: $F(A) = j(A) \cap \eta$. The ordinal, $\hat{\eta}$, could be greater than the critical point of j, in which case the extender, E, is a long extender.

The formal definition of the extender E specifies E as a family of ultrafilters. For each finite set $s \subseteq \eta$ let

$$E_s = \{A \subseteq [\hat{\eta}]^{|s|} \mid s \in j(A)\}.$$

Thus E_s is an ultrafilter. The set

$$E = \{(s, A) \mid s \in [\eta]^{<\omega} \text{ and } A \in E_s\}$$

is the extender of length η derived from j, it is also the (κ, η)-extender derived from j. If $\eta > j(\kappa)$ then the extender E is a *long extender*. Notice that if $\eta \leq j(\kappa)$ then $\hat{\eta} = \kappa$ and E is the usual (κ, η) extender derived from j.

Suppose that E is the (κ, η) extender derived from

$$j : V \to M.$$

The extender E gives an elementary embedding

$$j_E : V \to M_E$$

and there exists an elementary embedding (possibly the identity)

$$k : M_E \to M$$

such that $j = k \circ j_E$ and such that $k|\eta$ is the identity. Thus E is the extender of length η derived from

$$j_E : V \to M_E,$$

and so every extender contains all the information necessary to witness that E is an extender. Further, one can easily abstract out the relevant properties on E to ensure the construction of j_E yields an elementary embedding from which E can be derived.

The extender E is *α-strong* if $V_\alpha \subseteq M_E$. If $V_\alpha \subseteq M$ and $|V_\alpha|^M \leq \eta$ then E is α-strong.

Definition 56. Suppose that E is an extender of length η and that

$$j_E : V \to M_E$$

is the associated elementary embedding.

(1) An ordinal $\xi < \eta$, is a *generator* of E if

$$\xi \notin \{j_E(f)(s) \mid s \in [\hat{\xi}]^{<\omega}, f : [\eta]^{<\omega} \to [\hat{\eta}]^{<\omega}\}.$$

(2) If ξ is a generator of E and $\xi < j_E(\kappa)$, where κ is the critical point of j_E, then ξ is a *short generator* of E, otherwise ξ is a *long generator* of E.

Notice that the extender E is uniquely specified by $\mathrm{LTH}(E)$ and

$$E|X = \{(s, A) \mid s \in [X]^{<\omega} \text{ and } A \in E_s\}$$

where X is the set of generators of E.

Given the definition above, a long extender E may have no long generators. In this case, E is a *degenerate* long extender. However, since we shall primarily be concerned with (coarse) extender models, this distinction is not really an issue and so we shall ignore it.

Remark 57. When extracting a long extender of length η from j we shall require $V_{\eta+1} \subseteq M$. Otherwise the extender may encode too much.

If there exist a proper class of measurable cardinals then it is consistent that in all generic extensions of V, for any set a, there exist η and an elementary embedding

$$j : V \to M$$

such that $a \in L[E]$ where E is the (long) extender of length η derived from j.

This cannot happen in the case that E is not a long extender for then

$$L[E] = L[E_{\{\kappa\}}]$$

where κ is the critical point of j. Note

$$E_{\{\kappa\}} = \{A \subseteq \kappa \mid \kappa \in j(A)\},$$

which is simply the normal measure on κ derived from j. In particular, the inner model $L[E]$ just depends on $\mathrm{CRT}(E)$ and the family of inner models, $L[F]$, where F is a short extender, is wellordered under reverse containment with $L[F_2] \subseteq L[F_1]$ if $\mathrm{CRT}(F_1) < \mathrm{CRT}(F_2)$.

We fix some notation.

Definition 58. Suppose that E is an extender.

(1) $\mathrm{CRT}(E)$ is the critical point of the elementary embedding

$$j_E : V \to M_E$$

given by E.

(2) $\mathrm{LTH}(E)$ is the length of the extender E.

(3) For each $\alpha < \mathrm{LTH}(E)$ let $\mathrm{SPT}(E;\alpha)$ be the least ordinal β such that $j_E(\beta) > \alpha$ and let $\mathrm{SPT}(E) = \sup\{\mathrm{SPT}(E;\alpha) \mid \alpha < \mathrm{LTH}(E)\}$.

(4) $\rho(E) = \sup\{\eta \mid V_\eta \subseteq M_E\}$.

Remark 59. (1) $\mathrm{CRT}(E)$ is the completeness of the ultrafilters associated to the extender, E.

(2) $\mathrm{LTH}(E)$ is in essence the domain of the extender, E.

(3) $\mathrm{SPT}(E)$ is a minor variation of Steel's concept of the *space* of an extender, [16]. Note that $\mathrm{SPT}(E)$ is the least ordinal β such that $j_E(\beta) \geq \mathrm{LTH}(E)$.

(4) An extender, E, is ω-*huge* if $j_E(\rho(E)) = \rho(E)$ in which case $\rho(E)$ is the least ordinal ξ such that $j_E(\xi) = \xi$ and $\mathrm{CRT}(E) < \xi$.

(5) Define $\rho^*(E) = \sup\{\eta \mid \mathcal{P}(\eta) \subset M_E\}$. One could use this instead of $\rho(E)$ for many of the definitions we shall give.

3.2. *Iteration trees*

Following the definitions of Martin and Steel [7], a *premouse* is a pair (M, δ) such that M is transitive, $\delta \in M$, and

(1) $M \vDash \mathrm{ZC} + \Sigma_2$-Replacement.

(2) Suppose that $F : M_\delta \to M \cap \mathrm{Ord}$ is definable from parameters in M, then F is bounded in M.

(3) δ is strongly inaccessible in M.

The only difference between the definition here and that of [7] is in (3), we require δ be strongly inaccessible. Requiring $M \vDash \text{ZFC}$ is a bit too much of a departure from the definition of [7] particularly since we shall be generalizing some rather technical constructions from [7] in the proof of our main theorem on iterability. It is not that the constructions require weakening the theory a premouse must satisfy, for they do not, but instead the reason for not requiring $M \vDash \text{ZFC}$ is simply to avoid adding large cardinal hypotheses to the statements of the theorems.

A central notion in inner model theory is that of an *iteration tree* which specifies how models should be iterated and this notion is the key tool of [7]. We generalize the definition to the case of long extenders. For iteration trees of length ω this is actually also due to Steel, [16].

There are many possible generalizations however finding a suitable one is a little subtle since the most natural generalization leads to a failure of iterability.

Suppose that M is a transitive model of ZFC, $E \in M$, and

$$M \vDash \text{“}E \text{ is an extender”}.$$

Then, $\text{CRT}(E; M), \text{LTH}(E; M)$, $\text{SPT}(E; M)$, and $\rho(E; M)$ each denote the ordinals associated to E as calculated in M. If by context this is unambiguous (and this will essentially always be the case) then we will use the notation $\text{CRT}(E), \text{LTH}(E)$, $\text{SPT}(E)$ and $\rho(E)$. Suppose that (N, δ) is a premouse. Then $\text{Ult}(N, E)$ *is defined* if $\eta < \delta$ and

$$N \cap V_{\eta+1} = M \cap V_{\eta+1}$$

where $\eta = \text{SPT}(E)$. In general this is much more agreement between N and M than is required, for example, in the case that $\rho(E) = \text{LTH}(E) = j_E(\text{SPT}(E))$ one really only needs

$$N \cap V_\eta \subseteq M \cap V_\eta$$

where $\eta = \text{SPT}(E)$. Finally if $\text{Ult}(N, E)$ is defined and if

$$j_E : N \to \text{Ult}(N, E)$$

is the associated embedding then j_E is an elementary embedding. This is the point of the restriction, $\text{SPT}(E) < \delta$, and the reason for condition (2) in the definition of a premouse.

We now give the definition of an iteration tree which will be used throughout this paper.

Definition 60. Suppose that (M, δ) is a premouse. An iteration tree, $\mathcal{T}$, on (M, δ) of length η is a tree order $<_\mathcal{T}$ on η with minimum element 0 and which is a suborder of the standard order, together with a sequence

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_\mathcal{T} \alpha \rangle$$

such that the following hold.

(1) $M_0 = M$.
(2) $j_{\gamma,\alpha} : M_\gamma \to M_\alpha$ for all $\gamma <_\mathcal{T} \alpha < \eta$.
(3) Suppose that $\alpha + 1 < \eta$. Then $\alpha + 1$ has an immediate predecessor, α^*, in the tree order $<_\mathcal{T}$ and:
   (a) $E_\alpha \in j_{0,\alpha}(M \cap V_\delta)$ and $M_\alpha \vDash$ "E_α is an extender which is not ω-huge";
   (b) If $\alpha^* < \alpha$ then $\mathrm{SPT}(E_\alpha) + 1 \leq \min\{\rho(E_\beta) \mid \alpha^* \leq \beta < \alpha\}$;
   (c) $M_{\alpha+1} = \mathrm{Ult}(M_{\alpha^*}, E_\alpha)$ and

$$j_{\alpha^*,\alpha+1} : M_{\alpha^*} \to M_{\alpha+1}$$

   is the associated embedding.
(4) If $0 < \beta < \eta$ is a limit ordinal then the set of α such that $\alpha <_\mathcal{T} \beta$ is cofinal in β and M_β is the limit of the M_α where $\alpha <_\mathcal{T} \beta$ relative to the embeddings; $j_{\alpha,\beta}$.

Remark 61. (1) This definition, in case that none of the extenders, E_α, are long extenders, coincides with the definition of an iteration tree in the sense of [7].
(2) The fundamental nature of the generalization of iteration tree in the sense of [7], that is provided by Definition 60, is *not* that long extenders are allowed, it lies in the cancellation rule; (3b).
(3) It follows by induction, using Lemma 68 which we prove below, that the ultrapower $\mathrm{Ult}(M_{\alpha^*}, E_\alpha)$ indicated in (3c) is defined; i.e. there is sufficient *agreement* between M_{α^*} and M_α (and this *certified* by the iteration tree). The argument is sketched just after the proof of Lemma 68 and a very similar argument works for the case of more general iteration trees where the requirement (3a) is dropped (in which case Lemma 68 fails).
(4) Suppose E is an extender. For each $s \in \mathrm{dom}(E)$ let

$$\kappa_s = \min\{|X| \mid X \in E(s)\}$$

and define

$$\mathrm{sp}(E) = \sup\{\kappa_s \mid s \in \mathrm{dom}(E)\}.$$

Then $\mathrm{sp}(E)$ coincides with Steel's notion of the *space of an extender*, [16]. Now define

$$\rho^*(E) = \sup\{\eta \mid \mathcal{P}(\eta) \in \mathrm{Ult}(V, E)\}.$$

One can modify the definition of an iteration tree, replacing ρ by ρ^* and revising (3) to:

If $\alpha^* < \alpha$ then $\mathrm{sp}(E_\alpha) < \min\{\rho^*(E_\beta) \mid \alpha^* \leq \beta < \alpha\}$,

this is the definition of [16] for iteration trees of length ω. We have chosen the definition based on the parameters $(\mathrm{SPT}(E), \rho(E))$ simply in order to conform to the definitions of [7]. Restricting to extenders for which $\rho(E) = |V_{\rho(E)}|$, the two definitions coincide. For purposes of backgrounding an extender sequence construction, increasing the strength of the background extenders is irrelevant.

(5) Even using the parameters of $\text{SPT}(E)$ and $\rho(E)$, Definition 60 is not the most general definition of an iteration tree one might consider. Here we allow α^* to be the immediate predecessor of $\alpha + 1$ if

$$(V_{\eta+1})^{M_\alpha} = (V_{\eta+1})^{M_{\alpha^*}}$$

where $\eta = \text{SPT}(E_\alpha)$. But one might consider weakening this to the requirement that for each $\gamma < \text{LTH}(E_\alpha)$,

$$(V_{\eta+1})^{M_\alpha} = (V_{\eta+1})^{M_{\alpha^*}}$$

where $\eta = \text{SPT}(E_\alpha; \gamma)$ (again with this agreement certified by the iteration tree). A priori this seems a reasonable candidate for an iteration tree in the case of long extenders. However, by a result of Neeman, this definition leads to a failure of iterability even if one requires for each $\gamma < \text{LTH}(E_\alpha)$,

$$(V_{\eta+\omega})^{M_\alpha} = (V_{\eta+\omega})^{M_{\alpha^*}}$$

where $\eta = \text{SPT}(E_\alpha; \gamma)$. We shall discuss this more later.

We come to the definition of a $(+\theta)$-iteration tree where $\theta \in \text{Ord}$. Though we give the definition for general θ in fact we shall mostly be concerned with $(+\theta)$-iteration trees where $\theta \leq 2$. The definition is a direct generalization of the corresponding definition in [7].

Definition 62. Suppose that (M, δ) is a premouse and that $\mathcal{T}$ is an iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

Suppose that $\theta \in \text{Ord}$. Then the iteration tree, $\mathcal{T}$, is a $(+\theta)$-*iteration tree* if for all $\alpha + 1 < \eta$,

$$\sup\{\text{SPT}(E_\beta) \mid \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\} + \theta \leq \rho(E_\alpha)$$

where for each $\beta + 1 < \eta$, β^* is the $\mathcal{T}$ predecessor of $\beta + 1$.

Remark 63. Suppose that (M, δ) is a premouse and that $\mathcal{T}$ is an iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

(1) It is easily verified that for all $\alpha + 1 < \eta$,

$$\sup\{\text{SPT}(E_\beta) + 1 \mid \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\} \leq \rho(E_\alpha).$$

So every iteration tree is a $(+0)$-iteration tree and every iteration tree of finite length is necessarily a $(+1)$-iteration tree.

(2) If $\mathcal{T}$ is a $(+1)$-iteration tree and if $\rho(E_\beta)$ is a limit ordinal for all $\beta + 1 < \eta$, then for all $n < \omega$, $\mathcal{T}$ is a $(+n)$-iteration tree.

Remark 64. Suppose that $\mathcal{T}$ is an iteration tree on (M, δ) of length η where condition (3b) of Definition 60 is replaced by the weaker condition:

If $\alpha^* < \alpha$ *then for all* $\gamma < \text{LTH}(E_\alpha)$,

$$\text{SPT}(E_\alpha; \gamma) + 1 \leq \min\{\rho(E_\beta) \,|\, \alpha^* \leq \beta < \alpha\}.$$

Then for $\mathcal{T}$ the natural definition that $\mathcal{T}$ be a $(+i)$-iteration tree would be that $\mathcal{T}$ satisfies the requirement that for all for all $\alpha + 1 < \eta$,

$$\sup\{\text{SPT}(E_\beta; \gamma) \,|\, \gamma < \text{LTH}(E_\beta), \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\} + i \leq \rho(E_\alpha)$$

where for each $\beta + 1 < \eta$, β^* is the $\mathcal{T}$ predecessor of $\beta + 1$.

Thus, since

$$\text{SPT}(E) = \sup\{\text{SPT}(E; \gamma) \,|\, \gamma < \text{LTH}(E)\},$$

if $\mathcal{T}$ is a $(+i)$-iteration tree in this sense and $i > 0$ then $\mathcal{T}$ is a $(+i)$-iteration tree in the sense of Definition 60, and so by restricting to $(+i)$-iteration trees for $i \geq 1$, the two definitions of iteration trees coincide. Since we shall almost always be dealing with iteration trees which are $(+\theta)$-iteration trees for $\theta \geq 1$, we could have used the more general definition.

Suppose that $\mathcal{T}$ is an iteration tree of length η on (M, δ) where η is a (nonzero) limit ordinal. A subset, $b \subseteq \eta$, defines a *maximal branch* of $\mathcal{T}$ if b is totally ordered under the tree relation, $<_\mathcal{T}$, and b is maximal. The maximal branch, b, is *wellfounded* if the limit of the models M_α over $\alpha \in b$ is wellfounded; the maximal branch, b, is *proper* if b has limit length.

Notice that if $0 < \eta' < \eta$ and η' is a limit ordinal then $\mathcal{T}|\eta'$ is an iteration tree of length η' and the set

$$\{\alpha \,|\, \alpha <_\mathcal{T} \eta'\}$$

is a cofinal wellfounded maximal branch of $\mathcal{T}|\eta'$.

Our main result on iterability is that the fundamental theorem of [7] generalizes to the case of long extenders, for this definition of an iteration tree. This extends earlier results of Steel [16] which deal with generalized iteration trees of length ω and for ultimately a different purpose (wellfoundedness of the Mitchell order), not for the purpose of identifying a notion of iteration that might be relevant to the inner model theory of coherent sequences of (long) extenders.

The proof of Theorem 66 easily adapts to the iteration trees defined using the parameters, $\text{sp}(E)$ and $\rho^*(E)$, as indicated in Remark 61(4). For iteration trees of length $\leq \omega$, this is precisely the main theorem of [16].

Definition 65. Suppose that (M, δ) is a premouse,

$$\pi : M \to V_\Theta$$

is an elementary embedding, $\mathcal{T}$ is an iteration tree on (M, δ), and that b is a maximal branch of $\mathcal{T}$. Let M_b be the direct limit given by b and let

$$j_b : M \to M_b$$

be the associated elementary embedding.

The branch, b, is *π-realizable* if there exists an elementary embedding,

$$\pi_b : M_b \to V_\Theta$$

such that $\pi = \pi_b \circ j_b$.

Theorem 66. *Suppose that (M, δ) is a countable premouse,*

$$\pi : M \to V_\Theta$$

is an elementary embedding,

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable $(+2)$-iteration tree on (M, δ) and that $\mathcal{T}$ has no proper maximal π-realizable branch. Then $\eta = \gamma + 1$ and for all extenders $E \in M_\gamma \cap V_{j_{0,\gamma}(\delta)}$, for all $\gamma^* \leq \gamma$, if $\gamma^* = \gamma$ or if $\gamma^* < \gamma$ and

$$\text{SPT}(E) + 2 \leq \min\{\rho(E_\alpha) \mid \gamma^* \leq \alpha < \gamma\},$$

then $\text{Ult}(M_{\gamma^*}, E)$ is wellfounded and moreover the corresponding maximal branch of the induced iteration tree of length $\gamma + 2$ is π-realizable.

Note that the requirement on the extender E is simply the requirement that if we extend the tree to an iteration tree of length $\gamma + 2$ by setting $E_\gamma = E$ then the extended iteration tree is a $(+2)$-iteration tree (modulo wellfoundedness of the ultrapower, $\text{Ult}(M_{\gamma^*}, E)$). This is the induced iteration tree of length $\gamma + 2$.

So this generalization of iteration trees to the case of long extenders seems reasonable. For example, any $(+1)$-iteration tree on V of length ω has an infinite branch. This implies that the Mitchell order on extenders is wellfounded which is the main theorem of Steel, [16], see [7] for some special cases of this. In contrast, Neeman has shown that with more general rules for the construction of an iteration tree (allowing extenders from one model to be applied to another whenever the ultrapower is defined), if there is a cardinal δ which is $\beth_\omega(\delta)$-supercompact then there is an iteration tree on V of length ω and *no* branch of length 4, cf. Remark 61(5). For an explanation see Remark 64.

Remark 67. The proof we shall give for Theorem 66 actually works for iteration trees with ω-huge extenders allowed provided one incorporates into the iteration tree a function, $\rho^{\mathcal{T}}$, such that for all $\alpha + 1 < \eta$,

$$\rho^{\mathcal{T}}(E_\alpha) = \rho(E_\alpha)$$

if E_α is not ω-huge and

$$\rho^{\mathcal{T}}(E_\alpha) < \rho(E_\alpha)$$

otherwise. (3b) in Definition 60 is replaced by the condition:

If $\alpha^* < \alpha$ then $\mathrm{SPT}(E_\alpha) + 1 \le \min\{\rho^{\mathcal{T}}(E_\beta) \,|\, \alpha^* \le \beta < \alpha\}$.

In fact it is probably straightforward to adapt the proof of Theorem 66 to the case of iteration trees with no restriction on the strength of the extenders. The point is that if E_α is an ω-huge extender with $\rho(E_\alpha) = \mathrm{LTH}(E_\alpha)$ then necessarily,

$$E_\alpha \in M_{\alpha^*},$$

and so such ω-huge extenders can only be used internally.

However even iteration trees as we have defined them are far more general (in allowing long extenders) than we have any use for.

We now turn to the proof of Theorem 66 which is in essence just a matter of reworking the proof from [7] and we shall follow the notation of that proof, as well as its organization, as closely as makes sense. The adaptation to our case of iteration trees with long extenders is largely enabled by the following simple lemma, and this is only needed in the transfinite case. As we have noted, the special case of Theorem 66 for iteration trees of length $\le \omega$ is due to Steel, [16].

Lemma 68. *Suppose that (M, δ) is a premouse and that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is an iteration tree on (M, δ). Suppose that $b \subseteq \eta$ is a branch of $\mathcal{T}$ of limit length such that the direct limit of the set

$$\{M_\alpha \,|\, \alpha \in b\}$$

under the embeddings $\{j_{\alpha,\beta} \,|\, \alpha \in b, \beta \in b, \alpha <_{\mathcal{T}} \beta\}$ is wellfounded. Let θ_b be the lim-inf of the critical points along b;

$$\theta_b = \sup\{\min\{\mathrm{CRT}(E_\beta) \,|\, \beta + 1 \in b, \alpha + 1 <_{\mathcal{T}} \beta + 1\} \,|\, \alpha + 1 \in b\}.$$

Then there exists $\alpha_0 + 1 \in b$ such that for all $\alpha + 1 \in b$, if $\alpha_0 + 1 \le \alpha + 1$ and if $\mathrm{CRT}(E_\alpha) \le \theta_b$ then $\rho(E_\alpha) < \theta_b$.

Proof. Since the limit given by the branch b is wellfounded, for all

$$\xi \in \sup\{M_\alpha \cap \mathrm{Ord} \,|\, \alpha \in b\},$$

there exists $\alpha_0 + 1 \in b$ such that for all $\alpha <_{\mathcal{T}} \beta$ if $\beta \in b$ and if $\alpha_0 + 1 \le \alpha$ then $j_{\alpha,\beta}(\xi) = \xi$. This is the key point.

There are two cases. First suppose that

$$\sup\{M_\alpha \cap \mathrm{Ord} \,|\, \alpha \in b\} \le \theta_b.$$

In this case for all $\alpha + 1 \in b$, $\mathrm{LTH}(E_\alpha) < \theta_b$ and so we are done.

Now suppose

$$\theta_b < \sup\{M_\alpha \cap \mathrm{Ord} \mid \alpha \in b\}.$$

Therefore by the remark above, there exists $\alpha_0 + 1 \in b$ such that for all $\alpha <_{\mathcal{T}} \beta$ if $\beta \in b$ and $\alpha_0 + 1 \leq \alpha$, then

$$\theta_b \in M_\alpha$$

and $j_{\alpha,\beta}(\theta_b) = \theta_b$.

We finish by showing that if $\alpha + 1 \in b$, $\alpha_0 + 1 \leq \alpha^*$, and if $\mathrm{CRT}(E_\alpha) \leq \theta_b$, then $\rho(E_\alpha) < \theta_b$. If not then

$$\theta_b \leq \rho(E_\alpha).$$

But $j_{\alpha^*,\alpha+1}(\theta_b) = \theta_b$ and so

$$\mathrm{CRT}(E_\alpha) < \theta_b$$

since $\mathrm{CRT}(E_\alpha) \leq \theta_b$. Therefore E_α is an ω-huge extender in M_α. This contradicts the requirement, Definition 60(3a), that E_α not be an ω-huge extender in M_α.

Finally if $\alpha+1 \in b$ and $\alpha_0+1 \leq \alpha+1$ then $\alpha_0+1 \leq \alpha^*$ and so α_0 is as required. □

Suppose that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle$$

is an iteration tree of length η on a premouse (M, δ). Suppose that $\eta^* < \eta$ and

$$\xi + 1 \leq \min\{\rho(E_\alpha) \mid \eta^* \leq \alpha < \eta\}.$$

Then by Lemma 68 it follows that for all $\beta < \eta$, if $\beta > \eta^*$ then

$$V_{\xi+1} \cap M_\beta = V_{\xi+1} \cap M_{\eta^*}.$$

We verify this which shows that the ultrapower $\mathrm{Ult}(M_{\alpha^*}, E_\alpha)$ indicated in (3c) of Definition 60 is defined. Fix $\beta < \eta$ such that $\eta^* \leq \beta$. By induction we can assume that

$$V_{\xi+1} \cap M_\alpha = V_{\xi+1} \cap M_{\eta^*}$$

for all $\eta^* \leq \alpha < \beta$. If $\beta = \alpha+1$ then the claim is immediate since $\xi+1 \leq \rho(E_\alpha)$ and so we can suppose that β is a limit ordinal with $\beta > \eta^*$. Let $b = \{\alpha < \beta \mid \alpha <_{\mathcal{T}} \beta\}$ be the branch of $\mathcal{T}$ given by b and let θ_b be the lim-inf of the critical points along b. Thus b is a cofinal wellfounded branch of $\mathcal{T}|\beta$ and so by Lemma 68, there exists $\alpha_0+1 \in b$ such that $\eta^* < \alpha_0$ and such that for all $\alpha+1 \in b\backslash\alpha_0+1$, if $\mathrm{CRT}(E_\alpha) \leq \theta_b$ then $\rho(E_\alpha) < \theta_b$. This implies that $\xi < \theta_b$ and moreover that the set

$$\{\alpha + 1 \in b \mid \alpha_0 \leq \alpha, \mathrm{CRT}(E_\alpha) \leq \xi + 1\}$$

is bounded in β. By increasing α_0 if necessary we can assume that $\alpha_0 + 1 \in b$ and that for all $\alpha + 1 \in b$, if $\alpha_0 \leq \alpha$ then $\mathrm{CRT}(E_\alpha) > \xi$. This implies that

$$V_{\xi+1} \cap M_{\alpha_0+1} = V_{\xi+1} \cap M_\beta$$

and so since $\alpha_0 + 1 < \beta$, by the induction hypothesis, $V_{\xi+1} \cap M_{\eta^*} = V_{\xi+1} \cap M_\beta$.

Lemma 68 shows that along wellfounded branches of an iteration tree which have limit length, there is a restriction on the amount of overlapping that can occur in the associated extenders. Here is a simple consequence. Suppose b and c are each cofinal wellfounded branches of an iteration $\mathcal{T}$ on a premouse (M, δ) with limit models, M_b and M_c respectively. Let θ_b be the lim-inf of the critical points along b and let θ_c be the lim-inf of the critical points along c. Then by Lemma 68, $\theta_b = \theta_c$ and

$$V_\theta \cap M_b = V_\theta \cap M_c$$

where $\theta = \theta_b = \theta_c$. To see this assume toward a contradiction that

$$\theta_b < \theta_c.$$

Then appealing to Lemma 68 and assuming without loss of generality that the two branches are maximal (and so each is a closed cofinal subset of η where η is the length of $\mathcal{T}$), there must exist $\alpha_1 + 1$ in b and $\beta_0 + 1 < \beta_1 + 1$ in c such that $\beta_0 < \alpha_1 < \beta_1$,

$$\rho(E_{\alpha_1}) < \theta_b < \mathrm{CRT}(E_{\beta_1}),$$

and such that $\beta_1^* = \beta_0 + 1$. But by the cancellation rule (3b) of Definition 60,

$$\mathrm{SPT}(E_{\beta_1}) + 1 \le \min\{\rho(E_\xi) \mid \beta_0 + 1 \le \xi < \beta_1\} \le \rho(E_{\alpha_1})$$

since $\beta_1^* = \beta_0 + 1$ and since $\beta_0 + 1 \le \alpha_1 < \beta_1$, which is a contradiction.

In the case of iteration trees which involve no long extenders the fact that

$$V_\theta \cap M_b = V_\theta \cap M_c,$$

where $\theta = \theta_b = \theta_c$, plays an essential role in iterability proofs, [7].

Our only use for Theorem 66 is to provide evidence for the iteration hypotheses which seem relevant to suitable extender sequences. Further these iteration hypotheses probably need only refer to iteration trees,

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

such that

(1) for all $\alpha < \eta$, $\rho(E_\alpha) = \mathrm{LTH}(E_\alpha) > \mathrm{CRT}(E_\alpha) + 2$,
(2) for all $\alpha < \beta$, $\rho(E_\alpha) < \rho(E_\beta)$,
(3) for all $\alpha + 1 < \eta$, the tree predecessor of $\alpha + 1$ is the least ordinal, α^* such that

$$\mathrm{SPT}(E_\alpha) + \omega \le \rho(E_{\alpha^*}).$$

In the case that for all $\alpha < \eta$, E_α is not a long extender, such iteration trees are non-overlapping; i.e. for all $\alpha + 1 <_{\mathcal{T}} \beta + 1 < \eta$; $\mathrm{LTH}(E_\alpha) \le \mathrm{CRT}(E_\beta)$, and for non-overlapping iteration trees the method of [17] yields a much simpler proof of the iteration theorem. However in the case of long extenders, these iteration trees need not be non-overlapping and this seems to be an obstacle to using the method of [17].

Following the presentations of [7] we shall shall proceed in two steps. First we will assume that $\mathcal{T}$ has length at most ω and then we will prove the theorem for iteration trees of arbitrary countable length. As in [7], for the case that $\mathcal{T}$ has finite length, the assumption that $\mathcal{T}$ is a (+2)-iteration tree is not required, and for the case that $\mathcal{T}$ has length ω, the proof only requires that $\mathcal{T}$ be a (+1)-iteration tree. Further in the case that $\mathcal{T}$ has length at most ω, the proof works for iteration trees which allow extenders which are ω-huge — as we have already noted. These two cases are the theorems of [16]; we include proofs here as an introduction to the transfinite case.

Suppose that (M_0, δ_0) and (M_1, δ_1) are premice and that

$$\pi_0 : (M_0)_{\delta_0} \to (M_1)_{\delta_1}$$

is an elementary embedding. Then an iteration tree $\mathcal{T}_0$, on (M_0, δ_0), can be copied to define an iteration tree on (M_1, δ_1). More precisely if

$$\langle M^0_\alpha, E^0_\beta, j^0_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is the iteration tree $\mathcal{T}_0$ then one defines M^1_α and E^1_α together with

$$\pi_\alpha : (j^0_{0,\alpha})((M_0)_{\delta_0}) \to (j^1_{0,\alpha})((M_1)_{\delta_1})$$

by induction on $\alpha < \eta$. If at some stage M^1_α is illfounded then one stops. The construction is uniquely specified by the following inductive requirements:

(1) $<_{\mathcal{T}_0}|\alpha = <_{\mathcal{T}_1}|\alpha$;
(2) $E^1_\alpha = \pi_\alpha(E^0_\alpha)$;
(3) If α^* is the tree predecessor of $\alpha+1$ relative to $\mathcal{T}_0$ then α^* is the tree predecessor of $\alpha + 1$ relative to $\mathcal{T}_1$ and $\pi_{\alpha+1}$ is the unique embedding such that

$$\pi_{\alpha+1}(j^0_{\alpha^*,\alpha+1}(f)(s)) = j^1_{\alpha^*,\alpha+1}(\pi_{\alpha^*}(f))(\pi_\alpha(s))$$

for all $f \in M^0_{\alpha^*}$ and for all $s \in [\mathrm{LTH}(E^0_\alpha)]^{<\omega}$;
(4) If $\alpha > 0$ and α is a limit ordinal then for all $\beta <_{\mathcal{T}_0} \alpha$ for all $a \in j_{0,\beta}(M_{\delta_0})$,

$$\pi_\alpha(j^0_{\beta,\alpha}(a)) = j^1_{\beta,\alpha}(\pi_\beta(a)).$$

The verification is routine and the generalization to our situation with long extenders is immediate. The construction is enabled by a "shift lemma" which we state below.

Lemma 69 (Shift Lemma). *Suppose that* (M_0, δ_0), (M_1, δ_1), (N_0, γ_0) *and* (N_1, γ_1) *are premice,*

$$\pi_0 : (M_0, \delta_0) \to (N_0, \gamma_0)$$

and

$$\pi_1 : (M_1, \delta_1) \to (N_1, \gamma_1)$$

are elementary embeddings, $\xi < \delta_0$,

$$M_0 \cap V_\xi = M_1 \cap V_\xi$$

and

$$\pi_0|(M_0 \cap V_\xi) = \pi_1|(M_1 \cap V_\xi).$$

Suppose that $E \in M_1 \cap V_{\delta_1}$ *is an extender such that either*;

(i) $\text{SPT}(E) + 1 \leq \xi$ *and* $N_0 \cap V_{\pi_0(\text{SPT}(E))+1} = N_1 \cap V_{\pi_1(\text{SPT}(E))+1}$, *or*
(ii) $\text{SPT}(E) + 1 < \xi$.

Let

$$\sigma_0 : M_0 \to \text{Ult}(M_0, E)$$

and

$$\tau_0 : N_0 \to \text{Ult}(N_0, \pi_1(E))$$

be the ultrapower embeddings. Define $\pi : \text{Ult}(M_0, E) \to \text{Ult}(N_0, \pi_1(E))$ *by*:

$$\pi(\sigma_0(f)(s)) = (\tau_0(\pi_0(f)))(\pi_1(s))$$

where $s \in [\text{LTH}(E)]^{<\omega}$, $f \in M_0$, *and* $f : [\text{SPT}(E)]^{<\omega} \to M_0$. *Then*:

(1) π *is an elementary embedding and* $\pi \circ \sigma_0 = \tau_0 \circ \pi_0$,
(2) $M_1 \cap V_{\rho(E)} = \text{Ult}(M_0, E) \cap V_{\rho(E)}$,
(3) $\pi_1|(M_1 \cap V_{\rho(E)}) = \pi|(\text{Ult}(M_0, E) \cap V_{\rho(E)})$;

where $\rho(E)$ *is as computed in* M_1.

Proof. First note that in both cases, $\text{SPT}(E)+1 \leq \xi$ or $\text{SPT}(E)+1 < \xi$, necessarily

$$N_0 \cap V_{\text{SPT}(\pi_0(E))+1} = N_1 \cap V_{\text{SPT}(\pi_1(E))+1}.$$

Thus $\text{Ult}(N_1, \pi_1(E))$ is defined.

The standard part of $\text{Ult}(M_0, E)$ must contain $\text{LTH}(E)$ and similarly the standard part of $\text{Ult}(N_0, \pi_1(E))$ contains $\text{LTH}(\pi_1(E)) = \pi_1(\text{LTH}(E))$. Thus the definition of π makes sense. It is straightforward to show that the definition is well defined and that

$$\pi \circ \sigma_0 = \tau_0 \circ \pi_0,$$

the details (see [7]) are essentially identical to the those in the case that E is not a long extender. □

We will only use this lemma in the situation that both ultrapowers, $\text{Ult}(M_0, E)$ and $\text{Ult}(N_0, \pi_1(E))$, are wellfounded. But the lemma makes sense without this assumption. The two cases, (i) and (ii), correspond with the cases arising in copying $(+1)$-iteration trees and in copying $(+2)$-iteration trees, respectively.

Suppose that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is an iteration tree on (M, δ). The key concept in the proof of Theorem 66 is that of an enlargement of an iteration tree $\mathcal{T}$, [7]. The definition requires some notation. For all $\alpha < \beta < \eta$,

$$\rho^{\mathcal{T}}[\alpha, \beta) = \min\{\rho(E_\gamma) \mid \alpha \leq \gamma < \beta\}$$

and

$$\mu^{\mathcal{T}}(\alpha) = \sup\{\mathrm{SPT}(E_\beta) + 1 \mid \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\}$$

where for all $\beta + 1 < \eta$, β^* is the $<_{\mathcal{T}}$ predecessor of $\beta + 1$.

We note the following:

(1) For all $\alpha < \beta < \eta$,

$$M_\beta \cap V_{\rho^{\mathcal{T}}[\alpha,\beta]} = M_\alpha \cap V_{\rho^{\mathcal{T}}[\alpha,\beta]}.$$

(2) Suppose that N is transitive and $N \cap V_{\mu^{\mathcal{T}}(\alpha)} = M_\alpha \cap V_{\mu^{\mathcal{T}}(\alpha)}$. Then for all $\beta + 1 < \eta$ if $\alpha + 1 \leq \beta$ and if $\beta^* \leq \alpha$ then $\mathrm{Ult}(N, E_\beta)$ is defined.
(3) $\mu^{\mathcal{T}}(\alpha) \leq \rho(E_\alpha)$.
(4) $\rho^{\mathcal{T}}[\alpha, \alpha + 1] = \rho(E_\alpha)$.
(5) Suppose that for all $\alpha + 1 < \eta$, $\mu^{\mathcal{T}}(\alpha) + i \leq \rho(E_\alpha)$. Then $\mathcal{T}$ is a $(+i)$-iteration tree.

Definition 70. Suppose that $\mathcal{T}$ is an iteration tree on (M, δ) and that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

Suppose that $g : \eta \to \mathrm{Ord}$ is a function such that for all $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) \leq \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

A *g-enlargement* of $\mathcal{T}$ is a sequence

$$\langle (\pi_\alpha, (\mathcal{P}_\alpha, \delta_\alpha), \nu_\alpha) : \alpha \leq \gamma \rangle$$

such that:

(1) $\gamma < \eta$;
(2) For all $\alpha \leq \gamma$, $(\mathcal{P}_\alpha, \delta_\alpha)$ is a premouse, $\nu_\alpha \in \mathcal{P}_\alpha \cap \mathrm{Ord}$ or $\nu_\alpha = \mathcal{P}_\alpha \cap \mathrm{Ord}$, and

$$\pi_\alpha : (M_\alpha, j_{0,\alpha}(\delta)) \to (\mathcal{P}_\alpha \cap V_{\nu_\alpha}, \delta_\alpha)$$

is an elementary embedding;
(3) For all $\alpha < \gamma$, $\pi_\alpha(M_\alpha \cap V_\xi) = \pi_\gamma(M_\gamma \cap V_\xi)$ and

$$\pi_\alpha | (M_\alpha \cap V_\xi) = \pi_\gamma | (M_\gamma \cap V_\xi)$$

where $\xi = \min\{g(\beta) \mid \alpha \leq \beta < \gamma\}$.

Remark 71. (1) The initial segment of a g-enlargement of $\mathcal{T}$ need not be a g-enlargement.

(2) Condition (3) in the definition makes sense: For each $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) \leq \rho^{\mathcal{T}}[\alpha, \alpha+1),$$

and so it follows that for each $\alpha < \eta$

$$\min\{g(\beta) \mid \alpha \leq \beta < \gamma\} \leq \rho^{\mathcal{T}}[\alpha, \gamma).$$

(3) Suppose that

$$\langle(\pi_\alpha, (\mathcal{P}_\alpha, \delta_\alpha), \nu_\alpha) : \alpha \leq \gamma\rangle$$

is a g-enlargement of $\mathcal{T}$ and that $\gamma + 1 < \eta$. Let γ^* be the tree predecessor of $\gamma + 1$. Then $\mathrm{Ult}(\mathcal{P}_{\gamma^*}, E)$ is defined where $E = \pi_\alpha(E_\gamma)$.

3.3. *Iteration trees of finite length*

We prove Theorem 66 in the case that the iteration tree is finite and without the assumption that the iteration tree be a (+2)-iteration tree, this is an immediate corollary of the results of [16].

Lemma 72. *Suppose $n + 1 < \omega$. The following are equivalent.*

(1) *Suppose that (V_θ, δ) is a premouse and suppose that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < n+1, \beta+1 < n+1, \gamma <_{\mathcal{T}} \alpha\rangle$$

is an iteration tree on (V_θ, δ) of length $n + 1$. Suppose that $E \in M_n \cap V_{j_{0,n}(\delta)}$,

$$M_n \vDash \text{“}E \text{ is an extender”},$$

$n^* \leq n$, and $\mathrm{SPT}(E) + 1 \leq \rho^{\mathcal{T}}[n^*, n)$. Then $\mathrm{Ult}(M_{n^*}, E)$ is wellfounded.

(2) *Suppose that (M, δ_M) is a premouse, (V_θ, δ) is a premouse, and that*

$$\pi : M \to V_\theta$$

is an elementary embedding. Suppose that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < n+1, \beta+1 < n+1, \gamma <_{\mathcal{T}} \alpha\rangle$$

is an iteration tree on (M, δ_M) of length $n+1$. Suppose that $E \in M_n \cap V_{j_{0,n}(\delta)}$,

$$M_n \vDash \text{“}E \text{ is an extender”},$$

$n^* \leq n$, and $\mathrm{SPT}(E) + 1 \leq \rho^{\mathcal{T}}[n^*, n)$. Then $\mathrm{Ult}(M_{n^*}, E)$ is wellfounded.

(3) *Suppose that (M, δ_M) is a countable premouse, (V_θ, δ) is a premouse, and that*

$$\pi : M \to V_\theta$$

is an elementary embedding. Suppose that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < n+1, \beta+1 < n+1, \gamma <_{\mathcal{T}} \alpha\rangle,$$

is an iteration tree on (M, δ_M) *of length* $n+1$. *Suppose that* $E \in M_n \cap V_{j_{0,n}(\delta)}$,

$$M_n \models \text{“}E \textit{ is an extender”},$$

$n^* \leq n$, *and* $\mathrm{SPT}(E) + 1 \leq \rho^{\mathcal{T}}[n^*, n)$. *Then* $\mathrm{Ult}(M_{n^*}, E)$ *is wellfounded and there exists an elementary embedding*

$$e : \mathrm{Ult}(M_{n^*}, E) \to V_\theta$$

such that $\pi = e \circ j \circ j_{0,n^*}$ *where*

$$j : M_{n^*} \to \mathrm{Ult}(M_{n^*}, E)$$

is the ultrapower embedding.

Proof. Trivially (2) implies (1). By copying (1) implies (2). By copying and absoluteness (1) implies (3). □

We now prove Theorem 66 for finite iteration trees. As we have noted this proof works for iteration trees with no restriction that the extenders not be ω-huge.

Proof of Theorem 66 for iteration trees of finite length. Fix a countable premouse (M, δ) and an elementary embedding

$$\pi : M \to V_\Theta.$$

We can suppose, toward a contradiction, that (M, δ), π and $\mathcal{T}$ give a counterexample to Theorem 66 with the length of $\mathcal{T}$ is $n+1$ and this is as short as possible.

Thus by the lemma we can suppose that there is an iteration tree on (M, δ)

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < n+1, \beta+1 < n+1, \gamma <_{\mathcal{T}} \alpha \rangle$$

of length $n+1$, $E \in M_n \cap V_{j_{0,n}(\delta)}$ such that

$$M_n \models \text{“}E \text{ is an extender”},$$

and $n^* \leq n$ such that $\mathrm{SPT}(E) + \omega \leq \rho^{\mathcal{T}}[n^*, n)$, such that $\mathrm{Ult}(M_{n^*}, E)$ is not wellfounded.

We define by induction on $i \leq n$ a sequence

$$\langle (\pi_i, (\mathcal{P}_i, \delta_i), \nu_i) : i \leq n \rangle$$

such that for all $k \leq n$,

(1.1) the sequence,

$$\langle (\pi_i, (\mathcal{P}_i, \delta_i), \nu_i) : i \leq k \rangle,$$

is a g-enlargement of $\mathcal{T}$ where for each $m \leq n$, if $m < n^*$ then $g(m) = \mu^{\mathcal{T}}(m)$ and if $n^* \leq m$ then $g(m) = \max(\mu^{\mathcal{T}}(m), \mathrm{SPT}(E) + 1)$.

(1.2) the sequence,

$$\langle ((\mathcal{P}_i, \delta_i), \pi_i(E_i)) : i \leq k \rangle,$$

defines an iteration tree, $\mathcal{S}|k+1$, on $(\mathcal{P}_0, \delta_0)$, such that $<_{\mathcal{S}} |k+1 = <_{\mathcal{T}} |k+1$.

(1.3) $\pi_k \in \mathcal{P}_k$.

Let $\xi > \eta$ be such that $(V_\xi, \pi(\delta))$ is a premouse and set

$$(\pi_0, (\mathcal{P}_0, \delta_0), \nu_0) = (\pi, (V_\xi, \pi(\delta)), \eta).$$

Now suppose

$$\mathcal{E}_k = \langle (\pi_i, (\mathcal{P}_i, \delta_i), \nu_i) : i \leq k \rangle,$$

is defined where $k < n$ and let k^* be the $\mathcal{T}$-predecessor of $k+1$. Since $\mathcal{E}$ is an enlargement of $\mathcal{T}$ of length $k+1$,

$$\mathcal{P}_{k^*} \cap V_{\pi_{k^*}(\theta)} = \mathcal{P}_k \cap V_{\pi_k(\theta)}$$

and

$$\pi_{k^*} | M_{k^*} \cap V_\theta = \pi_k | M_k \cap V_\theta$$

where $\theta = \text{SPT}(E_k) + 1$. This implies that

$$\text{SPT}(\pi_k(E_k)) + 1 \leq \rho^{\mathcal{S}}[k^*, k)$$

and so the extender, $\pi_k(E_k)$, can be used to extend $\mathcal{S}|(k+1)$ to $\mathcal{S}|(k+2)$.

Since $k < n$, and n was least, $\text{Ult}(\mathcal{P}_{k^*}, \pi(E_k))$ is wellfounded. Define $\mathcal{P}_{k+1}$ to be the transitive collapse of $\text{Ult}(\mathcal{P}_{k^*}, \pi(E_k))$ and let

$$j : \mathcal{P}_{k^*} \to \mathcal{P}_{k+1}$$

be the associated embedding (so $j = (j_{k^*,k+1})^{\mathcal{S}}$). Let $\nu_{k+1} = j(\nu_k)$.

It remains to define

$$\pi_{k+1} : M_{k+1} \to \mathcal{P}_{k+1} \cap V_{\nu_{k+1}}$$

which must satisfy the condition that for each $i < k+1$,

$$\pi_i | M_i \cap V_\beta = \pi_{k+1} | M_{k+1} \cap V_\beta$$

where $\beta = \min\{g(m) \,|\, i \leq m < k+1\}$.

There is a natural elementary embedding

$$\tau : M_{k+1} \to \mathcal{P}_{k+1} \cap V_{\nu_{k+1}},$$

this is defined to satisfy:

$$\tau(j_{k^*,k+1}(f)(s)) = j(\pi_{k^*}(f))(\pi_k(s))$$

for all $f \in \mathcal{M}_k$ and $s \in [\text{LTH}(E_k)]^{<\omega}$. Further one can easily check that for all $i < k+1$, τ and π_i are in sufficient agreement. The trouble is that τ need not be in $\mathcal{P}_{k+1}$.

Let $\sigma = \tau | M_{k+1} \cap V_{g(k)} = \pi_k | M_{k+1} \cap V_{g(k)}$. The key point is that $\sigma \in \mathcal{P}_{k+1}$. Thus, by absoluteness, there must exist an elementary embedding

$$\tau^* : M_{k+1} \to \mathcal{P}_{k+1} \cap V_{\nu_{k+1}},$$

such that $\tau^* \in \mathcal{P}_{k+1}$ and such that $\tau^* | M_{k+1} \cap V_{g(k)} = \sigma$. Let $\pi_{k+1} = \tau^*$.

This completes the definition of the sequence,

$$\langle(\pi_i, (\mathcal{P}_i, \delta_i), \nu_i) : i \leq n\rangle.$$

Let $F = \pi_n(E)$. $\text{Ult}(M_{n^*}, E)$ is not wellfounded and so there is an increasing sequence, $\langle s_i : i < \omega\rangle$, of finite subsets of $\text{LTH}(E)$ and a sequence of functions $\langle f_i : i < \omega\rangle$ such that for all $i < \omega$,

(2.1) $f_i \in M_{n^*}$ and $f_i : [\text{SPT}(E)]^{|s_i|} \to \text{Ord}$,
(2.2) $j_E(f_{i+1})(s_{i+1}) < j_E(f_i)(s_i)$,

where $j_E : M_{n^*} \to \text{Ult}(M_{n^*}, E)$ is the ultrapower map (the standard part of $\text{Ult}(M_{n^*}, E)$ contains $\text{LTH}(E)$ and so (2.2) makes sense).

The sequence, $\langle E(s_i) : i < \omega\rangle$, naturally forms a tower of ultrafilters, note that for each $i < \omega$, $E(s_i) \in M_n$ and $E(s_i)$ concentrates on $[\text{SPT}(E)]^{|s_i|}$.

Let $\langle A_i : i < \omega\rangle$ be a sequence such that for all $i < \omega$, $A_i \in E(s_i)$, and for all $i + 1 < \omega$, if $a \in A_{i+1}$ then $b \in A_i$ and $f_{i+1}(a) < f_i(b)$ where $b \subseteq a$ is such that $(a, b, \in) \cong (s_{i+1}, s_i, \in)$.

$\pi_n \in \mathcal{P}_n$ and so (since $H(\omega_1) \in \mathcal{P}_n$),

$$\langle(\pi_n(A_i), \pi_n(s_i)) : i < \omega\rangle \in \mathcal{P}_n.$$

Finally, $\pi_n(E)$ is an extender in $\mathcal{P}_n$ and so there exists an increasing sequence $\langle t_i : i < \omega\rangle \in \mathcal{P}_n$ such that for all $i < \omega$, $t_i \in \pi_n(A_i)$ and $(t_{i+1}, t_i, \in) \cong (s_{i+1}, s_i, \in)$.

Therefore for all $i < \omega$, $\pi_{n^*}(f_{i+1})(t_{i+1}) < \pi_{n^*}(f_i)(t_i)$, which contradicts the wellfoundedness of $\mathcal{P}_{n^*}$. □

3.4. *Iteration trees of length ω*

We next prove Theorem 66 in the case that $\mathcal{T}$ is a (+1)-iteration tree of length ω, this is a special case of the principal result of [16]. The details here are a useful introduction to the constructions which are necessary in the transfinite case.

Definition 73. Suppose that $\mathcal{T}$ is an iteration tree of length ω on a premouse (M, δ) and that

$$\mathcal{T} = \langle M_m, E_m, j_{m,n} : m < \omega, m <_{\mathcal{T}} n\rangle.$$

The iteration tree, $\mathcal{T}$, is *continuously illfounded* if there exists a sequence $\langle\theta_n : n < \omega\rangle$ such that for all $n < \omega$, $\theta_n \in M_n \cap \text{Ord}$ and for all $m <_{\mathcal{T}} n < \omega$, $j_{m,n}(\theta_m) > \theta_n$.

As was the case for iteration trees of finite length, a counterexample to Theorem 66 for iteration trees of length ω yields a counterexample satisfying a strong form of illfoundedness.

Lemma 74. *Suppose that there exists a premouse (M, δ) and a $(+1)$-iteration tree, $\mathcal{T}$, on (M, δ) of length ω, such that there exist V_Θ and an elementary embedding*

$$\pi : M \to V_\Theta$$

with the property that $\mathcal{T}$ has no cofinal π-realizable branch.

Then there exist a countable premouse (M^*, δ^*), *an elementary embedding*

$$\pi^* : M^* \to V_\Theta,$$

and a $(+1)$*-iteration tree* $\mathcal{T}^*$ *on* (M, δ^*) *of length* ω *such that* $\mathcal{T}^*$ *is continuously illfounded.*

Proof. The proof is exactly the same as the proof of the analogous lemma for iteration trees with no long extenders, [7]. Let $\Upsilon > \Theta$ be such that $(V_\Upsilon, \pi(\delta))$ is a premouse. One copies $\mathcal{T}$ to define a $(+1)$-iteration tree $\mathcal{S}$ on $(V_\Upsilon, \pi(\delta))$. By Theorem 66 for finite iteration trees (which we have proved), the copying of $\mathcal{T}$ succeeds to define a $(+1)$-iteration tree of length ω.

The key point is that $\mathcal{S}$ is continuously illfounded, a more complicated version of this argument is given in the proof of Lemma 81. By taking an elementary substructure of $V_{\Upsilon+1}$ of size $\pi(\delta)$ and which contains $V_{\pi(\delta)}$ one can transfer this to a continuously illfounded $(+1)$-iteration tree on $(V_\Theta, \pi(\delta))$. Again this argument is a special case of the argument given in the proof of Lemma 81. Finally by taking a countable elementary substructure one obtains a countable premouse (M^*, δ^*), an elementary embedding

$$\pi^* : M^* \to V_\Theta,$$

and a $(+1)$-iteration tree $\mathcal{T}^*$ on (M, δ^*) of length ω such that $\mathcal{T}^*$ is continuously illfounded. □

Thus to prove Theorem 66 for $(+1)$-iteration trees of length ω it suffices to show that if (M, δ) is a countable premouse and if

$$\pi : M \to V_\Theta$$

is an elementary embedding then there is no $(+1)$-iteration tree of length ω on (M, δ) which is continuously illfounded.

Here again the proof works for iteration trees with no restriction that the extenders not be ω-huge.

Proof of Theorem 66 for iteration trees of length ω. Fix $(M, \delta), \pi : M \to V_\Theta$, and suppose toward a contradiction that

$$\mathcal{T} = \langle M_m, E_m, j_{m,n} : m < \omega, m <_{\mathcal{T}} n \rangle$$

is a $(+1)$-iteration tree on (M, δ) which is continuously illfounded. Let $\langle \theta_n : n < \omega \rangle$ witness that $\mathcal{T}$ is continuously illfounded.

Let $\Xi > \Theta$ be such that $(V_\Xi, \pi(\delta))$ is a premouse and such that the set

$$\{\gamma < \Xi \mid \eta < \gamma \text{ and } (V_\gamma, \pi(\delta)) \text{ is a premouse}\}$$

has ordertype $\pi(\theta_0)$.

A $(+1)$-*enlargement* of $\mathcal{T}$ is a g-enlargement where for all $n < \omega$, $g(n) = \gamma + 1$ and γ is least such that $\mu^{\mathcal{T}}(n) \leq \gamma + 1$.

We construct by induction on $i < \omega$, $(\pi_i, (\mathcal{P}_i, \delta_i), \nu_i)$, such that

(1.1) $\langle(\pi_k, (\mathcal{P}_k, \delta_k), \nu_k) : k \leq i\rangle$ is a (+1)-enlargement of $\mathcal{T}$,
(1.2) $\mathcal{P}_{k+1} \in \mathcal{P}_k$ for all $k < i$,
(1.3) $\pi_i \in \mathcal{P}_i$,
(1.4) the set

$$\{\gamma \mid \gamma \in \mathcal{P}_i \cap \mathrm{Ord}, \pi_i(\nu_i) < \gamma, \text{ and } (\mathcal{P}_i \cap V_\gamma, \delta_i) \text{ is a premouse}\}$$

has ordertype $\pi_i(\theta_i)$.

Let $(\pi_0, (\mathcal{P}_0, \delta_0), \nu_0) = (\pi, (V_\Xi, \pi(\delta)), \eta)$ and suppose that $(\pi_i, (\mathcal{P}_i, \delta_i), \nu_i)$ satisfies (1.1)–(1.4). We define $(\pi_{i+1}, (\mathcal{P}_{i+1}, \delta_{i+1}), \nu_{i+1})$. Note that

$$\mathcal{P}_{i^*} \cap V_\theta = \mathcal{P}_i \cap V_\theta$$

where $\theta = \mathrm{SPT}(\pi_i(E_i)) + 1$. This is by the agreement between π_i and π_{i^*}. Thus $\mathrm{Ult}(\mathcal{P}_{i^*}, \pi_i(E_i))$ is defined. We claim that this ultrapower is wellfounded. We cannot appeal to Theorem 66 for iteration trees of finite length since the sequence $\langle(\mathcal{P}_m, \pi_m) : m \leq i\rangle$ does not define an iteration tree (because of (1.2)). By condition (1.2), $\pi_i(E_i) \in \mathcal{P}_{i^*}$ and by the agreement between $\mathcal{P}_{i^*}$ and $\mathcal{P}_i$, $\pi_i(E_i)$ is a *pre-extender*, which is to say it satisfies all the requirements of being an extender except possibly the wellfoundedness condition. Again by the agreement between $\mathcal{P}_{i^*}$ and $\mathcal{P}_i$ (and since $\pi_i(E_i) \in \mathcal{P}_i$), if $\mathrm{Ult}(\mathcal{P}_{i^*}, \pi_i(E_i))$ is not wellfounded then $\mathrm{Ult}(\mathcal{P}_i, \pi_i(E_i))$ is not wellfounded which is a contradiction.

Let $\mathcal{P}$ be the transitive collapse of $\mathrm{Ult}(\mathcal{P}_{i^*}, \pi_i(E_i))$ and let

$$j : \mathcal{P}_{i^*} \to \mathcal{P}$$

be the induced elementary embedding. Again (by shifting) there is an elementary embedding

$$\tau : M_{i+1} \to \mathcal{P} \cap V_{j(\nu_i)}$$

defined by

$$\tau(j_{i^*,i+1}(f)(s)) = j(\pi_{i^*}(f))(\pi_i(s))$$

where $f \in M_{i^*}$ and $s \in [\mathrm{LTH}(E_i)]^{<\omega}$.

We now use the hypothesis that $\mathcal{T}$ is a (+1)-iteration tree. Let $\gamma + 1 \geq \mu^{\mathcal{T}}(i)$ be least. Thus

$$\mathrm{SPT}(E_i) + 1 \leq \gamma + 1 \leq \rho(E_i)$$

and so

$$\mathcal{P} \cap V_{\pi_i(\gamma)+1} = \mathcal{P}_i \cap V_{\pi_i(\gamma)+1}.$$

Therefore $\tau | M_{i+1} \cap V_\gamma \in \mathcal{P}$. Thus by absoluteness there exists an elementary embedding,

$$\sigma : (M_{i+1}, j_{0,i+1}(\delta)) \to (\mathcal{P} \cap V_{j(\nu_i)}, j(\delta_i))$$

such that $\sigma \in \mathcal{P}$, $\sigma(\gamma) = \pi_i(\gamma)$, and such that

$$\sigma | M_{i+1} \cap V_\gamma = \tau | M_{i+1} \cap V_\gamma.$$

Finally $j_{i^*,i+1}(\theta_{i^*}) > \theta_{i+1}$ and so the set,

$$\{\upsilon < j(\nu_i) \,|\, j(\delta_i) < \upsilon \text{ and } (\mathcal{P} \cap V_\upsilon, j(\delta_i)) \text{ is a premouse}\}$$

has ordertype greater than $\sigma(\theta_{i+1})$. Let $\upsilon_0 \in \mathcal{P} \cap \mathrm{Ord}$ be such that $(\mathcal{P} \cap V_{\upsilon_0}, j(\delta_i))$ is a premouse and such that the set

$$\{\upsilon < \upsilon_0 \,|\, j(\delta_i) < \upsilon \text{ and } (\mathcal{P} \cap V_\upsilon, j(\delta_i)) \text{ is a premouse}\}$$

has ordertype $\sigma(\theta_{i+1})$. Let

$$X \prec \mathcal{P} \cap V_{\upsilon_0}$$

be an elementary substructure such that

(2.1) $X \in \mathcal{P}$,
(2.2) $\{\sigma, j(\nu_i)\} \subseteq X$,
(2.3) $\mathcal{P} \cap V_{\sigma(\gamma)+1} \subseteq X$,
(2.4) $|X|^{\mathcal{P}} = |\mathcal{P} \cap V_{\sigma(\gamma)+1}|^{\mathcal{P}}$.

Let $\mathcal{P}_{i+1}$ be the transitive collapse of X and let $(\pi_{i+1}, \delta_{i+1}, \nu_{i+1})$ be the image of $(\sigma, j(\delta_i), j(\nu_i))$ under the transitive collapse of X. Finally we verify that $\mathcal{P}_{i+1} \in \mathcal{P}_i$. For this it suffices to verify that

$$\mathcal{P} \cap V_{\sigma(\gamma)+2} = \mathrm{Ult}(\mathcal{P}_i, \pi_i(E_i)) \cap V_{\sigma(\gamma)+2}$$

(where we identify the ultrapower with its transitive collapse). But

$$\mathcal{P}_{i^*} \cap V_{\pi(\gamma)+1} = \mathcal{P}_i \cap V_{\pi(\gamma)+1} = N$$

and $\mathrm{SPT}(\pi_i(E_i)) + 1 \leq \pi_i(\gamma) < \rho(\pi(E_i))$. Thus the ultrapowers of N are the same computed in $\mathcal{P}_i$ or in $\mathcal{P}_{i^*}$. This implies

$$\mathrm{Ult}(\mathcal{P}_{i^*}, \pi_i(E_i)) \cap V_{\pi(\gamma)+2} = \mathrm{Ult}(\mathcal{P}_i, \pi_i(E_i)) \cap V_{\pi(\gamma)+2}.$$

Thus $\mathcal{P}_{i+1} \in \mathcal{P}_i$.

This completes the definition of the sequence,

$$\langle (\pi_i, (\mathcal{P}_i, \delta_i), \nu_i) : i < \omega \rangle,$$

and this is a contradiction by (1.2). □

3.5. *Iteration trees of length α*

We shall prove that a counterexample to Theorem 66 gives a countable premouse, (M, δ), an elementary embedding,

$$\pi : M \to V_\Theta,$$

and a $(+2)$-iteration tree $\mathcal{T}$ on (M, δ) of countable length which in a suitable sense is continuously illfounded. The major part of the proof of Theorem 66 will be to show that this latter condition implies a certain kind of enlargement exists and from this we will obtain a contradiction.

Definition 75. Suppose η is an ordinal. An *iteration-template on* η is a tree order, $<_{\mathcal{T}}$, on η such that

(1) $<_{\mathcal{T}}$ is a suborder of the standard order,
(2) $0 <_{\mathcal{T}} \alpha$ for all $\alpha < \eta$,
(3) for each limit ordinal $\gamma < \eta$, the set
$$\{\xi < \gamma \,|\, \xi <_{\mathcal{T}} \gamma\}$$
is cofinal in γ.

Definition 76 (Martin–Steel [7]). Suppose that $\eta < \omega_1$ and suppose that $<_{\mathcal{T}}$ is an iteration-template on η. A function
$$p : \eta \to \eta$$
is a *preservation function for* $<_{\mathcal{T}}$ if $p(0) = 0$ and

(1) for all $\gamma < \eta$, if $\gamma > 0$ then $p(\gamma) = \beta + 1$ for some $\beta < \gamma$,
(2) for all $\gamma \leq \eta$, if γ is a limit ordinal then
$$\gamma = \sup\{\min\{p(\beta) \,|\, \alpha \leq \beta < \gamma\} \,|\, \alpha < \gamma\},$$
(3) for all $\gamma < \eta$, if γ is a limit ordinal then for all sufficiently large $\alpha < \gamma$, if $\alpha <_{\mathcal{T}} \gamma$ then:
   (a) for all $\xi < \gamma$, if $\alpha < \xi$ and if $\xi \not<_{\mathcal{T}} \gamma$, then $p(\xi) > \alpha$;
   (b) if $\xi < \gamma$ and if α is the $<_{\mathcal{T}}$-predecessor of ξ then $p(\xi) = \alpha + 1$.

Lemma 77 (Martin–Steel [7]). *Suppose that $\eta < \omega_1$ and $<_{\mathcal{T}}$ is an iteration-template on η. Then there exist a function*
$$p : \eta \to \eta$$
such that

(1) *p is a preservation function for $<_{\mathcal{T}}$,*
(2) *if η is a limit ordinal then there exists a sequence, $\langle \alpha_i : i < \omega\rangle$, such that*
   (a) $\sup\{\alpha_i \,|\, i < \omega\} = \eta$,
   (b) *for all $i < \omega$, $p(\alpha_{i+1} + 1) = \alpha_i + 1$,*
   (c) *for all $i < \omega$, for all ξ, if $\alpha_i + 1 < \xi \leq \alpha_{i+1}$, then $\alpha_i + 1 < p(\xi)$.*

Definition 78 (Martin–Steel [7]). Suppose that $\eta < \omega_1$, $<_{\mathcal{T}}$ is an iteration-template on η, and suppose that
$$p : \eta \to \eta$$
is a preservation function for $<_{\mathcal{T}}$ and suppose that $\alpha <_{\mathcal{T}} \beta$. Then α *survives at* β if

(1) for all $\xi <_{\mathcal{T}} \beta$, if $\alpha \leq \xi < \gamma < \beta$ and if $\gamma \not<_{\mathcal{T}} \beta$, then $p(\gamma) > \xi$, and
(2) for all $\gamma \leq_{\mathcal{T}} \beta$, if $\alpha < \gamma$ and if ξ is the $<_{\mathcal{T}}$-predecessor of γ then $p(\gamma) = \xi + 1$.

The key point of the definition of α survives at β is the following. Suppose that $\gamma < \eta$ and that γ is a limit ordinal. Then there exists $\xi < \gamma$, such that for all $\alpha <_{\mathcal{T}} \beta \leq_{\mathcal{T}} \gamma$, if $\xi < \alpha$ then α survives at β. This is immediate from the definitions noting that since γ is a limit ordinal, γ cannot have a $<_{\mathcal{T}}$-predecessor.

Suppose that $b \subseteq \eta$ is a maximal set which is totally ordered by $<_{\mathcal{T}}$. Define a sequence, $\langle \gamma_i^b : i < \omega \rangle$, by induction on $i < \omega$ as follows. Let γ_0^b be the least $\alpha \in b$ such that for some $\beta \in b$, $\alpha < \beta$ and α does not survive at β, if no such α exists then $\gamma_0^b = 0$. Given γ_i^b, define γ_{i+1}^b to be the least $\beta \in b$ such that $\gamma_i^b < \beta$ and γ_i^b does not survive at β if such an ordinal β exists, otherwise $\gamma_{i+1}^b = \gamma_i^b$.

(1) If $\{\gamma_i^b \mid i < \omega\}$ is infinite then $\sup(\{\gamma_i^b \mid i < \omega\}) = \sup(b)$;
(2) Suppose $\alpha, \beta \in b$, $\alpha < \beta$ and α does not survive at β. Then for some $i < \omega$ either, $\alpha = \gamma_i^b$ and $\gamma_i^b < \gamma_{i+1}^b \leq \beta$, or $\gamma_i^b < \alpha < \gamma_{i+1}^b \leq \beta$.

This motivates the following definition from [7].

Definition 79. Suppose that (M, δ) is a premouse and that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is an iteration tree on (M, δ) of length $\eta < \omega_1$. Suppose that

$$p : \eta \to \eta$$

is a preservation function for $<_{\mathcal{T}}$. Then $\mathcal{T}$ is *p-continuously illfounded* if there exists a function $f : \eta \to \mathrm{Ord}$ such that

(1) for all $\alpha < \eta$, $f(\alpha) \in M_\alpha$,
(2) for all $\alpha <_{\mathcal{T}} \beta$ if α does not survive at β then

$$j_{\alpha,\beta}(f(\alpha)) > f(\beta)$$

and if α survives at β then

$$j_{\alpha,\beta}(f(\alpha)) = f(\beta)$$

Remark 80. Suppose (M, δ) is a premouse. Then any iteration tree on (M, δ) of finite length is p-continuously illfounded. Further, it is clear from the definition that if $\mathcal{T}$ is an iteration tree on (M, δ) of length η which is p-continuously illfounded, then for all $\eta' < \eta$, $\mathcal{T}|\eta'$ is p'-continuously illfounded where $p' = p|\eta'$. Thus p-continuously illfounded iteration trees can exist.

However suppose that $\mathcal{T}$ is an iteration tree on (M, δ) of length η, η is a limit ordinal, and $\mathcal{T}$ is p-continuously illfounded. Suppose that $\mathcal{T}$ has a wellfounded cofinal branch. Then the preservation function p cannot be arbitrary, there must exist a cofinal branch b of $\mathcal{T}$ and $\beta \in b$ such that for all $\alpha \in b$, if $\beta <_{\mathcal{T}} \alpha$ then β survives at α, (take b to be any cofinal, wellfounded, branch of $\mathcal{T}$). Further and perhaps more relevant to the definition, if $\mathcal{T}$ is p-continuously illfounded then $\mathcal{T}$ *cannot* have two cofinal wellfounded branches. We verify this last claim.

Suppose that b and c are distinct cofinal wellfounded branches of $\mathcal{T}$. We can suppose that each is a maximal branch and so each is a closed cofinal subset of η.

Thus there exist $\alpha_0 \in b$ and $\beta_0 \in c$ such that for all $\alpha_0 \leq \alpha_1 < \alpha_2$ in b, α_1 survives at α_2 and for all $\beta_0 \leq \beta_1 < \beta_2$ in c, β_1 survives at β_2.

Since b and c are distinct there must exist $\gamma < \eta$ such that $b \cap c \subseteq \gamma$. By increasing γ if necessary we can also suppose that $\alpha_0 < \gamma$ and that $\beta_0 < \gamma$.

The branches b and c are each closed cofinal subsets of η, and so one of the following must hold.

(1) There exist $\alpha_1 < \alpha_2$ in $b \backslash \gamma$ such that

$$[\alpha_1, \alpha_2] \cap c = \emptyset.$$

(2) For all $\alpha_1 < \alpha_2$ in $b \backslash \gamma$ if α_2 is the b-successor of α_1 then

$$[\alpha_1, \alpha_2] \cap c \neq \emptyset.$$

In each case one obtains a contradiction. Suppose that $\alpha_1 < \alpha_2$ in $b \backslash \gamma$ witness that (1) holds. Let β_1 be the maximum element of $c \cap \alpha_1$ and let β_2 be the minimum element of $c \backslash \alpha_2$. Then β_2 must be the c-successor of β_1 and so since $\beta_0 \leq \beta_1$, β_1 survives at β_2 which implies that $p(\beta_2) = \beta_1 + 1$. But $\alpha_2 < \beta_2$ and $\alpha_0 \leq \alpha_1$. Therefore α_1 survives at α for all $\alpha \in b$ such that $\alpha_2 < \alpha$. This implies that $p(\beta_2) > \alpha_2$ which is a contradiction. Now suppose that (2) holds. Fix $\alpha_1 < \alpha_2$ in $b \cap \gamma$ such that α_2 is the b-successor of α_1. Let

$$\beta_1 = \max(c \cap \alpha_2)$$

and let β_2 be the c-successor of β_1. Thus

$$\gamma \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2.$$

But α_1 survives at α_2 which implies that $p(\alpha_2) = \alpha_1 + 1$ and β_1 survives at β_2 which implies that $p(\alpha_2) > \beta_1$, which is again a contradiction.

As an immediate corollary we obtain that if $\mathcal{T}$ is p-continuously illfounded then $\mathcal{T}$ can have at most one proper maximal wellfounded branch.

Lemma 81. *Suppose that (M, δ) is a countable premouse,*

$$\pi : M \to V_\Theta$$

is an elementary embedding and that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable iteration tree on (M, δ). Suppose that $\mathcal{T}$ has no proper maximal π-realizable branch, $\mathcal{T}$ can be copied to define an iteration tree $\mathcal{S}$ on $(V_\Theta, \pi(\delta))$ with length η, and $p : \eta \to \eta$ is a preservation function for $<_{\mathcal{S}}$. Then $\mathcal{S}$ is p-continuously illfounded.

Proof. The proof is essentially the same as the proof of the analogous claim of [7]. The generalization to iteration trees with long extenders allowed does not really affect the proof in any essential way.

We sketch the argument. First we reduce to the case that $\mathcal{T}$ has no wellfounded maximal branches of limit length. In fact we claim that $\mathcal{S}$ has no maximal wellfounded branches of limit length. To prove this claim, assume toward a contradiction that b is such a branch and let (N, δ_N) be the limit of $\mathcal{S}$ along b and let

$$k_b : (V_\Theta, \pi(\delta)) \to (N, \delta_N)$$

be the associated elementary embedding.

Let $\Theta' > \Theta$ be such that $(V_{\Theta'}, \pi(\delta))$ is a premouse. Since $\mathcal{T}$ has countable length, and since M is countable, if $\mathcal{S}'$ is the result of copying $\mathcal{T}$ to an iteration tree on $(V_{\Theta'}, \pi(\delta))$, then b is a wellfounded branch of $\mathcal{S}'$. Further letting

$$k'_b : (V_{\Theta'}, \pi(\delta)) \to (N', \delta_N)$$

be the associated embedding, then

$$k_b = k'_b | V_\Theta.$$

By the shift lemma, Lemma 69, if (M_b, δ_b) is the limit of $\mathcal{T}$ along b then there is an elementary embedding,

$$\sigma : (M_b, \delta_b) \to (N' \cap V_{j'_b(\Theta)}, \delta_N)$$

such that $\sigma \circ j_b = k_b \circ \pi$ where

$$j_b : (M, \delta) \to (M_b, \delta_b)$$

is the elementary embedding arising from $\mathcal{T}$. By absoluteness there exists an elementary embedding

$$\sigma' : (M_b, \delta_b) \to (N' \cap V_{j'_b(\Theta)}, \delta_N)$$

such that $\sigma' \in N'$ and such that $\sigma' \circ j_b | M = k_b(\pi)$. Therefore there must exist an elementary embedding

$$\tau : (M_b, \delta_b) \to (V_\Theta, \pi(\delta))$$

such that $\tau \circ j_b = \pi$ and so b is π-realizable which is a contradiction.

Therefore $\mathcal{S}$ has no maximal wellfounded branches.

Since $\mathcal{T}$ has only countably many non-maximal branches, this same argument shows that $\mathcal{S}'$ has no maximal wellfounded branches in $V[G]$, where G is V-generic for $\mathrm{Coll}(\omega, \Theta')$. The point is that even though $b \in V[G]$, and with notation as above, by absoluteness there exists a maximal branch d of limit length and an elementary embedding

$$\sigma' : (M_d, \delta_d) \to (N' \cap V_{j'_b(\Theta)}, \delta_N)$$

such that $d \in N'$, $\sigma' \in N'$, and such that $\sigma' \circ e_d | M = j_b(\pi)$.

Now let $\Upsilon > \Theta'$ be such that $V_\Upsilon \prec_{\Sigma_2} V$ and let $X \prec V_\Upsilon$ be a countable elementary substructure such that

$$\{\mathcal{T}, M, \pi, \Theta, \mathcal{S}\} \subseteq X.$$

Let M' be the transitive collapse of $X \cap V_\Theta$ and let

$$\pi_X : (M, \delta) \to (M', \delta')$$

be the image of π under the transitive collapse and let

$$\pi' : (M', \delta') \to (V_\Theta, \pi(\delta))$$

invert the transitive collapse. The image of $\mathcal{S}$ under the transitive collapse is an iteration tree $\mathcal{T}'$ on (M', δ'). Further the iteration tree $\mathcal{T}'$ is the result of copying $\mathcal{T}$ via π_X. It follows that every maximal branch of $\mathcal{T}'$ which is of limit length, is illfounded.

Finally copying $\mathcal{T}'$ to an iteration tree on $(V_\Theta, \pi(\delta))$ via π' is the same as copying $\mathcal{T}$ to $(V_\Theta, \pi(\delta))$ via π.

Thus we can assume that every maximal branch of $\mathcal{T}$ of limit length is illfounded.

Let $\langle(\beta_i, x_i) : i < \omega\rangle$ enumerate, with infinite repetition, the set of all pairs (β, x) such that $\beta < \eta$ and such that $x \in M_\beta$.

Define a tree, $\mathcal{U} \subseteq V_\Theta$, of finite sequences as follows: $\mathcal{U}$ is the set of all finite sequences,

$$\langle(\alpha_0, y_0), \ldots, (\alpha_n, y_n)\rangle$$

such that:

(1.1) $\alpha_0 = 0$;
(1.2) For all $i < n$, $\alpha_i <_\mathcal{T} \alpha_{i+1}$ and α_i does not survive at α_{i+1};
(1.3) For each $i \leq n$ let $z_i = j_{\beta_i, \alpha_n}(x_i)$ if $\beta_i <_\mathcal{T} \alpha_i$ and let $z_i = \emptyset$ otherwise. Then

$$(M_{\alpha_n}, j_{0,\alpha_n}(\delta), \langle z_i : i \leq n\rangle) \equiv (V_\Theta, \pi(\delta), \langle y_i : i \leq n\rangle).$$

Clearly $\mathcal{U}$ is wellfounded since any infinite branch of $\mathcal{U}$ produces a maximal branch of $\mathcal{T}$ which is wellfounded (and proper).

Fix $\Theta_0 > \Theta$ such that $(V_{\Theta_0}, \pi(\delta))$ is a premouse. Since $\mathcal{T}$ can be copied via π to define an iteration tree on $(V_\Theta, \pi(\delta))$, $\mathcal{T}$ can be copied via π to define an iteration tree on $(V_{\Theta_0}, \pi(\delta))$, Let

$$\mathcal{S}_0 = \langle N^0_\alpha, F^0_\alpha, k^0_{\alpha,\beta} : \alpha < \eta, \alpha <_\mathcal{T} \beta\rangle$$

be this iteration tree. We first show that $\mathcal{S}_0$ is p-continuously illfounded. Then we will by reflection show that $\mathcal{S}$ is p-continuously illfounded.

For each $\alpha < \eta$, let

$$\pi^0_\alpha : (M_\alpha, \delta_\alpha) \to (N^0_\alpha, \kappa^0_\alpha)$$

be the embedding from the copying of $\mathcal{T}$ where for $\alpha < \eta$, $\delta_\alpha = j_{0,\alpha}(\delta)$ and $\kappa_\alpha = k^0_{0,\alpha}(\pi(\delta))$.

We define $f_p : \eta \to \mathrm{Ord}$ to witness that $\mathcal{S}_0$ is p-continuously illfounded. Fix $\gamma < \eta$. Define by induction on $i < \omega$, $\alpha_0 = 0$, and

$$\alpha_{i+1} = \min\{\beta \leq_\mathcal{T} \gamma \mid \alpha_i <_\mathcal{T} \beta \text{ and } \alpha_i \text{ does not survive at } \beta\};$$

and let n be the largest i for which α_i is defined (the sequence must be finite).

For each $i \leq n$ if $\beta_i \leq_\mathcal{T} \alpha_i$ then set $y_i = k^0_{\beta_i,\alpha_i}(\pi_{\beta_i}(x_i))$, otherwise set $y_i = \emptyset$. Here (β_i, x_i) refers to the enumeration used to define the tree $\mathcal{U}$.

Note

$$\langle(\alpha_i, y_i) : i \le n\rangle \in k^0_{0,\gamma}(\mathcal{U}).$$

Define $f_p(\gamma)$ to be the rank of $\langle(\alpha_i, y_i) : i \le n\rangle$ within the wellfounded tree, $k^0_{0,\gamma}(\mathcal{U})$.

It follows that f_p witnesses that $\mathcal{S}_0$ is p-continuously illfounded.

We finish by showing that $\mathcal{S}$ is p-continuously illfounded. Note that $f_p \in V_{\Theta_0}$ since $\mathcal{U}$ has cardinality Θ and $\mathcal{T}$ has countable length.

Let $X \prec V_{\Theta_0}$ be an elementary substructure such that

$$V_{\pi(\delta)} \cup \{\pi, \Theta, f_p\} \subseteq X$$

and such that $|X| = \pi(\delta)$.

Let N be the transitive collapse of X. Let

$$\pi_N : (M, \delta) \to (N, \pi(\delta))$$

be the image of π under the transitive collapse. Let f^N_p be the image of f_p under the transitive collapse.

It follows that f^N_p witnesses that $\mathcal{S}_N$ is p-continuously illfounded where $\mathcal{S}_N$ is the iteration tree on $(N, \pi(\delta))$ obtained by copying $\mathcal{T}$ via π_N.

Since $V_{\pi(\delta)} \subseteq N$ and since

$$\pi|(M \cap V_\delta) = \pi_N|(M \cap V_\delta),$$

it follows that f^N_p witnesses that $\mathcal{S}$ is p-continuously illfounded. □

The proof of Theorem 66 involves the construction of a sequence, $\langle\mathcal{E}^\alpha : \alpha < \eta\rangle$, of enlargements of $\mathcal{T}$ which are *connected* in a certain rather technical sense involving a preservation function p. The notion of a connected sequence of enlargements is from [7] and adapted to our situation of iteration trees with long extenders. The *existence* of such a sequence of enlargements will follow if the iteration tree, $\mathcal{T}$, is p-continuously illfounded and Lemma 81 will allow us to reduce to this case.

The definition requires some more notation. Suppose that $\mathcal{T}$ is an iteration tree of length η on a premouse, (M, δ), and that $g : \eta \to \mathrm{Ord}$ is a function such that for all $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \le g(\alpha) \le \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Then for each $\alpha < \beta \le \eta$,

$$\mu^{\mathcal{T}}_g[\alpha, \beta] = \min\{g(\gamma) \mid \alpha \le \gamma < \beta\}.$$

Definition 82. Suppose that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle$$

is a countable iteration tree on (M, δ), $p : \eta \to \eta$ is a preservation function for $<_{\mathcal{T}}$ and that $g : \eta \to \mathrm{Ord}$ is a function such that for all $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \le g(\alpha) \le \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Suppose that $\lambda \leq \eta$ and that for each $\alpha < \lambda$,

$$\mathcal{E}^\alpha = \langle (\pi^\alpha_\beta, (\mathcal{P}^\alpha_\beta, \delta^\alpha_\beta), \nu^\alpha_\beta) : \beta \leq \alpha \rangle$$

is a g-enlargement of $\mathcal{T}$.

The sequence $\langle \mathcal{E}^\alpha : \alpha < \lambda \rangle$ is *p-connected* if there exists a sequence of elementary embeddings,

$$\langle e_{\alpha,\beta} : (\mathcal{P}^\alpha_\alpha, \delta^\alpha_\alpha) \to (\mathcal{P}^\beta_\beta, \delta^\beta_\beta) \mid \alpha <_{\mathcal{T}} \beta < \lambda \text{ and } \alpha \text{ survives at } \beta \rangle$$

such that the following hold.

(1) For all $\beta \leq \alpha < \lambda$, $\nu^\alpha_\beta \in \mathcal{P}^\alpha_\beta$ and for all $\alpha+1 < \lambda$, $\mathcal{E}^{\alpha+1}|(\alpha+1)$ is a g-enlargement of $\mathcal{T}$ and

(2) Suppose $\alpha + 1 < \lambda$ and $p(\alpha + 1) = \xi + 1$. Then

   (a) $\mathcal{E}^{\alpha+1}|(\xi + 1) = \mathcal{E}^\alpha|(\xi + 1)$.

   (b) If $\xi < \alpha$ and $\theta = \pi^\alpha_\alpha(\mu^{\mathcal{T}}_g[\xi, \alpha))$ then

      (i) $\mathcal{P}^\alpha_\alpha \cap V_{\theta+1} \subseteq \mathcal{P}^{\alpha+1}_\alpha$,

      (ii) $\mathcal{P}^{\alpha+1}_\alpha \in \mathcal{P}^\alpha_\alpha$ and

$$|\mathcal{P}^{\alpha+1}_\alpha|^{\mathcal{P}^\alpha_\alpha} = |\mathcal{P}^\alpha_\alpha \cap V_{\theta+1}|^{\mathcal{P}^\alpha_\alpha},$$

      (iii) there exist $\delta^{\alpha+1}_\alpha < \nu \in \mathcal{P}^\alpha_\alpha$ and an elementary embedding

$$\pi : (\mathcal{P}^{\alpha+1}_\alpha, \delta^{\alpha+1}_\alpha) \to (\mathcal{P}^\alpha_\alpha \cap V_\nu, \delta^\alpha_\alpha)$$

      such that $\pi|(\mathcal{P}^\alpha_\alpha \cap V_{\theta+1})$ is the identity, $\text{range}(\pi^\alpha_\alpha) \subseteq \text{range}(\pi)$, and $\pi^{\alpha+1}_\alpha = \pi^{-1} \circ \pi^\alpha_\alpha$.

   (c) If $\xi + 1 < \alpha$ then there is a sequence

$$\langle (\mathcal{R}_\gamma, \eta^i_\gamma) : \xi < \gamma < \alpha, i < \omega \rangle \in \mathcal{P}^{\alpha+1}_\alpha$$

   such that for all $\xi < \gamma < \alpha$, for all $i < \omega$,

      (i) $\eta^i_\gamma < \eta^{i+1}_\gamma \in \mathcal{R}_\gamma$,

      (ii) $\mathcal{P}^{\alpha+1}_\gamma = \mathcal{R}_\gamma \cap V_{\eta^0_\gamma}$,

      (iii) $(\mathcal{R}_\gamma, \delta^{\alpha+1}_\gamma)$ and $(\mathcal{R}_\gamma \cap V_{\eta^i_\gamma}, \delta^{\alpha+1}_\gamma)$ are premice,

      (iv) $\pi^{\alpha+1}_\gamma|(M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha)}) = \pi^{\alpha+1}_\alpha|(M_\alpha \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha)})$,

      (v) η^i_γ has uncountable cofinality in $\mathcal{R}_\gamma$.

   (d) Let $\delta = \alpha^*$. Then

$$\mathcal{P}^{\alpha+1}_{\alpha+1} = \text{Ult}(\mathcal{P}^{\alpha+1}_\delta, \pi^{\alpha+1}_\alpha(E_\alpha))$$

   and if $k : \mathcal{P}^{\alpha+1}_\delta \to \mathcal{P}^{\alpha+1}_{\alpha+1}$ is the associated embedding, $\nu^{\alpha+1}_{\alpha+1} = k(\nu^{\alpha+1}_\delta)$. Further if δ survives at $\alpha + 1$ then $e_{\delta,\alpha+1} = k$ and

$$e_{\gamma,\alpha+1} = e_{\delta,\alpha+1} \circ e_{\gamma,\delta}$$

   for all γ such that γ survives at δ.

(3) Suppose that $\alpha < \lambda$ is a limit ordinal. Then

(a) $\mathcal{P}^\alpha_\alpha$ is the direct limit of

$$\langle \mathcal{P}^\beta_\beta : \beta <_{\mathcal{T}} \alpha,\ \beta \text{ survives at } \alpha\rangle$$

under the embeddings,

$$\langle e_{\beta,\delta} : \beta <_{\mathcal{T}} \delta <_{\mathcal{T}} \alpha,\ \beta \text{ survives at } \alpha\rangle$$

and for $\beta <_{\mathcal{T}} \alpha$ such that β survives at α, $e_{\beta,\alpha}$ is the induced embedding.

(b) if $\beta <_{\mathcal{T}} \alpha$ and $x \in M_\beta$, then there exists $\beta_x <_{\mathcal{T}} \alpha$ such that

$$e_{\gamma,\delta}(\pi^\gamma_\gamma(j_{\beta,\gamma}(x))) = \pi^\delta_\delta(j_{\beta,\delta}(x))$$

for all $\beta_x <_{\mathcal{T}} \gamma <_{\mathcal{T}} \delta <_{\mathcal{T}} \alpha$.

(c) for all $\beta < p(\alpha)$,

$$(\pi^\alpha_\beta, (\mathcal{P}^\alpha_\beta, \delta^\alpha_\beta), \nu^\alpha_\beta) = (\pi^\gamma_\beta, (\mathcal{P}^\gamma_\beta, \delta^\gamma_\beta), \nu^\gamma_\beta)$$

for all sufficiently large $\gamma < \alpha$.

(d) There is a sequence

$$\langle (\mathcal{R}_\gamma, \eta^i_\gamma) : p(\alpha) \le \gamma < \alpha, i < \omega\rangle \in \mathcal{P}^\alpha_\alpha$$

such that for all $p(\alpha) \le \gamma < \alpha$, for all $i < \omega$,

(i) $\eta^i_\gamma < \eta^{i+1}_\gamma \in \mathcal{R}_\gamma$,
(ii) $\mathcal{P}^\alpha_\gamma = \mathcal{R}_\gamma \cap V_{\eta^0_\gamma}$,
(iii) $(\mathcal{R}_\gamma, \delta^\alpha_\gamma)$ and $(\mathcal{R}_\gamma \cap V_{\eta^i_\gamma}, \delta^\alpha_\gamma)$ are premice,
(iv) η^i_γ has uncountable cofinality in $\mathcal{R}_\gamma$.

We note that the sequence of elementary embeddings,

$$\langle e_{\alpha,\beta} : (\mathcal{P}^\alpha_\alpha, \delta^\alpha_\alpha) \to (\mathcal{P}^\beta_\beta, \delta^\beta_\beta) \mid \alpha <_{\mathcal{T}} \beta < \lambda \text{ and } \alpha \text{ survives at } \beta\rangle$$

is uniquely determined by the p-connected sequence,

$$\langle \mathcal{E}^\alpha : \alpha < \lambda\rangle,$$

of g-enlargements (together with p and $\mathcal{T}$).

Remark 83. Besides the adaptation to iteration trees with long extenders (which is completely straightforward), Definition 82 differs from the corresponding definition of [7] in one other aspect. This are the requirements, 2c(v) and 3d(iv), that the ordinals η^i_γ have uncountable cofinality (in $\mathcal{R}_\gamma$). This is really a "place holder" for more general requirements one can impose which could be relevant to proving further generalizations of Theorem 66. These additional requirements also allow for stronger induction hypotheses in main construction of the proof of Lemma 88 and do not seriously compromise the proof of Lemma 89 (any potential complications are mitigated by introducing *quasi-premice*).

We now prove two lemmas concerning the agreement between the models in a connected sequence of enlargements. These will be needed for the upcoming constructions of connected sequences of enlargements. The second of these lemmas corresponds to Remark 3 after [7, Definition 4.11].

Lemma 84. *Suppose that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable iteration tree on (M, δ), $p : \eta \to \eta$ *is a preservation function for* $<_{\mathcal{T}}$ *and that* $g : \eta \to \mathrm{Ord}$ *is a function such that for all* $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) + 1 \leq \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Suppose that $\lambda \leq \eta$ *and that for each* $\alpha < \lambda$,

$$\mathcal{E}^\alpha = \langle (\pi^\alpha_\beta, (\mathcal{P}^\alpha_\beta, \delta^\alpha_\beta), \nu^\alpha_\beta) : \beta \leq \alpha \rangle$$

is a g*-enlargement of* $\mathcal{T}$ *such that the sequence* $\langle \mathcal{E}^\alpha : \alpha < \lambda \rangle$ *is* p*-connected.*

Suppose that $\alpha < \lambda$ *and that* $\gamma < \alpha$. *Then*

$$\mathcal{P}^\alpha_\gamma \cap V_{\pi^\alpha_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} \subset \mathcal{P}^\alpha_\alpha.$$

Proof. Note that since $\mathcal{E}^\alpha$ is a g-enlargement of $\mathcal{T}$ of length $\alpha + 1$, for all $\gamma < \alpha$,

$$\mathcal{P}^\alpha_\gamma \cap V_\theta \subset \mathcal{P}^\alpha_\alpha$$

where $\theta = \pi^\alpha_\gamma(\mu^{\mathcal{T}}_g[\gamma, \alpha))$. The point of this lemma is to establish,

$$\mathcal{P}^\alpha_\gamma \cap V_{\theta+1} \subset \mathcal{P}^\alpha_\alpha,$$

though we shall only need this for $\gamma \geq p(\alpha + 1)$.

Fixing, γ and α, we prove by induction on $\gamma \leq \beta \leq \alpha$

$$\mathcal{P}^\beta_\gamma \cap V_{\pi^\beta_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} \subset \mathcal{P}^\beta_\beta.$$

If $\beta = \gamma$ this is immediate.

We next suppose that $\beta = \gamma + 1$.

By (2d) of Definition 82,

$$\mathcal{P}^{\gamma+1}_{\gamma+1} = \mathrm{Ult}(\mathcal{P}^{\gamma+1}_{\gamma^*}, \pi^{\gamma+1}_\gamma(E_\gamma)).$$

This implies that

$$\mathcal{P}^{\gamma+1}_\gamma \cap V_{\pi^{\gamma+1}_\gamma(\rho(E_\gamma))} = \mathcal{P}^{\gamma+1}_{\gamma+1} \cap V_{\pi^{\gamma+1}_\gamma(\rho(E_\gamma))}$$

and so

$$\mathcal{P}^{\gamma+1}_\gamma \cap V_{\pi^{\gamma+1}_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} \subset \mathcal{P}^{\gamma+1}_{\gamma+1}.$$

We now suppose that β is a successor ordinal, $\gamma + 1 < \beta \leq \alpha$, and that for all $\gamma \leq \xi < \beta$,

$$\mathcal{P}^\xi_\gamma \cap V_{\pi^\xi_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} \subset \mathcal{P}^\xi_\xi.$$

There are several subcases. First suppose that $p(\beta) = \beta$. Then by (2a) of Definition 82,

$$(\pi^{\beta}_{\beta-1}, (\mathcal{P}^{\beta}_{\beta-1}, \delta^{\beta}_{\beta-1}), \nu^{\beta}_{\beta-1}) = (\pi^{\beta-1}_{\beta-1}, (\mathcal{P}^{\beta-1}_{\beta-1}, \delta^{\beta-1}_{\beta-1}), \nu^{\beta-1}_{\beta-1})$$

and

$$(\pi^{\beta}_{\gamma}, (\mathcal{P}^{\beta}_{\gamma}, \delta^{\beta}_{\gamma}), \nu^{\beta}_{\gamma}) = (\pi^{\beta-1}_{\gamma}, (\mathcal{P}^{\beta-1}_{\gamma}, \delta^{\beta-1}_{\gamma}), \nu^{\beta-1}_{\gamma}).$$

By (2d) of Definition 82,

$$\mathcal{P}^{\beta}_{\beta} = \mathrm{Ult}(\mathcal{P}^{\beta}_{(\beta-1)^*}, \pi^{\beta}_{\beta-1}(E_{\beta-1})).$$

Now $\mu^{\mathcal{T}}_{g}[\gamma, \alpha) < \rho(E_{\beta-1})$ and so by the agreements given above, we are done.

Next suppose $p(\beta) < \beta$. There are two further subcases. First suppose $p(\beta) \leq \gamma$. Then by (2c) of Definition 82, since $\gamma + 1 < \beta$, $P^{\beta}_{\gamma} \in \mathcal{P}^{\beta}_{\beta-1}$. But again by (2d) of Definition 82,

$$\mathcal{P}^{\beta}_{\beta} = \mathrm{Ult}(\mathcal{P}^{\beta}_{(\beta-1)^*}, \pi^{\beta}_{\beta-1}(E_{\beta-1})).$$

As above, $\mu^{\mathcal{T}}_{g}[\gamma, \alpha) < \rho(E_{\beta-1})$, and since $\mathcal{E}^{\beta}$ is a g-enlargement of $\mathcal{T}$ of length $\beta+1$,

$$\pi^{\beta}_{\gamma}(\mu^{\mathcal{T}}_{g}[\gamma, \beta)) = \pi^{\beta}_{\beta}(\mu^{\mathcal{T}}_{g}[\gamma, \beta)).$$

However, $\mu^{\mathcal{T}}_{g}[\gamma, \alpha) \leq \mu^{\mathcal{T}}_{g}[\gamma, \beta)$ and so it follows that

$$\mathcal{P}^{\beta}_{\gamma} \cap V_{\pi^{\beta}_{\gamma}(\mu^{\mathcal{T}}_{g}[\gamma,\alpha))+1} \subset \mathcal{P}^{\beta}_{\beta}.$$

Finally suppose that $\gamma < p(\beta)$. By (2a) of Definition 82,

$$(\pi^{\beta}_{\gamma}, (\mathcal{P}^{\beta}_{\gamma}, \delta^{\beta}_{\gamma}), \nu^{\beta}_{\gamma}) = (\pi^{\beta-1}_{\gamma}, (\mathcal{P}^{\beta-1}_{\gamma}, \delta^{\beta-1}_{\gamma}), \nu^{\beta-1}_{\gamma}).$$

and by (2b) of Definition 82 (since $p(\beta) < \beta$),

$$\mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^{\mathcal{T}}_{g}[p(\beta)-1,\beta-1))+1} \subset \mathcal{P}^{\beta}_{\beta-1}.$$

But $\gamma \leq p(\beta) - 1$ and so $\mu^{\mathcal{T}}_{g}[p(\beta) - 1, \beta - 1) \geq \mu^{\mathcal{T}}_{g}[\gamma, \alpha)$. By (2d) of Definition 82,

$$\mathcal{P}^{\beta}_{\beta} = \mathrm{Ult}(\mathcal{P}^{\beta}_{(\beta-1)^*}, \pi^{\beta}_{\beta-1}(E_{\beta-1}))$$

and by the agreement above we are done.

Now suppose $\gamma < \beta \leq \alpha$ is a limit ordinal and that for all $\gamma \leq \xi < \beta$,

$$\mathcal{P}^{\xi}_{\gamma} \cap V_{\pi^{\xi}_{\gamma}(\mu^{\mathcal{T}}_{g}[\gamma,\alpha))+1} \subset \mathcal{P}^{\xi}_{\xi}.$$

There are two subcases. Suppose that $\gamma < p(\beta)$. Then by (3c) of Definition 82,

$$(\pi^{\beta}_{\gamma}, (\mathcal{P}^{\beta}_{\gamma}, \delta^{\beta}_{\gamma}), \nu^{\beta}_{\gamma}) = (\pi^{\xi}_{\gamma}, (\mathcal{P}^{\xi}_{\gamma}, \delta^{\xi}_{\gamma}), \nu^{\xi}_{\gamma})$$

for all sufficiently large $\xi < \beta$. By Lemma 68, for all sufficiently large $\xi <_{\mathcal{T}} \beta$,

$$\mu^{\mathcal{T}}_{g}[\gamma, \alpha) \leq \mu^{\mathcal{T}}_{g}[\gamma, \beta) < \mathrm{CRT}(j_{\xi,\beta}).$$

Thus (by (3a) and (3b)) of Definition 82, for all sufficiently large $\xi <_{\mathcal{T}} \beta$,

$$\pi^\beta_\gamma(\mu^{\mathcal{T}}_g[\gamma, \alpha)) < \mathrm{CRT}(e_{\xi,\beta}).$$

But this implies that for all sufficiently large $\xi <_{\mathcal{T}} \beta$,

$$\mathcal{P}^\xi_\xi \cap V_{\pi^\beta_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} = \mathcal{P}^\beta_\beta \cap V_{\pi^\beta_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1},$$

and from the agreement cited above,

$$\pi^\xi_\gamma(\mu^{\mathcal{T}}_g[\gamma, \alpha)) = \pi^\beta_\gamma(\mu^{\mathcal{T}}_g[\gamma, \alpha)).$$

Thus since the induction hypothesis holds for $\xi < \beta$, we obtain

$$\mathcal{P}^\beta_\gamma \cap V_{\pi^\beta_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} \subset \mathcal{P}^\beta_\beta$$

as desired.

Finally suppose that $p(\beta) \leq \gamma$. Then by (3d) of Definition 82,

$$\mathcal{P}^\beta_\gamma \in \mathcal{P}^\beta_\beta$$

and again we are done. □

Lemma 85. *Suppose that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle$$

is a countable iteration tree on (M, δ), $p : \eta \to \eta$ *is a preservation function for* $<_{\mathcal{T}}$ *and that* $g : \eta \to \mathrm{Ord}$ *is a function such that for all* $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) + 1 \leq \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Suppose that $\lambda \leq \eta$ *and that for each* $\alpha < \lambda$,

$$\mathcal{E}^\alpha = \langle(\pi^\alpha_\beta, (\mathcal{P}^\alpha_\beta, \delta^\alpha_\beta), \nu^\alpha_\beta) : \beta \leq \alpha\rangle$$

is a g*-enlargement of* $\mathcal{T}$ *such that the sequence* $\langle\mathcal{E}^\alpha : \alpha < \lambda\rangle$ *is* p*-connected.*

Suppose that $\beta < \alpha < \lambda$ *and that for all* $\gamma \leq \alpha$, *if* $\gamma > \beta$ *then* $p(\gamma) > \beta$. *Let*

$$\theta = \pi^\alpha_\beta(\mu^{\mathcal{T}}_g[\beta, \alpha)) = \pi^\alpha_\alpha(\mu^{\mathcal{T}}_g[\beta, \alpha)).$$

Then

(1) $\theta = \pi^\beta_\beta(\mu^{\mathcal{T}}_g[\beta, \alpha))$.
(2) $\pi^\beta_\beta|(M_\beta \cap V_{\mu^{\mathcal{T}}_g[\beta,\alpha)}) = \pi^\alpha_\beta|(M_\beta \cap V_{\mu^{\mathcal{T}}_g[\beta,\alpha)})$.
(3) $\mathcal{P}^\alpha_\alpha \cap V_{\theta+1} = \mathcal{P}^\beta_\beta \cap V_{\theta+1}$.
(4) $\mathcal{P}^\alpha_\alpha \cap V_{\theta+2} \subseteq \mathcal{P}^\beta_\beta \cap V_{\theta+2}$.
(5) *for all* $\beta < \gamma < \alpha$, $\mathcal{P}^\alpha_\gamma \cap V_{\theta+1} \subseteq \mathcal{P}^\beta_\beta \cap V_{\theta+1}$.

Proof. We fix β and prove (1)–(5) by induction on α.

Note since $\mathcal{E}^\alpha$ is a g-enlargement of $\mathcal{T}$ of length $\alpha + 1$,

$$\theta = \pi^\alpha_\beta(\mu^{\mathcal{T}}_g[\beta, \alpha)) = \pi^\alpha_\alpha(\mu^{\mathcal{T}}_g[\beta, \alpha)).$$

Further,

$$\pi^\alpha_\beta | M_\beta \cap V_{\mu^{\mathcal{T}}_g[\beta,\alpha)} = \pi^\alpha_\alpha | M_\alpha \cap V_{\mu^{\mathcal{T}}_g[\beta,\alpha)}.$$

We first suppose that α is a successor ordinal, (1)–(5) hold for $\alpha - 1$, and that $\beta < \alpha - 1$. We prove that (1)–(5) hold for α. (The case that $\alpha - 1 = \beta$ is similar though simpler).

Let $\delta = (\alpha - 1)^*$ (so δ is the $<_{\mathcal{T}}$-predecessor of α). Thus

$$\mathcal{P}^\alpha_\alpha = \mathrm{Ult}(\mathcal{P}^\alpha_\delta, \pi^\alpha_{\alpha-1}(E_{\alpha-1})).$$

Since $\textsc{spt}(E_{\alpha-1}) + 1 < g(\alpha - 1) + 1 \leq \rho^{\mathcal{T}}[\alpha - 1, \alpha)$,

$$\mathcal{P}^\alpha_\alpha \cap V_{\theta^*+1} = \mathcal{P}^\alpha_{\alpha-1} \cap V_{\theta^*+1}$$

and

$$\mathcal{P}^\alpha_\alpha \cap V_{\theta^*+2} \subseteq \mathcal{P}^\alpha_{\alpha-1} \cap V_{\theta^*+2}$$

where $\theta^* = \pi^\alpha_{\alpha-1}(g(\alpha - 1))$. But

$$\theta = \pi^\alpha_{\alpha-1}(\mu^{\mathcal{T}}_g[\beta, \alpha)) = \pi^\alpha_{\alpha-1}(g^{\mathcal{T}}_g[\beta, \alpha)) \leq \theta^*,$$

since $g(\alpha - 1) \leq \mu^{\mathcal{T}}[\beta, \alpha)$ (and since $\mathcal{E}^\alpha$ is a g-enlargement).

If $p(\alpha) = \alpha$ then (by (2) of Definition 82)

$$(\pi^\alpha_{\alpha-1}, (\mathcal{P}^\alpha_{\alpha-1}, \delta^\alpha_{\alpha-1}), \nu^\alpha_{\alpha-1}) = (\pi^{\alpha-1}_{\alpha-1}, (\mathcal{P}^{\alpha-1}_{\alpha-1}, \delta^{\alpha-1}_{\alpha-1}), \nu^{\alpha-1}_{\alpha-1}).$$

If $p(\alpha) < \alpha$ then

$$\mathcal{P}^\alpha_{\alpha-1} \cap V_{\theta^{**}+1} = \mathcal{P}^{\alpha-1}_{\alpha-1} \cap V_{\theta^{**}+1}$$

and

$$\mathcal{P}^\alpha_{\alpha-1} \cap V_{\theta^{**}+2} = \mathcal{P}^{\alpha-1}_{\alpha-1} \cap V_{\theta^{**}+2}$$

where $\theta^{**} = \pi^{\alpha-1}_{\alpha-1}(\mu^{\mathcal{T}}[p(\alpha) - 1, \alpha - 1))$. By (2b) of Definition 82,

$$\pi^\alpha_{\alpha-1}(\mu^{\mathcal{T}}[p(\alpha) - 1, \alpha - 1)) = \pi^{\alpha-1}_{\alpha-1}(\mu^{\mathcal{T}}[p(\alpha) - 1, \alpha - 1))$$

and

$$\pi^\alpha_{\alpha-1} | M_{\alpha-1} \cap V_{\mu^{\mathcal{T}}_g[p(\alpha)-1,\alpha-1)} = \pi^{\alpha-1}_{\alpha-1} | M_{\alpha-1} \cap V_{\mu^{\mathcal{T}}_g[p(\alpha)-1,\alpha-1)}.$$

Finally $\beta < p(\alpha)$ and so $\beta \leq p(\alpha) - 1$ which implies that

$$\mu^{\mathcal{T}}_g[\beta, \alpha) \leq \mu^{\mathcal{T}}_g[p(\alpha) - 1, \alpha - 1).$$

Thus $\theta \leq \theta^{**}$.

Therefore in either case ($p(\alpha) = \alpha$ or $p(\alpha) < \alpha$), we have:

(1.1) $\pi^\alpha_{\alpha-1} | M_{\alpha-1} \cap V_{\mu^{\mathcal{T}}_g[p(\alpha)-1,\alpha-1)} = \pi^{\alpha-1}_{\alpha-1} | M_{\alpha-1} \cap V_{\mu^{\mathcal{T}}_g[p(\alpha)-1,\alpha-1)}$;

(1.2) $\mathcal{P}^\alpha_{\alpha-1} \cap V_{\theta+1} = \mathcal{P}^{\alpha-1}_{\alpha-1} \cap V_{\theta+1}$;

(1.3) $\mathcal{P}^\alpha_{\alpha-1} \cap V_{\theta+2} = \mathcal{P}^{\alpha-1}_{\alpha-1} \cap V_{\theta+2}$;

and therefore since (1)–(4) hold for $\alpha - 1$, (1)–(4) hold for α.

To prove (5) for α suppose that $\beta < \gamma < \alpha$. Thus $\beta < \gamma \leq \alpha - 1$. If $\gamma = \alpha - 1$ we are done by (1.1) and (1.2) since (3) holds for $\alpha - 1$. Therefore we can suppose that $\gamma < \alpha - 1$. If $\gamma < p(\alpha)$ then by (2a) of Definition 82,

$$(\pi^\alpha_\gamma, (\mathcal{P}^\alpha_\gamma, \delta^\alpha_\gamma), \nu^\alpha_\gamma) = (\pi^{\alpha-1}_\gamma, (\mathcal{P}^{\alpha-1}_\gamma, \delta^{\alpha-1}_\gamma), \nu^{\alpha-1}_\gamma)$$

and if $p(\alpha) \leq \gamma < \alpha$ then by (2c) of Definition 82,

$$\mathcal{P}^\alpha_\gamma \in \mathcal{P}^{\alpha-1}_\gamma$$

and $\pi^\alpha_\gamma | M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha-1)} = \pi^\alpha_{\alpha-1} | M_{\alpha-1} \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha-1)}$, and in either case (5) follows since (5) holds for $\alpha - 1$.

Now we suppose that α is a limit ordinal, $\alpha > \beta$, and that (1)–(5) hold for all $\xi < \alpha$.

We use Lemma 68. Fix $\alpha_0 <_{\mathcal{T}} \alpha$ such that for all $\alpha_0 <_{\mathcal{T}} \gamma + 1 <_{\mathcal{T}} \alpha$,

$$\mu^{\mathcal{T}}_g[\beta, \alpha) < \mathrm{CRT}(E_\gamma).$$

By increasing α_0 if necessary we can suppose that $\beta < \alpha_0$ and that α_0 survives at α. Thus $\mathcal{P}^\alpha_\alpha$ is the direct limit of

$$\{\mathcal{P}^\xi_\xi \mid \alpha_0 <_{\mathcal{T}} \xi <_{\mathcal{T}} \alpha\}$$

under the embeddings, $\langle e_{\xi_1,\xi_2} : \alpha_0 <_{\mathcal{T}} \xi_1 <_{\mathcal{T}} \xi_2 <_{\mathcal{T}} \alpha \rangle$.

Suppose that $\alpha_0 \leq_{\mathcal{T}} \xi <_{\mathcal{T}} \delta + 1 <_{\mathcal{T}} \alpha$ and that $\xi = \delta^*$ (the $\mathcal{T}$-predecessor of $\delta + 1$). Therefore

$$\mathcal{P}^{\delta+1}_{\delta+1} = \mathrm{Ult}(\mathcal{P}^\xi_\xi, \pi^{\delta+1}_\delta(E_\delta))$$

and $\mathrm{CRT}(E_\delta) > \mu^{\mathcal{T}}_g[\beta, \alpha)$. But

$$\mu^{\mathcal{T}}_g[\beta, \alpha) \leq \mu^{\mathcal{T}}_g[\beta, \delta)$$

(by definition of $\mu^{\mathcal{T}}_g[\beta, \alpha)$ and $\mu^{\mathcal{T}}_g[\beta, \delta)$) and $p(\gamma) > \beta$ for all $\beta < \gamma \leq \alpha$. Thus

$$\mathrm{CRT}(\pi^{\delta+1}_\delta(E_\delta)) > \pi^{\delta+1}_\delta(\mu^{\mathcal{T}}_g[\beta, \alpha)) = \pi^{\delta+1}_\beta(\mu^{\mathcal{T}}_g[\beta, \alpha)) = \pi^\beta_\beta(\mu^{\mathcal{T}}_g[\beta, \alpha)) = \theta.$$

Thus by (2d) of the definition of a p-connected sequence of g-enlargements, $\mathrm{CRT}(e_{\alpha_0,\alpha}) > \theta$ and so

$$\mathcal{P}^{\alpha_0}_{\alpha_0} \cap V_{\theta+2} = \mathcal{P}^\alpha_\alpha \cap V_{\theta+2}.$$

However $\beta < \alpha_0 < \alpha$ and so by induction,

$$\mathcal{P}^{\alpha_0}_{\alpha_0} \cap V_{\theta+2} \subseteq \mathcal{P}^\beta_\beta \cap V_{\theta+2}$$

and

$$\mathcal{P}^{\alpha_0}_{\alpha_0} \cap V_{\theta+1} = \mathcal{P}^\beta_\beta \cap V_{\theta+1}$$

Thus (3) and (4) hold for α. We next prove (5). Suppose $\beta < \gamma < \alpha$. There are two possibilities (we use (3) of Definition 82). If $\gamma < p(\alpha)$ then

$$\mathcal{P}^\alpha_\gamma = \mathcal{P}^\delta_\gamma$$

for all sufficiently large $\delta < \alpha$ in which case we can apply the induction hypothesis. The second possibility is that $p(\alpha) \leq \gamma < \alpha$. But then

$$\mathcal{P}^{\alpha}_{\gamma} \in \mathcal{P}^{\alpha}_{\alpha}.$$

We have already proved,

$$\mathcal{P}^{\alpha_0}_{\alpha_0} \cap V_{\theta+2} = \mathcal{P}^{\alpha}_{\alpha} \cap V_{\theta+2},$$

and so in this case,

$$\mathcal{P}^{\alpha}_{\gamma} \cap V_{\theta+1} \subseteq \mathcal{P}^{\alpha}_{\alpha} \cap V_{\theta+1} = \mathcal{P}^{\alpha_0}_{\alpha_0} \cap V_{\theta+1} = \mathcal{P}^{\beta}_{\beta} \cap V_{\theta+1}$$

again applying the induction hypothesis to α_0. Finally we prove (1) and (2). By (3b) of Definition 82, for all $\xi <_{\mathcal{T}} \alpha$ and for all $x \in M_\xi$ there exists $\alpha(\xi, x) <_{\mathcal{T}} \alpha$ such that for all $\alpha(\xi, x) <_{\mathcal{T}} \gamma_1 <_{\mathcal{T}} \gamma_2 <_{\mathcal{T}} \alpha$,

$$e_{\gamma_1,\gamma_2}(\pi^{\gamma_1}_{\gamma_1}(j_{\xi,\gamma_1}(x))) = \pi^{\gamma_2}_{\gamma_2}(j_{\gamma_1,\gamma_2}(x)).$$

By Lemma 68, it follows that for all sufficiently large $\gamma_1 <_{\mathcal{T}} \gamma_2 <_{\mathcal{T}} \alpha$,

$$\mu^{\mathcal{T}}_g[\beta, \alpha) < \mathrm{CRT}(j_{\gamma_1,\gamma_2}).$$

Now we use that $\beta < p(\alpha)$. By (3c) of Definition 82,

$$(\pi^{\alpha}_{\beta}, (\mathcal{P}^{\alpha}_{\beta}, \delta^{\alpha}_{\beta}), \nu^{\alpha}_{\beta}) = (\pi^{\gamma}_{\beta}, (\mathcal{P}^{\gamma}_{\beta}, \delta^{\gamma}_{\beta}), \nu^{\gamma}_{\beta})$$

for all sufficiently large $\gamma <_{\mathcal{T}} \alpha$. (1) and (2) now follow using the induction hypothesis for $\gamma <_{\mathcal{T}} \alpha$. □

Corollary 86. *Suppose that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable iteration tree on (M, δ), $p : \eta \to \eta$ *is a preservation function for* $<_{\mathcal{T}}$ *and that* $g : \eta \to \mathrm{Ord}$ *is a function such that for all* $\alpha < \eta$,

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) + 1 \leq \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Suppose that $\lambda \leq \eta$ *and that for each* $\alpha < \lambda$,

$$\mathcal{E}^{\alpha} = \langle (\pi^{\alpha}_{\beta}, (\mathcal{P}^{\alpha}_{\beta}, \delta^{\alpha}_{\beta}), \nu^{\alpha}_{\beta}) : \beta \leq \alpha \rangle$$

is a g*-enlargement of* $\mathcal{T}$ *such that the sequence* $\langle \mathcal{E}^{\alpha} : \alpha < \lambda \rangle$ *is* p*-connected.*

Suppose that β *is the* $<_{\mathcal{T}}$*-predecessor of* $\alpha + 1$, β *survives at* $\alpha + 1$, *and that* $\alpha + 1 < \lambda$. *Then*

$$\pi^{\alpha+1}_{\alpha}(E_\alpha) \in \mathcal{P}^{\beta}_{\beta}.$$

Proof. Let $F = \pi^{\alpha+1}_{\alpha}(E_\alpha)$. Thus

$$\mathcal{P}^{\alpha+1}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}^{\beta}_{\beta}, F)$$

and $e_{\beta,\alpha}$ is the induced embedding, by (3a) of Definition 82.

There are two cases, $\beta = \alpha$ and $\beta < \alpha$.

If $\beta = \alpha$. But then

$$\mathcal{P}^{\alpha+1}_{\beta} = \mathcal{P}^{\beta}_{\beta}$$

and so $F \in \mathcal{P}^{\beta}_{\beta} \cap V_{\delta^{\beta}_{\beta}}$.

If $\beta < \alpha$ then since $p(\alpha+1) = \beta+1$ (which must be the case since β survives at $\alpha+1$) it follows from part (2a) of Definition 82 that $\mathcal{P}^{\alpha+1}_{\alpha}$ is coded by a set $A \in \mathcal{P}^{\alpha}_{\alpha} \cap V_{\theta+1}$ where $\theta = \pi^{\alpha}_{\alpha}(\mu^{\mathcal{T}}_{g}[\beta,\alpha))$. By Lemma 85, $A \in \mathcal{P}^{\beta}_{\beta}$. This implies

$$\mathcal{P}^{\alpha+1}_{\alpha} \in \mathcal{P}^{\beta}_{\beta}$$

and so $F \in \mathcal{P}^{\beta}_{\beta}$. $\square$

Corollary 86 shows that the direct limit which must define $\mathcal{P}^{\alpha}_{\alpha}$ in the case that α is a limit ordinal (see (3a) of Definition 82) is an *internal* iteration and so necessarily it is wellfounded (provided the transitive sets, $\mathcal{P}^{\beta}_{\beta}$, where $\beta <_{\mathcal{T}} \alpha$ and β survives at α are iterable in the usual (linear) sense which is easy to arrange). This will be relevant when we actually construct connected sequences of enlargements.

Remark 87. We shall only consider connected sequences of g-enlargements when

(1) η (the length of $\mathcal{T}$) is a limit ordinal and

$$g(\beta) = \min\{\xi + 1 \mid \mu^{\mathcal{T}}(\beta) \le \xi + 1\}$$

for all $\beta < \eta$, or when

(2) $\eta = \gamma + 1$, $\gamma^* \le \gamma$, $E \in M_\gamma$ is an extender such that if $\gamma^* < \gamma$ then

$$\text{SPT}(E) + 2 \le \rho^{\mathcal{T}}[\gamma^*, \gamma)$$

and for all $\beta < \gamma$, if $\beta < \gamma^*$ then

$$g(\beta) = \min\{\xi + 1 \mid \mu^{\mathcal{T}}(\beta) \le \xi + 1\}$$

and if $\gamma^* \le \beta < \gamma$ then

$$g(\beta) = \min\{\xi + 1 \mid \max\{\mu^{\mathcal{T}}(\beta), \text{SPT}(E)\} \le \xi + 1\}.$$

Thus we are really considering at most a slight variation of (+1)-enlargements and if $\mathcal{T}$ is a (+2)-iteration tree then the hypotheses of Lemma 84, Lemma 85 and Corollary 86 are automatically satisfied for these possibilities of g.

Lemma 88. *Suppose that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable iteration tree on (M, δ), $p : \eta \to \eta$ *is a preservation function for* $<_{\mathcal{T}}$ *and that* $g : \eta \to \text{Ord}$ *is a function such that for all* $\alpha < \eta$, $g(\alpha)$ *is a successor ordinal and*

$$\mu^{\mathcal{T}}(\alpha) \le g(\alpha) < g(\alpha) + 1 \le \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Suppose that $\lambda \leq \eta$ *and that for each* $\alpha < \lambda$,

$$\mathcal{E}^\alpha = \langle (\pi^\alpha_\beta, (\mathcal{P}^\alpha_\beta, \delta^\alpha_\beta), \nu^\alpha_\beta) : \beta \leq \alpha \rangle$$

is a g*-enlargement of* $\mathcal{T}$ *such that the sequence* $\langle \mathcal{E}^\alpha : \alpha < \lambda \rangle$ *is* p*-connected.*

Suppose that there exist Θ *and an elementary embedding*

$$\pi : (\mathcal{P}^0_0, \delta^0_0) \to V_\Theta.$$

Suppose that $\theta < \lambda$, $E \in \mathcal{P}^\theta_\theta \cap V_{\delta^\theta_\theta}$,

$$\mathcal{P}^\theta_\theta \vDash \text{“}E \textit{ is an extender”},$$

$\xi < \theta$ *and* $\mathrm{SPT}(E) + 1 \leq \pi^\theta_\theta(\mu^{\mathcal{T}}_g[\xi, \theta))$. *Then* $\mathrm{Ult}(\mathcal{P}^\theta_\xi, E)$ *is wellfounded.*

Proof. This lemma is a transfinite version of the special case of Theorem 66 for finite iteration trees and the idea of the proof is to obtain a contradiction in a similar fashion. However whereas before we only needed to construct a finite sequence of enlargements, now we shall need to construct a p-connected sequence of g-enlargements of length $\theta + 1$.

We suppose that θ is a small as possible (for any p-connected sequence of g-enlargements of $\mathcal{T}$ of length $\theta + 1$). Thus θ is countable. By taking a countable elementary substructure of a large enough V_Υ if necessary we can suppose that $\mathcal{P}^\alpha_\beta$ is countable for all $\beta \leq \alpha \leq \theta$.

Let $\langle i_\xi : \xi < \theta \rangle$ enumerate ω and let $\langle (\beta_i, x_i) : i < \omega \rangle$ enumerate the set

$$\{(\beta, x) \mid \beta \leq \theta, x \in \mathcal{P}^\beta_\beta\}$$

with each enumeration being 1-to-1 (so we have just fixed two bijections).

We define by induction on $\beta \leq \theta$ a sequence

$$\langle (\sigma^\beta_\alpha, (\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha), \mu^\beta_\alpha) : \alpha \leq \beta \rangle$$

such that

(1.1) $\mu^\beta_\alpha \in \mathcal{Q}^\beta_\alpha$, $\sigma^\beta_\alpha \in \mathcal{Q}^\beta_\alpha$, and σ^β_α is an elementary embedding,

$$\sigma^\beta_\alpha : (\mathcal{P}^\beta_\alpha, \delta^\beta_\alpha) \to (\mathcal{Q}^\beta_\alpha \cap V_{\mu^\beta_\alpha}, \gamma^\beta_\alpha),$$

(1.2) $\langle \mathcal{E}^\xi_Q : \xi \leq \beta \rangle$ is a p-connected sequence of g-enlargements of $\mathcal{T}$ where for each $\xi \leq \beta$,

$$\mathcal{E}^\xi_Q = \langle (\sigma^\xi_\alpha \circ \pi^\xi_\alpha, (\mathcal{Q}^\xi_\alpha, \gamma^\xi_\alpha), \sigma^\xi_\alpha(\nu^\xi_\alpha)) : \alpha \leq \xi \rangle,$$

(1.3) For $\alpha < \beta$,

$$\sigma^\beta_\alpha(\mathcal{P}^\beta_\alpha \cap V_\rho) = \sigma^\beta_\beta(\mathcal{P}^\beta_\beta \cap V_\rho)$$

and

$$\sigma^\beta_\alpha|(\mathcal{P}^\beta_\alpha \cap V_\rho) = \sigma^\beta_\beta|(\mathcal{P}^\beta_\beta \cap V_\rho),$$

where $\rho = \pi^\beta_\alpha(\mu^{\mathcal{T}}_g[\alpha, \beta)) = \pi^\beta_\beta(\mu^{\mathcal{T}}_g[\alpha, \beta))$, (1.4) Suppose α survives at β, $\alpha \leq_{\mathcal{T}} \gamma <_{\mathcal{T}} \beta$, $m < \omega$,

$$m \leq \min\{i_\xi \mid \gamma <_{\mathcal{T}} \xi \leq_{\mathcal{T}} \beta\},$$

and that $\beta_m = \alpha$. Let $y = e_{\alpha,\gamma}(x_m)$. Then

$$e^Q_{\gamma,\beta}(\sigma^\gamma_\gamma(y)) = \sigma^\beta_\beta(e_{\gamma,\beta}(y))$$

where

$$e^Q_{\gamma,\beta} : \mathcal{Q}^\gamma_\gamma \to \mathcal{Q}^\beta_\beta$$

and

$$e_{\gamma,\beta} : \mathcal{P}^\gamma_\gamma \to \mathcal{P}^\beta_\beta$$

are the embeddings given by p-connectedness.

(1.5) Suppose that α survives at β and

$$\delta < \min\{p(\gamma) \mid \alpha < \gamma \leq \beta\}.$$

Let

$$\rho = \pi^\alpha_\alpha(\mu^{\mathcal{T}}_g[\delta, \beta)) = \pi^\beta_\beta(\mu^{\mathcal{T}}_g[\delta, \beta))$$

and suppose $x \in \mathcal{P}^\alpha_\alpha \cap V_\rho$. If $\mathrm{CRT}(e_{\alpha,\beta}) > \rho$ and if $\mathrm{CRT}(e^Q_{\alpha,\beta}) > \sigma^\alpha_\alpha(\rho)$ then

$$e^Q_{\alpha,\beta}(\sigma^\alpha_\alpha(x)) = \sigma^\beta_\beta(e_{\alpha,\beta}(x)).$$

(1.6) $H(\omega_1) \in \mathcal{Q}^\beta_\alpha$ and $(\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha)$ is (linearly) iterable.

It is the requirement that $\sigma^\beta_\alpha \in \mathcal{Q}^\beta_\alpha$ for all $\alpha \leq \beta$ (instead of just $\sigma^\beta_\beta \in \mathcal{Q}^\beta_\beta$) for which we shall use the requirements, Definition 82(2c(v)) and Definition 82(3d(iv)), that the ordinals η^i_γ have uncountable cofinality (in $\mathcal{R}_\gamma$), in the definition of a p-connected sequence of g-enlargements, see Remark 83.

<u>Case 0:</u> $\beta = 0$.

We have an elementary embedding,

$$\pi : (\mathcal{P}^0_0, \delta^0_0) \to V_\Theta.$$

Let $\Upsilon > \Theta$ be such that $(V_\Upsilon, \pi(\delta^0_0))$ is a premouse and set

$$(\sigma^0_0, (\mathcal{Q}^0_0, \gamma^0_0), \mu^0_0) = (\pi, (V_\Upsilon, \pi(\delta^0_0)), \Theta).$$

Note that $(\mathcal{Q}^0_0, \gamma^0_0)$ is linearly iterable.

<u>Case 1:</u> β is a successor ordinal.

We first define

$$(\sigma^\beta_\alpha, (\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha), \mu^\beta_\alpha)$$

for all $\alpha < \beta$. This we will do in three steps, first for $\alpha < p(\beta)$, second for $\alpha = \beta - 1$ assuming $p(\beta) < \beta$, and third for $p(\beta) \leq \alpha < \beta - 1$ assuming $p(\beta) < \beta - 1$.

For $\alpha < p(\beta)$ set,

$$(\sigma^\beta_\alpha, (\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha), \mu^\beta_\alpha) = (\sigma^{\beta-1}_\alpha, (\mathcal{Q}^{\beta-1}_\alpha, \gamma^{\beta-1}_\alpha), \mu^{\beta-1}_\alpha).$$

Next we suppose $p(\beta) < \beta$. Let π_P be the elementary embedding ("π") and $\nu_P \in \mathcal{P}^{\beta-1}_{\beta-1}$ be the ordinal ("ν") which witnesses (2b(iii)) of Definition 82 for the successor ordinal β,

$$\pi_P : (\mathcal{P}^\beta_{\beta-1}, \delta^\beta_{\beta-1}) \to (\mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\nu_P}, \delta^{\beta-1}_{\beta-1}).$$

Let $\nu_Q = \sigma^{\beta-1}_{\beta-1}(\nu_P)$ and let $\rho_Q = \sigma^{\beta-1}_{\beta-1}(\rho_P)$ where $\rho_P = \pi^{\beta-1}_{\beta-1}(\mu^{\mathcal{T}}_g[p(\beta)-1, \beta-1))$. Choose

$$X \prec \mathcal{Q}^{\beta-1}_{\beta-1} \cap V_{\mu^{\beta-1}_{\beta-1}}$$

such that

$$\{\nu_Q, \sigma^{\beta-1}_{\beta-1}|(\mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\nu_P})\} \cup (\mathcal{Q}^{\beta-1}_{\beta-1} \cap V_{\rho_Q+1}) \subseteq X,$$

$X \in \mathcal{Q}^{\beta-1}_{\beta-1}$ and such that in $\mathcal{Q}^{\beta-1}_{\beta-1}$, X and $\mathcal{Q}^{\beta-1}_{\beta-1} \cap V_{\rho_Q+1}$ have the same cardinality.

Define $\mathcal{Q}^\beta_{\beta-1}$ to be the transitive collapse of X and let

$$\tau_Q : \mathcal{Q}^\beta_{\beta-1} \to \mathcal{Q}^{\beta-1}_{\beta-1} \cap V_{\mu^{\beta-1}_{\beta-1}}$$

be the inverse of the collapsing map. Let

$$\mu^\beta_{\beta-1} = \tau_Q^{-1}(\nu_Q),$$

let $\gamma^\beta_{\beta-1} = \gamma^{\beta-1}_{\beta-1}$ and let

$$\sigma^\beta_{\beta-1} = \tau_Q^{-1}(\sigma^{\beta-1}_{\beta-1}|\mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\nu_P}) \circ \pi_P = \tau_Q^{-1} \circ \sigma^{\beta-1}_{\beta-1} \circ \pi_P.$$

Finally we suppose $p(\beta) < \beta - 1$. Let

$$\langle (\mathcal{R}_\gamma, \eta^i_\gamma) : p(\beta) \le \gamma < \beta - 1, i < \omega \rangle \in \mathcal{P}^\beta_{\beta-1}$$

witness (2c) of Definition 82 for β.

For $p(\beta) \le \gamma < \beta - 1$ let

$$\mathcal{Q}^\beta_\gamma = \sigma^\beta_{\beta-1}(\mathcal{R}_\gamma \cap V_{\eta^1_\gamma}),$$

let $\gamma^\beta_\gamma = \sigma^\beta_{\beta-1}(\delta^\beta_\gamma)$, let

$$\mu^\beta_\gamma = \sigma^\beta_{\beta-1}(\eta^0_\gamma)$$

and let $\sigma^\beta_\gamma = \sigma^\beta_{\beta-1}|\mathcal{P}^\beta_\gamma$.

This completes the definition of

$$(\sigma^\beta_\alpha, (\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha), \mu^\beta_\alpha)$$

for all $\alpha < \beta$. Since the ordinals η^i_γ each have uncountable cofinality in the relevant $\mathcal{R}_\gamma$, it follows by the induction hypothesis that for all $\alpha < \beta$, $\sigma^\beta_\alpha \in \mathcal{Q}^\beta_\alpha$. The point of course is that $\mathcal{P}^\beta_\alpha$ is countable and $H(\omega_1) \in \mathcal{Q}^\beta_\alpha$.

We must also produce the witness for (2c) of Definition 82. For

$$p(\beta) - 1 < \gamma < \beta - 1$$

and for $i < \omega$, the (γ, i)th pair of the required sequence is $(\sigma^\beta_{\beta-1}(\mathcal{R}_\gamma), \sigma^\beta_{\beta-1}(\eta^{i+1}_\gamma))$ where $(\mathcal{R}_\gamma, \eta^{i+1}_\gamma)$ is as above.

We finish this case by defining

$$(\sigma^\beta_\beta, (\mathcal{Q}^\beta_\beta, \gamma^\beta_\beta), \mu^\beta_\beta).$$

Let ξ be the $<_\mathcal{T}$-predecessor of β. Thus

$$\mathcal{P}^\beta_\beta = \mathrm{Ult}(\mathcal{P}^\beta_\xi, \pi^\beta_{\beta-1}(E_{\beta-1})).$$

Define

$$\mathcal{Q}^\beta_\beta = \mathrm{Ult}(\mathcal{Q}^\beta_\xi, \sigma^\beta_{\beta-1}(\pi^\beta_{\beta-1}(E_{\beta-1})))$$

and let $e_Q : \mathcal{Q}^\beta_\xi \to \mathcal{Q}^\beta_\beta$ be the associated embedding (if ξ survives at β then $e_Q = e^Q_{\xi,\beta}$).

We must verify that this ultrapower is defined and that it is wellfounded.

First we show that this ultrapower is defined. The key point is that

$$M_\xi \cap V_{\mu^\mathcal{T}_g[\xi,\beta-1)} = M_{\beta-1} \cap V_{\mu^\mathcal{T}_g[\xi,\beta-1)}$$

and that (setting $N = M_\xi \cap V_{\mu^\mathcal{T}_g[\xi,\beta-1)}$)

$$\sigma^\beta_\xi \circ \pi^\beta_\xi | N = \sigma^\beta_{\beta-1} \circ \pi^\beta_{\beta-1} | N.$$

We verify this key point which is really immediate from the definitions. First the two embeddings, π^β_ξ and $\pi^\beta_{\beta-1}$, are from a p-connected sequence of g-enlargements. Therefore

$$M_\xi \cap V_{\mu^\mathcal{T}_g[\xi,\beta-1)} = M_{\beta-1} \cap V_{\mu^\mathcal{T}_g[\xi,\beta-1)}$$

and (with N as above)

$$\pi^\beta_\xi | N = \pi^\beta_{\beta-1} | N.$$

If $p(\beta) \le \xi < \beta - 1$ then $\sigma^\beta_\xi = \sigma^\beta_{\beta-1} | \mathcal{P}^\beta_\xi$ and so the claimed agreement holds. If $\xi < p(\beta)$ then $\sigma^\beta_\xi = \sigma^{\beta-1}_\xi$,

$$\mathcal{P}^{\beta-1}_\xi \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^\mathcal{T}_g[\xi,\beta-1))} = \mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^\mathcal{T}_g[\xi,\beta-1))}$$

and setting $N^* = \mathcal{P}^{\beta-1}_\xi \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^\mathcal{T}_g[\xi,\beta-1))}$,

$$\sigma^{\beta-1}_\xi | N^* = \sigma^{\beta-1}_{\beta-1} | N^*.$$

However, by Definition 82(2a(i)–2a(ii)),

$$\mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^\mathcal{T}_g[p(\beta)-1,\beta-1))} = \mathcal{P}^\beta_{\beta-1} \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^\mathcal{T}_g[p(\beta)-1,\beta-1))},$$

and setting $N^{**} = \mathcal{P}^{\beta-1}_{\beta-1} \cap V_{\pi^{\beta-1}_{\beta-1}(\mu^{\mathcal{T}}_g[p(\beta)-1,\beta-1))}$, it follows from Definition 70(3) that

$$\sigma^{\beta-1}_{\xi}|N^{**} = \sigma^{\beta-1}_{\beta-1}|N^{**}.$$

Further,

$$\pi^{\beta-1}_{\beta-1}(\mu^{\mathcal{T}}_g[p(\beta)-1,\beta-1)) \geq \pi^{\beta-1}_{\beta-1}(\mu^{\mathcal{T}}_g[\xi,\beta-1))$$

and so again the claimed agreement holds.

Thus the ultrapower,

$$\mathrm{Ult}(\mathcal{Q}^{\beta}_{\xi}, \sigma^{\beta}_{\beta-1}(\pi^{\beta}_{\beta-1}(E_{\beta-1}))),$$

is defined. We finish by showing that it is wellfounded. If $\xi = \beta - 1$ then this is an internal ultrapower and so necessarily wellfounded. Similarly if $p(\beta) \leq \xi < \beta - 1$ then $\mathcal{Q}^{\beta}_{\xi} \in \mathcal{Q}^{\beta}_{\beta-1}$ and so again the ultrapower is necessarily wellfounded (it embeds into an internal ultrapower). We have reduced to the case that $\xi < p(\beta)$ which implies that $\mathcal{Q}^{\beta}_{\xi} = \mathcal{Q}^{\beta-1}_{\xi}$. We now appeal to the minimal choice of θ, $\beta - 1 < \theta$ and by induction,

$$\langle \mathcal{E}^{\gamma}_{\mathcal{Q}} : \gamma \leq \beta - 1 \rangle$$

is a p-connected sequence of g-enlargements of $\mathcal{T}$. Therefore,

$$\mathrm{Ult}(\mathcal{Q}^{\beta-1}_{\xi}, \sigma^{\beta-1}_{\beta-1}(\pi^{\beta-1}_{\beta-1}(E_{\beta-1}))),$$

is wellfounded. If $p(\beta) = \beta$ then

$$\mathrm{Ult}(\mathcal{Q}^{\beta-1}_{\xi}, \sigma^{\beta-1}_{\beta-1}(\pi^{\beta-1}_{\beta-1}(E_{\beta-1}))), = \mathrm{Ult}(\mathcal{Q}^{\beta}_{\xi}, \sigma^{\beta}_{\beta-1}(\pi^{\beta}_{\beta-1}(E_{\beta-1}))),$$

and so we may reduce to the case that $p(\beta) < \beta$. But,

$$\sigma^{\beta-1}_{\beta-1} \circ \pi^{\beta-1}_{\beta-1} = \tau_{\mathcal{Q}} \circ \sigma^{\beta}_{\beta-1} \circ \pi^{\beta}_{\beta-1}$$

and $\mathrm{CRT}(\tau_{\mathcal{Q}}) > \sigma^{\beta}_{\beta-1} \circ \pi^{\beta}_{\beta-1}(\mathrm{SPT}(E_{\beta-1}) + 1)$. This implies that the ultrapower,

$$\mathrm{Ult}(\mathcal{Q}^{\beta}_{\xi}, \sigma^{\beta}_{\beta-1}(\pi^{\beta}_{\beta-1}(E_{\beta-1}))),$$

embeds into the ultrapower,

$$\mathrm{Ult}(\mathcal{Q}^{\beta-1}_{\xi}, \sigma^{\beta-1}_{\beta-1}(\pi^{\beta-1}_{\beta-1}(E_{\beta-1}))),$$

and so again the ultrapower is wellfounded.

We have defined $\mathcal{Q}^{\beta}_{\beta}$. Let $\mu^{\beta}_{\beta} = e_{\mathcal{Q}}(\mu^{\beta}_{\xi})$. Since ξ is the $<_{\mathcal{T}}$-predecessor of β,

$$\mathcal{P}^{\beta}_{\beta} = \mathrm{Ult}(\mathcal{P}^{\beta}_{\xi}, \pi^{\beta-1}_{\beta-1}(E_{\beta-1})).$$

Let $e_{\mathcal{P}} : \mathcal{P}^{\beta}_{\xi} \to \mathcal{P}^{\beta}_{\beta}$ be the associated elementary embedding (so $e_{\mathcal{P}} = e_{\xi,\beta}$ if ξ survives at β). Thus by the shift lemma, Lemma 69, there is an elementary embedding

$$\sigma : \mathcal{P}^\beta_\beta \to \mathcal{Q}^\beta_\beta$$

such that $\sigma \circ e_{\mathcal{P}} = e_{\mathcal{Q}} \circ \sigma^\beta_\xi$ (if ξ survives at β then $e_{\mathcal{Q}} = e^{\mathcal{Q}}_{\xi,\beta}$). Note

$$\mathcal{P}^\beta_{\beta-1} \cap V_{\pi^\beta_{\beta-1}(\rho^{\mathcal{T}}[\beta-1,\beta))} = \mathcal{P}^\beta_\beta \cap V_{\pi^\beta_{\beta-1}(\rho^{\mathcal{T}}[\beta-1,\beta))}$$

and for all $a \in \mathcal{P}^\beta_{\beta-1} \cap V_{\pi^\beta_{\beta-1}(\rho^{\mathcal{T}}[\beta-1,\beta))}$, $\sigma(a) = \sigma^\beta_{\beta-1}(a)$.

By hypothesis, $g(\beta-1)+1 \leq \rho^{\mathcal{T}}[\beta-1,\beta) = \rho(E_{\beta-1})$. Thus for all $a \in \mathcal{P}^\beta_{\beta-1} \cap V_{\pi^\beta_{\beta-1}(g(\beta-1))}$, $\sigma(a) = \sigma^\beta_{\beta-1}(a)$. Further, $\sigma^\beta_{\beta-1} \in \mathcal{Q}^\beta_{\beta-1}$ and

$$\sigma^\beta_{\beta-1}|\mathcal{P}^\beta_{\beta-1} \cap V_{\pi^\beta_{\beta-1}(g(\beta-1))}$$

can be coded by a set in $\mathcal{Q}^\beta_{\beta-1} \cap V_{\sigma^\beta_{\beta-1}\circ\pi^\beta_{\beta-1}(g(\beta-1))}$ (here we use that $g(\beta-1)$ is a successor ordinal). Since

$$\mathcal{Q}^\beta_\beta \cap V_{\sigma^\beta_{\beta-1}\circ\pi^\beta_{\beta-1}(g(\beta-1))} = \mathcal{Q}^\beta_{\beta-1} \cap V_{\sigma^\beta_{\beta-1}\circ\pi^\beta_{\beta-1}(g(\beta-1))},$$

it follows that

$$\sigma|\mathcal{P}^\beta_\beta \cap V_{\pi^\beta_{\beta-1}(g(\beta-1))} \in \mathcal{Q}^\beta_\beta.$$

Our potential problem of course is that in general we cannot set $\sigma^\beta_\beta = \sigma$ since σ may not belong to $\mathcal{Q}^\beta_\beta$. But by absoluteness there exists

$$\sigma' : (\mathcal{P}^\beta_\beta, \delta^\beta_\beta) \to (\mathcal{Q}^\beta_\beta \cap V_{\mu^\beta_\beta}, \gamma^\beta_\beta)$$

such that $\sigma' \in \mathcal{Q}^\beta_\beta$,

$$\sigma'|\mathcal{P}^\beta_\beta \cap V_{\pi^\beta_{\beta-1}(g(\beta-1))} = \sigma|\mathcal{P}^\beta_\beta \cap V_{\pi^\beta_{\beta-1}(g(\beta-1))},$$

and such that for all $i \leq i_\beta$ if α_i survives at β then $\sigma'(x) = \sigma(x)$ where $x = e_{\alpha_i,\beta}(y_i)$. Here we are using our two bijections, $\langle i_\xi : \xi < \theta\rangle$ and $\langle(\beta_i, x_i) : i < \omega\rangle$.

This completes the definition of $\mathcal{E}^\beta_{\mathcal{Q}}$, the verification that (1.1)–(1.6) hold is immediate from the construction.

Case 2: β is a limit ordinal.

We first define $\mathcal{Q}^\beta_\beta$. It is simply the direct limit of

$$\{\mathcal{Q}^\alpha_\alpha \,|\, \alpha <_{\mathcal{T}} \beta, \alpha \text{ survives at } \beta\}$$

under the embeddings; $e^{\mathcal{Q}}_{\alpha_1,\alpha_2}$ which are defined for each pair (α_1, α_2) such that $\alpha_1 <_{\mathcal{T}} \alpha_2 <_{\mathcal{T}} \beta$ and such that α_1 survives at β. $\mathcal{Q}^\beta_\beta$ is wellfounded since it is the limit of an internal iteration (by (1.6)).

For each $\alpha <_{\mathcal{T}} \beta$ such that α survives at β let

$$e^{\mathcal{Q}}_{\alpha,\beta} : \mathcal{Q}^\alpha_\alpha \to \mathcal{Q}^\beta_\beta$$

be the associated embedding, and define

$$(\mu_\beta^\beta, \gamma_\beta^\beta) = e_{\alpha,\beta}^{\mathcal{Q}}((\mu_\alpha^\alpha, \gamma_\alpha^\alpha))$$

where $\alpha <_{\mathcal{T}} \beta$ and α survives at β. Clearly $(\mu_\beta^\beta, \gamma_\beta^\beta)$ is well defined.

We now appeal to property (1.4) and our induction hypothesis that $\langle \mathcal{E}_{\mathcal{Q}}^\xi : \xi < \beta \rangle$ satisfies (1.1)–(1.6). For each $x \in \mathcal{P}_\beta^\beta$ there exists $\alpha_x <_{\mathcal{T}} \beta$ such that for all $\alpha_x <_{\mathcal{T}} \xi <_{\mathcal{T}} \gamma <_{\mathcal{T}} \beta$,

$$e_{\xi,\gamma}^{\mathcal{Q}}(\sigma_\xi^\xi(x_\xi)) = \sigma_\gamma^\gamma(x_\gamma)$$

where $e_{\xi,\beta}(x_\xi) = x$ and $e_{\gamma,\beta}(x_\gamma) = x$; (i.e. where x_ξ and x_γ are the relevant preimages of x in the direct limit which defines $\mathcal{P}_\beta^\beta$).

Thus we can define

$$\sigma : (\mathcal{P}_\beta^\beta, \delta_\beta^\beta) \to (\mathcal{Q}_\beta^\beta \cap V_{\mu_\beta^\beta}, \gamma_\beta^\beta)$$

by

$$\sigma(x) = e_{\alpha,\beta}^{\mathcal{Q}}(\sigma_\alpha^\alpha(y))$$

where $y \in \mathcal{P}_\alpha^\alpha$, $e_{\alpha,\beta}(y) = x$, and where $\alpha_x <_{\mathcal{T}} \alpha <_{\mathcal{T}} \beta$ (i.e. where $\alpha <_{\mathcal{T}} \beta$ and α and α is sufficiently large).

Let $\xi = p(\beta) - 1$. For all $\alpha \leq \xi$ let

$$(\sigma_\alpha^\beta, \mathcal{Q}_\alpha^\beta, \gamma_\alpha^\beta, \mu_\alpha^\beta) = (\sigma_\alpha^\epsilon, \mathcal{Q}_\alpha^\epsilon, \gamma_\alpha^\epsilon, \mu_\alpha^\epsilon)$$

where $\epsilon < \beta$ is sufficiently large. This is well defined by (1.5) and the fact that

$$\sup\{\min\{p(\epsilon) \mid \epsilon_0 < \epsilon < \beta\} \mid \epsilon_0 < \beta\} = \beta.$$

Let $\rho = \pi_\xi^\beta(\mu_g^{\mathcal{T}}[\xi, \beta)) = \pi_\beta^\beta(\mu_g^{\mathcal{T}}[\xi, \beta))$. We next show that

$$\mathcal{P}_\xi^\beta \cap V_\rho = \mathcal{P}_\beta^\beta \cap V_\rho$$

and that, setting $N = \mathcal{P}_\beta^\beta \cap V_\rho$,

$$\sigma|N = \sigma_\xi^\beta|N.$$

Fix $\alpha_0 <_{\mathcal{T}} \beta$ such that

(2.1) α_0 survives at β,
(2.2) $p(\alpha) > \xi$ for all $\alpha_0 \leq \alpha < \beta$,
(2.3) $\mathrm{CRT}(e_{\alpha_0,\beta}) > \rho$,
(2.4) $\mathrm{CRT}(e_{\alpha_0,\beta}^{\mathcal{Q}}) > \sigma_\xi^{\alpha_0}(\rho)$.

That α_0 exists follows from Lemma 68. We give the details (the only issue is satisfying (2.3) and (2.4)). For all $\xi < \epsilon < \beta$,

$$\mu_g^{\mathcal{T}}[\xi, \beta) \leq g(\epsilon) < \rho^{\mathcal{T}}[\epsilon, \epsilon + 1).$$

Let

$$\theta_0 = \sup\{\min\{\mathrm{CRT}(E_\epsilon) \mid \epsilon + 1 <_{\mathcal{T}} \beta, \epsilon_0 + 1 <_{\mathcal{T}} \epsilon + 1\} \mid \epsilon_0 + 1 <_{\mathcal{T}} \beta\}$$

By Lemma 68, since the limit of

$$\{M_\epsilon \mid \epsilon <_{\mathcal{T}} \beta\}$$

is wellfounded, for all sufficiently large $\epsilon + 1 <_{\mathcal{T}}$, if $\mathrm{CRT}(E_\epsilon) \leq \theta_0$ then $\rho(E_\epsilon) < \theta_0$. But for all sufficiently large $\epsilon + 1 <_{\mathcal{T}} \beta$

$$\mu_g^{\mathcal{T}}[\xi, \beta) < \rho(E_\epsilon).$$

Thus for all sufficiently large $\epsilon + 1 <_{\mathcal{T}} \beta$,

$$\mu_g^{\mathcal{T}}[\xi, \beta) < \mathrm{CRT}(E_\epsilon).$$

Now, for all sufficiently large $\epsilon + 1 <_{\mathcal{T}} \beta$,

$$M_\epsilon \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)} = M_\xi \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)},$$

$$\pi_\epsilon^{\epsilon+1}|M_\epsilon \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)} = \pi_\xi^{\epsilon+1}|M_\xi \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)},$$

$\mu_g^{\mathcal{T}}[\xi, \beta) \leq \mu_g^{\mathcal{T}}[\xi, \epsilon)$ and $\pi_\xi^{\epsilon+1} = \pi_\xi^\beta$. Putting everything together we obtain that for all sufficiently large $\epsilon + 1 <_{\mathcal{T}} \beta$,

$$\mathrm{CRT}(\pi_\epsilon^{\epsilon+1}(E_\epsilon)) > \rho$$

(since $\rho = \pi_\xi^\beta(\mu_g^{\mathcal{T}}[\xi, \beta))$). This implies that $\mathrm{CRT}(e_{\alpha,\beta}) > \rho$ for all sufficiently large $\alpha <_{\mathcal{T}} \beta$. This shows we can choose α_0 to satisfy (2.3).

Similarly for sufficiently large $\epsilon + 1 <_{\mathcal{T}} \beta$,

$$M_\epsilon \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)} = M_\xi \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)},$$

$$\sigma_\epsilon^{\epsilon+1} \circ \pi_\epsilon^{\epsilon+1}|M_\epsilon \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)} = \sigma_\xi^{\epsilon+1} \circ \pi_\xi^{\epsilon+1}|M_\xi \cap V_{\mu_g^{\mathcal{T}}[\xi,\beta)},$$

and so exactly as above, for all sufficiently large $\epsilon + 1 <_{\mathcal{T}} \beta$,

$$\mathrm{CRT}(\sigma_\epsilon^{\epsilon+1} \circ \pi_\epsilon^{\epsilon+1}(E_\epsilon)) > \sigma_\xi^{\epsilon+1}(\rho) = \sigma_\xi^\beta(\rho).$$

Thus we can choose α_0 to satisfy (2.1)–(2.4).

The point of this (existence of α_0) is that it follows that

$$\sigma|N = \sigma_{\alpha_0}^{\alpha_0}|N$$

where N is as defined above,

$$N = \mathcal{P}_\xi^\beta \cap V_\rho = \mathcal{P}_\beta^\beta \cap V_\rho.$$

But $\sigma_\xi^{\alpha_0} = \sigma_\xi^\beta$ and

$$\sigma_{\alpha_0}^{\alpha_0}|N = \sigma_\xi^{\alpha_0}|N.$$

This proves that

$$\sigma|N = \sigma_\xi^\beta|N.$$

We also have that

$$\sigma|N \in \mathcal{Q}_\beta^\beta.$$

This is because

$$\sigma^{\alpha_0}_{\alpha_0} \in \mathcal{Q}^{\alpha_0}_{\alpha_0}$$

(by (1.5)), because $\text{CRT}(e^{\mathcal{Q}}_{\alpha_0,\beta}) > \sigma^{\alpha_0}_{\alpha_0}(\rho) = \sigma^{\alpha_0}_{\xi}(\rho)$, and because of the agreement,

$$\sigma|N = \sigma^{\alpha_0}_{\alpha_0}|N.$$

We next define the sequence to witness (3d) of Definition 82. For $\xi < \alpha < \beta$ and for $i < \omega$ let the (α, i)th member of the sequence be:

$$(\sigma(\mathcal{R}_\alpha), \sigma(\eta^{i+1}_\alpha))$$

where

$$\langle \mathcal{R}_\alpha, \eta^i_\alpha : \xi < \alpha < \beta, i < \omega\rangle \in \mathcal{P}^\beta_\beta$$

witnesses (3d) of Definition 82.

For each $\xi < \alpha < \beta$ let

$$(\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha, \mu^\beta_\alpha) = (\sigma(\mathcal{R}_\alpha) \cap V_{\sigma(\eta^0_\alpha)}, \sigma(\delta^\beta_\alpha), \sigma(\eta^0_\alpha)).$$

We would like to define

$$\sigma^\beta_\alpha = \sigma|\mathcal{P}^\beta_\alpha$$

for all $\xi < \alpha \leq \beta$. This would satisfy all the requirements except possibly the requirement that $\sigma^\beta_\beta \in \mathcal{Q}^\beta_\beta$. We appeal to absoluteness; there exists an elementary embedding

$$\sigma' : \mathcal{P}^\beta_\beta \to \mathcal{Q}^\beta_\beta \cap V_{\mu^\beta_\beta}$$

such that:

(3.1) $\sigma' \in \mathcal{Q}^\beta_\beta$;
(3.2) $\sigma'|N = \sigma|N$;
(3.3) $\sigma'|\mathcal{P}^\beta_\alpha$ is an elementary embedding of $(\mathcal{P}^\beta_\alpha, \delta^\beta_\alpha)$ to $(\mathcal{Q}^\beta_\alpha \cap V_{\mu^\beta_\alpha}, \gamma^\beta_\alpha)$;
(3.4) for all $i \leq i_\beta$ if α_i survives at β then $\sigma'(x) = \sigma(x)$ where $x = e_{\alpha_i,\beta}(y_i)$. (Here we are again using our two bijections, $\langle i_\xi : \xi < \theta\rangle$ and $\langle (\beta_i, x_i) : i < \omega\rangle$.)

Let $\sigma^\beta_\beta = \sigma'$ and for all $\xi < \alpha < \beta$ let $\sigma^\beta_\alpha = \sigma'|\mathcal{P}^\beta_\alpha$. This completes the construction of the sequence,

$$\langle (\sigma^\beta_\alpha, (\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha), \mu^\beta_\alpha) : \alpha \leq \beta\rangle,$$

in this case β is a limit ordinal which was the last case. Again the verification that (1.1)–(1.6) hold is immediate from the construction.

We use the sequence,

$$\langle (\sigma^\beta_\alpha, (\mathcal{Q}^\beta_\alpha, \gamma^\beta_\alpha), \mu^\beta_\alpha) : \alpha \leq \theta\rangle,$$

to complete the proof of the lemma.

Suppose that $E \in \mathcal{P}^\theta_\theta \cap V_{\delta^\theta_\theta}$,

$$\mathcal{P}^\theta_\theta \vDash \text{“}E \text{ is an extender”},$$

$\xi < \theta$ and $\text{SPT}(E) + 1 \leq \pi^\theta_\theta(\mu^{\mathcal{T}}_g[\xi, \theta))$.

We must show that $\mathrm{Ult}(\mathcal{P}^\theta_\xi, E)$ is wellfounded.

We have constructed

$$\sigma^\theta_\theta : (\mathcal{P}^\theta_\theta, \delta^\theta_\theta) \to (\mathcal{Q}^\theta_\theta \cap V_{\mu^\theta_\theta}, \gamma^\theta_\theta)$$

and

$$\sigma^\theta_\xi : (\mathcal{P}^\theta_\xi, \delta^\theta_\xi) \to (\mathcal{Q}^\theta_\xi \cap V_{\mu^\theta_\xi}, \gamma^\theta_\xi)$$

with the property that

$$\sigma^\theta_\xi | \mathcal{P}^\theta_\xi \cap V_\rho = \sigma^\theta_\theta | \mathcal{P}^\theta_\theta \cap V_\rho$$

where $\rho = \pi^\theta_\theta(\mu^{\mathcal{T}}_g[\xi, \theta))$. Further $\sigma^\theta_\xi \in \mathcal{Q}^\theta_\xi$,

$$\mathcal{P}(\mathcal{P}^\theta_\xi) \in H(\omega_1) \subseteq \mathcal{Q}^\theta_\xi,$$

and $\mathrm{SPT}(E) + 1 \leq \rho$.

Suppose toward a contradiction that $\mathrm{Ult}(\mathcal{P}^\theta_\xi, E)$ is not wellfounded. Choose an increasing sequence, $\langle s_i : i < \omega \rangle$, of finite subsets of $\mathrm{LTH}(E)$ and a sequence of functions $\langle f_i : i < \omega \rangle$ such that for all $i < \omega$,

(4.1) $f_i \in \mathcal{P}^\theta_\xi$ and $f_i : [\mathrm{SPT}(E)]^{|s_i|} \to \mathrm{Ord}$,
(4.2) $j_E(f_{i+1})(s_{i+1}) < j_E(f_i)(s_i)$,

where $j_E : \mathcal{P}^\theta_\xi \to \mathrm{Ult}(\mathcal{P}^\theta_\xi, E)$ is the ultrapower map (again the standard part of $\mathrm{Ult}(\mathcal{P}^\theta_\xi, E)$ contains $\mathrm{LTH}(E)$ and so (4.2) makes sense).

The sequence, $\langle E(s_i) : i < \omega \rangle$, naturally forms a tower of ultrafilters, for each $i < \omega$, $E(s_i) \in \mathcal{P}^\theta_\theta$, $E(s_i)$ concentrates on $[\mathrm{SPT}(E)]^{|s_i|}$ and $E(s_i) \subseteq \mathcal{P}^\theta_\xi$.

Let $\langle A_i : i < \omega \rangle$ be a sequence such that for all $i < \omega$, $A_i \in E(s_i)$, and for all $i + 1 < \omega$, if $a \in A_{i+1}$ then $b \in A_i$ and $f_{i+1}(a) < f_i(b)$ where $b \subseteq a$ is such that $(a, b, \in) \cong (s_{i+1}, s_i, \in)$.

We have that $\sigma^\theta_\theta \in \mathcal{Q}^\theta_\theta$, $\mathcal{P}^\theta_\theta$ is countable, and $H(\omega_1) \in \mathcal{Q}^\theta_\theta$, so

$$\langle (\sigma^\theta_\theta(A_i), \sigma^\theta_\theta(s_i)) : i < \omega \rangle \in \mathcal{Q}^\theta_\theta.$$

Finally, E is an extender in $\mathcal{Q}^\theta_\theta$ and so there exists an increasing sequence $\langle t_i : i < \omega \rangle \in \mathcal{Q}^\theta_\theta$ such that for all $i < \omega$, $t_i \in \sigma^\theta_\theta(A_i)$ and $(t_{i+1}, t_i, \in) \cong (s_{i+1}, s_i, \in)$.

Therefore for all $i < \omega$, $\sigma^\theta_\xi(f_{i+1})(t_{i+1}) < \sigma^\theta_\xi(f_i)(t_i)$, which contradicts the wellfoundedness of $\mathcal{Q}^\theta_\xi$. $\square$

Now we have to prove the existence of a connected sequence of enlargements. The details are in many ways similar to the construction we just did.

Lemma 89. *Suppose that* (M, δ) *is a countable premouse,*

$$\pi : M \to V_\Theta$$

is an elementary embedding,

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable iteration tree on (M, δ), $p : \eta \to \eta$ *is a preservation function for* $<_{\mathcal{T}}$ *and that* $g : \eta \to \mathrm{Ord}$ *is a function such that for all* $\alpha < \eta$, $g(\alpha)$ *is a successor* *ordinal and*

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) < g(\alpha) + 1 \leq \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Suppose that $\mathcal{T}$ is p-continuously illfounded. Then there exists a p-connected sequence

$$\langle \mathcal{E}^\alpha : \alpha < \eta \rangle$$

of g-enlargements of $\mathcal{T}$ such that there exist Υ and an elementary embedding

$$e : \mathcal{P}_0^0 \to V_\Upsilon$$

where $\mathcal{E}^0 = (\pi_0^0, (\mathcal{P}_0^0, \delta_0^0), \nu_0^0)$.

Proof. It is (for reasons of notation) convenient to introduce the following definition. A pair (N, κ) is a *quasi-premouse* if for some $\eta \in N \cap \mathrm{Ord}$,

(1.1) N is transitive set, $N \vDash \mathrm{ZC}$,
(1.2) $N = N \cap V_{\eta+\omega}$,
(1.3) $(N \cap V_\eta, \kappa)$ is a premouse,
(1.4) $N \vDash$ "$\mathrm{cof}(\eta) > \omega$".

This is simply to facilitate dealing with the requirements, Definition 82(2c(v)) and Definition 82(3d(iv)), that the ordinals η_γ^i have uncountable cofinality (in $\mathcal{R}_\gamma$), in the definition of a p-connected sequence of g-enlargements, see Remark 83.

Fix a function $f_p : \eta \to \mathrm{Ord}$ witnessing that $\mathcal{T}$ is p-continuously illfounded.

Let $\langle i_\xi : \xi < \theta \rangle$ enumerate ω and let $\langle (\beta_i, x_i) : i < \omega \rangle$ enumerate the set

$$\{(\beta, x) \mid \beta \leq \eta, x \in M_\beta\}$$

with each enumeration being 1-to-1 (so as in the proof of Lemma 88, we have just fixed two bijections).

We shall construct,

$$\mathcal{E}^\alpha = \langle \pi_\beta^\alpha, (\mathcal{P}_\beta^\alpha, \delta_\beta^\alpha), \nu_\beta^\alpha : \beta \leq \alpha \rangle$$

by induction on $\alpha < \eta$ such that $\langle \mathcal{E}^\xi : \xi \leq \alpha \rangle$ is a p-connected sequence of g-enlargements and such that the following hold where for each $\beta \leq \alpha$,

$$c(\beta, \alpha) = |\{\gamma \mid \alpha < \gamma < \eta \text{ and } p(\gamma) \leq \beta\}|.$$

(Note that since p is a preservation function for $<_{\mathcal{T}}$, $c(\beta, \alpha) < \omega$ for all $\beta \leq \alpha$).

(2.1) For all $\beta \leq \alpha$, $\pi_\beta^\alpha \in \mathcal{P}_\beta^\alpha$.
(2.2) For all $\beta \leq \alpha$, $V_{\omega+1} \subseteq \mathcal{P}_\beta^\alpha$ and $(\mathcal{P}_\beta^\alpha, \delta_\beta^\alpha)$ is linearly iterable.
(2.3) For all $\beta \leq \alpha$, the set

$$\{\theta \in \mathcal{P}_\beta^\alpha \mid \nu_\beta^\alpha < \theta \text{ and } (\mathcal{P}_\beta^\alpha \cap V_{\theta+\omega}, \delta_\beta^\alpha) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi_\beta^\alpha(f_p(\beta)) + \omega \cdot c(\beta, \alpha) + 1$ if $\beta < \alpha$, and the set has ordertype at least $\omega^2 \cdot \pi_\alpha^\alpha(f_p(\alpha)) + \omega^2$ if $\beta = \alpha$.

(2.4) Suppose $\beta <_{\mathcal{T}} \alpha$, β survives at α, and that $\beta <_{\mathcal{T}} \gamma <_{\mathcal{T}} \alpha$.

Suppose that $i < i_\xi$ for all ξ such that $\gamma <_{\mathcal{T}} \xi \leq_{\mathcal{T}} \alpha$, $\beta = \beta_i$, and that $y = j_{\beta,\gamma}(x_i)$. Then $e_{\gamma,\alpha}(\pi^\gamma_\gamma(y)) = \pi^\alpha_\alpha(j_{\gamma,\alpha}(y))$.

(2.5) Suppose $\beta <_{\mathcal{T}} \alpha$ and β survives at α. Suppose $\xi < p(\gamma)$ for all γ such that $\beta < \gamma \leq \alpha$,

$$x \in M_\beta \cap V_{\mu^{\mathcal{T}}_g[\xi,\alpha)},$$

and that $\text{CRT}(j_{\beta,\alpha}) > \mu^{\mathcal{T}}_g[\xi,\alpha)$. Then $e_{\beta,\alpha}(\pi^\beta_\beta(x)) = \pi^\alpha_\alpha(j_{\beta,\alpha}(x))$.

<u>Definition of $\mathcal{E}^0$:</u>

Let Υ be such that the ordertype of the set

$$\{\theta < \Upsilon \mid \Theta < \theta \text{ and } (V_{\theta+\omega}, \pi(\delta)) \text{ is a quasi-premouse}\}$$

is $\omega^2 \cdot \pi(f_p(0)) + \omega^2$. Define

$$(\pi^0_0, (\mathcal{P}^0_0, \delta^0_0), \nu^0_0) = (\pi, (V_\Upsilon, \pi(\delta)), \Theta).$$

<u>Definition of $\mathcal{E}^{\alpha+1}$:</u>

Let $\beta = p(\alpha+1) - 1$.

We shall define

$$(\pi^{\alpha+1}_\gamma, (\mathcal{P}^{\alpha+1}_\gamma, \delta^{\alpha+1}_\gamma), \nu^{\alpha+1}_\gamma)$$

first for $\gamma \leq \beta$. Then assuming $\beta + 1 = \alpha$ we will define

$$(\pi^{\alpha+1}_\alpha, (\mathcal{P}^{\alpha+1}_\alpha, \delta^{\alpha+1}_\alpha), \nu^{\alpha+1}_\alpha)$$

Next assuming $\beta + 1 < \alpha$ we will define the sequence which will be used to construct the witness for (2c) of Definition 82 and we will define

$$(\pi^{\alpha+1}_\alpha, (\mathcal{P}^{\alpha+1}_\alpha, \delta^{\alpha+1}_\alpha), \nu^{\alpha+1}_\alpha).$$

We then will construct the witness for (2c) which will give the definition of

$$(\pi^{\alpha+1}_\gamma, (\mathcal{P}^{\alpha+1}_\gamma, \delta^{\alpha+1}_\gamma), \nu^{\alpha+1}_\gamma)$$

for $\beta + 1 \leq \gamma < \alpha$.

We will finish this case by defining

$$(\pi^{\alpha+1}_{\alpha+1}, (\mathcal{P}^{\alpha+1}_{\alpha+1}, \delta^{\alpha+1}_{\alpha+1}), \nu^{\alpha+1}_{\alpha+1}).$$

This describes our strategy and now we proceed with the construction.

For $\gamma \leq \beta$ define

$$(\pi^{\alpha+1}_\gamma, (\mathcal{P}^{\alpha+1}_\gamma, \delta^{\alpha+1}_\gamma), \nu^{\alpha+1}_\gamma) = (\pi^\alpha_\gamma, (\mathcal{P}^\alpha_\gamma, \delta^\alpha_\gamma), \nu^\alpha_\gamma).$$

Now suppose $\beta + 1 = \alpha$. By (2.3), the set

$$S = \{\theta \in \mathcal{P}^\alpha_\alpha \mid \nu^\alpha_\alpha < \theta \text{ and } (\mathcal{P}^\alpha_\alpha \cap V_{\theta+\omega}, \delta^\alpha_\alpha) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi^\alpha_\alpha(f_p(\alpha)) + \omega \cdot c(\alpha,\alpha) + 1$. Let $\eta_\alpha \in S$ be such that $S \cap \eta_\alpha$ has ordertype exactly $\omega^2 \cdot \pi^\alpha_\alpha(f_p(\alpha)) + \omega \cdot c(\alpha,\alpha)$. Let

$$X \prec \mathcal{P}^\alpha_\alpha \cap V_{\eta_\alpha}$$

be an elementary substructure such that

$$\mathcal{P}^\alpha_\alpha \cap V_{\pi^\alpha_\alpha(\mu^{\mathcal{F}}_g[\beta,\alpha))+1} \cup \{\pi^\alpha_\alpha, \nu^\alpha_\alpha\} \subseteq X,$$

$X \in \mathcal{P}^\alpha_\alpha$ and such that

$$|X|^{\mathcal{P}^\alpha_\alpha} = |\mathcal{P}^\alpha_\alpha \cap V_{\pi^\alpha_\alpha(\mu^{\mathcal{F}}_g[\beta,\alpha))+1}|^{\mathcal{P}^\alpha_\alpha}.$$

Let $\mathcal{P}^{\alpha+1}_\alpha$ be the transitive collapse of X and let $(\pi^{\alpha+1}_\alpha, \delta^{\alpha+1}_\alpha, \nu^{\alpha+1}_\alpha)$ be the image of $(\pi^\alpha_\alpha, \delta^\alpha_\alpha, \nu^\alpha_\alpha)$ under the transitive collapse.

We now suppose that $\beta + 1 < \alpha$. We shall define for each $\gamma < \alpha$ such that $\beta + 1 \leq \gamma$,

$$(\sigma_\gamma, (\mathcal{Q}_\gamma, \kappa_\gamma), \mu_\gamma)$$

such that

(3.1) $\mathcal{Q}_\gamma \in \mathcal{P}^\alpha_\alpha$.

(3.2) $\sigma_\gamma \in \mathcal{Q}_\gamma$ and

$$\sigma_\gamma : (M_\gamma, \delta_\gamma) \to (\mathcal{Q}_\gamma \cap V_{\mu_\gamma}, \kappa_\gamma)$$

is an elementary embedding.

(3.3) $\sigma_\gamma(M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha)}) = \pi^\alpha_\alpha(M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha)})$ and

$$\sigma_\gamma|(M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha)}) = \pi^\alpha_\alpha|(M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\gamma,\alpha)}).$$

(3.4) The set,

$$\{\xi \in \mathcal{Q}_\gamma \cap \mathrm{Ord} \mid \mu_\gamma < \xi \text{ and } (\mathcal{Q} \cap V_{\xi+\omega}, \kappa_\gamma) \text{ is a quasi-premouse}\},$$

has ordertype $\omega^2 \cdot \sigma_\gamma(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha)$.

We are working toward constructing the sequence which will witness (2c) of Definition 82 for $\alpha + 1$. The difficulty is ensuring that (3.1) holds and Lemma 84 will play a crucial role here.

Suppose that $\beta + 1 \leq \gamma < \alpha$. By (2.3), the set

$$S = \{\theta \in \mathcal{P}^\alpha_\gamma \mid \nu^\alpha_\gamma < \theta \text{ and } (\mathcal{P}^\alpha_\gamma \cap V_{\theta+\omega}, \delta^\alpha_\gamma) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi^\alpha_\gamma(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha) + 1$. Let $\eta_\gamma \in S$ be such that $S \cap \eta_\gamma$ has ordertype exactly $\omega^2 \cdot \pi^\alpha_\gamma(f_p(\beta)) + \omega \cdot c(\gamma, \alpha)$. Let

$$X \prec \mathcal{P}^\alpha_\gamma \cap V_{\eta_\gamma}$$

be an elementary substructure such that

$$\mathcal{P}^\alpha_\gamma \cap V_{\pi^\alpha_\gamma(\mu^{\mathcal{F}}_g[\gamma,\alpha))} \cup \{\pi^\alpha_\gamma(\mu^{\mathcal{F}}_g[\gamma,\alpha)), \pi^\alpha_\gamma, \nu^\alpha_\gamma\} \subseteq X,$$

$X \in \mathcal{P}^\alpha_\gamma$ and such that

$$|X|^{\mathcal{P}^\alpha_\gamma} = |\mathcal{P}^\alpha_\gamma \cap V_{\pi^\alpha_\gamma(\mu^{\mathcal{F}}_g[\gamma,\alpha))}|^{\mathcal{P}^\alpha_\gamma}.$$

Let $\mathcal{Q}_\gamma$ be the transitive collapse of X and let $(\sigma_\gamma, \kappa_\gamma, \mu_\gamma)$ be the image of $(\pi^\alpha_\gamma, \delta^\alpha_\gamma, \mu^\alpha_\gamma)$ under the transitive collapse of X.

By Lemma 84,

$$\mathcal{P}^\alpha_\gamma \cap V_{\pi^\alpha_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))+1} \subseteq \mathcal{P}^\alpha_\alpha$$

and so it follows that $\mathcal{Q}_\gamma \in \mathcal{P}^\alpha_\alpha$.

This finishes the construction of

$$(\sigma_\gamma, (\mathcal{Q}_\gamma, \kappa_\gamma), \mu_\gamma)$$

for $\beta + 1 \leq \gamma < \alpha$ satisfying (3.1)–(3.4).

By absoluteness there must exist such a sequence,

$$\langle(\sigma_\gamma, (\mathcal{Q}_\gamma, \kappa_\gamma), \mu_\gamma) : \beta + 1 \leq \gamma < \alpha\rangle \in \mathcal{P}^\alpha_\alpha,$$

since $\pi^\alpha_\alpha \in \mathcal{P}^\alpha_\alpha$.

Let

$$\mathcal{S} = \langle(\sigma'_\gamma, (\mathcal{Q}'_\gamma, \kappa'_\gamma), \mu'_\gamma) : \beta + 1 \leq \gamma < \alpha\rangle \in \mathcal{P}^\alpha_\alpha$$

be such a sequence. By (2.3), the set

$$S = \{\theta \in \mathcal{P}^\alpha_\alpha \mid \nu^\alpha_\alpha < \theta \text{ and } (\mathcal{P}^\alpha_\alpha \cap V_{\theta+\omega}, \delta^\alpha_\alpha) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi^\alpha_\alpha(f_p(\alpha)) + \omega \cdot c(\alpha, \alpha) + 1$. Let $\eta_\alpha \in S$ be such that $S \cap \eta_\alpha$ has ordertype exactly $\omega^2 \cdot \pi^\alpha_\alpha(f_p(\alpha)) + \omega \cdot c(\alpha, \alpha)$. Let

$$X \prec \mathcal{P}^\alpha_\alpha \cap V_{\eta_\alpha}$$

be an elementary substructure such that

$$\mathcal{P}^\alpha_\alpha \cap V_{\pi^\alpha_\alpha(\mu^{\mathcal{F}}_g[\beta,\alpha))+1} \cup \{\pi^\alpha_\alpha, \nu^\alpha_\alpha, \mathcal{S}\} \subseteq X,$$

$X \in \mathcal{P}^\alpha_\alpha$ and such that

$$|X|^{\mathcal{P}^\alpha_\alpha} = |\mathcal{P}^\alpha_\alpha \cap V_{\pi^\alpha_\alpha(\mu^{\mathcal{F}}_g[\beta,\alpha))+1}|^{\mathcal{P}^\alpha_\alpha}.$$

Let $\mathcal{P}^{\alpha+1}_\alpha$ be the transitive collapse of X and let $(\pi^{\alpha+1}_\alpha, \delta^{\alpha+1}_\alpha, \nu^{\alpha+1}_\alpha)$ be the image of $(\pi^\alpha_\alpha, \delta^\alpha_\alpha, \nu^\alpha_\alpha)$ under the transitive collapse.

We finish by defining the sequence,

$$\langle(\mathcal{R}_\gamma, \eta^i_\gamma) : \beta + 1 \leq \gamma < \alpha, i < \omega\rangle \in \mathcal{P}^{\alpha+1}_\alpha$$

which will witness (2c) of Definition 82 and defining

$$(\pi^{\alpha+1}_\gamma, \delta^{\alpha+1}_\gamma, \nu^{\alpha+1}_\gamma)$$

for $\beta + 1 \leq \gamma < \alpha$.

For $\beta + 1 \leq \gamma < \alpha$, let $\mathcal{R}_\gamma$ be the transitive collapse of $\mathcal{Q}'_\gamma$ and let

$$(\pi^{\alpha+1}_\gamma, \delta^{\alpha+1}_\gamma, \nu^{\alpha+1}_\gamma)$$

be the image of $(\sigma'_\gamma, \kappa'_\gamma, \mu'_\gamma)$ under the transitive collapse.

Finally for each $\beta + 1 \leq \gamma < \alpha$ and for each $i < \omega$ let $\eta^i_\gamma \in \mathcal{R}_\gamma \cap \mathrm{Ord}$ be such that

(4.1) $\nu^{\alpha+1}_\gamma < \eta^i_\gamma$,
(4.2) $(\mathcal{R}_\gamma \cap V_{\eta^i_\gamma+\omega}, \delta^{\alpha+1}_\gamma)$ is a quasi-premouse,

(4.3) the set

$$\{\theta \mid \nu_\gamma^{\alpha+1} < \theta < \eta_\gamma^i \text{ and } (\mathcal{R}_\gamma \cap V_{\theta+\omega}, \delta_\gamma^{\alpha+1}) \text{ is a quasi-premouse}\}$$

has ordertype, $\omega^2 \cdot \pi_\gamma^{\alpha+1}(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha+1) + i + 2$.

Finally set $\mathcal{P}_\gamma^{\alpha+1} = \mathcal{R}_\gamma \cap V_{\eta_\gamma^0}$ (as required by (2c) of Definition 82).

We finish the definition of $\mathcal{E}^{\alpha+1}$ by defining,

$$(\pi_{\alpha+1}^{\alpha+1}, (\mathcal{P}_{\alpha+1}^{\alpha+1}, \delta_{\alpha+1}^{\alpha+1}), \nu_{\alpha+1}^{\alpha+1}).$$

Let $\epsilon = \alpha^*$ and let

$$\mathcal{P}_{\alpha+1}^{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\epsilon^{\alpha+1}, \pi_\alpha^{\alpha+1}(E_\alpha)).$$

It is straightforward to verify that this ultrapower is defined, we verify that it is wellfounded.

If $\epsilon < p(\alpha+1)$ then by (2a) of Definition 82,

$$\mathcal{P}_\epsilon^{\alpha+1} = \mathcal{P}_\epsilon^\alpha.$$

Either $p(\alpha+1) = \alpha+1$ in which case again by (2a) of Definition 82,

$$\mathcal{P}_\alpha^{\alpha+1} = \mathcal{P}_\alpha^\alpha,$$

or $p(\alpha+1) < \alpha+1$ in which case by (2b) of Definition 82,

$$\mathcal{P}_\alpha^{\alpha+1} \in \mathcal{P}_\alpha^\alpha.$$

In either case $\pi_\alpha^{\alpha+1}(E_\alpha) \in \mathcal{P}_\alpha^\alpha$. Therefore by Lemma 88, $\mathrm{Ult}(\mathcal{P}_\epsilon^\alpha, \pi_\alpha^{\alpha+1}(E_\alpha))$ is wellfounded and so $\mathcal{P}_{\alpha+1}^{\alpha+1}$ is wellfounded.

If $p(\alpha+1) \leq \epsilon$ then by (2c) of Definition 82,

$$\mathcal{P}_\epsilon^{\alpha+1} \in \mathcal{P}_\alpha^{\alpha+1}.$$

But $\mathrm{Ult}(\mathcal{P}_\alpha^{\alpha+1}, \pi_\alpha^{\alpha+1}(E_\alpha))$ is an internal ultrapower and therefore wellfounded. This implies

$$\mathrm{Ult}(\mathcal{P}_\epsilon^\alpha, \pi_\alpha^{\alpha+1}(E_\alpha)) = \mathcal{P}_{\alpha+1}^{\alpha+1}$$

is wellfounded.

Let

$$e : \mathcal{P}_\epsilon^{\alpha+1} \to \mathcal{P}_{\alpha+1}^{\alpha+1}$$

be the ultrapower embedding.

Let $\langle \delta_{\alpha+1}^{\alpha+1}, \nu_{\alpha+1}^{\alpha+1} \rangle = e(\langle \delta_\epsilon^{\alpha+1}, \nu_\epsilon^{\alpha+1} \rangle)$.

Suppose that ϵ survives at $\alpha+1$. Then $\epsilon < p(\xi)$ for all $\epsilon < \xi \leq \alpha+1$ and so by induction on ξ,

$$\mathcal{P}_\epsilon^\xi = \mathcal{P}_\epsilon^\epsilon$$

for all $\epsilon < \xi \leq \alpha+1$. In particular, $\mathcal{P}_\epsilon^\epsilon = \mathcal{P}_\epsilon^{\alpha+1}$. Thus if ϵ survives at $\alpha+1$ then we define $e_{\epsilon,\alpha+1} = e$ (as we must).

It remains to define $\pi^{\alpha+1}_{\alpha+1}$.

Let

$$\sigma : (M_{\alpha+1}, \delta_{\alpha+1}) \to (\mathcal{P}^{\alpha+1}_{\alpha+1} \cap V_{\nu^{\alpha+1}_{\alpha+1}}, \delta^{\alpha+1}_{\alpha})$$

be the canonical embedding from the shift lemma, Lemma 69.

This embedding satisfies all the requirements that $\pi^{\alpha+1}_{\alpha+1}$ must satisfy except possibly the requirement that $\pi^{\alpha+1}_{\alpha+1} \in \mathcal{P}^{\alpha+1}_{\alpha+1}$.

By absoluteness there must exist an elementary embedding,

$$\sigma' : (M_{\alpha+1}, \delta_{\alpha+1}) \to (\mathcal{P}^{\alpha+1}_{\alpha+1} \cap V_{\nu^{\alpha+1}_{\alpha+1}}, \delta^{\alpha+1}_{\alpha}),$$

such that

(5.1) $\sigma' \in \mathcal{P}^{\alpha+1}_{\alpha+1}$,
(5.2) $\sigma'(M_{\alpha+1} \cap V_{\mu^{\mathcal{T}}_g[\alpha,\alpha+1)}) = \sigma(M_{\alpha+1} \cap V_{\mu^{\mathcal{T}}_g[\alpha,\alpha+1)})$,
(5.3) $\sigma'|(M_{\alpha+1} \cap V_{\mu^{\mathcal{T}}_g[\alpha,\alpha+1)}) = \sigma|(M_{\alpha+1} \cap V_{\mu^{\mathcal{T}}_g[\alpha,\alpha+1)})$,
(5.4) $\sigma'(j_{\beta_i,\alpha+1}(x_i)) = \sigma(j_{\beta_i,\alpha+1}(x_i))$ for all $i \leq i_{\alpha+1}$,
(5.5) $\sigma'(j_{\epsilon,\alpha+1}(f_p(\epsilon))) = \sigma(j_{\epsilon,\alpha+1}(f_p(\epsilon)))$,

where (β_i, x_i) and $i_{\alpha+1}$ refer to the two enumerations we fixed at the beginning of the proof. Let $\pi^{\alpha+1}_{\alpha+1} = \sigma'$.

This completes the definition of $\mathcal{E}^{\alpha+1}$ and it is evident from the construction that $\langle \mathcal{E}^{\xi} : \xi \leq \alpha+1 \rangle$ is a p-connected sequence of g-enlargements of $\mathcal{T}$. It remains to verify that the additional inductive requirements, (2.1)–(2.5), are satisfied for all $\gamma \leq \alpha+1$ (we are continuing to let $\beta = p(\alpha+1) - 1$).

That the first two, (2.1) and (2.2), hold is immediate.

We show that (2.3) holds for all $\gamma \leq \alpha+1$. If $\gamma \leq \beta$ then

$$(\pi^{\alpha+1}_{\gamma}, (\mathcal{P}^{\alpha+1}_{\gamma}, \delta^{\alpha+1}_{\gamma}), \nu^{\alpha+1}_{\gamma}) = (\pi^{\alpha}_{\gamma}, (\mathcal{P}^{\alpha}_{\gamma}, \delta^{\alpha}_{\gamma}), \nu^{\alpha}_{\gamma})$$

and (2.3) holds by induction.

If $\beta < \gamma \leq \alpha$, then (2.3) holds by definition of $\mathcal{P}^{\alpha+1}_{\gamma}$.

Finally if $\gamma = \alpha+1$ there are two cases; we must show that the set,

$$\{\theta \in \mathcal{P}^{\alpha+1}_{\alpha+1} \mid \nu^{\alpha+1}_{\alpha+1} < \theta \text{ and } (\mathcal{P}^{\alpha+1}_{\alpha+1} \cap V_{\theta+\omega}, \delta^{\alpha+1}_{\alpha+1}) \text{ is a quasi-premouse}\},$$

has ordertype at least $\omega^2 \cdot \pi^{\alpha+1}_{\alpha+1}(f_p(\alpha+1)) + \omega^2$.

We set $\epsilon = \alpha^*$ as in the construction used to define $\mathcal{P}^{\alpha+1}_{\alpha+1}$ and consider several cases.

The first case is that ϵ survives at $\alpha+1$. In this case

$$(\pi^{\alpha+1}_{\epsilon}, (\mathcal{P}^{\alpha+1}_{\epsilon}, \delta^{\alpha+1}_{\epsilon}), \nu^{\alpha+1}_{\epsilon}) = (\pi^{\epsilon}_{\epsilon}, (\mathcal{P}^{\epsilon}_{\epsilon}, \delta^{\epsilon}_{\epsilon}), \nu^{\epsilon}_{\epsilon})$$

and we have defined $e_{\epsilon,\alpha+1}$ to be the ultrapower embedding,

$$e : \mathcal{P}^{\alpha+1}_{\epsilon} \to \mathcal{P}^{\alpha+1}_{\alpha+1}.$$

By the induction hypothesis (2.3) applied to ϵ, the set

$$\{\theta \in \mathcal{P}_\epsilon^\epsilon \mid \nu_\epsilon^\epsilon < \theta \text{ and } (\mathcal{P}_\epsilon^\epsilon \cap V_{\theta+\omega}, \delta_\epsilon^{\alpha+1}) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi_\epsilon^\epsilon(f_p(\epsilon)) + \omega^2$. Therefore by the agreement above, the set,

$$\{\theta \in \mathcal{P}_\epsilon^{\alpha+1} \mid \nu_\epsilon^{\alpha+1} < \theta \text{ and } (\mathcal{P}_\epsilon^{\alpha+1} \cap V_{\theta+\omega}, \delta_\epsilon^{\alpha+1}) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi_\epsilon^{\alpha+1}(f_p(\epsilon)) + \omega^2$.

By the definition of $\pi_{\alpha+1}^{\alpha+1}$ (condition (5.5)),

$$\pi_{\alpha+1}^{\alpha+1} \circ j_{\epsilon,\alpha+1}(f_p(\epsilon)) = \sigma \circ j_{\epsilon,\alpha+1}(f_p(\epsilon))$$

where $e \circ \pi_\epsilon^{\alpha+1} = \sigma \circ j_{\epsilon,\alpha+1}$ (from the shift lemma).

Thus the set

$$\{\theta \in \mathcal{P}_{\alpha+1}^{\alpha+1} \mid \nu_{\alpha+1}^{\alpha+1} < \theta \text{ and } (\mathcal{P}_{\alpha+1}^{\alpha+1} \cap V_{\theta+\omega}, \delta_{\alpha+1}^{\alpha+1}) \text{ is a quasi-premouse}\}$$

has ordertype at least

$$\omega^2 \cdot e \circ \pi_\epsilon^{\alpha+1}(f_p(\epsilon)) + \omega^2 = \omega^2 \cdot \pi_{\alpha+1}^{\alpha+1} \circ j_{\epsilon,\alpha+1}(f_p(\epsilon)) + \omega^2.$$

But $j_{\epsilon,\alpha+1}(f_p(\epsilon)) = f_p(\alpha+1)$ since ϵ survives at $\alpha+1$ and so this set has ordertype, $\omega^2 \cdot \pi_{\alpha+1}^{\alpha+1}(f_p(\alpha+1)) + \omega^2$.

The second case is that ϵ does not survive at $\alpha+1$. The set

$$\{\theta \in \mathcal{P}_\epsilon^{\alpha+1} \mid \nu_\epsilon^{\alpha+1} < \theta \text{ and } (\mathcal{P}_\epsilon^{\alpha+1} \cap V_{\theta+\omega}, \delta_\epsilon^{\alpha+1}) \text{ is a quasi-premouse}\}$$

has ordertype at least $\omega^2 \cdot \pi_\epsilon^{\alpha+1}(f_p(\epsilon))$ (since (2.3) holds for $(\epsilon, \alpha+1)$).

This implies that the set

$$Z = \{\theta \in \mathcal{P}_{\alpha+1}^{\alpha+1} \mid \nu_{\alpha+1}^{\alpha+1} < \theta \text{ and } (\mathcal{P}_{\alpha+1}^{\alpha+1} \cap V_{\theta+\omega}, \delta_{\alpha+1}^{\alpha+1}) \text{ is a quasi-premouse}\}$$

has ordertype at least

$$\omega^2 \cdot e \circ \pi_\epsilon^{\alpha+1}(f_p(\epsilon)) = \omega^2 \cdot \pi_{\alpha+1}^{\alpha+1} \circ j_{\epsilon,\alpha+1}(f_p(\epsilon)),$$

where again we have used that

$$\pi_{\alpha+1}^{\alpha+1} \circ j_{\epsilon,\alpha+1}(f_p(\epsilon)) = \sigma \circ j_{\epsilon,\alpha+1}(f_p(\epsilon)).$$

Since ϵ does not survive at $\alpha+1$,

$$f_p(\alpha+1) < j_{\epsilon,\alpha+1}(f_p(\epsilon))$$

and the set Z has ordertype at least $\omega^2 \cdot \pi_{\alpha+1}^{\alpha+1}(f_p(\alpha+1)) + \omega^2$ as required.

This proves that (2.3) holds for all $(\gamma, \alpha+1)$ for all $\gamma \leq \alpha+1$.

It remains to show that (2.4) and (2.5) hold for all such pairs $(\gamma, \alpha+1)$.

Suppose that $\gamma <_\mathcal{T} \alpha+1$ and γ survives at $\alpha+1$. Thus $\gamma \leq_\mathcal{T} \epsilon$ since $\epsilon = \alpha^*$. Thus ϵ survives at $\alpha+1$ and so

$$(\pi_\epsilon^{\alpha+1}, (\mathcal{P}_\epsilon^{\alpha+1}, \delta_\epsilon^{\alpha+1}), \nu_\epsilon^{\alpha+1}) = (\pi_\epsilon^\epsilon, (\mathcal{P}_\epsilon^\epsilon, \delta_\epsilon^\epsilon), \nu_\epsilon^\epsilon)$$

and we have defined $e_{\epsilon,\alpha+1}$ to be the ultrapower embedding,

$$e : \mathcal{P}_\epsilon^{\alpha+1} \to \mathcal{P}_{\alpha+1}^{\alpha+1}.$$

We verify (2.4) for $(\gamma, \alpha+1)$. Suppose $\gamma <_{\mathcal{T}} \lambda <_{\mathcal{T}} \alpha+1$, $i < i_\xi$ for all $\gamma <_{\mathcal{T}} \xi \leq_{\mathcal{T}} \alpha+1$, $\gamma = \beta_i$, and let $y = j_{\gamma,\lambda}(x_i)$. We must show that

$$e_{\lambda,\alpha+1} \circ \pi^{\lambda}_{\lambda}(y) = \pi^{\alpha+1}_{\alpha+1} \circ j_{\lambda,\alpha+1}(y).$$

But $i < i_{\alpha+1}$ and so by (5.4)

$$\pi^{\alpha+1}_{\alpha+1} \circ j_{\gamma,\alpha+1}(x_i) = \sigma \circ j_{\gamma,\alpha+1}(x_i).$$

Further

$$\sigma : (M_{\alpha+1}, \delta_{\alpha+1}) \to (\mathcal{P}^{\alpha+1}_{\alpha+1} \cap V_{\nu^{\alpha+1}_{\alpha+1}}, \delta^{\alpha+1}_{\alpha})$$

is the canonical embedding from the shift lemma. The desired equality follows.

Finally we verify (2.5) for $(\gamma, \alpha+1)$. Suppose that $\xi < p(\lambda)$ for all $\gamma < \lambda \leq \alpha+1$,

$$x \in M_\gamma \cap V_{\mu^{\mathcal{T}}_g[\xi,\alpha+1)},$$

and that $\mu^{\mathcal{T}}_g[\xi, \alpha+1) < \mathrm{CRT}(j_{\gamma,\alpha+1})$. Now we must verify that

$$e_{\gamma,\alpha+1} \circ \pi^{\gamma}_{\gamma}(x) = \pi^{\alpha+1}_{\alpha+1} \circ j_{\gamma,\alpha+1}(x).$$

This follows by (5.3), the shift lemma, and induction.

Definition of $\mathcal{E}^\alpha$ (α a limit ordinal):

Define $(\mathcal{P}^{\alpha}_{\alpha}, \delta^{\alpha}_{\alpha})$ to be the direct limit of

$$\{(\mathcal{P}^{\xi}_{\xi}, \delta^{\xi}_{\xi}) \,|\, \xi <_{\mathcal{T}} \alpha \text{ and } \xi \text{ survives at } \alpha\}$$

under the embeddings,

$$e_{\xi_1,\xi_2} : (\mathcal{P}^{\xi_1}_{\xi_1}, \delta^{\xi_1}_{\xi_1}) \to (\mathcal{P}^{\xi_2}_{\xi_2}, \delta^{\xi_2}_{\xi_2})$$

where $\xi_1 <_{\mathcal{T}} \xi_2 <_{\mathcal{T}} \alpha$, ξ_1 survives at α, and ξ_2 survives at α (and so ξ_1 survives at ξ_2).

The direct limit is the limit of an internal iteration (by Corollary 86) and so it is wellfounded, by induction hypothesis (2.2).

For each $\xi <_{\mathcal{T}} \alpha$ such that ξ survives at α, let

$$e_{\xi,\alpha} : (\mathcal{P}^{\xi}_{\xi}, \delta^{\xi}_{\xi}) \to (\mathcal{P}^{\alpha}_{\alpha}, \delta^{\alpha}_{\alpha})$$

be the induced embedding. Let

$$\nu^{\alpha}_{\alpha} = e_{\xi,\alpha}(\nu^{\xi}_{\xi})$$

for sufficiently large $\xi <_{\mathcal{T}} \alpha$. This is well defined since $\langle \mathcal{E}^\xi : \xi < \alpha \rangle$ is a p-connected sequence of g-enlargements of $\mathcal{T}$ (by (2a), (3a) and (3b) of Definition 82).

Note that

$$\sup\{\min\{i_\xi \,|\, \gamma < \xi < \alpha\} \,|\, \gamma < \alpha\} = \omega.$$

Thus by (2.4), for all $\xi <_{\mathcal{T}} \alpha$ such that ξ survives at α, for all $x \in \mathcal{M}_\xi$, there exists $\xi_x <_{\mathcal{T}} \alpha$ such that for all $\xi_x <_{\mathcal{T}} \gamma <_{\mathcal{T}} \alpha$,

$$\pi^\gamma_\gamma \circ j_{\gamma,\xi_x}(x) = e_{\xi_x,\gamma} \circ \pi^{\xi_x}_{\xi_x} \circ j_{\xi,\xi_x}(x),$$

and so for all $\xi_x <_{\mathcal{T}} \gamma <_{\mathcal{T}} \alpha$,

$$e_{\gamma,\alpha} \circ \pi^\gamma_\gamma \circ j_{\gamma,\xi_x}(x) = e_{\xi_x,\alpha} \circ \pi^{\xi_x}_{\xi_x} \circ j_{\xi,\xi_x}(x).$$

This defines an embedding,

$$\sigma : (M_\alpha, \delta_\alpha) \to (\mathcal{P}^\alpha_\alpha \cap V_{\nu^\alpha_\alpha}, \delta^\alpha_\alpha).$$

(This construction was precisely the reason for (2.4).)

Let $\beta = p(\alpha) - 1$. For $\gamma < \beta + 1$ we define

$$(\pi^\alpha_\gamma, (\mathcal{P}^\alpha_\gamma, \delta^\alpha_\gamma), \nu^\alpha_\gamma) = (\pi^\xi_\gamma, (\mathcal{P}^\xi_\gamma, \delta^\xi_\gamma), \nu^\xi_\gamma)$$

for sufficiently large $\xi < \alpha$. As above, this is well defined since $\langle \mathcal{E}^\xi : \xi < \alpha \rangle$ is a p-connected sequence of g-enlargements of $\mathcal{T}$.

We now define the sequence which will witness (3d) of Definition 82. The construction is quite similar to the analogous construction in the successor case, a key step involves constructing elementary substructures and Lemma 84 will again be used to show the corresponding transitive collapses are in the required model.

For $\beta < \gamma < \alpha$, let

$$(\tau_\gamma, (\mathcal{P}_\gamma, \lambda_\gamma), \mu_\gamma) = (\pi^\xi_\gamma, (\mathcal{P}^\xi_\gamma, \delta^\xi_\gamma), \nu^\xi_\gamma)$$

for sufficiently large $\xi < \alpha$ (this is well defined).

Let

$$S_\gamma = \{\theta \in \mathcal{P}_\gamma \cap \mathrm{Ord} \,|\, \nu_\gamma < \theta \text{ and } (\mathcal{P}_\gamma \cap V_{\theta+\omega}, \kappa_\gamma) \text{ is a quasi-premouse}\}$$

so by induction, S_γ has ordertype at least $\omega^2 \cdot \pi^\alpha_\gamma(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha) + 1$ noting that for all $\gamma < \xi_1 < \xi_2 < \alpha$,

$$c(\gamma, \alpha) \leq c(\xi_1, \alpha) \leq c(\xi_1, \xi_2).$$

Let $\eta \in S_\gamma$ be such that the ordertype of $S_\gamma \cap \eta$ is

$$\omega^2 \cdot \tau_\gamma(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha) + 1,$$

η exists since for all sufficiently large $\gamma < \xi_1 < \alpha$, for all $\xi_1 < \xi_2 < \alpha$,

$$c(\gamma, \alpha) < c(\xi_1, \alpha) \leq c(\xi_1, \xi_2).$$

Let

$$X_\gamma \prec \mathcal{P}_\gamma \cap V_\eta$$

be an elementary substructure such that

$$\mathcal{P}_\gamma \cap V_{\tau_\gamma(\mu^{\mathcal{T}}_g[\gamma,\alpha))} \cup \{\tau_\gamma(\mu^{\mathcal{F}}_g[\gamma, \alpha)), \tau_\gamma, \nu_\gamma\} \subseteq X_\gamma,$$

$X_\gamma \in \mathcal{P}_\gamma$ and such that

$$|X|^{\mathcal{P}_\gamma} = |\mathcal{P}_\gamma \cap V_{\tau_\gamma(\mu_g^{\mathcal{F}}[\gamma,\alpha))}|^{\mathcal{P}_\gamma}.$$

Let $\mathcal{Q}_\gamma$ be the transitive collapse of X_γ and let $(\sigma_\gamma, \kappa_\gamma, \mu_\gamma)$ be the image of $(\tau_\gamma, \lambda_\gamma, \nu_\gamma)$ under the transitive collapse of X.

By Lemma 68, for all $\epsilon < \alpha$,

$$\mu_g^{\mathcal{T}}[\epsilon, \alpha) < \mathrm{CRT}(E_\xi)$$

for all sufficiently large $\xi + 1 <_{\mathcal{T}} \alpha$.

Now

$$\tau_\gamma(\mu_g^{\mathcal{T}}[\gamma,\alpha)) = \pi_\gamma^\xi(\mu_g^{\mathcal{T}}[\gamma,\alpha))$$

for all sufficiently large $\xi < \alpha$ (by definition of τ_γ), and so for all sufficiently large $\xi <_{\mathcal{T}} \alpha$, ξ survives at α and

$$\tau_\gamma(\mu_g^{\mathcal{T}}[\gamma,\alpha)) < \mathrm{CRT}(e_{\xi,\alpha}).$$

By Lemma 84, if $\gamma < \xi < \alpha$ then

$$\mathcal{P}_\gamma^\xi \cap V_{\pi_\gamma^\xi(\mu_g^{\mathcal{T}}[\gamma,\xi))+1} \subset \mathcal{P}_\xi^\xi \cap V_{\pi_\gamma^\xi(\mu_g^{\mathcal{T}}[\gamma,\xi))+1}.$$

Thus for all sufficiently large $\xi < \alpha$,

$$\mathcal{P}_\gamma^\xi \cap V_{\pi_\gamma^\xi(\mu_g^{\mathcal{T}}[\gamma,\xi))+1} \subset \mathcal{P}_\alpha^\alpha$$

and so $\mathcal{Q}_\gamma \in \mathcal{P}_\alpha^\alpha$ since for all sufficiently large $\xi < \alpha$,

$$(\tau_\gamma, (\mathcal{P}_\gamma, \lambda_\gamma), \mu_\gamma) = (\pi_\gamma^\xi, (\mathcal{P}_\gamma^\xi, \delta_\gamma^\xi), \nu_\gamma^\xi).$$

Thus for all $\beta < \gamma < \alpha$ we have defined

$$(\sigma_\gamma, (\mathcal{Q}_\gamma, \kappa_\gamma), \mu_\gamma)$$

and we have defined an elementary embedding,

$$\sigma : (M_\alpha, \delta_\alpha) \to (\mathcal{P}_\alpha^\alpha \cap V_{\nu_\alpha^\alpha}, \delta_\alpha^\alpha).$$

such that

(6.1) $\mathcal{Q}_\gamma \in \mathcal{P}_\alpha^\alpha$.

(6.2) $\sigma_\gamma \in \mathcal{Q}_\gamma$ and

$$\sigma_\gamma : (M_\gamma, \delta_\gamma) \to (\mathcal{Q}_\gamma \cap V_{\mu_\gamma}, \kappa_\gamma)$$

is an elementary embedding.

(6.3) $\sigma_\gamma(M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)}) = \sigma(M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)})$ and

$$\sigma_\gamma|(M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)}) = \sigma|(M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)}).$$

(6.4) The set,

$$\{\xi \in \mathcal{Q}_\gamma \cap \mathrm{Ord} \mid \mu_\gamma < \xi \text{ and } (\mathcal{Q} \cap V_{\xi+\omega}, \kappa_\gamma) \text{ is a quasi-premouse}\},$$

has ordertype at least $\omega^2 \cdot \sigma_\gamma(f_p(\gamma)) + \omega \cdot c(\gamma,\alpha) + \omega$.

(6.5) $\sigma|(M_\alpha \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)}) = \pi_\beta^\alpha|(M_\beta \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)})$.

We have $M_\beta \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)} = M_\alpha \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)}$ and

$$\pi_\beta^\alpha | (M_\beta \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)}) \in \mathcal{P}_\alpha^\alpha,$$

since $\pi_\beta^\alpha \in \mathcal{P}_\beta^\alpha$ (the latter holds by definition of $(\pi_\beta^\alpha, \mathcal{P}_\beta^\alpha)$ and the induction hypothesis).

By absoluteness there exists a sequence

$$\langle (\sigma'_\gamma, (\mathcal{Q}'_\gamma, \kappa'_\gamma), \mu'_\gamma) : \beta < \gamma < \alpha \rangle \in \mathcal{P}_\alpha^\alpha$$

and an elementary embedding,

$$\sigma' : (M_\alpha, \delta_\alpha) \to (\mathcal{P}_\alpha^\alpha \cap V_{\nu_\alpha^\alpha}, \delta_\alpha^\alpha).$$

such that $\sigma' \in \mathcal{P}_\alpha^\alpha$ and such that

(7.1) $\mathcal{Q}'_\gamma \in \mathcal{P}_\alpha^\alpha$.

(7.2) $\sigma'_\gamma \in \mathcal{Q}'_\gamma$ and

$$\sigma'_\gamma : (M_\gamma, \delta_\gamma) \to (\mathcal{Q}'_\gamma \cap V_{\mu'_\gamma}, \kappa'_\gamma)$$

is an elementary embedding.

(7.3) $\sigma'_\gamma(M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)}) = \sigma'(M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)})$ and

$$\sigma'_\gamma | (M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)}) = \sigma' | (M_\gamma \cap V_{\mu_g^{\mathcal{T}}[\gamma,\alpha)}).$$

(7.4) The set,

$$S'_\gamma = \{\xi \in \mathcal{Q}'_\gamma \cap \mathrm{Ord} \mid \mu'_\gamma < \xi \text{ and } (\mathcal{Q}' \cap V_{\xi+\omega}, \kappa'_\gamma) \text{ is a quasi-premouse}\},$$

has ordertype at least $\omega^2 \cdot \sigma'_\gamma(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha) + \omega$.

(7.5) $\sigma' | (M_\alpha \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)}) = \pi_\beta^\alpha | (M_\beta \cap V_{\mu_g^{\mathcal{T}}[\beta,\alpha)})$.

(7.6) $\sigma'(f_p(\alpha)) = \sigma(f_p(\alpha))$ and $\sigma'(x_i) = \sigma(x_i)$ for all $i \le i_\alpha$ (x_i and i_α refer to the bijections fixed at the beginning of the proof).

We now define the sequence which witnesses (3d) of Definition 82. For $\beta < \gamma < \alpha$,

$$\mathcal{R}_\gamma = \mathcal{Q}'_\gamma.$$

For $\beta < \gamma < \alpha$ and for $i < \omega$, $\eta_\gamma^i \in S'_\gamma$ is such that $S'_\gamma \cap \eta_\gamma^i$ has ordertype

$$\omega^2 \cdot \sigma'(f_p(\gamma)) + \omega \cdot c(\gamma, \alpha) + i + 2.$$

For all $\beta < \gamma < \alpha$, define

$$(\pi_\gamma^\alpha, (\mathcal{P}_\gamma^\alpha, \delta_\gamma^\alpha), \nu_\gamma^\alpha) = (\sigma'_\gamma, (\mathcal{R}_\gamma \cap V_{\eta_\gamma^0}, \kappa'_\gamma), \mu'_\gamma),$$

and define $\pi_\alpha^\alpha = \sigma'$.

This completes the definition of $\mathcal{E}^\alpha$ and the verification that (2.1)–(2.5) hold is routine. □

We now prove Theorem 66.

Proof of Theorem 66 for iteration trees of length α. Assume toward a contradiction that $(\pi, (M, \delta), \mathcal{T})$ is counterexample with η, the length of $\mathcal{T}$, as small as possible. Let $p : \eta \to \eta$ be a preservation function for $\mathcal{T}$. By Lemma 81, we can suppose both that $\mathcal{T}$ is p-continuously illfounded and that $\mathcal{T}$ can be copied via π to define an iteration tree on $(V_\Theta, \pi(\delta))$ of length η (if the latter fails we obtain a counterexample to the theorem with a (+2)-iteration tree of length $< \eta$).

We show that η is a limit ordinal. Suppose not and let $\theta = \eta - 1$. Let

$$E \in M_\gamma \cap V_{j_{0,\gamma}(\delta)}$$

and $\gamma^* \leq \gamma$, be such that

(1.1) if $\gamma^* < \gamma$ then $\mathrm{SPT}(E) + 2 \leq \min\{\rho(E_\alpha) \mid \gamma^* \leq \alpha < \gamma\}$,
(1.2) $\mathrm{Ult}(M_{\gamma^*}, E)$ is not π-realizable.

By copying the tree to $(V_\Theta, \pi(\delta))$ and taking a countable countable elementary substructure, we can suppose that $\mathrm{Ult}(M_{\gamma^*}, E)$ is illfounded.

Define $g : \eta \to \mathrm{Ord}$ by

$$g(\beta) = \min\{\xi + 1 \mid \max\{\mu^{\mathcal{T}}(\beta), \mathrm{SPT}(E)\} \leq \xi + 1\}.$$

Thus, since $\mathcal{T}$ continued by setting $E_\gamma = E$ is a (+2)-iteration tree (modulo the wellfoundedness of $\mathrm{Ult}(M_{\gamma^*}, E)$), for all $\beta < \eta$, $g(\beta)$ is a successor ordinal and

$$\mu^{\mathcal{T}}(\beta) \leq g(\beta) \leq g(\beta) + 1 \leq \rho^{\mathcal{T}}[\beta, \beta + 1).$$

By Lemma 89 there exists a p-connected sequence

$$\langle \mathcal{E}^\alpha : \alpha < \eta \rangle$$

of g-enlargements of $\mathcal{T}$ such that there is an elementary embedding,

$$e : \mathcal{P}_0^0 \to V_\Upsilon$$

for some $\Upsilon > \Theta$.

By Lemma 88, $\mathrm{Ult}(\mathcal{P}_{\theta^*}^\theta, \pi_\theta^\theta(E))$ is wellfounded. However $\mathrm{Ult}(M_{\theta^*}, E)$ embeds into $\mathrm{Ult}(\mathcal{P}_{\theta^*}^\theta, \pi_\theta^\theta(E))$ which is a contradiction.

Thus η is a limit ordinal. By Lemma 77, we can assume that the preservation function, p, is such that there exists a sequence, $\langle \alpha_i : i < \omega \rangle$, such that

(2.1) $\sup\{\alpha_i \mid i < \omega\} = \eta$,
(2.2) for all $i < \omega$, $p(\alpha_{i+1} + 1) = \alpha_i + 1$,
(2.3) for all $i < \omega$, for all $\alpha_i + 1 < \xi \leq \alpha_{i+1}$, $\alpha_i + 1 < p(\xi)$.

Let $g : \eta \to \mathrm{Ord}$ be defined by;

$$g(\beta) = \min\{\xi + 1 \mid \mu^{\mathcal{T}}(\beta) \leq \xi + 1\}$$

for all $\beta < \eta$. Again since $\mathcal{T}$ is a (+2)-iteration tree, for all $\alpha < \eta$, $g(\alpha)$ is a successor ordinal and

$$\mu^{\mathcal{T}}(\alpha) \leq g(\alpha) < g(\alpha) + 1 \leq \rho^{\mathcal{T}}[\alpha, \alpha + 1).$$

Again, by Lemma 89 there exists a p-connected sequence

$$\langle \mathcal{E}^\alpha : \alpha < \eta \rangle$$

of g-enlargements of $\mathcal{T}$.

We claim that for all $i + 1 < \omega$, $\mathcal{P}^{\alpha_{i+1}+1}_{\alpha_{i+1}} \in \mathcal{P}^{\alpha_i+1}_{\alpha_i}$, and this is a contradiction.

Let $\beta = \alpha_i$ and let $\alpha = \alpha_{i+1}$. Thus $p(\alpha + 1) = \beta + 1 \leq \alpha$. Let

$$\theta = \pi^\alpha_\alpha(\mu^T_g[\beta, \alpha)).$$

Thus by (2b(ii)) of Definition 82,

$$\mathcal{P}^{\alpha+1}_\alpha \in \mathcal{P}^\alpha_\alpha,$$

and

$$|\mathcal{P}^{\alpha+1}_\alpha|^{\mathcal{P}^\alpha_\alpha} = |\mathcal{P}^\alpha_\alpha \cap V_{\theta+1}|^{\mathcal{P}^\alpha_\alpha}.$$

We can apply Lemma 85 to $(\beta+1, \alpha)$, since for all ordinals ξ if $\beta+1 < \xi \leq \alpha$ then $\beta + 1 < p(\xi)$. Thus, since $\theta \leq \pi^\alpha_\alpha(\mu^{\mathcal{T}}_g[\beta + 1, \alpha))$,

$$\mathcal{P}^\alpha_\alpha \cap V_{\theta+2} \subseteq \mathcal{P}^{\beta+1}_{\beta+1}.$$

We finish by showing that

$$\mathcal{P}^{\beta+1}_{\beta+1} \cap V_{\theta+2} \subseteq \mathcal{P}^{\beta+1}_\beta$$

for this will show that $\mathcal{P}^{\alpha+1}_\alpha \in \mathcal{P}^{\beta+1}_\beta$ (since $\mathcal{P}^{\alpha+1}_\alpha \in \mathcal{P}^\alpha_\alpha$ and $|\mathcal{P}^{\alpha+1}_\alpha|^{\mathcal{P}^\alpha_\alpha} = |\mathcal{P}^\alpha_\alpha \cap V_{\theta+1}|^{\mathcal{P}^\alpha_\alpha}$).

Let $\xi = \beta^*$. Thus

$$\mathcal{P}^{\beta+1}_{\beta+1} = \mathrm{Ult}(\mathcal{P}^{\beta+1}_\xi, \pi^{\beta+1}_\beta(E_\beta)),$$

and so

$$\mathcal{P}^{\beta+1}_{\beta+1} \cap V_{\pi^{\beta+1}_\beta(\rho(E_\beta))} = \mathcal{P}^{\beta+1}_\beta \cap V_{\pi^{\beta+1}_\beta(\rho(E_\beta))}.$$

Since

$$\mathcal{P}^{\beta+1}_\xi \cap V_{\mathrm{SPT}(\pi^{\beta+1}_\beta(E_\beta))+1} = \mathcal{P}^{\beta+1}_\beta \cap V_{\mathrm{SPT}(\pi^{\beta+1}_\beta(E_\beta))+1},$$

we have that

$$\mathcal{P}^{\beta+1}_{\beta+1} \cap V_{\pi^{\beta+1}_\beta(\rho(E_\beta))+1} = \mathcal{P}^{\beta+1}_\beta \cap V_{\pi^{\beta+1}_\beta(\rho(E_\beta))+1}.$$

The iteration tree, $\mathcal{T}$, is a $(+2)$-iteration tree and so (from the definition of g)

$$\mu^{\mathcal{T}}_g[\beta, \alpha) \leq g(\beta) < \rho(E_\beta).$$

Further, since $\beta + 1 < p(\gamma)$ for all $\gamma \leq \alpha$ such that $\beta + 1 < \gamma$,

$$\pi^{\beta+1}_\beta = \pi^\alpha_\beta,$$

(by induction on γ, using (2a) and (3c) of Definition 82, if $\beta + 1 < \gamma \leq \alpha$ then $\pi^{\beta+1}_\beta = \pi^\gamma_\beta$).

But $\theta = \pi^\alpha_\alpha(\mu^{\mathcal{T}}_g[\beta,\alpha]) = \pi^\alpha_\beta(\mu^{\mathcal{T}}_g[\beta,\alpha])$ and

$$\pi^\alpha_\beta(\mu^{\mathcal{T}}_g[\beta,\alpha]) + 1 \leq \pi^\alpha_\beta(\rho(E_\beta)).$$

Putting everything together,

$$\mathcal{P}^{\beta+1}_{\beta+1} \cap V_{\theta+2} \subseteq \mathcal{P}^{\beta+1}_{\beta},$$

as desired.

In summary, we have shown that

$$\mathcal{P}^{\alpha+1}_{\alpha} \in \mathcal{P}^{\beta+1}_{\beta},$$

i.e. for all $i+1<\omega$,

$$\mathcal{P}^{\alpha_{i+1}+1}_{\alpha_{i+1}} \in \mathcal{P}^{\alpha_i+1}_{\alpha_i},$$

which contradicts the wellfoundedness of these models. □

3.6. *Strong realizability*

Suppose that (M,δ) is a countable premouse,

$$\pi : M \to V_\Theta$$

is an elementary embedding, $\mathcal{T}$ is a countable (+2)-iteration tree on (M,δ), and that b is a wellfounded branch of $\mathcal{T}$. Let

$$j_b : M \to M_b$$

be the associated embedding. Let $\langle \alpha_i : i<\omega\rangle$ be an enumeration of $j_b(\delta)$ and for each $k<\omega$ define for $A \in \mathcal{P}(\delta^k) \cap M$, $\mu_k(A) = 1$ if $\langle \alpha_i : i<k\rangle \in j_b(A)$. Since $\mathcal{T}$ is a (+2)-iteration tree it follows that $\mu_k \in M$ for all k. Therefore $\langle \pi(\mu_k) : k<\omega\rangle$ is a tower of measures in V. Define a tower of measures to be a (b,π)-tower if the tower arises as above from some enumeration of $j_b(\delta)$. The wellfoundedness of a (b,π)-tower is independent of the choice of the enumeration (any two such towers are equivalent in the natural sense of the equivalence of towers).

Note that if the (b,π)-towers are wellfounded then the branch b is π-realizable.

This suggests the following definition and the subsequent variation of Theorem 66.

Definition 90. Suppose that (M,δ) is a countable premouse,

$$\pi : M \to V_\Theta$$

is an elementary embedding, $\mathcal{T}$ is a countable (+2)-iteration tree on (M,δ), and that b is a branch of $\mathcal{T}$. The branch, b, is *strongly-π-realizable* if the corresponding (π,b)-towers are wellfounded.

Suppose that (M,δ) is a countable premouse,

$$\pi : M \to V_\Theta$$

is an elementary embedding, $(V_\Upsilon, \pi(\delta))$ is a premouse with $\Theta < \Upsilon$ and $X \prec V_\Upsilon$ is a countable elementary substructure such that $\pi \in X$. Let N be the transitive collapse, let $\pi_N : (N, \delta_N) \to (V_\Upsilon, \pi(\delta))$ invert the collapse and let

$$\pi_N^M : M_\delta \to N_{\delta_n}$$

be the induced embedding. With this notation we have the following lemma.

Lemma 91. *Suppose $\mathcal{T}$ is a countable $(+2)$-iteration tree on M which copies by π_N^M to an iteration tree, $\mathcal{S}$, on N. Suppose b is a branch of $\mathcal{T}$ for which the corresponding branch of $\mathcal{S}$ is π_N-realizable. Then b is strongly-π-realizable.*

Proof. M is countable so there is a function $\rho : M \to V_{\pi(\delta)}$ such that

(1.1) if $\nu \in M_\delta$ is a measure then $\rho(\nu) \in \pi(\nu)$,

(1.2) if $\langle \nu_i : i < \omega \rangle$ is a tower of measures with $\nu_i \in M_\delta$ for all $i < \omega$ and the image tower $\langle \pi(\nu_i) : i < \omega \rangle$ is not wellfounded on V then the sequence $\langle \rho(\nu_i) : i < \omega \rangle$ witnesses that the tower, $\langle \pi(\nu_i) : i < \omega \rangle$, is not countably complete.

By elementarity we can suppose that $\rho \in X$.

Let b be a branch of $\mathcal{T}$ for which the corresponding branch, c, of $\mathcal{S}$ is π_N-realizable.

Let

$$j_b : M \to M_b$$

and let

$$k_c : N \to N_c$$

be the corresponding embeddings. Let

$$e : (M_b)_{j_b(\delta)} \to (N_c)_{k_c(\delta_N)}$$

be elementary embedding induced by π and copying.

Let $\langle \alpha_i : i < \omega \rangle$ be an enumeration of $j_b(\delta)$ and let $\langle \nu_i : i < \omega \rangle$ be the corresponding tower.

Then $\langle e(\nu_i) : i < \omega \rangle$ is the tower given by initial segments of $\langle e(\alpha_i) : i < \omega \rangle$.

Finally let $\pi_c : N_c \to V_\Upsilon$ be such that $\pi_N = \pi_c \circ k_c$. Thus for all $m < \omega$,

$$\langle \pi_c(e(\alpha_i)) : i < m \rangle \in \rho(\nu_m)$$

which implies by the definition of ρ that the tower

$$\langle \pi(\nu_i) : i < \omega \rangle$$

is wellfounded on V. □

Theorem 92. *Suppose that (M, δ) is a countable premouse,*

$$\pi : M \to V_\Theta$$

is an elementary embedding,

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable $(+2)$-iteration tree on (M, δ) and that $\mathcal{T}$ has no proper maximal strongly-π-realizable branch. Then $\eta = \gamma + 1$ and for all extenders $E \in M_\gamma \cap V_{j_{0,\gamma}(\delta)}$, *for all* $\gamma^* \leq \gamma$, *if* $\gamma^* = \gamma$ *or if* $\gamma^* < \gamma$ *and*

$$\text{SPT}(E) + 2 \leq \min\{\rho(E_\alpha) \mid \gamma^* \leq \alpha < \gamma\},$$

then $\mathrm{Ult}(M_{\gamma^*}, E)$ *is wellfounded and moreover, the corresponding branch is strongly-*π*-realizable.*

Proof. If $\mathcal{T}$ does not copy under $\pi|M_\delta$ to define an iteration tree on V then choose a premouse $(V_\Upsilon, \pi(\delta))$ such that $\mathcal{T}$ does not copy by π to define an iteration tree on $(V_\Upsilon, \pi(\delta))$. Let

$$X \prec (V_\Upsilon, \pi(\delta))$$

be a countable elementary substructure with $\pi \in X$.

Let N be the transitive collapse, let $\pi_N : (N, \delta_N) \to (V_\Upsilon, \pi(\delta))$ invert the collapse and let

$$\pi_N^M : M_\delta \to N_{\delta_n}$$

be the induced embedding.

Thus $\mathcal{T}$ does not copy by π_N^M to define an iteration tree on N. Therefore by Theorem 66, there exists a proper maximal branch, b, of $\mathcal{T}$ of length η such that $\mathcal{T}|\eta$ copies by π_N^M to an iteration tree on N and such that if c is the corresponding branch given by b, then c is π_N-realizable. Therefore by Lemma 91, b is strongly-π-realizable, a contradiction.

Therefore $\mathcal{T}$ copies by π to define an iteration tree on V. If $\mathcal{T}$ has limit length then repeating the argument just given, we again get a contradiction.

Thus $\mathcal{T}$ has successor length $\gamma + 1$. If the continuation of the tree by choice of (E, γ^*) does not copy by π to an iteration tree on V then once more, repeating the construction above, we get a contradiction.

Therefore the continuation of the tree $\mathcal{T}$ by the choice of (E, γ^*) must copy to an iteration tree on V. But this implies that $\mathrm{Ult}(M_{\gamma^*}, E)$ is wellfounded and moreover that the corresponding branch is strongly-π-realizable. □

3.7. *Strongly closed iteration trees*

Definition 93. An iteration tree, $\mathcal{T}$, is *strongly closed* if:

(1) $\mathcal{T}$ is a $(+1)$-iteration tree; and
(2) each extender, E, occurring in $\mathcal{T}$ is $\text{LTH}(E)$-strong in the model from which it is selected and $\text{LTH}(E)$ is strongly inaccessible in that model.

Suppose that $\mathcal{T}$ is an iteration tree on a premouse (M, δ) and that for each extender E occurring in $\mathcal{T}$, E is $\text{LTH}(E)$-strong in the model from which it is selected. Suppose in addition that $\text{LTH}(E)$ is not a limit of inaccessible cardinals in that model. If every extender occurring in $\mathcal{T}$ is a short extender then $\mathcal{T}$ is necessarily a $(+\omega)$-iteration tree (and more).

This can fail for iteration trees with long extenders. But as the next lemma demonstrates it does hold for strongly closed iteration trees. This is the reason for requiring that $\mathcal{T}$ be a (+1)-iteration tree in the definition above instead of simply requiring something like $\text{LTH}(E)$ not be a limit of inaccessible cardinals in the model from which it is selected.

Lemma 94. *Suppose that* (M, δ) *is a premouse and that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a strongly closed iteration tree on (M, δ). *Let*

$$\theta = \min\{\text{CRT}(E_\beta) \mid \beta + 1 < \eta\}.$$

Then $\mathcal{T}$ *is a* $(+\theta)$*-iteration tree.*

Proof. This is immediate from the definitions. For each $\alpha + 1 < \eta$,

$$\sup\{\text{SPT}(E_\beta) \mid \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\} + 1 \leq \rho(E_\alpha)$$

since $\mathcal{T}$ is a (+1)-iteration tree. But then

$$\sup\{\text{SPT}(E_\beta) \mid \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\} + \rho(E_\alpha) \leq \rho(E_\alpha)$$

since $\rho(E_\alpha)$ is strongly inaccessible in M_α. □

The next lemma is a reformulation of the natural generalization of [7, Theorem (5.6)] to the case of iteration trees with long extenders. The proof in the case of iteration trees with long extenders is the same as the proof in the case of iteration trees with no long extenders.

Our purpose is to prove the generalization of Theorem 66 to the case of countable strongly closed iteration trees on V, see Definition 93 on p. 198. As a corollary, every strongly closed iteration tree on V of length ω has a cofinal wellfounded branch, which is a special case of the results of [16].

Lemma 95. *Suppose* (V_Θ, δ) *is a premouse*, $\mathcal{T}$ *is a countable*, *strongly closed iteration tree on* (V_Θ, δ) *of length* η, *and that* $\mathcal{T}$ *has no proper*, *maximal*, *wellfounded branches. Suppose that* $p : \eta \to \eta$ *is a preservation function for* $\leq_{\mathcal{T}}$. *Then* $\mathcal{T}$ *is* p*-continuously illfounded.*

Proof. By increasing Θ if necessary we can suppose that $\text{cof}(\Theta) > \delta$.

Let $(M_0, \delta_0) = (V_\Theta, \delta)$ and let

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

Let $[\mathcal{T}]$ be the set of all $a \subseteq \eta$ such that a is a maximal subset of η which is totally ordered by the tree order.

For each $a \in [\mathcal{T}]$, define a sequence, $\langle \gamma_i^a : i < \omega \rangle$, by induction on $i < \omega$ as follows. Let γ_0^a be the least $\alpha \in a$ such that for some $\beta \in a$, $\alpha < \beta$ and α does not survive at β, if no such α exists then $\gamma_0^a = 0$. Given γ_i^a, define γ_{i+1}^a to be the least

$\beta \in a$ such that $\gamma_i^a < \beta$ and γ_i^a does not survive at β if such an ordinal β exists, otherwise $\gamma_{i+1}^a = \gamma_i^a$.

We note the following.

(1.1) If $\{\gamma_i^a \mid i < \omega\}$ is infinite then $\sup(\{\gamma_i^a \mid i < \omega\}) = \sup(a)$.
(1.2) If $\sup(a) < \eta$ then $\{\gamma_i^a \mid i < \omega\}$ is infinite.
(1.3) Suppose $\alpha, \beta \in a$, $\alpha < \beta$ and α does not survive at β. Then for some $i < \omega$ either, $\alpha = \gamma_i^a$ and $\gamma_i^a < \gamma_{i+1}^a \leq \beta$, or $\gamma_i^a < \alpha < \gamma_{i+1}^a \leq \beta$.

Therefore to show that $\mathcal{T}$ is p-continuously illfounded it suffices to construct a function

$$\pi : \eta \to \mathrm{Ord}$$

such that for all $\alpha < \eta$, $\pi(\alpha) \in M_\alpha$, and such that for all $a \in [\mathcal{T}]$,

(2.1) for all $i < \omega$, if $\gamma_i^a < \gamma_{i+1}^a$ then $j_{\gamma_i^a, \gamma_{i+1}^a}(\pi(\gamma_i^a)) > \gamma_{i+1}^a$,
(2.2) for all $\alpha \in a$, if $\alpha \notin \{\gamma_i^a \mid i < \omega\}$ then

$$\pi(\alpha) = j_{\gamma_k^a, \alpha}(\beta)$$

where $\beta = \max\{\gamma_i^a \mid \gamma_i^a < \alpha\}$ and $k = \min\{i < \omega \mid \gamma_i^a = \beta\}$.

Let $[\mathcal{T}]^*$ be the set of $a \in [\mathcal{T}]$ such that the set

$$\{\gamma_i^a \mid i < \omega\}$$

is infinite. Reducing still further it suffices to construct a function

$$\pi : \{\gamma_{k+1}^a \mid a \in [\mathcal{T}]^*, k < \omega\} \to \Theta$$

such that for all $a \in [\mathcal{T}]^*$,

(3.1) for all $k + 1 < \omega$, $j_{\gamma_{k+1}^a, \gamma_{k+2}^a}(\pi(\gamma_{k+1}^a)) > \gamma_{k+2}^a$.

Let

$$\mathcal{S} = \{\langle \gamma_i^a : i \leq k+1\rangle \mid a \in [\mathcal{T}], k < \omega, \text{ and } \gamma_k^a < \gamma_{k+1}^a\}$$

and for each $s \in \mathcal{S}$, let $\max(s) = \gamma_{k+1}^a$ where $a \in [\mathcal{T}]$, $k < \omega$ and

$$s = \langle \gamma_i^a : i \leq k+1\rangle.$$

We claim that there exists $\langle X_s : s \in \mathcal{S}\rangle$ such that for all $s \in \mathcal{S}$

(4.1) $X_s \prec M_{\max(s)}$.
(4.2) $(X_s)^c \subseteq X_s$.
(4.3) For all $t \in \mathcal{S}$, if s is a proper initial segment of t then for all $z \in X_s$,

$$j_{\max(s),\max(t)}(z) \in X_t,$$

and such that

(5.1) for all $a \in [\mathcal{T}]^*$,

$$M_a = \{j_{\max(s),a}(z) \mid z \in X_s \text{ and for some } k < \omega, s = \langle \gamma_i^a : i \leq k+1\rangle\},$$

where M_a is the limit given by the maximal branch a and where for each $\alpha \in a$,

$$j_{\alpha,a} : M_\alpha \to M_a$$

is the corresponding embedding.

The existence of the sequence $\langle X_s : s \in \mathcal{S} \rangle$ follows from the fact that $\mathcal{T}$ is a countable, strongly closed, iteration tree. First we define by induction on $\alpha < \eta$ sequences,

$$\langle X_i^\alpha : i < \omega \rangle,$$

such that for all $\alpha < \eta$,

(6.1) for all $i < \omega$, $X_i^\alpha \prec M_\alpha$, $X_i^\alpha \subseteq X_{i+1}^\alpha$, and $(X^\alpha)^c \subseteq X_i^\alpha$,
(6.2) $M_\alpha = \cup\{X_i^\alpha \,|\, i < \omega\}$,
(6.3) for all $\beta <_{\mathcal{T}} < \alpha$, for all $i < \omega$, there exists $k < \omega$ such that

$$\{j_{\beta,\alpha}(z) \,|\, z \in X_i^\beta\} \subseteq X_k^\alpha.$$

For $\alpha < \omega$ and for $i < \omega$ define $X_i^\alpha = M_\alpha$. It is easily verified by induction on $\alpha < \omega$ that

$$(M_\alpha)^c \subseteq M_\alpha.$$

Now suppose that $\langle X_i^\beta : i < \omega \rangle$ is defined for all $\beta \leq \alpha$. Suppose that $\alpha + 1 < \eta$ and let α^* be the $\leq_{\mathcal{T}}$-predecessor of $\alpha + 1$. Define

$$X_i^{\alpha+1} = \{j_{\alpha^*,\alpha+1}(f)(z) \,|\, z \in V_{\mathrm{LTH}(E_\alpha)} \cap X_{i_0+i}^\alpha, f \in X_i^{\alpha^*}\},$$

where i_0 is such that $E_\alpha \in X_{i_0}^\alpha$.

Finally suppose that $\langle X_i^\beta : i < \omega \rangle$ is defined for all $\beta < \alpha$, $\alpha < \eta$, and α is a limit ordinal. Let

$$a = \{\beta < \alpha \,|\, \beta <_{\mathcal{T}} \alpha\},$$

let $\langle \alpha_i : i < \omega \rangle$ enumerate a and let $\langle \beta_i : i < \omega \rangle$ be an increasing sequence cofinal in a.

Let $\langle m_i : i < \omega \rangle$ be an increasing sequence such that for all $i < \omega$, for all $m \leq m_i$, for all $k \leq i$, if $\alpha_k \leq \beta_i$ then

$$\{j_{\alpha_k,\beta_{i+1}}(z) \,|\, z \in X_m^{\alpha_k}\} \subseteq X_{m_{i+1}}^{\beta_{i+1}}.$$

For all $i < \omega$ define

$$X_i^\alpha = \{j_{\beta_{i+1},\alpha}(z) \,|\, z \in X_{m_{i+1}}^{\beta_{i+1}}\}.$$

Then $\langle X_i^\alpha : i < \omega \rangle$ is as required.

Let $\langle m_s : s \in \mathcal{S} \rangle$ be such that for all $s_0, s_1 \in \mathcal{S}$ if s_0 is a proper initial segment of s_1 then

(7.1) $m_{s_0} < m_{s_1} < \omega$,
(7.2) if t is an initial segment of s_0 and if $m \leq m_{s_0}$ then for all $z \in X_m^{\max(t)}$, $j_{\max(t),\max(s_1)}(z) \in X_{m_{s_1}}^{\max(s_1)}$.

Suppose that $a \in [\mathcal{T}]^*$ and for each $k < \omega$, let $m_k = m_s$ where $s = \langle \gamma_i^a : i \leq k \rangle$. Then

$$M_a = \{ j_{\gamma_k^a, a}(z) \,|\, k < \omega \text{ and } z \in X_{m_k}^{\gamma_k^a} \}$$

where M_a is the limit of $\{M_\alpha \,|\, \alpha \in a\}$ given by $\mathcal{T}$ and for each $\alpha \in a$,

$$j_{\alpha,a} : M_\alpha \to M_a$$

is corresponding elementary embedding. Finally, for each $s \in \mathcal{S}$ define

$$X_s = X_{m_s}^{\max(s)}.$$

The sequence $\langle X_s : s \in \mathcal{S} \rangle$ is as required.

For each $a \in [\mathcal{T}]$, let

$$f_a : \{ \gamma_{k+1}^a \,|\, k < \omega \text{ and } \gamma_k^a < \gamma_{k+1}^a \} \to \Theta$$

be a function which satisfies (2.1) and such that for all $k < \omega$, if $\gamma_k^a < \gamma_{k+1}^a$ then

$$f_a(\gamma_{k+1}^a) \in X_s$$

where $s = \langle \gamma_i^a : i \leq k+1 \rangle$.

Clearly f_a exists. If $\{\gamma_i^a \,|\, i < \omega\}$ is finite this is trivial. If $\{\gamma_i^a \,|\, i < \omega\}$ is infinite then a is a proper maximal branch of $\mathcal{T}$ and $\langle \gamma_i^a : i < \omega \rangle$, is an increasing cofinal sequence in a. By hypothesis, M_a is not wellfounded where M_a is the limit of the $\{M_\alpha \,|\, \alpha \in a\}$ given by $\mathcal{T}$. Therefore by (5.1), the function f_a exists.

Fix $\Theta_0 < \Theta$ such that

(8.1) for all $\alpha < \eta$, $j_{0,\alpha}(\Theta_0) = \Theta_0$,
(8.2) for all $a \in [\mathcal{T}]$, $f_a \in V_{\Theta_0}$.

Define a tree $\mathcal{U}$ as follows where for each $s \in \eta^{<\omega}$,

$$[\mathcal{T}]_s = \{ a \in [\mathcal{T}] \,|\, s = \langle \gamma_i^a : i \in \operatorname{dom}(s) \rangle \}.$$

Let $\mathcal{U}$ be the set of all finite sequences,

$$\langle (\alpha_i, F_i) : i \leq k \rangle,$$

such that the following hold.

(9.1) For all $i < k$, $\alpha_i < \alpha_{i+1}$.
(9.2) $[\mathcal{T}]_s \neq \emptyset$ where $s = \langle \alpha_i : i \leq k \rangle$.
(9.3) For all $i \leq k$, $F_i : [\mathcal{T}]_s \to \Theta_0$, where $s = \langle \alpha_m : m \leq i \rangle$.
(9.4) For all $i < k$, for all $a \in \operatorname{dom}(F_{i+1})$, $F_i(a) > F_{i+1}(a)$.

Clearly the tree, $\mathcal{U}$, is wellfounded.

For each $a \in [\mathcal{T}]$ and for each $k < \omega$ such that $\gamma_k^a < \gamma_{k+1}^a$, let

$$u_{k+1}^a = \langle (\gamma_i^a, F_i^a, m_i^a) : i \leq k+1 \rangle$$

be the finite sequence such that the following hold.

(10.1) For all $i \leq k+1$, $\operatorname{dom}(F_i^a) = [\mathcal{T}]_s$ where $s = \langle \gamma_m^a : m \leq i \rangle$.

(10.2) For all $i < k+1$, for each $b \in \mathrm{dom}(F_i^a)$, $F_i^a(b) = j_{\gamma_i^a, \gamma_{k+1}^a}(f_b(\gamma_i^a))$.
(10.3) For all $b \in \mathrm{dom}(F_{k+1}^a)$, $F_{k+1}^a(b) = f_b(\gamma_{k+1}^a)$.

By the choice of $\langle f_a : a \in [\mathcal{T}]\rangle$, and the properties of the sequence, $\langle X_s : s \in \mathcal{S}\rangle$, it follows that for all $a \in [\mathcal{T}]$, for all $k < \omega$, if $\gamma_k^a < \gamma_{k+1}^a$ then

$$u_{k+1}^a \in M_{\gamma_{k+1}^a}$$

which implies that $u_{k+1}^a \in j_{0,\gamma_{k+1}^a}(\mathcal{U})$.

Finally define

$$\pi : \{\gamma_{k+1}^a \mid a \in [\mathcal{T}]^*, k < \omega\} \to \Theta$$

such that for all $a \in [\mathcal{T}]^*$, for all $k < \omega$, $\pi(\gamma_{k+1}^a)$ is the rank of u_{k+1}^a in the tree, $j_{0,\gamma_{k+1}^a}(\mathcal{U})$.

Since for all $a \in [\mathcal{T}]^*$, for all $k < \omega$,

$$j_{\gamma_{k+1}^a, \gamma_{k+2}^a}(u_{k+1}^a) = u_{k+2}^a | (k+2),$$

it follows that for all $a \in [\mathcal{T}]^*$, π satisfies (3.1) and so is as required. □

Lemma 95 yields the generalization of Theorem 66 to countable strongly closed iteration trees on V.

Theorem 96. *Suppose that (V_Θ, δ) is a premouse and that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle$$

is a countable iteration tree on (V_Θ, δ) which is strongly closed and with no proper, maximal, wellfounded branches. Then $\eta = \gamma + 1$ and for all extenders $E \in M_\gamma \cap V_{j_{0,\gamma}(\delta)}$ such that $\rho(E)$ is strongly inaccessible in M_γ, for all $\gamma^* \leq \gamma$, if $\gamma^* = \gamma$ or if $\gamma^* < \gamma$ and

$$\mathrm{SPT}(E) + 1 \leq \min\{\rho(E_\alpha) \mid \gamma^* \leq \alpha < \gamma\},$$

then $\mathrm{Ult}(M_{\gamma^*}, E)$ *is wellfounded.*

Proof. By Lemma 94, $\mathcal{T}$ is a (+2)-iteration tree. Similarly,

$$\mathrm{SPT}(E) + 2 \leq \min\{\rho(E_\alpha) \mid \gamma^* \leq \alpha < \gamma\},$$

With these observations, the proof is identical to the proof of Theorem 66, using the main technical lemmas, Lemmas 88 and 89, but using Lemma 95 in place of Lemma 81; see pp. 193–196. □

Corollary 97. *Suppose (V_Θ, δ) is a premouse and that $\mathcal{T}$ is a countable, strongly closed iteration tree on (V_Θ, δ) of limit length. Then $\mathcal{T}$ has a proper, maximal, wellfounded branch.*

Corollary 98 (Steel, [16]). *Suppose (V_Θ, δ) is a premouse and that $\mathcal{T}$ is a strongly closed iteration tree on (V_Θ, δ) of length ω. Then $\mathcal{T}$ has a cofinal, wellfounded, branch.*

3.8. *Branch conjectures*

Martin and Steel, [7], proposed two hypotheses with regard to iteration trees on V.

(UBH) *The Unique Branch Hypothesis*:

Suppose $\mathcal{T}$ is an iteration tree on a premouse (V_Θ, δ). Then $\mathcal{T}$ does not have two distinct cofinal wellfounded branches.

(CBH) *The Cofinal Branch Hypothesis*:

Suppose $\mathcal{T}$ is an iteration tree on a premouse (V_Θ, δ). Then:

(1) If $\mathcal{T}$ has limit length then $\mathcal{T}$ has a cofinal wellfounded branch;
(2) If $\mathcal{T}$ has successor length, $\eta + 1$, then $\mathcal{T}$ can be freely extended to an iteration tree of length $\eta + 2$.

Unfortunately if there is a supercompact cardinal then these hypotheses are each false in essentially the simplest cases. Define an iteration tree on V to be *short* if no extender occurring in the iteration tree is a long extender. Both UBH and CBH refer only to iteration trees which are short.

An iteration tree, $\mathcal{T}$, is *non-overlapping* if

$$\text{LTH}(E_\alpha) \leq \text{CRT}(E_\beta)$$

for all $\alpha + 1 <_{\mathcal{T}} \beta + 1$. The iteration tree, $\mathcal{T}$, is *totally non-overlapping* if

$$j_{E_\alpha}(\text{SPT}(E_\alpha)) < \text{CRT}(E_\beta)$$

for all $\alpha + 1 <_{\mathcal{T}} \beta + 1$.

In [10], Neeman and Steel give a much simpler construction for counterexamples to both UBH and CBH than the construction given here in the proof of Theorem 99. Their construction requires much weaker large cardinal hypotheses and the counterexamples produced have the same underlying tree orders as the examples constructed here, but their counterexamples are not iteration trees which are non-overlapping. For the special case of non-overlapping iteration trees, Steel has shown that hypotheses below the level of superstrong are probably not sufficient, [18].

Theorem 99. *Suppose that there is a supercompact cardinal.*

(1) *There is a short, totally non-overlapping, $(+2)$-iteration tree on V of length ω with only two cofinal branches and each is wellfounded.*
(2) *There is a short, totally non-overlapping, $(+2)$-iteration tree on V of length $\omega \cdot \omega$ with only one cofinal branch and this branch is illfounded.*

Proof. We sketch the proof which involves some material which is a little outside the scope of this paper.

Fix $\delta_0 < \delta$ and an elementary embedding,

$$j : V \to M$$

with $\text{CRT}(j) = \delta_0$ such that $j(\delta_0) = \delta$ and such that $V_{\delta+\omega} \subseteq M$.

Since δ is supercompact, (δ_0, j) exists. It is the existence of (δ_0, j) which is all that we require. In fact we only require that $V_{\delta+2} \subseteq M$ and even this can be weakened.

It is useful to introduce some notation. Suppose that $\kappa_0 \leq \kappa_1$ are measurable cardinals. Suppose that $X_0 \subseteq V_{\kappa_0+2}$ is a set of κ_0-complete ultrafilters on V_{κ_0} and $X_1 \subseteq V_{\kappa_1+2}$ is a set of κ_1-complete ultrafilters on V_{κ_1} and that $|X_0| = |X_1|$.

A bijection

$$\pi : X_0 \to X_1$$

is a *tower isomorphism* if the following hold for all sequences

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from X_0.

(1.1) $\langle U_i : i < \omega \rangle$ is a tower if and only if $\langle \pi(U_i) : i < \omega \rangle$ is a tower.

(1.2) If $\langle U_i : i < \omega \rangle$ is a tower then the tower $\langle U_i : i < \omega \rangle$ is wellfounded if and only if the tower, $\langle \pi(U_i) : i < \omega \rangle$, is wellfounded.

The sets X_0 and X_1 are *tower isomorphic* if there exists a tower isomorphism,

$$\pi : X_0 \to X_1.$$

Suppose that there exists a Woodin cardinal γ such that

$$|X_0| < \gamma < \kappa_0$$

and that

$$\pi : X_0 \to X_1$$

is a tower isomorphism. Suppose $G \subseteq \mathbb{P}$ is V-generic for a partial order $\mathbb{P} \in V_\gamma$. Then in $V[G]$, π is a tower isomorphism where we identify the elements $U \in X_0 \cup X_1$ with the ultrafilters they generate in $V[G]$. The verification uses Lemma 147 (see p. 256) and the generic elementary embeddings associated to the stationary tower.

Another preliminary fact we shall need is the following. Again suppose $X_0 \subseteq V_{\kappa_0+2}$ is a set of κ_0-complete ultrafilters on V_{κ_0}. Suppose that κ_0 is a limit of Woodin cardinals and that $|X_0| < \kappa_0$. Then there is a set $Y_0 \subseteq V_{\kappa_0+2}$ of κ_0-complete ultrafilters on V_{κ_0} and a function,

$$e : X_0 \to Y_0$$

such that all sequences

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from X_0:

(2.1) $\langle U_i : i < \omega \rangle$ is a tower if and only if $\langle e(U_i) : i < \omega \rangle$ is a tower.

(2.2) If $\langle U_i : i < \omega \rangle$ is a tower then the tower $\langle U_i : i < \omega \rangle$ is wellfounded if and only if the tower, $\langle e(U_i) : i < \omega \rangle$, is illfounded.

As above this property of e persists to all generic extensions of V given by partial orders $\mathbb{P} \in V_{\kappa_0}$.

We fix some more notation and isolate what is really the key to the proof.

Using G which we regard as a surjection,

$$G : \omega \to V_{\delta_0+2},$$

we define a reduction map

$$\mathcal{R}_G : V_\delta \to V_{\delta_0+2}$$

as follows. Suppose that $\langle U_i : i < \omega \rangle$ is a tower of δ_0^+-complete ultrafilters from V_δ. To simplify notation and with no essential loss of generality we can suppose that for some $\kappa < \delta$, each ultrafilter U_i concentrates on $(V_\kappa)^i$. In any case necessarily (by our conventions on towers), U_0 is the principal ultrafilter concentrating on $\{\emptyset\} = (V_\kappa)^0$.

Set $\mathcal{R}_G(U_0) = U_0$. Fix $i + 1 < \omega$ and suppose

$$s = \langle G(0), a_0, G(1), a_1, \ldots, G(i), a_i \rangle$$

is such that for all $n \leq i$, $a_n \in V_\kappa$. Let

$$U_s = \{A \subseteq V_{\delta_0} \mid s \in j(A)\}.$$

Thus U_s is a δ_0-complete ultrafilter on V_{δ_0}. Since U_{i+1} is δ_0^+-complete there must exist a set $A \in U_{i+1}$ and $U \in V_{\delta_0+2}$ such that for all

$$\langle a_0, \ldots, a_i \rangle \in A,$$

if $s = \langle G(0), a_0, G(1), a_1, \ldots, G(i), a_i \rangle$ then $U_s = U$. Define for $B \subseteq (V_{\delta_0})^i$,

$$B \in \mathcal{R}_G(U_i)$$

if $\{t \in (V_{\delta_0})^{2i} \mid t|i \in B\} \in U$. Thus $\mathcal{R}_G(U_i)$ is simply the projection of U to $(V_{\delta_0})^i$ (and the only reason for not setting $\mathcal{R}_G(U_i) = U$ is in order to conform with our conventions on towers). Note that since $\langle U_i : i < \omega \rangle$ is a tower, $\langle \mathcal{R}_G(U_i) : i < \omega \rangle$ is a tower.

Let

$$\mathcal{X}_G = \{j(f)(G|i) \mid f : V_{\delta_0} \to V, i < \omega\} \prec M.$$

Thus

$$\mathcal{X}_G = \{j(f)(a) \mid f : V_{\delta_0} \to V, a \in V_{\delta_0+2}\}.$$

We claim, and this claim follows easily from the definitions, that for all towers, $\langle U_i : i < \omega \rangle$, from $V_\delta \cap \mathcal{X}_G$ consisting of δ_0^+-complete ultrafilters, the following are equivalent where N is the transitive collapse of $\mathcal{X}_G$ and where for each $i < \omega$, U_i^N is the image of U_i under the collapsing map.

(3.1) The tower $\langle \mathcal{R}_G(U_i) : i < \omega \rangle$ is wellfounded.
(3.2) The direct limit of $\langle \mathrm{Ult}(N, U_i^N) : i < \omega \rangle$ is wellfounded.

The definition of the reduction map, $\mathcal{R}_G$, only requires that $V_\delta \subseteq M$. The key consequence of $V_{\delta+2} \subseteq M$ is that there exist cofinally many $\kappa < \delta$ such that κ is measurable and such that there exists a set $Y \subset V_{\kappa+2}$ of κ-complete ultrafilters such that Y is tower isomorphic to X where X is the set of all δ_0-complete ultrafilters on V_{δ_0}. This follows by reflection in M since

$$\{j(U) \mid U \in X\} \in M$$

and since δ is superstrong in M. This is all (beyond $V_\delta \subseteq M$) that is required for the construction.

Let X be the set of all δ_0-complete ultrafilters on V_{δ_0} and let $\kappa < \delta$ be least such that

(4.1) $\delta_0 < \kappa$,
(4.2) $V_\kappa \prec V_\delta$,
(4.3) κ measurable and there exists a set $Y \subset V_{\kappa+2}$ of κ-complete ultrafilters on V_κ such that Y is tower isomorphic to X.

By the definability of κ, $\kappa \in \mathcal{X}_G$. Fix

$$Y \subseteq V_{\kappa+2}$$

such that Y is a set of κ-complete ultrafilters on V_κ such that Y is tower isomorphic to X and such that $Y \in \mathcal{X}_G$. Fix a tower isomorphism,

$$\pi : X \to Y,$$

such that $\pi \in \mathcal{X}_G$.

Since $V_\kappa \prec V_\delta$,

$$V_\kappa[G] \vDash \text{ZFC} + \text{“There is a proper class of Woodin cardinals”}.$$

Therefore in $V[G]$, if $A \subseteq \mathbb{R}^{V[G]}$ is $(<\kappa)$-weakly homogeneously Suslin then $(A, \mathbb{R}^{V[G]})^\#$ is $(<\kappa)$-weakly homogeneously Suslin.

For each partial function,

$$F : V_{\delta_0+2} \to V_{\kappa+2}$$

there is a canonical set $A_F^G \subseteq \mathbb{R}^{V[G]}$ such that A_F^G is $(<\kappa)$-weakly homogeneously Suslin in $V[G]$. The set A_F^G is the set of all $x \in \omega^\omega$ such that

$$\langle G(x(i)) : i < \omega \rangle$$

is a wellfounded tower of κ complete ultrafilters on V_κ from the range of F and here we identify G with the corresponding surjection,

$$G : \omega \to V_{\delta_0+2},$$

and as above we identify each κ-complete ultrafilter in V with the ultrafilter it generates in $V[G]$. Notice that

(5.1) if $G^* \subseteq \text{Coll}(\omega, V_{\delta_0+2})$ is V-generic and

$$V[G] = V[G^*]$$

then A_F^G and $A_F^{G^*}$ are continuously reducible to each other.

Since κ is a measurable limit of Woodin cardinals, for each set $A \subseteq \mathbb{R}^{V[G]}$ such that A is $({<}\kappa)$-weakly homogeneously Suslin in $V[G]$, there exists an injective function

$$F : V_{\delta_0+2} \to V_{\kappa+2}$$

in V such that A is continuously reducible to A^G_F.

Fix a function

$$F : V_{\delta_0+2} \to V_{\kappa+2}$$

such that $F \in \mathcal{X}_G$ and such that $((A^G_\pi, \mathbb{R}^{V[G]})^\#)^\#$ is continuously reducible to A^G_F and let Y_F be the set of κ-complete ultrafilters U on V_κ such that U is in the range of F.

Fix a set $Z \subseteq V_{\kappa+2}$ of κ-complete ultrafilters on V_κ and a surjection

$$e : Y_F \to Z$$

such that $(Z, e) \in V$ and such that in V, for all sequences

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from Y_F:

(6.1) $\langle U_i : i < \omega \rangle$ is a tower if and only if $\langle e(U_i) : i < \omega \rangle$ is a tower,
(6.2) If $\langle U_i : i < \omega \rangle$ is a tower then the tower $\langle U_i : i < \omega \rangle$ is wellfounded if and only if the tower, $\langle e(U_i) : i < \omega \rangle$, is illfounded.

As indicated above, this property of e must hold in $V[G]$. Again we can and do choose $(e, Z) \in \mathcal{X}_G$.

We now come to the key claim.

(7.1) There exists in $V[G]$ a tower,

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from Y_F such that both the towers,

$$\langle j(\mathcal{R}_G(U_i)) : i < \omega \rangle \text{ and } \langle j(\mathcal{R}_G(e(U_i))) : i < \omega \rangle,$$

are wellfounded in $V[G]$.

To verify this assume toward a contradiction that in $V[G]$ no such tower exists. We work in $V[G]$. Since (7.1) fails to hold it must be the case that for all towers

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from Y_F, the following are equivalent:

(8.1) The tower $\langle U_i : i < \omega \rangle$ is wellfounded.
(8.2) The tower $\langle j(\mathcal{R}_G(U_i)) : i < \omega \rangle$ is wellfounded.

However the following hold in V:

(9.1) For each tower, $\langle U_i : i < \omega \rangle$, of ultrafilters from Y_F, either the tower, $\langle U_i : i < \omega \rangle$, is wellfounded or the tower $\langle e(U_i) : i < \omega \rangle$ is wellfounded.

(9.2) For all towers $\langle W_i : i < \omega \rangle$ of ultrafilters from $Y_F \cup Z$, if the tower $\langle W_i : i < \omega \rangle$ is wellfounded then so is the tower, $\langle \mathcal{R}_G(W_i) : i < \omega \rangle$.

(9.3) For all towers $\langle W_i : i < \omega \rangle$ of δ_0-complete ultrafilters on V_{δ_0}, the tower $\langle W_i : i < \omega \rangle$ is wellfounded if and only if the tower $\langle j(W_i) : i < \omega \rangle$ is wellfounded.

(9.4) For all towers $\langle W_i : i < \omega \rangle$ of ultrafilters from $Y_F \cup Z$, if the tower $\langle W_i : i < \omega \rangle$ is wellfounded then so is the tower, $\langle j(\mathcal{R}_G(W_i)) : i < \omega \rangle$.

Note that (9.4) follows from (9.1)–(9.3).

For any set $W \in V$ of γ-complete ultrafilters such that $|W| < \gamma$ there exists a function

$$H : W \to V$$

in V such that for all $U \in W$, $H(U) \in U$, and such that for all towers $\langle U_i : i < \omega \rangle$ of ultrafilters from W, the tower, $\langle U_i : i < \omega \rangle$, is wellfounded if and only if there exists a function,

$$h : \omega \to V$$

such that $h|i \in H(U_i)$ for all $i < \omega$. By absoluteness this property of H must hold in all generic extensions of V given by partial orders $\mathbb{P} \in V_\gamma$.

Therefore again by absoluteness, both (9.1) and (9.4) must hold in $V[G]$ and the equivalence of (8.1) and (8.2) follows. But in V, X, Y and $\{j(U) \,|\, U \in X\}$ are mutually tower isomorphic. Therefore Y and $\{j(U) \,|\, U \in X\}$ are tower isomorphic in $V[G]$. The equivalence of (8.1) and (8.2) implies that in $V[G]$, the set A_F^G is definable in the structure,

$$\langle V[G]_{\omega+1}, A_\pi^G, \in \rangle$$

from parameters and this contradicts the choice of F.

Therefore (7.1) must hold and we will use the witness to give the counterexample for part (1) of the theorem. Before giving the details we establish a variant of (7.1) which will give the counterexample for part (2) of the theorem.

We require a definition. Suppose that

$$\langle \langle U_i^n : i < \omega \rangle : n < \omega \rangle$$

is a sequence of towers of κ-complete ultrafilters on V_κ. For each $0 < i < \omega$ let U_i^* be the κ-complete ultrafilter on V_κ given by the product,

$$U_i^0 \times \cdots \times U_i^{i-1}.$$

Thus $\langle U_i^* : i < \omega \rangle$ is naturally regarded as a tower of ultrafilters using the natural projection maps:

$$p_{i_2,i_1} : V_\kappa^{i_2 \cdot i_2} \to V_\kappa^{i_1 \cdot i_1}$$

where for $0 < i_1 < i_2 < \omega$,

$$p_{i_2,i_1}(s_1 + s_2 + \cdots + s_{i_2}) = (s_1|i_1) + (s_2|i_1) + \cdots + (s_{i_2}|i_2),$$

and so here we are violating our notational convention on towers.

It is straightforward to show that the tower $\langle U_i^* : i < \omega\rangle$ is wellfounded if and only if each of the towers, $\langle U_i^n : i < \omega\rangle$, is wellfounded. But we caution that this equivalence requires that each of the towers, $\langle U_i^n : i < \omega\rangle$, belongs to V.

The second key claim is the following.

(10.1) There exists in $V[G]$ a sequence,

$$\langle\langle U_i^n : i < \omega\rangle : n < \omega\rangle$$

of towers of ultrafilters from Y_F such that

(a) For all $n < \omega$, the tower

$$\langle j(\mathcal{R}_G(U_i^n)) : i < \omega\rangle$$

is wellfounded in $V[G]$,

(b) The tower,

$$\langle j(\mathcal{R}_G(U_i^*)) : i < \omega\rangle,$$

is not wellfounded in $V[G]$.

We work in $V[G]$. Let $B \subseteq \mathbb{R}^{V[G]}$ be the set of all $x \in \mathbb{R}^{V[G]}$ such that x codes a countable elementary substructure,

$$\sigma \prec (V[G]_{\omega+1}, (A_\pi^G, \mathbb{R}^{V[G]})^\#).$$

Since $((A_\pi^G, \mathbb{R}^{V[G]})^\#)^\#$ is continuously reducible to A_F^G, there exists a function,

$$h : \omega^{<\omega} \to Y_F,$$

which witnesses that B is homogeneously Suslin, therefore for all $x \in \mathbb{R}^{V[G]}$, $x \in B$ if and only if

$$\langle U_i : i < \omega\rangle$$

is a wellfounded tower where for each $i < \omega$, $U_i = h(x|i)$. Since $(\pi, F) \in \mathcal{X}_G$ we can choose $h \in \mathcal{X}_G[G]$. Let $T \in \mathcal{X}_G[G]$ be a tree such that h witnesses the homogeneity of T. Thus $p[T] = B$.

Let $\hat{M}$ be the transitive collapse of $\mathcal{X}_G$ and let

$$\hat{j} : V \to \hat{M}$$

be the induced elementary embedding. Let

$$\hat{k} : \hat{M} \to M$$

invert the collapsing map. Thus $\hat{k}$ is an elementary embedding, $j = \hat{k} \circ \hat{j}$, and $\mathrm{CRT}(\hat{k}) > \delta_0$ and $V_{\delta_0+2} \subseteq \hat{M}$.

For each $U \in Y_F$ let $\hat{U}$ be the image of U under the transitive collapse of $\mathcal{X}_G$ (since $F \in \mathcal{X}_G$, $Y_F \subseteq \mathcal{X}_G$ and so this makes sense).

Suppose that

$$\langle U_i : i < \omega\rangle \in V[G]$$

is a tower of ultrafilters from Y_F.

(11.1) If the tower $\langle U_i : i < \omega\rangle$ is wellfounded then so is the tower, $\langle \mathcal{R}_G(U_i) : i < \omega\rangle$.

(11.2) If the tower, $\langle \mathcal{R}_G(U_i) : i < \omega\rangle$, is wellfounded then the direct limit of

$$\langle \mathrm{Ult}(\hat{M}, \hat{U}_i) : i < \omega\rangle$$

under the natural embeddings is wellfounded.

Similarly, suppose that

$$\langle\langle U_i^n : i < \omega\rangle : n < \omega\rangle \in V[G]$$

is a sequence of towers of ultrafilters from Y_F.

(12.1) If the tower $\langle U_i^* : i < \omega\rangle$ is wellfounded then so is the tower,

$$\langle \mathcal{R}_G(U_i^*) : i < \omega\rangle.$$

(12.2) If the tower, $\langle \mathcal{R}_G(U_i^*) : i < \omega\rangle$, is wellfounded then the direct limit of

$$\langle \mathrm{Ult}(\hat{M}, \widehat{U_i^*}) : i < \omega\rangle$$

under the natural embeddings is wellfounded.

We come to the key point. Suppose $x \in \mathbb{R}^{V[G]}$ and that the direct limit of

$$\langle \mathrm{Ult}(\hat{M}, \hat{U}_i) : i < \omega\rangle$$

under the natural embeddings is wellfounded, where for each $i < \omega$, $U_i = \hat{U}$ and $U = h(x|i)$. Then x codes the sharp of a countable set $a \subseteq \mathbb{R}^{V[G]}$ such that $a = \mathbb{R}^{V[G]} \cap L(a)$. This follows by absoluteness since h witnesses that B is homogeneously Suslin and since $h \in \mathcal{X}_G$. The point is that $\hat{k}$ lifts to an elementary embedding

$$\hat{k}_G : \hat{M}[G] \to M[G]$$

and there exists $\hat{h} \in \hat{M}[G]$ such that $\hat{k}_G(\hat{h}) = h$.

Let B^* be the set of all finite sequences,

$$\langle x_0, \ldots, x_n\rangle \in (\mathbb{R}^{V[G]})^{<\omega}$$

such that for all $k \le n$, the tower

$$\langle j(\mathcal{R}_G(h(x_k|i))) : i < \omega\rangle$$

is wellfounded. This implies that the towers,

$$\langle \hat{h}(x_k|i) : i < \omega\rangle = \langle \widehat{h(x_k|i)} : i < \omega\rangle,$$

for $k \le n$, are jointly wellfounded over $\hat{M}$ (in the obvious sense).

The set B^* is definable from parameters in

$$\langle V[G]_{\omega+1}, A_\pi^G, \in\rangle.$$

We claim there must exist an infinite sequence,

$$\langle x_k : k < \omega\rangle \in V[G]$$

such that

(13.1) for all $n < \omega$, $\langle x_k : k \le n\rangle \in B^*$,

(13.2) for all $k_1 < k_2 < \omega$, $(a_{k_1})^\# \subseteq (a_{k_2})^\#$,
(13.3) $\cup\{(a_k)^\# \mid k < \omega\} \neq (\cup\{a_k \mid k < \omega\})^\#$,

where for each $k < \omega$, $a_k \subseteq \mathbb{R}^{V[G]}$ is the countable set such that x_k codes $(a_k)^\#$.

If no such sequence $\langle x_k : k < \omega\rangle$ exists then there is a wellfounded relation which definable in

$$\langle V[G]_{\omega+1}, B^*, \in\rangle,$$

and which has rank greater that $\Theta^{L(A^G_\pi)}$ and this contradicts that B^* is projective in A^G_π. Here the relevant point is that if $\langle \sigma_k : k < \omega\rangle \in V[G]$ is an increasing sequence of countable elementary substructures of

$$(V[G]_{\omega+1}, (A^G_\pi, \mathbb{R}^{V[G]})^\#)$$

then there exists a sequence $\langle x_k : k < \omega\rangle$ such that for each $k < \omega$, x_k codes σ_k and such that for all $n < \omega$, $\langle x_k : k \leq n\rangle \in B^*$.

Therefore the sequence $\langle x_k : k < \omega\rangle$ exists as specified. For each $n < \omega$ let

$$\langle U^n_i : i < \omega\rangle = \langle h(x_n|i) : i < \omega\rangle.$$

Let $\langle U^*_i : i < \omega\rangle$ be the associated tower as defined above. We claim that the tower,

$$\langle j(\mathcal{R}_G(U^*_i)) : i < \omega\rangle$$

is not wellfounded. If not then the direct limit of the iteration of $\hat{M}$ given by the sequence of towers,

$$\big\langle\langle \widehat{U^n_i} : i < \omega\rangle : n < \omega\big\rangle$$

is wellfounded and this yields an elementary embedding,

$$j^* : \hat{M}[G] \to M^*[G] \subseteq V[G]$$

such that for all $k < \omega$, $x_k \in p[j^*(\hat{T})]$, where $\hat{T}$ is the image of T under the transitive collapse of $\mathcal{X}_G[G]$. But then by the elementary of j^* and the wellfoundedness of $M^*[G]$ there must exist a sequence $\langle y_k : k < \omega\rangle \in V[G]$ such that

(14.1) for all $k_1 < k_2 < \omega$, $(a_{k_1})^\# \subseteq (a_{k_2})^\#$,
(14.2) $\cup\{(a_k)^\# \mid k < \omega\} \neq (\cup\{a_k \mid k < \omega\})^\#$,

where for each $k < \omega$, $a_k \subseteq \mathbb{R}^{V[G]}$ is the countable set such that y_k codes $(a_k)^\#$.

For each $k < \omega$, $y_k \in B$ and so $a_k \subseteq (A^G_\pi, \mathbb{R}^{V[G]})^\#$ and so

$$\cup\{(a_k)^\# \mid k < \omega\} \subseteq (A^G_\pi, \mathbb{R}^{V[G]})^\#,$$

and this is a contradiction. Therefore the tower,

$$\langle j(\mathcal{R}_G(U^*_i)) : i < \omega\rangle,$$

is not wellfounded and so the sequence of towers,

$$\langle\langle U^n_i : i < \omega\rangle : n < \omega\rangle,$$

witnesses that (10.1) holds.

We finish by outlining how (7.1) and (10.1) yield the counterexamples for the theorem. In V, for each function

$$g : \omega \to V_{\delta_0+2}$$

there is a reduction map

$$\mathcal{R}_g : V_\delta \to V_{\delta_0+2}$$

which is defined from g exactly as $\mathcal{R}_G$ is defined in $V[G]$ from G. For each such function g, let

$$\mathcal{X}_g = \{j(f)(g|i) \mid f : V_{\delta_0} \to V, i < \omega\} \prec M$$

and let M_g be the transitive collapse of $\mathcal{X}_g$. Let

$$j_g : V \to M_g$$

and

$$k_g : M_g \to M$$

be the associated elementary embeddings.

By absoluteness, using (7.1) and (10.1), there exists in V a function

$$G_0 : \omega \to V_{\delta_0+2}$$

such that

(15.1) $(F, e, Y_F, Z) \in \mathcal{X}_{G_0}$.

(15.2) There exists a tower,

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from $Y_F \cap \mathcal{X}_{G_0}$ such that both the towers,

$$\langle j(\mathcal{R}_{G_0}(U_i)) : i < \omega \rangle \text{ and } \langle j(\mathcal{R}_{G_0}(e(U_i))) : i < \omega \rangle,$$

are wellfounded.

(15.3) There exists a sequence,

$$\langle \langle U_i^n : i < \omega \rangle : n < \omega \rangle$$

of towers of ultrafilters from $Y_F \cap \mathcal{X}_{G_0}$ such that

(a) For all $n < \omega$, the tower $\langle j(\mathcal{R}_{G_0}(U_i^n)) : i < \omega \rangle$ is wellfounded.

(b) The tower $\langle j(\mathcal{R}_{G_0}(U_i^*)) : i < \omega \rangle$ is not wellfounded.

Let $(F_{G_0}, e_{G_0}, Y_F^{G_0}, Z_{G_0})$ be the image of (F, e, Y_F, Z) under the transitive collapse of $\mathcal{X}_{G_0}$. Thus we established the following where for a tower

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from M_{G_0}, we say the tower is wellfounded if the induced direct limit of the ultrapowers, $\mathrm{Ult}(M_{G_0}, U_i)$, is wellfounded. The point of course is that we are not requiring that the sequence $\langle U_i : i < \omega \rangle$ be an element of M_{G_0} (and so the equivalence with countable completeness need not hold).

(16.1) There exists a tower,

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from $Y_F^{G_0}$ such that both the towers,

$$\langle U_i : i < \omega \rangle \text{ and } \langle e_{G_0}(U_i) : i < \omega \rangle,$$

are wellfounded.

(16.2) There exists a sequence,

$$\langle \langle U_i^n : i < \omega \rangle : n < \omega \rangle$$

of towers of ultrafilters from $Y_F^{G_0}$ such that:

(a) For all $n < \omega$, the towers $\langle \langle U_i^m : i < \omega \rangle : m \leq n \rangle$ are jointly wellfounded;
(b) The tower $\langle U_i^* : i < \omega \rangle$ is not wellfounded.

From this we obtain the counterexamples witnessing the theorem, though first we produce counterexamples where the iteration trees are short and non-overlapping (not totally non-overlapping).

From [7], an iteration tree

$$\mathcal{T} = \langle N_m, E_m, j_{m,n} : m < \eta, m <_{\mathcal{T}} n \rangle.$$

on a premouse (N, δ_N) is an *alternating chain* if $\eta \leq \omega$ and if for all $0 < n < m < \eta$, $n <_{\mathcal{T}} m$ if and only if n and m are either both even or both odd. Thus if $\eta = \omega$ then $\mathcal{T}$ has exactly two cofinal branches, an even branch and an odd branch.

Fix $\delta_0 < \gamma < \Theta < \kappa_{G_0}$ such that

$$(M_{G_0} \cap V_\Theta, \gamma)$$

is a premouse such that γ is a limit of Woodin cardinals in M_{G_0} and where κ_{G_0} is the image of κ under the transitive collapse of $\mathcal{X}_{G_0}$. Note that if $\mathcal{T} \in M_{G_0}$ is a countable iteration tree on the premouse $(M_{G_0} \cap V_\Theta, \gamma)$ then $\mathcal{T}$ defines a iteration tree on M_{G_0}.

Fix a function,

$$H : Y_F^{G_0} \cup Z_{G_0} \to M_{G_0}$$

such that

(17.1) $H \in M_{G_0}$.
(17.2) $H(U) \in U$ for all $U \in Y_F^{G_0} \cup Z_{G_0}$.
(17.3) For all towers $\langle U_i : i < \omega \rangle \in M_{G_0}$ of ultrafilters from $Y_F^{G_0} \cup Z_{G_0}$, the tower is wellfounded if and only if there exists a function

$$f : \omega \to M_{G_0}$$

such that for all $i < \omega$, $f|i \in H(U_i)$.

The existence of H follows from the fact,

$$|Y_F^{G_0} \cup Z_{G_0}|^{M_{G_0}} < \kappa_{G_0},$$

since each ultrafilter $U \in Y_F^{G_0} \cup Z_{G_0}$ is κ_{G_0}-complete in M_{G_0}. One property that H has and which will need need is the following.

(18.1) Suppose that $j^* : M_{G_0} \to N^*$ is an elementary embedding and that

$$\langle\langle U_i^k : i < \omega\rangle : k < \omega\rangle$$

is a sequence of towers of ultrafilters from $j^*(Y_F^{G_0} \cup Z_{G_0})$ and that for each $k < \omega$ there is a function

$$f_k^* : \omega \to N^*$$

such that for all $i < \omega$, $f_k^*|i \in j^*(H)(U_i^k)$. Then the tower,

$$\langle U_i^* : i < \omega\rangle,$$

is wellfounded over N^*.

There are Woodin cardinals in M_{G_0} in the interval, (δ_0, κ_{G_0}) and so applying the basic construction of [7] within M_{G_0} to the set of ultrafilters, $Y_F^{G_0}$, one obtains a function

$$I : (Y_F^{G_0} \cup Z_{G_0})^{<\omega} \to M_{G_0}$$

such that $I \in M_{G_0}$ and such that the following hold.

(19.1) For all finite towers $s \in (Y_F^{G_0} \cup Z_{G_0})^{<\omega}$, $I(s)$ is a finite alternating chain on the premouse $(M_{G_0} \cap V_\Theta, \gamma)$ with all associated critical points above δ_0 and which is totally non-overlapping.

(19.2) If s_1 and s_2 are finite towers from $Y_F^{G_0} \cup Z_{G_0}$ and s_2 extends s_1 then $I(s_2)$ extends $I(s_1)$.

(19.3) For all infinite towers $\langle U_i : i < \omega\rangle \in M_{G_0}$ of ultrafilters from $Y_F^{G_0} \cup Z_{G_0}$,

  (a) the tower is wellfounded if and only if the even branch of the alternating chain of length ω given by $\{I(\langle U_i : i \leq n\rangle) \,|\, n < \omega\}$ is wellfounded,

  (b) the tower is not wellfounded if and only if the odd branch of the alternating chain of length ω given by $\{I(\langle U_i : i \leq n\rangle) \,|\, n < \omega\}$ is wellfounded.

(19.4) For all infinite towers $\langle U_i : i < \omega\rangle$ of ultrafilters from $Y_F^{G_0} \cup Z_{G_0}$, if the even branch of the alternating chain of length ω given by $\{I(\langle U_i : i \leq n\rangle) \,|\, n < \omega\}$ is wellfounded and

$$j^* : M_{G_0} \to M^*$$

is the induced elementary embedding, then there exists a function

$$f^* : \omega \to M^*$$

such that for all $i < \omega$, $f^*|i \in j^*(H(U_i))$.

We emphasize that (19.4) holds for all towers from $Y_F^{G_0} \cup Z_{G_0}$, even those towers which are not elements of M_{G_0}.

Let $\langle U_i : i < \omega \rangle$ be a tower of ultrafilters from $Y_F^{G_0}$ which witnesses that (16.1) holds. Let $\mathcal{T}_0$ be the alternating chain on $(M_{G_0} \cap V_\Theta, \gamma)$ given by applying I to the initial segments of $\langle U_i : i < \omega \rangle$. We claim that both the even and odd branches of $\mathcal{T}_0$ are wellfounded acting on M_{G_0}. Suppose not and we first suppose that the even branch is not wellfounded acting on M_{G_0}.

Let N_0 be the direct limit of $\mathrm{Ult}(M_{G_0}, U_i)$ and let

$$j_0 : M_{G_0} \to N_0$$

be the associated elementary embedding. Since $\Theta < \kappa_{G_0}$ and since each of the ultrafilters, U_i, is κ_{G_0}-complete, $j_0(\mathcal{T}_0) = \mathcal{T}_0$.

There is a function

$$f : \omega \to N_0$$

such that for all $i < \omega$, $f|i \in j_0(H)(e_{G_0}(U_i))$ and even branch of $j_0(\mathcal{T}_0)$ is not wellfounded acting on N_0. This implies that the tree of attempts to refute the property that $j_0(I)$ must have in N_0, is not wellfounded and this is a contradiction.

We next suppose that the odd branch of $\mathcal{T}_0$ is not wellfounded acting on M_{G_0}. Now let N_0 be the direct limit of $\mathrm{Ult}(M_{G_0}, e_{G_0}(U_i))$ and let

$$j_0 : M_{G_0} \to N_0$$

be the associated elementary embedding. Again we have $j_0(\mathcal{T}_0) = \mathcal{T}_0$ and in this case there is a function

$$f : \omega \to N_0$$

such that for all $i < \omega$, $f|i \in j_0(H)(e_{G_0}(U_i))$. Thus in N_0 the tree of attempts to find a tower, $\langle U'_i : i < \omega \rangle$, of ultrafilters from $j_0(Y_F^{G_0})$ and a function

$$f' : \omega \to N_0,$$

such that

(20.1) the odd branch of the alternating chain given by applying $j_0(I)$ to the initial segments of $\langle U'_i : i < \omega \rangle$ is not wellfounded acting on N_0,

(20.2) for all $i < \omega$, $f'|i \in j_0(e_{G_0}(U'_i))$.

This again contradicts the properties that $j_0(I)$ must have in N_0, since by the properties of $j_0(e_{G_0})$ and $j_0(H)$, the tower $\langle U'_i : i < \omega \rangle$ must be illfounded in N_0

The same argument shows that the odd branch of $\mathcal{T}_0$ must be wellfounded acting on M_{G_0}. The iteration tree given by j_{G_0} followed by $\mathcal{T}_0$ is a non-overlapping tree on V with exactly two cofinal branched each of which is wellfounded.

Let $\langle \langle U_i^n : i < \omega \rangle : n < \omega \rangle$ be a sequence of towers of ultrafilters from $Y_F^{G_0}$ which witnesses (16.2).

Let

$$j_0 : M_{G_0} \to N_0$$

be the elementary embedding given by $\langle U_i^0 : i < \omega\rangle$ and by induction on $k < \omega$, let

$$j_{k+1} : N_k \to N_{k+1}$$

be the elementary embedding given by the tower,

$$\langle (j_k \circ \cdots \circ j_0)(U_i^{k+1}) : i < \omega\rangle.$$

Since for all $n < \omega$, the towers

$$\langle\langle U_i^k : i < \omega\rangle : k \leq n\rangle$$

are jointly wellfounded, for each $k < \omega$, the tower

$$\langle (j_k \circ \cdots \circ j_0)(U_i^{k+1}) : i < \omega\rangle$$

is wellfounded over N_k.

Since the tower $\langle U_i^* : i < \omega\rangle$ is not wellfounded over M_{G_0}, the direct limit of the N_k under the maps, j_{k+1}, is not wellfounded.

For each $k < \omega$, there is a function

$$f_k : \omega \to N_k$$

such that for all $i < \omega$,

$$f|i \in (j_k \circ \cdots \circ j_0)(H(U_i^k)).$$

Define an iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j^{\mathcal{T}}_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \omega \cdot \omega, \gamma <_{\mathcal{T}} \alpha\rangle$$

as follows. We define $\mathcal{T}|(\omega \cdot (k+1) + 1)$ by induction on k.

Let $\mathcal{T}|(\omega + 1)$ be the iteration tree given by applying I to the initial segments of $\langle U_i^0 : i < \omega\rangle$ and then taking the even branch. Then by induction, let

$$\mathcal{T}|(\omega \cdot (k+2) + 1) = \mathcal{T}|(\omega \cdot (k+1) + 1) + \mathcal{S},$$

where $\mathcal{S}$ is the iteration tree given by applying $j^{\mathcal{T}}_{0,\omega\cdot(k+1)}(I)$ to the initial segments of the tower,

$$\langle j^{\mathcal{T}}_{0,\omega\cdot(k+1)}(U_i^{k+1}) : i < \omega\rangle$$

and then taking the even branch.

If at some stage k, the definition of $\mathcal{T}|(\omega \cdot (k+1) + 1)$ fails (because of illfoundedness) then for all $n \geq 0$, the construction must fail over N_n using

$$j_n \circ \cdots \circ j_0(I)$$

and the towers, $\langle\langle j_n \circ \cdots \circ j_0(U_i^m) : i < \omega\rangle : m \leq k\rangle$.

But (taking $n = k+1$) this yields that for some $m \leq k$ there exists an elementary embedding,

$$j^* : N_{k+1} \to N^*,$$

such that even branch of the iteration tree on N^* given by applying

$$j^* \circ j_{k+1} \circ \cdots \circ j_0(I)$$

to the initial segments of the tower,

$$\langle j^* \circ j_{k+1} \circ \cdots \circ j_0(U_i^m) : i < \omega \rangle,$$

is not wellfounded (over N^*). But for all $i < \omega$,

$$j^* \circ j_{k+1} \circ \cdots \circ j_{m+1}(f_m)|i \in j^* \circ j_{k+1} \circ \cdots \circ j_0(H(U_i^m))$$

and this is a contradiction.

Thus the definition of $\mathcal{T}$ succeeds to define an iteration tree on M_{G_0} of length $\omega \cdot \omega$. The tree $\mathcal{T}$ has only one cofinal branch. We must show that this branch is not wellfounded. Assume toward a contradiction that this branch is wellfounded and let

$$j^* : M_{G_0} \to M^*$$

be the associated elementary embedding. By the key property (19.4) of I, applied in $M_{\omega\cdot(k+1)}$ to $j^{\mathcal{T}}_{0,\omega\cdot(k+1)}(I)$, it follows that for each $k < \omega$, there is a function

$$f_k^* : \omega \to M^*$$

such that for all $i < \omega$,

$$f_k^*|i \in j^*(H)(U_i^k).$$

By (18.1) this implies that the tower,

$$\langle j^*(U_i^*) : i < \omega \rangle$$

is wellfounded over M^* and this contradicts that the tower,

$$\langle U_i^* : i < \omega \rangle,$$

is not wellfounded over M_{G_0}.

The trees $\mathcal{T}_0$ and $\mathcal{T}$ are each short totally non-overlapping trees on M_{G_0} with all critical points above δ_0. Further

$$M_{G_0} = \mathrm{Ult}(V, E_0)$$

where E_0 is an extender with $\text{CRT}(E_0) = \delta_0$ and with

$$\text{LTH}(E_0) \leq |M_{G_0} \cap V_{\delta_0+2}|^{M_{G_0}}.$$

The difficulty is that $j_{G_0}(\delta_0) > \kappa_{G_0}$ and these trees are each based on the premouse, $(M_{G_0} \cap V_\Theta, \gamma)$ and so while the induced iteration trees on V are necessarily non-overlapping, the induced iteration trees on V are not totally non-overlapping.

There is a function

$$f : \delta_0 \to \delta_0$$

such that for all $\alpha < \delta_0$ and for all $\gamma < \delta_0$, for all $W \subseteq V_{\gamma+2}$ of γ-complete ultrafilters on V_γ, $\gamma < \delta_0$ and for all $W \subseteq V_{\gamma+2}$ of γ-complete ultrafilters on V_γ if

$|W| \leq |V_{\alpha+2}|$ and if $f(\alpha) < \gamma$ then for cofinally many $\gamma^* < \delta_0$ there exists a set W^* of γ^*-complete ultrafilters on V_{γ^*} such that W is tower isomorphic to W^*. Clearly we can choose f such that f is definable in V_{δ_0} and so by choice of κ,

$$\kappa_{G_0} > j_{G_0}(f)(\delta_0).$$

Note that $j_{G_0}(\delta) = \delta$ and so

$$j_{G_0}(V_{\delta_0}) \prec j_{G_0}(V_\delta) = M_{G_0} \cap V_\delta.$$

This implies that in M_{G_0}, for cofinally many $\gamma < \delta$ there exist sets $W_0^\gamma, W_1^\gamma \in M_{G_0}$ such that

(21.1) $W_0^\gamma \subseteq M_{G_0} \cap V_{\gamma+2}$ and W_0^γ is tower isomorphic in M_{G_0} with $Y_F^{G_0}$,
(21.2) $W_1^\gamma \subseteq M_{G_0} \cap V_{\gamma+2}$ and W_1^γ is tower isomorphic in M_{G_0} with Z_{G_0}.

By choosing $\gamma > j_{G_0}(\delta_0)$ sufficiently large so that there are Woodin cardinals in M_{G_0} in the interval, $(j_{G_0}(\delta_0), \gamma)$, and using (W_0^γ, W_1^γ) in place of $(Y_F^{G_0}, Z_{G_0})$ one produces $\mathcal{T}_0$, $\mathcal{T}$ such that the induced iteration trees on V are each totally non-overlapping. □

If δ is supercompact then the counterexamples of Theorem 99 can easily be constructed (following the proof of Theorem 99) such that for a given set $\mathcal{E} \subset V_\delta$ of extenders which witnesses that δ is a Woodin cardinal and which is closed under initial segments, each extender, E, of the iteration tree, except for the first extender E_0, has the following properties in the model from which E is selected.

(1) $E \in \mathcal{E}^*$;
(2) $\text{LTH}(E) = \rho(E)$ and $\text{LTH}(E)$ is strongly inaccessible;

where $\mathcal{E}^*$ is the image of $\mathcal{E}$ in that model. Further (as in the proof of Theorem 99) E_0 can be chosen to be very "short":

(3) $\text{LTH}(E_0) \leq (2^{2^\kappa})^{\text{Ult}(V,E_0)}$ where $\kappa = \text{CRT}(E_0)$.

Let $\mathcal{F}_\mathcal{E}$ be the set of all short extenders $F \in V_\delta$ such that F satisfies (2) and such that if $\gamma = \rho(F)$, then

$$j_F(\mathcal{E}) \cap V_\gamma = \mathcal{E} \cap V_\gamma$$

and $(V_\gamma, \mathcal{E} \cap V_\gamma) \prec (V_\delta, \mathcal{E})$.

In the case of the counterexample to UBH still more can be required and this also follows from the proof of Theorem 99. If $\mathcal{T}$ is the iteration tree on $\text{Ult}(V, E_0)$ with exactly two cofinal branches, b and c, each of which are wellfounded, then there exists an extender $F_0 \in \mathcal{F}_\mathcal{E}$, and elementary embeddings,

$$\pi_b : \text{Ult}(V, E_0) \to \text{Ult}(V, F_0)$$

and

$$\pi_c : \text{Ult}(V, E_0) \to \text{Ult}(V, F_0)$$

each determined by their restrictions to $\text{LTH}(E_0)$ such that

(4) $\mathcal{T}$ copies by π_b to an iteration tree on $\text{Ult}(V, F_0)$ for which c copies to an illfounded branch,
(5) $\mathcal{T}$ copies by π_c to an iteration tree on $\text{Ult}(V, F_0)$ for which b copies to an illfounded branch,
(6) $\mathcal{T}$ copied by π_b yields an iteration tree on V (with first extender given by F_0) which is totally non-overlapping,
(7) $\mathcal{T}$ copied by π_c yields an iteration tree on V (with first extender given by F_0) which is totally non-overlapping.

The point here is that in the counterexample to UBH, the two wellfounded branches are wellfounded in essentially the strongest possible sense (their wellfoundedness is *certified* by the additional extender F_0 together with the embeddings, π_b and π_c). In particular if (V_Θ, δ) is a premouse, these branches copy to wellfounded branches on any premouse (M, δ_M) under any Σ_0-embedding,

$$\pi : V_\delta \to M_{\delta_M}$$

(chosen in any extension of V). This is inherited by all elementary substructures of (V_Θ, δ) which contain $(E_0, \mathcal{T}, F_0)$.

In contrast to this stronger version of Theorem 99, Steel [18] has proved that in any iterable Mitchell–Steel extender model, if δ is strongly inaccessible and $\mathcal{E} \subseteq V_\delta$ is the set of the extenders in V_δ which are the initial segment of some extender on the sequence, then UBH *holds* for all countable non-overlapping $(+2)$-iteration trees based on $(V_\Theta, \delta, \mathcal{E})$ where $\Theta > \delta$ and (V_Θ, δ) is a premouse. Because of the shortness of E_0, Steel's proof easily adapts to rule out the kind of strong counterexample to UBH indicated above.

It is not known if UBH can fail for strongly closed iteration trees even in the case of iteration trees with no long extenders and which are totally non-overlapping.

We state three iteration hypotheses for V in increasing strength. It is possible that these hypotheses require restricting to a special class of strongly closed iteration trees, which we define below.

Definition 100. Suppose that (M, δ) is a premouse. An iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

on (V_Θ, δ) is a *restricted iteration tree* if for all $\alpha < \eta$, if $\alpha^* < \alpha$ then

$$\text{SPT}(E_\alpha) + 1 \leq \min\{j_{E_\beta}(\text{CRT}(E_\beta)) \mid \alpha^* \leq \beta < \alpha\}$$

Definition 101 (Strong $(\omega_1 + 1)$-Iteration Hypothesis). Suppose that (M, δ) is a countable premouse and that

$$\pi : M \to V_\Theta$$

is an elementary embedding. Then (M, δ) has an iteration strategy of order $\omega_1 + 1$ for strongly closed iteration trees on (M, δ).

A natural maximal version of the Strong $(\omega_1 + 1)$-Iteration Hypothesis is the following Strong Iteration Hypothesis.

Definition 102 (Strong Iteration Hypothesis). Suppose that (M, δ) is a premouse, $\kappa < \delta$, and that

$$\pi : M \to V_\Theta$$

is an elementary embedding such that there is a strong cardinal below $\pi(\kappa)$.

Suppose that there is a proper class of Woodin cardinals. Then (M, δ) has an iteration strategy of order ω_1 which is universally Baire in the codes, for strongly closed iteration trees with all critical points above κ.

Definition 103 (Strong Unique Branch Hypothesis). Suppose that (V_Θ, δ) is a premouse that $\mathcal{T}$ is a countable strongly closed iteration tree on (V_Θ, δ) of limit length. Then $\mathcal{T}$ as at most one cofinal wellfounded branch.

Theorem 104. *Suppose that (V_Θ, δ) is a premouse and that Strong Unique Branch Hypothesis holds.*

(1) *Suppose that $\mathcal{T}$ is a countable strongly closed iteration tree on (V_Θ, δ) of limit length. Then $\mathcal{T}$ has a cofinal wellfounded branch.*
(2) *Suppose that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta + 1, \beta + 1 < \eta + 1, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable strongly closed iteration tree on (V_Θ, δ). Suppose $\eta^* < \eta$ and that

$$\text{SPT}(E_\eta) + 1 \leq \min\{j_{E_\beta}(\text{CRT}(E_\beta)) \mid \eta^* \leq \beta < \eta\}.$$

Then $\mathrm{Ult}(M_{\eta^*}, E_\eta)$ *is wellfounded.*

Definition 105. Suppose that (V_Θ, κ) is a premouse and that $X \prec (V_\Theta, \kappa)$ is a countable elementary substructure with transitive collapse (M, κ_M). Let

$$\pi : M \to V_\Theta$$

invert the transitive collapse.

Suppose $\delta_M < \kappa_M$. Then $\mathcal{G}(\pi, M, \delta_M, \kappa_M)$ is the set of all $(\mathcal{T}, b)$ such that

(1) $\mathcal{T}$ is a strongly closed iteration tree on (M, κ_M) of countable length and with all critical points above δ_M,
(2) $\mathcal{T}$ can be copied by π to an iteration tree, $\mathcal{T}_\pi$, on (V_Θ, κ),
(3) b is a cofinal branch of $\mathcal{T}$ such that the induced cofinal branch b_π of $\mathcal{T}_\pi$ is wellfounded.

Note that (3) is trivially implied by (2) if $\mathcal{T}$ has successor length.

The following lemma is a corollary of the proof of Lemma 147 on p. 256.

Lemma 106. *Suppose that δ is a strong cardinal, (V_Θ, κ) is a premouse with $\kappa > \delta$, and that $X \prec (V_\Theta, \kappa, \delta)$ is a countable elementary substructure with transitive collapse (M, κ_M, δ_M). Let*

$$\pi : M \to V_\Theta$$

invert the transitive collapse and let A be the set of $x \in \mathbb{R}$ such that x codes an element of $\mathcal{G}(\pi, M, \kappa_M, \delta_M)$. Then A is universally Baire.

Proof. Let $\gamma < \delta$ be a Woodin cardinal such that for all $A \subseteq \mathbb{R}$, A is $(<\gamma)$-universally Baire if and only if A is $(<\delta)$-universally Baire. The least such γ is definable in V_δ and so we can suppose that $\gamma \in X$.

Let $\mathbb{Q}_{<\gamma}$ be the stationary tower given by the (generalized) stationary sets $a \in V_\gamma$ such that $a \subseteq \mathcal{P}_{\omega_1}(\cup a)$; [4].

Suppose that $g \subseteq \mathbb{Q}_{<\gamma}$ is V-generic and let

$$j : V \to N \subseteq V[g]$$

be the associated generic elementary embedding. By decreasing Θ if necessary we can suppose that $j(\Theta) = \Theta$ (if Θ_0 is least such that (V_{Θ_0}, κ) is a premouse then $j(\Theta_0) = \Theta_0$ and if $\Theta_0 < \Theta$ then $\Theta_0 \in X$).

In $V[g]$, let

$$(\mathcal{G}(\pi, M, \kappa_M, \delta_M))^{V[g]}$$

denote the set of $(\mathcal{T}, b)$ such that

(1.1) $\mathcal{T}$ is a strongly closed iteration tree on (M, κ_M) of countable length and with all critical points above δ_M,

(1.2) $\mathcal{T}$ can be copied by π to an iteration tree, $\mathcal{T}_\pi$, on (V_Θ, κ),

(1.3) b is a cofinal branch of $\mathcal{T}$ such that the induced cofinal branch b_π of $\mathcal{T}_\pi$ is wellfounded — this is trivially implied by (2) if $\mathcal{T}$ has successor length.

So we are using the definition of $\mathcal{G}(\pi, M, \kappa_M, \delta_M)$ but applying it to iteration trees on (M, δ_M) which lie in $V[g]$.

The key claim is the following:

(2.1) $j(\mathcal{G}(\pi, M, \kappa_M, \delta_M)) = (\mathcal{G}(\pi, M, \kappa_M, \delta_M))^{V[g]}$.

We sketch why (2.1) holds. Suppose that $\mathcal{T}$ is an iteration tree on (M, κ_M, δ_M) and that $\mathcal{T}$ copies by π to an iteration tree, $\mathcal{T}_\pi$, on $(V_\Theta, \kappa, \delta)$. Then it is easily verified that $\mathcal{T}_\pi$ lifts to define an iteration tree, $\mathcal{T}_\pi^g$, on $(V_\Theta[g], \kappa, \delta)$.

Now suppose that $\mathcal{T}$ is strongly closed. Then by induction of the length of $\mathcal{T}$ it follows from the proof of Lemma 147, that $\mathcal{T}$ copies by $j(\pi)$ to an iteration tree,

$\mathcal{T}_{j(\pi)}$, which is simply the restriction of $\mathcal{T}^g_\pi$ to $j(V_\Theta)$ in the following sense: For each $\beta + 1 < \text{length}(\mathcal{T})$, if $\alpha = \beta^*$, the $\mathcal{T}$-predecessor of $\alpha + 1$, and if

$$j^{\mathcal{T}^g_\pi}_{\alpha,\beta+1} : M^{\mathcal{T}^g_\pi}_\alpha \to M^{\mathcal{T}^g_\pi}_{\beta+1}$$

and

$$j^{\mathcal{T}_{j(\pi)}}_{\alpha,\beta+1} : M^{\mathcal{T}_{j(\pi)}}_\alpha \to M^{\mathcal{T}_{j(\pi)}}_{\beta+1}$$

are the associated embeddings then

$$M^{\mathcal{T}_{j(\pi)}}_\alpha = j^{\mathcal{T}^g_\pi}_{0,\alpha}(N \cap V[g]_\Theta)$$

and

$$j^{\mathcal{T}_{j(\pi)}}_{\alpha,\beta+1} = j^{\mathcal{T}^g_\pi}_{\alpha,\beta+1} \,|\, j^{\mathcal{T}^g_\pi}_{0,\alpha}(N \cap V[g]_\Theta).$$

This is where Lemma 147 and strong closure is used. The only difference between the situation here and that of the hypothesis of Lemma 147, is that the strong extender, E, is not being applied to the model from which it is selected. But there is enough agreement between the two models so that exactly the same analysis applies. The point is that since $\mathcal{T}$ is a strongly closed iteration tree, every possible measure generated by E and an element of $V_{\text{LTH}(E)}$ *is* an element of the model to which E is being applied.

It follows from (2.1) that the set A is $(<\gamma)$-universally Baire in V and so by choice of γ, A is universally Baire in V. The argument for this is a relatively standard one for establishing that sets are $(<\gamma)$-universally Baire using the stationary towers associated to the Woodin cardinal, γ; [19] and [4]. □

Theorem 107. *Suppose that the Strong Unique Branch Hypothesis holds and that δ_0 is a supercompact cardinal. Then the Strong Iteration Hypothesis holds at all strong cardinals $\delta \geq \delta_0$.*

Proof. Suppose that $\delta > \delta_0$ is a strong cardinal, (V_Θ, κ) is a premouse with $\kappa > \delta$, and that $X_0 \prec (V_\Theta, \kappa, \delta)$ is a countable elementary substructure with transitive collapse $(M_0, \kappa_{M_0}, \delta_{M_0})$. Let

$$\pi_0 : M_0 \to V_\Theta$$

invert the transitive collapse. We must produce an ω_1-iteration strategy I_0 for (M_0, κ_{M_0}) for countable strongly closed iteration trees on $(M_0, \kappa_{M_0}, \delta_{M_0})$ with all critical points above δ_{M_0} such that $I_0 \in L(A, \mathbb{R})$ for some universally Baire set A. This suffices since there is a supercompact cardinal.

Let $X \prec (V_\Theta, \kappa, \delta, \delta_0)$ be a countable elementary substructure with transitive collapse $(M, \kappa_M, \delta_M, \delta_0^M)$ such that $\pi_0 \in X$. Let

$$\pi : M \to V_\Theta$$

invert the transitive collapse and let A be the set of $x \in \mathbb{R}$ such that x codes an element of $\mathcal{G}(\pi, M, \kappa_M, \delta_M)$. Then by Lemma 106, A is universally Baire.

Suppose that $\mathcal{T}$ is a countable strongly closed iteration tree on $(V_\Theta, \kappa, \delta)$ with all critical point above δ_0.

By the Strong Unique Branch Hypothesis and Theorem 104, $\mathcal{T}$ has a unique cofinal wellfounded branch. This specifies an ω_1-iteration strategy I for (M, κ_M, δ_M). Clearly I is definable

$$(H(\omega_1), A)$$

from $(M, \kappa_M, \delta_M, \delta_0^M)$ and so $I \in L(A, \mathbb{R})$. Since $\pi_0 \in X$, there is an ω_1-iteration strategy I_0 as required which is definable from I and the image of π_0 under the transitive collapse of X. Clearly $I_0 \in L(A, \mathbb{R})$. $\square$

Theorem 108. *Suppose that δ_0 is supercompact and that the Strong Unique Branch Hypothesis hold. Suppose that (V_Θ, κ) is a premouse with $\delta_0 < \kappa$. Then for each ordinal γ there there is an iteration strategy for (V_Θ, κ) of order γ restricting to iteration trees with all critical points above δ_0.*

Proof. Fix $\gamma > \delta_0$ such that $\gamma > \Theta$ and fix an elementary embedding,

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \delta_0$, $j(\delta_0) > \gamma$, and such that $M^\gamma \subset M$.

Thus $j|V_\Theta \in M$. It suffices to show that the Strong Unique Branch Hypothesis holds in $M[G]$ relative to $j(\delta_0)$, in the obvious sense, for all countable strongly closed iteration trees on $(j(V_\Theta), j(\kappa))$ with all critical points above $j(\delta_0)$ where $G \subset \mathrm{Coll}(\omega, \Theta)$ is M-generic.

Fix a Woodin cardinal δ of M such that $\gamma < \delta < j(\delta_0)$. Let $g \subset (\mathbb{Q}_{<\delta})^M$ be V-generic with $G \in M[g]$ and let

$$j_g : M \to M_g \subset M[g]$$

be the associated generic elementary embedding.

Exactly as in the proof of Lemma 106 using Lemma 147, in $M[g]$ all countable strongly closed iteration trees on $(j(V_\Theta), j(\kappa))$ with critical points above $j(\delta_0)$ copy under j_g to strongly closed iteration trees on M_g. Since M_g is closed under ω-sequences in $M[g]$ and since the Strong Unique Branch Hypothesis holds in M, it follows by the elementarity of j_g that the Strong Cofinal Branch Hypothesis holds in $M[G]$ for strongly closed restricted iteration trees with all critical points above $j(\delta_0)$. This gives the required iteration strategy of order γ for the premouse (V_Θ, κ), restricting to strongly closed iteration trees with all critical points above δ_0. $\square$

The following theorem was one of the original motivations for the Ω Conjecture, the proof is straightforward using the genericity iterations associated to extender algebras.

Theorem 109. *Suppose that there is a proper class of Woodin cardinals, there is a strong cardinal, and that the Strong Iteration Hypothesis holds. Then the* Ω *Conjecture holds.*

Neeman [11] has proved an interesting genericity result which suggests the following genericity conjecture. This conjecture could be relevant to proving Strong Unique Branch Hypothesis.

Definition 110 (Genericity Hypothesis). Suppose that (V_Θ, δ) is a premouse and that Θ is a limit of Woodin cardinals. Suppose M is an iterate of V_Θ by a countable strongly closed iteration tree on (V_Θ, δ) and $a \subset \Theta$ is bounded and countable. Then a is set generic over M.

The results of [11] prove the Genericity Hypothesis for non-overlapping short iteration trees whose initial segments satisfy UBH.

We end this section by stating a technical special case of the Strong Unique Branch Hypothesis (which we shall prove in the sequel to this paper) and two further variations of unique branch hypotheses.

The proof of Theorem 112 is a relatively standard argument using term relations and the fact if b and c are two cofinal wellfounded branches of an iteration tree $\mathcal{T}$ on a premouse (M, δ) with limit models M_b and M_c respectively, then θ is a Woodin cardinal in

$$M_b \cap M_c,$$

where $\theta = \theta_b = \theta_c$. This is the fundamental connection between the failure of the existence of unique cofinal wellfounded branches and Woodin cardinals. For iteration trees with short extenders this is proved in [7] and using Lemma 68, the same proof works in the case of iteration trees with long extenders. The reduction to the situation of term relations uses Lemma 106 on p. 222.

The statement of Theorem 112, requires the following definition where, as above, if b is a maximal wellfounded branch of an iteration tree,

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle,$$

on a countable premouse, (M, κ_M), then θ_b denotes the lim-inf of the critical points along b,

$$\theta_b = \sup\{\min\{\mathrm{CRT}(E_\alpha) \mid \alpha + 1 \in b, \beta < \alpha\} \mid \beta \in b\}.$$

Definition 111. Suppose that (M, κ_M) is a countable premouse and that

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is a countable iteration tree on (M, κ_M) of limit length. A cofinal wellfounded branch b of $\mathcal{T}$ is of *Type I* if there exists $\theta \in M \cap \mathrm{Ord}$ such that the following hold where

$$j_b : M \to M_b$$

is the elementary embedding given by b.

(1) $j_b(\theta) = \theta_b$.
(2) For all $\alpha < \eta$, $E_\alpha \in M_\alpha \cap V_{j_{0,\alpha}(\theta)}$.

Otherwise the branch b is of *Type II*.

The proof of Theorem 99 easily adapts to produce a counterexample to UBH where the two wellfounded branches are each of Type I.

Suppose that δ is supercompact and that $\mathcal{T}$ is a countable iteration tree on V which is strongly closed and with critical points above δ. Suppose that b and c are cofinal wellfounded branches of $\mathcal{T}$ with limit models M_b and M_c respectively. Let

$$M = M_b \cap V_\theta = M_c \cap V_\theta$$

where $\theta = \theta_b = \theta_c$. Then b and c are *similar* if for every strongly closed iteration tree, $\mathcal{T}^*$, on M with critical points above δ, $\mathcal{T}^*$ induces an iteration tree on M_b if and only if $\mathcal{T}^*$ induces an iteration tree on M_c.

Theorem 112. *Suppose that (V_Θ, κ) is a premouse, $\delta < \kappa$, δ is a supercompact cardinal, and that Θ is a limit of Woodin cardinals. Suppose the Genericity Hypothesis holds for strongly closed iteration trees on (V_Θ, κ) and cofinal wellfounded branches of Type I. Then the Strong Unique Branch Hypothesis holds for all strongly closed iteration trees on (V_Θ, κ) for all pairs of similar cofinal wellfounded branches of Type I.*

We end this section with two more versions of the unique branch hypothesis which might hold for all strongly closed iteration trees on V even if the results of [22] can be extended to refute, assuming the existence of a proper class of supercompact cardinals, that the unique branch hypothesis holds for all strongly closed iteration trees on V.

Definition 113 (Revised Unique Branch Hypothesis). Suppose that $\mathcal{T}$ is a countable strongly closed iteration tree of limit length on a premouse (V_Θ, δ). Suppose that b and c are cofinal wellfounded branches of $\mathcal{T}$ of Type I. Then $b = c$.

Definition 114 (Strong Revised Unique Branch Hypothesis). Suppose that $\mathcal{T}$ is a countable strongly closed iteration tree of limit length on a premouse (V_Θ, δ). Suppose that b and c are cofinal wellfounded branches of $\mathcal{T}$ of the same type. Then $b = c$.

4. Generalized Martin–Steel Extender Sequences

4.1. *Martin–Steel extender sequences*

An important precursor to the fine structural models of Mitchell–Steel of [8] are the Martin–Steel inner models of [7], these extender models are of the form $L[\tilde{E}]$ where

$$\tilde{E} \subseteq (\mathrm{Ord} \times \mathrm{Ord}) \times V$$

is a predicate defining a sequence of (total) extenders. The predicate $\tilde{E}$ is defined such that for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$, the set,

$$\{a \in V \mid ((\alpha, \beta), a) \in \tilde{E}\},$$

is an extender which we denote by E^α_β. In the case of the Martin–Steel inner models, the extender E^α_β is the (κ, α)-extender derived from an elementary embedding

$$j : V \to M$$

such that $\mathcal{P}^\omega(\alpha) \subseteq M$ and such that $\alpha < j(\kappa)$.

For $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$, $\tilde{E}|(\alpha, \beta)$ is the extender sequence given by restricting $\tilde{E}$ to the set of all (η, γ) such that $(\eta, \gamma) <_{\mathcal{L}} (\alpha, \beta)$ in the lexicographical ordering of pairs of ordinals:

$$\tilde{E}|(\alpha, \beta) = \{((\eta, \gamma), a) \in \tilde{E} \mid (\eta, \gamma) <_{\mathcal{L}} (\alpha, \beta)\}.$$

Definition 115. An extender sequence,

$$\tilde{E} = \langle E^\alpha_\beta : (\alpha, \beta) \in \mathrm{dom}(\tilde{E})\rangle$$

is a *Martin–Steel extender sequence* if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$:

(1) (Coherence) There exists an extender F such that
   (a) $\alpha + \omega \leq \rho(F)$,
   (b) $E^\alpha_\beta = F|\alpha$,
   (c) (shortness) $\alpha \leq j_F(\mathrm{CRT}(F))$,
   (d) $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha, \beta)$.

(2) (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^\alpha_{\beta^*} \cap L[\tilde{E}|(\alpha, \beta)] \neq E^\alpha_\beta \cap L[\tilde{E}|(\alpha, \beta)].$$

(3) (Initial Segment Condition) Suppose that

$$\kappa < \alpha^* < \alpha$$

where κ is the critical point associated to E^α_β.

Then there exists β^* such that $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{E})$ and such that

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^* + 1, 0)] = (E^\alpha_\beta|\alpha^*) \cap L[\tilde{E}|(\alpha^* + 1, 0)].$$

The Martin–Steel extender models are actually defined in [7] as $L[P]$ where P is a predicate defined from a sequence of sets of extenders. Such sequences are called Doddages and the approach of constructing extender models from Doddages has the advantage that the resulting inner model can be ordinal definable.

Definition 116. A *Doddage* is a sequence $\tilde{\mathcal{E}}$ such that

$$\mathrm{dom}(\tilde{\mathcal{E}}) \subseteq \mathrm{Ord} \times \mathrm{Ord}$$

and such that for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, $\tilde{\mathcal{E}}(\alpha, \beta)$ is a set of extenders of length α.

Definition 117. Suppose that $\tilde{\mathcal{E}}$ is a Doddage. Then $L[\tilde{\mathcal{E}}]$ denotes $L[P_{\tilde{\mathcal{E}}}]$ where $P_{\tilde{\mathcal{E}}}$ is the set of all (α, β, s, a) such that

(1) $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$,
(2) $s \in [\alpha]^{<\omega}$,
(3) $a \in E(s)$ for all $E \in \tilde{\mathcal{E}}(\alpha, \beta)$.

Suppose $\tilde{\mathcal{E}}$ is a Doddage. For each $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ we denote $\tilde{\mathcal{E}}(\alpha, \beta)$ by $\mathcal{E}^\alpha_\beta$. For each ordinal, δ, $o^{\tilde{\mathcal{E}}}_{\text{LONG}}(\delta)$ is the supremum of the ordinals, $\mathrm{LTH}(E)$, where

(1) $j_E(\mathrm{CRT}(E)) = \delta$,
(2) $\delta < \mathrm{LTH}(E)$,
(3) $E \in \mathcal{E}^\alpha_\beta$ for some $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$.

Definition 118. A Doddage,

$$\tilde{\mathcal{E}} = \langle \mathcal{E}^\alpha_\beta : (\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}) \rangle$$

is a *Martin–Steel Doddage* if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and for each extender $E \in \mathcal{E}^\alpha_\beta$,

(1) (Coherence) There exists an extender F such that
   (a) $\mathrm{CRT}(F) < \alpha \leq j_F(\mathrm{CRT}(F))$ and $\alpha + \omega \leq \rho(F)$.
   (b) $E = F|\alpha$.
   (c) $j_F(\tilde{\mathcal{E}})|(\alpha + 1, 0) = \tilde{\mathcal{E}}|(\alpha, \beta)$.
(2) (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\mathcal{E})$ and for all $E^* \in \mathcal{E}^\alpha_{\beta^*}$,
$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)] \neq E \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)].$$
(3) (Initial Segment Condition) Suppose that
$$\mathrm{CRT}(E) < \alpha^* < \alpha.$$
Then there exists $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and there exists $E^* \in \mathcal{E}^{\alpha^*}_{\beta^*}$ such that
$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha^* + 1, 0)] = (E|\alpha^*) \cap L[\tilde{\mathcal{E}}|(\alpha^* + 1, 0)].$$

Definition 119. Suppose $\tilde{\mathcal{E}}$ is a Martin–Steel Doddage. Then $\tilde{\mathcal{E}}$ is *good* if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, for all $E_0, E_1 \in \mathcal{E}^\alpha_\beta$, $E_0 \cap L[\tilde{\mathcal{E}}] = E_1 \cap L[\tilde{\mathcal{E}}]$.

Definition 120. Suppose $\tilde{\mathcal{E}}$ is a Martin–Steel Doddage. Then $\tilde{\mathcal{E}}$ is *strongly backgrounded* if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and for all $\gamma \in \mathrm{Ord}$, there is an extender F such that $\rho(F) > \gamma$ and such that F witnesses the coherence condition for $\mathcal{E}(\alpha, \beta)$.

The following theorem is an immediate corollary of the results of [7]. The requirement that $\tilde{\mathcal{E}} \in V_\delta$ for some strongly inaccessible cardinal δ is necessary only because of the precise formulation of the Strong $(\omega_1 + 1)$-Iteration Hypothesis.

Theorem 121 (Martin–Steel). *Suppose that the Strong $(\omega_1 + 1)$-Iteration Hypothesis holds, $\tilde{\mathcal{E}}$ is a strongly backgrounded Martin–Steel Doddage and that $\tilde{\mathcal{E}} \in V_\delta$ for some strongly inaccessible cardinal δ. Then $\tilde{\mathcal{E}}$ is good.*

We prove a stronger version of this theorem. Theorem 122 is the generalization to Martin–Steel Doddages of the uniqueness of the structure,

$$(L[\mu], \mu \cap L[\mu]),$$

at κ where κ is the measurable cardinal associated to the normal measure μ.

Theorem 122. *Suppose that the Strong* $(\omega_1 + 1)$*-Iteration Hypothesis holds. Suppose that* $\tilde{\mathcal{E}}_0$ *and* $\tilde{\mathcal{E}}_1$ *are strongly backgrounded Martin–Steel Doddages such that*

$$\mathrm{dom}(\tilde{\mathcal{E}}_0) = \mathrm{dom}(\tilde{\mathcal{E}}_1).$$

Then

$$L[\tilde{\mathcal{E}}_0] = L[\tilde{\mathcal{E}}_1],$$

and moreover for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}_0)$*, for all* $E_0 \in \tilde{\mathcal{E}}_0(\alpha, \beta)$*, for all* $E_1 \in \tilde{\mathcal{E}}_1(\alpha, \beta)$*,*

$$E_0 \cap L[\tilde{\mathcal{E}}_0] = E_1 \cap L[\tilde{\mathcal{E}}_1].$$

Proof. We sketch the proof. It is convenient to fix some notation. Suppose that $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ are Martin–Steel Doddages such that $\mathrm{dom}(\tilde{\mathcal{E}}) = \mathrm{dom}(\tilde{\mathcal{F}})$. Define $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{F}}$ if

(1.1) $L[\tilde{\mathcal{E}}] = L[\tilde{\mathcal{F}}]$,
(1.2) for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, for all $E \in \tilde{\mathcal{E}}(\alpha, \beta)$, for all $F \in \tilde{\mathcal{F}}(\alpha, \beta)$,

$$E \cap L[\tilde{\mathcal{E}}] = F \cap L[\tilde{\mathcal{F}}].$$

Fix $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \delta)$ and suppose toward a contradiction that the theorem fails.

Suppose that (V_Θ, δ) is a premouse such that $\tilde{\mathcal{E}}_0 \in V_\delta$ and such that there exists a countable elementary substructure,

$$X \prec (V_\Theta, \delta)$$

such that (M, δ_M) has an $(\omega_1 + 1)$-iteration strategy where (M, δ_M) is the transitive collapse of X. Suppose that $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$ each contain only short extenders.

Thus

$$V_\delta \vDash \tilde{\mathcal{E}}_0 \not\equiv \tilde{\mathcal{E}}_1.$$

Fix a countable elementary substructure,

$$X \prec (V_\Theta, \delta),$$

such that (M, δ_M) has an $(\omega_1 + 1)$-iteration strategy where (M, δ_M) is the transitive collapse of X.

By the elementarity of X we can suppose without loss of generality that $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \delta) \in X$. Let $(\tilde{\mathcal{E}}_0^M, \tilde{\mathcal{E}}_1^M) \in M$ be the image of $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1)$ under the collapsing map. Thus

$$M \cap V_{\delta_M} \vDash \tilde{\mathcal{E}}_0^M \not\equiv \tilde{\mathcal{E}}_1^M.$$

Fix an (ω_1+1)-iteration strategy for (M,δ_M) and following this strategy we shall define two iteration trees

$$\mathcal{T} = \langle M^{\mathcal{T}}_\alpha, E^{\mathcal{T}}_\beta, j^{\mathcal{T}}_{\gamma,\alpha} : \alpha \le \omega_1, \beta < \omega_1, \gamma <_{\mathcal{T}} \alpha\rangle$$

and

$$\mathcal{S} = \langle M^{\mathcal{S}}_\alpha, E^{\mathcal{S}}_\beta, j^{\mathcal{S}}_{\gamma,\alpha} : \alpha \le \omega_1, \beta < \omega_1, \gamma <_{\mathcal{S}} \alpha\rangle$$

on (M,δ_M) each of length ω_1+1 such that for all $\beta<\omega_1$, the predecessor of $\beta+1$ relative to each of the two iteration trees is as small as possible for that iteration tree.

To define $\mathcal{S}$ and $\mathcal{T}$, we define a continuous increasing sequence

$$\langle(\beta_{\mathcal{S}},\beta_{\mathcal{T}}) : \beta\le\omega_1\rangle$$

of pairs of ordinals and define $(\mathcal{S}|\beta_{\mathcal{S}},\mathcal{T}|\beta_{\mathcal{T}})$ by induction on β with $(0_{\mathcal{S}},0_{\mathcal{T}})=(0,0)$. The limit stages are immediate. Therefore we can suppose that $\beta<\omega_1$ and that

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}} : M \to M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$$

and

$$j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}} : M \to M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$$

are given. We define $((\beta+1)_{\mathcal{S}},(\beta+1)_{\mathcal{T}})$ and at the same time we will define $E^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ if $(\beta+1)_{\mathcal{S}}\neq\beta_{\mathcal{S}}$ and define $E^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ if $(\beta+1)_{\mathcal{T}}\neq\beta_{\mathcal{T}}$. It is convenient to use the following notation. Suppose A,B are subsets of $\mathrm{Ord}\times\mathrm{Ord}$, then

$$A \le_{\mathcal{L}} B$$

if $A=B$ or if A is an initial segment of B relative to the lexicographical order on $\mathrm{Ord}\times\mathrm{Ord}$.

Case 1. Suppose that there exists

$$(\eta,\gamma)\in j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))\cap j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))$$

such that

(2.1) $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))|(\eta,\gamma) = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))|(\eta,\gamma)$,
(2.2) there exist

$$E_{\mathcal{S}}\in(j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0))(\eta,\gamma)\cup(j^{\mathcal{S}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1))(\eta,\gamma)$$

and

$$E_{\mathcal{T}}\in(j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0))(\eta,\gamma)\cup(j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1))(\eta,\gamma)$$

such that

$$E_{\mathcal{S}}\cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}}\cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}} \neq E_{\mathcal{T}}\cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}}\cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}}.$$

Let (η, γ) be the least such pair (relative the lexicographical order) and define $E^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ to be an extender in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ which witnesses the coherence condition for $E_{\mathcal{S}}$ relative to $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0)$ if

$$E_{\mathcal{S}} \in (j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0))(\eta, \gamma)$$

or witnesses coherence condition for $E_{\mathcal{S}}$ relative to $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$, with $\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$ as small as possible such that

$$\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}}) = \rho(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$$

and such that $\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$ is strongly inaccessible in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$. Since both $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0)$ and $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$ are strongly backgrounded in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ and since $\mathcal{E}_0, \mathcal{E}_1 \in V_\delta$, it follows that $E^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ exists.

Similarly, define $E^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ to be an extender in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ which witnesses the coherence condition for $E_{\mathcal{T}}$ relative to either $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0)$ if

$$E_{\mathcal{T}} \in (j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0))(\eta, \gamma)$$

or witnesses coherence condition for $E_{\mathcal{T}}$ relative to $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)$ otherwise, with $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ and small as possible such that

$$\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}}) = \rho(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$$

and such that $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ is strongly inaccessible in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$. Exactly as above, since both $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0)$ and $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)$ are strongly backgrounded in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ and since $\mathcal{E}_0, \mathcal{E}_1 \in V_\delta$, it follows that $E^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ exists.

Define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\beta_{\mathcal{S}} + 1, \beta_{\mathcal{T}} + 1)$.

Case 2. Otherwise. Then

$$j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\text{dom}(\tilde{\mathcal{E}}^M_0)) \not\leq_{\mathcal{L}} j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\text{dom}(\tilde{\mathcal{E}}^M_0))$$

and

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\text{dom}(\tilde{\mathcal{E}}^M_0)) \not\leq_{\mathcal{L}} j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\text{dom}(\tilde{\mathcal{E}}^M_0)).$$

Let $(\eta, \gamma)_{\mathcal{T}} = \min(j^{\mathcal{T}}_{0,\beta}(\text{dom}(\tilde{\mathcal{E}}^M_0)) \backslash j^{\mathcal{S}}_{0,\beta}(\text{dom}(\tilde{\mathcal{E}}^M_0)))$ and let

$$(\eta, \gamma)_{\mathcal{S}} = \min(j^{\mathcal{S}}_{0,\beta}(\text{dom}(\tilde{\mathcal{E}}^M_0))) \backslash j^{\mathcal{T}}_{0,\beta}(\text{dom}(\tilde{\mathcal{E}}^M_0)),$$

where in each case the minimum is relative to the lexicographical order.

Thus $(\eta, \gamma)_{\mathcal{S}} \neq (\eta, \gamma)_{\mathcal{T}}$. There are two subcases. If $(\eta, \gamma)_{\mathcal{S}} < (\eta, \gamma)_{\mathcal{T}}$ then let $E^{\mathcal{S}}_{\beta_{\mathcal{S}}} \in M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ be an extender which witnesses the coherence condition for some extender

$$E \in (j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0))((\eta, \gamma)_{\mathcal{S}})$$

with $\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$ as small as possible such that

$$\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}}) = \rho(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$$

and such that $\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$ is strongly inaccessible in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$. Exactly as above, since $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0)$ is strongly backgrounded in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ and since $\mathcal{E}_0 \in V_\delta$ it follows that $E^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ exists.

Define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\beta_{\mathcal{S}}+1, \beta_{\mathcal{T}})$.

If $(\eta,\gamma)_{\mathcal{T}} < (\eta,\gamma)_{\mathcal{S}}$ then let $E^{\mathcal{T}}_{\beta_{\mathcal{T}}} \in M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ be an extender which witnesses the coherence condition for some extender

$$E \in (j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0))((\eta,\gamma)_{\mathcal{T}})$$

with $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ as small as possible such that

$$\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}}) = \rho(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$$

and such that $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ is strongly inaccessible in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$.

Define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\beta_{\mathcal{S}}, \beta_{\mathcal{T}}+1)$.

This completes the definition of $\mathcal{S}$ and $\mathcal{T}$. If at some stage β neither case applies then it follows that (interchanging $\mathcal{S}$ and $\mathcal{T}$ if necessary):

(3.1) $j^{\mathcal{S}}_{0,\beta_{\mathcal{T}}}(\text{dom}(\tilde{\mathcal{E}}^M_0)) \leq_{\mathcal{L}} j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\text{dom}(\tilde{\mathcal{E}}^M_0))$,
(3.2) for all $(\eta,\gamma) \in j^{\mathcal{T}}_{0,\beta_{\mathcal{S}}}(\text{dom}(\tilde{\mathcal{E}}^M_0))$,

$$E \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}} = F \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}}.$$

for all

$$E \in (j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0))(\eta,\gamma) \cup (j^{\mathcal{S}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1))(\eta,\gamma)$$

and for all

$$F \in (j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0))(\eta,\gamma) \cup (j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1))(\eta,\gamma).$$

If

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\text{dom}(\tilde{\mathcal{E}}^M_0)) = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\text{dom}(\tilde{\mathcal{E}}^M_0)),$$

then either

$$M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap V_{j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)} \vDash j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$$

or

$$M^{\mathcal{T}}_{\beta_{\mathcal{T}}} \cap V_{j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M)} \vDash j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)$$

(depending on whether $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M) \leq j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M)$ or whether $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M) \leq j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)$) and this contradicts the choice of $(M, \tilde{\mathcal{E}}^M_0, \tilde{\mathcal{E}}^M_1, \delta_M)$.

If $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))$ is a proper initial segment of $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))$ then it follows that

$$M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap V_{j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)} \vDash j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$$

and this again is a contradiction.

To see this latter claim fix $(\eta_0, \gamma_0) \in j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))$ such that

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)) = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\mathrm{dom}(\tilde{\mathcal{E}}^M_0))|(\eta_0, \gamma_0).$$

Since $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0)(\eta_0, \gamma_0)$ is defined it follows that

$$M^{\mathcal{T}}_{\beta} \cap V_{j^{\mathcal{T}}_{0,\beta}(\delta_M)} \vDash \text{“}(L[\tilde{\mathcal{E}}])^{\#} \text{ and } (L[\tilde{\mathcal{F}}])^{\#} \text{ exist”}$$

where $\tilde{\mathcal{E}} = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0)|(\eta_0, \gamma_0)$ and where $\tilde{\mathcal{F}} = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)|(\eta_0, \gamma_0)$. Further since $(M^{\mathcal{T}}_{\beta_{\mathcal{T}}}, j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M))$ is iterable,

$$((L[\tilde{\mathcal{E}}])^{\#})^{M^{\mathcal{T}}_{\beta_{\mathcal{T}}}} = (L[\tilde{\mathcal{E}}])^{\#}$$

and

$$((L[\tilde{\mathcal{F}}])^{\#})^{M^{\mathcal{T}}_{\beta_{\mathcal{T}}}} = (L[\tilde{\mathcal{F}}])^{\#}.$$

Now by (3.2), it follows that

$$M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap V_{j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)} \vDash j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$$

as claimed. Therefore at every stage $\beta < \omega_1$, either Case 1 holds or Case 2 holds.

Note that for each extender, E, occurring in either $\mathcal{S}$ or $\mathcal{T}$, in the model from which E is chosen there exists λ such that

(4.1) $\lambda = |V_\lambda|$ and $\rho(E) = \mathrm{LTH}(E) = \lambda$,
(4.2) $\mathrm{CRT}(E) = \mathrm{SPT}(E)$,
(4.3) λ is not a limit of inaccessible cardinals.

To see that (4.2) holds it suffices to see that if $\mathcal{E}$ is a Martin–Steel Doddage, $(\alpha, \beta) \in \mathrm{dom}(\mathcal{E})$ and if F is an extender which witnesses the coherence condition for $\mathcal{E}(\alpha, \beta)$ then necessarily $(\alpha, \beta) \in j_F(V_\kappa)$ where $\kappa = \mathrm{CRT}(F)$.

This has two consequences. First, (4.1)–(4.3) imply that both $\mathcal{S}$ and $\mathcal{T}$ are non-overlapping; in fact, for all $\beta_1 < \beta_2$ if $\beta_1 + 1 <_{\mathcal{S}} \beta_2 + 1$ then $\mathrm{LTH}(E^{\mathcal{S}}_{\beta_1}) < \mathrm{CRT}(E^{\mathcal{S}}_{\beta_2})$, and similarly for $\mathcal{T}$. This is a slightly stronger condition. Second, by (4.2) both $\mathcal{S}$ and $\mathcal{T}$ are iteration trees involving only short extenders, and so (4.1)–(4.3) imply that both $\mathcal{S}$ and $\mathcal{T}$ are $(+1)$-iteration trees (which implies that they are each $(+\theta)$-iteration trees where θ is the least measurable cardinal of M). Therefore the iteration strategy fixed for (M, δ_M) must supply cofinal, wellfounded, branches at all limit stages $\beta \leq \omega_1$.

We note that unlike the usual comparison arguments, it is not obviously the case that the lengths of the extenders in these iteration trees are nondecreasing, more precisely it is not obvious that for all $\beta_1 < \beta_2$, $\mathrm{LTH}(E^{\mathcal{S}}_{\beta_1}) \leq \mathrm{LTH}(E^{\mathcal{S}}_{\beta_2})$. For example, suppose that $E^{\mathcal{S}}_{\beta_1}$ is chosen to witness the coherence condition relative to $j^{\mathcal{S}}_{0,\beta_1}(\tilde{\mathcal{E}}^M_0)$. Then there is no reason to expect that $E^{\mathcal{S}}_{\beta_1}$ coheres $j^{\mathcal{S}}_{0,\beta_1}(\tilde{\mathcal{E}}^M_1)$ and so at the next stage of the construction of $(\mathcal{S},\mathcal{T})$ there may be an "earlier" disagreement.

We obtain a contradiction is the usual fashion. Let

$$Z \prec H(\omega_2)$$

be a countable elementary substructure such that $\{\mathcal{S},\mathcal{T}\} \in Z$. Let

$$b_{\mathcal{S}} = \{\beta < \omega_1 \mid \beta <_{\mathcal{S}} \omega_1\}$$

and let $b_{\mathcal{T}} = \{\beta < \omega_1 \mid \beta <_{\mathcal{T}} \omega_1\}$. Thus $b_{\mathcal{S}}$ and $b_{\mathcal{T}}$ are each closed cofinal subsets of ω_1. Let $\beta_Z = Z \cap \omega_1$. The image of $(\mathcal{S},\mathcal{T})$ under the transitive collapse of Z is $(\mathcal{S}|(\beta_Z+1), \mathcal{T}|(\beta_Z+1))$.

Let N be the transitive collapse of X and let

$$\pi : N \to H(\omega_2)$$

invert the transitive collapse. Thus $\beta_Z \in b_{\mathcal{S}} \cap b_{\mathcal{T}}$ and

(5.1) $\pi(M^{\mathcal{S}}_{\beta_Z}) = M^{\mathcal{S}}_{\omega_1}$ and $\pi|M^{\mathcal{S}}_{\beta_Z} = j^{\mathcal{S}}_{\beta_Z,\omega_1}$,
(5.2) $\pi(M^{\mathcal{S}}_{\beta_Z}) = M^{\mathcal{S}}_{\omega_1}$ and $\pi|M^{\mathcal{S}}_{\beta_Z} = j^{\mathcal{S}}_{\beta_Z,\omega_1}$.

We now come to the key points. Let $\alpha^{\mathcal{S}}_Z$ be such that $\beta_Z = (\alpha^{\mathcal{S}}_Z)^*$ computed relative to $<_{\mathcal{S}}$, and let $\alpha^{\mathcal{T}}_Z$ be such that $\beta_Z = (\alpha^{\mathcal{T}}_Z)^*$ computed relative to $<_{\mathcal{T}}$.

By (5.1)–(5.2) and since the iteration trees are non-overlapping:

(6.1) For all $\beta > \beta_Z$, $\mathrm{LTH}(E^{\mathcal{S}}_\beta) > \beta_Z$ and $\mathrm{LTH}(E^{\mathcal{T}}_\beta) > \beta_Z$;
(6.2) For all $\beta > \beta_Z$,

$$M^{\mathcal{S}}_\beta \cap V_{\beta_Z+\omega} = M^{\mathcal{S}}_{\beta_Z} \cap V_{\beta_Z+\omega}$$

and

$$M^{\mathcal{T}}_\beta \cap V_{\beta_Z+\omega} = M^{\mathcal{T}}_{\beta_Z} \cap V_{\beta_Z+\omega};$$

(6.3) Either

$$E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z} \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z} = (E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}|\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z})) \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z},$$

or

$$E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z} \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z} = (E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}|\mathrm{LTH}(E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z})) \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z};$$

(6.4) For each α such that $\alpha^{\mathcal{S}}_Z < \alpha < \omega_1$, $\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}) < \mathrm{LTH}(E^{\mathcal{S}}_\alpha)$,
(6.5) For each α such that $\alpha^{\mathcal{T}}_Z < \alpha < \omega_1$, $\mathrm{LTH}(E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}) < \mathrm{LTH}(E^{\mathcal{T}}_\alpha)$.

The third of these claims, (6.3), follows from (5.1) and (5.2) since both $\mathcal{S}$ and $\mathcal{T}$ are non-overlapping.

To see that (6.4) holds, suppose toward a contradiction that $\alpha^{\mathcal{S}}_Z < \alpha < \omega_1$ and that $\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}) \geq \mathrm{LTH}(E^{\mathcal{S}}_\alpha)$. Let $\hat{\alpha}$ be such that

$$(\hat{\alpha})^* = \sup\{\beta \leq \alpha \mid \beta \in b_{\mathcal{S}}\},$$

and such that $\hat\alpha + 1 \in b_{\mathcal{S}}$, where $(\hat\alpha)^*$ is computed relative to $<_{\mathcal{S}}$. Then $\hat\alpha \geq \alpha$ and $(\hat\alpha)^* \geq \alpha_Z^{\mathcal{S}} + 1$ since $\alpha > \alpha_Z^{\mathcal{S}}$ and $\alpha_Z^{\mathcal{S}} + 1 \in b_{\mathcal{S}}$. But

$$\textsc{crt}(E_{\hat\alpha}^{\mathcal{S}}) < \min\{\rho(E_\beta^{\mathcal{S}}) \,|\, (\hat\alpha)^* \leq \beta < \hat\alpha\} \leq \textsc{lth}(E_\alpha^{\mathcal{S}}) \leq \textsc{lth}(E_{\alpha_Z^{\mathcal{S}}}^{\mathcal{S}})$$

and since $\mathcal{S}$ is non-overlapping, $\textsc{lth}(E_{\alpha_Z^{\mathcal{S}}}^{\mathcal{S}}) \leq \textsc{crt}(E_{\hat\alpha}^{\mathcal{S}})$. This is a contradiction. The proof that (6.5) holds is similar as is the proof of (6.1). Finally (6.2) follows from (6.1) since each of the extenders, $E_\beta^{\mathcal{S}}$ and $E_\beta^{\mathcal{T}}$, is an extender of minimum possible length which witnesses the coherence condition for a Martin–Steel Doddage (such extenders cannot have length which is a limit of inaccessible cardinals).

We fix some notation. Suppose that $\beta \leq \omega_1$ and that $(\eta, \gamma) \in j_{0,\beta}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$. Let $\mathcal{M}_{\tilde{\mathcal{E}}_0,\beta}^{\mathcal{S}}(\eta,\gamma)$ denote the structure,

$$(L[j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_0^M)|(\eta,\gamma)], j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_0^M)|(\eta,\gamma) \cap L[j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_0^M)|(\eta,\gamma)]),$$

and let $\mathcal{M}_{\tilde{\mathcal{E}}_1,\beta}^{\mathcal{S}}(\eta,\gamma)$ denote structure,

$$(L[j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_1^M)|(\eta,\gamma)], j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_1^M)|(\eta,\gamma) \cap L[j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_1^M)|(\eta,\gamma)]).$$

Similarly, suppose that $\beta \leq \omega_1$, and that $(\eta,\gamma) \in j_{0,\beta}^{\mathcal{T}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$. Let $\mathcal{M}_{\tilde{\mathcal{E}}_0,\beta}^{\mathcal{T}}(\eta,\gamma)$ and $\mathcal{M}_{\tilde{\mathcal{E}}_1,\beta}^{\mathcal{T}}(\eta,\gamma)$ denote the analogous structures defined relative to $\mathcal{T}$.

Let $(\eta,\gamma)_{\mathcal{S}} \in j_{0,\alpha_Z^{\mathcal{S}}}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$ be the element involved in the definition of $E_{\alpha_Z^{\mathcal{S}}}^{\mathcal{S}}$. By (6.4) and the fact that the extenders $E_\alpha^{\mathcal{S}}$ are chosen of minimal length to witness the coherence condition:

(7.1) Suppose that $\alpha_Z^{\mathcal{S}} < \alpha < \omega_1$. Let (η,γ) be the element of $j_{0,\alpha}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$ involved in the definition of $E_\alpha^{\mathcal{S}}$. Then $\eta^{\mathcal{S}} < \eta$ where $(\eta^{\mathcal{S}}, \gamma^{\mathcal{S}}) = (\eta,\gamma)_{\mathcal{S}}$.

We claim that for all β such that $\alpha_Z^{\mathcal{S}} \leq \beta \leq \omega_1$:

(8.1) $j_{0,\alpha_Z^{\mathcal{S}}}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{S}} = j_{0,\beta}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{S}} = j_{0,\beta}^{\mathcal{T}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{S}}$;

(8.2) Let $(\eta^{\mathcal{S}}, \gamma^{\mathcal{S}}) = (\eta,\gamma)_{\mathcal{S}}$, then if $\alpha_Z^{\mathcal{S}} < \beta$,

$$j_{0,\beta}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{S}} = j_{0,\beta}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{S}}+1, 0)$$

and

$$j_{0,\beta}^{\mathcal{T}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{S}} = j_{0,\beta}^{\mathcal{T}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{S}}+1, 0);$$

(8.3) For all $(\eta^*, \gamma^*) \in j_{0,\beta}^{\mathcal{S}}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{S}}$,

$$E \cap M_\beta^{\mathcal{S}} \cap M_\beta^{\mathcal{T}} = F \cap M_\beta^{\mathcal{S}} \cap M_\beta^{\mathcal{T}},$$

for all

$$E \in (j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_0))(\eta^*,\gamma^*) \cup (j_{0,\beta}^{\mathcal{S}}(\tilde{\mathcal{E}}_1))(\eta^*,\gamma^*)$$

and for all

$$F \in (j_{0,\beta}^{\mathcal{T}}(\tilde{\mathcal{E}}_0))(\eta^*,\gamma^*) \cup (j_{0,\beta}^{\mathcal{T}}(\tilde{\mathcal{E}}_1))(\eta^*,\gamma^*);$$

(8.4) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\beta}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\beta}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}})$;
(8.5) $\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\beta}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\beta}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}})$;
(8.6) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}})$;
(8.7) $(\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta,\gamma)_{\mathcal{S}}))^{\#} \in M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta$.

The only potential issue is (8.7); (8.1)–(8.6) follow from (6.1)–(6.5) and (7.1) by relatively standard arguments. The proof of (8.7) uses (8.1)–(8.6) and the definition of $\mathcal{S}$ and $\mathcal{T}$. There are two additional relevant points. First,

$$\omega_1 \subseteq M^{\mathcal{S}}_{\omega_1}$$

and so for all $a \in M^{\mathcal{S}}_{\omega_1}$, if

$$M^{\mathcal{S}}_{\omega_1} \vDash \text{“}a^{\#} \text{ exists”}$$

then $a^{\#} \in M^{\mathcal{S}}_{\omega_1}$ (and similarly for $M^{\mathcal{T}}_{\omega_1}$). Second, if $\tilde{\mathcal{E}}$ is a Martin–Steel Doddage and if $(\eta,\gamma) \in \mathrm{dom}(\tilde{\mathcal{E}})$ then since $\tilde{\mathcal{E}}(\eta,\gamma)$ is defined, necessarily $(L[\tilde{\mathcal{E}}|(\eta,\gamma)])^{\#}$ exists.

Similarly, let $(\eta,\gamma)_{\mathcal{T}} \in j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$ be the element involved in the definition of $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$. By (6.5), for all β such that $\alpha^{\mathcal{T}}_Z \le \beta \le \omega_1$;

(9.1) $j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}}$;
(9.2) Let $(\eta^{\mathcal{T}},\gamma^{\mathcal{T}}) = (\eta,\gamma)_{\mathcal{T}}$, then if $\alpha^{\mathcal{T}}_Z < \beta$,

$$j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{T}}+1,0),$$

and

$$j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{T}}+1,0);$$

(9.3) For all $(\eta^*,\gamma^*) \in j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}}$,

$$E \cap M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta = F \cap M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta,$$

for all

$$E \in (j^{\mathcal{S}}_{0,\beta}(\tilde{\mathcal{E}}_0))(\eta^*,\gamma^*) \cup (j^{\mathcal{S}}_{0,\beta}(\tilde{\mathcal{E}}_1))(\eta^*,\gamma^*)$$

and for all

$$F \in (j^{\mathcal{T}}_{0,\beta}(\tilde{\mathcal{E}}_0))(\eta^*,\gamma^*) \cup (j^{\mathcal{T}}_{0,\beta}(\tilde{\mathcal{E}}_1))(\eta^*,\gamma^*);$$

(9.4) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})$;
(9.5) $\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})$;
(9.6) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})$;
(9.7) $(\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}))^{\#} \in M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta$.

Using (8.1)–(8.7) and (9.1)–(9.7), the argument is now very much like the standard arguments in a comparison proof.

By the definition of $\mathcal{S}$, $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$ witnesses in $M^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$ the coherence condition for $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}|\eta^{\mathcal{S}}$ relative to either $j^{\mathcal{S}}_{0,\alpha^{\mathcal{S}}_Z}(\tilde{\mathcal{E}}_0)$ or $j^{\mathcal{S}}_{0,\alpha^{\mathcal{S}}_Z}(\tilde{\mathcal{E}}_1)$ where as in (8.2), $\eta^{\mathcal{S}}$ is the first coordinate of $(\eta,\gamma)_{\mathcal{S}}$.

Similarly, by the definition of $\mathcal{T}$, $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$ witnesses in $M^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$ the coherence condition for $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}|\eta^{\mathcal{T}}$ relative to either $j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\tilde{\mathcal{E}}_0)$ or $j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\tilde{\mathcal{E}}_1)$ where as in (9.2), $\eta^{\mathcal{T}}$ is the first coordinate of $(\eta,\gamma)_{\mathcal{T}}$.

By the novelty and initial segment conditions for Martin–Steel Doddages, (6.3), (8.1)–(8.7), and (9.1)–(9.7),

$$\eta^{\mathcal{S}} = \eta^{\mathcal{T}}$$

and $(\eta,\gamma)_{\mathcal{S}} = (\eta,\gamma)_{\mathcal{T}}$. This implies that both $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$ and $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$ were chosen according to (Case 1) in the construction of $\mathcal{S}$ and $\mathcal{T}$ and moreover the corresponding stages of the construction are the same, i.e. for some $\beta < \omega_1$,

$$(\beta_{\mathcal{S}}, \beta_{\mathcal{T}}) = (\alpha^{\mathcal{S}}_Z, \alpha^{\mathcal{T}}_Z).$$

and $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\alpha^{\mathcal{S}}_Z + 1, \alpha^{\mathcal{T}}_Z + 1)$. But

$$(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}|\eta) \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}} = (E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}|\eta) \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$$

where $\eta = \eta^{\mathcal{S}} = \eta^{\mathcal{T}}$, and this contradicts the disagreement which must have been satisfied in the definition of $(E^{\mathcal{S}}_{\beta_{\mathcal{S}}}, E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$. □

4.2. *Martin–Steel extender sequences with long extenders*

Eliminating the shortness requirement in Definition 115 one obtains the natural extension of Martin–Steel extender sequences to the case of long extenders.

Definition 123. An extender sequence,

$$\tilde{E} = \langle E^{\alpha}_{\beta} : (\alpha,\beta) \in \mathrm{dom}(\tilde{E})\rangle$$

is a *generalized Martin–Steel extender sequence* if for each pair $(\alpha,\beta) \in \mathrm{dom}(\tilde{E})$:

(1) (Coherence) There exists an extender F such that

 (a) $\alpha + \omega \leq \rho(F)$,
 (b) $E^{\alpha}_{\beta} = F|\alpha$,
 (c) $j_F(\tilde{E})|(\alpha+1, 0) = \tilde{E}|(\alpha,\beta)$.

(2) (Novelty) For all $\beta^* < \beta$, $(\alpha,\beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^{\alpha}_{\beta^*} \cap L[\tilde{E}|(\alpha,\beta)] \neq E^{\alpha}_{\beta} \cap L[\tilde{E}|(\alpha,\beta)].$$

(3) (Initial Segment Condition) Suppose that

$$\kappa < \alpha^* < \alpha$$

where κ is the critical point associated to E^α_β.

Then there exists β^* such that $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{E})$ and such that

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^* + 1, 0)] = (E^\alpha_\beta|\alpha^*) \cap L[\tilde{E}|(\alpha^* + 1, 0)].$$

Remark 124. If one requires in condition 1(a) that $\rho(F) < j_F(\mathrm{CRT}(F))$ then one obtains a Martin–Steel extender sequence. So the generalization, which is an obvious one, is simply to allow long extenders on the sequence.

A *generalized Martin–Steel premouse* is a structure, $\langle M, \tilde{E}, \delta\rangle$, such that:

(1) (M, δ) is a premouse;
(2) $\tilde{E} \in M_\delta$ and $M_\delta \vDash$ "$\tilde{E}$ is a generalized Martin–Steel extender sequence".

Suppose that $\langle M_0, \tilde{E}_0, \delta_0\rangle$ and $\langle M_1, \tilde{E}_1, \delta_1\rangle$ are generalized Martin–Steel premice. Then

$$\tilde{E}_0 \trianglelefteq \tilde{E}_1$$

if either

$$\tilde{E}_1 \cap (L[\tilde{E}_0])^{M_0} = \tilde{E}_0 \cap (L[\tilde{E}_0])^{M_0},$$

or for some $(\alpha, \beta) \in \mathrm{dom}(\tilde{E}_1)$,

$$(\tilde{E}_1|(\alpha, \beta)) \cap (L[\tilde{E}_0])^{M_0} = \tilde{E}_0 \cap (L[\tilde{E}_0])^{M_0}.$$

These two possibilities are not in general mutually exclusive, though if $\mathrm{dom}(\tilde{E}_1) \subseteq M_0$ then they are.

The attempts to generalize the theorems on comparison even granting iterability have failed to date because of several obstacles. Suppose that $\langle M_0, \tilde{E}_0, \delta_0\rangle$ and $\langle M_1, \tilde{E}_1, \delta_1\rangle$ are countable generalized Martin–Steel premice such that there exist iteration strategies of order $\omega_1 + 1$ for both (M_0, δ_0) and (M_1, δ_1).

To prove *comparison* one attempts to construct iteration maps

$$\pi_0 : (M_0, \delta_0) \to (M_0^*, \delta_0^*)$$

and

$$\pi_1 : (M_1, \delta_1) \to (M_1^*, \delta_1^*)$$

such that either $\pi_0(\tilde{E}_0) \trianglelefteq \pi_1(\tilde{E}_1)$ or $\pi_1(\tilde{E}_1) \trianglelefteq \pi_0(\tilde{E}_0)$.

Generally the iteration is constructed not using extenders on the image of the extender sequence but the corresponding "background" extenders, these are extenders (in the image model) which witness the coherence condition for some extender on the image sequence (usually with minimum possible length). However one also achieves a much stronger version of comparison, producing iterations,

$$\pi_0 : (M_0, \delta_0) \to (M_0^*, \delta_0^*)$$

and

$$\pi_1 : (M_1, \delta_1) \to (M_1^*, \delta_1^*),$$

with the property that (in addition to achieving comparison as defined above) for all $(\alpha, \beta) \in \pi_0(\mathrm{dom}(\tilde{E}_0)) \cap \pi_1(\mathrm{dom}(\tilde{E}_1))$,

$$(\pi_0(\tilde{E}_0))(\alpha, \beta) \cap M_0^* \cap M_1^* = (\pi_1(\tilde{E}_1))(\alpha, \beta) \cap M_0^* \cap M_1^*.$$

Martin and Steel proved that this is possible for Martin–Steel premice. The iteration trees are constructed by choosing the extenders of *least disagreement* at every stage and choosing the models to which the corresponding background extenders are to be applied based on the requirement of not *moving the generators* of previously selected extenders. The doctrine of not moving generators is central to the theory of the comparison of extender models.

The attempts to generalize the Martin–Steel construction to the context of extender sequences with long extenders have failed because of two obstacles each of which arises because of the constraint of not moving generators.

(Steel) The moving spaces problem:

> *In the case of generalized Martin–Steel premice, the requirement of not moving generators is not compatible with the rules for constructing iteration trees; even allowing the most general rules possible for constructing iteration trees.*

The difficulty, for example, is that if

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

is an iteration tree on (M, δ) then there could exist (a limit ordinal) $\alpha < \eta$ such that for all $\beta < \alpha$,

(1) $M_\beta \cap \mathrm{Ord} < \mathrm{SPT}(E_\alpha)$.
(2) $\mathrm{CRT}(E_\alpha) \leq \mathrm{CRT}(E_\beta)$.

Not moving generators would demand that $\alpha^* = 0$, but the rules for iteration trees require $\alpha^* = \alpha$. In fact in this situation, $\mathrm{Ult}(M_\beta, E_\alpha)$ simply cannot be defined for any $\beta < \alpha$.

There is a restricted class of generalized Martin–Steel premice for which the requirement of not moving generators is compatible with the rules for forming an iteration tree in a comparison iteration *if* one allows a more general notion of an iteration tree. However this approach is also doomed.

(Neeman) Iteration trees cannot be generalized:

> *Allowing the most general possible rules for constructing iterations trees leads to a total failure of iterability.*

4.3. *The failure of comparison*

We prove that for generalized Martin–Steel extender sequences comparison fails far below where anything like the moving spaces problem arises. The failure is extreme enough that there is no need to define an abstract form of comparison: we prove in Theorem 127 below that by passing to a generic extension of V, there can exist relatively short generalized Martin–Steel sequences $\tilde{E} \subset V_\kappa$ where κ is supercompact such that

$$V_\kappa \subset L[\tilde{E}].$$

The shortness condition is just past Martin–Steel sequences: for each δ the set of all β such that $(\delta+1,\beta) \in \operatorname{dom}(\tilde{E})$ and $\tilde{E}(\delta+1,\beta)$ is not a short extender, has ordertype at most δ. The construction of counterexample is compatible with essentially any assumption on the strength of the extenders witnessing coherence or any additional coherence requirements.

We also prove as Theorem 128 another version of the failure of comparison which arguably rules out avoiding the problem by any possible refinement of the definition of a generalized Martin–Steel extender sequences.

Remark 125. Notice that this shortness condition implies that for all $(\gamma,\alpha) \in \operatorname{dom}(\tilde{E})$, if $\tilde{E}(\gamma,\alpha)$ is not a short extender then

$$\gamma = j_E(\mathrm{CRT}(E)) + 1$$

where $E = \tilde{E}(\gamma,\alpha)$. To see this let (γ,α) be the least counterexample and let F be an extender which witnesses the coherence condition for $\tilde{E}(\gamma,\alpha)$. Let

$$j_F : V \to M$$

be the associated elementary embedding. We have $\rho(F) \geq \gamma + \omega$ and so

$$V_{\gamma+\omega} \subset M.$$

We also have that $j_F(\tilde{E})|(\gamma+1,0) = \tilde{E}|(\gamma,\alpha)$. Let $\kappa = \mathrm{CRT}(j_F)$. Thus since (γ,α) is a counterexample, necessarily $j_F(\kappa) + 2 \leq \gamma$.

We have $\tilde{E}$ satisfies the initial segment condition at (γ,α) and so there must exist $(j_F(\kappa)+1,\alpha_0) \in \operatorname{dom}(\tilde{E})$ such that

$$(F|(j_F(\kappa)+1)) \cap L[\tilde{E}|(j_F(\kappa)+2,0)] = \tilde{E}(j_F(\kappa)+1,\alpha_0) \cap L[\tilde{E}|(j_F(\kappa)+2,0)] = E_0.$$

Let

$$j_{E_0} : L[\tilde{E}|(j_F(\kappa)+2,0)] \to \mathrm{Ult}(L[\tilde{E}|(j_F(\kappa)+2,0)], E_0)$$

be the embedding where the ultrapower is computed using only functions in $L[\tilde{E}|(j_F(\kappa)+2,0)]$ and as usual we identify the ultrapower with its transitive collapse.

There is an elementary embedding,

$$\pi : \mathrm{Ult}(L[\tilde{E}|(j_F(\kappa)+2,0)], E_0) \to j_F(L[\tilde{E}|(j_F(\kappa)+2,0)])$$

such that

$$j_F|\ L[\tilde{E}|(j_F(\kappa)+2,0)] = (\pi \circ j_{E_0})|\ L[\tilde{E}|(j_F(\kappa)+2,0)]$$

and such that $\mathrm{CRT}(\pi) > j_F(\kappa)$.

We claim that α_0 is not in the range of π. Suppose toward a contradiction that $\alpha_0 = \pi(\beta_0)$. Then $E_0 = \pi(F_0)$ where

$$F_0 = \tilde{E}(j_F(\kappa)+1, \beta_0) \cap L[\tilde{E}|(j_F(\kappa)+2,0)].$$

It follows that

$$E_0 \in \mathrm{Ult}(L[\tilde{E}|(j_F(\kappa)+2,0)], E_0)$$

which is a contradiction, and so α_0 is not in the range of π as claimed.

Let S be the set of all β such that $(j_F(\kappa)+1, \beta) \in \mathrm{dom}(\tilde{E})$ and such that

$$j_F(\kappa) = j_G(\mathrm{CRT}(G))$$

where $G = \tilde{E}(j_F(\kappa)+1, \beta)$. Then $\alpha_0 \in \pi(S)$. Since α_0 is not in the range of π and since $\mathrm{CRT}(\pi) > j_F(\kappa)$, it follows that S must have ordertype at least $\mathrm{CRT}(\pi) > j_F(\kappa)$. This implies that the shortness condition fails for $\tilde{E}|(j_F(\kappa)+2,0)$.

We fix some notation. For each strongly inaccessible cardinal δ, let $\mathbb{Q}_\delta$ be the following partial order (which adds a fast club at δ). Conditions are pairs (c, X) where c is a bounded closed subset of δ and X is a set of closed cofinal subsets of δ with $|X| < \delta$ and such that $c \subset \cap X$.

Suppose $(d, Y), (c, X) \in \mathbb{Q}_\delta$. Then $(d, Y) \leq (c, X)$ if the following hold.

(1) $c = d \cap (\sup(c)+1)$ and $d \backslash c \subset \cap X$,
(2) $X \subseteq Y$.

Thus $\mathbb{Q}_\delta$ is $(<\!\delta)$-closed. Suppose $G \subset \mathbb{Q}_\delta$ is V-generic and let

$$C_G = \cup\{c \,|\, (c, X) \in G\}.$$

Then C_G is a closed cofinal subset of δ such that for all closed cofinal sets $D \subset \delta$ with $D \in V$, $C_G \backslash D$ is bounded in δ (so C_G is a fast club in δ).

Lemma 126. *Suppose κ is strongly inaccessible and $A \subset \kappa$. Suppose $G \subset \mathbb{Q}_\kappa$ is V-generic and in $V[G]$ there is a club $D \subset C_G$ such that*

$$D \cap \gamma \in L[A]$$

for all $\gamma < \kappa$. Then $V_\kappa \subset L[A]$.

Proof. Fix a term τ for D. By the homogeneity of $\mathbb{Q}_\kappa$, we can suppose

$$1 \Vdash \text{“}\tau \cap \gamma \in L[A] \text{ for all } \gamma < \kappa\text{”}$$

and that

$$1 \Vdash \text{“}\tau \text{ is closed, cofinal in } C_G\text{”}.$$

For each $\gamma < \kappa$, let D_γ be the set of $(c, X) \in \mathbb{Q}_\kappa$ such that

(1.1) $\gamma < \sup(c)$,
(1.2) for all $\alpha < \sup(c)$, either $(c, X) \Vdash$ "$\alpha \in \tau$" or $(c, X) \Vdash$ "$\alpha \notin \tau$",
(1.3) $\{\alpha < \sup(c) \,|\, (c, X) \Vdash \text{“}\alpha \in \tau\text{”}\}$ is cofinal in $\sup(c)$.

Thus for each $\gamma < \kappa$, D_γ is dense in $\mathbb{Q}_\kappa$. Further D_γ is ($<\kappa$)-closed. More precisely if

$$\langle (c_\alpha, X_\alpha) : \alpha < \gamma \rangle$$

is a decreasing sequence in D_γ where $\gamma < \kappa$, then

$$(\cup\{c_\alpha \,|\, \alpha < \gamma\}, \cup\{X_\alpha \,|\, \alpha < \gamma\}) \in D_\gamma.$$

Let $\mathbb{D} = \{D_\gamma \,|\, \gamma < \kappa\}$. Thus a filter $\mathcal{F} \subset \mathbb{Q}_\kappa$ is $\mathbb{D}$-generic if and only if for each $\gamma < \kappa$ there exists $(c, X) \in D_0 \cap \mathcal{F}$ such that $\gamma < \sup(c)$.

If $\mathcal{F}$ is a $\mathbb{D}$-generic filter let $D_{\mathcal{F}}$ be the interpretation of τ by $\mathcal{F}$. Thus $D_{\mathcal{F}}$ is closed cofinal in κ and for all $\gamma < \kappa$, $D_{\mathcal{F}} \cap \gamma \in L[A]$. The key claim is the following.

(2.1) For each $B \subset \kappa$, there exists a pair $(\mathcal{F}_0, \mathcal{F}_1)$ of $\mathbb{D}$-generic filters such that if

$$\langle \eta_\alpha : \alpha < \kappa \rangle$$

is the increasing enumeration of the limit points of $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1}$ then for all $\alpha < \kappa$, $\alpha \in B$ if and only if

$$\min\{\eta \in D_{\mathcal{F}_0} \,|\, \eta_\alpha < \eta\} < \min\{\eta \in D_{\mathcal{F}_1} \,|\, \eta_\alpha < \eta\}.$$

Since for all $\gamma < \kappa$, $(D_{\mathcal{F}_0} \cap \gamma, D_{\mathcal{F}_1} \cap \gamma) \in L[A]$, (2.1) implies that for all $\gamma < \kappa$, $B \cap \gamma \in L[A]$ and the lemma follows.

The proof of (2.1) follows by noting the following. Suppose $(c_0, X_0) \in \mathbb{Q}_\kappa$ and that either $(c_0, X_0) \in \mathbb{D}$ or $c_0 = \emptyset$. Then for each $\eta < \kappa$ such that $\sup(c_0) < \eta$, there exists $(c_1, X_1) \in \mathbb{D}$ such that

(3.1) $(c_1, X_1) < (c_0, X_0)$,
(3.2) $\eta < \sup(c_1)$,
(3.3) $c_1 \cap \eta = c_0$.

One uses this to construct decreasing sequences

$$\langle (c^0_\alpha, X^0_\alpha) : \alpha < \kappa \rangle$$

and

$$\langle (c^1_\alpha, X^1_\alpha) : \alpha < \kappa \rangle$$

of conditions in D_0 by induction on α such that for all α,

(4.1) $c^0_0 \cap c^1_0 = \emptyset$,
(4.2) $c^0_{\alpha+1} \cap c^1_{\alpha+1} = c^0_\alpha \cap c^1_\alpha$,
(4.3) if $\alpha > 0$ and α is a limit then

(a) $c^0_\alpha = \cup\{c^0_\beta \,|\, \beta < \alpha\} \cup \sup(\cup\{c^0_\beta \,|\, \beta < \alpha\})$

(b) $c^1_\alpha = \cup\{c^1_\beta \mid \beta < \alpha\} \cup \sup(\cup\{c^1_\beta \mid \beta < \alpha\})$
(c) $\max(c^0_\alpha) = \max(c^1_\alpha)$,
(d) if α is the ηth nonzero limit ordinal then $\eta \in B$ if and only if

$$\min(c^0_{\alpha+1} \backslash c^0_\alpha) < \min(c^1_{\alpha+1} \backslash c^1_\alpha).$$

The filters

(5.1) $\mathcal{F}_0$ generated by $\{(c^0_\alpha, X^0_\alpha) : \alpha < \kappa\}$,
(5.2) $\mathcal{F}_1$ generated by $\{(c^1_\alpha, X^1_\alpha) : \alpha < \kappa\}$,

witness (2.1). □

Theorem 127. *Suppose that κ a supercompact cardinal. Then there is a partial order $\mathbb{P}$ such that if $G \subset \mathbb{P}$ is V-generic then in $V[G]$ the following hold.*

(1) *κ is supercompact.*
(2) *There is a generalized Martin–Steel extender sequence $\tilde{E} \subset V[G]_\kappa$ such that*
  (a) *for all δ, the set of β such that $(\delta + 1, \beta) \in \mathrm{dom}(\tilde{E})$ and such that*

$$j_E(\mathrm{CRT}(E)) = \delta$$

  where $E = \tilde{E}(\delta + 1, \beta)$, has ordertype at most δ.
  (b) *$V[G]_\kappa \subset L[\tilde{E}]$.*

Proof. Let $\mathbb{P}_{\kappa+1}$ be the backward Easton iteration of length $\kappa + 1$ such that at each strongly inaccessible cardinal $\delta \leq \kappa$, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \mathbb{Q}_\delta$ and we set $\mathbb{P} = \mathbb{P}_{\kappa+1}$.

Suppose $G \subset \mathbb{P}$ is V-generic. For each strongly inaccessible cardinal $\delta \leq \kappa$, let C^G_δ be the $V[G \cap \mathbb{P}_\delta]$-generic fast club in δ given by $G \cap \mathbb{P}_{\delta+1}$. We note that the following holds in $V[G]$, see Lemma 129 for the connection to the supercompactness of κ in $V[G]$. (The proof is just by the usual master-condition argument which one uses to show that κ is supercompact in $V[G]$, see for example the proof of Theorem 226 for a similar kind of argument.)

(1.1) For each $\lambda > \kappa$ and for each $a \in V[G]_\lambda$, there exist $(\bar{a}, \bar{\kappa}, \bar{\lambda}) \in V[G]_\kappa$ and an elementary embedding

$$j : V[G]_{\bar{\lambda}+1} \to V[G]_{\lambda+1}$$

such that

(a) $\mathrm{CRT}(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$ and $j(\bar{a}) = a$,
(b) $j(G \cap \mathbb{P}_{\bar{\kappa}}) = G \cap \mathbb{P}_\kappa$,
(c) $j(C^G_{\bar{\kappa}}) = C^G_\kappa$.

Let $\mathcal{E}$ be the set of all extenders F in $V[G]$ such that

(2.1) $\mathrm{CRT}(F) < \kappa$ and $j_F(\mathrm{CRT}(F)) \leq \kappa$.
(2.2) $\rho(F) = \mathrm{LTH}(F) = \lambda$ where λ is least such that $|V[G]_\lambda| = \lambda$ and such that $j_F(\mathrm{CRT}(F)) < \lambda$.

(2.3) $j_F(G \cap \mathbb{P}_{\mathrm{CRT}(F)}) = G \cap \mathbb{P}_{j_F(\mathrm{CRT}(F))}$.
(2.4) $j_F(C^G_{\mathrm{CRT}(F)}) = C^G_{j_F(\mathrm{CRT}(F))}$.

Let $\mathcal{M}$ be the set of all Martin–Steel extender sequences $\tilde{E}$ such that:

(3.1) For all $\delta < \kappa$, the set of β such that $(\delta+1, \beta) \in \mathrm{dom}(\tilde{E})$ and such that

$$j_E(\mathrm{CRT}(E)) = \delta$$

where $E = \tilde{E}(\delta+1, \beta)$, has ordertype at most δ.
(3.2) For all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$ if $\tilde{E}(\alpha, \beta)$ is a long extender then there is an extender $F \in \mathcal{E}$ which witnesses the coherence condition for $\tilde{E}(\alpha, \beta)$.

Note that $V[G]_\kappa = V[G \cap \mathbb{P}_\kappa]_\kappa$ and so $\mathcal{E} \cap V[G]_\kappa \in V[G \cap \mathbb{P}_\kappa]$ which implies that $\mathcal{M} \cap V[G]_\kappa \in V[G \cap \mathbb{P}_\kappa]$. Let $\tilde{E} \in V[G \cap \mathbb{P}_\kappa]$ be a generalized Martin–Steel extender sequence such that

(4.1) $\tilde{E} \subset V[G \cap \mathbb{P}_\kappa]_\kappa$,
(4.2) for all $(\alpha, \beta) \in \kappa \times \kappa$ if there exists $\tilde{E}^* \in \mathcal{M} \cap V[G]_\kappa$ such that

$$\tilde{E}|(\alpha, \beta) = \tilde{E}^*|(\alpha, \beta)$$

and such that $(\alpha, \beta) \in \mathrm{dom}(\tilde{E}^*)$ then $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$.

Such a sequence is easily constructed in $V[G \cap \mathbb{P}_\kappa]$. One simply constructs $\tilde{E}|(\alpha, \beta)$ by induction. Note that at any limit stage, the union of that which has been constructed is necessarily a generalized Martin–Steel extender sequence. In other words, an extender sequence of limit length all of whose proper initial segments are generalized Martin–Steel extender sequences, is itself a generalized Martin–Steel extender sequence.

We now apply (1.1). Let $\lambda > \kappa$ be least such that $|V[G]_\lambda| = \lambda$ By (1.1) there exists $(\bar{\kappa}, \bar{\lambda}, \tilde{F}) \in V[G]_\kappa$ and an elementary embedding,

$$j : V[G]_{\bar{\lambda}+\omega+1} \to V[G]_{\lambda+\omega+1}$$

such that

(5.1) $\mathrm{CRT}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$,
(5.2) $j(\tilde{F}) = \tilde{E}$,
(5.3) $j(G \cap \mathbb{P}_{\bar{\kappa}}) = G \cap \mathbb{P}_\kappa$,
(5.4) $j(C^G_{\bar{\kappa}}) = C^G_\kappa$.

An important point is that the strength condition for witnessing coherence for an extender E of length α relative to a generalized Martin–Steel extender sequence, is only $\alpha + \omega$.

We have that $\tilde{E} \cap V[G]_{\bar{\lambda}} \in \mathcal{M}$ and by the point just made, the extenders in $\mathcal{E} \cap V[G]_{\bar{\lambda}}$ suffice to witness the coherence condition for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E}) \cap V[G]_{\bar{\lambda}}$. Therefore it follows that $j(\tilde{E}) \cap V[G]_\lambda \in \mathcal{M}$.

Let $\tilde{E}_0 = j(\tilde{E}) \cap V[G]_\lambda$ and let F be the extender of length λ given by j. Thus $F \in j(\mathcal{E} \cap V[G]_{\lambda+\omega})$ and so by the maximality of $\tilde{E}$ it follows that the set of β such that $(\kappa+1, \beta) \in \mathrm{dom}(\tilde{E}_0)$ and such that

$$j_E(\text{CRT}(E)) = \kappa$$

where $E = \tilde{E}_0(\kappa+1, \beta)$, has ordertype κ. The argument here is just like the argument given in Remark 125 on p. 240.

We summarize the three key points.

(6.1) $\tilde{E}_0|\kappa \in V[G \cap \mathbb{P}_\kappa]$.
(6.2) $L[\tilde{E}_0] \cap V[G]_\kappa = L[\tilde{E}_0|\kappa] \cap V[G]_\kappa$.
(6.3) For each $(\kappa+1, \beta) \in \mathrm{dom}(\tilde{E}_0)$, if

$$j_E(\text{CRT}(E)) = \kappa$$

then $\text{CRT}(E) \in C^G_\kappa$, where $E = \tilde{E}_0(\kappa+1, \beta)$.

Both (6.1) and (6.3) are immediate from the remarks above since $\tilde{E}_0|\kappa = \tilde{E}$. The remaining claim, (6.2), follows by reflection using j.

Let D be the closure of the set of all $\text{CRT}(\tilde{E}_0(\kappa+1, \beta))$ such that $(\kappa+1, \beta) \in \mathrm{dom}(\tilde{E}_0)$ and $\tilde{E}_0(\kappa+1, \beta)$ is a long extender. Then D is a closed cofinal subset of C^G_κ and by (6.2), $D \cap \gamma \in L[\tilde{E}]$ for all $\gamma < \kappa$. But $\tilde{E} \in V[G \cap \mathbb{P}_\kappa]$ and C^G_κ is generic over $V[G \cap \mathbb{P}_\kappa]$ for $\mathbb{Q}_\kappa$. Therefore by Lemma 126, $V[G]_\kappa \subset L[\tilde{E}]$. ☐

The following variation of Theorem 127 essentially rules out avoiding the failure of comparison for generalized Martin–Steel extender sequences by restricting the class of such sequences by adding additional conditions beyond the novelty and initial segment conditions.

Theorem 128. *Suppose there is a proper class of supercompact cardinals. Then there is a class-generic extension $V[G]$ of V in which the following hold.*

(1) $V[G] = (\mathrm{HOD})^{V[G]}$.
(2) *There is a proper class of supercompact cardinals.*
(3) *Suppose $\tilde{E} = \langle E_\alpha : \alpha \in \mathrm{Ord}\rangle$ is an extender sequence and that δ is a strong cardinal such that*
  (a) *$\tilde{E}$ is Σ_2-definable and for all α, there exists an extender F such that $\rho(F) \geq \kappa$ and $E_\alpha = F|\alpha$ where κ is the least Mahlo cardinal above* $\text{LTH}(E_\alpha)$,
  (b) $\delta = \sup\{\text{CRT}(E_\alpha) \mid E_\alpha(\text{CRT}(E_\alpha)) = \delta < \text{LTH}(E_\alpha)\}$.

  Then $V[G]_\delta \subset L[\tilde{E}]$.

Proof. We assume that the GCH holds (by passing to a class forcing extension if necessary).

We fix some notation. Suppose κ is strongly inaccessible, $A \subset \kappa$, and the GCH holds above κ. Then $\mathbb{H}_{(A,\kappa)}$ denotes the partial order,

$$\mathbb{H}_{(A,\kappa)} = \Pi_{\alpha \in A} \mathrm{Add}(\gamma_\alpha, \gamma_\alpha^{++}),$$

where γ_α is the αth strongly inaccessible cardinal above δ and $\mathrm{Add}(\gamma_\alpha, \gamma_\alpha^{++})$ is the partial order for adding γ_α^{++} subsets of γ_α: the conditions are partial functions

$$f : \gamma_\alpha^{++} \to \{0, 1\}$$

with $|\mathrm{dom}(f)| < \gamma_\alpha$.

Let $\mathbb{P}_{\kappa+1}$ be the backward Easton iteration of length $\kappa + 1$ such that at each Mahlo cardinal $\delta \leq \kappa$, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \mathbb{Q}_\delta * \mathbb{H}_{(A,\delta)}$ and we set $\mathbb{P} = \mathbb{P}_{\kappa+1}$. Here $A \in V^{\mathbb{P}_\delta * \mathbb{Q}_\delta}$ and with Boolean value 1, A is a subset of δ which (canonically) codes $(V_\delta, G_\delta, C_\delta)$ where G_δ is the $\mathbb{P}_\delta$-generic filter and C_δ is the fast club given by the $V[G_\delta]$-generic filter for $(\mathbb{Q}_\delta)^{V[G_\delta]}$.

The class forcing is just $\mathbb{P}_\infty$. Suppose $G \subset \mathbb{P}_\infty$ is V-generic. Then since there is a proper class of Mahlo cardinals:

(1.1) $V[G] = (\mathrm{HOD})^{V[G]}$.
(1.2) If δ is supercompact in V then δ is supercompact in $V[G]$.
(1.3) Suppose κ is a Mahlo cardinal in V then $V[G]_\kappa = V_\kappa[G|\kappa]$.

Suppose that in $V[G]$, $\tilde{E} = \langle E_\alpha : \alpha \in \mathrm{Ord}\rangle$ is an extender sequence, δ is a strong cardinal, and that in $V[G]$:

(2.1) $\tilde{E}$ is Σ_2-definable and for all α, there exists an extender F such that $\rho(F) \geq \alpha$ and such that $E_\alpha = F|\alpha$,
(2.2) $\delta = \sup\{\mathrm{CRT}(E_\alpha) \mid E_\alpha(\mathrm{CRT}(E_\alpha)) = \delta < \mathrm{LTH}(E_\alpha) = \alpha\}$.

Then since δ is a strong cardinal,

(3.1) $V[G]_\delta = V_\delta[G|\delta]$,
(3.2) $(\tilde{E})^{V[G]_\delta} = \tilde{E}|\delta$,
(3.3) $L[\tilde{E}] \cap V[G]_\delta = L_\delta[\tilde{E}] = L_\delta[\tilde{E}|\delta]$,

where

$$(\tilde{E})^{V[G]_\delta} = \{a \mid V[G]_\delta \vDash \phi[a]\}$$

and $\phi(x)$ is the Σ_2-formula which defines $\tilde{E}$ in $V[G]$.

Let

$$D = \{\mathrm{CRT}(E_\alpha) \mid j_{E_\alpha}(\mathrm{CRT}(E_\alpha)) = \delta \text{ and } \delta < \mathrm{LTH}(E_\alpha)\},$$

and let C_δ be the fast club added by G at stage $\delta + 1$. Thus it follows that

(4.1) $D \subset C_\delta$ and $\sup(D) = \delta$,
(4.2) $D \in L[\tilde{E}]$,
(4.3) for all $\xi < \delta$, $D \cap \xi \in L_\delta[\tilde{E}] \subset V[G|\delta]$.

The remainder of the proof is exactly like the proof of Theorem 127. □

5. Closure Properties and Supercompactness

We begin with a simple lemma which gives a useful reformulation of supercompactness. This lemma is a trivial variant of an analogous reformulation due to Magidor.

Lemma 129 (Magidor). *The following are equivalent.*

(1) δ *is supercompact.*
(2) *For all* $\gamma > \delta$ *there exists an extender* E *such that*
   (a) $\text{SPT}(E) < \delta$ *and* $\rho(E) \geq \gamma$,
   (b) $j_E(\text{CRT}(E)) = \delta$.

Proof. We first suppose that δ is supercompact and prove that (2) holds. Fix $\gamma > \delta$. By increasing γ if necessary we can suppose that $\gamma = |V_\gamma|$. Since δ is supercompact there exists an elementary embedding,

$$j : V \to M,$$

such that $\text{CRT}(j) = \delta$, $j(\delta) > \gamma$, and such that

$$M^{V_{\gamma+1}} \subseteq M.$$

Let E be the extender of length $j(\gamma)$ given by j. Thus $E \in M$ and in M,

(1.1) $\text{SPT}(E) < j(\delta)$ and $\rho(E) = j(\gamma)$,
(1.2) $j_E(\text{CRT}(E)) = j(\delta)$.

Thus by the elementarity of j, there must exist an extender $F \in V$ such that

(2.1) $\text{SPT}(F) < \delta$ and $\rho(F) = \gamma$,
(2.2) $j_F(\text{CRT}(F)) = \delta$,

and this proves (2).

We now suppose that (2) holds and prove that δ is supercompact. Fix $\gamma_0 > \delta$. We must prove that there exists a normal fine δ-complete measure on $\mathcal{P}_\delta(\gamma_0)$. We assume toward a contradiction that there is no such measure and that γ_0 is as small as possible.

Let $\gamma > \gamma_0$ be such that $\gamma = |V_\gamma|$. By (2) there exists an extender E such that

(3.1) $\text{SPT}(E) < \delta$ and $\rho(E) \geq \gamma$,
(3.2) $j_E(\text{CRT}(E)) = \delta$.

By replacing E with $E|\gamma$ we can suppose that $\rho(E) = \text{LTH}(E) = \gamma$. Let

$$j_E : V \to M$$

be the elementary embedding given by E. Since $V_\gamma \subseteq M$, by the choice of γ_0, γ_0 is definable in M from δ. Therefore there must exist $\bar{\gamma}_0$ such that $j_E(\bar{\gamma}_0) = \gamma_0$. Let $\bar{\delta} = \text{CRT}(E)$. Necessarily $j_E(\text{SPT}(E)) \geq \rho(E)$ (this is true for any extender) and so since $j_E(\text{CRT}(E)) = \delta$, the elementary embedding, j_E, induces a normal fine $\bar{\delta}$-complete measure, μ, on $\mathcal{P}_{\bar{\delta}}(\bar{\gamma}_0)$. Finally, $j_E(\mu)$ is a normal fine δ-complete measure on $\mathcal{P}_\delta(\gamma_0)$ which contradicts the choice of γ_0. □

Lemma 129 motivates the next definition which is a central theme of this paper.

Definition 130. Suppose $\mathcal{E}$ is a class.

(1) $o^{\mathcal{E}}_{\text{mLONG}}(\delta) = \infty$ if for all $\gamma > \delta$ there exists an extender $E \in \mathcal{E}$ such that
 (a) $\text{SPT}(E) < \delta$ and $\rho(E) > \gamma$,
 (b) $j_E(\text{CRT}(E)) = \delta$.

(2) $o^{\mathcal{E}}_{\text{sLONG}}(\delta) = \infty$ if for all $\gamma > \delta$ there exists an extender $E \in \mathcal{E}$ such that
 (a) $\text{CRT}(E) = \delta$,
 (b) $\text{SPT}(E) > \gamma$.

Remark 131. By Lemma 129, $o^{V}_{\text{mLONG}}(\delta) = \infty$ if and only if $o^{V}_{\text{sLONG}}(\delta) = \infty$.

5.1. *Closure properties of N*

We define a slight variation of supercompactness.

Definition 132. Suppose that N is transitive, $N \subseteq V$, $N \vDash \text{ZFC}$, and $\text{Ord} \subseteq N$. A supercompact cardinal, δ, is N-supercompact if for all $\lambda > \delta$ there exists an elementary embedding,

$$j : V \to M$$

such that

(1) $\text{CRT}(j) = \delta$,
(2) $\lambda < j(\delta)$,
(3) $M^{V_\lambda} \subseteq M$,
(4) $j(N \cap V_\delta) \cap V_\lambda = N \cap V_\lambda$.

The following lemma gives an equivalent and more useful formulation.

Lemma 133. *Suppose that N is transitive, $N \subseteq V$, $N \vDash \text{ZFC}$, $\text{Ord} \subseteq N$, and that δ is N-supercompact. Suppose $\gamma > \delta$ and that $a \in V_\gamma$. Then there exists an elementary embedding,*

$$j : V \to M$$

with critical point $\bar{\delta} < \delta$ such that

(1) $j(\bar{\delta}) = \delta$,
(2) *there exists* $(\bar{a}, \bar{\gamma}) \in V_\delta$ *such that* $j(\langle \bar{a}, \bar{\gamma} \rangle) = \langle a, \gamma \rangle$,
(3) $V_\gamma \subseteq M$ *and* $j(N \cap V_{\bar{\gamma}}) = N \cap V_\gamma$.

Proof. Fix $\gamma > \delta$ and $a \in V_\gamma$. By increasing γ if necessary and replacing a by (a, γ), we can suppose that $\gamma = |V_\gamma|$. Let

$$j_0 : V \to M_0$$

be an elementary embedding with critical point δ such that

$$M_0^{V_{\gamma+1}} \subseteq M_0,$$

such that $\gamma < j_0(\delta)$ and such that

$$V_\gamma \cap N = V_\gamma \cap j_0(N \cap V_\delta).$$

Let E be the $(\delta, j_0(\gamma))$-extender derived from j_0. Since

$$M_0^{V_{\gamma+1}} \subseteq M_0$$

it follows that $E \in M_0$. But since $|V_\gamma| = \gamma$,

$$j_E(b) = (j_E)^{M_0}(b)$$

for all $b \subseteq V_\gamma$. Thus $(j_E)^{M_0}$ witnesses that the lemma holds in M_0 at $j_0(\delta)$ for $(j_E(\gamma), j_E(a))$. Therefore the lemma holds in V at δ for (γ, a). $\square$

Remark 134. Suppose that κ is a huge cardinal. Suppose that $N \subseteq V_\kappa$,

$$N \vDash \mathrm{ZFC}$$

and $\kappa \subseteq N$. Then there exists $\delta < \kappa$ such that

$$(V_\kappa, N) \vDash \text{“}\delta \text{ is } N\text{-supercompact”}.$$

Definition 135. Suppose that N is transitive, $N \subseteq V$, $N \vDash \mathrm{ZFC}$, and $\mathrm{Ord} \subseteq N$. Then

$$o^N_{\mathrm{LONG}}(\delta) = \infty$$

if for all $\lambda > \delta$ there exists a normal fine δ-complete measure, μ, on $\mathcal{P}_\delta(\lambda)$ such that

(1) $\mu(N \cap \mathcal{P}_\delta(\lambda)) = 1$,
(2) $\mu \cap N \in N$.

The following easy lemma shows that if $o^N_{\mathrm{LONG}}(\delta) = \infty$, then δ is N-supercompact. The notions are not in general equivalent (note that if δ is supercompact then δ is L-supercompact).

Lemma 136. *Suppose that N is transitive, $N \subseteq V$, $N \vDash \mathrm{ZFC}$, and $\mathrm{Ord} \subseteq N$. Suppose $\delta < \kappa$, $|V_\kappa| = \kappa$, and that μ is a normal fine measure on $\mathcal{P}_\delta(\kappa)$ such that*

(i) $\mu(N \cap \mathcal{P}_\delta(\kappa)) = 1$,
(ii) $\mu \cap N \in N$.

Let $j : V \to M$ be the elementary embedding given by the ultrapower of V by μ. Then $j(N \cap V_\delta) \cap V_\kappa = N \cap V_\kappa$.

Proof. Note that

$$|N_\kappa|^N = \kappa$$

since $|V_\kappa| = \kappa$.

Let

$$\pi : \kappa \to N \cap V_\kappa$$

be a bijection with $\pi \in N$.

For each $X \subseteq \kappa$ let

$$N_X = \{\pi(\alpha) \mid \alpha \in X\}.$$

To verify the conclusion of the lemma, it suffices to prove the claim that the set of $X \in \mathcal{P}_\delta(\kappa)$ such that

$$N \cap V_{\kappa_X} \cong N_X$$

is of μ-measure 1, where κ_X is the ordertype of X. The key point is that in N, $\mu \cap N$ is a normal fine measure on $\mathcal{P}_\delta(\kappa)$ and $\pi \in N$. The claim follows, simply use the ultrapower embedding computed in N using $\mu \cap N$. □

For now we are primarily interested in the case that N is transitive, $N \subseteq V$,

$$N \vDash \text{ZFC},$$

$\text{Ord} \subseteq N$ and $o^N_{\text{LONG}}(\delta) = \infty$. The first theorem we shall prove is an immediate consequence of the definition, using the following theorem of Solovay.

Theorem 137 (Solovay). *Suppose that $\lambda > \delta$, λ is a regular cardinal and that μ is a normal fine measure on $\mathcal{P}_\delta(\lambda)$. Then there exists $X \in \mu$ such that for all $\sigma, \tau \in X$ if*

$$\sup(\sigma) = \sup(\tau)$$

then $\sigma = \tau$.

Proof. Remarkably, the set X does not depend on μ. Let

$$\langle S_\alpha : \alpha < \lambda \rangle$$

be a partition of $\{\alpha < \lambda \mid \text{cof}(\alpha) = \omega\}$ into pairwise disjoint stationary sets. For each $\eta < \lambda$ such that $\delta > \text{cof}(\eta) > \omega$, let Z_η be the of $\alpha < \eta$ such that $S_\alpha \cap \eta$ is stationary in η (in the sense that for all closed cofinal sets $C \subset \eta$, $S_\alpha \cap C \neq \emptyset$). Let

$$X = \{Z_\eta \mid \eta < \lambda, \omega < \text{cof}(\eta) < \delta, \text{ and } \sup(Z_\eta) = \eta\}.$$

It suffices to show that $X \in \mu$. Let

$$j : V \to M$$

be elementary embedding given by μ. Thus

(1.1) $\text{CRT}(j) = \delta$ and $j(\delta) > \lambda$,
(1.2) $\{j(\alpha) \mid \alpha < \lambda\} \in M$,
(1.3) for all $Y \subset \mathcal{P}_\delta(\lambda)$, $Y \in \mu$ if and only if $\{j(\alpha) \mid \alpha < \lambda\} \in j(Y)$.

Let $\eta = \sup\{j(\alpha) \mid \alpha < \lambda\}$. Since $\{j(\alpha) \mid \alpha < \lambda\}$ is ω-closed, for each $\alpha < \lambda$,

$$M \vDash \text{“}j(S_\alpha) \cap \eta \text{ is stationary in } \eta\text{”}.$$

Since $\{j(\alpha) \mid \alpha < \lambda\} \in M$, it follows that $\{j(\alpha) \mid \alpha < \lambda\} \in j(X)$ and so $X \in \mu$. □

Theorem 138. *Suppose that $o^N_{\text{LONG}}(\delta) = \infty$. Then*:

(1) *Suppose $a \in [\text{Ord}]^{<\delta}$, then there exists $b \in [\text{Ord}]^{<\delta} \cap N$ such that $a \subseteq b$.*
(2) *Suppose $\lambda > \delta$ and λ is a singular cardinal. Then λ is a singular cardinal in N and $\lambda^+ = (\lambda^+)^N$.*
(3) *Suppose $\gamma > \delta$ and γ is a regular cardinal in N. Then $|\gamma| = \text{cof}(\gamma)$.*

Proof. (1) is an immediate from the definitions and (3) implies (2). We prove (3). Let μ be a normal fine measure on $\mathcal{P}_\delta(\gamma)$ such that

(a) $\mu(N \cap \mathcal{P}_\delta(\gamma)) = 1$,
(b) $\mu \cap N \in N$.

By Theorem 137, there exists a set $X \in \mu \cap N$ such that for all $\sigma, \tau \in X$ if

$$\sup(\sigma) = \sup(\tau)$$

then $\sigma = \tau$.

Assume toward a contradiction that $\text{cof}(\gamma) < |\gamma|$. By (1) and since γ is regular in N, $\text{cof}(\gamma) \geq \delta$. Let ν be the uniform measure on γ induced by μ; i.e. for all $A \subseteq \gamma$, $A \in \nu$ if and only if

$$\{\sigma \in \mathcal{P}_\delta(\gamma) \mid \sup(\sigma) \in A\} \in \mu.$$

Suppose $C \subseteq \gamma$ is closed and unbounded in γ, then $C \in \nu$. Therefore since $\text{cof}(\gamma) < |\gamma|$, there exists $A \subseteq \gamma$ such that $A \in \nu$ and $|A| < |\gamma|$. But ν induces a measure μ^* on $\mathcal{P}_\delta(\gamma)$ using X and necessarily, $\mu^* = \mu$. Therefore μ^* is a fine measure on $\mathcal{P}_\delta(\gamma)$ and this is a contradiction. □

The next lemma is the version of Lemma 133 for the case that $o^N_{\text{LONG}}(\delta) = \infty$. It will simplify the subsequent proof we shall give on the closure of N under extenders. From the perspective of extender sequences this lemma explains the notation, $o^N_{\text{LONG}}(\delta) = \infty$. We note that given our convention on extenders, if E is an extender then the condition

$$E \cap N \in N$$

is equivalent to the condition that the function $\pi_E | N \in N$ where

$$\pi_E : \mathcal{P}(\eta) \to V$$

is the function given by $\pi_E(a) = j_E(a) \cap \gamma$, $\eta = \text{SPT}(E)$ and $\gamma = \text{LTH}(E)$.

Remark 139. Notice that the converse of the next lemma holds as well. More precisely, suppose that N is transitive, $N \subseteq V$, $N \vDash \text{ZFC}$, and $\text{Ord} \subseteq N$. Suppose that $\delta \in \mathrm{Ord}$ and for each $\gamma > \delta$ there exists an elementary embedding

$$j : V_{\bar{\gamma}+\omega} \to V_{\gamma+\omega}$$

such that the following hold where $\bar{\delta} = \mathrm{CRT}(j)$ and where E is the extender of length γ given by j.

(1) $j(\bar{\delta}) = \delta$,
(2) $j(N \cap V_{\bar{\gamma}}) = N \cap V_\gamma$,
(3) $E \cap N \in N$.

Then $o^N_{\mathrm{LONG}}(\delta) = \infty$.

Lemma 140. *Suppose that $o^N_{\mathrm{LONG}}(\delta) = \infty$. Suppose $\gamma > \delta$, $|V_\gamma| = \gamma$, and that $a \in V_\gamma$. Then there exists an elementary embedding*

$$j : V_{\bar{\gamma}+\omega} \to V_{\gamma+\omega}$$

such that the following hold where $\bar{\delta} = \mathrm{CRT}(j)$ and where E is the extender of length γ given by j.

(1) $j(\bar{\delta}) = \delta$,
(2) *there exists $\bar{a} \in V_\delta$ such that $j(\langle \bar{a}, \bar{\gamma}\rangle) = \langle a, \gamma\rangle$,*
(3) $j(N \cap V_{\bar{\gamma}}) = N \cap V_\gamma$,
(4) $E \cap N \in N$.

Proof. Fix a cardinal $\kappa > \gamma$ such that $|V_\kappa| = \kappa$. Let μ be a normal fine δ-complete measure on $\mathcal{P}_\delta(\kappa)$ such that

(1.1) $\mu(N \cap \mathcal{P}_\delta(\kappa)) = 1$.
(1.2) $\mu \cap N \in N$.

Let

$$j : V \to M$$

be the elementary embedding given by the ultrapower of V by μ. By Lemma 136,

$$j(N \cap V_\delta) \cap V_\kappa = N \cap V_\kappa.$$

Thus $j|V_{\gamma+\omega} \in M$ and witnesses the conclusion of the lemma holds in M for $(j(\gamma), j(a))$ relative to $j(N)$. $\square$

Corollary 141. *Suppose that $o^N_{\mathrm{LONG}}(\delta) = \infty$. Let $\mathcal{E}$ be the class of all extenders, E, such that $\rho(E) = \mathrm{LTH}(E)$ and let*

$$\mathcal{E}_N = \{E \cap N \mid E \in \mathcal{E} \text{ and } E \cap N \in N\}.$$

Then $o^{\mathcal{E}_N}_{\mathrm{mLONG}}(\delta) = \infty$.

Theorem 142. *Suppose that $o^N_{\mathrm{LONG}}(\delta) = \infty$. Suppose that $\gamma > \delta$ and γ is a cardinal of N. Suppose that M is a transitive set, and that*

$$j : (H(\gamma^+))^N \to M$$

is an elementary embedding with $\mathrm{CRT}(j) \geq \delta$. *Suppose that* $\lambda \leq j(\gamma)$ *and* $\mathcal{P}(\lambda) \cap M \subseteq N$. *Let* F *be the* N*-pre-extender of length* λ *given by* j. *Then* $\mathrm{Ult}(N, F)$ *is wellfounded and* $F \in N$.

Proof. Let F be the N-pre-extender of length λ given by j. If $F \in N$ then necessarily $\mathrm{Ult}(N, F)$ is wellfounded and so it suffices to just prove that $F \in N$.

Fix $\kappa > \gamma$ such that

$$|V_\kappa| = \kappa.$$

By Lemma 140 there exist

$$\bar{\delta} < \bar{\gamma} \leq \bar{\lambda} < \bar{\kappa} < \delta,$$

a transitive set $\bar{M} \in N$, an elementary embedding,

$$\bar{j} : (H(\bar{\gamma}^+))^N \to \bar{M}$$

with $\bar{j} \in V_{\bar{\kappa}}$, and an elementary embedding

$$\pi : V_{\bar{\kappa}+1} \to V_{\kappa+1},$$

such that the following hold.

(1.1) $\mathrm{CRT}(\pi) = \bar{\delta}$.
(1.2) $\pi(\bar{\delta}) = \delta$ and $\pi(\bar{\lambda}) = \lambda$.
(1.3) $\pi(\bar{M}) = M$ and $\pi(\bar{j}) = j$.
(1.4) $\pi(N \cap V_{\bar{\kappa}}) = N \cap V_\kappa$.
(1.5) $\pi|(N \cap V_{\bar{\kappa}+1}) \in N$.

Let $\bar{F}$ be the N-pre-extender of length $\bar{\lambda}$ derived from $\bar{j}$. Thus $\pi(\bar{F}) = F$.

We prove that $\bar{F} \in N$. Suppose that $s \in [\bar{\lambda}]^{<\omega}$ and that

$$a \in \mathcal{P}([\bar{\gamma}]^{|s|}) \cap N.$$

Then $a \in \bar{F}(s)$ if and only if $s \in \bar{j}(a)$. But the latter holds if and only if,

$$\pi(s) \in \pi(\bar{j}(a)).$$

But $\pi(\bar{j}) = j$ and so $a \in \bar{F}(s)$ if and only if $\pi(s) \in j(\pi(a))$.

Let E be the N-extender of length κ given by π. Thus $E|\gamma \in (H(\gamma^+))^N$.

Let $H = j(E|\gamma)|\lambda$. Since $\mathrm{SPT}(E) < \delta$ and since

$$\mathcal{P}(\lambda) \cap M \subseteq N,$$

it follows that $H \in N$ and that H is a pre-extender in N of length λ. If $\mathrm{Ult}(N, H)$ is wellfounded then H is an extender in N. Note that $N \cap V_\delta \subseteq M$ and $\mathrm{SPT}(H) \leq \mathrm{SPT}(E) < \delta$. Suppose toward a contradiction that $\mathrm{Ult}(N, H)$ is not wellfounded. Then there exists

$$f : \omega \to \lambda$$

such that $f \in N$ and such that the tower,

$$\langle H(f|k) : k < \omega \rangle$$

is not wellfounded. But $\text{CRT}(j) \geq \delta$ and $H = j(E|\gamma)|\lambda$. Therefore there must exist

$$g : \omega \to \gamma$$

such that $g \in N$ and such that

$$\langle E(g|k) : k < \omega \rangle = \langle H(f|k) : k < \omega \rangle$$

and this is a contradiction. Thus $\text{Ult}(N, H)$ is wellfounded and so H is an extender in N. Let

$$j_H : N \to M_H$$

be the associated elementary embedding where $M_H \cong \text{Ult}(N, H)$.

We come to the key point. For each $a \in \mathcal{P}([\bar{\gamma}]^{|s|}) \cap N$,

$$j(\pi(a)) \cap \lambda^{|s|} = j_H(j(a)) \cap \lambda^{|s|} = j_H(a) \cap \lambda^{|s|}.$$

This follows from the definition of H and how j_H approximates $j(\pi|N \cap V_{\bar{\gamma}+1})$ noting that since $a \in V_\delta$, $\delta \leq \text{CRT}(j)$ and so $j(a) = a$.

Putting everything together, for all $s \in [\bar{\lambda}]^{<\omega}$, for all

$$a \in \mathcal{P}([\bar{\gamma}]^{|s|}) \cap N$$

$a \in \bar{F}(s)$ if and only if $\pi(s) \in j_H(a)$.

This implies that $\bar{F} \in N$. But $\pi(\bar{F}) = F$ and so $F \in N$. □

The next theorem is a special case of Theorem 142.

Theorem 143. *Suppose that $o^N_{\text{LONG}}(\delta) = \infty$. Suppose that $\gamma > \delta$ and γ is a cardinal of N. Suppose that $M \in N$, M is a transitive set, and that*

$$j : (H(\gamma^+))^N \to M$$

is an elementary embedding with critical point $\kappa \geq \delta$. Then $j \in N$.

Proof. Let $\lambda = j(\gamma)$. Then since $M \in N$,

$$\mathcal{P}(\lambda) \cap M \subseteq N$$

and so by Theorem 142, $F \in N$ where F is the N-extender of length λ given by j. But this implies that

$$j|(\mathcal{P}(\gamma) \cap N) \in N$$

which implies that $j \in N$. □

There are two useful special cases of Theorem 143.

Theorem 144. *Suppose that $o^N_{\text{LONG}}(\delta) = \infty$. Suppose that $\gamma > \delta$ and γ is a cardinal of N. Suppose that*

$$j : (H(\gamma^+))^N \to (H(j(\gamma)^+))^N$$

is an elementary embedding with critical point $\kappa \geq \delta$. Then $j \in N$.

The second useful version of Theorem 143 is an immediate corollary of Theorem 144, noting that for all ordinals $\gamma \geq \omega$,

$$V_{\gamma+1} \sim H(|V_\gamma|^+).$$

Theorem 145. *Suppose that* $o^N_{\text{LONG}}(\delta) = \infty$. *Suppose that* $\gamma \in \text{Ord}$,

$$j : N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)+1}$$

is an elementary embedding with critical point $\kappa \geq \delta$. *Then* $j \in N$.

An immediate corollary of Theorem 145 is that with N and δ as in the statement of Theorem 145, there can be no non-trivial elementary embedding,

$$j : N \to N,$$

with $\text{CRT}(j) \geq \delta$. The point of course is that for all $\gamma \in \text{Ord}$, $j|N_\gamma \in N$ and so by Kunen's theorem on the nonexistence of a non-trivial elementary embedding,

$$j : V_{\lambda+2} \to V_{\lambda+2},$$

(assuming the Axiom of Choice), j must be trivial.

At this point a natural conjecture or concern (especially in light of Theorem 127) is that if N and δ are as in the statement of Theorem 145, then something like $V = N$ must hold, so that the conclusion of Theorem 145 holds for trivial reasons. Such a strong version of Theorem 145 would essentially eliminate the possibility of an extender based inner model theory at the level of one supercompact cardinal just as Theorem 127 eliminates the possibility of any coarse theory of coherent extender models at this level.

We shall show that this is not the case, in fact we shall show that it is possible for there to exist N and δ as in the statement of Theorem 145, for which there is a non-trivial elementary embedding,

$$j : N \to N.$$

We require a technical lemma (Lemma 147 below) which in turn requires the following definition.

Definition 146. An extender, E, of length η is λ-*complete* if

$$\eta^\lambda \subseteq M$$

where $M = \text{Ult}(V, E)$.

Suppose that E is a (κ, η)-extender, $\mathbb{P} \in V_\kappa$, and $G \subseteq \mathbb{P}$ is V-generic. Then E naturally defines a (κ, η)-extender in $V[G]$ and

$$(j_E)^{V[G]}|V = (j_E)^V.$$

The special case of Lemma 147 where the extender E has length $\text{CRT}(E) + 1$ (i.e. for measures) is due to Steel and is a key part of the proof of Theorem 15 on p. 120.

Lemma 147. *Suppose that $\delta < \kappa$, E is an extender which is δ-complete with critical point κ, and that*

$$j : V \to M \subseteq V[G]$$

is a generic elementary embedding such that

(i) $M = \{j(f)(\alpha) \mid \alpha < \delta\}$,
(ii) *G is V-generic for some partial order $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \delta$ in V.*

Then $(j_E)^{V[G]}|M = (j_F)^M$ where $F = j(E)$.

Proof. By (i), $M = \mathrm{Ult}(V, H)$ where H is a V-extender of length δ.

Let $\eta = \mathrm{LTH}(E)$ and for each $a \in [\eta]^\delta$ let E_a be the ultrafilter,

$$E_a = \{A \subseteq [\hat{\eta}]^\delta \mid a \in j_E(A)\},$$

where $\hat{\eta} = \min\{\gamma \mid \eta \leq j_E(\gamma)\}$.

Since E is δ-complete for each $a \in [\eta]^\delta$, $a \in \mathrm{Ult}(V, E)$ and so E_a is defined.

Suppose that $a \subseteq b$ and $b \in [\eta]^\delta$. Then there is a natural elementary embedding,

$$j_{a,b} : \mathrm{Ult}(V, E_a) \to \mathrm{Ult}(V, E_b).$$

This defines a directed system indexed by the directed set, $([\eta]^\delta, \subseteq)$ with limit, $\mathrm{Ult}(V, E)$.

This is just the usual analysis of $\mathrm{Ult}(V, E)$ as the limit of a directed system of ultrapowers except here the underlying directed set is $([\eta]^\delta, \subseteq)$ instead of the directed set, $([\eta]^{<\omega}, \subseteq)$.

Let $X = [\hat{\eta}]^\delta$. For each $a \in [\eta]^\delta$, $E_a \subseteq \mathcal{P}(X)$ and E_a is an ultrafilter on X. Fix $a \in [\eta]^\delta$.

We first show the following. Suppose that

$$f : X \to M$$

is a function in $V[G]$. Then there exists a function

$$f^* : j(X) \to M$$

such that $f^* \in M$ and such that

$$\{y \in X \mid f(y) = f^*(j(y))\} \in (E_a)_G$$

where $(E_a)_G$ is the ultrafilter in $V[G]$ generated by E_a.

Fix f and work in $V[G]$. For each $y \in X$ there exists a pair (g_y, α_y) such that

(1.1) $\alpha_y < \delta$,
(1.2) $g_y \in V$,
(1.3) $f(y) = j(g_y)(\alpha_y)$.

This defines a function

$$F : X \to V$$

where for all $y \in X$, $F(y) = (g_y, \alpha_y)$.

Since E_a is κ-complete and since $|\mathbb{P}|^V \leq \delta < \kappa$, it follows that there exists $Z \in E_a$ and there exists $\alpha < \delta$ such that

(2.1) $F|Z \in V$,
(2.2) $\alpha_y = \alpha$ for all $y \in Z$.

Define

$$f^* : j(X) \to M$$

by $f^*(t) = 0$ if $t \notin j(Z)$ and if $t \in j(Z)$ then

$$f^*(t) = j(F)_t(\alpha)$$

where for each $y \in X$, $F_y = g_y$.

Thus for each $y \in Z$,

$$f^*(j(y)) = j(F)_{j(y)}(\alpha) = (j(F_y))(\alpha) = (j(g_y))(\alpha) = (j(g_y))(\alpha_y) = f(y),$$

and so f^* is as required.

What we have done is show that for each $a \in [\eta]^\delta$ the lemma holds with E replaced by E_a. This special case is due to Steel.

Now we use the hypothesis that E is δ-complete. The key point is that the set

$$\{j(a) \mid a \in [\eta]^\delta\}$$

is cofinal in the directed set,

$$\{a \mid a \in j([\eta]^\delta)\}.$$

To see this suppose $b \in j([\eta]^\delta)$. Then there exists $\alpha < \delta$ and a function

$$g : \delta \to [\eta]^\delta$$

such that $j(g)(\alpha) = b$ noting that $\delta \leq j(\delta)$. Let $a = \cup\{g(\beta) \mid \beta < \delta\}$. Thus $a \in [\eta]^\delta$, $a \in V$ and $b \subseteq j(a)$.

Thus

$$\{j(a) \mid a \in [\eta]^\delta\}$$

is cofinal in the directed set,

$$\{a \mid a \in j([\eta]^\delta)\},$$

and so $\mathrm{Ult}(M, j(E))$ is the limit of $\mathrm{Ult}(M, j(E_a))$ over the directed set $([\eta]^\delta, \subseteq)^V$ and the lemma follows by the correspondence of functions established above. □

There is a useful corollary of Lemma 147.

Corollary 148. *Suppose that* $\delta < \kappa$, κ *is supercompact, and that*

$$j : V \to M \subseteq V[G]$$

is a generic elementary embedding such that

(i) $M = \{j(f)(\alpha) \mid \alpha < \delta\}$,
(ii) G *is* V*-generic for some partial order* $\mathbb{P} \in V$ *such that* $|\mathbb{P}| \leq \delta$ *in* V.

Then in $V[G]$, $o^M_{\text{LONG}}(\kappa) = \infty$.

Proof. By Lemma 147, for each extender $E \in V$, if (in V),

(1.1) $\mathbb{P} \in V_{\text{CRT}(E)}$,
(1.2) $\rho(E) = \text{LTH}(E)$,
(1.3) $\text{cof}(\text{LTH}(E)) > \delta$,

then in $V[G]$, $E_G \cap M \in M$ where E_G is the extender in $V[G]$ generated by E. The point is that by Lemma 147,

$$j(E) = E_G \cap M.$$

Since κ is supercompact in V, the class of all such extenders, E_G, witnesses that κ is supercompact in $V[G]$. The corollary follows. □

There is a more general version of Corollary 148 which is worth noting.

Theorem 149. *Suppose that* $o^N_{\text{LONG}}(\kappa) = \infty$. *Suppose that* $\mathbb{P} \in V_\kappa$, $G \subseteq \mathbb{P}$ *is* V*-generic and in* $V[G]$,

$$j : N \to M \subseteq V[G]$$

is an elementary embedding such that for some $\delta < \kappa$,

$$M = \{j(f)(s) \mid s \in [\delta]^{<\omega}\}.$$

Then in $V[G]$, $o^M_{\text{LONG}}(\kappa) = \infty$.

Proof. Suppose E is an extender such that

(1.1) $\text{LTH}(E) = \rho(E)$,
(1.2) $E \cap N \in N$,
(1.3) $|\mathbb{P}| < \text{CRT}(E) \leq \text{cof}(\rho(E))$.

Arguing essentially as in the proof of Lemma 147,

$$j(E \cap N) = E_G \cap N$$

where E is the extender in $V[G]$ generated by E.

The class, $\mathcal{E}$, of all extenders E which satisfy (1.1)–(1.3) witnesses that $o^N_{\text{LONG}}(\kappa) = \infty$ and so the class

$$\mathcal{E}_G = \{E_G \mid E \in \mathcal{E}\}$$

witnesses in $V[G]$ that $o^M_{\text{LONG}}(\kappa) = \infty$. □

The following lemma provides evidence that Theorem 145 is not true for trivial reasons which could be the case if necessarily N has much stronger closure properties such as those analogous to Theorem 127.

Lemma 150. *Suppose δ is a supercompact cardinal. Then there exists an inner model N of* ZFC *such that*

(1) $o^N_{\text{LONG}}(\delta) = \infty$ *and* $N^\omega \subseteq N$,
(2) *there is a non-trivial elementary embedding*, $j : N \to N$,
(3) V *is not a generic extension of* N.

Proof. Let $\kappa < \delta$ be a measurable cardinal and let μ be a normal measure on κ. Let

$$j_0 : V \to M_0$$

be the associated elementary embedding.

For each $i \leq \omega$, let M_i be given by ith iterate of j_0 and let

$$j_{0,i} : V \to M_i$$

be the associated elementary embedding. By Corollary 148, $o^{M_\omega}_{\text{LONG}}(\delta) = \infty$.

Let $\langle \kappa_i : i < \omega \rangle$ be the critical sequence of j_0 and let $N = \cap_i M_i$. Then $N = M_\omega[G]$ where $G = \langle \kappa_i : i < \omega \rangle$ and moreover G is M_ω-generic for the Prikry forcing given by $j_{0,\omega}(\mu)$. Thus $N^\kappa \subseteq N$ and $o^N_{\text{LONG}}(\delta) = \infty$.

For each $i < \omega$, let

$$j_{i,i+1} : M_i \to M_{i+1}$$

be iteration embedding, $j_{i,i+1} = j_{0,i}(j_0)$. Thus for all $i < \omega$, $j_{i,i+1}(N) = N$.

We finish by showing that $N[\mu]$ is not a set-generic extension of N, this will prove that (3) holds since (3) is equivalent to the assertion that for some $a \subseteq N$, $N[a]$ is not a set-generic extension of N.

We have that $N^\kappa \subseteq N$ and so for each $\gamma \in \text{Ord}$, $j|\gamma \in N[\mu]$. But this implies that for every set $a \subseteq \text{Ord}$, $a \in N[\mu]$, since for each $a \subseteq \text{Ord}$,

$$a = \{\alpha \,|\, j(\alpha) \in j(a)\}.$$

Thus $V = N[\mu]$. Finally there exists a proper class of $\gamma \in \text{Ord}$ such that γ is a cardinal of N but not a cardinal in V. Therefore $N[\mu]$ is not a set-generic extension of N. $\square$

Another corollary of Theorem 145 is the following equivalence and this is immediate by Lemma 129.

Theorem 151. *Suppose that $o^N_{\text{LONG}}(\delta) = \infty$ and $\kappa > \delta$. Then the following are equivalent.*

(1) κ *is N-supercompact.*
(2) $o^N_{\text{LONG}}(\kappa) = \infty$.

The following is an immediate corollary of Theorems 138 and 151.

Theorem 152. *Suppose that $o^N_{\text{LONG}}(\delta) = \infty$. Suppose that E is an extender such that*

(i) $\delta < \text{CRT}(E) < \text{SPT}(E)$,
(ii) $\rho(E) = \text{LTH}(E)$ *and both* $\text{SPT}(E)$ *and* $\text{LTH}(E)$ *are strongly inaccessible*,
(iii) $\text{SPT}(E) = \sup\{\gamma < \text{SPT}(E) \mid \gamma$ *is N-supercompact*$\}$,
(iv) $E \cap N \in N$.

Then $j_E|N = (j_F)^N$ where $F = E \cap N$.

Proof. By (ii), $\text{SPT}(E)$ is a limit ordinal and by (iii), $\text{SPT}(E)$ is a limit of supercompact cardinals. Therefore $|V_{\text{SPT}(E)}| = \text{SPT}(E)$ and so $j_N|(N \cap V_{\text{SPT}(E)}) = (j_F)^N(N \cap V_{\text{SPT}(E)})$.

By Theorems 138, 151 and (iii), for each $\sigma \subseteq N$ such that $|\sigma| < \text{SPT}(E)$, there exists $\tau \in N$ such that $\sigma \subseteq \tau$ and $|\tau| < \text{SPT}(E)$. It follows that $j_E|N = (j_F)^N$. We sketch the argument. It suffices to show that for each $s \in [\text{LTH}(E)]^{<\omega}$ and for each function

$$f : V_{\text{SPT}(E)} \to N$$

there exist $t \in [\text{LTH}(E)]^{<\omega}$ and a function

$$g : V_{\text{SPT}(E)} \to N$$

such that $g \in N$, $s \subseteq t$ and such that in the direct limit which defines $\text{Ult}(V, E)$, f and g define the same element.

Fix (s, f). Choose $Y \in N$ such that $|Y|^N < \text{SPT}(E)$ and such that Y covers the range of f. Let $\gamma = |Y|^N$ and fix a bijection

$$h : Y \to \gamma$$

such that $h \in N$. Let $f^* = h \circ f$. Since

$$j_E|(N \cap V_{\text{SPT}(E)}) = (j_E)^N|(N \cap V_{\text{SPT}(E)})$$

it follows that there exists $t \in [\text{LTH}(E)]^{<\omega}$ and a function

$$g^* : V_{\text{SPT}(E)} \to N$$

such that $g^* \in N$, $s \subseteq t$ and such that in the direct limit which defines $\text{Ult}(V, E)$, f^* and g^* define the same element of $j_E(\gamma)$. Finally let $g = h^{-1} \circ g^*$. It follows that t and g are as required. $\square$

We now turn to discuss the connection with iteration trees.

Definition 153. Suppose that (M, δ) is a premouse, $\mathcal{E} \in M$ and $\mathcal{E} \subseteq M_\delta$. Then an iteration tree,

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

on (M, δ) is an *iteration tree on* $(M, \delta, \mathcal{E})$ if for all $\beta < \eta$, $E_\beta \in j_{0,\beta}(\mathcal{E})$.

For the statement of Theorem 154 it is convenient to introduce the following notation. Suppose that N is an inner model such that $o^N_{\text{LONG}}(\delta) = \infty$ and $\kappa > \delta$. Let $\mathcal{E}^N_\kappa$ be the set of all extenders, $E \in V_\kappa$ such that

(1) $\delta < \text{CRT}(E)$,
(2) $\text{SPT}(E) > \text{CRT}(E)$ and $\rho(E) = \text{LTH}(E)$,
(3) both $\rho(E)$ and $\text{SPT}(E)$ are strongly inaccessible,
(4) $\text{SPT}(E) = \sup\{\gamma < \text{SPT}(E) \mid o^N_{\text{LONG}}(\gamma) = \infty\}$,
(5) $\rho(E) = \sup\{\gamma < \rho(E) \mid o^N_{\text{LONG}}(\gamma) = \infty\}$,
(6) $E \cap N \in N$ and $j_E(N) \cap V_{\text{LTH}(E)} = N \cap V_{\text{LTH}(E)}$.

A key property of $\mathcal{E}_N$ is that for all $E \in \mathcal{E}_N$,

$$j_E(I \cap \text{SPT}(E)) \cap \rho(E) = I \cap \rho(E),$$

where I is the class of all γ such that $o^N_{\text{LONG}}(\gamma) = \infty$. This follows from (2) and (6).

Suppose that N is an inner model such that $o^N_{\text{LONG}}(\delta) = \infty$. Then for $\gamma > \delta$,

$$o^N_{\text{LONG}}(\gamma) = \infty$$

if and only if γ is N-supercompact. This is by Theorem 145.

Similarly by Theorem 145, the requirement, $E \cap N \in N$, necessarily holds if $E = F|\text{LTH}(E)$ where F is an extender (with $\text{CRT}(F) > \delta$) which coheres N past the image of $\text{SPT}(E)$, which is to say if

$$j_F(N) \cap V_{\gamma+1} = N \cap V_{j_F(\gamma)+1}$$

where $\gamma = \text{SPT}(E)$.

Theorem 154. *Suppose that $o^N_{\text{LONG}}(\delta) = \infty$. Suppose (V_Θ, κ) is a premouse and that*

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle$$

is an iteration tree on $(V_\Theta, \kappa, \mathcal{E}_N \cap V_\kappa)$. Then

$$\langle M^*_\alpha, E^*_\beta, j^*_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle$$

is an iteration tree on $(N \cap V_\Theta, \kappa)$ where

(1) *for each $\beta + 1 < \eta$, $E^*_\beta = E_\beta \cap M^*_\beta$,*
(2) *for each $\gamma <_{\mathcal{T}} \alpha$, $j^*_{\gamma,\alpha} = j_{\gamma,\alpha}|M^*_\gamma$.*

Proof. For each $\alpha < \eta$, $\text{SPT}(E_\alpha)$ is the limit of $\gamma < \text{SPT}(E_\alpha)$ such that

$$M_\alpha \vDash o^{M^*_\alpha}_{\text{LONG}}(\gamma) = \infty.$$

Therefore by induction on $\alpha < \eta$, if $\alpha + 1 < \eta$ and if $\alpha^* < \alpha$ then $\text{SPT}(E_\alpha)$ must be the limit of all $\gamma < \text{SPT}(E_\alpha)$ such that

$$M_{\alpha^*} \vDash o^{M^*_{\alpha^*}}_{\text{LONG}}(\gamma) = \infty.$$

This is because for each $E \in \mathcal{E}_N$,

$$j_E(I \cap \text{SPT}(E)) \cap \rho(E) = I \cap \rho(E),$$

where I is the class of all γ such that $o^N_{\text{LONG}}(\gamma) = \infty$.

By induction on $\alpha < \eta$ using the proof of Theorem 152, it follows that if $\alpha+1 < \eta$ if α^* is the $\mathcal{T}$-predecessor of $\alpha + 1$, then

$$M^*_{\alpha+1} = \text{Ult}(M^*_{\alpha^*}, F)$$

and

$$k : M^*_{\alpha^*} \to M^*_{\alpha+1}$$

is the associated elementary embedding where

$$F = E_\alpha \cap M^*_\alpha$$

and where $k = j_{\alpha^*,\alpha+1}|M^*_{\alpha^*}$. The theorem follows easily from this. □

5.2. *Where comparison must fail*

A more general family of coarse coherent extender models are the family of coarse coherent *sub-extender* models. Such an inner model is constructed from a sequence of sub-extenders rather than from a sequence of extenders. Suppose E is an extender of length α. Then a sub-extender is obtained from E by restricting E to a set $[X]^{<\omega}$ where $X \subseteq \alpha$ and we require that if $X \neq \alpha$ then

$$j_E(\text{CRT}(E)) \subset X.$$

Assuming the $(\omega_1 + 1)$-iterability of countable substructures of V for iteration trees which are strongly closed and satisfy the cancellation rules for short extender, the conclusion of Theorem 127 fails for sub-extender models even if one requires that the inner model have cardinals κ which are n-extendible for all $n < \omega$ with this witnessed by the sub-extenders on the sequence from which that sub-extender model is constructed. Nevertheless we shall show that comparison must fail even for sub-extender models if one simply requires that all critical points of sub-extenders on the sequence be strong cardinals in V. However, in this situation the failure of comparison is at the level predicted by the problem of moving spaces.

There are two equivalent definitions that δ be supercompact, the second definition being that for each $\gamma > \delta$ there exists an extender E such that

(1) $\rho(E) \geq \gamma$,
(2) $\text{SPT}(E) < \delta$,
(3) $j_E(\text{CRT}(E)) = \delta$.

That this is equivalent to the usual definition follows from Lemma 129. The usual definition that δ is supercompact, states that for each $\lambda > \delta$ there exists a normal fine δ-complete ultrafilter on $\mathcal{P}_\delta(\lambda)$.

Our convention, as specified in Definition 130, is that a collection $\mathcal{E}$ of extenders witnesses δ is supercompact if for each $\gamma > \delta$ there exists an extender $E \in \mathcal{E}$ which satisfies (1)–(3) above.

Similarly if $\mathcal{E}$ is a set of extenders then $\mathcal{E}$ witnesses that δ is a Woodin cardinal if for each $A \subseteq V_\delta$ there exists $\kappa < \delta$ such that for all $\gamma < \delta$ there exists $E \in \mathcal{E}$ such that

(1) $\text{CRT}(E) = \kappa$ and $j_E(\kappa) > \gamma$,
(2) $\rho(E) \geq \gamma$,
(3) $j_E(A) \cap V_\gamma = A \cap V_\gamma$.

Such a cardinal κ is $(<\delta)$-A-strong.

Definition 155. Let $\mathcal{M}_S$ be the set of all transitive sets M such that

(1) $M \vDash \text{ZFC}$,
(2) $M \cap \text{Ord}$ is a strong cardinal,
(3) there exists $\delta \in M$ such that the set, $\{E \cap M \mid E \in \mathcal{E} \text{ and } E \cap M \in M\}$, witnesses that the cardinal δ is supercompact in M, where (here and below) $\mathcal{E}$ is the set of all initial segments of extenders E such that $\rho(E) = \text{LTH}(E)$ and $\rho(E)$ is strongly inaccessible,
(4) there exists δ_0 and there exists

$$\mathcal{E}_0 \subseteq \{E \cap M \mid E \in \mathcal{E} \text{ and } E \cap M \in M\}$$

such that $\mathcal{E}_0 \in M$, $\mathcal{E}_0$ witnesses in M that δ_0 is a Woodin cardinal, and such that there is a strong cardinal below δ_0.

The purpose of (4) is to obtain the following which is an immediate corollary of Theorem 173 on p. 273.

Lemma 156. *Suppose $M \in \mathcal{M}_S$ and let κ_0 be the least strong cardinal. Then for every set $a \subseteq \kappa_0$, $M[a]$ is a set-generic extension of M.*

From the perspective of current inner model theory, one would expect that comparison (when it can be established) be proved as follows. Suppose $L_{\alpha_0}[\mathbb{E}_0]$ and $L_{\alpha_1}[\mathbb{E}_1]$ are generalized premice, so $\mathbb{E}_0$ and $\mathbb{E}_1$ are sequences of partial extenders each constructed in V. The weakest requirement of comparison is the following. Let T_0 be the Σ_1-theory of the structure,

$$(L_{\alpha_0}[\mathbb{E}_0], \mathbb{E}_0 \cap L_{\alpha_0}[\mathbb{E}_0])$$

and let T_1 be the Σ_1-theory of the structure,

$$(L_{\alpha_1}[\mathbb{E}_1], \mathbb{E}_1 \cap L_{\alpha_1}[\mathbb{E}_1]).$$

Then $T_0 \subseteq T_1$ or $T_1 \subseteq T_0$. Call this *comparison*.

To prove comparison for the pair $(L_{\alpha_0}[\mathbb{E}_0], L_{\alpha_1}[\mathbb{E}_1])$ one proceeds in two steps. One step is to identify a local property (that is a property which is Δ_2^{ZFC}-expressible) and prove that this property holds for the countable generalized premice which arise from transitive collapses of countable elementary substructures of the given generalized premice. The other (and generally easier) step is to prove comparison for all pairs of countable generalized premice which satisfy this local property. For Mitchell–Steel premice the property is iterability which is a first order property in $V_{\omega+2}$ and so clearly Δ_2^{ZFC}-expressible.

Theorem 160 which we prove below and which is a corollary of Theorem 145 shows that one cannot really avoid the problem of the failure of comparison at the level of the moving spaces problem and we will explain why this is the case, in the comments just before Theorem 160.

Remark 157. Suppose that N is an inner model such that $o^N_{\text{LONG}}(\delta) = \infty$ for a proper class of δ. Suppose that for a proper class of δ there exists a set $\mathcal{E}$ of extenders such that

(1) $\{E \cap N \mid E \in \mathcal{E}\} \in N$,
(2) $\{E \cap N \mid E \in \mathcal{E}\}$ witnesses in N that δ is a Woodin cardinal.

Then for every set $a \subseteq \mathrm{Ord}$, $N[a]$ is a set-generic extension of N. Without the requirement, (1), this can fail, see Theorem 176 on p. 278. This kind of genericity property is discussed on p. 273.

So this provides an important example distinguishing between the assumption that $E \cap N \in N$ for various extenders E versus the assumption that $\tilde{E} \cap N \in N$ for various extender sequences $\tilde{E}$.

Definition 158. Suppose that $N \subseteq V_\kappa$ and κ is strongly inaccessible. Then

$$(V_\kappa, N) \vDash \text{“There is an } N\text{-extendible cardinal.”},$$

if there exists $\delta < \kappa$ such that for all $\alpha < \kappa$, there exists an elementary embedding,

$$j : V_{\delta+\alpha} \to V_{j(\delta)+j(\alpha)},$$

such that $\mathrm{CRT}(j) = \delta$, $\alpha < j(\delta) < \kappa$, and such that for all $\beta < \alpha$,

$$j(N \cap V_{\delta+\beta}) = N \cap V_{j(\delta)+j(\beta)}.$$

Lemma 159. *Suppose that κ is a huge cardinal. Then for each set $N \subseteq V_\kappa$,*

$$(V_\kappa, N) \vDash \text{“There is an } N\text{-extendible cardinal.”}$$

Proof. Since κ is a huge cardinal there exists an elementary embedding,

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$ and such that $M^{j(\kappa)} \subseteq M$. This implies that

$$V_{j(\kappa)} \subseteq M$$

and that

$$j|V_{j(\kappa)} \in M.$$

By the elementarity of j, it suffices to show that

$$(V_{j(\kappa)}, j(N)) \models \text{“}\kappa \text{ is a } j(N)\text{-extendible cardinal.”}$$

If this fails then there must exist $\alpha_0 < j(\kappa)$ such that there does not exist an elementary embedding,

$$k : V_{\kappa+\alpha_0} \to V_{k(\kappa+\alpha_0)}$$

such that

(1.1) $\mathrm{CRT}(k) = \kappa$,
(1.2) $\kappa < k(\alpha_0) < j(\kappa)$,
(1.3) for all $\beta < \alpha_0$, $k(j(N) \cap V_{\kappa+\beta}) = j(N) \cap V_{k(\kappa)+k(\beta)}$.

Clearly $(V_\kappa, N) \prec (V_{j(\kappa)}, j(N))$, and so applying j,

$$j((V_\kappa, N)) \prec j((V_{j(\kappa)}, j(N))),$$

which implies that

$$(V_{j(\kappa)}, j(N)) \prec (M \cap V_{j(j(\kappa))}, j(j(N))).$$

But $j|V_{\kappa+\alpha_0} \in M$ and this contradicts the choice of α_0. $\square$

Suppose that $L_\alpha[\mathbb{E}]$ is a generalized premouse

$$L_\alpha[\mathbb{E}] \models \mathrm{ZFC}$$

and that for some $\delta < \kappa$, $\mathbb{E}$ witness in $L_\alpha[\mathbb{E}]$ that both that δ is $(<\kappa)$-supercompact and that κ is measurable. We can assume that for all $\beta < \alpha$, $L_\beta[\mathbb{E}]$ fails to have this property and moreover that $L_\beta[\mathbb{E}]$ is strongly acceptable — recall that this is the property that for all $\gamma < \beta$, if

$$\mathcal{P}(\gamma) \cap L_{\beta+1}(\mathbb{E}) \neq \mathcal{P}(\gamma) \cap L_\beta(\mathbb{E})$$

then there is a surjection

$$\pi : \gamma \to L_\beta(\mathbb{E})$$

such that $\pi \in L_{\beta+1}(\mathbb{E})$ (in the natural sense if β is not a limit ordinal).

If $\mathbb{E}$ is constructed by backgrounding with extenders whose critical point is a strong cardinal, then $L_\kappa[\mathbb{E}] \in \mathcal{M}_S$ and κ is definable in

$$(L_\alpha[\mathbb{E}], \mathbb{E} \cap L_\alpha[\mathbb{E}]).$$

Therefore κ is Σ_1-definable in $(L_{\alpha+1}[\mathbb{E}], \mathbb{E} \cap L_{\alpha+1}[\mathbb{E}])$ and so the first order theory of

$$(L_\kappa[\mathbb{E}], \mathbb{E} \cap L_\kappa[\mathbb{E}]).$$

is reducible to the Σ_1-theory of $(L_{\alpha+1}[\mathbb{E}], \mathbb{E} \cap L_{\alpha+1}[\mathbb{E}])$.

This explains why the following theorem refutes comparison in the sense that we have abstractly defined on p. 263 at the level of the moving spaces problem, assuming the generalized premice satisfy the fairly abstract conditions we have specified above.

Theorem 160. *Suppose that there is a proper class of huge cardinals. Then for each* $\gamma \in \mathrm{Ord}$ *such that* γ *is* Σ_2*-definable there exists a transitive set* N *such that*

(1) $N \vDash \mathrm{ZFC}$ *and* $V_\gamma \in N$,
(2) $N \vDash$ "*There is an extendible cardinal*",
(3) *for all* $M_0 \in \mathcal{M}_S$, *for all* $M_1 \in (\mathcal{M}_S)^N$, $M_0 \not\equiv M_1$.

Proof. By increasing γ if necessary we can suppose that for some sentence ϕ, γ is least such that $V_\gamma \vDash \phi$.

Let N be transitive such that

(1.1) $V_\gamma \in N$,
(1.2) $N \vDash \mathrm{ZFC} +$ "There is an extendible cardinal",
(1.3) $N \cap \mathrm{Ord}$ is as small as possible.

Suppose that $M_0 \in \mathcal{M}_S$ and that $M_1 \in (\mathcal{M}_S)^N$ (of course it could be that $\mathcal{M}_S = \emptyset$ but then the theorem is vacuously true).

Thus $M_0 \cap \mathrm{Ord}$ is a strong cardinal of V which implies that $M_0 \cap \mathrm{Ord}$ is a limit of huge cardinals. By Lemma 159 and Theorem 145, for cofinally many $\kappa_0 < M_0 \cap \mathrm{Ord}$ there exists $\kappa_1 \in M_0 \cap \mathrm{Ord}$ such that $\kappa_0 < \kappa_1$ and such that

$$M_0 \cap V_{\kappa_1} \vDash \mathrm{ZFC} + \text{“}\kappa_0 \text{ is an extendible cardinal”}.$$

This property of M_0 is invariant under set forcing.

Suppose toward a contradiction that $M_0 \equiv M_1$. Let $a \subseteq |V_\gamma|^N$ be a set in N such that a codes V_γ. Necessarily γ is below the least strong cardinal of N. This is because for some sentence ϕ, γ is least such that $V_\gamma \vDash \phi$. Therefore by Lemma 156, $M_1[a]$ is a set-generic extension of M_1. But then, since $M_0 \equiv M_1$, there exists $\kappa \in M_1 \cap \mathrm{Ord}$ such that $a \in V_\kappa$ and such that

$$M_1[a] \cap V_\kappa \vDash \mathrm{ZFC} + \text{“There is an extendible cardinal”}.$$

This contradicts the minimality of $N \cap \mathrm{Ord}$ since $V_\gamma \in M_1[a] \cap V_\kappa$. $\square$

6. Suitable Extender Models

It seems reasonable to expect that inner models constructed as solutions to the inner model problem take the form of $L[\mathbb{E}]$ where

$$\mathbb{E} \subset \mathrm{Ord} \times V$$

is a sequence of partial extenders and that these partial extenders have an ancestry in extenders E of V.

Now suppose that $L[\mathbb{E}]$ is maximal in the sense that if

$$j : V \to M$$

is an elementary embedding such that $V_{j(\kappa)+\omega} \subset M$ and such that $j(\mathbb{E}|\kappa) = \mathbb{E}|j(\kappa)$, then one of the following hold.

(1) Let $N = L[\mathbb{E}] \cap V_\kappa$. Then for some $\delta < \kappa$, $V_\kappa \vDash$ "$o^N_{\text{LONG}}(\delta) = \infty$".
(2) There exists $\alpha \in \text{dom}(\mathbb{E})$ such that

$$\mathbb{E}_\alpha | j(\kappa) = E \cap L[\mathbb{E}]$$

where E is the extender of length $j(\kappa)$ given by j.

Suppose that in addition

$$o^{L[\mathbb{E}]}_{\text{LONG}}(\delta) = \infty$$

for some δ and let δ_0 be the least such δ. Assuming that for each $\alpha > \delta_0$, if $\alpha \in \text{dom}(\mathbb{E})$ then

$$\mathbb{E}_\alpha | \delta_0 = (E|\delta_0) \cap L[\mathbb{E}]$$

for some extender $E \in V$ with $\rho(E) \geq \delta_0$, then using $\mathbb{E}$ there exists a sequence $\langle E_\alpha : \alpha < \delta_0 \rangle$ of extenders in V_{δ_0} such that

$$\langle E_\alpha \cap L[\mathbb{E}] : \alpha < \delta_0 \rangle \in L[\mathbb{E}]$$

and such that this sequence witnesses δ_0 is a Woodin cardinal in $L[\mathbb{E}]$. This motivates the definition of a suitable extender model. If the sequence $\mathbb{E}$ is constructed by backgrounding using a coherent sequence of extenders in V then one will obtain that Definition 161(2a) holds for $L[\mathbb{E}]$ at δ_0 under very general assumptions about the nature of the construction. The formulation of Definition 161(2a) is stronger than is needed for the analysis of suitable extender models but it simplifies a number of details, see for example Theorem 175 and Remark 174.

Definition 161. Suppose that $\mathbb{M}$ is a transitive class such that for some δ, $o^{\mathbb{M}}_{\text{LONG}}(\delta) = \infty$.

(1) $\delta_{\mathbb{M}}$ denotes the least $\kappa \leq \delta$ such that $o^{\mathbb{M}}_{\text{LONG}}(\kappa) = \infty$.
(2) $\mathbb{M}$ is a *suitable extender model* if the following hold.
    (a) There exist a cofinal set $I_{\mathbb{M}} \subset \delta_{\mathbb{M}}$ and a sequence $\langle E_\alpha : \alpha \in I_{\mathbb{M}} \rangle$ of extenders in $V_{\delta_{\mathbb{M}}}$ witnessing that $\delta_{\mathbb{M}}$ is a Woodin cardinal (in V) such that

$$\langle E_\alpha \cap \mathbb{M} : \alpha \in I_{\mathbb{M}} \rangle \in \mathbb{M}$$

and such that for all $\alpha \in I_{\mathbb{M}}$,

$$j_{E_\alpha}(\langle E_\beta : \beta \in \text{CRT}(E_\alpha) \cap I_{\mathbb{M}} \rangle) | \text{LTH}(E_\alpha) = \langle E_\beta : \beta \in \text{LTH}(E_\alpha) \cap I_{\mathbb{M}} \rangle,$$

$\rho(E_\alpha) = \text{LTH}(E_\alpha) = \alpha$, and such that $\alpha = \text{CRT}(E_\beta)$ for some $\beta \in I_{\mathbb{M}}$.

(b) (Weak Σ_2-definablity) There exists $X \in V_{\delta_{\mathbb{M}}+1}$ and a formula $\phi(x_0, x_1)$ such that for all $\beta < \eta_1 < \eta_2 < \eta_3$, if $X \in V_\beta$ and if

$$(\mathbb{M})^{V_{\eta_1}} \cap V_\beta = (\mathbb{M})^{V_{\eta_3}} \cap V_\beta$$

then $(\mathbb{M})^{V_{\eta_1}} \cap V_\beta = (\mathbb{M})^{V_{\eta_2}} \cap V_\beta = (\mathbb{M})^{V_{\eta_3}} \cap V_\beta$,

where for all $\gamma > \delta_{\mathbb{M}}$, $(\mathbb{M})^{V_\gamma} = \{a \in V_\gamma \mid V_\gamma \vDash \phi[a, X]\}$.

Remark 162. The definability requirement, Definition 161(2b), is motivated by the results of [22] which suggest that modulo iterability hypotheses (such as the Strong Unique Branch Hypothesis) the natural construction will actually yield a suitable extender model with this definability property. The reason for imposing such a rather strong requirement is that it simplifies the downward transference from V to $\mathbb{M}$ of large cardinals at the level of ω-huge and beyond. Though this is at the price of strengthening the axioms by requiring that $V_\lambda \prec_{\Sigma_2} V$. Notice (see the proof of Lemma 169) that if $X \in V_\lambda$ where X is the parameter witnessing that $\mathbb{M}$ is a suitable extender model using formula $\phi(x_0, x_1)$, then

$$\mathbb{M} \cap V_\lambda = \{a \in V_\lambda \mid V_\lambda \vDash \phi[a, X]\}.$$

Since our goal is the exploration of the extent of transference arguably this strengthening is not really relevant to the question of what large cardinal axioms can hold in all suitable extender models. Of course we are really only interested in the fine-structural versions of suitable extender models but by analyzing the question in terms of the transference to all suitable extender models we are clearly dealing with this question in a more general setting.

Suppose that $\mathbb{M}$ is a suitable extender model and $E \in V_{\delta_{\mathbb{M}}}$ is an extender. Let

$$M = j_E(\mathbb{M}).$$

Thus by Corollary 148,

$$o^M_{\text{LONG}}(\delta_{\mathbb{M}}) = \infty.$$

However one can show that there can be no cofinal set $X \subset \delta_{\mathbb{M}}$ such that $X \in M$ and such that every element of X is a regular cardinal in V and so necessarily M is *not* a suitable extender model. This raises the possibility that V is the only possible suitable extender model. The following remarkable lemma due independently to Laver [5], shows that suitable extender models are not completely trivial. See Reitz [12] and [21] for other applications.

Lemma 163. *Suppose that $\mathbb{B}$ is a complete Boolean algebra and that $G \subseteq \mathbb{B}$ is V-generic. Then V is Σ_2-definable in $V[G]$ from $(G, \mathbb{B})$.*

Lemma 164. *Suppose that δ_0 is the least supercompact cardinal, $\mathbb{P} \in V_{\delta_0}$ is a partial order and $G \subset \mathbb{P}$ is V-generic. Suppose that δ_0 is the least supercompact cardinal in $V[G]$. Then V is a suitable extender model in $V[G]$.*

Proof. Clearly, $o^V_{\text{LONG}}(\delta)(\delta_0) = \infty$ in $V[G]$ and V satisfies the requirement Definition 161(2a) in $V[G]$. By Lemma 163, V satisfies the requirement of weak Σ_2-definability in $V[G]$. $\square$

Remark 165. One can arrange that there exist suitable extender models $\mathbb{M}$ for which V is not a set generic extension of $\mathbb{M}$. However it is not clear if there can exist a suitable extender model $\mathbb{M}$ together with a nontrivial elementary embedding

$$j : \mathbb{M} \to \mathbb{M}.$$

This potentially distinguishes suitable extender models from arbitrary transitive inner models $N \subset V$ for which $o^N_{\text{LONG}}(\delta) = \infty$ for some δ, see Lemma 150 on p. 259.

The following variation of Theorem 128 explains the necessity of dealing with such a general notion as that of a suitable extender model. We require the following definition which is a variant of the notion of definablity given in Definition 161(2b).

Definition 166. Suppose that $\mathbb{E} = \langle \mathbb{E}_\alpha : \alpha \in \rangle$ is a sequence of sets. Then $\mathbb{E}$ is *weakly* Σ_2*-definable* if there is a formula $\phi(x)$ such that for all $\beta < \eta_1 < \eta_2 < \eta_3$, if $(\mathbb{E})^{V_{\eta_1}}|\beta = (\mathbb{E})^{V_{\eta_3}}|\beta$ then

$$(\mathbb{E})^{V_{\eta_1}}|\beta = (\mathbb{E})^{V_{\eta_2}}|\beta = (\mathbb{E})^{V_{\eta_3}}|\beta,$$

where for all γ, $(\mathbb{E})^{V_\gamma} = \{a \in V_\alpha \mid V_\gamma \vDash \phi[a]\}$.

Remark 167. (1) The sequence $\langle S_\alpha : \alpha \in \text{Ord}\rangle$ is Σ_2-definable where for each $\alpha \in \text{Ord}$, S_α is the Σ_2-theory of V with ordinal paramters from α. However the sequence $\langle T_\alpha : \alpha \in \text{Ord}\rangle$ is weakly Σ_2-definable where for each $\alpha \in \text{Ord}$, T_α is the Σ_3-theory of V with ordinal paramters from α.

(2) The canonical enumeration of HOD is Σ_2, more precisely there is a class $E \subset \text{Ord}$ such that E and $\text{Ord}\backslash E$ are each Σ_2-definable and such that $\text{HOD} = L[E]$. Let

$$\langle \kappa_\alpha : \alpha \in \text{Ord}\rangle$$

be the increasing enumeration of the HOD-cardinals. Then $\langle \kappa_\alpha : \alpha \in \text{Ord}\rangle$ is weakly Σ_2-definable.

Theorem 168. *Suppose there is a proper class of supercompact cardinals. Then there is a class-generic extension $V[G]$ of V in which the following hold.*

(1) $V[G] = (\text{HOD})^{V[G]}$.
(2) *There is a proper class of supercompact cardinals.*
(3) *Suppose $\tilde{E} = \langle E_\alpha : \alpha \in \text{Ord}\rangle$ is an extender sequence such that*
   (a) *$\tilde{E}$ is weakly Σ_2-definable and for all α, there exists an extender F such that $\rho(F) \geq \text{LTH}(E_\alpha)$ and such that $E_\alpha = F|\text{LTH}(E_\alpha)$,*
   (b) $o^{\tilde{E}}_{\text{mLONG}}(\delta) = \infty$.

   Then $V[G] \subset L[\tilde{E}]$.

Proof. Following the proof of Theorem 128, without loss of generality we can assume that the GCH holds. Let $\mathbb{P}_\infty$ be the class partial order as defined in the proof of Theorem 128.

Suppose $G \subset \mathbb{P}_\infty$ is V-generic. Then since there is a proper class of Mahlo cardinals:

(1.1) $V[G] = (\mathrm{HOD})^{V[G]}$.
(1.2) If δ is supercompact in V then δ is supercompact in $V[G]$.
(1.3) Suppose κ is a Mahlo cardinal in V then $V[G]_\kappa = V_\kappa[G|\kappa]$.

Suppose that in $V[G]$, $\tilde{E} = \langle E_\alpha : \alpha \in \mathrm{Ord}\rangle$ is an extender sequence, δ is a cardinal, and that in $V[G]$:

(2.1) $\tilde{E}$ is weakly Σ_2-definable and for all α, there exists an extender F such that $\rho(F) \geq \mathrm{LTH}(E_\alpha)$ and such that $E_\alpha = F|\mathrm{LTH}(E_\alpha)$,
(2.2) $o^{\tilde{E}}_{\mathrm{mLONG}}(\delta) = \infty$.

Since $o^{\tilde{E}}_{\mathrm{mLONG}}(\delta) = \infty$, δ is a supercompact cardinal in $V[G]$, and so

$$V[G]_\delta \prec_{\Sigma_2} V[G].$$

Therefore it follows that:

(3.1) $V[G]_\delta = V_\delta[G|\delta]$,
(3.2) $(\tilde{E})^{V[G]_\delta} = \tilde{E}|\delta$,
(3.3) $L[\tilde{E}] \cap V[G]_\delta = L_\delta[\tilde{E}] = L_\delta[\tilde{E}|\delta]$,

where

$$(\tilde{E})^{V[G]_\delta} = \{a \mid V[G]_\delta \vDash \phi[a]\}$$

and $\phi(x)$ is a formula which witnesses in $V[G]$ that $\tilde{E}$ is weakly Σ_2-definable.

Let κ be the least Mahlo cardinal of V above δ, let

$$D = \{\mathrm{CRT}(E_\alpha) \mid j_{E_\alpha}(\mathrm{CRT}(E_\alpha)) = \delta \text{ and } \kappa < \mathrm{LTH}(E_\alpha)\},$$

and let C_δ be the fast club added by G at stage $\delta + 1$. Thus it follows that

(4.1) $D \subset C_\delta$ and $\sup(D) = \delta$,
(4.2) $D \in L[\tilde{E}]$,
(4.3) for all $\xi < \delta$, $D \cap \xi \in L_\delta[\tilde{E}] \subset V[G|\delta]$.

The remainder of the proof is exactly like the proof of Theorem 127. □

6.1. *Closure properties of suitable extender models*

The results of Sec. 5 show that if $\mathbb{M}$ and δ are such that $o^{\mathbb{M}}_{\mathrm{LONG}}(\delta) = \infty$ then V is remarkably close to $\mathbb{M}$ in several ways analogous to the closeness of V to L in the situation that $0^\#$ does not exist. In particular, there is a fairly strong transference of large cardinals from V to $\mathbb{M}$. If $\mathbb{M}$ is actually a suitable extender model then, as we shall show in this section, one actually gets much stronger covering properties and transference properties. In fact, there is an entire hierarchy of large cardinals much stronger than any previously known and these also transfer down to suitable extender models.

Theorem 169. *Suppose* $\mathbb{M}$ *is a suitable extender model and*

$$j : V_\lambda \to V_\lambda$$

is an elementary embedding such that $\delta_{\mathbb{M}} < \text{CRT}(j)$ *and such that* $V_\lambda \prec_{\Sigma_2} V$*. Then* $j(\mathbb{M} \cap V_\lambda) = \mathbb{M} \cap V_\lambda$ *and for all* $\gamma < \lambda$,

$$j|(\mathbb{M} \cap V_\gamma) \in \mathbb{M}.$$

Proof. Since $\mathbb{M}$ is a suitable extender model, there exists $X_0 \in V_{\delta_{\mathbb{M}}+1}$ and a formula $\phi_0(x_0, x_1)$ witnessing that $\mathbb{M}$ is a suitable extender model.

Since $V_\lambda \prec_{\Sigma_2} V$ it follows that

$$\mathbb{M} \cap V_\lambda = \{a \in V_\lambda \mid V_\lambda \vDash \phi_0[a, X]\}.$$

We verify this. Fix $\beta < \lambda$ such that $X_0 \in V_\beta$. Then since $V_\lambda \prec_{\Sigma_2}$, the sets

$$\{\eta < \lambda \mid \beta < \eta \text{ and } (\mathbb{M})^{V_\eta} \cap V_\beta = (\mathbb{M})^{V_\lambda} \cap V_\beta\}$$

and

$$\{\eta < \lambda \mid \beta < \eta \text{ and } (\mathbb{M})^{V_\eta} \cap V_\beta = \mathbb{M} \cap V_\beta\}$$

must each be cofinal in λ where for all $\gamma > \beta$, $(\mathbb{M})^{V_\gamma} = \{a \in V_\gamma \mid V_\eta \vDash \phi[a, X]\}$. It follows that

$$\mathbb{M} \cap V_\beta = \{a \in V_\lambda \mid V_\lambda \vDash \phi_0[a, X]\}.$$

and so $\mathbb{M} \cap V_\lambda = \{a \in V_\lambda \mid V_\lambda \vDash \phi_0[a, X]\}$.

By Theorem 145, it suffices to show that for each $\gamma < \lambda$, $j(\mathbb{M} \cap V_\gamma) = \mathbb{M} \cap V_{j(\gamma)}$. Fix γ and choose $\kappa < \lambda$ such that $\delta_{\mathbb{M}} + \gamma < \kappa$ and such that $V_\kappa \prec V_\lambda$. For example let $\langle \kappa_n : n < \omega \rangle$ be the critical sequence of j and take $\kappa = \kappa_n$ where n is large enough so that $(X_0, \gamma) \in V_{\kappa_n}$. Thus $V_\kappa \prec V_{j(\kappa)} \prec V_\lambda$ and so

$$j(\mathbb{M} \cap V_\kappa) = j(\{a \in V_\kappa \mid V_\kappa \vDash \phi_0[a, X]\}) = \{a \in V_{j(\kappa)} \mid V_{j(\kappa)} \vDash \phi_0[a, X]\} = \mathbb{M} \cap V_{j(\kappa)}.$$

□

The following is an immediate corollary of Theorem 169 by absoluteness.

Corollary 170. *Suppose* $\mathbb{M}$ *is a suitable extender model and*

$$j : V_\lambda \to V_\lambda$$

is an elementary embedding such that $\delta_{\mathbb{M}} < \text{CRT}(j)$ *and such that* $V_\lambda \prec_{\Sigma_2} V$*. Then there exists* $\lambda' \leq \lambda$ *and a nontrivial elementary embedding*

$$j' : \mathbb{M} \cap V_{\lambda'} \to \mathbb{M} \cap V_{\lambda'}$$

such that $j' \in \mathbb{M}$.

We also obtain the following variation.

Corollary 171. *Suppose* $2 < k < \omega$, $\mathbb{M}$ *is a suitable extender model, and*

$$j : V_\lambda \to V_\lambda$$

is an elementary embedding such that $\delta_{\mathbb{M}} < \mathrm{CRT}(j)$ *and such that* $V_\lambda \prec_{\Sigma_k} V$. *Then there exists* $\lambda' \leq \lambda$ *and a nontrivial elementary embedding*

$$j' : \mathbb{M} \cap V_{\lambda'} \to \mathbb{M} \cap V_{\lambda'}$$

such that $\mathbb{M} \cap V_{\lambda'} \prec_{\Sigma_k} \mathbb{M}$ *and such that* $j' \in \mathbb{M}$.

Proof. Fix $X \in V_{\delta_{\mathbb{M}}+1}$ and a formula $\phi(x_0, x_1)$ such that $\mathbb{M}$ is weakly Σ_2-definable in V from X, using $\phi(x_0, x_1)$, in the sense of Definition 161(2b).

We have that $V_\lambda \prec_{\Sigma_k} V$ and that $V_\lambda \vDash \mathrm{ZFC}$. Since $k \geq 2$ it follows that

$$\mathbb{M} \cap V_\lambda = \{a \in V_\lambda \mid V_\lambda \vDash \phi[a, X]\}.$$

Therefore, it follows by induction on $i \leq k$ that for all $i \leq k$, $\mathbb{M} \cap V_\lambda \prec_{\Sigma_i} \mathbb{M}$.

Let I be the set of all $\gamma < \lambda$ such that

$$V_\gamma \prec V_\lambda$$

and such that $X \in V_\gamma$. Then I is cofinal in λ and $j(I) = I$. Note that for each $\gamma \in I$,

$$\mathbb{M} \cap V_\gamma \prec \mathbb{M} \cap V_\lambda$$

and so for each $\gamma \in I$,

$$\mathbb{M} \cap V_\gamma \prec_{\Sigma_k} \mathbb{M}.$$

The corollary follows by absoluteness. □

Thus Theorem 169 easily gives the downward transference of axioms at the level of ω-huge from V to $\mathbb{M}$. However, Theorem 169 also suggests difficulties arise in trying to transfer the strongest large cardinal hypotheses from V to $\mathbb{M}$. The point is that if $\mathbb{M}$ is a suitable extender model and

$$j : \mathbb{M} \cap V_{\lambda+1} \to \mathbb{M} \cap V_{j(\lambda)+1}$$

is an elementary embedding with critical point $\kappa \geq \delta_{\mathbb{M}}$, then $j \in \mathbb{M}$ (since the induced $\mathbb{M}$-extender is in $\mathbb{M}$). In particular, Theorem 169 essentially rules out directly transferring hypotheses such as the following which are among the strongest currently known (and not known to refute the Axiom of Choice).

(1) There exists a non-trivial elementary embedding,

$$j : V_{\lambda+1} \to V_{\lambda+1}.$$

(2) There exists a non-trivial elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ.

The problem is that the corresponding requirement;

$$j(\mathbb{M} \cap V_{\lambda+1}) = \mathbb{M} \cap V_{\lambda+1},$$

is not a reasonable requirement, indeed if

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

is an elementary embedding with critical point above δ such that $(\mathrm{cof}(\lambda))^{\mathbb{M}} > \omega$ then by Theorem 145, the requirement *cannot* hold.

One could speculate it is the requirement that $V_\lambda \prec_{\Sigma_2} V$ which is the issue. However the following theorem, which we shall prove in Part II, suggests this is not the case and Laver has proved much stronger versions.

Theorem 172. *Suppose that*

$$j_0 : L(V_{\lambda_0+1}) \to L(V_{\lambda_0+1})$$

is an elementary embedding with critical point below λ_0. *Then there there exist* $\lambda < \lambda_0$ *and a nontrivial elementary embedding*,

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

such that $V_\lambda \prec V_{\lambda_0}$.

Nevertheless the large cardinal hypotheses listed above do transfer to $\mathbb{M}$ under reasonable conditions. To show this we must prove a stronger version of Theorem 169 and for this we use the additional feature, Definition 161(2a), that distinguishes suitable extender models from arbitrary transitive classes for which

$$o^N_{\text{LONG}}(\delta) = \infty$$

for some δ (and which have the definability property, Definition 161(2b)).

We will need the following genericity theorem. The statement of this theorem involves the notion that a set of extenders witnesses that a cardinal is a Woodin cardinal, which we defined on p. 263.

Theorem 173. *Suppose that* δ *is a strong limit cardinal*, $\mathcal{E} \subseteq V_\delta$ *is a set of short extenders*, N *is an inner model of* V, $\{E \cap N \mid E \in \mathcal{E}\} \in N$, *and that*

$$\{E \cap N \mid E \in \mathcal{E}\}$$

witnesses in N *that* δ *is a Woodin cardinal.*

(1) *Suppose that* $A \subseteq \delta$ *and* $\sup(A) < \delta$. *Then* A *is* N*-generic for a partial order* $\mathbb{P} \in N$ *such that* $\mathbb{P}$ *is* δ*-cc in* N.

(2) *Suppose that* $A \subseteq \delta$ *and that for each* $E \in \mathcal{E}$,

$$j_E(A \cap \text{CRT}(E)) \cap \text{LTH}(E) = A \cap \text{LTH}(E).$$

Then A *is* N*-generic for a partial order* $\mathbb{P} \in$ *such that* $\mathbb{P}$ *is* δ*-cc in* N.

Proof. Note that (2) implies (1) since for any $\alpha < \delta$,

$$\{E \cap N \mid E \in \mathcal{E} \text{ and } \text{CRT}(E) > \alpha\} \in N$$

and also witnesses in N that δ is Woodin.

We sketch the proof of (2). The proof uses the *Extender Algebra* which is a Boolean algebra naturally constructed from a set of extenders. The difference in the situation here is that we will not have to *iterate* the ground model (which here is N) to achieve genericity.

Let

$$\mathcal{E}_N = \{(F|\eta) \cap N \mid F \in \mathcal{E} \text{ and } \eta \leq \text{LTH}(F)\}$$

and let $\mathcal{E}^*$ be the set of all extenders $E \in \mathcal{E}_N$ such that $\text{LTH}(E)$ is strongly inaccessible in N. Clearly $\mathcal{E}^*$ witnesses in N that δ is a Woodin cardinal.

We work in the inner model N. Let $\mathbb{W}$ be the set of all formal terms, τ, for elements of a complete Boolean algebra with generators from $\{\alpha \mid \alpha < \delta\}$ such that $\tau \in V_\delta$. If $\mathbb{B}$ is a complete Boolean algebra and

$$\pi : \delta \to \mathbb{B}$$

then π induces a map

$$\pi^* : \mathbb{W} \to \mathbb{B}.$$

The logical identities of $\mathbb{W}$ are the equivalences,

$$\sigma_1 = \sigma_2,$$

such that for all complete Boolean algebras, $\mathbb{B}$, for all

$$\pi : \delta \to \mathbb{B},$$

$\pi^*(\sigma_1) = \pi^*(\sigma_2)$. This defines an equivalence relation, $\sim$, on $\mathbb{W}$ and the quotient, $\mathbb{W}/\sim$ is a δ-complete Boolean algebra.

Let $\mathbb{I}_{\mathcal{E}^*}$ be the set of all pairs (σ_1, σ_2) such that there exist a sequence, $\langle \tau_\alpha : \alpha < \kappa \rangle$, and an extender $E \in \mathcal{E}^*$ such that

(1.1) $\kappa = \text{CRT}(E)$ and $\{\tau_\alpha \mid \alpha < \kappa\} \subseteq V_\kappa$,
(1.2) $\sigma_1 = \vee\{\tau_\alpha \mid \alpha < \kappa\}$,
(1.3) for some $\gamma < \text{LTH}(E)$,

  (a) $\kappa \leq \gamma$,
  (b) $\sigma_2 = \vee\{\tau^*_\alpha \mid \alpha < \gamma\}$,
  (c) $\{\tau^*_\alpha \mid \alpha < \gamma\} \subset V_{\text{LTH}(E)}$,

  where $\langle \tau^*_\alpha : \alpha < \gamma \rangle = j_E(\langle \tau_\alpha : \alpha < \kappa \rangle)|\gamma$.

Define an equivalence relation, $\sim_{\mathcal{E}^*}$, on $\mathbb{W}$ as follows.

$$\tau_1 \sim_{\mathcal{E}^*} \tau_2$$

if for all complete Boolean algebras, $\mathbb{B}$, for all

$$\pi : \delta \to \mathbb{B},$$

if $\pi^*(\sigma_1) = \pi^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in \mathbb{I}_{\mathcal{E}^*}$ then $\pi^*(\tau_1) = \pi^*(\tau_2)$, where

$$\pi^* : \mathbb{W} \to \mathbb{B}$$

is the map induced by π. Note that the trivial map

$$\pi_0 : \delta \to \mathbb{B}$$

where $\pi_0(\alpha) = 0$ for all $\alpha < \delta$ satisfies the requirement that $\pi_0^*(\sigma_1) = \pi_0^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in \mathbb{I}_{\mathcal{E}^*}$ and so $\sim_{\mathcal{E}^*}$ extends the logical equivalence relation, $\sim$.

Let $\mathbb{B}_{\mathcal{E}^*} = \mathbb{W}/{\sim_{\mathcal{E}^*}}$. We now come to the key points.

The first key point is that since $\mathcal{E}^*$ witnesses δ is a Woodin cardinal, it follows that $\mathbb{B}_{\mathcal{E}^*}$ is a δ-cc Boolean algebra and therefore it is a complete Boolean algebra. For this it suffices to show that if $\langle \tau_\alpha : \alpha < \delta \rangle$ is a sequence of elements from $\mathbb{W}$ then there exist $\kappa < \gamma < \delta$ such that $(\sigma_1, \sigma_2) \in \mathbb{I}_{\mathcal{E}^*}$ where

$$\sigma_1 = \vee\{\tau_\alpha \mid \alpha < \kappa\}$$

and $\sigma_2 = \vee\{\tau_\alpha \mid \alpha < \gamma\}$. Since $\mathcal{E}^*$ witnesses that δ is a Woodin cardinal there exists $E \in \mathcal{E}^*$ such that

(2.1) $\{\tau_\alpha \mid \alpha < \mathrm{CRT}(E)\} \subset V_{\mathrm{CRT}(E)}$,
(2.2) $\{\tau_\alpha \mid \alpha < \mathrm{LTH}(E)\} \subset V_{\mathrm{LTH}(E)}$,
(2.3) $\langle \tau_\alpha : \alpha < \mathrm{LTH}(E)\rangle = j_E(\langle \tau_\alpha : \alpha < \mathrm{CRT}(E)\rangle)|\ \mathrm{LTH}(E)$.

Let $\kappa = \mathrm{CRT}(E)$ and let $\gamma = \kappa + 1$.

The second key point is that

$$(\sim)^N = (\sim)^V \cap N$$

and that

$$(\sim_{\mathcal{E}^*})^N = (\sim_{\mathcal{E}^*})^V \cap N$$

where $\sim^*$ is computed in V from $(\mathbb{I}_{\mathcal{E}^*})^N$. This is equivalent to the claim that in V, if $\mathbb{B}_0$ is the two element Boolean algebra then for all

$$\pi : \delta \to \mathbb{B}_0$$

if $\pi^*(\sigma_1) = \pi^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in (\mathbb{I}_{\mathcal{E}^*})^N$ then

$$\pi^*(\sigma_1) = \pi^*(\sigma_2)$$

for all $\sigma_1, \sigma_2 \in (\mathbb{W})^N$ such that $\sigma_1 \sim_{\mathcal{E}^*} \sigma_2$. This second key point follows by absoluteness. By $\utilde{\Sigma}^1_2$-absoluteness it suffices to prove this absoluteness claim for all generic extensions of N; i.e. for the purpose of verifying this point, by passing if necessary to a set-generic extension of V, we can reduce to the case that π is set-generic over N. Now (working still in N) if $\mathbb{B}$ is a complete Boolean algebra and if $t \in N^{\mathbb{B}}$, is a term for a function

$$t : \delta \to \mathbb{B}_0,$$

then there is a canonical function

$$\pi_t : \delta \to \mathbb{B}$$

which is induced by the pair $(t, \mathbb{B})$; for each $\alpha < \delta$,

$$\pi_t(\alpha) = [[\text{“}t(\alpha) = 1\text{”}]].$$

Now if

$$1 \Vdash \text{“For all } (\sigma_1, \sigma_2) \in \mathbb{I}_{\mathcal{E}^*},\ t^*(\sigma_1) = t^*(\sigma_2)\text{”}$$

then in N, for all $(\sigma_1, \sigma_2) \in \mathbb{I}_{\mathcal{E}^*}$, $\pi_t^*(\sigma_1) = \pi_t^*(\sigma_2)$. By then by the definition of $\sim_{\mathcal{E}^*}$, for all $\sigma_1, \sigma_2 \in \mathbb{W}$, if

$$\sigma_1 \sim_{\mathcal{E}^*} \sigma_2$$

then $(\pi_t)^*(\sigma_1) = (\pi_t)^*(\sigma_2)$. But then for all $\sigma_1, \sigma_2 \in \mathbb{W}$, if

$$\sigma_1 \sim_{\mathcal{E}^*} \sigma_2$$

then $1 \Vdash$ “$t^*(\sigma_1) = t^*(\sigma_2)$”.

The last key point is that since for each $E \in \mathcal{E}^*$,

$$(j_E)^V(A \cap \mathrm{CRT}(E)) \cap \mathrm{LTH}(E) = A \cap \mathrm{LTH}(E),$$

the set

$$\{b_\alpha \mid \alpha \in A\} \cup \{b'_\alpha \mid \alpha \notin A\}$$

generates an N-generic filter on $\mathbb{B}_{\mathcal{E}^*}$, where for each $\alpha < \delta$, b_α is the element of $\mathbb{B}_{\mathcal{E}^*}$ given by the generator α.

To see this let

$$\pi_A : \delta \to \mathbb{B}_0$$

be the map defined as follows where as above $\mathbb{B}_0$ is the two element Boolean algebra. For each $\alpha \in A$, $\pi_A(\alpha) = 1$ and for each $\alpha \in \delta \backslash A$, $\pi_A(\alpha) = 0$.

Let

$$\pi_A^* : (\mathbb{W})^N \to \mathbb{B}_0$$

be the induced function.

We claim that for all $(\sigma_1, \sigma_2) \in (\mathbb{I}_{\mathcal{E}^*})^N$,

$$\pi_A^*(\sigma_1) = \pi_A^*(\sigma_2).$$

This is immediate. Let $E \in \mathcal{E}^*$ witness that $(\sigma_1, \sigma_2) \in \mathbb{I}_{\mathcal{E}^*}$. Since

$$(j_E)^V(A \cap \mathrm{CRT}(E)) \cap \mathrm{LTH}(E) = A \cap \mathrm{LTH}(E),$$

it follows easily that $\pi_A^*(\sigma_1) = \pi_A^*(\sigma_2)$.

Thus π_A^* induces a boolean homomorphism,

$$\pi_A^{**} : (\mathbb{B}_{\mathcal{E}^*})^N \to \mathbb{B}_0$$

and the corresponding ultrafilter,

$$G = \{b \in (\mathbb{B}_{\mathcal{E}^*})^N \mid \pi_A^{**}(b) = 1\},$$

is N-generic. Finally, $N[A] = N[G]$, and so $(\mathbb{B}_{\mathcal{E}^*})^N$ gives $\mathbb{P} \in N$ as required. $\square$

Remark 174. Suppose that $\mathbb{M}$ is a suitable extender model where one weakens the requirement Definition 161(2a) by just requiring there is a sequence

$$\langle E_\alpha : \alpha < \delta_{\mathbb{M}} \rangle$$

of extenders in V such that

$$\langle E_\alpha \cap \mathbb{M} : \alpha < \delta_{\mathbb{M}} \rangle \in \mathbb{M}$$

and such that $\langle E_\alpha \cap \mathbb{M} : \alpha < \delta_{\mathbb{M}} \rangle$ witnesses in $\mathbb{M}$ that $\delta_{\mathbb{M}}$ is a Woodin cardinal. Then one obtains the weaker conclusion that for each bounded set $a \subset \delta_{\mathbb{M}}$, a is $\mathbb{M}$-generic for a partial order $\mathbb{P} \in \mathbb{M} \cap V_{\delta_{\mathbb{M}}}$. This weaker conclusion (as compared to the conclusion of Theorem 175) suffices for most applications but is more cumbersome to deal with. This is why we defined suitable extender models with the stronger formulation regarding $\langle E_\alpha : \alpha < \delta_{\mathbb{M}} \rangle$.

Theorem 175. *Suppose* $\mathbb{M}$ *is a suitable extender model. Then there exists* $\mathcal{F} \subseteq \delta_{\mathbb{M}}$ *such that*:

(1) $V_{\delta_{\mathbb{M}}} \in L[\mathcal{F}]$;
(2) $\mathbb{M}[\mathcal{F}]$ *is a generic extension of* $\mathbb{M}$ *for a partial order*

$$\mathbb{P} = (\delta, <_{\mathbb{P}})$$

such that $\mathbb{P} \in \mathbb{M}$ *and such that* $\mathbb{P}$ *is* $\delta_{\mathbb{M}}$*-cc in* $\mathbb{M}$;
(3) $(\mathbb{M}[\mathcal{F}])^{<\delta_{\mathbb{M}}} \subseteq \mathbb{M}[\mathcal{F}]$.

Proof. Let $\langle E_\alpha : \alpha \in I_{\mathbb{M}} \rangle$ witness Definition 161(2a) for $\mathbb{M}$. Thus

(1.1) $\langle E_\alpha : \alpha \in I_{\mathbb{M}} \rangle$ witnesses that $\delta_{\mathbb{M}}$ is a Woodin cardinal.
(1.2) $\langle E_\alpha \cap \mathbb{M} : \alpha \in I_{\mathbb{M}} \rangle \in \mathbb{M}$ and so necessarily witnesses in $\mathbb{M}$ that $\delta_{\mathbb{M}}$ is a Woodin cardinal.
(1.3) For each $\alpha \in I_{\mathbb{M}}$, $\mathrm{LTH}(E_\alpha) = \rho(E_\alpha) = \alpha$ and α strongly inaccessible (and in fact $\alpha = \mathrm{CRT}(E_\beta)$ for some $\beta > \alpha$).
(1.4) (coherence) For each $\alpha \in I_{\mathbb{M}}$, $j_{E_\alpha}(\langle E_\beta : \beta \in I_{\mathbb{M}} \cap \mathrm{CRT}(E_\alpha)\rangle)|\alpha = \langle E_\beta : \beta \in I_{\mathbb{M}} \cap \alpha \rangle$.

Let δ_0 be least such that

$$\delta_{\mathbb{M}} = \sup\{\alpha \in I_{\mathbb{M}} \mid \mathrm{CRT}(E_\alpha) = \delta_0\},$$

and let $I_0 = \{\alpha \in I_{\mathbb{M}} \mid \mathrm{CRT}(E_\alpha) = \delta_0\}$.

Fix a set $\mathcal{F}_0 \subset \delta_0$ such that for all $\lambda < \delta_0$ if $\lambda = |V_\lambda|$ then $V_\lambda \subset L_\lambda[\mathcal{F}_0]$. Define

$$\mathcal{F} = \{(\beta, \alpha) \mid \alpha \in I_0, \beta \in j_{E_\alpha}(\mathcal{F}_0) \cap \alpha\}.$$

By the coherence condition (1.4), for each $\beta \in I_{\mathbb{M}}$, if $\text{CRT}(E_\beta) > \delta_0$ then

$$j_{E_\beta}(\mathcal{F}) \cap \beta = \mathcal{F} \cap \beta.$$

By the choice of $\mathcal{F}_0$, for each $\alpha \in I_0$,

$$V_\alpha \subset L[j_{E_\alpha}(\mathcal{F}_0) \cap \alpha] \subset L[\mathcal{F}]$$

and so $V_{\delta_{\mathbb{M}}} \subset L[\mathcal{F}]$. Let $\mathbb{B}$ be the extender algebra as defined in $\mathbb{M}$ using the set of extenders:

$$\mathcal{E} = \{E_\beta \cap \mathbb{M} \mid \beta \in I_{\mathbb{M}} \text{ and } \delta_0 < \text{CRT}(E_\beta)\}.$$

By Theorem 173, since necessarily $\mathcal{E}$ witnesses in $\mathbb{M}$ that $\delta_{\mathbb{M}}$ is a Woodin cardinal, the set $\mathcal{F} \subset \delta_{\mathbb{M}}$ is $\mathbb{M}$-generic for $\mathbb{B}$. □

The following lemma shows that the requirement 2(a) in Definition 161 is necessary for the conclusion of Theorem 175.

Lemma 176. *Suppose there is a proper class of supercompact cardinals. Let δ be a measurable cardinal, let U be a normal, uniform, ultrafilter on δ, and let $j : V \to N$ be the associated elementary embedding. Then the following hold.*

(1) *There exist a proper class of cardinals κ such that*

$$o^N_{\text{LONG}}(\kappa) = \infty.$$

(2) *There exists a set $a \subseteq \text{Ord}$ such that $N[a]$ is not a set-generic extension of N.*

Proof. By Corollary 148, for all $\kappa > \delta$, if κ is a supercompact cardinal in V then $o^N_{\text{LONG}}(\kappa) = \infty$.

We finish by showing that $N[U]$ is not a set-generic extension of N, this will prove that (2) holds since (2) is equivalent to the assertion that for some $a \subseteq N$, $N[a]$ is not a set-generic extension of N.

Note that $N^\delta \subseteq N$. Therefore for each $\gamma \in \text{Ord}$, $j|\gamma \in N[U]$. But this implies that for every set $a \subseteq \text{Ord}$, $a \in N[U]$, since for each $a \subseteq \text{Ord}$,

$$a = \{\alpha \mid j(\alpha) \in j(a)\}.$$

Thus $V = N[U]$. Finally there exists a proper class of $\gamma \in \text{Ord}$ such that γ is a cardinal of N but not a cardinal in V. Therefore $N[U]$ is not a set-generic extension of N. □

Theorem 175 shows that suitable extender models are necessarily quite *close* to V. We shall use Theorem 175 to transfer the existence of large cardinals from V to $\mathbb{M}$ which are essentially as strong as possible; well beyond ω-huge cardinals. As we have previously noted, Theorem 145 does not obviously suffice.

The following theorem is how we shall exploit Theorem 175. This theorem is a corollary of Lemma 147.

Theorem 177. *Suppose that $\mathbb{P}$ is a partial order, $G \subseteq \mathbb{P}$ is V-generic and in $V[G]$ there exists an elementary embedding,*

$$j_0 : L(V[G]_{\lambda_0+1}) \to L(V[G]_{\lambda_0+1})$$

such that $\text{CRT}(j_0) < \lambda_0$ *and such that* $\mathbb{P} \in V_{\text{CRT}(j_0)}$. *Suppose that in $V[G]$, $(L(V[G]_{\lambda_0+1}))^{\#}$ exists. Then in V there exist $\lambda \leq \lambda_0$ and an elementary embedding,*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

such that $\text{CRT}(j) = \text{CRT}(j_0)$ *and* $\text{CRT}(j) < \lambda$.

Proof. Note that $V_{\lambda_0}[G] = (V[G])_{\lambda_0}$.

Let $\kappa_0 = \text{CRT}(j_0)$. Since $\mathbb{P} \in V_{\kappa_0}$ it follows that for each $\gamma < \lambda_0$, $j_0|V_\gamma \in V$ and moreover,

$$j_0|V_{\lambda_0} : V_{\lambda_0} \to V_{\lambda_0}.$$

Thus κ_0 is a limit of Woodin cardinals in V. Let $\delta < \text{CRT}(j_0)$ be a Woodin cardinal such that

$$\mathbb{P} \in V_\delta.$$

Let $\mathbb{Q}_{<\delta}$ be the stationary tower (given by the (generalized) stationary sets $a \in V_\delta$ such that $a \subseteq \mathcal{P}_{\omega_1}(\cup a)$).

Since $\mathbb{P} \in V_\delta$, $\text{RO}(\mathbb{P})$ is isomorphic to a complete subalgebra of $\text{RO}(\mathbb{Q}_{<\delta})$. Thus by forcing over $V[G]$ with the appropriate factor, we can find a generic extension, $V[G][H]$, of $V[G]$ and a V-generic filter $g \subseteq \mathbb{Q}_{<\delta}$ such that

$$V[G][H] = V[g].$$

Let

$$j_g : V[g] \to M_g$$

be the generic elementary embedding given by g. Thus

(1.1) $\text{CRT}(j_g) = \omega_1^V$,
(1.2) $j_g(\omega_1^V) = \delta$,
(1.3) $M_g = \{j_g(f)(\alpha) \mid f \in V, \alpha < \delta\}$,
(1.4) $M_g^\omega \subseteq M_g$ in $V[g]$.

Let $\langle \kappa_n : n < \omega \rangle$ be the critical sequence of j_0 (so for all $n \geq 0$, $j_0(\kappa_n) = \kappa_{n+1}$).

For each $0 < n < \omega$ let E_n be the extender of length κ_n from $j_0|V_{\lambda_0}$. Thus $E_n \in V$ and E_n is δ-complete.

By (1.4), $\langle j_g(E_n) : 0 < n < \omega \rangle \in M_g$, and this defines an elementary embedding,

$$j_0^* : (M_g)_{\lambda_0} \to (M_g)_{\lambda_0}.$$

But $V_{\lambda_0} \in (V[g])_{\lambda_0+1}$ and the elementary embedding j_0 lifts to give an elementary embedding

$$j_0^H : L(V[G]_{\lambda_0+1})[H] \to L(V[G]_{\lambda_0+1})[H]$$

such that $j_0^H((M_g)_{\lambda_0}) = (M_g)_{\lambda_0}$ (by the definition of M_g as a generic ultrapower and since $j_0(V_{\lambda_0}) = V_{\lambda_0}$).

By Lemma 147 it follows that $j_0^* = j_0^H|(M_g)_{\lambda_0}$ (this is the key point).

By (1.4),

$$(M_g)_{\lambda_0+1} \in L(V[g]_{\lambda_0+1})$$

and

$$j_0^H((M_g)_{\lambda_0+1}) = (M_g)_{\lambda_0+1}$$

since $j_0^H((M_g)_{\lambda_0}) = (M_g)_{\lambda_0}$. Thus $j_0^H|L((M_g)_{\lambda_0+1})$ defines an elementary embedding,

$$j^* : L((M_g)_{\lambda_0+1}) \to L((M_g)_{\lambda_0+1}).$$

Let Θ be the supremum of the ordinals, α, such that there exists a surjection,

$$\pi : (M_g)_{\lambda_0+1} \to \alpha$$

such that $\pi \in L((M_g)_{\lambda_0+1})$. Since $(L(V[g]_{\lambda_0+1}))^\#$ exists and since

$$(M_g)_{\lambda_0+1} \in L(V[g]_{\lambda_0+1}),$$

$(L((M_g)_{\lambda_0+1}))^\#$ exists in $V[g]$ and so Θ has cofinality ω.

Since $j_0^H|(M_g)_{\lambda_0} = j_0^*$,

$$j^*|(M_g)_{\lambda_0+1} \in M_g.$$

But this implies that for each $\eta < \Theta$,

$$j^*|L_\eta((M_g)_{\lambda_0+1}) \in M_g.$$

To see this choose $\alpha < \Theta$ such that $\eta < \alpha$ and such that there exists a surjection

$$\rho : (M_g)_{\lambda_0+1} \to L_\alpha((M_g)_{\lambda_0+1})$$

which is definable in $L_\alpha((M_g)_{\lambda_0+1})$ from a parameter $a \in (M_g)_{\lambda_0+1}$. Thus for all $b \in (M_g)_{\lambda_0+1}$,

$$j^*(\rho(b)) = j^*(\rho)(j^*(b)).$$

But $j^*(\rho) \in M_g$, since it is definable from $j^*(a)$ in $L_{j^*(\alpha)}((M_g)_{\lambda_0+1})$, and $j^*|(M_g)_{\lambda_0+1} \in M_g$. Therefore

$$j^*|L_\alpha((M_g)_{\lambda_0+1}) \in M_g.$$

Since $(M_g)^\omega \subseteq M_g$ in $V[g]$ and since $\mathrm{cof}(\Theta) = \omega$, it follows that

$$j^*|L_\Theta((M_g)_{\lambda_0+1}) \in M_g.$$

Note that for all $a \in (M_g)_{\lambda_0+1}$ there exists a function

$$F_a : (M_g)_{\lambda_0+1} \to (M_g)_{\lambda_0+1}$$

such that $a = j^*(F)(j^*|(M_g)_{\lambda_0})$ and such that F is definable from parameters in $(M_g)_{\lambda_0+1}$. For $k \in (M_g)_{\lambda_0+1}$ if k is an elementary embedding,

$$k : (M_g)_{\lambda_0} \to (M_g)_{\lambda_0},$$

and if $k(c) = a$ then $F(k) = c$. Otherwise $F(k) = \emptyset$. Thus $j^*(F_a)(j^*|(M_g)_{\lambda_0}) = a$.

Define

$$X = \{j^*(F)(j^*|(M_g)_{\lambda_0}) \mid F \in L((M_g)_{\lambda_0+1})\}.$$

Thus $(M_g)_{\lambda_0+1} \subseteq X$. Since there is a surjection,

$$\rho : \mathrm{Ord} \times (M_g)_{\lambda_0+1} \to L((M_g)_{\lambda_0+1}),$$

which is Σ_1-definable in $L((M_g)_{\lambda_0+1})$ from $(M_g)_{\lambda_0+1}$, it follows that

$$X \prec L((M_g)_{\lambda_0+1})$$

and so $L((M_g)_{\lambda_0+1})$ is the transitive collapse of X. This gives an elementary embedding,

$$j^{**} : L((M_g)_{\lambda_0+1}) \to L((M_g)_{\lambda_0+1})$$

such that

(2.1) $j^{**}|L_\Theta((M_g)_{\lambda_0+1}) = j^*|L_\Theta((M_g)_{\lambda_0+1})$,
(2.2) $L((M_g)_{\lambda_0+1}) = \{j^*(F)(j^*|(M_g)_{\lambda_0}) \mid F \in L((M_g)_{\lambda_0+1})\}$.

Define

$$U \subseteq \mathcal{P}((M_g)_{\lambda_0+1}) \cap L((M_g)_{\lambda_0+1})$$

by $Z \in U$ if $j^{**}|(M_g)_{\lambda_0} \in j^{**}(Z)$. Thus U is an $L((M_g)_{\lambda_0+1})$-ultrafilter and by (2.2), j^{**} is simply the ultrapower embedding,

$$j_U : L((M_g)_{\lambda_0+1}) \to \mathrm{Ult}(L((M_g)_{\lambda_0+1}), U),$$

where we identify $\mathrm{Ult}(L((M_g)_{\lambda_0+1}), U)$ with its transitive collapse. Finally by (2.1), $U \in M_g$, since

$$\mathcal{P}((M_g)_{\lambda_0+1}) \cap L((M_g)_{\lambda_0+1}) \subseteq L_\Theta((M_g)_{\lambda_0+1}),$$

and so $j^{**} \in M_g$ (as a definable class).

Thus in M_g, since $j_g(\kappa_0) = \kappa_0$, there exist $\lambda \leq j_g(\lambda_0)$ and an elementary embedding,

$$j : L((M_g)_{\lambda+1}) \to L((M_g)_{\lambda+1})$$

such that $\mathrm{CRT}(j) = \kappa_0$ and $\mathrm{CRT}(j) < \lambda$. But this implies that in V there exist $\lambda \leq \lambda_0$ and an elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

such that $\mathrm{CRT}(j) = \kappa_0$ and $\mathrm{CRT}(j) < \lambda$. $\square$

Theorem 178. *Suppose $\mathbb{M}$ is a suitable extender model $\lambda_0 > \delta_{\mathbb{M}}$ and*

$$j_0 : L(V_{\lambda_0+1}) \to L(V_{\lambda_0+1})$$

is an elementary embedding with critical point below λ_0 such that $V_{\lambda_0} \prec_{\Sigma_2} V$. Then there exists $\lambda \leq \lambda_0$ and an elementary embedding

$$j : L(\mathbb{M} \cap V_{\lambda+1}) \to L(\mathbb{M} \cap V_{\lambda+1})$$

with critical point below λ such that $j \in \mathbb{M}$.

Proof. By Theorem 175, there exists $\mathcal{F} \subset \delta_{\mathbb{M}}$ such that

(1.1) $\mathcal{F}$ is $\mathbb{M}$-generic for Boolean algebra $\mathbb{B}$ of cardinality $\delta_{\mathbb{M}}$ in $\mathbb{M}$ (and which is $\delta_{\mathbb{M}}$-cc),
(1.2) $(\mathbb{M}[\mathcal{F}])^{<\delta_{\mathbb{M}}} \subset \mathbb{M}[\mathcal{F}]$.

By Theorem 169,

$$j_0(\mathbb{M} \cap V_{\lambda_0}) = \mathbb{M} \cap V_{\lambda_0}$$

and for each $\gamma < \lambda_0$, $j_0|(\mathbb{M} \cap V_\gamma) \in \mathbb{M}$. We have that $(\mathbb{M}[\mathcal{F}])^{<\delta_{\mathbb{M}}} \subset \mathbb{M}[\mathcal{F}]$ and so since for each $a \in \mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}$,

$$a = \cup\{a \cap V_\gamma \mid \gamma < \lambda_0\}$$

and since $(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0})^\omega \subset \mathbb{M}[\mathcal{F}]$, it follows that

(2.1) $\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1} \in L(V_{\lambda_0+1})$,
(2.2) $j_0(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) = \mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}$,
(2.3) $j_0|(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \in \mathbb{M}[\mathcal{F}]$.

noting that $j_0|(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$ is uniquely determined by $j_0|(\mathbb{M} \cap V_{\lambda_0})$ and $j_0|(\mathbb{M} \cap V_{\lambda_0}) \in (\mathbb{M})^\omega \subset \mathbb{M}[\mathcal{F}]$.

Note that $(L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}))^\# \in \mathbb{M}[\mathcal{F}]$. Let I be the class of all Silver indiscernibles of $L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$. Thus every set in $L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$ is Σ_1-definable in $L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$ with parameters from $\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1} \cup I$.

Therefore since $j_0(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) = \mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}$, j_0 induces uniquely an elementary embedding

$$j_0^* : L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \to L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$$

as follows: Suppose $A \in L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$. Then A is Σ_1-definable in $L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$ with parameters from $\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1} \cup I$. Thus there is a Σ_1-formula, $\phi(x_0, x_1, x_2, \ldots, x_n)$, such that for some $c \in \mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}$ and for some $\eta_2, \ldots, \eta_n \in I$,

$$A = \{b \in L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \mid L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \vDash \phi[b, c, \eta_2, \ldots, \eta_n]\}.$$

Now define

$$j_0^*(A) = \{b \in L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \mid L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \vDash \phi[b, j_0(c), \eta_2, \ldots, \eta_n]\}.$$

It is easily checked that j_0^* is well-defined and that j_0^* is an elementary embedding,

$$j_0^* : L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}) \to L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1})$$

such that $j_0^*|(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0}) = j_0|(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0})$. Note that j_0^* is uniquely specified by $j_0|(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0})$ together with $(L(\mathbb{M}[\mathcal{F}] \cap V_{\lambda_0+1}))^\#$. Therefore $j_0^* \in \mathbb{M}[\mathcal{F}]$. Finally $\mathcal{F}$ is $\mathbb{M}$-generic for a partial order $\mathbb{P} \in \mathbb{M} \cap V_{\delta_{\mathbb{M}}+1}$ and $\delta_{\mathbb{M}} < \mathrm{CRT}(j_0^*)$. Therefore by Theorem 177 applied within $\mathbb{M}$, there exists $\lambda \leq \lambda_0$ and an elementary embedding

$$j : L(\mathbb{M} \cap V_{\lambda+1}) \to L(\mathbb{M} \cap V_{\lambda+1})$$

with critical point below λ such that $j \in \mathbb{M}$. $\square$

The detailed structure theory of $L(V_{\lambda+1})$ under the assumption that there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ is somewhat of a mystery. For example, while it is known that

$$\mathcal{P}(\lambda^+) \cap L(V_{\lambda+1})/\mathcal{I}$$

is atomic where $\mathcal{I}$ is the nonstationary ideal of $L(V_{\lambda+1})$ on λ^+, it is not known if $\mathcal{I}$ can be a maximal ideal in $L(V_{\lambda+1})$ on any fixed cofinality. Here is another example. By a theorem of Martin, there exists a single set

$$A \in \mathcal{P}(\lambda^+) \cap L(V_{\lambda+1})$$

such that every set in $\mathcal{P}(\lambda^+) \cap L(V_{\lambda+1})$ is definable from parameters in the structure,

$$(H(\lambda^+), A, \in).$$

It is not known if it is consistent that $A = \emptyset$, or if it is consistent that necessarily $A \neq \emptyset$.

Since such elementary embeddings can exist in inner models under very general conditions, the appropriate inner models could provide the correct setting in which to examine these questions and ultimately understand $L(V_{\lambda+1})$. If comparison holds for these inner models then the questions above are interesting and natural questions one can ask about the least λ such that in the inner model there exists an elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ.

This suggests a more general problem for the fine-structural versions of suitable extender models; i.e. where $\mathbb{M} = L[\mathbb{E}]$ or its strategic variation. The more general problem is to determine for which $\mathbb{M}$-cardinals, $\lambda > \omega$:

$$(L(\mathcal{P}(\lambda)))^{\mathbb{M}} \not\models \mathrm{ZFC}.$$

In a *canonical* model this problem is quite interesting. Schlutzenberg [14] has shown, in the case where $\mathbb{M}$ is an (iterable) Mitchell–Steel model, that for all $\lambda > \omega$,

$$(L(\mathcal{P}(\lambda)))^{\mathbb{M}} \models \mathrm{ZFC}.$$

The case where $\mathbb{M}$ is a suitable extender model is of course entirely different (by the univerality properties of such models). Suppose that it is necessarily the case that for all λ ,if

$$(L(\mathcal{P}(\lambda)))^{\mathbb{M}} \not\models \mathrm{ZFC}$$

then λ is ω-huge in $\mathbb{M}$. This would be striking evidence that the program of understanding the hierarchy of ω-huge cardinals could be the key to the validation of large cardinal axioms beyond the level of supercompact cardinals, *for their existence within the fine-structural suitable extender models could be correlated with natural structural features.*

The following theorem shows that for λ which are singular in $\mathbb{M}$, it is only the cases where $\mathrm{cof}(\lambda) = \omega$ that are relevant.

Theorem 179 (Shelah). *Suppose that λ is a singular strong limit cardinal of uncountable cofinality. Then $L(\mathcal{P}(\lambda)) \models \mathrm{ZFC}$.*

There are a number of ways one might strengthen the axiom that there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ. We shall deal with a family of axioms of this type, motivated both by the problem of transference from V to N and by analogies with determinacy axioms, but we defer this analysis until Part II. There we shall identify what is arguably the analog of the minimum model of $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$ at the level of $V_{\lambda+1}$ and so we shall identify an axiom at the level of $V_{\lambda+1}$ which is analogous to the axiom, $\mathrm{AD}_{\mathbb{R}}$. We shall then prove that this axiom (if it holds at a proper class of λ for which $V_\lambda \prec_{\Sigma_2} V$), transfers down to N where N is any suitable extender model.

The analysis leads to various structural conjectures and these conjectures if true suggest that the correct setting for the analysis of the hierarchy of ω-huge cardinals is *not* a fine-structural version of N. The only alternative is the fine-structural strategic version of N. Such an inner model is of the form $L[\mathbb{E}, I]$ where $\mathbb{E}$ is a sequence of partial extenders and I is the iteration strategy on the levels of $L[\mathbb{E}, I]$. This will require stronger iterability assumptions in V for the construction but the Strong Unique Branch Hypothesis should suffice (unless, of course, the strategic suitable extender model cannot exist).

6.2. *The Ω-logic of suitable extender models*

The main result of this section is that if there is an extendible cardinal then Ω-logic is absolute between V and $\mathbb{M}$ where $\mathbb{M}$ is a suitable extender model. In the sequel to this paper we will prove (subject to fairly general constraints) that if $\mathbb{M}$ is a fine-structural extender model then the Ω Conjecture must hold in $\mathbb{M}$. By the univerality properties of suitable extender models any such result provides strong evidence that the Ω Conjecture cannot be refuted by any large cardinal axiom.

The following theorem is a corollary of Theorem 175.

Theorem 180. *Suppose* $\mathbb{M}$ *is a suitable extender sequence. Suppose* $A \in \mathcal{P}(\mathbb{R}) \cap \mathbb{M}$ *and*

$$\mathbb{M} \vDash \text{“}A \textit{ is universally Baire”}.$$

Then there exists a universally Baire set $A^* \subseteq \mathbb{R}$ *and an elementary embedding,*

$$j : L(A, \mathbb{R} \cap \mathbb{M}) \to L(A^*, \mathbb{R}).$$

Proof. Let S and T be trees on $\omega \times \delta_{\mathbb{M}}$ such that $(S, T) \in \mathbb{M}$ and witnesses in $\mathbb{M}$ that A is $(<\delta_{\mathbb{M}})$-universally Baire.

By Theorem 175, every bounded set $a \subseteq \delta_{\mathbb{M}}$ is $\mathbb{M}$-generic for a partial order $\mathbb{P} \in \mathbb{M} \cap V_{\delta_{\mathbb{M}}}$.

Thus, since the trees S, T witness in $\mathbb{M}$ that A is $(<\delta_{\mathbb{M}})$-universally Baire, for all partial orders $\mathbb{P} \in V_{\delta_{\mathbb{M}}}$ if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$:

$$p[S] = \mathbb{R}^{V[G]} \backslash p[T].$$

Let $A^* = p[S]$. Thus the trees S, T witness in V that A^* is $(<\delta_{\mathbb{M}})$-universally Baire. Since $\delta_{\mathbb{M}}$ is supercompact, A^* is universally Baire.

Let $G \subseteq \mathrm{Coll}(\omega, <\delta_{\mathbb{M}})$ be V-generic and let $\mathbb{R}_G = \mathbb{R}^{V[G]}$ (to simplify notation). Therefore by Theorem 175 again,

$$\mathbb{M}(\mathbb{R}_G)$$

is a symmetric extension of $\mathbb{M}$ for $\mathrm{Coll}(\omega, <\delta_{\mathbb{M}})$. Further,

$$(A_G)^{\mathbb{M}(\mathbb{R}_G)} = (A^*)_G.$$

Finally $\delta_{\mathbb{M}}$ is supercompact both in $\mathbb{M}$ and in V and so there exist in $V[G]$, elementary embeddings

$$j_0 : L(A^*, \mathbb{R}^V) \to L((A^*)_G, \mathbb{R}_G)$$

and

$$j_1 : L(A, \mathbb{R} \cap \mathbb{M}) \to L((A_G)^{\mathbb{M}(\mathbb{R}_G)}, \mathbb{R}_G).$$

The theorem follows. □

As an immediate corollary we obtain the upward absoluteness of Ω-logic from $\mathbb{M}$ to V.

Corollary 181. *Suppose $\mathbb{M}$ is a suitable extender model. Suppose that ϕ is a sentence, T is a theory with $T \in \mathbb{M}$, and that*

$$\mathbb{M} \vDash \text{“}T \vdash_\Omega \phi\text{”}.$$

Then $T \vdash_\Omega \phi$.

To show that Ω-logic is downward absolute to $\mathbb{M}$ one must show that if A is universally Baire and if B is Δ^2_1 in $L(A, \mathbb{R})$ then $\mathbb{M}$ is B-closed. This relativized to parameters in $\mathbb{R} \cap \mathbb{M}$ implies (and is equivalent to) the statement that Ω-logic is absolute between $\mathbb{M}$ and V. What makes this both plausible and ultimately provable is the strong closure of $\mathbb{M}$ with respect to extenders with critical point above $\delta_{\mathbb{M}}$. But to leverage this fact we shall need to assume that fairly large cardinals exist in V above $\delta_{\mathbb{M}}$ and we shall need to represent universally Baire sets as homogeneously Suslin sets in an *embedding normal form* relative to extenders which cohere $\mathbb{M}$ and in a way that the representation is absolute to $\mathbb{M}$. We prove a much stronger version of this than is necessary for the application of showing that Ω-logic is downward absolute to $\mathbb{M}$, but at this stage of the analysis of $\mathbb{M}$ there is no need to economize on large cardinal assumptions.

The proof of Lemma 182 uses the stationary tower, $\mathbb{Q}_{<\delta}$, associated to a Woodin cardinal δ, and the representation of universally Baire sets involves a version of extendible cardinals. (See [4] for an account of the stationary tower.)

Recall that a cardinal, κ, is *α-extendible*, where $\alpha < \kappa$, if there exists an elementary embedding,

$$j : V_{\kappa+\alpha} \to V_{j(\kappa)+\alpha}$$

with critical point κ. If δ is supercompact then for each $\alpha < \delta$,

$$V_\delta \vDash \text{“There is a proper class of } \alpha\text{-extendible cardinals”}.$$

Lemma 182 gives a representation of a homogeneously Suslin set in terms of elementary embeddings,

$$j : V_{\delta+\omega} \to V_{j(\delta)+\omega}$$

with critical point δ. Such elementary embeddings correspond uniquely to elementary embeddings,

$$j : H(|\mathcal{P}^\omega(\delta)|) \to H(|\mathcal{P}^\omega(j(\delta))|),$$

with critical point δ.

Lemma 182. *Suppose that κ is $(\omega + 2)$-extendible, $P \subseteq V_\kappa$ and $B \subseteq \omega^\omega$ is κ-homogeneously Suslin. Suppose there is a supercompact cardinal below κ. Then there exists $\pi : \omega^{<\omega} \to V_\kappa$ such that:*

(1) *For all $s \in \omega^{<\omega}$,*

$$\pi(s) : H(|\mathcal{P}^\omega(\kappa^s_0)|) \to H(|\mathcal{P}^\omega(\kappa^s_1)|)$$

is an elementary embedding with critical point κ_0^s *such that*

$$\pi(s)(P \cap V_{\kappa_0^s+\omega}) = P \cap V_{\kappa_1^s+\omega};$$

(2) *For all* $s, t \in \omega^{<\omega}$, *if* $s \subseteq t$ *and* $\mathrm{dom}(t) = \mathrm{dom}(s) + 1$ *then* $\kappa_0^t = \kappa_1^s$;

(3) *For all* $x \in \omega^\omega$, $x \in B$ *if and only if the image of* $(\kappa_0^{x|0})^+$ *is wellfounded in the direct limit of*

$$\{H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \mid n < \omega\}$$

under the maps, $\pi_{n,m}$, *where for* $n < m$,

$$\pi_{n,m} : H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \to H(|\mathcal{P}^\omega(\kappa_1^{x|m})|)$$

and $\pi_{n,m} = \pi(x|m) \circ \cdots \circ \pi(x|k) \circ \cdots \circ \pi(x|n+1)$.

Proof. Let $\delta < \kappa$ be supercompact and let $\mathcal{E}$ be the set of all elementary embeddings,

$$k : V_{\gamma+\omega} \to V_{k(\gamma)+\omega}$$

such that $\mathrm{CRT}(k) = \gamma$, $\delta < \gamma < \kappa$ and such that

$$k(P \cap V_{\gamma+\omega}) = P \cap V_{k(\gamma)+\omega}.$$

For each set $\sigma \subseteq \mathcal{E}$ such that let G_σ be the game where Player I plays $x \in \omega^\omega$, specifying $x(i)$ in move i, and Player II plays $\langle (k_i, \gamma_i) : i < \omega \rangle$. Play II wins if

(1.1) $k_i \in \sigma$,

(1.2) $k_i : V_{\gamma_i+\omega} \to V_{\gamma_{i+1}+\omega}$,

(1.3) $x \in B$ if and only if the image of $(\gamma_0)^+$ is wellfounded in the direct limit of

$$\{H(|V_{\gamma_i+\omega}|) \mid i < \omega\}$$

under the maps given by $\langle k_i : i < \omega \rangle$.

Clearly it suffices to show that Player II has a winning strategy in G_σ for some $\sigma \subseteq \mathcal{E}$.

Our first claim is that if $\mathcal{S}$ is a winning strategy in G_σ (for either player) then this property of $\mathcal{S}$ is absolute to $V[g]$ where $g \subseteq \mathbb{P}$ is V-generic and $\mathbb{P} \in V_\delta$. To see this let $\delta_0 < \delta$ be a Woodin cardinal with $\mathbb{P} \in V_{\delta_0}$. Suppose that $G \subset \mathbb{Q}_{<\delta_0}$ is V-generic and let

$$j : V \to M \subseteq V[G]$$

be the corresponding generic elementary embedding where $\mathbb{Q}_{<\delta_0}$ is the stationary tower (given by the (generalized) stationary sets $a \in V_{\delta_0}$ such that $a \subseteq \mathcal{P}_{\omega_1}(\cup a)$). Thus $M^\omega \subseteq M$ in $V[G]$ and by the elementarity of j, $j(\mathcal{S})$ is a winning strategy in the game $j(G_\sigma)$. The set B is κ-homogeneously Suslin and $\delta_0 < \kappa$ and so it follows that $j(B) = B_G$. By Lemma 147, for each $k \in \sigma$,

$$j(k)(\eta) = k(\eta)$$

for all $\eta < |V_{\gamma+\omega}|$ where $\gamma = \text{CRT}(k)$. Since $M^\omega \subseteq M$ in $V[G]$ it follows that $\mathcal{S}$ is a winning strategy in $(G_\sigma)^{V[G]}$. Finally since $\delta_0 = (\omega_1)^{V[G]}$ and since $\mathbb{P} \in V_{\delta_0}$, we can suppose that $g \in V[G]$. But we just proved that $\mathcal{S}$ is a winning strategy in $(G_\sigma)^{V[G]}$ and so it follows easily that $\mathcal{S}$ is a winning strategy in $(G_\sigma)^{V[g]}$, again we are using that the set B is κ-homogeneously Suslin. This proves the claim.

We now suppose that $\sigma \subseteq \mathcal{E}$ is countable and we prove that the game G_σ is determined. Let $e : \omega \to \sigma$ be a surjection and let A be the set of $y \in \omega^\omega$ such that

(2.1) for all $i < \omega$,
$$(e(y(i)))(\gamma) = \text{CRT}(e(y(i+1)))$$
where $\gamma = \text{CRT}(e(y(i)))$,

(2.2) the image of $\text{CRT}(e(y(0)))^+$ under the direct limit given by
$$\langle e(y(i)) : i < \omega\rangle$$
is wellfounded.

We claim that A is $({<}\delta)$-universally Baire. To see this suppose that $\delta_0 < \delta$ is a Woodin cardinal,
$$G \subseteq \mathbb{Q}_{<\delta_0}$$
is V-generic and that
$$j : V \to M \subseteq V[G]$$
is the corresponding generic elementary embedding.

Each embedding $k \in \mathcal{E}$ lifts to define an embedding $k_G \in V[G]$: if
$$k : V_{\gamma+\omega} \to V_{k(\gamma)+\omega}$$
then $k_G : V[G]_{\gamma+\omega} \to V[G]_{k(\gamma)+\omega}$.

Let $\sigma_G = \{k_G \mid k \in \sigma\}$ and let $e_G : \omega \to \sigma_G$ be the induced surjection: $e_G(i) = k_G$ where $k = e(i)$.

By Lemma 147, it follows that
$$j(A) = (A)^{V[G]} \cap M$$
where $(A)^{V[G]}$ is A computed in $V[G]$ from e_G. Since δ is a limit of Woodin cardinals it follows by *tree production*, [4], that A is $({<}\delta)$-universally Baire. By Theorem 11, A is $({<}\delta)$-homogeneously Suslin. Finally δ is supercompact, B is κ-homogeneously Suslin, and $\delta < \kappa$, therefore the game G_σ is determined since every set in $L(A, B, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is $({<}\delta)$-homogeneously Suslin and as a consequence determined.

Now suppose that $|\sigma| < \delta$. If $V[G]$ is a generic extension of V for some partial order $\mathbb{P} \in V_\delta$ then each $k \in \sigma$ lifts to define an elementary embedding
$$k_G : V[G]_{\gamma+\omega} \to V[G]_{k(\gamma)+\omega}$$

where $\gamma = \text{CRT}(k)$. Further since B is κ-homogeneously Suslin and since $\delta < \kappa$, B_G is κ-homogeneously Suslin in $V[G]$. Therefore if σ is countable in $V[G]$, the game G_σ as interpreted in $V[G]$ is determined.

Let

$$\tau \in V^{\text{Coll}(\omega,|\sigma|)}$$

be a term for a winning strategy for in the game G_σ as interpreted in $V^{\text{Coll}(\omega,|\sigma|)}$. By the homogeneity of $\text{Coll}(\omega, |\sigma|)$, either τ is a winning strategy for Player I (with boolean value 1) or τ is a winning strategy for Player II.

Let $\delta_0 < \delta$ be a Woodin cardinal such that $|\sigma| < \sigma_0$. Let $\kappa_0 < \delta_0$ be a strongly inaccessible cardinal such that $|\sigma| < \kappa_0$.

Fix a bijection $e : |\sigma| \to \sigma$. Suppose $X \prec V_{\kappa_0}$ is a countable elementary substructure with $|\sigma| \in X$. Then X is τ-*good* if for every X-generic filter $g \subseteq X \cap \text{Coll}(\omega, |\sigma|)$, the interpretation of τ by g is a winning strategy for G_{σ_X} where

$$\sigma_X = \{e(\alpha) \,|\, \alpha \in X \cap |\sigma|\}.$$

The key claim is that the set of τ-good elementary substructures is closed unbounded in $\mathcal{P}_{\omega_1}(V_{\kappa_0})$. To see this let S be the set of all countable $X \prec V_{\kappa_0}$ such that X is not τ-good. Assume toward a contradiction that S is stationary in $\mathcal{P}_{\omega_1}(V_{\kappa_0})$. Let $G \subseteq \mathbb{Q}_{<\delta_0}$ be V-generic with $S \in G$ and let

$$j : V \to M \subseteq V[G]$$

be the corresponding generic elementary embedding. Thus $M^\omega \subseteq M$ in $V[G]$ and

$$\{j(a) \,|\, a \in V_{\kappa_0}\} \in j(S).$$

Let $X = \{j(a) \,|\, a \in V_{\kappa_0}\}$. Thus X is not $j(\tau)$-good in M and so there exists an X-generic filter g for $j(\text{Coll}(\omega, |\sigma|))$ such that the interpretation of $j(\tau)$ by g is not a winning strategy in the game $(G_{\sigma^*})^M$ where

$$\sigma^* = \{j(e)(\alpha) \,|\, \alpha \in X \cap j(|\sigma|)\} = \{j(k) \,|\, k \in \sigma\}.$$

But $\{p \in \text{Coll}(\omega, |\sigma|) \,|\, j(p) \in g\}$ is V-generic for $\text{Coll}(\omega, |\sigma|)$ and the interpretation of $j(\tau)$ by g is isomorphic to the interpretation of τ by g which is a winning strategy in the game $(G_\sigma)^{V[g]}$. Further $V[G]$ is a generic extension of $V[g]$ for a partial order in $V_\delta[g]$ and so by the first claim we proved above, this strategy is a winning strategy in $V[G]$ for $(G_\sigma)^{V[G]}$. Finally by Lemma 147, for each $k \in \sigma$, $j(k)$ and k agree on all ordinals η such that $|\eta| < |V_{\gamma+\omega}|$ where $\gamma = \text{CRT}(k)$. Therefore the interpretation of $j(\tau)$ by g is a winning strategy in the game $(G_{\sigma^*})^M$ which is a contradiction.

Thus there exists a function

$$F : V_{\kappa_0}^{<\omega} \to V_{\kappa_0}$$

such that if $X \prec V_{\kappa_0}$ is countable and closed under F then X is τ-good. This gives a winning strategy for G_σ.

Thus for all $\sigma \in \mathcal{P}_\delta(\mathcal{E})$ the game G_σ is determined. Since δ is supercompact it follows that the game $G_\mathcal{E}$ is determined.

Assume toward a contradiction that Player I has a winning strategy in $G_\mathcal{E}$. Let $\mathcal{S} \subseteq V_\kappa$ be a winning strategy for Player I.

Since κ is $(\omega+2)$-extendible there exists $\delta < \kappa_0 < \kappa$ such that κ_0 is $|V_{\kappa_0+\omega}|$-supercompact. Let

$$j_0 : V \to M_1$$

be an elementary embedding with $\text{CRT}(j_0) = \kappa_0$ and such that $M_1^{V_{\kappa_0+\omega}} \subseteq M_1$.

Let $\langle \kappa_i : i \leq \omega \rangle$ be the critical sequence of j_0, for each $i \leq \omega$ let

$$j_i : M_i \to M_{i+1}$$

be the ith iterate of j_0 and let

$$j_{i,\omega} : M_i \to M_\omega$$

be the limit embedding, where $M_0 = V$.

For each $i < \omega$ let

$$k_i = j_i|(M_i \cap V_{\kappa_i+\omega}) = j_i \cap M_i \cap V_{\kappa_{i+1}+\omega}.$$

Thus for all $i < \omega$,

(3.1) $M_i \cap V_{\kappa_i+\omega} = M_\omega \cap V_{\kappa_i+\omega}$,
(3.2) $k_i \in M_\omega$,
(3.3) for all $a \subseteq V_{\kappa_0}$, $j_i(j_{0,\omega}(a \cap V_{\kappa_0})) = j_{0,\omega}(a \cap V_{\kappa_0})$,
(3.4) for all $a \subseteq V_{\kappa_0}$,

$$k_i(j_{0,\omega}(a \cap V_{\kappa_0}) \cap V_{\kappa_i+\omega}) = j_{0,\omega}(a \cap V_{\kappa_0}) \cap V_{\kappa_{i+1}+\omega},$$

(3.5) $k_i \in j_\omega(\mathcal{E})$.

(3.1)–(3.4) follow immediately since for each $i < \omega$, j_i is the ith iterate of j_0, and (3.5) follows from (3.4).

Let $x \in \omega^\omega$ be the result of playing $\langle (k_i, \kappa_i) : i < \omega \rangle$ against $j_{0,\omega}(\mathcal{S})$. We claim that by absoluteness, $x \notin B$. To see this first note that for each $i < \omega$, $\pi_i \in M_\omega$ where

$$\pi_i = j_{i+1,\omega}|(M_{i+1} \cap V_{\kappa_{i+1}+\omega}).$$

Let $T \in M_\omega$ be the tree of all finite sequences

$$\langle (k_i^*, \kappa_i^*, \pi_i^*) : i \leq m \rangle$$

such that for all $i < m$,

(4.1) $k_i^* : V_{\kappa_i^*+\omega} \cap M_\omega \to V_{\kappa_{i+1}^*+\omega} \cap M_\omega$,
(4.2) $\pi_i^* : V_{\kappa_{i+1}^*+\omega} \cap M_\omega \to V_{\kappa_\omega+\omega} \cap M_\omega$,
(4.3) $\pi_{i+1}^* \circ k_{i+1}^* = \pi_i^*$,
(4.3) $k_i^* \in j_\omega(\mathcal{E})$.

and such that $x|(m+1)$ is the result of playing $\langle (k_i^*, \kappa_i^*) : i < m \rangle$ against $j_{0,\omega}(\mathcal{S})$.

Thus $\langle (k_i, \kappa_i, \pi_i) : i < \omega\rangle$ is an infinite branch of T. Note for every infinite branch,

$$\langle (k_i^*, \kappa_i^*, \pi_i^*) : i < \omega\rangle,$$

of T, the image of $M_\omega \cap V_{\kappa_0^*+\omega}$ under the direct limit given by the sequence, $\langle k_i^* : i < \omega\rangle$ is strongly wellfounded in the sense that all (internally) wellfounded relations of the limit structure are wellfounded. By absoluteness, there must exist an infinite branch of T with

$$\langle (k_i^*, \kappa_i^*, \pi_i^*) : i < \omega\rangle \in M_\omega.$$

If $x \in B$ then this branch defeats $j_{0,\omega}(\mathcal{S})$ and so $x \notin B$.

Now let T^* be the tree of finite sequences,

$$\langle (k_i^*, \kappa_i^*) : i \le m\rangle \in M_\omega$$

such that $\langle (k_i^*, \kappa_i^*) : i \le m\rangle$ is a legal play for Player II against $j_{0,\omega}(\mathcal{S})$ such that $x|m$ is the response of $j_{0,\omega}(\mathcal{S})$.

Therefore $\langle (k_i, \kappa_i) : i < \omega\rangle$ is an infinite branch of T^*. Since $x \notin B$, by absoluteness if

$$\langle (k_i^*, \kappa_i^*) : i < \omega\rangle$$

is an infinite branch of T^* then the image of $((\kappa_0^*)^+)^{M_\omega}$ is wellfounded. However $M_\omega \cap V_{\kappa_\omega+\omega}$ is the image of $V_{\kappa_0+\omega}$ in the direct limit given by $\langle k_i : i < \omega\rangle$. So T^* yields a wellfounded relation,

$$R_T \subseteq M_\omega \cap V_{\kappa_\omega},$$

of rank at least $((\kappa_\omega)^+)^{M_\omega}$ which is a contradiction since

$$|M_\omega \cap V_{\kappa_\omega}|^{M_\omega} = \kappa_\omega.$$

Thus for all sufficiently large σ, Player II has a winning strategy in the game G_σ and this proves the lemma. $\square$

We shall need that the representations given by Lemma 182 are absolute for small forcing extensions, this in effect we proved as the first claim within the proof of Lemma 182.

Lemma 183. *Suppose that* $\pi : \omega^{<\omega} \to V_\kappa$, $B \subseteq \omega^\omega$, *and that*:

(i) *For all* $s \in \omega^{<\omega}$,

$$\pi(s) : H(|\mathcal{P}^\omega(\kappa_0^s)|) \to H(|\mathcal{P}^\omega(\kappa_1^s)|)$$

is an elementary embedding with critical point κ_0^s;

(ii) *For all* $s, t \in \omega^{<\omega}$, *if* $s \subseteq t$ *and* $\mathrm{dom}(t) = \mathrm{dom}(s) + 1$ *then* $\kappa_0^t = \kappa_1^s$;

(iii) *For all* $x \in \omega^\omega$, $x \in B$ *if and only if the image of* $(\kappa_0^{x|0})^+$ *is wellfounded in the direct limit of*

$$\{H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \mid n < \omega\}$$

under the maps, $\pi_{n,m}$, where for $n < m$,

$$\pi_{n,m} : H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \to H(|\mathcal{P}^\omega(\kappa_1^{x|m})|)$$

and $\pi_{n,m} = \pi(x|m) \circ \cdots \circ \pi(x|k) \circ \cdots \circ \pi(x|n+1)$.

Suppose that $\delta < \kappa_0^{\emptyset}$, δ is a limit of Woodin cardinals. Then:

(1) *B is $(<\delta)$-homogeneously Suslin;*
(2) *Suppose $\mathbb{P} \in V_\delta$ and that $G \subseteq \mathbb{P}$ is V-generic. Then in $V[G]$, $x \in B_G$ if and only if the image of $(\kappa_0^{x|0})^+$ is wellfounded in the direct limit of*

$$\{(H(|\mathcal{P}^\omega(\kappa_1^{x|n})|))^{V[G]} \mid n < \omega\}$$

under the maps, $\pi^G_{n,m}$, where for $n < m$,

$$\pi^G_{n,m} : (H(|\mathcal{P}^\omega(\kappa_1^{x|n})|))^{V[G]} \to (H(|\mathcal{P}^\omega(\kappa_1^{x|m})|))^{V[G]},$$

$\pi^G_{n,m} = \pi_G(x|m) \circ \cdots \circ \pi_G(x|k) \circ \cdots \circ \pi_G(x|n+1)$, and where for all $s \in \omega^{<\omega}$, $\pi_G(s)$ is the elementary embedding

$$\pi_G(s) : (H(|\mathcal{P}^\omega(\kappa_0^{s})|))^{V[G]} \to (H(|\mathcal{P}^\omega(\kappa_1^{s})|))^{V[G]}$$

given by $\pi(s)$.

Proof. Let $\delta_0 < \delta$ be a Woodin cardinal with $\mathbb{P} \in V_{\delta_0}$. Suppose that $G_0 \subset \mathbb{Q}_{<\delta_0}$ is V-generic and let

$$j : V \to M \subseteq V[G_0]$$

be the corresponding generic elementary embedding where $\mathbb{Q}_{<\delta_0}$ is the stationary tower (given by the (generalized) stationary sets $a \in V_{\delta_0}$ such that $a \subseteq \mathcal{P}_{\omega_1}(\cup a)$). Thus $M^\omega \subseteq M$ in $V[G_0]$ and $\delta_0 = j(\omega_1)$. Since $\mathbb{P} \in V_{\delta_0}$ we can suppose that $G \in V[G_0]$.

The set B is κ-homogeneously Suslin and $\delta_0 < \kappa$. Therefore, $B_{G_0} = j(B)$ and $B_G = B_{G_0} \cap V[G]$.

Suppose that $x \in (\omega^\omega)^{V[G_0]}$. Then $x \in j(B)$ if and only if the image of $j((\kappa_0^{x|0})^+)$ is wellfounded in the direct limit of

$$\{j(H(|\mathcal{P}^\omega(\kappa_1^{x|n})|)) \mid n < \omega\}$$

under the maps, $j(\pi_{n,m})$, where for $n < m$,

$$j(\pi_{n,m}) : j(H(|\mathcal{P}^\omega(\kappa_1^{x|n})|)) \to j(H(|\mathcal{P}^\omega(\kappa_1^{x|m})|))$$

and $j(\pi_{n,m}) = j(\pi(x|m)) \circ \cdots \circ j(\pi(x|k)) \circ \cdots \circ j(\pi(x|n+1))$. Therefore by Lemma 147 and since $j(B) = B_{G_0}$, $x \in B_{G_0}$ if and only if the image of $(\kappa_0^{x|0})^+$ is wellfounded in the direct limit of

$$\{H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \mid n < \omega\}$$

under the maps, $\pi_{n,m}$, where for $n < m$,

$$\pi_{n,m} : H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \to H(|\mathcal{P}^\omega(\kappa_1^{x|m})|)$$

and $\pi_{n,m} = \pi(x|m) \circ \cdots \circ \pi(x|k) \circ \cdots \circ \pi(x|n+1)$.

Note that for each $s \in \omega^{<\omega}$, if $k = \pi(s)$ then k lifts in $V[G_0]$ to an elementary embedding,

$$k_{G_0} : (H(|\mathcal{P}^\omega(\kappa_0^s)|))^{V[G_0]} \to (H(|\mathcal{P}^\omega(\kappa_1^s)|))^{V[G_0]},$$

and $((\kappa_0^s)^+)^V = ((\kappa_0^s)^+)^{V[G_0]}$.

Finally since $V[G] \subseteq V[G_0]$ and since $B_G = B_{G_0} \cap V[G]$, the lemma follows. □

Lemma 184. *Suppose that N is transitive, $N \subseteq V$, $N \vDash$ ZFC, Ord $\subseteq N$, and that δ is N-supercompact. Suppose that $A \subseteq \mathbb{R}$ is universally Baire and that $N \cap V_\delta$ is A-closed. Then N is A-closed.*

Proof. Let S, T be trees on $\omega \times \delta$ which witness that A is $(<\delta)$-universally Baire with $A = p[T]$. Thus for all partial orders $\mathbb{P} \in V_\delta$, if $g \subseteq \mathbb{P}$ is V-generic then in $V[g]$:

$$p[T] = \mathbb{R}^{V[g]} \backslash p[S].$$

Fix $\eta > \delta$ and suppose that $G \subseteq \mathrm{Coll}(\omega, \eta)$ is V-generic. It suffices to show that in $V[G]$:

$$A_G \in N[G],$$

where A_G is the interpretation of A in $V[G]$.

Let

$$j : V \to M$$

be an elementary embedding with $\mathrm{CRT}(j) = \delta$ such that

(1.1) $j(\delta) > \eta$,
(1.2) $j(N \cap V_\delta) \cap V_{\eta+2} = N \cap V_{\eta+2}$.

By elementarity, in $M[G]$:

$$p[j(T)] = \mathbb{R}^{M[G]} \backslash p[j(S)].$$

Therefore, since $\mathcal{P}(\eta) \subseteq M$,

$$A_G = p[j(T)].$$

Further, since $N \cap V_\delta$ is A-closed, it follows that in $V[G]$:

$$p[j(T)] \in j(N)[G],$$

and so by (1.2), $A_G \in N[G]$. □

Theorem 185. *Suppose $\mathbb{M}$ is a suitable extender model and there exists an $(\omega+2)$-extendible cardinal above $\delta_{\mathbb{M}}$. Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then there exists a countable set, σ, of ordinals such that*

(1) *σ is set generic over $\mathbb{M} \cap V_{\delta_{\mathbb{M}}}$;*
(2) *$\mathbb{M}[\sigma]$ is A-closed.*

Proof. Suppose that $\kappa > \alpha$ and that κ is an $(\omega+2)$-extendible cardinal. Let $P \subset V_\kappa$ be the set of all $a \in V_\kappa$ such that a codes a transitive set of $\mathbb{M}$; i.e. $a = (M, E)$, $E \subset M \times M$, and for some transitive set $x \in \mathbb{M}$, $(M, E) \cong (x, \in)$.

For any γ if

$$\pi : H(|V_{\gamma+\omega}|) \to H(|V_{\pi(\gamma)+\omega}|)$$

is an elementary embedding such that for all $i < \omega$, $\pi(P \cap V_{\gamma+i}) = P \cap V_{\pi(\gamma)+i}$ then

$$\pi(\mathbb{M} \cap H(|V_{\gamma+\omega}|)) = \mathbb{M} \cap H(|V_{\pi(\gamma)+\omega}|).$$

Therefore by Lemma 182, there exists $\pi : \omega^{<\omega} \to V_\kappa$ such that:

(1.1) For all $s \in \omega^{<\omega}$,

$$\pi(s) : H(|\mathcal{P}^\omega(\kappa_0^s)|) \to H(|\mathcal{P}^\omega(\kappa_1^s)|)$$

is an elementary embedding with critical point κ_0^s such that

$$\pi(s)(\mathbb{M} \cap H(|\mathcal{P}^\omega(\kappa_0^s)|)) = \mathbb{M} \cap H(|\mathcal{P}^\omega(\kappa_1^s)|);$$

(1.2) For all $s, t \in \omega^{<\omega}$, if $s \subseteq t$ and $\mathrm{dom}(t) = \mathrm{dom}(s) + 1$ then $\kappa_0^t = \kappa_1^s$;

(1.3) For all $x \in \omega^\omega$, $x \in A$ if and only if the image of $(\kappa_0^{x|0})^+$ is wellfounded in the direct limit of

$$\{H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \mid n < \omega\}$$

under the maps, $\pi_{n,m}$, where for $n < m$,

$$\pi_{n,m} : H(|\mathcal{P}^\omega(\kappa_1^{x|n})|) \to H(|\mathcal{P}^\omega(\kappa_1^{x|m})|)$$

and $\pi_{n,m} = \pi(x|m) \circ \cdots \circ \pi(x|k) \circ \cdots \circ \pi(x|n+1)$.

For each $s \in \omega^{<\omega}$, let

$$\pi^*(s) = \pi(s)|(\kappa_0^s)^+$$

so that

$$\pi^*(s) : (\kappa_0^s)^+ \to (\kappa_1^s)^+$$

Suppose that $s \in \omega^{<\omega}$ and that γ is a cardinal of $\mathbb{M}$ such that

$$\gamma < |\mathcal{P}^\omega(\kappa_0^s)|.$$

Then by Theorem 151,

$$\pi(s)|(H(\gamma^+))^{\mathbb{M}} \in \mathbb{M}$$

and so $\pi(s)|\gamma \in \mathbb{M}$.

Therefore for each $s \in \omega^{<\omega}$, $\pi^*(s) \in \mathbb{M}$. Thus by Theorem 175, there exist a partial order

$$\mathbb{P} \in V_{\delta_{\mathbb{M}}} \cap \mathbb{M}$$

and an $\mathbb{M}$-generic filter $g \subseteq \mathbb{P}$ such that $\pi^* \in \mathbb{M}[g]$.

This implies by Lemma 183 that $V_{\delta_{\mathbb{M}}} \cap \mathbb{M}[\pi^*]$ is A-closed. Finally, since $\delta_{\mathbb{M}}$ is $\mathbb{M}[\pi^*]$-supercompact, by Lemma 184, $\mathbb{M}[\pi^*]$ is A-closed. □

Theorem 186. *Suppose $\mathbb{M}$ is a suitable extender model and there exists an $(\omega+2)$-extendible cardinal above $\delta_{\mathbb{M}}$. Suppose $x \in \mathbb{R} \cap \mathbb{M}$, $B \subseteq \mathbb{R}$,*

$$L(B, \mathbb{R}) \vDash \mathrm{AD}^+$$

and every set $C \subseteq \mathbb{R}$ which is $\Delta^2_1(x)$ in $L(B, \mathbb{R})$, is universally Baire. Suppose that $A \subseteq \mathbb{R}$ is $\Delta^2_1(x)$ in $L(B, \mathbb{R})$. Then

(1) *$\mathbb{M}$ is A-closed,*
(2) *$A \cap \mathbb{M}$ is universally Baire in $\mathbb{M}$.*

Proof. Let Γ_0 be the collection of all sets $Z \subseteq \omega^\omega$ such that Z such that Z is Δ^2_1-definable in $L(B, \mathbb{R})$ with parameter x.

Since $L(B, \mathbb{R}) \vDash \mathrm{AD}^+$, by the Σ^2_1-Basis Theorem, Theorem 25, every set in Γ_0 has a scale in Γ_0.

By Theorem 185, there exists a countable set $\sigma \subseteq \mathrm{Ord}$ such that $\mathbb{M}[\sigma]$ is Z-closed for each $Z \in \Gamma_0$. By Theorem 175, $\mathbb{M}[\sigma]$ is a generic extension of $\mathbb{M}$ for some partial order $\mathbb{P} \in \mathbb{M} \cap V_{\delta_{\mathbb{M}}}$.

Suppose that $G \subseteq \mathrm{Coll}(\omega, <\delta_{\mathbb{M}})$ is V-generic. Then in $V[G]$, $\mathbb{M}[\sigma][G]$ is Z_G-closed for each $Z \in \Gamma_0$.

By reflection and Lemma 38,

$$\langle V_{\omega+1}, Z : Z \in \Gamma_0 \rangle \prec \langle V[G]_{\omega+1}, Z_G : Z \in \Gamma_0 \rangle.$$

Let $\mathbb{R}_G = (\mathbb{R})^{V[G]}$. By Theorem 175, $\mathbb{M}[\sigma](\mathbb{R}_G)$ is a symmetric extension of $\mathbb{M}[\sigma]$ for $\mathrm{Coll}(\omega, <\delta_{\mathbb{M}})$ and so again by reflection and Lemma 38, for each $Z \in \Gamma_0$, Z_G is the same computed either from (V, G) or from $(\mathbb{M}[\sigma], \mathbb{R}_G)$; i.e. there exists a tree $T \in \mathbb{M}[\sigma]$ on $\omega \times \delta_{\mathbb{M}}$ such that in $V[G]$,

$$Z_G = p[T].$$

Since $\delta_{\mathbb{M}}$ is supercompact in $\mathbb{M}[\sigma]$, it follows by Theorem 5 that T is $(<\delta_{\mathbb{M}})$-weakly homogeneous in $\mathbb{M}[\sigma]$.

Note that $\mathbb{M}[\sigma](\mathbb{R}_G) = \mathbb{M}(\mathbb{R}_G)$ since $\mathbb{M}[\sigma]$ is a generic extension of $\mathbb{M}$ for some partial order $\mathbb{P} \in \mathbb{M} \cap V_{\delta_{\mathbb{M}}}$.

Let Γ be the set of all $Z \subseteq \mathbb{R}_G$ such that $Z \in \mathbb{M}(\mathbb{R}_G)$ and such that Z is Suslin and co-Suslin in $\mathbb{M}(\mathbb{R}_G)$. Then by the Derived Model Theorem, Theorem 31,

$$L(\Gamma, \mathbb{R}_G) \vDash \mathrm{AD}^+$$

and $\Gamma_0^G \subseteq \Gamma$. Since

$$\langle V_{\omega+1}, Z : Z \in \Gamma_0 \rangle \prec \langle V[G]_{\omega+1}, Z_G : Z \in \Gamma_0 \rangle,$$

it follows by the definition of Γ_0 and the Σ^2_1-Basis Theorem, Theorem 25, that for each $Z \in \Gamma_0$ there exists a tree T_Z on $\omega \times \delta_{\mathbb{M}}$ such that T is definable in $\mathbb{M}(\mathbb{R}_G)$ and such that in $V[G]$, $Z_G = p[T_Z]$.

Thus for each $Z \in \Gamma_0$, $\mathbb{M} \cap V_{\delta_{\mathbb{M}}}$ is Z-closed and so by Lemma 184, $\mathbb{M}$ is Z-closed. Finally the set of trees, $\{T_Z \mid Z \in \Gamma_0\}$, witnesses that for each $Z \in \Gamma_0$, $Z \cap \mathbb{M}$ is $<\delta_{\mathbb{M}}$-universally Baire in $\mathbb{M}$. Finally $\delta_{\mathbb{M}}$ is supercompact in $\mathbb{M}$ and so for each $Z \in \Gamma_0$, $Z \cap \mathbb{M}$ is universally Baire in $\mathbb{M}$. □

Combining the previous theorem with Corollary 181 we obtain the following theorem.

Theorem 187. *Suppose* $\mathbb{M}$ *is a suitable extender model and there exists an* $(\omega+2)$*-extendible cardinal above* $\delta_{\mathbb{M}}$. *Suppose that* ϕ *is a sentence and that* T *is a theory with* $T \in \mathbb{M}$. *Then the following are equivalent.*

(1) $T \vdash_\Omega \phi$.
(2) $\mathbb{M} \models$ "$T \vdash_\Omega \phi$".

Proof. Since $\delta_{\mathbb{M}}$ is supercompact in $\mathbb{M}$, by Theorem 186 and the definition of $T \vdash_\Omega \phi$, it follows that (1) implies (2). The point is that in general, if $\delta_{\mathbb{M}}$ is supercompact then

$$T \vdash_\Omega \phi$$

if and only if $V_{\delta_{\mathbb{M}}} \models$ "$T \vdash_\Omega \phi$". By Corollary 181, (2) implies (1). □

Thus the proof relation, $\vdash_\Omega$, is absolute to $\mathbb{M}$. The theorem is also true with real parameters.

7. HOD and Supercompact Cardinals

7.1. *Closure properties of* HOD

We shall examine the situation that there exists a cardinal δ such that δ is HOD-supercompact, see Definition 132 on p. 248. Of course if $V = \text{HOD}$ then every supercompact cardinal is HOD-supercompact. However the notions are not equivalent, [13]. Nevertheless the existence of HOD-supercompact cardinals follows from the existence of an extendible cardinal.

Lemma 188. *Suppose that* δ *is an extendible cardinal. Then the cardinal* δ *is* HOD*-supercompact.*

Proof. Fix $\gamma_0 > \delta$. We must produce an elementary embedding

$$j_0 : V \to M_0$$

with critical point δ such that $M_0^{V_{\gamma_0}} \subseteq M_0$, such that $\gamma_0 < j_0(\delta)$, and such that

$$(\text{HOD})^{M_0} \cap V_{\gamma_0} = \text{HOD} \cap V_{\gamma_0}.$$

Since δ is an extendible cardinal the displayed requirement is equivalent to the requirement that $j_0(\text{HOD} \cap V_\delta) \cap V_{\gamma_0} = \text{HOD} \cap V_{\gamma_0}$.

Let $\gamma > \gamma_0$ be such that

(1.1) $|V_\gamma| = \gamma$,
(1.2) $\mathrm{cof}(\gamma) > |V_{\gamma_0}|$,
(1.3) $(\mathrm{HOD})^{V_\gamma} = \mathrm{HOD} \cap V_\gamma$.

Let

$$j : V_{\gamma+1} \to V_{j(\gamma)+1}$$

be an elementary embedding with critical point, δ, such that $\gamma < j(\delta)$.

Let E be the extender of length $j(\gamma)$ from j and let

$$j_E : V \to M$$

be the associated embedding where $M = \mathrm{Ult}(V, E)$. Thus

$$j_E | V_{\gamma+1} = j$$

which implies that

$$V_{j_E(\gamma)} \subseteq M.$$

Since $\mathrm{cof}(\gamma) > |V_{\gamma_0}| > \delta$ and since $V_{j(\gamma)} \subset M$, it follows that

$$j(\gamma)^\kappa \subset M$$

where $\kappa = |V_{\gamma_0}|$, and this implies that

$$M^{V_{\gamma_0}} \subseteq M,$$

since $M = \mathrm{Ult}(V, E)$ and since $\mathrm{LTH}(E) = j(\gamma)$.

By choice of γ, $(\mathrm{HOD})^{V_\gamma} = \mathrm{HOD} \cap V_\gamma$. Therefore, applying j to this equality we have

$$(\mathrm{HOD})^M \cap V_{j_E(\gamma)} = (\mathrm{HOD})^{j_E(V_\gamma)} = (\mathrm{HOD})^{V_{j_E(\gamma)}}$$

and so since $\mathrm{HOD} \cap V_\gamma = (\mathrm{HOD})^{V_\gamma}$,

$$j_E(\mathrm{HOD} \cap V_\delta) \cap V_\gamma = \mathrm{HOD} \cap V_\gamma.$$

Thus j_E witnesses the existence of j_0 as required. $\square$

Suppose that κ is an uncountable regular cardinal which is not measurable in HOD and $S \subseteq \kappa$ is a stationary set such that $S \in \mathrm{HOD}$. Then for each $\gamma < \kappa$ such that $(2^\gamma)^{\mathrm{HOD}} < \kappa$, there exists a sequence

$$\langle S_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$$

of pairwise disjoint subsets of S such that for each $\alpha < \gamma$, S_α is stationary in S. This motivates the next definition.

Definition 189. Suppose that κ is an uncountable regular cardinal. Then κ is *ω-strongly measurable in* HOD if there exists $\gamma < \kappa$ such that:

(1) γ is an infinite cardinal in HOD and $(2^\gamma)^{\mathrm{HOD}} < \kappa$;

(2) There does not exist a sequence

$$\langle S_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$$

of pairwise disjoint subsets of κ such that for each $\alpha < \gamma$, S_α is stationary in $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}$.

Lemma 190. *Suppose that*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

is an elementary embedding such that $\mathrm{CRT}(j) < \lambda$. *Suppose that*

$$G \subseteq \mathrm{Coll}(\lambda^+, V_{\lambda+1})$$

is V-generic. Then in $L(V_{\lambda+1})[G]$, λ^+ is ω-strongly measurable in HOD.

Proof. Let $\gamma = \mathrm{CRT}(j)$. Thus in V, $2^\gamma < \lambda$. We shall show that γ is the witness in $L(V_{\lambda+1})[G]$, that λ^+ is ω-strongly measurable in HOD. First note that by the homogeneity of $\mathrm{Coll}(\lambda^+, V_{\lambda+1})$, it follows that

$$(\mathrm{HOD})^{L(V_{\lambda+1})[G]} \subseteq V$$

and so necessarily $2^\gamma < (\lambda^+)^V$ in $(\mathrm{HOD})^{L(V_{\lambda+1})[G]}$.

We claim that in $L(V_{\lambda+1})$ there is no partition,

$$\langle S_\alpha : \alpha < \gamma \rangle \in L(V_{\lambda+1})$$

of $\{\xi < \lambda^+ \mid \mathrm{cof}(\xi) = \omega\}$ into stationary sets. We sketch the proof which will adapted in Part II to a more general setting and to prove much stronger statements. The key points are that $j|\lambda^+ \in L(V_{\lambda+1})$ and $(\lambda^+)^\omega \in L(V_{\lambda+1})$.

Assume toward a contradiction that there is a partition

$$\langle S_\alpha : \alpha < \gamma \rangle \in L(V_{\lambda+1})$$

of $\{\xi < \lambda^+ \mid \mathrm{cof}(\xi) = \omega\}$ into sets each of which is stationary in $L(V_{\lambda+1})$. Let

$$\langle T_\beta : \beta < j(\gamma) \rangle = j(\langle S_\alpha : \alpha < \gamma \rangle).$$

Thus for each $\beta < j(\gamma)$, $T_\beta \subset \{\xi < \lambda^+ \mid \mathrm{cof}(\xi) = \omega\}$ and T_β is stationary in $L(V_{\lambda+1})$. Let

$$C = \{\xi < \lambda^+ \mid \mathrm{cof}(\xi) = \omega \text{ and } j(\xi) = \xi\}.$$

Thus C is ω-closed and $C \in L(V_{\lambda+1})$. Since T_γ is stationary in $L(V_{\lambda+1})$ and since $C \in L(V_{\lambda+1})$, $C \cap T_\gamma \neq \emptyset$. Fix $\xi_0 \in C \cap T_\gamma$. Since $\langle S_\alpha : \alpha < \gamma \rangle$ is a partition of $\{\xi < \lambda^+ \mid \mathrm{cof}(\xi) = \omega\}$, there must exist $\gamma_0 < \gamma$ such that $\xi \in S_{\gamma_0}$. This implies that $j(\xi_0) \in j(S_{\gamma_0})$, but $j(\xi_0) = \xi_0$ and $j(S_{\gamma_0}) = T_{j(\gamma_0)} = T_{\gamma_0}$ since $\gamma_0 < \gamma$ and since $\gamma = \mathrm{CRT}(j)$. This implies $T_{\gamma_0} \cap T_\gamma \neq \emptyset$ and so $\gamma_0 = \gamma$ which is a contradiction.

Again by the homogeneity of $\mathrm{Coll}(\lambda^+, V_{\lambda+1})$,

$$(\mathrm{HOD})^{L(V_{\lambda+1})} = (\mathrm{HOD})^{L(V_{\lambda+1})[G]},$$

since by the closure of $\mathrm{Coll}(\lambda^+, V_{\lambda+1})$,

$$\mathcal{P}(\lambda) \cap L(V_{\lambda+1}) = \mathcal{P}(\lambda) \cap L(V_{\lambda+1})[G].$$

The lemma now follows from the definitions. □

The following conjecture is a strong version of a conjecture from [19].

Definition 191 (HOD Conjecture). There is a proper class of uncountable regular cardinals κ which are not ω-strongly measurable in HOD.

Remark 192. (1) If one can prove the weaker conjecture that there is at least one cardinal $\kappa > \omega_1$ that is not ω-strongly measurable in HOD then one can prove the HOD Conjecture. The point is that if in every generic extension of V there exists a cardinal $\kappa \geq \omega_2$ which is not ω-strongly measurable in HOD, then *in* V there must exist a proper class of uncountable regular cardinals κ which are not ω-strongly measurable in HOD.

(2) It is unknown whether there can even exist *one* singular strong limit cardinal κ of uncountable cofinality such that κ^+ is ω-strongly measurable in HOD.

(3) Suppose that κ is ω-strongly measurable in HOD. Then there must exist a stationary set $S \subset \{\xi < \kappa \mid \mathrm{cof}(\xi) = \omega\}$ such that $S \in \mathrm{HOD}$ and such that if $\mathcal{F}$ is the club filter at κ then $\mathcal{F}|S$ is an ultrafilter on $\mathcal{P}(S) \cap \mathrm{HOD}$; i.e. there is no partition of S in HOD into two sets each of which is stationary in V. Thus κ is in fact a measurable cardinal in HOD and in a very strong sense.

(4) Suppose that γ is a singular strong limit cardinal such that

$$(\gamma^+)^{\mathrm{HOD}} < \gamma^+,$$

then one can show that either $\mathrm{cof}((\gamma^+)^{\mathrm{HOD}}) = \omega$ or $\mathrm{cof}((\gamma^+)^{\mathrm{HOD}}) = \mathrm{cof}(\gamma)$. A natural conjecture is that necessarily $\mathrm{cof}((\gamma^+)^{\mathrm{HOD}}) = \omega$.

Theorem 193 (HOD Conjecture). *Suppose that δ is a supercompact cardinal and $\gamma > \delta$. Then there is a normal fine measure μ on $\mathcal{P}_\delta(\gamma)$ such that*:

(1) $\mu(\mathrm{HOD} \cap \mathcal{P}_\delta(\gamma)) = 1$;
(2) *if δ is* HOD-*supercompact then* $\mu \cap \mathrm{HOD} \in \mathrm{HOD}$.

Proof. Fix a regular cardinal $\kappa > |V_{\gamma+\omega+1}|$ such that κ is not ω-strongly measurable in HOD and fix $\lambda > |V_{\kappa+\omega}|$ such that

$$\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}.$$

Thus

$$\mathrm{HOD} \cap V_{\kappa+2} = (\mathrm{HOD})^{V_\lambda} \cap V_{\kappa+2}.$$

Let $S^\kappa_\omega = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}$. Since κ is not ω-strongly measurable in HOD there exists a sequence,

$$\langle S_\alpha : \alpha < |V_{\gamma+\omega}|\rangle \in \mathrm{HOD},$$

of pairwise disjoint sets such that for all $\alpha < |V_{\gamma+\omega}|$, S_α is stationary (in V) and such that

$$S^\kappa_\omega = \cup\{S_\alpha \mid \alpha < |V_{\gamma+\omega}|\}.$$

Let $a = (\gamma, \kappa, \langle S_\alpha : \alpha < |V_{\gamma+\omega}|\rangle)$. Note that since δ is supercompact, δ is a strong cardinal and so

$$\mathrm{HOD} \cap V_\delta = (\mathrm{HOD})^{V_\delta}.$$

Therefore by Lemma 133, since δ is V-supercompact, there exists an elementary embedding,

$$j : V \to M$$

with critical point $\bar\delta$ such that

(1.1) $j(\bar\delta) = \delta$,
(1.2) there exists $(\bar a, \bar\lambda) \in V_\delta$ such that $j(\langle \bar a, \bar\lambda\rangle) = \langle a, \lambda\rangle$,
(1.3) $V_\lambda \subseteq M$,
(1.4) $\mathrm{HOD} \cap V_\lambda \subseteq j(\mathrm{HOD} \cap V_{\bar\lambda})$.

Note that by the choice of λ, $\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}$, and by the choice of j, $V_\lambda \subset M$. This implies (1.4).

If δ is HOD-supercompact then we also assume that

$$j(\mathrm{HOD} \cap V_{\bar\lambda}) = \mathrm{HOD} \cap V_\lambda,$$

or equivalently, that

$$\mathrm{HOD} \cap V_{\bar\lambda} = (\mathrm{HOD})^{V_{\bar\lambda}}.$$

Let $(\bar\gamma, \bar\kappa, \bar\lambda)$ be the preimage of $(\gamma, \kappa, \lambda)$ under j and let

$$\langle \bar S_\alpha : \alpha < |V_{\bar\gamma+\omega}|\rangle$$

be the preimage of $\langle S_\alpha : \alpha < |V_{\gamma+\omega}|\rangle$ under j.

Thus $\langle \bar S_\alpha : \alpha < |V_{\bar\gamma+\omega}|\rangle \in (\mathrm{HOD})^{V_{\bar\lambda}}$ and for all $\alpha < |V_{\bar\gamma+\omega}|$,

$$\bar S_\alpha \subseteq \{\eta < \bar\kappa \mid \mathrm{cof}(\eta) = \omega\}$$

and $\bar S_\alpha$ is stationary.

For $\eta < \bar\kappa$ such that $\mathrm{cof}(\eta) > \omega$ let

$$\sigma_\eta = \{\alpha < |V_{\bar\gamma+\omega}| \mid \bar S_\alpha \cap \eta \text{ is stationary in } \eta\}.$$

Since $\langle \bar S_\alpha : \alpha < |V_{\bar\gamma+\omega}|\rangle \in (\mathrm{HOD})^{V_{\bar\lambda}}$, it follows that

$$\langle \sigma_\eta : \eta < \bar\kappa, \mathrm{cof}(\eta) > \omega\rangle \in (\mathrm{HOD})^{V_{\bar\lambda}}.$$

Let

$$\langle \tau_\eta : \eta < \kappa, \mathrm{cof}(\eta) > \omega\rangle = j(\langle \sigma_\eta : \eta < \bar\kappa, \mathrm{cof}(\eta) > \omega\rangle).$$

Thus for each $\eta < \kappa$ if $\mathrm{cof}(\eta) > \omega$ then

$$\tau_\eta = \{\alpha < |V_{\gamma+\omega}| \mid S_\alpha \cap \eta \text{ is stationary in } \eta\}.$$

Further

$$\langle \tau_\eta : \eta < \kappa, \mathrm{cof}(\eta) > \omega \rangle \in (\mathrm{HOD})^{V_\lambda} \subseteq \mathrm{HOD}.$$

Let

$$\eta_0 = \sup\{j(\xi) \mid \xi < \bar{\kappa}\}.$$

Thus $\eta_0 < \kappa$ and $\mathrm{cof}(\eta_0) = \bar{\kappa} > \omega$. We claim that

$$\tau_{\eta_0} = \{j(\alpha) \mid \alpha < |V_{\bar{\gamma}+\omega}|\}.$$

It is easy to verify that for *any* club $C \subseteq \eta_0$ there exists a club $D \subseteq \bar{\kappa}$ such that

$$\{j(\xi) \mid \xi \in D, \mathrm{cof}(\xi) = \omega\} \subseteq \{\xi \in C \mid \mathrm{cof}(\xi) = \omega\}.$$

Notice also that for each $\alpha < |V_{\bar{\gamma}+\omega}|$,

$$\bar{S}_\alpha \subseteq \{\xi < \bar{\kappa} \mid \mathrm{cof}(\xi) = \omega\}$$

and $\bar{S}_\alpha$ is stationary in $\bar{\kappa}$.

Assume toward a contradiction $\alpha \notin \tau_{\eta_0}$ and there exists $\bar{\alpha}$ such that $j(\bar{\alpha}) = \alpha$. Thus there exists a club C_α such that

$$C_\alpha \cap S_\alpha \cap \eta_0 = \emptyset.$$

Let (by the above fact) $D_\alpha \subseteq \bar{\kappa}$ be a club such that

$$\{j(\xi) \mid \xi \in D_\alpha, \mathrm{cof}(\xi) = \omega\} \subseteq \{\xi \in C_\alpha \mid \mathrm{cof}(\xi) = \omega\}.$$

Since $\bar{S}_\alpha$ is stationary in $\bar{\kappa}$ there must exist $\xi \in D \cap \bar{S}_\alpha$ such that $j(\xi) \in C_\alpha$. But then $S_{j(\alpha)} \cap C_\alpha \neq \emptyset$, a contradiction. This shows that

$$\{j(\alpha) \mid \alpha < |V_{\bar{\gamma}+\omega}|\} \subseteq \tau_{\eta_0},$$

Now suppose that $\beta \in \tau_{\eta_0}$. Let $\xi \in C$ be such that $\mathrm{cof}(\xi) = \omega$ and such that $\xi \in S_\beta$. Since $\beta \in \tau_{\eta_0}$, $S_\beta \cap \eta_0$ is stationary in η_0 and so ξ exists. Let $\bar{\xi}$ be the preimage of ξ under j. Since

$$j(\langle \bar{S}_\alpha : \alpha < |V_{\bar{\gamma}+\omega}|\rangle) = \langle S_\alpha : \alpha < |V_{\gamma+\omega}|\rangle,$$

it follows that

$$\{\epsilon < \bar{\kappa} \mid \mathrm{cof}(\epsilon) = \omega\} = \cup\{\bar{S}_\alpha \mid \alpha < \bar{\kappa}\}$$

and so there exists $\alpha < \bar{\kappa}$ such that $\bar{\xi} \in \bar{S}_\alpha$. But then

$$\xi \in S_{j(\alpha)}$$

and so $j(\alpha) = \beta$ since otherwise $S_{j(\alpha)} \cap S_\beta = \emptyset$.

This shows

$$\tau_{\eta_0} \subseteq \{j(\alpha) \mid \alpha < |V_{\bar{\gamma}+\omega}|\},$$

and so $\tau_{\eta_0} = \{j(\alpha) \mid \alpha < |V_{\bar{\gamma}+\omega}|\}$.

Let ν be the normal measure on $\mathcal{P}_{\bar{\delta}}(\bar{\gamma})$ induced by j and let $\mu = j(\nu)$. Since $\{j(\xi) \mid \xi < \bar{\gamma}\} \in \mathrm{HOD} \cap V_\lambda$ and since (by choice of λ),

$$\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda},$$

it follows that

$$\nu(\mathcal{P}_{\bar{\delta}}(\bar{\gamma}) \cap \mathrm{HOD}) = 1,$$

and so

$$\mu(\mathcal{P}_\delta(\gamma) \cap \mathrm{HOD}) = 1.$$

This proves that μ witnesses (1) holds for γ.

We now assume δ is HOD-supercompact and hence that j satisfies the stronger version of (1.4):

$$j(\mathrm{HOD} \cap V_{\bar{\lambda}}) = \mathrm{HOD} \cap V_\lambda.$$

We prove that μ witnesses (2) holds for γ.

Let $\pi \in V_{\bar{\lambda}} \cap \mathrm{HOD}$ be a surjection,

$$\pi : \theta \to \mathcal{P}(\mathcal{P}(\bar{\gamma})) \cap \mathrm{HOD}$$

for some $\theta < |V_{\bar{\gamma}+\omega}|$. Since

$$(\mathrm{HOD})^{V_{\bar{\lambda}}} = V_{\bar{\lambda}} \cap \mathrm{HOD},$$

it follows by the definition of λ that $j(\pi) \in \mathrm{HOD} \cap V_\lambda$ is a surjection

$$j(\pi) : j(\theta) \to \mathcal{P}(\mathcal{P}(\gamma)) \cap \mathrm{HOD}.$$

Finally,

$$\{j(\xi) \mid \xi < |V_{\bar{\gamma}+\omega}|\} \in \mathrm{HOD},$$

and so it follows

$$\{j(x) \mid x \in \mathcal{P}(\mathcal{P}(\bar{\gamma})) \cap \mathrm{HOD}\} \in \mathrm{HOD}.$$

This implies that $\nu \cap \mathrm{HOD} \in \mathrm{HOD}$ and so $\mu \cap \mathrm{HOD} \in \mathrm{HOD}$. □

The proof of Theorem 193 actually yields the following variation which will be useful.

Theorem 194. *Suppose that δ is a supercompact cardinal and for all λ there exists a regular cardinal $\gamma > \lambda$ and a partition,*

$$\langle T_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD}$$

of $\{\eta < \gamma \mid \mathrm{cof}(\eta) = \omega\}$ into stationary sets. Then there is a normal fine measure μ on $\mathcal{P}_\delta(\gamma)$ such that:

(1) $\mu(\mathrm{HOD} \cap \mathcal{P}_\delta(\gamma)) = 1$;
(2) *if δ is* HOD*-supercompact then $\mu \cap \mathrm{HOD} \in \mathrm{HOD}$.*

Similarly, the proof of Theorem 193 easily adapts to prove the following variation. The point of course is that ω can be replaced by any infinite regular cardinal $\lambda < \delta$.

Theorem 195. *Suppose that δ is a supercompact cardinal, $\lambda < \delta$, λ is an infinite regular cardinal and for all $\gamma > \delta$ there exists a regular cardinal κ and there exists an* OD *partition, $\langle S_\alpha : \alpha < \gamma \rangle$, of the set, $\{\eta < \kappa \,|\, \mathrm{cof}(\eta) = \lambda\}$, into stationary subsets of κ. Then there is a normal fine measure μ on $\mathcal{P}_\delta(\gamma)$ such that*:

(1) $\mu(\mathrm{HOD} \cap \mathcal{P}_\delta(\gamma)) = 1$;
(2) *if δ is* HOD*-supercompact then $\mu \cap \mathrm{HOD} \in \mathrm{HOD}$.*

As an immediate corollary of Theorem 193 and the definition of $o^N_{\text{LONG}}(\delta) = \infty$, see Definition 135 on p. 249, we obtain the following theorem.

Theorem 196 (HOD Conjecture). *Suppose that δ is an* HOD*-supercompact cardinal. Then $o^{\mathrm{HOD}}_{\text{LONG}}(\delta) = \infty$.*

The next theorem is an immediate corollary by Theorem 138.

Theorem 197 (HOD Conjecture). *Suppose that δ is an* HOD*-supercompact cardinal. Then*:

(1) *For each $a \in [\mathrm{Ord}]^{<\delta}$, then there exists $b \in [\mathrm{Ord}]^{<\delta} \cap \mathrm{HOD}$ such that $a \subseteq b$.*
(2) *Suppose that $\lambda > \delta$, λ is a singular cardinal, and that δ is* HOD*-supercompact. Then λ is singular in* HOD *and λ^+ is correctly computed by* HOD.
(3) *Suppose $\gamma > \delta$, γ is a regular cardinal in* HOD, *and that δ is* HOD*-supercompact. Then $|\gamma| = \mathrm{cof}(\gamma)$.*

Theorem 197(1) immediately gives the following theorem.

Theorem 198 (HOD Conjecture). *Suppose that $|V_\lambda| = \lambda$, $\mathrm{cof}(\lambda) = \omega$, and there is an* HOD*-supercompact cardinal below λ. Then $\mathrm{HOD}_{V_{\lambda+1}} \vDash \mathrm{ZFC}$.*

Proof. Fix $\delta < \lambda$ such that δ is supercompact. By Theorem 197(1),

$$\mathrm{Ord}^{<\delta} \subseteq \mathrm{HOD}[\delta^{<\delta}].$$

Let $a \subseteq \lambda$ be such that $V_\lambda \subseteq L[a]$. Therefore

$$V_{\lambda+1} \subseteq \mathrm{HOD}_{\{a\}}$$

and so since $\delta^{<\delta} \subset L[a]$,

$$\mathrm{Ord}^\omega \subset \mathrm{HOD}_{\{a\}}$$

which implies

$$\mathrm{HOD}_{V_{\lambda+1}} = \mathrm{HOD}_{\{a\}}$$

since $V_\lambda \subset L[a]$, $\lambda = |V_\lambda|$, and since $\mathrm{cof}(\lambda) = \omega$.

This implies

$$\mathrm{HOD}_{V_{\lambda+1}} \vDash \mathrm{ZFC}$$

since $\mathrm{HOD}_{\{a\}} \vDash \mathrm{ZFC}$. □

Two immediate corollaries of Theorem 198 block natural attempts to find (strong) variations of the axiom that there exists a non-trivial elementary embedding

$$j : V_{\lambda+1} \to V_{\lambda+1}.$$

Theorem 199 (HOD Conjecture). *Suppose there is an* HOD*-supercompact cardinal below λ. Then there is no non-trivial elementary embedding,*

$$j : \mathrm{HOD}_{V_{\lambda+1}} \cap V_{\lambda+2} \to \mathrm{HOD}_{V_{\lambda+1}} \cap V_{\lambda+2}.$$

Theorem 200 (HOD Conjecture). *Suppose there is an* HOD*-supercompact cardinal below λ. Then there is no non-trivial elementary embedding,*

$$j : V_{\lambda+1} \to V_{\lambda+1},$$

such that for all Σ_2-formulas $\phi(x)$, for all $a \in V_{\lambda+1}$,

$$V \vDash \phi[a]$$

if and only if $V \vDash \phi[j(a)]$.

Note that if it is consistent for there to exist an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ then it is consistent for there to exist a non-trivial elementary embedding,

$$j : V_{\lambda+1} \to V_{\lambda+1},$$

such that for all Σ_2-formulas $\phi(x)$, for all $a \in V_{\lambda+1}$,

$$V \vDash \phi[a]$$

if and only if $V \vDash \phi[j(a)]$, for if $G \subseteq \mathrm{Coll}(\lambda^+, \lambda^+)$ is $L(V_{\lambda+1})$-generic then this latter axiom holds in $L(V_{\lambda+1})[G]$ and

$$L(V_{\lambda+1})[G] \vDash \mathrm{ZFC}.$$

Further the cardinals of $(\mathrm{HOD})^{L(V_{\lambda+1})[G]}$ coincide exactly with the cardinals of $L(V_{\lambda+1})[G]$ above λ^+. This example shows that in Theorem 200, the assumption that there is an HOD-supercompact cardinal below λ is necessary.

A natural conjecture is that for all λ, if there is a supercompact cardinal below λ then there is no non-trivial elementary embedding

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

such that for all formulas $\phi(x)$, for all $a \in V_{\lambda+1}$,

$$V_{\lambda+2} \models \phi[a]$$

if and only if $V_{\lambda+2} \models \phi[j(a)]$.

By Theorem 145, the next theorem is an immediate corollary of Theorem 201.

Theorem 201 (HOD Conjecture). *Suppose that δ is an* HOD-*supercompact cardinal and that*

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with critical point $\kappa \geq \delta$. Then $j \in \mathrm{HOD}$.

Another version of Theorem 201 is the following.

Theorem 202. *Suppose that δ is a supercompact cardinal and for all λ there exists a regular cardinal $\gamma > \lambda$ and a partition,*

$$\langle T_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD}$$

of $\{\eta < \gamma \mid \mathrm{cof}(\gamma) = \omega\}$ into stationary sets. Suppose that $\gamma \in \mathrm{Ord}$ and that

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with critical point $\kappa \geq \delta$. Then $j \in \mathrm{HOD}$.

Remark 203. If δ is an extendible cardinal then the conclusions of the three theorems, Theorems 196, 197 and 201, follow just assuming there is just *one* regular cardinal above δ which is not ω-strongly measurable HOD, see Theorem 212.

The following lemma is a definable variation of the results of Solovay, [15].

Lemma 204. *Suppose κ is an infinite cardinal such that*

$$(\kappa^+)^{\mathrm{HOD}} = \kappa^+$$

and that $S \subseteq \kappa^+$ is a stationary set with $S \in \mathrm{HOD}$. Then there is a partition

$$\langle S_\alpha : \alpha < \kappa^+ \rangle \in \mathrm{HOD}$$

of S into κ^+ many stationary sets.

Proof. Let $\mathcal{I}_S$ be the nonstationary ideal at κ^+ restricted to S, this is the ideal generated by the sets, $(S \backslash C) \cup T$, where C is closed and unbounded in κ^+, $T \subseteq \kappa^+$, and $T \cap S = \emptyset$. Thus $\mathcal{I}_S \cap \mathrm{HOD} \in \mathrm{HOD}$.

Suppose toward a contradiction that there is no partition,

$$\langle S_\alpha : \alpha < \kappa^+ \rangle \in \mathrm{HOD}$$

of S into κ^+ many stationary sets (and we make no assumption about the relationship of $(2^\gamma)^{\mathrm{HOD}}$ and κ^+ for any $\gamma < \kappa^+$).

Then in HOD, $\mathcal{I}_S \cap \mathrm{HOD}$ is a κ^+-saturated, κ^+-complete, uniform ideal on κ^+. Such an ideal can never exist on a successor cardinal and so this is a contradiction. $\square$

An immediate corollary of Theorem 201 is that if δ is HOD-supercompact and if the HOD Conjecture holds, then there is no elementary embedding,

$$j : \mathrm{HOD} \to \mathrm{HOD}$$

with $\mathrm{CRT}(j) \geq \delta$. We can strengthen this a bit and in particular show (from the hypotheses above) that there exists an ordinal λ such that there is no non-trivial elementary embedding

$$j : \mathrm{HOD} \to \mathrm{HOD}$$

with the property that $j(\lambda) = \lambda$ *without* requiring the assumption that $\mathrm{CRT}(j) \geq \delta$. This shows that there is no non-trivial sequence $\langle j_k : k < \omega \rangle$ of elementary embeddings,

$$j_k : \mathrm{HOD} \to \mathrm{HOD},$$

with wellfounded limit and the details also show there can be no non-trivial elementary embedding

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T)$$

where T is the Σ_2-theory of V in ordinal parameters (see Theorem 206).

All that is involved is the following improvement of Theorem 201. This theorem gives an example where the special nature of HOD allows one to infer consequences for HOD of the assumption that there exists a cardinal δ such that

$$o^{\mathrm{HOD}}_{\mathrm{LONG}}(\delta) = \infty$$

which do not obviously follow for N just assuming

$$o^{N}_{\mathrm{LONG}}(\delta) = \infty;$$

compare Theorem 205 with Lemma 208.

Theorem 205 (HOD Conjecture). *Suppose that there is an* HOD-*supercompact cardinal. Then there is an ordinal λ such that for all $\gamma > \lambda$, if*

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with $j(\lambda) = \lambda$ then $j \in \mathrm{HOD}$.

Proof. Let δ be HOD-supercompact and let $\lambda_0 = \delta^{+\omega}$ be the ωth cardinal above δ. Clearly $(\mathrm{cof}(\lambda_0))^{\mathrm{HOD}} = \omega$. Further by Theorem 197,

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+.$$

Therefore if $\eta < \lambda_0^+$ then $(\mathrm{cof}(\eta))^{\mathrm{HOD}} < \lambda_0$. Let κ_0 be least such that

$$\{\eta < \lambda_0^+ \mid \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\}$$

is stationary in λ_0^+.

Define $\lambda = \lambda_0 + \kappa_0$. We show that λ is as required. Suppose $\gamma > \lambda$ and

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding such that $j(\lambda) = \lambda$. By Theorem 201, if $j|\delta$ is the identity then $j \in \mathrm{HOD}$. Therefore we have only to prove that $j|\delta$ is the identity. Since $\kappa_0 < \lambda_0$ and since $j(\lambda) = \lambda$, $j(\lambda_0) = \lambda_0$ and $j(\kappa_0) = \kappa_0$.

Clearly j induces canonically an elementary embedding

$$j^* : (H(\lambda_0^{++}))^{\mathrm{HOD}} \to (H(\lambda_0^{++}))^{\mathrm{HOD}}$$

with the property that $j|\lambda_0 = j^*|\lambda_0$.

Let

$$S = \{\eta < \lambda_0^+ \,|\, \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\}.$$

Thus since S is stationary in λ_0^+ and since

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+,$$

by Lemma 204, there is a partition

$$\langle S_\alpha : \alpha < \lambda_0^+\rangle \in \mathrm{HOD}$$

of S into stationary sets. Let

$$\langle T_\beta : \beta < \lambda_0^+\rangle = j^*(\langle S_\alpha : \alpha < \lambda_0^+\rangle).$$

Note that if $\eta \in S$ and if η is closed under j^* then $j^*(\eta) = \eta$. This is because $(\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0$ and because $j^*(\kappa_0) = \kappa_0$.

Therefore for all $\beta < \lambda_0^+$, $T_\beta \cap S$ is stationary in λ_0^+ if and only if $\beta = j^*(\alpha)$ for some $\alpha < \lambda_0^+$. This implies that

$$\{j^*(\alpha) \,|\, \alpha < \lambda_0^+\} \in \mathrm{HOD}$$

since $\{\beta < \lambda_0^+ \,|\, T_\beta \cap S \text{ is stationary in } \lambda_0^+\} \in \mathrm{HOD}$. But by the elementarity of j^* and since $j^*(S) = S$, for all $\beta < \lambda_0^+$,

$$\mathrm{HOD} \vDash \text{“}T_\beta \cap S \text{ is stationary in } \lambda_0^+\text{”},$$

which implies (since $\{j^*(\alpha) \,|\, \alpha < \lambda_0^+\} \in \mathrm{HOD}$) that $j^*|\lambda_0^+$ is the identity. Thus $\mathrm{CRT}(j) > \delta$ and so by Theorem 201, $j \in \mathrm{HOD}$. □

Theorem 206 (HOD Conjecture). *Suppose that there is an* HOD*-supercompact cardinal. Let T be the Σ_2-theory of V with ordinal parameters. Then there is no non-trivial elementary embedding,*

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T).$$

Proof. By Theorem 205, there exists $\lambda \in \mathrm{Ord}$ such that for all $\gamma > \lambda$, if

$$k : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{k(\gamma)+1}$$

is an elementary embedding with $k(\lambda) = \lambda$, then $k \in \mathrm{HOD}$. Let λ_0 be the least such λ. Clearly λ_0 is definable in V and so λ_0 is definable in (HOD, T).

Suppose toward a contradiction that

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T)$$

is a non-trivial elementary embedding. Therefore $j(\lambda_0) = \lambda_0$ and so for all $\gamma > \lambda_0$,

$$j|\ \mathrm{HOD} \cap V_{\gamma+1} \in \mathrm{HOD},$$

which is a contradiction. $\square$

Remark 207. Let T be the Σ_2-theory of V with ordinal parameters and suppose the HOD Conjecture holds. It seems plausible that there might be a direct proof that there is no non-trivial elementary embedding,

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T).$$

The proof of Theorem 205 would suffice for this if one assumes in addition that for some $\gamma > \mathrm{CRT}(j)$, $j(\gamma) = \gamma$.

We conjecture that as a theorem of ZFC there is no non-trivial elementary embedding

$$j : \mathrm{HOD} \to \mathrm{HOD}.$$

A weaker conjecture is simply that HOD Conjecture implies that there is no non-trivial elementary embedding

$$j : \mathrm{HOD} \to \mathrm{HOD}.$$

The following lemma suggests these questions are more subtle than one might first expect and so raises an interesting question:

> *Suppose that* $\mathbb{M}$ *is a suitable extender model. Can there exist a non-trivial elementary embedding,*
>
> $$j : \mathbb{M} \to \mathbb{M}?$$

Again we conjecture the answer is no and here is a potential example where suitable extender models are distinguished from the more general notion of a transitive class $N \vDash \mathrm{ZFC}$ such that $o^N_{\mathrm{LONG}}(\delta) = \infty$ for some δ. This is by the following lemma which is an immediate corollary of the proof of Lemma 150.

Lemma 208. *Suppose* δ *is supercompact cardinal. Then there exists an inner model* N *of* ZFC *such that*

(1) $o^N_{\mathrm{LONG}}(\delta) = \infty$ *and* $N^\omega \subseteq N$,
(2) *for each* λ*, there is a non-trivial elementary embedding,*

$$j : N \to N,$$

such that $j(\lambda) = \lambda$.

We now continue the analysis of HOD assuming the HOD Conjecture.

Theorem 209 (HOD Conjecture). *Suppose that* δ *is an* HOD*-supercompact cardinal. Then there exist* $\mathbb{P} \in \mathrm{HOD} \cap V_{\delta+\omega}$ *and an* HOD*-generic filter* $G \subseteq \mathbb{P}$

such that

$$(\mathrm{HOD}[G])^{<\delta} \subseteq \mathrm{HOD}[G].$$

Proof. Fix a set $A \subseteq \delta$ such that $V_\delta \subseteq L[A]$.

Let $\mathbb{B}$ be the boolean subalgebra of $\mathcal{P}(\mathcal{P}(\delta))$ given by

$$\{X \subseteq \mathcal{P}(\delta) \mid X \text{ is OD}\}.$$

Then there is a boolean algebra $\mathbb{B}_0 \in \mathrm{HOD}$ and an isomorphism,

$$\pi : \mathbb{B}_0 \to \mathbb{B}$$

such that π is OD.

Let $G \subseteq \mathbb{B}_0$ be the set of all $b \in \mathbb{B}_0$ such that $A \in \pi(b)$. By Vopenka's Theorem, G is an HOD-generic filter and further

$$\mathrm{HOD}[G] = \mathrm{HOD}_{\{A\}}$$

where $\mathrm{HOD}_{\{A\}}$ is class of all sets which are hereditarily ordinal definable from $\{A\}$.

Thus $V_\delta \subseteq \mathrm{HOD}[G]$ and by Theorem 197,

$$(\mathrm{HOD}[G])^{<\delta} \subseteq \mathrm{HOD}[G]. \qquad \square$$

Suppose δ is HOD-supercompact and the HOD Conjecture holds. Then HOD has closure properties which are very similar to those which $\mathbb{M}$ has, where $\mathbb{M}$ is a suitable extender model such that $\delta_{\mathbb{M}} = \delta$. The next theorem explains why the covering properties hold. We require a definition.

Definition 210. HOD is a suitable extender model at δ if the following hold.

(1) $o^{\mathrm{HOD}}_{\mathrm{LONG}}(\delta) = \infty$.
(2) There exists a sequence $\langle E_\alpha : \alpha < \delta \rangle$ of extenders in V_δ witnessing that δ is a Woodin cardinal in V such that

$$\langle E_\alpha \cap \mathrm{HOD} : \alpha < \delta \rangle \in \mathrm{HOD}$$

and such that for all $\alpha < \delta$, $\rho(E_\alpha) = \mathrm{LTH}(E_\alpha)$.

Remark 211. Suppose that HOD is a suitable extender model at δ. Let $\mathcal{E}$ be the set of all extenders $E \in V_\delta$ such that

(1) $\mathrm{SPT}(E) = \mathrm{CRT}(E)$,
(2) $\mathrm{LTH}(E) = \rho(E)$ and $\mathrm{LTH}(E)$ is strongly inaccessible,
(3) $E \cap \mathrm{HOD} \in \mathrm{HOD}$.

Since HOD is a suitable extender model at δ, $\mathcal{E}$ witnesses in V that δ is a Woodin cardinal and

$$\{E \cap \mathrm{HOD} \mid E \in \mathcal{E}\} \in \mathrm{HOD}.$$

Let $\mathcal{E}^*$ be the set of all $E \in \mathcal{E}$ such that

$$j_E(\mathcal{E} \cap V_{\mathrm{CRT}(E)}) \cap V_{\mathrm{LTH}(E)} = \mathcal{E} \cap V_{\mathrm{LTH}(E)}.$$

Then $\mathcal{E}^*$ witnesses in V that δ is a Woodin cardinal,

$$\{E \cap \mathrm{HOD} \mid E \in \mathcal{E}^*\} \in \mathrm{HOD},$$

and for each $E \in \mathcal{E}^*$,

$$j_E(\mathcal{E}^* \cap V_{\mathrm{CRT}(E)}) \cap V_{\mathrm{LTH}(E)} = \mathcal{E}^* \cap V_{\mathrm{LTH}(E)}.$$

This gives that at δ a very good approximation of (2a) in Definition 161 holds, and this approximation suffices for all the applications (such as Theorem 175).

Theorem 212. *Suppose that δ is an extendible cardinal. Then the following are equivalent.*

(1) HOD *is a suitable extender model at δ.*
(2) *There is a suitable extender model $\mathbb{M} \subseteq \mathrm{HOD}$ such that $\delta_{\mathbb{M}} = \delta$.*
(3) *There exists a regular cardinal $\kappa \geq \delta$ such that κ is not measurable in* HOD.
(4) *The* HOD *Conjecture.*
(5) *There exists a regular cardinal $\kappa \geq \delta$ and a partition*

$$\langle S_\alpha : \alpha < \delta \rangle \in \mathrm{HOD}$$

of $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}$ into pairwise disjoint stationary sets.

Proof. By Remark 211 it follows easily that (1) implies (2), (2) implies (3) by Theorem 138, and (4) trivially implies (5). Therefore we only have to show that (3) implies (4) and that (5) implies (1).

Assume (3). Let I be the class of all regular cardinals γ such that there exists $\eta > \gamma$ such that

(1.1) $V_\eta \vDash \mathrm{ZFC}$,
(1.2) $V_\eta \vDash$ "γ is not ω-strongly measurable in HOD".

Note that if $\gamma \in I$ then γ is not ω-strongly measurable in HOD. We verify this. Assume toward a contradiction that γ is ω-strongly measurable in HOD and let η witness that $\gamma \in I$. Since γ is ω-strongly measurable in HOD there must exist $\lambda < \gamma$ such that

(2.1) $(2^\lambda)^{\mathrm{HOD}} < \gamma$,
(2.2) There is no partition $\langle S_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD}$ of $\{\xi < \gamma \mid \mathrm{cof}(\xi) = \omega\}$ into stationary sets.

But $(\mathrm{HOD})^{V_\eta} \subset \mathrm{HOD}$ and so 2^λ as computed in $(\mathrm{HOD})^{V_\eta}$ is necessarily smaller than γ and this contradicts (1.2).

Note that κ is not ω-strongly measurable in V_η for all sufficiently large η (since κ is not measurable in HOD) and since δ is extendible there is a proper class of strongly inaccessible cardinals. Therefore $\kappa \in I$. Since δ is extendible and since $\kappa > \delta$, $I \cap \delta$ is cofinal in δ. Finally again since δ is extendible it follows that I is a proper class. This proves (4).

Finally we assume (5) and prove (1). By Lemma 188, δ is HOD-supercompact, and by an easy reflection argument (since δ is extendible) δ is a limit of HOD-supercompact cardinals. We first show that for every cardinal κ, if κ is HOD-supercompact then

$$o^{\mathrm{HOD}}_{\text{LONG}}(\kappa) = \infty$$

and for this we will use Theorem 195.

Let $\kappa \geq \delta$ be a regular cardinal such that there exists a sequence $\langle S_\alpha : \alpha < \delta\rangle \in$ HOD of pairwise disjoint stationary subsets of $\{\eta < \kappa \,|\, \mathrm{cof}(\eta) = \omega\}$, and let $\theta > \kappa$ be large enough such that

$$\mathrm{HOD} \cap \mathcal{P}(\kappa) = (\mathrm{HOD})^{V_\theta} \cap \mathcal{P}(\kappa).$$

Let

$$j : V_{\theta+1} \to V_{j(\theta)+1}$$

be an elementary embedding with critical point δ such that $j(\delta) > \kappa$. Since $\kappa \geq \delta$ and since

$$(\mathrm{HOD})^{V_{j(\theta)}} \subseteq \mathrm{HOD},$$

it follows by the elementarity of j that for all $\lambda < j(\delta)$ there exists a regular cardinal $\gamma > \lambda$ and a partition,

$$\langle T_\alpha : \alpha < \lambda\rangle \in \mathrm{HOD}$$

of $\{\eta < \gamma \,|\, \mathrm{cof}(\gamma) = \omega\}$ into stationary sets. Since j can be chosen with $j(\delta)$ arbitrarily large, it follows that for all λ there exists a regular cardinal $\gamma > \lambda$ and a partition,

$$\langle T_\alpha : \alpha < \lambda\rangle \in \mathrm{HOD}$$

of $\{\eta < \gamma \,|\, \mathrm{cof}(\gamma) = \omega\}$ into stationary sets. Thus by Theorem 195, for every cardinal κ, if κ is HOD-supercompact then

$$o^{\mathrm{HOD}}_{\text{LONG}}(\kappa) = \infty.$$

Thus

$$o^{\mathrm{HOD}}_{\text{LONG}}(\delta) = \infty.$$

Thus to show that (1) holds we only have to show that condition (2) of Definition 210 holds at δ.

Let δ_0 be the least HOD-supercompact cardinal. Thus $\delta_0 < \delta$ and by Theorem 145, for all $\gamma > \delta_0$, if

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with $\mathrm{CRT}(j) > \delta_0$ then $j \in \mathrm{HOD}$.

From this and since δ is extendible it follows for each $A \subset \delta$ with $A \in \mathrm{HOD}$ there exists $\delta_0 < \bar{\delta} < \delta$ and an elementary embedding

$$j : V_{\bar{\delta}+2} \to V_{\delta+2}$$

such that $\mathrm{CRT}(j) = \bar{\delta}$ and such that

(3.1) $j(A \cap \bar{\delta}) = A$,
(3.2) $j(\mathrm{HOD} \cap V_{\bar{\delta}+1}) = \mathrm{HOD} \cap V_{\delta+1}$,
(3.3) $j|(\mathrm{HOD} \cap V_{\bar{\delta}+1}) \in \mathrm{HOD}$.

It follows easily that there exists a sequence $\langle E_\alpha : \alpha < \delta \rangle$ of extenders in V_δ witnessing that δ is a Woodin cardinal in V such that

$$\langle E_\alpha \cap \mathrm{HOD} : \alpha < \delta \rangle \in \mathrm{HOD}$$

and such that for all $\alpha < \delta$, $\rho(E_\alpha) = \mathrm{LTH}(E_\alpha)$. This finishes the proof that (5) implies (1). $\square$

Theorem 213. *Suppose that δ is an extendible cardinal. Then the following are equivalent.*

(1) *There exists a regular cardinal $\kappa \geq \delta$ and a partition*

$$\langle S_\alpha : \alpha < \delta \rangle \in \mathrm{HOD}$$

of $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}$ into pairwise disjoint stationary sets.

(2) *There exist regular cardinals*

$$\gamma < \delta \leq \kappa$$

and a partition

$$\langle S_\alpha : \alpha < \delta \rangle \in \mathrm{HOD}$$

of $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \gamma\}$ into pairwise disjoint stationary sets.

Proof. Trivially, (1) implies (2). Assume (2) holds. Therefore, since δ is an extendible cardinal, there exists an (infinite) regular cardinal $\lambda < \delta$ such that for all $\eta \in \mathrm{Ord}$ there exist regular cardinals $\kappa > \gamma > \eta$ and a partition,

$$\langle S_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$$

of $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \lambda\}$ into pairwise disjoint stationary sets.

By Lemma 188, δ is HOD-supercompact and so by Theorem 195,

$$o^{\mathrm{HOD}}_{\mathrm{LONG}}(\delta) = \infty.$$

Therefore by Theorem 197, for all singular cardinals $\gamma > \delta$,

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}.$$

This implies that (1) holds. $\square$

The next theorem is also a corollary of Theorem 212. The proof also requires Lemma 163.

Theorem 214. *Suppose there is a proper class of extendible cardinals. Then*

$$V \vDash \text{“}\textit{The}\ \mathrm{HOD}\ \textit{Conjecture.}\text{”},$$

if and only if for all complete Boolean algebras, $\mathbb{B}$,

$$V^{\mathbb{B}} \models \text{“\textit{The} HOD \textit{Conjecture}.”}.$$

Proof. We first prove that if the HOD Conjecture holds in V then for all complete Boolean algebras, $\mathbb{B}$,

$$V^{\mathbb{B}} \models \text{“The HOD Conjecture.”}.$$

Here we use Lemma 163. Fix $\mathbb{B}$ and suppose that $G \subseteq \mathbb{B}$ is V-generic.

Fix a Σ_2-formula, $\phi(x_0, x_1)$ and fix $b \subseteq \mathrm{Ord}$ such that

$$V = \{a \mid V[G] \models \phi[a, b]\}.$$

These exist by Lemma 163.

Let κ be an extendible cardinal in V such that $\mathbb{B} \in V_\kappa$ and such that $b \in V_\kappa[G]$. Fix a cardinal $\gamma > \kappa$ such that $\mathrm{cof}(\gamma) = \omega$ in V. By Theorem 212 and since $\mathbb{B} \in V_\kappa$,

$$(\gamma^+)^{V[G]} = (\gamma^+)^V = (\gamma^+)^{\mathrm{HOD}^V}.$$

The key point is that by Vopenka's Theorem and since κ is strongly inaccessible, $\mathrm{HOD}^{V[G]}_{\{b\}}$ is a κ-cc generic extension of $\mathrm{HOD}^{V[G]}$. But

$$V = \{a \mid V[G] \models \phi[a, b]\}.$$

and so it follows that $\mathrm{HOD}^V \subseteq \mathrm{HOD}^{V[G]}_{\{b\}}$. This implies that

$$(\gamma^+)^{V[G]} = (\gamma^+)^{\mathrm{HOD}^{V[G]}}$$

and so by Theorem 212, the HOD Conjecture holds at κ in $V[G]$.

Now suppose that $\mathbb{B}$ is a complete Boolean algebra and that

$$V^{\mathbb{B}} \models \text{“The HOD Conjecture.”}.$$

Let $\gamma = |\mathbb{B}|$. Then by the generic persistence of the HOD Conjecture (which we have just proved) and since in $V^{\mathbb{B}}$ there exists a proper class of extendible cardinals,

$$V^{\mathrm{RO}(\mathbb{B} \times \mathrm{Coll}(\omega,\gamma))} \models \text{“The HOD Conjecture.”}.$$

But $\mathrm{RO}(\mathbb{B} \times \mathrm{Coll}(\omega, \gamma)) \cong \mathrm{RO}(\mathrm{Coll}(\omega, \gamma))$ and so

$$V^{\mathrm{RO}(\mathrm{Coll}(\omega,\gamma))} \models \text{“The HOD Conjecture.”}.$$

By the equivalence of (1) and (5) in Theorem 212, this implies that

$$V \models \text{“The HOD Conjecture.”},$$

since $\mathrm{Coll}(\omega, \gamma)$ is OD and homogeneous. $\square$

Should the HOD Conjecture be consistently false, an obvious question is what large cardinal hypothesis (consistent with the Axiom of Choice) might suffice to establish this. The following theorem, which is immediate corollary of Theorem 200 and Theorem 212, does indicate a sufficient hypothesis but at present there is no basis from which one can argue that this is a large cardinal hypothesis. Whether this approach can lead to a genuine large cardinal hypothesis which is also sufficient is far from clear.

Theorem 215 (ZFC). *Suppose that* $V_\lambda \prec_{\Sigma_4} V$ *and*

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

is an elementary embedding with $\kappa = \mathrm{CRT}(j)$ *such that for all* Σ_2*-formulas,* $\phi(x)$*, for all* $a \in V_{\lambda+1}$*,*

$$V \vDash \phi[a] \text{ if and only if } V \vDash \phi[j(a)].$$

Then the HOD *Conjecture fails in* V*.*

One can use the hypothesis of Theorem 215 (which implies the existence of a proper class of extendible cardinals) to obtain the strong failure of the HOD Conjecture, specifically one can prove that the hypothesis of Theorem 215 is equiconsistent with the statement of the hypothesis together with the assertion that *every* regular (uncountable) cardinal is ω-strongly measurable in HOD. The ultimate relevance of this theorem is entirely unclear and so we simply state it and leave the proof to the dedicated reader.

Theorem 216 (ZFC). *Suppose that* $V_\lambda \prec_{\Sigma_4} V$ *and*

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

is an elementary embedding with $\kappa = \mathrm{CRT}(j)$ *such that for all* Σ_2*-formulas,* $\phi(x)$*, for all* $a \in V_{\lambda+1}$*,* $V \vDash \phi[a]$ *if and only if* $V \vDash \phi[j(a)]$*. Then there is a homogeneous partial order* $\mathbb{P}$ *such that* $\mathbb{P}$ *is* OD *and such that if* $G \subseteq \mathbb{P}$ *is* V*-generic then in* $V[G]$ *the following hold.*

(1) $VV[G]_\lambda \prec_{\Sigma_4} V[G]$.
(2) j *lifts to an elementary embedding,*

$$j_G : V[G]_{\lambda+1} \to V[G]_{\lambda+1},$$

such that for all Σ_2*-formulas,* $\phi(x)$*, for all* $a \in V[G]_{\lambda+1}$*,* $V[G] \vDash \phi[a]$ *if and only if* $V[G] \vDash \phi[j(a)]$*.*
(3) *Every regular cardinal* $\gamma > \omega$ *is measurable in* HOD*.*
(4) *There is no set* $S \subseteq \omega_1$ *such that* $S \in \mathrm{HOD}$ *and such that* S *is both stationary and co-stationary.*
(5) *There is an ordinal* $\eta < \omega_2$ *such that for each regular cardinal* $\gamma \geq \omega_2$*, there is no partition*

$$\langle S_\alpha : \alpha < \eta \rangle \in \mathrm{HOD}$$

of $\{\xi < \gamma \mid \mathrm{cof}(\xi) = \omega\}$ *into pairwise disjoint stationary sets.*

Assuming the existence of a proper class of extendible cardinals, the HOD Conjecture is connected to the problem of whether the Ω Conjecture can be refuted from some large cardinal hypothesis.

Lemma 217. *Suppose that* $S \subseteq \mathrm{Ord}$ *and that*

(i) $\mathrm{HOD}_{\{S\}}(\mathbb{R}) \vDash \mathrm{AD}^+$,
(ii) *every set*

$$A \in \mathcal{P}(\mathbb{R}) \cap \mathrm{HOD}_{\{S\}}(\mathbb{R})$$

which is Δ^2_1 *in* $\mathrm{HOD}_{\{S\}}(\mathbb{R})$, *is universally Baire.*

Then $\mathrm{HOD}_{\{S\}} \vDash$ "*The* Ω *Conjecture*".

Proof. We sketch the proof which requires some basic facts from the theory of AD^+ as summarized in the Σ^2_1-Basis Theorem, Theorem 25 on p. 124.

Clearly we can suppose that there exist a proper class of Woodin cardinals in $\mathrm{HOD}_{\{S\}}$ (for otherwise the Ω Conjecture is vacuously true in $\mathrm{HOD}_{\{S\}}$) and so for each $A \in \mathcal{P}(\mathbb{R}) \cap \mathrm{HOD}_{\{S\}}$, $(A, \mathbb{R})^\# \in \mathrm{HOD}_{\{S\}}(\mathbb{R})$.

Fix a sentence, ϕ, such that $\emptyset \not\vdash_\Omega \phi$ in $\mathrm{HOD}_{\{S\}}$. We must show that $\emptyset \not\vDash_\Omega \phi$ in $\mathrm{HOD}_{\{S\}}$; i.e. that for some partial order $\mathbb{P} \in \mathrm{HOD}_{\{S\}}$ and for some (nonzero) $\alpha \in \mathrm{Ord}$,

$$V^{\mathbb{P}}_\alpha \cap \mathrm{HOD}_{\{S\}} \vDash (\neg\phi).$$

We claim that for all $A \in \mathcal{P}(\mathbb{R}) \cap \mathrm{HOD}_{\{S\}}(\mathbb{R})$, there exists a countable transitive set M such that

(1.1) $M \vDash \mathrm{ZFC}$,
(1.2) M is A-closed,
(1.3) $M \not\vDash$ "$\emptyset \vDash_\Omega \phi$".

To verify this claim suppose that A is a counterexample. Without loss of generality we can suppose that A is Δ^2_1 in $\mathrm{HOD}_{\{S\}}(\mathbb{R})$. This is by the Σ^2_1-Basis Theorem, Theorem 25, which shows that assuming AD^+ for any sentence ψ if there exists a set $B \subseteq \mathbb{R}$ such that

$$(H(\omega_1), B) \vDash \psi$$

then there exists such a set B which is Δ^2_1.

Since A is Δ^2_1 in $\mathrm{HOD}_{\{S\}}(\mathbb{R})$ and since $(A, \mathbb{R})^\# \in \mathrm{HOD}_{\{S\}}(\mathbb{R})$, $(A, \mathbb{R})^\#$ is Δ^2_1 in $\mathrm{HOD}_{\{S\}}(\mathbb{R})$. Since

$$\mathrm{HOD}_{\{S\}}(\mathbb{R}) \vDash \mathrm{AD}^+,$$

by Theorem 25 again, the set $(A, \mathbb{R})^\#$ admits a scale each norm of which is Δ^2_1 in $\mathrm{HOD}_{\{S\}}(\mathbb{R})$. By hypothesis every set which is Δ^2_1 in $\mathrm{HOD}_{\{S\}}(\mathbb{R})$, is universally Baire.

Therefore the following hold.

(2.1) $\mathrm{HOD}_{\{S\}}$ is A-closed;
(2.2) $\mathrm{HOD}_{\{S\}}$ is $(A, \mathbb{R})^\#$-closed;

(2.3) $(A, \mathbb{R})^{\#} \cap \mathrm{HOD}_{\{S\}} = (A \cap \mathrm{HOD}_{\{S\}}, \mathbb{R} \cap \mathrm{HOD}_{\{S\}})^{\#}$;
(2.4) $A \cap \mathrm{HOD}_{\{S\}}$ and $(A, \mathbb{R})^{\#} \cap \mathrm{HOD}_{\{S\}}$ are each universally Baire in $\mathrm{HOD}_{\{S\}}$.

Since there is a proper class of Woodin cardinals in $\mathrm{HOD}_{\{S\}}$ it follows that $A \cap \mathrm{HOD}_{\{S\}}$ witnesses that $\emptyset \vdash_\Omega \phi$ in $\mathrm{HOD}_{\{S\}}$ which contradicts the choice of ϕ.

Fix $T \subseteq \mathrm{Ord}$ such that $L[T] = \mathrm{HOD}_{\{S\}}$ (so T is in general a class) and work in $L(T, \mathbb{R})$ (this is to simplify notation). So below (until we return to V), $\mathrm{HOD}_{\{T\}}$ and $\mathrm{HOD}_{\{T,x\}}$ are as computed in $L(T, \mathbb{R})$.

Since $V = L(T, \mathbb{R})$, for all $x \in \mathbb{R}$,

$$\mathrm{HOD}_{\{T,x\}} = \mathrm{HOD}_{\{T\}}[x].$$

Therefore by the remarks above, and taking A to be the Σ_2-theory of

$$(L(T, \mathbb{R}), T)$$

with real parameters, there exists a countable transitive set, M, such that

(3.1) $M \vDash \mathrm{ZFC}$,
(3.2) M is A-closed,
(3.3) for all $\alpha \in M \cap \mathrm{Ord}$, $M \cap V_\alpha = \mathrm{HOD}_{\{T\}}(M_\alpha) \cap V_\alpha$,
(3.4) $M \nvDash$ "$\emptyset \vDash_\Omega \phi$".

By the choice of A and since every set has the property of Baire, (3.2) implies (3.3).

Let $\mathbb{P} \in M$ be a partial order, let $G \subseteq \mathbb{P}$ be M-generic, and let $\alpha \in M \cap \mathrm{Ord}$ be such that $M[G] \cap V_\alpha \vDash \neg\phi$. By (3.4), these exist.

Fix $\beta > \alpha$ such that $\mathbb{P} \in M \cap V_\beta$ and let $a \subseteq \mathrm{Ord}$ be a set such that $a \in M$ and $M_\beta \subseteq L[a]$.

By (3.3), $\mathrm{HOD}_{\{T\}}[a] \cap V_\beta = M \cap V_\beta$ and so G is $\mathrm{HOD}_{\{T\}}[a]$-generic for $\mathbb{P}$ and

$$\mathrm{HOD}_{\{T\}}[a][G] \cap V_\alpha \vDash \neg\phi.$$

However $\mathrm{HOD}_{\{T\}}[a]$ is a set generic extension of $\mathrm{HOD}_{\{T\}}$ and so

$$\mathrm{HOD}_{\{T\}} \nvDash \text{“}\emptyset \vDash_\Omega \phi\text{”}.$$

Finally (returning to V), $(\mathrm{HOD}_{\{T\}})^{L(T,\mathbb{R})} = L[T]$ since $L[T] = (\mathrm{HOD}_{\{S\}})^V$. □

As an immediate corollary of Lemma 217 we obtain the following theorem.

Theorem 218. *Suppose that there is a proper class of Woodin cardinals and that for every set $A \subseteq \mathbb{R}$, if A is* OD *then A is universally Baire. Then*

$$\mathrm{HOD} \vDash \text{“}\Omega \textit{ Conjecture.}\text{”}.$$

Thus if some large cardinal hypothesis such as: for a proper class of cardinals λ there exists a proper elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1});$$

refutes the Ω Conjecture then assuming sufficient cardinal hypotheses hold in V either:

(1) There is an OD set $A \subseteq \mathbb{R}$ such that A is not universally Baire; or
(2) there exists α such that every regular cardinal $\delta > \alpha$ is ω-strongly measurable in HOD.

Recall that the assertion that an uncountable regular cardinal κ is ω-strongly measurable in HOD is a very strong assertion (much stronger that the assertion that κ is measurable in HOD) — for example it is *not known* if there can exist more than three uncountable regular cardinals which are ω-strong measurable in HOD.

This would be a rather pathological dichotomy from the viewpoint of inner model theory. Further, if the conjecture that there is a proper class of uncountable regular cardinals which are not ω-strongly measurable in HOD is provable then (1) must hold. In either case this is the kind of phenomenon which is evidence for the claim that a large cardinal hypothesis which refutes the Ω Conjecture cannot have a reasonable inner model theory.

Finally we note the following dichotomy theorem. Here again, that this is a non-trivial dichotomy (i.e. that one cannot rule out one of the possibilities) seems unlikely and this is the motivation for the conjecture that (as theorem of ZFC) there is an uncountable regular cardinal which is not ω-strongly measurable in HOD.

Theorem 219. *Suppose that κ is a huge cardinal. Then for each set $A \subseteq \kappa$ one of the following holds.*

(1) *There exists $\delta < \kappa$ such that if γ is a regular cardinal with $\delta < \gamma < \kappa$, then γ is ω-strongly measurable in $(\mathrm{HOD}_{\{A\}})^{V_\kappa}$; or*
(2) *There exists $\delta < \kappa$ such that if γ is singular cardinal and $\delta < \gamma < \kappa$, then γ is singular in $(\mathrm{HOD}_{\{A\}})^{V_\kappa}$ and γ^+ is correctly computed by $(\mathrm{HOD}_{\{A\}})^{V_\kappa}$.*

Proof. Since κ is a huge cardinal, by Lemma 159 on page 264 there exists $\delta < \kappa$ such that

$$(V_\kappa, N) \models \text{“}\delta \text{ is an } N\text{-extendible cardinal”}$$

where $N = (\mathrm{HOD}_{\{A\}})^{V_\kappa}$, and so

$$(V_\kappa, A) \models \text{“}\delta \text{ is } \mathrm{HOD}_{\{A\}}\text{-supercompact”}.$$

If (1) fails then the proof of Theorem 193 adapts to show that

$$(V_\kappa, A) \models \text{“}o_{\mathrm{LONG}}^{\mathrm{HOD}_{\{A\}}}(\delta) = \infty\text{”}.$$

This implies that (2) holds, by Theorem 138. $\square$

7.2. *Ramifications for the strongest hypotheses*

Whether or not the HOD Conjecture holds is a key question and for this it is only necessary to work in the context that there is a supercompact cardinal. One indication is given by the following theorem.

If the HOD Conjecture is provable then the corollaries are striking. We list three the last two of which are theorems of ZF and argue for the Axiom of Choice just from the existence of a proper class of extendible cardinals by showing that for each ordinal α, if $\delta > \alpha$ and δ is an extendible cardinal, then there exists a definable surjection,

$$\pi : \mathrm{Ord} \times \delta^\alpha \to \mathrm{Ord}^\alpha.$$

Working in just ZF, δ is an extendible cardinal if for all α there is an elementary embedding,

$$j : V_{\delta+\alpha} \to V_{j(\delta)+j(\alpha)}$$

such that $\delta = \mathrm{CRT}(j)$ and such that $j(\delta) > \alpha$. Assuming the Axiom of Choice this is equivalent to the assertion that for all α there is an elementary embedding,

$$j : V_{\delta+\alpha} \to V_{j(\delta)+j(\alpha)}$$

such that $\delta = \mathrm{CRT}(j)$.

Suppose that the HOD Conjecture is provable. Then:

(I) Suppose that δ is an extendible cardinal. Then:
   (a) For each $a \in [\mathrm{Ord}]^{<\delta}$, then there exists $b \in [\mathrm{Ord}]^{<\delta} \cap \mathrm{HOD}$ such that $a \subseteq b$.
   (b) If $\lambda > \delta$ and λ is a singular cardinal then λ is a singular cardinal in HOD and λ^+ is correctly computed by HOD.

(II) (ZF) Suppose that $\delta_0 < \delta_1$ are extendible cardinals. Then for each

$$a \in [\mathrm{Ord}]^{<\delta_0} \cap V_{\delta_1},$$

there exists $b \in [\mathrm{Ord}]^{<\delta_0} \cap \mathrm{HOD}$ such that $a \subseteq b$.

(III) (ZF) Suppose that λ is a limit of extendible cardinals, $\mathrm{cof}(\lambda) < \lambda$, and there is an extendible cardinal above λ. Then λ^+ is correctly computed by HOD.

By Lemma 188 and Theorem 197, the HOD Conjecture implies (I). Theorem 226 shows that (II) and (III) are corollaries of (I) and Theorem 228 shows that (III) implies (in ZF) that if there is a proper class of extendible cardinals then there is no non-trivial elementary embedding, $j : V \to V$.

It will be convenient to define the notion that κ is a supercompact cardinal in just ZF. If the HOD Conjecture is provable then (I)–(III) actually hold with "extendible" replaced by "supercompact".

Definition 220 (ZF). A cardinal κ is supercompact if for each $\alpha > \kappa$ there exist $\beta > \alpha$ and an elementary embedding, $j : V_\beta \to N$ such that

(1) N is a transitive set and $N^{V_\alpha} \subseteq N$,
(2) j has critical point κ,
(3) $\alpha < j(\kappa)$.

The cardinal κ is HOD-supercompact if for each $\alpha > \kappa$, j can be chosen such that in addition to (1)–(3), $\mathrm{HOD} \cap V_\alpha = j(\mathrm{HOD} \cap V_\kappa) \cap V_\alpha$.

Remark 221. (1) It is clear that assuming the Axiom of Choice this definition coincides with the usual definition.

(2) Note that it is not clear what the optimal complexity of assertion that δ is a supercompact cardinal is. Assuming the Axiom of Choice then it is evidently a Π_2-assertion about δ. Without the Axiom of Choice it is clearly a Π_3-assertion about δ but not obviously (equivalent to) a Π_2-assertion about δ.

(3) (ZF) Suppose that

$$j : V \to M$$

is an elementary embedding such that $\mathrm{CRT}(j) = \kappa$, $M^{V_\alpha} \subseteq M$, and such that $\alpha < j(\kappa)$. Let $\beta = j(\kappa)$, let $N = M \cap V_{j(\beta)}$ and let $k = j|V_\beta$. Then $N^{V_\alpha} \subseteq N$ and

$$k : V_\beta \to N$$

is an elementary embedding.

(4) (ZF) If κ is an extendible cardinal then κ is HOD-supercompact.

The following lemma gives a useful reformulation of the property that δ is supercompact; cf. Lemma 133. We use the following notation. Suppose $V_\gamma \prec_{\Sigma_1} V$. Then $V_\gamma \prec_{\Sigma_1^*} V$ if for all $\alpha < \gamma$, for all $a \in V_\gamma$ and for all Σ_0-formulas, $\phi(x_0, x_1)$, if there exists $b \in V$ such that

$$V \vDash \phi[a, b]$$

and such that $b^{V_\alpha} \subseteq b$ then there exists $b \in V_\gamma$ such that

$$V \vDash \phi[a, b]$$

and such that $b^{V_\alpha} \subseteq b$.

Note that assuming the Axiom of Choice, for each ordinal γ, if $V_\gamma \prec_{\Sigma_1} V$ then $V_\gamma \prec_{\Sigma_1^*} V$ (recall that assuming the Axiom of Choice, $V_\gamma \prec_{\Sigma_1} V$ if and only if $\gamma > \omega$ and $|V_\gamma| = \gamma$). Without assuming the Axiom of Choice this can fail.

Lemma 222 (ZF). *The following equivalent.*

(1) *δ is supercompact.*

(2) *For all $\gamma > \delta$ such that $V_\gamma \prec_{\Sigma_1^*} V$, for all $a \in V_\gamma$, there exists $\bar{\gamma} < \delta$, $\bar{a} \in V_{\bar{\gamma}}$, and an elementary embedding,*

$$j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

with critical point $\bar{\delta} < \delta$ such that $j(\bar{\delta}) = \delta$, $j(\bar{a}) = a$ and such that $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$.

Proof. We first prove that (1) implies (2).

Since δ is supercompact there exist $\beta > \gamma + \omega$ and an elementary embedding,

$$j : V_\beta \to N$$

such that

(1.1) N is transitive and $N^{V_{\gamma+1}} \subseteq N$,
(1.2) j has critical point δ,
(1.3) $\gamma < j(\delta)$.

Since $V_\gamma \prec_{\Sigma_1^*} V$ and since $N^{V_\gamma} \subseteq N$, $V_\gamma \prec_{\Sigma_1^*} j(V_\gamma)$. By reflection, since $j|V_{\gamma+1} \in N$, there exists $\bar{\gamma} < \delta$, $\bar{a} \in V_{\bar{\gamma}}$, and an elementary embedding,

$$\bar{j} : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

with critical point $\bar{\delta} < \delta$ such that $\bar{j}(\bar{\delta}) = \delta$, $\bar{j}(\bar{a}) = a$ and such that

$$V_{\bar{\gamma}} \prec_{\Sigma_1^*} V_\gamma.$$

But $V_\gamma \prec_{\Sigma_1^*} V$ and so $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$. Thus $\bar{j}$ is as required.

We finish by proving that (2) implies (1). We first assume (2) and prove that for all $\alpha > \delta$:

(2.1) There exist transitive sets M and N, and an elementary embedding,

$$j : M \to N$$

such that

(a) $\mathrm{CRT}(j) = \delta$ and $j(\delta) > \alpha$,
(b) $M^{V_\alpha} \subseteq M$ and $N^{V_\alpha} \subseteq N$,
(c) $M \vDash \mathrm{Z}$ and $N \vDash \mathrm{Z}$, where $Z = \mathrm{ZF}\backslash\mathrm{Replacement}$.

Fix $\alpha > \delta$. Let $\gamma > \alpha$ be such that $V_\gamma \prec_{\Sigma_1^*} V$. Since $V_\gamma \prec_{\Sigma_1^*} V$, the existence of (j, M, N) which satisfies (2.1) for α is absolute to V_γ.

By (1), there exist $\bar{\gamma} < \delta$, $\bar{\alpha} < \bar{\gamma}$, and an elementary embedding,

$$k : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

with critical point $\bar{\delta} < \delta$ such that $k(\bar{\delta}) = \delta$, $k(\bar{\alpha}) = \alpha$, and such that $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$.

Since $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$ and since $V_\delta^{V_{\bar{\alpha}}} \subseteq V_\delta$, there exists $\bar{\beta} < \bar{\gamma}$ such that $V_{\bar{\beta}}^{V_{\bar{\alpha}}} \subseteq V_{\bar{\beta}}$. Let $\beta = k(\bar{\beta})$. Thus $V_\beta^{V_\alpha} \subseteq V_\beta$ which trivially implies that $V_\beta^{V_{\bar{\alpha}}} \subseteq V_\beta$.

Thus $(k|V_{\bar{\beta}}, V_{\bar{\beta}}, V_\beta)$ witnesses (2.1) at $\bar{\delta}$ for $\bar{\alpha}$. Since $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$ there must exist a witness $(\bar{j}, \bar{M}, \bar{N}) \in V_{\bar{\gamma}}$. Let $(j, M, N) = (k(\bar{j}), k(\bar{M}), k(\bar{N}))$. By the elementarity of k, (j, M, N) witnesses (2.1) at δ for α.

Therefore (2.1) holds at δ for all α. We finish by showing that this implies that δ is supercompact. Fix $\alpha > \delta$.

We must show that there exist $\beta > \alpha$ and an elementary embedding

$$j : V_\beta \to N$$

such that N is transitive, $N^{V_\alpha} \subseteq N$, $\mathrm{CRT}(j) = \delta$, and such that $j(\delta) > \alpha$.

By (2.1) there exists (j_0, M_0, N_0) such that

(3.1) M_0, N_0 are transitive models of Z,
(3.2) $N_0^{V_\alpha} \subseteq N_0$,

(3.3) j_0 is an elementary embedding, $j_0 : M_0 \to N_0$,
(3.4) $\mathrm{CRT}(j_0) = \delta$ and $j_0(\delta) > \alpha$.

Let $\beta = j_0(\delta)$. Thus $V_\beta^{V_\alpha} \subset V_\beta$. By (2.1) again (this time applied to β), there exists (j_1, M_1, N_1) such that

(4.1) M_1, N_1 are transitive models of Z,
(4.2) $M_1^{V_\beta} \subseteq M_1$ and $N_1^{V_\beta} \subseteq N_1$,
(4.3) j_1 is an elementary embedding, $j_1 : M_1 \to N_1$,
(4.4) $\mathrm{CRT}(j_1) = \delta$ and $j_1(\delta) > \beta$.

Thus $V_\beta \in M_1$. Further $V_\beta^{V_\alpha} \subseteq V_\beta$ and $N_1^{V_\beta} \subseteq N_1$. Therefore $j_1(V_\beta)^{V_\alpha} \subseteq j_1(V_\beta)$ and so

$$j : V_\beta \to N$$

is an elementary embedding as required where $j = j_1 | V_\beta$ and $N = j_1(V_\beta)$. □

Lemma 222 motivates the following definition of a strong version of HOD-supercompactness which we shall need.

Definition 223 (ZF). A cardinal δ is a *strongly*-HOD-*supercompact cardinal* if for all $\gamma > \delta$ such that $V_\gamma \prec_{\Sigma_2} V$, there exists $\bar{\gamma} < \delta$ and an elementary embedding,

$$j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

with critical point $\bar{\delta} < \delta$ such that

(a) $j(\bar{\delta}) = \delta$,
(b) $V_{\bar{\gamma}} \prec_{\Sigma_2} V$.

Remark 224. (1) δ is HOD-supercompact if and only if Definition 223 holds with the condition (b) weakened to just

$$\mathrm{HOD} \cap V_{\bar{\gamma}} = (\mathrm{HOD})^{V_{\bar{\gamma}}}$$

together with the requirement that $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$.

(2) If δ is an extendible cardinal then δ is strongly-HOD-supercompact.
(3) If $V = \mathrm{HOD}$ then every supercompact cardinal is an HOD-supercompact cardinal but note that assuming the Axiom of Choice, if δ is a strongly-HOD-supercompact cardinal then δ must be a limit of supercompact cardinals.

The proof of Theorem 226 requires the following preliminary lemma which is really the key to the proof.

It is convenient to introduce the following notation for both Lemma 225 and Theorem 226. For each pair, (δ, λ), of regular infinite cardinals such that $\delta < \lambda$, let $\mathbb{P}_\delta^\lambda$ be the partial order of all partial functions,

$$f : \delta \times \lambda \to V_\lambda$$

such that $|\mathrm{dom}(f)| < \delta$ and such that for all $(\alpha, \eta) \in \mathrm{dom}(f)$, $f(\alpha, \eta) \in V_{1+\eta}$. The order on $\mathbb{P}^\delta_\lambda$ is by extension; $f \leq g$ if $g \subseteq f$. Thus in the natural interpretation, $\mathbb{P}^\lambda_\delta = \mathrm{Coll}(\delta, {<}V_\lambda)$.

Lemma 225 (ZF). *Suppose that κ is an infinite regular cardinal such that $({<}\kappa)$-*DC *holds and suppose that δ is a supercompact cardinal with $\delta > \kappa$. Suppose that $G \subseteq \mathbb{P}^\delta_\kappa$ is V-generic. Then $({<}\delta)$-*DC *holds in $V[G]$.*

Proof. Let $(R, <) \in V[G]$ be a partial order which is $({<}\delta)$-closed. For each $\xi < \delta$ we must produce a decreasing sequence in $(R, <)$ of length ξ.

Let τ be a term for $(R, <)$, suppose $p \in G$, and let $\gamma > \delta$ such that $\tau \in V_\gamma$. By Lemma 222 there exist $\bar{\gamma} < \delta$, $\bar{\tau} \in V_{\bar{\gamma}}$, $\bar{\delta} < \bar{\gamma}$, and an elementary embedding

$$j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

such that

(1.1) $\bar{\delta}$ is the critical point of j,
(1.2) $j(\bar{\delta}) = \delta$,
(1.3) $j(\bar{\tau}) = \tau$,
(1.4) $\kappa < \bar{\delta}$, $p \in V_{\bar{\delta}}$ and $\xi < \bar{\delta}$.

Let $\bar{G} = G \cap V_{\bar{\delta}}$. Thus j lifts to define an elementary embedding,

$$j_G : V_{\bar{\gamma}}[\bar{G}] \to V_\gamma[G].$$

Let $(\bar{R}, \bar{<})$ be the interpretation of $\bar{\tau}$ by $\bar{G}$. Thus $j_G(\bar{R}) = R$ and $j_G(\bar{<}) = <$. Therefore $(\bar{R}, \bar{<})$ is $({<}\bar{\delta})$-closed in $V[\bar{G}]$. Since $({<}\kappa)$-DC holds in V, it follows that $({<}\kappa)$-DC holds in $V[\bar{G}]$ which implies that

$$(V[\bar{G}])^{<\kappa} \subseteq V[\bar{G}]$$

in $V[G]$. Therefore $(\bar{R}, \bar{<})$ is $({<}\kappa)$-closed in $V[G]$. However $|V_{\bar{\gamma}}| = \kappa$ in $V[G]$ (since $\bar{\gamma} < \delta$). Therefore in $V[G]$ there exists a decreasing sequence in $(\bar{R}, \bar{<})$ of length ξ. Applying j pointwise to this decreasing sequence gives a decreasing sequence in $(R, <)$ of length ξ. □

Suppose

$$j : V_\gamma \to V_\gamma$$

is a non-trivial elementary embedding. Then $\kappa_\omega(j)$ is the ωth element of the critical sequence of j; i.e.

$$\kappa_\omega(j) = \sup\{\kappa_n(j) \mid n < \omega\}$$

where $\kappa_0(j) = \mathrm{CRT}(j)$ and for all $n \geq 0$, $\kappa_{n+1}(j) = j(\kappa_n(j))$.

Theorem 226 (ZF). *Suppose that δ_0 is a supercompact cardinal. Then there is a homogeneous partial order $\mathbb{Q}$ such that $\mathbb{Q}$ is Σ_3-definable in V_{δ_0} and such that if*

$G \subseteq \mathbb{Q}$ is V-generic then:

(1) $V[G]_{\delta_0} \vDash \mathrm{ZFC}$.

(2) *If δ is a supercompact cardinal in V and $\delta < \delta_0$ then δ is a supercompact cardinal in $V[G]_{\delta_0}$ and for each $\delta < \gamma < \delta_0$ there exists $\bar{\gamma} < \delta$ and an elementary embedding*

$$j : V[G]_{\bar{\gamma}+1} \to V[G]_{\gamma+1}$$

such that $j(\mathrm{CRT}(j)) = \delta$ and such that $j(V_{\bar{\gamma}}) = V_\gamma$, and such that $j|V_{\bar{\gamma}+1} \in V$.

(3) *If δ is a strongly-*HOD*-supercompact cardinal in V and $\delta < \delta_0$ then δ is a strongly-*HOD*-supercompact cardinal in $V[G]_{\delta_0}$ and for each $\delta < \gamma < \delta_0$ there exists $\bar{\gamma} < \delta$ and an elementary embedding*

$$j : V[G]_{\bar{\gamma}+1} \to V[G]_{\gamma+1}$$

such that $j(\mathrm{CRT}(j)) = \delta$, $j(V_{\bar{\gamma}}) = V_\gamma$, $j|V_{\bar{\gamma}+1} \in V$, and such that

$$j((\mathrm{HOD})^{V[G]} \cap V[G]_{\bar{\gamma}+1}) = (\mathrm{HOD})^{V[G]} \cap V[G]_{\gamma+1}.$$

(4) *Suppose $V_\lambda \prec V_{\delta_0}$ and*

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

is an elementary embedding with $\lambda = \kappa_\omega(j)$. Then $V[G]_\lambda \prec V[G]_{\delta_0}$ and in $V[G]$, j lifts to an elementary embedding,

$$j_G : V[G]_{\lambda+1} \to V[G]_{\lambda+1}.$$

Proof. A cardinal κ is strongly inaccessible if for all $\alpha < \kappa$ there is no function

$$f : V_\alpha \to \kappa$$

with range unbounded in κ. Note that if κ is strongly inaccessible then for all partial orders, $\mathbb{P} \in V_\kappa$, κ is strongly inaccessible in $V^{\mathbb{P}}$.

Clearly we can suppose that for all $\alpha < \delta_0$, if $G \subseteq \mathrm{Coll}(\omega, V_\alpha)$ is V-generic then in $V_{\delta_0}[G]$, the Axiom of Choice fails.

By Lemma 225, if δ is supercompact and if γ is an uncountable regular cardinal such that $({<}\gamma)$-DC holds and if $G \subseteq \mathbb{P}^\delta_\gamma$ is V-generic then in $V[G]$, $({<}\delta)$-DC holds. This implies, by Lemma 222, that there exists a strongly inaccessible cardinal $\bar{\delta} < \delta$ such that $\gamma < \bar{\delta}$ and such that if $G \subseteq \mathbb{P}^{\bar{\delta}}_\gamma$ is V-generic then in $V[G]$, $({<}\bar{\delta})$-DC holds.

Define by induction an increasing sequence, $\langle \kappa_\alpha : \alpha \leq \delta_0 \rangle$, of strongly inaccessible cardinals and an iteration, $\langle \mathbb{Q}_\alpha : \alpha < \delta_0 \rangle$, as follows. Let γ_0 be the least infinite regular cardinal such that $\gamma_0 < \delta_0$, $({<}\gamma_0)$-DC holds and γ_0-DC fails.

κ_0 is the least strongly inaccessible cardinal such that if $G \subseteq \mathbb{P}^{\kappa_0}_{\gamma_0}$ is V-generic then in $V[G]$, $({<}\kappa_0)$-DC holds. Define $\mathbb{Q}_0 = \mathbb{P}^{\kappa_0}_\omega$.

Suppose $(\mathbb{Q}_\beta, \kappa_\beta)$ is defined, $\mathbb{Q}_\beta$ is homogeneous, and that

$$V^{\mathbb{Q}_\beta} \vDash \text{“}\kappa_\beta \text{ is an uncountable regular cardinal such that } ({<}\kappa_\beta)\text{-DC holds”.}$$

Let $\kappa_{\beta+1}$ be the least κ such that

(1.1) $\kappa_\beta < \kappa$ and κ is strongly inaccessible,
(1.2) $V^{\mathbb{Q}_\beta * \mathbb{P}^\kappa_{\kappa_\beta}} \vDash$ "($<\kappa$)-DC holds".

Define $\mathbb{Q}_{\beta+1} = \mathbb{Q}_\beta * \mathbb{P}^\kappa_{\kappa_\beta}$.

Finally, suppose that $\langle \kappa_\alpha : \alpha < \beta \rangle$ and $\langle \mathbb{Q}_\alpha : \alpha \leq \beta \rangle$ have been defined. Let $\gamma = \sup\{\kappa_\alpha \,|\, \alpha < \beta\}$. If γ is strongly inaccessible then $\kappa_\beta = \gamma$ and $\mathbb{Q}_\beta$ is the direct limit of the sequence, $\langle \mathbb{Q}_\alpha : \alpha < \beta \rangle$. Otherwise let $\mathbb{Q}_\beta$ be the inverse limit of $\langle \mathbb{Q}_\alpha : \alpha < \beta \rangle$. It follows that

$$V^{\mathbb{Q}} \vDash \text{"(}{<}\gamma^+\text{)-DC holds"}.$$

Define κ_β be the least κ such that

(2.1) $\gamma < \kappa$ and κ is strongly inaccessible,
(2.2) $V^{\mathbb{Q} * \mathbb{P}^\kappa_\delta} \vDash$ "($<\kappa$)-DC holds",

and define $\mathbb{Q}_\beta = \mathbb{Q} * \mathbb{P}^\kappa_\delta$ where $\delta = (\gamma^+)^{V^{\mathbb{Q}}}$.

This defines $\langle \kappa_\alpha : \alpha \leq \delta_0 \rangle$ and $\langle \mathbb{Q}_\alpha : \alpha < \delta_0 \rangle$. Since δ_0 is supercompact it follows that $\delta_0 = \kappa_{\delta_0}$. The key claims are the following and each is straightforward to verify.

(3.1) Suppose that $\delta < \delta_0$ and that δ is supercompact. Then $\delta = \kappa_\delta$.
(3.2) Suppose that $\delta \leq \delta_0$ and that $\delta = \kappa_\delta$. Then for each $\alpha < \delta$,

  (a) $\mathbb{Q}_\delta \cong \mathbb{Q}_\kappa * \mathbb{P}$
  (b) $\mathbb{P}$ is homogeneous and ($<\kappa$)-directed-closed in $V^{\mathbb{Q}_\kappa}$,

  where $\kappa = \kappa_\alpha$ and $\mathbb{P} = (\mathbb{Q}_\delta)^{V^{\mathbb{Q}_\kappa}}$.

(3.3) Suppose that $\kappa < \delta_0$, κ is strongly inaccessible and $V_\kappa \prec_{\Sigma_1^*} V$. Suppose that $\alpha < \kappa$, $\mathbb{Q} \in V_\kappa$ and that

$$\mathbb{Q} = (\mathbb{Q}_\alpha)^{V_\kappa}.$$

  Then $\mathbb{Q} = \mathbb{Q}_\alpha$.

Thus $\mathbb{Q}_{\delta_0}$ is homogeneous. Clearly, $\mathbb{Q}_{\delta_0}$ is OD and moreover

$$\langle \mathbb{Q}_\alpha : \alpha < \delta_0 \rangle = (\langle \mathbb{Q}_\alpha : \alpha < \delta_0 \rangle)^{V_{\delta_0}}.$$

To verify that $\mathbb{Q}_{\delta_0}$ is Σ_3-definable in V_{δ_0} note that for all $\eta < \delta_0$,

$$\langle \mathbb{Q}_\alpha : \alpha < \eta \rangle = (\langle \mathbb{Q}_\alpha : \alpha < \eta \rangle)^{V_\gamma},$$

for all sufficiently large strongly inaccessible cardinals $\gamma < \delta_0$.

We finish by showing that $\mathbb{Q}_{\delta_0}$ satisfies the requirements (1)–(4) of the lemma.

Suppose that $G \subseteq \mathbb{Q}_{\delta_0}$ is V-generic. By (3.2), δ_0 is strongly inaccessible in $V[G]$ and so it follows that $(V[G])_{\delta_0} \vDash$ ZFC.

We now prove (2). Suppose that $\delta < \delta_0$ is supercompact in V. We prove that δ is supercompact in $(V[G])_{\delta_0}$.

Fix $\gamma < \delta_0$ such that $\delta < \gamma$, $\kappa_\gamma = \gamma$ and such that $V_\gamma \prec_{\Sigma_1^*} V$. By Lemma 222 there exists there exist $\bar{\gamma} < \delta$, $\bar{\delta} < \bar{\gamma}$, $\bar{\mathbb{Q}}$, and an elementary embedding

$$j : V_{\bar{\gamma}+\omega} \to V_{\gamma+\omega}$$

such that

(4.1) $\bar{\delta}$ is the critical point of j,
(4.2) $j(\bar{\delta}) = \delta$,
(4.3) $j(\bar{\mathbb{Q}}) = \mathbb{Q}_\gamma$,
(4.4) $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$.

By (4.3), $\bar{\mathbb{Q}}|\bar{\delta} = \mathbb{Q}_{\bar{\delta}}$. Therefore j lifts to define an elementary embedding

$$j : V_{\bar{\gamma}+\omega}[G|\bar{\delta}] \to V_{\gamma+\omega}[G|\delta].$$

Note that since $\gamma = \kappa_\gamma$, γ is strongly inaccessible and so $\bar{\gamma}$ is strongly inaccessible. Since $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$, it follows that $\bar{\mathbb{Q}} = \mathbb{Q}_{\bar{\gamma}}$.

Since $\kappa_\gamma = \gamma$, $\mathbb{Q}_\gamma = \mathbb{Q}_\delta * \mathbb{P}$ where $\mathbb{P} \in V_{\gamma+\omega}[G|\delta]$ and such that $\mathbb{P}$ is homogeneous and $(<\delta)$-directed-closed in $V_{\gamma+\omega}[G|\delta]$.

Let $\bar{\mathbb{P}} \in V_{\bar{\gamma}+\omega}[G|\bar{\delta}]$ be such that $j(\bar{\mathbb{P}}) = \mathbb{P}$. Thus

$$\bar{\mathbb{Q}} = \mathbb{Q}_{\bar{\delta}} * \bar{\mathbb{P}} = \mathbb{Q}_{\bar{\gamma}},$$

and so $V[G|\bar{\gamma}] = V[G|\bar{\delta}][g]$ where $g \subseteq \bar{\mathbb{P}}$ is $V[G|\bar{\delta}]$-generic.

The partial order $\mathbb{P}$ is $(<\delta)$-directed-closed in $V[G|\delta]$ and so there exists a condition $p \in \mathbb{P}$ such that $p < j(q)$ for each $q \in g$. By homogeneity there exists a generic filter $G^* \subseteq \mathbb{Q}_{\delta_0}$ such that

(5.1) $G|\delta = G^*|\delta$,
(5.2) $V[G] = V[G^*]$,
(5.3) $p \in G^*$,

and so we can suppose that $p \in G$. Therefore j lifts to an elementary embedding,

$$j : V_{\bar{\gamma}+\omega}[G|\bar{\gamma}] \to V_{\gamma+\omega}[G|\gamma].$$

However $\kappa_{\bar{\gamma}} = \bar{\gamma}$ and so

$$V_{\bar{\gamma}}[G|\bar{\gamma}] = (V[G])_{\bar{\gamma}}.$$

Similarly since $\kappa_\gamma = \gamma$,

$$V_\gamma[G|\gamma] = (V[G])_\gamma.$$

Thus in $V[G]$, for each $\gamma > \delta$ such that $\gamma < \delta_0$ there exist $\bar{\delta} < \delta$, $\bar{\gamma} < \delta$ and an elementary embedding,

$$j : (V[G])_{\bar{\gamma}} \to (V[G])_\gamma$$

with critical point $\bar{\delta}$ such that $j(\bar{\delta}) = \delta$. This shows that δ is a supercompact cardinal in $(V[G])_{\delta_0}$ and verifies the second part of (2). The proof of (3) is similar, note that since δ is strongly-HOD-supercompact and since δ_0 is supercompact it follows that δ is strongly-HOD-supercompact in V_{δ_0}. Now one follows the proof of (2), but given $\gamma > \delta$ such that $\gamma < \delta_0$ and such that $V_\gamma \prec_{\Sigma_2} V_{\delta_0}$, one chooses $\bar{\delta} < \bar{\gamma} < \delta$ and an elementary embedding

$$j : V_{\bar{\gamma}} \to V_\gamma$$

such that

(6.1) $\bar{\delta}$ is the critical point of j,
(6.2) $j(\bar{\delta}) = \delta$,
(6.3) $V_{\bar{\gamma}} \prec_{\Sigma_2} V$.

Argung as above j lifts to an elementary embedding,

$$j : V_{\bar{\gamma}}[G|\bar{\gamma}] \to V_\gamma[G|\gamma],$$

and the situation here is actually simpler since $V_{\bar{\gamma}} \prec_{\Sigma_2} V_{\delta_0}$. Finally, from the definablity of $\mathbb{Q}_{\delta_0}$,

$$V_{\bar{\gamma}}[G|\bar{\gamma}] \prec_{\Sigma_2} V[G]_{\delta_0}.$$

Finally we prove (4). Since $V_\lambda \prec V_{\delta_0}$,

(7.1) $\langle \mathbb{Q}_\alpha : \alpha < \lambda \rangle \subseteq V_\lambda$,
(7.2) $\mathbb{Q}_\lambda$ is the inverse limit of $\langle \mathbb{Q}_\alpha : \alpha < \lambda \rangle$,
(7.3) $\mathbb{Q}/\mathbb{Q}_\lambda$ is λ-closed in $V^{\mathbb{Q}_\lambda}$.

Therefore in $V[G]$,

$$(V[G \cap \mathbb{Q}_\lambda])^\lambda \subseteq V[G \cap \mathbb{Q}_\lambda].$$

Further as above, it follows that there is a condition $p_\lambda \in \mathbb{Q}_\lambda$ such that if $p_\lambda \in G$ then

$$G \cap \mathbb{Q}_\lambda = \{p \in \mathbb{Q}_\lambda \mid j(p) \in G\}.$$

By (7.1)–(7.3), we can suppose that $p_\lambda \in G$. Therefore

$$V[G]_\lambda \prec V_{\delta_0}$$

and j lifts in $V[G \cap \mathbb{Q}_\lambda]$ to an elementary embedding,

$$j_G : V[G \cap \mathbb{Q}_\lambda]_{\lambda+1} \to V[G \cap \mathbb{Q}_\lambda]_{\lambda+1}.$$

This proves (4). □

As another corollary of Lemma 225, we obtain the following theorem which shows that non-trivial consequences of the Axiom of Choice must hold at λ^+ where λ is a singular limit of supercompact cardinals.

Theorem 227 (ZF). *Suppose that λ is a singular limit of supercompact cardinals and let $\mathcal{F}$ be the filter generated by the closed unbounded subsets of λ^+. Then*

(1) *λ^+ is a regular cardinal,*
(2) *$\mathcal{F}$ is λ^+-complete.*

Proof. The filter $\mathcal{F}$ is λ^+-complete if for all sequences, $\langle S_\alpha : \alpha \leq \lambda \rangle$, of elements of $\mathcal{F}$,

$$\cap\{S_\alpha \,|\, \alpha < \lambda\} \in \mathcal{F}.$$

The key claim is the following. Suppose $\mathbb{P} \in V_\lambda$ is a partial order and that $G \subseteq \mathbb{P}$ is V-generic. Then for all sets $A \subseteq \mathrm{Ord}$, with $A \in V[G]$, there exists a (finite) set $Z \in V$ such that

$$\mathrm{HOD}^V_Z[A]$$

is a κ-cc generic extension of HOD^V_Z for some $\kappa < \lambda$.

We sketch the proof of this claim and for this we work in V. Fix a term τ for A and let

$$Z = \{(\mathbb{P}, \leq_{\mathbb{P}}), \tau\}.$$

Thus Z is finite which implies that $\mathrm{HOD}_Z \vDash \mathrm{ZFC}$. Let $\delta < \lambda$ be such that $\mathbb{P} \in V_\delta$ and let κ be the supremum of the ordinals α such that there is a surjection (in V) of $V_{\delta+1}$ onto α. Note that since λ is a limit of supercompact cardinals, necessarily $\kappa < \lambda$.

For each $n < \omega$, let $\mathbb{Q}_n$ be the set of all nonempty sets $X \subseteq (V_\delta)^n$ such that X is ordinal definable from Z. For each $n \leq m < \omega$, let

$$\pi_{m,n} : \mathbb{Q}_m \to \mathbb{Q}_n$$

be the projection map,

$$\pi_{m,n}(X) = \{s|n \,|\, s \in X\}.$$

Let $\mathbb{Q} = \cup\{\mathbb{Q}_n \,|\, n < \omega\}$ and define for $X, Y \in \mathbb{Q}$,

$$X \leq Y$$

if $\pi_{m,n}(Y) \subseteq X$ where $X \in \mathbb{Q}_n$, $Y \in \mathbb{Q}_m$ and $n \leq m$.

Thus $(\mathbb{Q}, \leq)$ is ordinal definable from Z and there exists a partial order

$$(\mathbb{Q}^*, \leq^*) \in \mathrm{HOD}$$

and an isomorphism

$$e : (\mathbb{Q}^*, \leq^*) \to (\mathbb{Q}, \leq)$$

such that e is ordinal definable from Z.

Now suppose $g_0 \subseteq \mathrm{Coll}(\omega, V_\delta)$ is V-generic and in $V[g_0]$, let

$$f_{g_0} : \omega \to V_\delta$$

be the induced surjection. Let

$$G_0 = \{X \in \mathbb{Q} \,|\, f_{g_0}|n \in X \text{ where } X \in \mathbb{Q}_n\}$$

and let

$$G_0^* = \{p \in \mathbb{Q}^* \,|\, e(p) \in G_0\}.$$

Then by Vopenka's basic argument,

(1.1) G_0^* is $(\mathrm{HOD}_Z)^V$-generic for $(\mathbb{Q}^*, \leq^*)$,
(1.2) $(\mathrm{HOD}_Z)^V[G_0^*] = (\mathrm{HOD}_{Z\cup V_\delta})^V[g_0]$.

There is a surjection of $V_{\delta+1}$ onto $\mathbb{Q}$ and so necessarily $(\mathbb{Q}^*, \leq^*)$ is κ-cc in $(\mathrm{HOD}_Z)^V$. Note that

$$V_\delta \in (\mathrm{HOD}_Z)^V[G_0^*]$$

and so $\mathbb{P} \in (\mathrm{HOD}_Z)^V[G_0^*]$. But

$$(\mathrm{HOD}_Z)^V[G_0^*] \vDash \mathrm{ZFC}$$

and V_δ is countable in $(\mathrm{HOD}_Z)^V[G_0^*]$. Thus $\mathbb{P}$ is countable in $(\mathrm{HOD}_Z)^V[G_0^*]$.

Finally, returning to $V[G]$, we can suppose that G is $(\mathrm{HOD}_Z)^V[G_0^*]$-generic for $\mathbb{P}$ (by choosing g_0 to be $V[G]$-generic). Thus

$$A \in (\mathrm{HOD}_{Z\cup V_\delta})^V[g_0][G].$$

The iteration,

$$(\mathbb{Q}^*, \leq^*) * \mathbb{P},$$

is κ-cc in $(\mathrm{HOD}_Z)^V$ and so since

$$(\mathrm{HOD}_Z)^V[G_0^*] = (\mathrm{HOD}_{Z\cup V_\delta})^V[g_0],$$

$\mathrm{HOD}_Z^V[A]$ is a κ-cc forcing extension of HOD_Z^V as claimed.

Let $\delta < \lambda$ be supercompact, we show that $\mathrm{cof}(\lambda^+) \geq \delta$ and that $\mathcal{F}$ is δ-complete. Let $\mathbb{P}$ be the partial order of finite partial functions,

$$f : \omega \times \delta \to V_\delta$$

such that for all $(k, \alpha) \in \mathrm{dom}(f)$, $f(k, \alpha) \in V_{1+\alpha}$, ordered by extension. Thus $\mathbb{P} = \mathbb{P}_\omega^\delta$ (as defined before Lemma 225) and we suppose that $G \subseteq \mathbb{P}$ is V-generic. By Lemma 225,

$$V[G] \vDash \mathrm{DC}.$$

By the claim we proved above, if $A \subseteq \lambda^+$ and $A \in V[G]$ there exists a finite set $Z \in V$ such that

$$\mathrm{HOD}_Z^V[A]$$

is a generic extension of HOD_Z^V for a partial order which is κ-cc in HOD_Z^V for some $\kappa < \lambda$.

Therefore

$$(\lambda^+)^V = (\lambda^+)^{V[G]}.$$

This proves that in V, $\mathrm{cof}(\lambda^+) \geq \delta$. Further since λ is a limit of supercompact cardinals, it follows (by varying the choice of δ) that λ^+ is a regular cardinal in $V[G]$.

Suppose that $C \subseteq \lambda^+$ is closed and unbounded in λ^+ and that $C \in V[G]$. Then again we have that for some finite set $Z \in V$,

$$\mathrm{HOD}_Z^V[C]$$

is a generic extension of HOD_Z^V for a partial order which is κ-cc in HOD_Z^V for some $\kappa < \lambda$. But this implies that there exists a closed unbounded set $D \subseteq C$ such that $D \in V$. But

$$V[G] \vDash \mathrm{DC}$$

and $\delta = (\omega_1)^{V[G]}$. Therefore since

$$(\lambda^+)^V = (\lambda^+)^{V[G]},$$

it follows that in V, $\mathcal{F}$ is δ-complete.

Again since λ is a limit of supercompact cardinals, this proves that the filter $\mathcal{F}$ is $(<\lambda)$-complete and so since λ is singular, $\mathcal{F}$ is λ^+-complete. $\square$

A consequence of (III) is that if there exist a proper class of extendible cardinals, then there exists no non-trivial elementary embedding $j : V \to V$ (as a theorem of ZF). This is an immediate corollary of the following theorem, see [2] for a discussion of Kunen's theorem on the nonexistence of a non-trivial elementary embedding, $j : V \to V$, assuming the Axiom of Choice.

Theorem 228 (ZF). *Suppose that* $\mathrm{cof}(\lambda) = \omega$, λ *is a limit of supercompact cardinals and that for some set* $A \subseteq \mathrm{Ord}$,

$$\lambda^+ = (\lambda^+)^{L[A]}.$$

Then there is no non-trivial elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Proof. Let M be the set of all sets a such that there exists a surjection

$$\pi : V_{\lambda+1} \to b$$

where b is the transitive closure of a. Thus if there is a wellordering of $V_{\lambda+1}$, $M = H(\gamma^+)$ where $\gamma = |V_{\lambda+1}|$. Note $\mathcal{P}(\lambda^+) \subseteq M$.

There is a non-trivial elementary embedding,

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

if and only if there exists a non-trivial elementary embedding,

$$j : M \to M,$$

and so we assume toward a contradiction that

$$j : M \to M$$

is a non-trivial elementary embedding. Thus $j(\lambda) = \lambda$, $j(\lambda^+) = \lambda^+$ and $\mathrm{CRT}(j) < \lambda$.

Let $\mathcal{F}$ be the filter on λ^+ generated by the closed unbounded subsets of λ^+ and let $S = \{\eta < \lambda^+ \mid \mathrm{cof}(\eta) = \omega\}$. By Theorem 227,

$$\mathcal{F} \cap \mathrm{HOD}_{\{A\}}$$

is λ^+-complete in $\mathrm{HOD}_{\{A\}}$ and so, noting that $S \in \mathrm{HOD}_{\{A\}}$, there must exist a sequence,

$$\langle S_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD}_{\{A\}}$$

of pairwise disjoint stationary subsets of S. Otherwise λ^+ is a measurable cardinal in $\mathrm{HOD}_{\{A\}}$ which contradicts that $\lambda^+ = (\lambda^+)^{L[A]}$.

Let

$$\langle T_\beta : \beta < \lambda \rangle = j(\langle S_\alpha : \alpha < \lambda \rangle).$$

Since $j(S) = S$ it follows that $\langle T_\beta : \beta < \lambda \rangle$ is a sequence of pairwise disjoint stationary subsets of S. Let κ_0 be the critical point of j and let $C \subseteq \lambda^+$ be the set of all limit ordinals, η, such that $j(\xi) < \eta$ for all $\xi < \eta$. Since $j(\lambda^+) = \lambda^+$, C is closed unbounded in λ^+ and so since T_{κ_0} is stationary in S there must exist an ordinal $\eta_0 < \lambda^+$ such that $\eta_0 \in T_{\kappa_0} \cap C$. For each $\eta \in C \cap S$, $j(\eta) = \eta$ and so $j(\eta_0) = \eta_0$. But

$$S = \cup\{S_\alpha \mid \alpha < \lambda\}$$

and so $\eta_0 \in S_{\alpha_0}$ for some $\alpha_0 < \lambda$. Therefore $\eta_0 \in j(S_{\alpha_0}) = T_{j(\alpha_0)}$ which implies that $\kappa_0 = j(\alpha_0)$ and this is a contradiction since $\kappa_0 = \mathrm{CRT}(j)$. □

A sentence ϕ is Ω-valid from $\mathrm{ZFC} + \Phi$ if for all complete Boolean algebras, $\mathbb{B}$, and for all $\alpha \in \mathrm{Ord}$ if

$$V_\alpha^{\mathbb{B}} \vDash \mathrm{ZFC} + \Phi$$

then $V_\alpha^{\mathbb{B}} \vDash \phi$. This definition is in the context of ZF. We next prove a theorem which shows that if the HOD Conjecture is Ω-valid from

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”}$$

then one obtains a dramatic fragment of the Axiom of Choice from just ZF and the existence of at least two supercompact cardinals. As a corollary, we obtain a strong version of Theorem 228 but under the hypothesis that the HOD Conjecture is Ω-valid from

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”}.$$

Theorem 229 (ZF). *Suppose the* HOD *Conjecture is* Ω*-valid from*

$$\mathrm{ZFC} + \text{“\textit{There is a supercompact cardinal}”},$$

δ_0 *is a supercompact cardinal, and that there is a supercompact cardinal below* δ_0. *Then there exists a transitive class* $N \subset V$ *and* $X \in V_{\delta_0}$ *such that the following hold.*

(1) $N \vDash \mathrm{ZFC}$.
(2) N *is* Σ_2*-definable from* X.

(3) *There exists a partial order $\mathbb{P} \in N \cap V_{\delta_0}$ such that for all $A \subset \mathrm{Ord}$, $A \in N[G]$ for some N-generic filter $G \subset \mathbb{P}$.*

Proof. Let δ be the least supercompact cardinal. Let $\mathbb{Q}_{\delta_0}$ be the partial order constructed in the proof of Theorem 226 which witnesses that the conclusions of Theorem 226 hold.

Suppose $G \subset \mathbb{Q}_{\delta_0}$ be V-generic. Thus

$$V[G]_{\delta_0} \vDash \mathrm{ZFC} + \text{“}\delta \text{ is supercompact”}$$

and δ_0 is Mahlo in $V[G]$.

Therefore since the HOD-Conjecture is Ω-valid from

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”},$$

necessarily $V[G]_{\delta_0} \vDash$ “The HOD Conjecture”.

Suppose that $\delta < \kappa$, κ is strongly inaccessible in $V[G]$ and that $V[G]_\kappa \prec V[G]_{\delta_0}$. By Theorem 226, there exists $\bar{\kappa} < \delta$ and an elementary embedding

$$j : V[G]_{\bar{\kappa}+1} \to V[G]_{\kappa+1}$$

such that $j(V_{\bar{\kappa}}) = V_\kappa$, $j(\mathrm{CRT}(j)) = \delta$, and such that $j|V_{\bar{\kappa}} \in V$.

The key point is the following where $\bar{\delta} = \mathrm{CRT}(j)$.

(1.1) For each $\gamma < \bar{\kappa}$ and for each $A \in \mathcal{P}(\gamma) \cap V$, there exists $B \subset \bar{\delta}$ such that $A \in L[j(B)]$.

We verify this. Let λ be the least regular cardinal of $V[G]$ such that $j(2^\gamma) < \lambda$ and such that in $V[G]_{\delta_0}$, λ is not ω-strongly measurable in HOD. Since $V[G]_\kappa \prec V[G]_{\delta_0}$, $\lambda < \kappa$. Since λ is definable from $j(\gamma)$ in $V[G]_\kappa$, necessarily $\lambda = j(\bar{\lambda})$ for some $\bar{\lambda} < \bar{\kappa}$.

Thus in $V[G]_{\bar{\kappa}}$, $\bar{\lambda}$ is not ω-strongly measurable in HOD. Therefore since $2^\gamma < \bar{\lambda}$, there is a partition

$$\langle S_\alpha : \alpha < \gamma \rangle \in (\mathrm{HOD})^{V[G]_{\bar{\kappa}}}$$

of $\{\eta < \bar{\lambda} \,|\, \mathrm{cof}(\eta) = \omega\}$ into disjoint stationary sets.

Let μ be the normal fine measure on $\mathcal{P}_{\bar{\delta}}(\bar{\lambda})$ given by j. Thus there is a set $X \in \mu$ such that X is definable in $V[G]_{\bar{\lambda}}$ from $\langle S_\alpha : \alpha < \gamma \rangle$ such that for all $\sigma, \tau \in X$ if $\sup(\sigma) = \sup(\tau)$ then $\sigma = \tau$. This is by the proof of Solovay's theorem, Theorem 137.

By the homogeneity of $\mathbb{Q}_{\delta_0}$, $X \in V$. Now suppose $A \in \mathcal{P}(\gamma) \cap V$. Let $B \subset \bar{\delta}$ be a set such that

(2.1) $B \in L[X, A]$,
(2.2) $L[B] \cap V_{\bar{\delta}} = L[A, X] \cap V_{\bar{\delta}}$.

Thus by the elementarity of j,

$$L[j(B)] \cap V_\delta = L[j(A), j(X)] \cap V_\delta.$$

But $X \in \mu$ and so $\{j(\alpha) \,|\, \alpha < \lambda\} \in j(X)$ which implies that $A \in L[j(A), j(X)]$. Finally $\bar{\lambda} < \delta$ and so $A \in L[j(B)]$. This verifies (1.1).

Let N_j be HOD as computed in

$$(V[G]_{\delta_0}, V_{\delta_0})$$

with parameter $j|(\mathcal{P}(\bar{\delta}) \cap V)$, and let $\mathbb{B}^j_{\bar{\kappa}}$ be the Boolean algebra of all subsets of $\mathcal{P}(\bar{\delta})$ which are definable in $(V[G]_{\delta_0}, V_{\delta_0})$ with parameters from $\delta_0 \cup \{j|(\mathcal{P}(\bar{\delta}) \cap V)\}$. Again since $\mathbb{Q}_{\delta_0}$ is homogeneous, $N_j \in V$.

Let γ_0 be the least strongly inaccessible cardinal in $V[G]$ above $\bar{\delta}$. Then setting $M = N_j \cap V_{\bar{\kappa}}$ we have (by Vopenka's Theorem):

(3.1) $M \vDash \mathrm{ZFC}$.
(3.2) $M \in V$.
(3.3) There is a partial order $\mathbb{P}_0 \in M \cap V_{\gamma_0}$ such that for every set $B \in V_{\bar{\kappa}}$ such that $B \subset \bar{\kappa}$, there exists an N-generic filter $G \subset \mathbb{P}_0$ such that $G \in V$ and such $B \subset M[G]$.
(3.4) Suppose $\gamma < \gamma_0$ and $g \subset \mathrm{Coll}(\omega, \mathcal{P}(\gamma) \cap V)$ is V-generic. Then g is set-generic over $M \cap V_{\lambda_0}$.

The key claim is that (3.1)–(3.4) together with $N_j \cap V_{\gamma_0}$ uniquely specifies M in V. This is by Lemma 163. Let $M' \in V$ satisfy (3.1)–(3.4) and be such that $M' \cap V_{\gamma_0} = M \cap V_{\gamma_0}$.

Let $\mathbb{P}^M_0$ witness (3.3) for N and let $\mathbb{P}^{M'}_0$ witness (3.3) for M'. Let

$$\gamma = \max\{|\mathbb{P}^M_0|^M, |\mathbb{P}^{M'}_0|^{M'}\}$$

and let $g \subset \mathrm{Coll}(\omega, \mathcal{P}(\gamma) \cap V)$ be V-generic. Thus g is set-generic over $M \cap V_{\gamma_0}$ and g is set-generic over $M' \cap V_{\gamma_0}$. Therefore $M' \subset M[g]$ and $M \subset M'[g]$ which implies that

$$M[g] = M'[g],$$

and so by Lemma 163, $M' = M$.

Let λ be the least strongly inaccessible cardinal above δ in $V[G]$. Let $M^* = j(M)$. Thus by the elementarity of j we have the following.

(4.1) $M^* \vDash \mathrm{ZFC}$.
(4.2) $M^* \in V$ and $M^* \cap \mathrm{Ord} = \kappa$.
(4.3) There is a partial order $\mathbb{P}_0 \in M^* \cap V_\lambda$ such that for every set $B \in V_\kappa$ such that $B \subset \kappa$, there exists an M^*-generic filter $G \subset \mathbb{P}_0$ such that $G \in V$ and such $B \subset M^*[G]$.
(4.4) Suppose $\gamma < \lambda$ and $g \subset \mathrm{Coll}(\omega, \mathcal{P}(\gamma) \cap V)$ is V-generic. Then g is set-generic over $M^* \cap V_\lambda$.

Let I be the set of all strongly inaccessible cardinals κ such that

$$V[G]_\kappa \prec V[G]_{\delta_0}$$

and such that $\kappa > \delta$, Note that $V[G]_{\delta_0}$ is wellordered by G in $V[G]$ and so there is a function

$$F : I \to V[G]_{\delta_0}$$

such that for each $\kappa \in I$, $F(\kappa)$ is a transitive set which satisfies (4.1)–(4.4).

Since I is cofinal in δ_0 and since δ_0 is strongly inaccessible, there exists a cofinal set $I_0 \subset I$ and $N_0 \subset V_\lambda$ such that for all $\kappa \in I_0$,

$$F(\kappa) \cap V_\lambda = N_0.$$

But then by uniqueness, for all $\kappa_1 < \kappa_2$ in I_0,

$$F(\kappa_1) \cap V_{\kappa_0} = F(\kappa_0).$$

This specifies $N_1 \subset V_{\delta_0}$. By uniqueness again, it follows that $\kappa > \lambda$, if

$$V_\kappa \vDash \mathrm{ZF}\backslash\mathrm{Replacement} + \Sigma_1\text{-Replacement}$$

then for all M, $M = N_1 \cap V_\kappa$ if and only if:

(5.1) $\kappa \subset M \subset V[G]_\kappa$.
(5.2) $M \vDash \mathrm{ZFC}\backslash\mathrm{Replacement} + \Sigma_1$-Replacement.
(5.3) $M \cap V_\lambda = N_0$.
(5.4) There is a partial order $\mathbb{P} \in M \cap V_\gamma$ such that for all bounded sets $A \subset \kappa$ there is an M-generic filter $G \subset \mathbb{P}$ such that $A \in M[G]$.

This is arguing exactly as above but noting that Laver's Lemma can be proved in the weaker theory:

$$\mathrm{ZFC}\backslash\mathrm{Replacement} + \Sigma_1\text{-Replacement}.$$

Since δ_0 is supercompact this extends N_1 to a proper class N and this proves the theorem. $\square$

Remark 230. With notation as in Theorem 229, it follows by the local version of Lemma 163, that for all limit $\alpha > \delta_0$, $N \cap V_\alpha$ is Σ_2-definable in V_α from X. The local version of Lemma 163 is simply the version in just the theory ZC; i.e. in the theory ZFC\Replacement. Also note that if

$$j : V_{\gamma+\omega} \to V_{j(\gamma)+\omega}$$

is any elementary embedding such that $(X, \mathbb{P}) \in V_{\mathrm{CRT}(j)}$ then $j|(N \cap V_{\lambda+1}) \in N$. This is because there exists an N-generic filter $G \subset \mathbb{P}$ such that $j|(N \cap V_{\lambda+\omega}) \in N[G]$.

Finally (with notation still as in Theorem 229), if $\gamma = |\mathbb{P}|^N$ and if $G \subset \mathrm{Coll}(\omega, \gamma)$ is V-generic, then in $V[G]$, δ_0 is supercompact and the conclusion of Theorem 229 holds witnessed by $N[G]$, except now $\mathbb{P}$ is just Cohen forcing.

Theorem 229 (especially in light of the previous remark) suggests the following Axiom of Choice Conjecture. This conjecture if true would arguably provide a compelling argument for the Axiom of Choice based solely on large cardinal axioms. Of course a proof of the HOD Conjecture would also give such an argument by Theorem 229 but perhaps not quite so a compelling one.

Definition 231 (ZF). (***The Axiom of Choice Conjecture***) Suppose that δ_0 is a supercompact cardinal and there is a supercompact cardinal below δ_0. Suppose $G \subset \mathrm{Coll}(\omega, V_{\delta_0})$ is V-generic. Then $V[G] \vDash$ Axiom of Choice.

Remark 232. Assume just ZF and that δ_0 is a supercompact cardinal such that there is a supercompact cardinal below δ_0. Assume that the HOD Conjecture is Ω-valid from

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”}.$$

Then by Theorem 229, there exists $(X, \gamma) \in V_{\delta_0}$ such that there is a surjection

$$\pi : \mathcal{P}(\gamma) \times \mathrm{Ord} \to \mathcal{P}(\mathrm{Ord})$$

which is Σ_2-definable from (X, γ). This gives that for each $\alpha \in \mathrm{Ord}$, a surjection,

$$\pi_\alpha : \mathcal{P}^\alpha(\gamma) \times \mathrm{Ord} \to \mathcal{P}^\alpha(\mathrm{Ord})$$

which is Σ_2-definable from (X, γ, α), where $\mathcal{P}^\alpha(Z)$ is the α-iterated powerset of Z. If for some $\alpha < \beta$ there is a surjection

$$e : \mathcal{P}^\alpha(\gamma) \times \beta \to \mathcal{P}^{\alpha+1}(\gamma)$$

then one gets a surjection

$$\pi^* : \mathcal{P}^\alpha(\gamma) \times \mathrm{Ord} \to V$$

which is Σ_2-definable from (X, γ, e). This would prove that

$$V^{\mathrm{Coll}(\omega, \mathcal{P}^\alpha(\gamma))} \vDash \text{Axiom of Choice}$$

and by reflection the Axiom of Choice Conjecture follows since the least α for which e exists as above must be below δ_0.

As a corollary of Theorem 229 we obtain a strong version of Theorem 228.

Corollary 233. *Suppose the* HOD *Conjecture is Ω-valid from*

$$\mathrm{ZFC} + \text{“\textit{There is a supercompact cardinal}”}$$

δ is a supercompact cardinal, and that there is a supercompact cardinal below δ. Then for all $\lambda > \delta$, there is no non-trivial elementary embedding,

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Assuming the HOD Conjecture is provable then one can in a natural hierarchy of large cardinal axioms give a threshold for inconsistency in just ZF which more closely parallels the Kunen inconsistency in ZFC. There is a very suggestive coincidence. Essentially same threshold arises from a completely different conjecture and by a completely different argument. This is the conjecture on the weak uniqueness of square roots from Part II of this paper. Combined with the AD-Conjecture from that section, exactly the same threshold arises.

Assuming the Axiom of Choice, Kunen's Theorem shows that if

$$j : V_\gamma \to V_\gamma$$

is a non-trivial elementary embedding then $\gamma = \kappa_\omega(j)$ or $\gamma = \kappa_\omega(j) + 1$, where as defined before Theorem 226, $\kappa_\omega(j)$ denotes the ωth element of the critical sequence of j.

The proof of Kunen's Theorem does not obviously adapt to rule out any of the following axioms (for $n < \omega$) which are therefore, in the classical view of the large cardinal hierarchy, among the strongest large cardinal axioms believed to be consistent with the Axiom of Choice:

There exists $\lambda \in \mathrm{Ord}$ *such that*

$$V_\lambda \prec_{\Sigma_n} V$$

and such that there is a non-trivial elementary embedding

$$j : V_{\lambda+1} \to V_{\lambda+1}.$$

By Kunen's Theorem, necessarily $\lambda = \kappa_\omega(j)$. Notice that the requirement,

$$V_\lambda \prec_{\Sigma_n} V,$$

is equivalent to the requirements:

(1) $V_{\mathrm{CRT}(j)} \prec_{\Sigma_n} V$;
(2) for all $\alpha < \lambda$, if $V_\alpha \prec_{\Sigma_n} V$ then $V_{j(\alpha)} \prec_{\Sigma_n} V$.

Remark 234. (1) These axioms are strictly increasing in strength as a function of n (for $n \geq 2$).

(2) One can show that these axioms are all weaker in consistency strength that the Axiom I0 which asserts the existence of an elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}),$$

with $\mathrm{CRT}(j) < \lambda$.

If the HOD Conjecture is provable then these axioms cannot be significantly strengthened even without the Axiom of Choice. We note that the axiom implicit in Theorem 235 can be shown to imply the consistency *with* ZFC of the existence of an elementary embedding

$$j : \mathrm{HOD}_{V_{\lambda+1}} \to \mathrm{HOD}_{V_{\lambda+1}}$$

with $\mathrm{CRT}(j) < \lambda$.

Theorem 235 (ZF). *Suppose the* HOD *Conjecture is* Ω*-valid from* ZFC *and that*

$$V_\lambda \prec_{\Sigma_4} V.$$

Then there is no non-trivial elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

such that $\lambda = \kappa_\omega(j)$.

Proof. Note that λ must be a limit of supercompact cardinals in V_λ and for all $\delta < \lambda$, if δ is supercompact in V_λ then δ is supercompact in V. This is because $V_\lambda \prec_{\Sigma_4} V$. Assuming the Axiom of Choice, the requirement $V_\lambda \prec_{\Sigma_3}$ suffcies for this but without the Axiom of Choice the situation is not clear.

Let $\delta < \lambda$ be the least supercompact cardinal. By Theorem 229, there exists a transitive class $N \subset V$ and $X \in V_\delta$ such that the following hold.

(1.1) $N \vDash$ ZFC.
(1.2) For all limit $\alpha > \delta$, $N \cap V_\alpha$ is definable (uniformly) in V_α from X.
(1.3) There exists a partial order $\mathbb{P} \in N \cap V_\delta$ and $\kappa < \delta$ such that for all $A \subset$ Ord, $A \in N[G]$ for some N-generic filter $G \subset \mathbb{P}$.

Thus $\lambda^+ = (\lambda^+)^N$ and so the theorem follows by Theorem 228. □

8. Conclusions

We briefly discuss the results of [22] which concern the fine-structure of suitable extender models. The failure of comparison for coherent extender models provides constraints on exactly how a fine-structural suitable extender model can be constructed using extenders in V. The emerging picture from [22] is that assuming the Strong $(\omega_1 + 1)$-Iteration Hypothesis and that there is an extendible cardinal then there is a fine-structural suitable extender model $\mathbb{M} \subseteq$ HOD. However, at several times during the development of [22] it seemed one might actually refute such a general hypothesis by in effect proving too much comparison, in other words, by proving comparison for structures from which one can define too many coarse extender models and thereby obtain a contradiction. But (so far) each time upon closer inspection this was not the case.

Subject to very general constaints if there is fine-structural suitable extender model, $L[\mathbb{E}]$, (which is not a strategic extender model) then the original analysis of the previous versions of this paper applies yielding a number of corollaries in a stronger form than could be obtained before.

(1) $L[\mathbb{E}] \subset$ HOD and so the HOD Conjecture must hold in V.
(2) The Ω Conjecture holds in $L[\mathbb{E}]$.
(3) $\mathbb{R} \cap L[\mathbb{E}]$ is definable in $H(c^+)$ without parameters.
(4) 0^Ω is definable in $H(c^+)$ without parameters.
(5) $L[\mathbb{E}] = L[\mathbb{F}][G]$ for some fine structural suitable extender model $L[\mathbb{F}]$ and for some G which is $L[\mathbb{F}]$-generic for a partial order $\mathbb{P} \in L_{\delta_0+1}[\mathbb{F}]$ where δ_0 is the least Woodin cardinal of $L[\mathbb{E}]$ (and therefore of $L[\mathbb{F}]$).

(6) Let δ_0 be the least Woodin cardinal of $L[\mathbb{E}]$. Then for all singular cardinals λ of $L[\mathbb{E}]$ above δ_0, there is in $L[\mathbb{E}]$ a partition of

$$\{\gamma < \lambda^+ \mid \mathrm{cof}(\alpha) = \delta_0\}$$

into δ_0^{++} many stationary sets which is definable from parameters in $H(\lambda^+)$.

The conclusions (5) and (6) in fact argue for a strategic version of $L[\mathbb{E}]$ since that is the only possible variation of $L[\mathbb{E}]$ in which natural structural principles arising from generalizations of Axiom I0 can possibly hold, this will be discussed quite a bit more in Part II and [22].

As remarked in the introduction, a fine-structural inner suitable extender model $L[\mathbb{E}] \subseteq V$ would be a compelling analog of L in the context of $0^\#$ does not exist, but with no corresponding limitation on the large cardinals of V. Of course the same applies to the strategic version $L[\mathbb{E}, I_{\mathbb{E}}]$ if that actually exists, however the key difference (should both $L[\mathbb{E}]$ and $L[\mathbb{E}, I_{\mathbb{E}}]$ exist) is that the connections with the generalization of determinacy axioms to $V_{\lambda+1}$ could provide a justification for these strongest of large cardinal axioms in conjunction with the axiom that $V = L[\mathbb{E}, I_{\mathbb{E}}]$. Moreover (as we indicated in the introduction) the axiom $V = L_S^\Omega$ (i.e. the axiom $V = L[\mathbb{E}, I_{\mathbb{E}}]$) would simply be the conjunction of the following:

(1) There is a supercompact cardinal.
(2) There exist a universally Baire set $A \subset \mathbb{R}$ and $\gamma < \Theta^{L(A,\mathbb{R})}$ such that

$$V \equiv (\mathrm{HOD})^{L(A,\mathbb{R})} \cap V_\gamma$$

for all Π_2-sentences (equivalently, for all Σ_2-sentences).

Is this the axiom for V? At this stage there are certainly other candidates. For example consider the conjunction of:

(1) There is a supercompact cardinal.
(2) There exist a universally Baire set $A \subset \mathbb{R}$, $\gamma < \Theta^{L(A,\mathbb{R})}$ and an $L(A,\mathbb{R})$-generic filter $G \subset \mathbb{P}_{\max}$ such that

$$V \equiv (\mathrm{HOD}_{\{G\}\cup\mathbb{R}})^{L(A,\mathbb{R})[G]} \cap V_\gamma$$

for all Π_2-sentences.

Here $\mathbb{P}_{\max}$ is the partial order from [19]. Thus $\mathbb{P}_{\max} \in L(\mathbb{R})$ and assuming $\mathrm{AD}^{L(\mathbb{R})}$, $\mathbb{P}_{\max}$ is both ω-closed and homogeneous (in $L(\mathbb{R})$).

The assumption that L_S^Ω exists includes the assumption that at least for the universally Baire sets $A \subset \mathbb{R}$, $\mathrm{HOD}^{L(A,\mathbb{R})}$ admits a certain analysis. This is known to be true for a substantial initial segment of the universally Baire sets. As a consequence of this assumption, the second axiom actually implies that V is a homogeneous generic extension of an inner model in which the first axiom holds (with the partial order explicitly definable from the extender sequence and iteration strategy of the inner model). It also implies this inner model is itself uniformly definable in any generic extension of V (in fact uniformly Δ_3-definable). Very likely, this inner model is simply the intersection of the HOD's of all the generic extensions of V (this would require that the first axiom implies $V = \mathrm{HOD}$).

Of course the second axiom gives a canonical theory for $\mathcal{P}(\omega_1)$ which is (essentially) characterized by maximizing the Π_2-theory of $(H(\omega_2), \mathcal{I}_{\mathrm{NS}})$ where $\mathcal{I}_{\mathrm{NS}}$ is the nonstationary ideal on ω_1, this is discussed at length in [19]. But this fact leads to another difficulty with the second axiom and this difficulty also makes the second axiom an unlikely candidate for specifying V at this stage. The point is that the second axiom is an instance of a series of axioms each giving the same theory of $\mathcal{P}(\omega_1)$, this series is obtained by specifying naturally increasing closure properties for Γ^∞. Assuming the first axiom, $L(\Gamma^\infty, \mathbb{R}) = L(B, \mathbb{R})$ where B is the complete Σ^2_1 set as given by Γ^∞ and moreover $L(\Gamma^\infty) \vDash \mathrm{ZFC}$. Assuming the second axiom,

$$\Gamma^\infty = L(\Gamma^\infty) \cap \mathcal{P}(\mathbb{R})$$

and $\Theta^{\Gamma^\infty} = (\Theta_0)^{L(\Gamma^\infty)}$. One can easily vary the second axiom and arrange that Γ^∞ more closely approximates the asymptotic version given by the Sealing Theorem, Theorem 33. For example one could naturally arrange both that

$$\Gamma^\infty = L(\Gamma^\infty) \cap \mathcal{P}(\mathbb{R})$$

and that

$$L(\Gamma^\infty) \vDash \mathrm{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}.$$

But then one has to require more than the existence of an $L(\Gamma^\infty)$-generic filter $G \subset \mathbb{P}_{\max}$ in order to have the Axiom of Choice; one needs something like an $L(\Gamma^\infty)$-generic filter $G \subset \mathbb{P}_{\max} * \mathrm{Coll}(\Theta, \Theta)$ and this particular choice has the advantage of yielding Martin's Maximum(c).

The scenario that one of these variations of the second axiom gets singled out by structural considerations arising from generalizations of the Axiom I0 (see part II) seems rather unlikely. This leaves the first axiom as the more likely contender at this stage.

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