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The name for Kojman–Shelah collapsing function

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Dedicated to our teacher and friend Petr Vopěnka

Abstract

In the previous paper of this volume, Kojman and Shelah solved our long standing problem of collapsing cardinal $\kappa^{\aleph_0}$ to ω_1 by the forcing $([\kappa]^\kappa, \subseteq)$ for singular κ with countable cofinality. The aim of the present paper is to give an explicit construction of the Boolean matrix for this collapse. © 2001 Published by Elsevier Science B.V.

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1. Introduction

In early 1970s, Petr Vopěnka initiated the study of infinities by examining structural properties of quotient Boolean algebras $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ with emphasis on the properties of generic extension via this forcing notion. First result [5, 6] concerns all uncountable regular cardinals under GCH and says that $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is forcing equivalent to the standard collapsing of 2^κ to ω by finite functions. By the way, it should be clear that $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is forcing equivalent to the (non-separative) partial order $([\kappa]^\kappa, \subseteq)$. For infinite cardinals $\kappa<\lambda$, κ regular, $Col(\kappa, \lambda)$ stands for the complete Boolean algebra containing a partial order $Fn(\kappa, \lambda, \kappa)$ (see [7, p. 271]) as a dense part. It is a standard algebra for adding a function from κ onto λ by conditions of size $<\kappa$.

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The general question what is collapsed to where when forcing with $([\kappa]^\kappa, \subseteq)$ is still not completely answered. Known results just in ZFC are:

(i) $([\omega]^\omega, \subseteq)$ collapses 2^ω to $\mathfrak{h}$ [1, 3], and this result cannot be improved;

(ii) for κ regular uncountable, $([\kappa]^\kappa, \subseteq)$ collapses $\mathfrak{b}_\kappa$ to ω [2]. Here, $\mathfrak{b}_\kappa$ is not known to be the best value. Other reasonable candidates to replace the unboundedness number $\mathfrak{b}_\kappa$, the almost disjoint number $\mathfrak{a}_\kappa$ and the cellularity of $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$, present an open problem;

(iii) for κ singular with $cf(\kappa) = \omega$, $([\kappa]^\kappa, \subseteq)$ collapses κ^ω to ω_1, a brandly new result [8] in this volume;

(iv) the case of κ singular with uncountable cofinality is announced in [8], namely, $([\kappa]^\kappa, \subseteq)$ collapses κ^+ to ω.

(Let us remark another open problem for uncountable κ: Is the cellularity of $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ always attained?)

The remarkable recent result of Menachem Kojman and Saharon Shelah (iii) proves the existence of collapsing function in generic extension by constructing a special type of a gap in $\prod_{n\in\omega} \kappa_n$.

In the present paper, we wish to construct a Boolean matrix showing that a partial order Q is $(\omega_1, \cdot, \kappa^\omega)$-nowhere distributive, which is in this case equivalent to the description of the name of a collapsing function from κ^ω to ω_1.

Let us complete the picture by adding that under the assumption $2^\kappa = \kappa^+$, the forcing $([\kappa]^\kappa, \subseteq)$ is forcing equivalent either to $Col(\omega, 2^\kappa)$ or to $Col(\omega_1, 2^\kappa)$, the latter possibility takes place if only if $cf(\kappa) = \omega$ [2]. Looking for an isomorphism between these two algebras is hopeless in ZFC, but the next best is true. $Col(\omega_1, \kappa^\omega)$ is a regular subalgebra of the completion of $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ if $\omega = cf(\kappa) < \kappa$ [8].

Notation and basic facts. In the rest of this paper, κ stands for an uncountable cardinal with countable cofinality. Following [8], denote by $P = ([\kappa]^\kappa, \leqslant)$ the quasi ordered set $[\kappa]^\kappa$ with an order $p \leqslant q$ if $|p \setminus q| < \kappa$. It is easy to see that the regular open algebra $RO(P)$ is isomorphic to the completion of the algebra $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$.

Denote by Q the suborder of P consisting of all closed members of P, where the closure is taken in the standard interval topology on a set $\{\alpha : \alpha \text{ is an ordinal}, \alpha < \kappa\} = \kappa$. Notice that Q is not dense in P, but the complete Boolean algebra $RO(Q)$ is regularly (= completely) embeddable into $RO(P)$ [8].

The crucial properties of Q are listed as follows:

(i) Q is homogeneous, i.e., whenever $q \in Q$, then $Q \restriction q$ is isomorphic to Q;

(ii) Q is σ-closed;

(iii) there is a disjoint system in Q of size κ^ω.

An important feature of Q, not shared by P, is that

(iv) Q has a dense subset of size κ^ω. The proof of this fact can be found in [8].

The result. We aim to show explicitly that the Boolean algebra $RO(Q)$ is $(\omega_1, \cdot, \kappa^\omega)$-nowhere distributive. Hence we need to prove:

Theorem. *There exists a family* $\{Q_\alpha : \alpha < \omega_1\}$ *with the following properties*:
(i) *for each* $\alpha < \omega_1$, Q_α *is a maximal almost disjoint* (*modulo* $< \kappa$) *family of members of* Q;
(ii) *for every* $q \in Q$ *there is some* $\alpha < \omega_1$ *with* $|\{p \in Q_\alpha : |p \cap q| = \kappa\}| = \kappa^\omega$.

This is less than to produce a name for a collapsing function from ω_1 onto κ^ω in V^Q. However, this statement together with (i)–(iv) above imply the existence of an isomorphism between the complete Boolean algebras $Compl(Q, \leqslant)$ and $Col(\omega_1, \kappa^\omega)$ by Corollary 1.15 in [3].

The proof will be given by a series of claims ending with an almost known proposition. Fix a strictly increasing sequence of regular cardinals $\langle \kappa_n : n \in \omega \rangle$ with $\sup_{n\in\omega} \kappa_n = \kappa$ and with $\kappa_0 > \omega$. Call a set $q \in P$ *a normal condition*, if there is a strictly increasing mapping $f \in {}^\omega\omega$ with $n < f(n)$ for all $n \in \omega$ and a sequence of closed sets $L_n(q)$ such that
(a) for each $n \in \omega$, $L_n(q) \subseteq [\kappa_{f(n)}, \kappa_{f(n)+1})$;
(b) for each $n \in \omega$, $\mathrm{otp}(L_n(q)) = \kappa_n + 1$;
(c) $q = \bigcup_{n\in\omega} L_n(q)$.

It should be clear that every normal condition belongs to Q and that any $q \in Q$ contains some normal condition as a subset.

Let Q_0 be an arbitrary maximal almost disjoint family in Q consisting of normal conditions.

Before proceeding further, let us introduce a relation $\lhd$ between normal conditions as follows.

Definition. If $p = \bigcup_{n\in\omega} L_n(p)$ and $q = \bigcup_{n\in\omega} L_n(q)$ are normal conditions, we shall write $p \lhd q$ if $p \leqslant q$ and for each $n, m \in \omega$ we have
(1) either $L_n(p) \cap L_m(q) = \emptyset$, or
(2) $L_n(p) \subseteq L_m(q)$ and $n < m$.

Notice that in general, $\lhd$ is not a transitive relation, because the relation of disjointness is not transitive. Clearly, if $p \lhd q$, then $|q \setminus p| = \kappa$.

Our aim is to find maximal almost disjoint families Q_α ($\alpha < \omega_1$), consisting of normal conditions, such that for all $\alpha < \beta < \omega_1$, $p \in Q_\alpha$, $q \in Q_\beta$, either $q \lhd p$ or $|q \cap p| < \kappa$.

Claim 1. *Suppose* $\alpha < \omega_1$ *and suppose that* $\{q_\beta : \beta < \alpha\}$ *is a set of normal conditions satisfying* $q_\gamma \leqslant q_\beta$ *whenever* $\beta < \gamma < \alpha$. *Let* $p \in Q$ *be such that* $|p \cap q_\beta| = \kappa$ *for all* $\beta < \alpha$. *Then there is a normal condition* q *such that* $q \leqslant p$ *and* $q \lhd q_\beta$ *for all* $\beta < \alpha$.

If, in addition, α *is limit and* α_n *is a sequence of type* ω *converging to* α, *then* q *can satisfy also* $|q \setminus q_{\alpha_n}| < \kappa_n$ *whenever* $n \in \omega$.

Proof. We shall show the case of a limit α only. Looking for the normal condition $q = \bigcup_{n\in\omega} L_n(q)$, proceed by an induction. Suppose that $n \in \omega$ is such that for all $k < n$ we already know the sets $L_k(q)$ and the values $f(k)$ of a mapping f; we have to find $f(n)$ and $L_n(q)$. Since the set $p \cap \bigcap_{k \leqslant n} q_{\alpha_k}$ is closed and has the size κ, there is some natural number $f(n) > f(n-1)$ such that $[\kappa_{f(n)}, \kappa_{f(n)+1}) \cap p \cap \bigcap_{k<n} q_{\alpha_k}$ contains a topological copy C_n of κ_{n+1}. Put $J_n = \{\beta < \alpha : q_\beta \cap C_n$ is bounded in $C_n\}$. Since J_n is countable, there is some $\xi < \sup C_n$ such that $(C_n \backslash \xi) \cap q_\beta = \emptyset$ for all $\beta \in J_n$. The set $C = (C_n \backslash \xi) \cap \bigcap \{q_\beta : \beta \in \alpha \backslash J_n\}$ is closed unbounded in C_n, since for each $\beta < \alpha$, $\beta \notin J_n$, the set $C_n \cap q_\beta$ is. Since C is homeomorphic to κ_{n+1}, it contains an initial segment $L_n(q)$ homeomorphic to $\kappa_n + 1$.

It is immediate from the construction that $q \lhd q_{\alpha_n}$ and that $|q \backslash q_{\alpha_n}| < \kappa_n$ for all $n \in \omega$. If $\beta < \alpha$ is arbitrary, choose $k \in \omega$ so that $\beta < \alpha_k$. Since $q_{\alpha_k} \leqslant q_\beta$, there is some $m \in \omega$ such that $|q_{\alpha_k} \backslash q_\beta| < \kappa_m$. Now, whenever $n \in \omega$, $n > \max\{k, m\}$, then $\beta \notin J_n$ and consequently $q \lhd q_\beta$. The claim follows. $\square$

To define the remaining Q_α's, we shall apply Claim 1 repeatedly. If Q_α is known and $r \in Q_\alpha$ is arbitrary, let $\mathscr{Q}(r)$ be a maximal almost disjoint collection of normal conditions $q \in Q$ such that $q \subseteq r$ and $q \lhd s$ whenever $s \in Q_\beta$, $\beta \leqslant \alpha$, $r \leqslant s$. Put $Q_{\alpha+1} = \bigcup \{\mathscr{Q}(r) : r \in Q_\alpha\}$. Note that for $q \in \mathscr{Q}(r)$, $|r \backslash q| = \kappa$ by (2) from the definition of $\lhd$.

If $\alpha < \omega_1$ is limit, fix some sequence $\langle \alpha_n : n \in \omega \rangle$ converging to α. For every $\lhd$-decreasing chain $\mathscr{C} = \{q_\beta : \beta < \alpha$ and $q_\beta \in Q_\beta\}$ choose a maximal almost disjoint collection $\mathscr{Q}(\mathscr{C})$ of $q \in Q$ which satisfy $q \lhd q_\beta$ for all $q_\beta \in \mathscr{C}$, and $|q \backslash q_{\alpha_n}| < \kappa_n$ for all $n \in \omega$. Put $Q_\alpha = \bigcup \{\mathscr{Q}(\mathscr{C}) : \mathscr{C}$ is a $\lhd$-decreasing chain in $\bigcup_{\beta<\alpha} Q_\beta\}$. This completes the inductive definitions of Q_α's.

It remains to verify that the family $\{Q_\alpha : \alpha < \omega_1\}$ is a witness of $(\omega_1, \cdot, \kappa^\omega)$-nowhere distributivity of Q. Other claims will successively approach this goal.

Claim 2. *For each $\alpha < \omega_1$, Q_α is a maximal almost disjoint family in Q, i.e., whenever $p \in Q$, then $|p \cap q| = \kappa$ for some $q \in Q_\alpha$.*

Proof. By our choice in the 0th step of the induction, Claim 2 holds true for Q_0. Suppose $\alpha < \omega_1$ and Q_α to be maximal almost disjoint in Q, let $p \in Q$ be arbitrary. By maximality of Q_α, there is some $q \in Q_\alpha$ with $|q \cap p| = \kappa$. Consider the family $\mathscr{Q}(q)$ for this q. By Claim 1 applied to the condition $q \cap p$, there is some normal condition $q_0 \subseteq q \cap p$ satisfying $q_0 \lhd q$ and $q_0 \lhd r$ for every $r \in Q_\beta$, $\beta < \alpha$, $q \leqslant r$. The family $\mathscr{Q}(q)$ was maximal, so $|q_0 \cap s| = \kappa$ for some $s \in \mathscr{Q}(q)$. Consequently $|p \cap s| = \kappa$, too, and $s \in \mathscr{Q}(q) \subseteq Q_{\alpha+1}$. Similar use of Claim 1 applies also for $\alpha < \omega_1$, α limit. $\square$

Claim 3. *Let $\alpha \leqslant \omega_1$ be a limit ordinal and suppose that $q_\beta \in Q_\beta$ for $\beta < \alpha$ were chosen such that $q_\gamma \lhd q_\beta$ whenever $\beta < \gamma < \alpha$. If $\xi \in \kappa$, then the set $\{\beta < \alpha : \xi \in q_\beta\}$ is finite.*

Proof. Since each q_β is normal, $q_\beta = \bigcup_{n\in\omega} L_n(q_\beta)$. Whenever $\beta < \alpha$ is such that $\xi \in q_\beta$, denote by $n(\beta)$ the unique integer satisfying $\xi \in L_{n(\beta)}(q_\beta)$. If $\beta < \gamma < \alpha$, $q_\gamma \lhd q_\beta$ and $\xi \in L_{n(\beta)}(q_\beta) \cap L_{n(\gamma)}(q_\gamma)$, then $L_{n(\beta)}(q_\beta) \cap L_{n(\gamma)}(q_\gamma)$ is non-empty and thus, by the definition of $\lhd$, $n(\gamma) < n(\beta)$. So the set $\{\beta < \alpha : \xi \in q_\beta\}$ must be finite, because there is no strictly decreasing infinite sequence of natural numbers. $\square$

Claim 4. *Let $p \in Q$ be arbitrary. Then there is some $\alpha \in \omega_1$ such that*

$$|\{q \in Q_\alpha : |q \cap p| = \kappa\}| \geqslant 2.$$

Proof. We may assume without any loss of generality that p is a normal condition. Using the maximality of each Q_α, select by a transfinite induction a $\lhd$-decreasing chain $\{q_\alpha \in Q_\alpha : \alpha < \omega_1\}$ such that $|q_\alpha \cap p| = \kappa$ for each $\alpha < \omega_1$.

The condition p is normal, $p = \bigcup_{n\in\omega} L_n(p)$. Put $\xi_n = \max L_n(p)$. By Claim 3, each set $I_n = \{\alpha < \omega_1 : \xi_n \in q_\alpha\}$ is finite, so there is some $\alpha \in \omega_1 \setminus \bigcup_{n\in\omega} I_n$. For this α, $\{\xi_n : n \in \omega\} \cap q_\alpha = \emptyset$. Since the set q_α is closed, for every $n \in \omega$ there is some $\eta_n < \xi_n$ such that the closed neighbourhood $[\eta_n, \xi_n]$ of a point ξ_n is disjoint with q_α.

Clearly, the set $\bigcup_{n\in\omega}(L_n(p) \cap [\eta_n, \xi_n])$ is a normal condition disjoint with q_α. By the maximality of Q_α, there is some $s \in Q_\alpha$ with $|s \cap (\bigcup_{n\in\omega} L_n(p) \cap [\eta_n, \xi_n])| = \kappa$. So p meets at least two members of Q_α, namely q_α and s, in size κ. $\square$

Claim 5. *Let $p \in Q$ be arbitrary. Then there is some $\alpha < \omega_1$ such that*

$$|\{q \in Q_\alpha : |q \cap p| = \kappa\}| \geqslant \kappa^+.$$

Proof. First, apply Claim 4 inductively: There is some $\alpha(0) < \omega_1$ and two distinct conditions $p_0, q_0 \in Q_{\alpha(0)}$ such that $|p_0 \cap p| = |q_0 \cap p| = \kappa$. If $\alpha(n) < \omega_1$ and $p_n, q_n \in Q_{\alpha(n)}$ are known, let $\alpha(n+1)$ be such that there are two distinct $p_{n+1}, q_{n+1} \in Q_{\alpha(n+1)}$ with $|p_{n+1} \cap q_n \cap p| = |q_{n+1} \cap q_n \cap p| = \kappa$. Put $\alpha = \sup_{n\in\omega} \alpha(n)$.

Second, let us prove that this α is as required. To this end, consider the family $\mathscr{C} = \{q \in \bigcup_{\beta<\alpha} Q_\beta : \text{for some } n \in \omega,\ q_n \lhd q\}$. It is enough to show that $|\{q \in \mathscr{Q}(\mathscr{C}) : |q \cap p| = \kappa\}| \geqslant \kappa^+$. Suppose not, then we may enumerate $\{q \in \mathscr{Q}(\mathscr{C}) : |q \cap p| = \kappa\} = \{s_\xi : \xi < \kappa\}$. According to the definition of the family $\mathscr{Q}(\mathscr{C})$, and by the fact that both sequences $\langle \alpha_n : n \in \omega \rangle$ and $\langle \alpha(n) : n \in \omega \rangle$ are cofinal in α, we are allowed to assume that for each $q \in \mathscr{Q}(\mathscr{C})$ and for each $n \in \omega$, $|q \setminus q_n| < \kappa_n$.

Let $r_0 = p_0 \cap p$, $r_{n+1} = p_{n+1} \cap q_n \cap p$. Each set r_n is closed and its size equals to κ, moreover, $|r_n \cap r_m| < \kappa$ whenever $n \neq m$. Thus there is a pairwise disjoint family of normal conditions, $\{t_n : n \in \omega\}$ with $t_n \subseteq r_n$ for all $n \in \omega$. Choose inductively a set $L_n \subseteq t_n$ such that L_n is a topological copy of κ_{n+1}, and $\sup L_n < \min L_{n+1}$.

Since for each $\xi < \kappa$ and $n \in \omega$ we have $|s_\xi \setminus q_n| < \kappa_n$, the set L_n contains a closed copy of $\kappa_n + 1$, say K_n, disjoint with $\bigcup\{s_\xi : \xi < \kappa_n\}$.

Consider now the condition $\bigcup_{n\in\omega} K_n$. It is a subset of p, and it is almost disjoint with each member of $\mathscr{Q}(\mathscr{C})$, which meets p. This, however, contradicts the assumption of the maximality of $\mathscr{Q}(\mathscr{C})$. $\square$

The Theorem follows now from the forthcoming more general Proposition applied to $Q = R$, $\lambda = \kappa$.

Proposition. *Let $\lambda \geqslant 2$ be a cardinal number. Assume that $R = (R, \leqslant)$ is a partial order which is*
(i) *σ-closed, and*
(ii) *$(\omega_1, \cdot, \lambda)$-nowhere distributive.*
Then R is $(\omega_1, \cdot, \lambda^\omega)$-nowhere distributive.

Proof. There is nothing to prove if $\lambda^\omega = \lambda$, so assume that $\lambda^\omega > \lambda$. Let $\tau \leqslant \lambda$ be the smallest cardinal satisfying $\tau^\omega \geqslant \lambda$. Clearly, $\tau^\omega = \lambda^\omega$. By Hausdorff formula from cardinal arithmetic, two cases are possible. Either $\tau = 2$ or $cf(\tau) = \omega$.

Let $\{A_\alpha : \alpha < \omega_1\}$ be a family of maximal disjoint (= consisting of pairwise incompatible elements) subsets of R witnessing to the fact that R is $(\omega_1, \cdot, \lambda)$-nowhere distributive. Since R is σ-closed, we are allowed to assume moreover that for $\alpha < \beta < \omega_1$ and $r \in A_\beta$, $s \in A_\alpha$, either $r \leqslant s$ or r, s are incompatible, i.e., A_β refines A_α.

Case 1: $\tau = 2$. Classical branching argument. Let $s \in R$ be arbitrary. Denote $s_\emptyset = s$. Choose by induction using the $(\omega_1, \cdot, 2)$-nowhere distributivity ordinals α_n, elements $r_\varphi \in A_{\alpha_n}$ and $s_\varphi \in R$ for $\varphi \in {}^n2$ so that for $n < m < \omega$, $\varphi \in {}^n2$, $\psi, \chi \in {}^m2$, $\varphi \subseteq \psi$ we have $r_\psi \leqslant r_\varphi$, $s_\psi \leqslant s_\varphi$, $s_\psi \leqslant r_\varphi$, s_ψ is compatible with r_ψ and if $\psi \neq \chi$, then r_ψ and r_χ are incompatible.

Then for $\alpha = \sup\{\alpha_n : n \in \omega\}$ and $f \in {}^\omega 2$, there is by the maximality of A_α an element $r_f \in A_\alpha$ compatible with all $s_{f \restriction n}$, $n \in \omega$, so s is compatible with all $s_f \in A_\alpha$, $f \in {}^\omega 2$, which shows that R is $(\omega_1, \cdot, 2^\omega)$-nowhere distributive.

Case 2: $cf(\tau) = \omega$. Choose a strictly increasing sequence $\{\tau_n : n \in \omega\}$ of regular cardinals converging to τ, $\tau_0 > \omega$. Let $s \in R$ be again arbitrary.

Analogously to the previous case, denote $s_\emptyset = s$. We shall find for $n \in \omega$ and $\varphi \in \prod_{i<n} \tau_i$ an ordinal $\alpha < \omega_1$ and elements $r_\varphi \in A_\alpha$ and $s_\varphi \in R$. We denote $\alpha = F(\varphi)$.

The induction step is as follows. Suppose s_φ for $\varphi \in \prod_{i<n} \tau_i$ is known, $s_\varphi \leqslant r_\varphi$ and $r_\varphi \in A_{F(\varphi)}$. Then there is some $\alpha < \omega_1$ such that the set $\{r \in A_\alpha : r \text{ is compatible with } s_\varphi\}$ is of size $\geqslant \tau_n$ because of the $(\omega_1, \cdot, \lambda)$-nowhere distributivity of R and by the fact that $\tau_n < \tau \leqslant \lambda$. Choose the smallest $\alpha < \omega_1$ with this property and put $\alpha = F(\varphi \cup \{\langle n, \xi \rangle\})$ for all $\xi < \tau_n$. Next, choose pairwise incompatible elements $r_{\varphi \cup \{\langle n, \xi \rangle\}} \in A_\alpha$ so that for each $\xi < \tau_n$, $r_{\varphi \cup \{\langle n, \xi \rangle\}}$ is compatible with s_φ. Finally, choose $s_{\varphi \cup \{\langle n, \xi \rangle\}}$ such that $s_{\varphi \cup \{\langle n, \xi \rangle\}} \leqslant r_{\varphi \cup \{\langle n, \xi, \rangle\}}$ and $s_{\varphi \cup \{\langle n, \xi \rangle\}} \leqslant s_\varphi$. This completes the inductive definitions.

Now, for each $f \in \prod_{n<\omega} \tau_n$, we have an increasing sequence $S(f)$ of countable ordinals, $S(f) = \{F(f \restriction n) : n \in \omega\}$. Notice that the definition of F immediately implies that S is a continuous mapping from $\prod_{n \in \omega} \tau_n$ to ω_1^ω, where all ordinals are considered with discrete topology, products equipped by Tychonoff topology.

We have $\omega_1^\omega = 2^\omega$ countable sequences in ω_1 and $2^\omega < \tau^\omega = |\prod_{n \in \omega} \tau_n|$. So there is a strictly increasing sequence of countable ordinals $\bar{\alpha} = \{\alpha_n : n \in \omega\}$ such that

$$\left| \left\{ f \in \prod_{n \in \omega} \tau_n : \bar{\alpha} = S(f) \right\} \right| > \tau.$$

The set $M = \{f \in \prod_{n\in\omega} \tau_n : \bar{\alpha} = S(f)\}$ is closed in the product $\prod_{n\in\omega} \tau_n$, since it is a preimage of a one-point set $\{\bar{\alpha}\}$ under a continuous mapping.

Put $M_0 = \{f \in M : (\exists n \in \omega)|\{g \in M : g \supseteq f \upharpoonright n\}| \leqslant \tau\}$, $M_1 = M \setminus M_0$. Clearly, $|M_0| \leqslant \tau$, M_1 is closed and $|M_1| > \tau$.

Let

$$T = \{f \upharpoonright n : n \in \omega,\ f \in M_1\} \subseteq \bigcup_{n\in\omega} \prod_{i<n} \tau_n.$$

Notice that the set M_1 equals to the set $[T]$ of all branches in the tree T. From the inequality $\tau_n^\omega < \tau$, which holds for all $n \in \omega$, we get that for every $\varphi \in T$ and for every $n \in \omega$ there is some $\psi \supseteq \varphi$, $\psi \in T$, such that ψ has at least τ_n immediate successors in T. Therefore $|[T]| = \tau^\omega$ and so $[M] = \tau^\omega = \lambda^\omega$.

To conclude the proof, put $\beta = \sup \bar{\alpha}$. For $f \in M$, let $s_f \in R$ be such that $s_f \leqslant s_{f\upharpoonright n}$ for all $n \in \omega$ and choose $r_f \in A_\beta$ compatible with s_f. Now, the members r_f and r_g are incompatible for distinct $f, g \in M$, but compatible with s_f and s_g respectively, hence with s, too. □

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