How to have more things by forgetting where you put them

Mike Oliver

March 17, 2010

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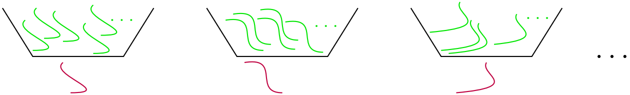
- In a picture like this:



can there ever be, in any sense, more purple squiggles than green ones?

Doubly infinite strings

The “things” I want to consider are doubly infinite strings on a finite alphabet, which might as well be the two letters 0 and 1:

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Formally, these can be represented as maps from the integers to the set $\{0,1\}$; each string is a function $f : \mathbb{Z} \to \{0,1\}$.

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Or... can they?

Actually, given such an f, there's a particular value for $f(0)$. So really one of these maps f is more like:

$$\ldots 10111101{\color{red}0}00100010100001\ldots$$

where $f(0)$ is marked in red, $f(1)$ is the letter to the right of the red one, $f(-1)$ is the one to the left of the red one, and so on.

Forgetting the origin

So, if we really want to model doubly infinite strings, we have to forget where the origin is. That is, we want

$$\ldots 111111111\underline{001110\textcolor{red}{0}}1111111111111\ldots$$

with the 1s repeating infinitely in both directions, to be the same string as

$$\ldots 1111\underline{0011100}1111\textcolor{red}{1}1111111111111\ldots$$

again with 1s stretching out to infinity.

The underlining is not meant to have any significance for the string — it's just to make it visually clear why the two strings are the same. Or rather, would be the same, if we could forget which letter is red.

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Forgetting is hard

Well, not really *hard*, but the formalities might not be obvious if you haven't done them before.

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- Given f and g, if, say, you can shift f by three spaces and get g; that is, $f(n+3) = g(n)$ for *every* integer n, then f and g are equivalent, $f \sim g$. Or, any other integer in place of the 3.
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- Then, formally, our objects are the quotient of the functions from $\mathbb{Z}$ to $\{0, 1\}$, modulo $\sim$.
- But mostly we won't get that formal. We'll just require that all "well-defined" questions about doubly infinite strings, *must give the same answer* for f as for g, if f and g are shifts of one another.

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These facts can be seen via very simple arguments due to Georg Cantor, which you have all likely seen.

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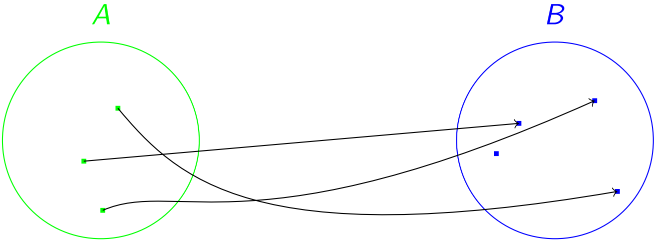

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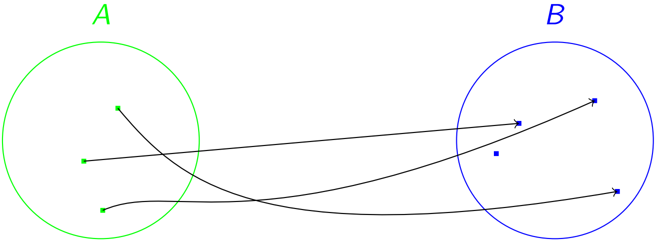

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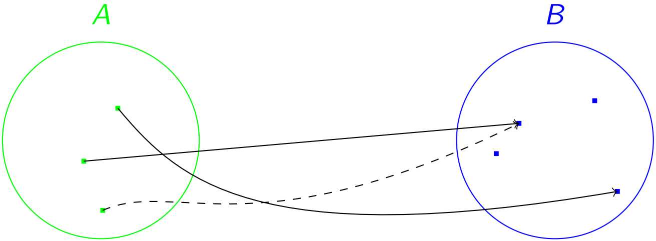

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  - That is, you can't do this.

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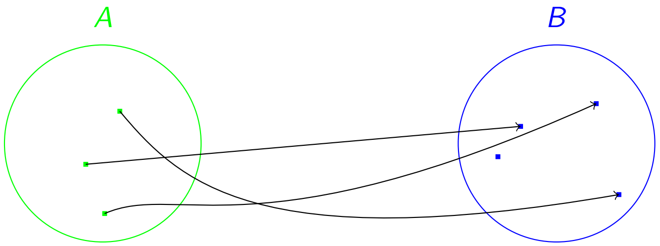

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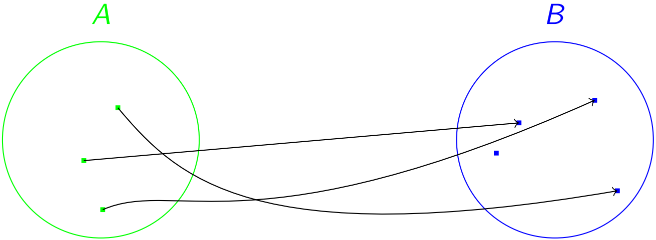

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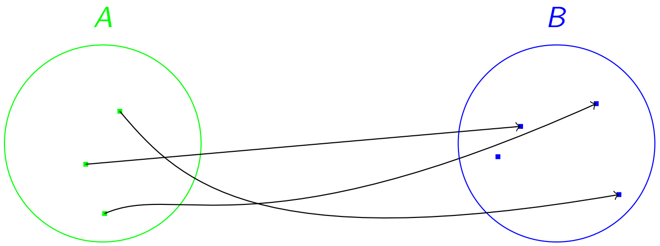

- The map doesn't have to be a *rule* of any sort. It can just be an arbitrary bunch of arrows.
- However, we *are* sometimes interested in whether there's a "reasonably definable" rule saying where the arrows go. (If we weren't, this talk would be very short.)

Probabilities, with origin

Suppose we make a doubly infinite string by flipping a coin. Heads, we set $f(0)$ to 1; tails, to 0. Flip again to get $f(1)$, then again for $f(-1)$, then $f(2)$, $f(-2)$, and so on:

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As it turns out, for any "reasonably definable" probability question we can ask about the forgotten-origin string, the answer is always exactly 0 or exactly 1. (This is called *ergodicity*.)

How many doubly-infinite strings?

It's easy to count the number of doubly-infinite strings. There are exactly as many as there are *singly*-infinite strings; namely, $2^{\aleph_0}$. Remember that to show that the number of doubly-infinite strings is $\leq$ the number of singly-infinite strings, we just have to find a map that converts a doubly-infinite string to a *unique* singly-infinite string.

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$$\textcolor{red}{1}1011000\underline{1} \qquad\qquad \ldots$$

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It's easy to count the number of doubly-infinite strings. There are exactly as many as there are *singly*-infinite strings; namely, $2^{\aleph_0}$. Remember that to show that the number of doubly-infinite strings is $\leq$ the number of singly-infinite strings, we just have to find a map that converts a doubly-infinite string to a *unique* singly-infinite string.

$$\ldots \quad 1010111000\underline{0} \quad \ldots$$
$$1101100010\underline{0} \qquad\qquad \ldots$$

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$$\begin{array}{lll} \ldots & 1010\textcolor{red}{1}11000 & \ldots \\ \textcolor{red}{1}101100010 & & \ldots \end{array}$$

...but of course this requires knowing which letter of the doubly-infinite string is red.

So what if we don't know the origin?

- If we don't have origins for the doubly-infinite strings, then we can't use the trick of the previous slide to assign a unique singly-infinite string to each doubly-infinite string, because we don't know where to start. But maybe it can still be done, just in some more complicated way.

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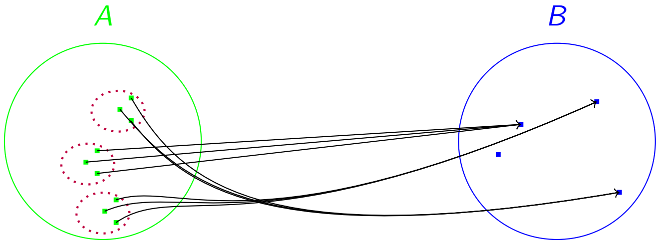
- But if f is not a shift of g, then $F(f) \neq F(g)$ (that's the uniqueness).

Revisiting cardinality, for sets of equivalence classes



- In this picture, the number of equivalence classes (dotted ovals) in A is $\leq$ the number of points in B

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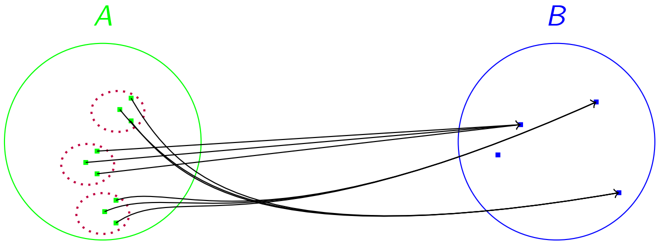

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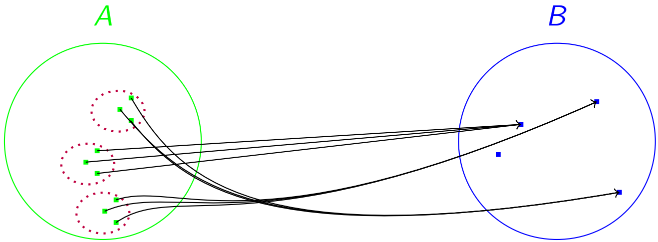

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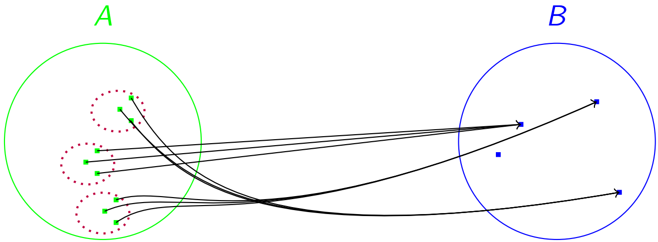

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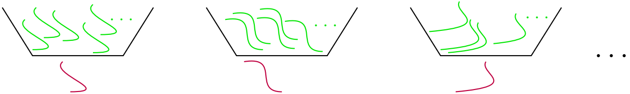
- Think of the ovals as representing strings-without-origin; each green dot is a string-with-origin
- Two green dots in the *same* oval must match to the *same* blue dot
- But green dots in *different* ovals must match to different blue dots

Is there such a map?



- Yes, there is.

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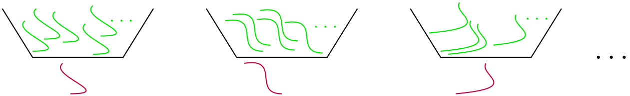

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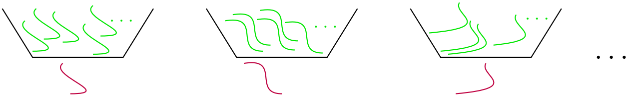

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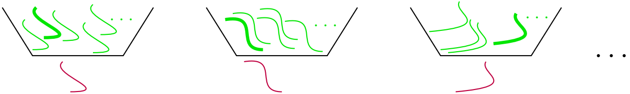

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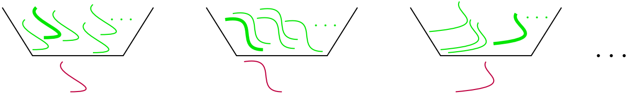

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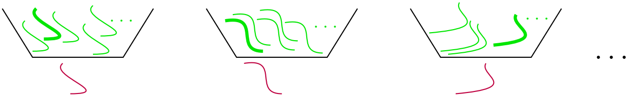

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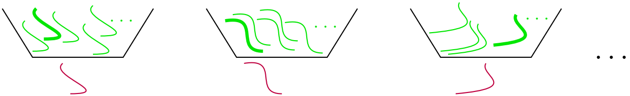

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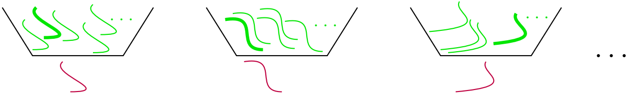

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However this doesn't appear to help us if we want a "reasonably definable" such map. The axiom just tells us that there *is* a way of distinguishing one string in each box; it doesn't tell us how to do it. Of course, it also doesn't tell us that there *isn't* a definable way to do it.

Tying things together (with strings)

So suppose we *do* have a "reasonably definable" map F that shows that there are only as many forgotten-origin doubly-infinite strings as there are singly-infinite strings. What can we find out about it? For example, we might want to know, if you put a doubly-infinite string s into F, giving a singly-infinite string $F(s)$, does $F(s)$ start with a 1? Of course, we expect the answer to depend on s....

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But! That's a "reasonably definable" question.

So the probability that $F(s)$ starts with 1, for a *random* string s, is either exactly 0 or exactly 1.

Building a special string, based on F

So now let's look at the probability that the nth letter of $F(s)$ is 1 (for a random doubly-infinite string s)

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```
Probability that              0
bit number 0 is 1:
Special string:               0       ...
```

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`Probability that bit number 1 is 1:`	0	
`Special string:`	0<u>0</u>	...

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`Special string:`	00<u>1</u> ...

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```
Probability that              0
bit number 3 is 1:
Special string:               0010    ...
```

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`Probability that bit number 4 is 1:`	`1`
`Special string:`	`00101 ...`

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```
Probability that              1
bit number 5 is 1:
Special string:               001011 ...
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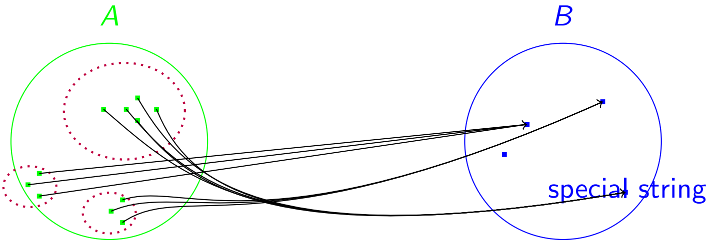
  - ... and probability is countably additive

Picture of F



- Again, the green dots are strings-with-origin; the ovals around them gather them into strings-without-origin

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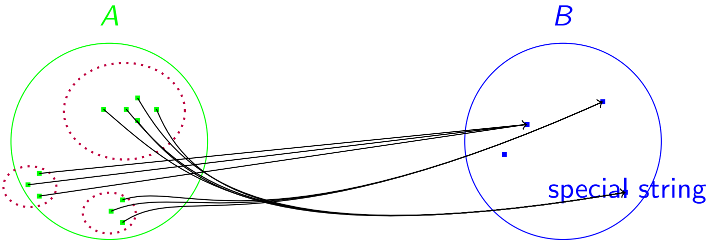

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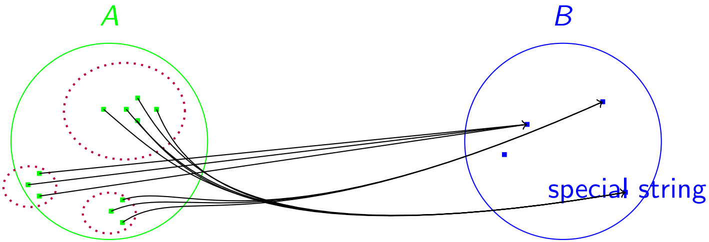

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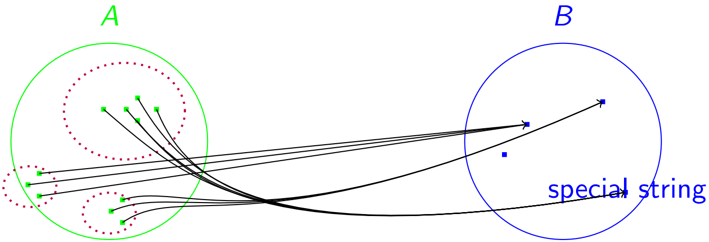

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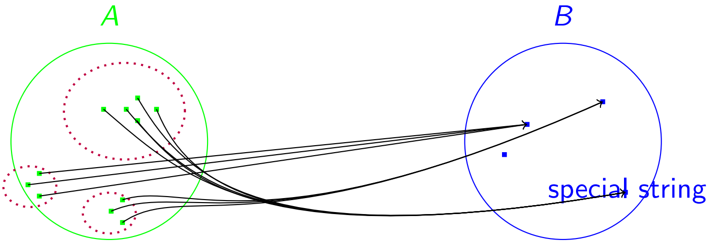
- *Almost everything* on the left, gets mapped to the special string
- So a random green dot has probability one of being in the big oval

Is this possible?



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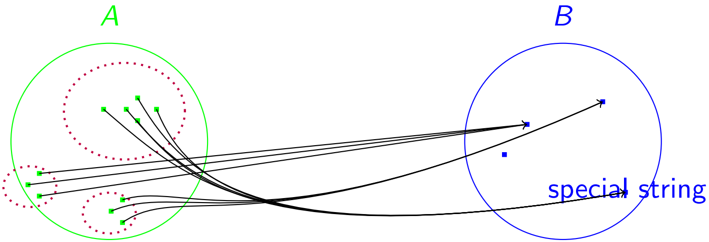

- No

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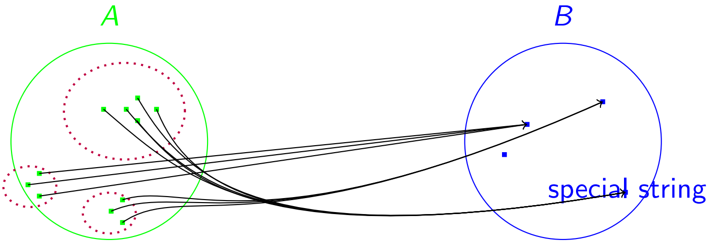

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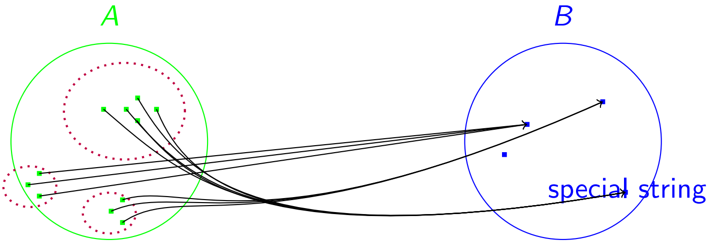

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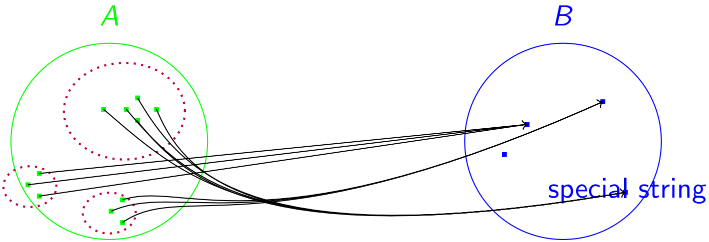

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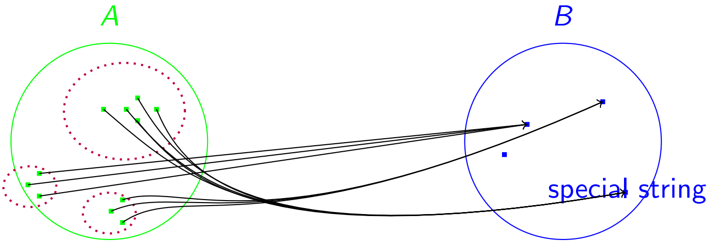

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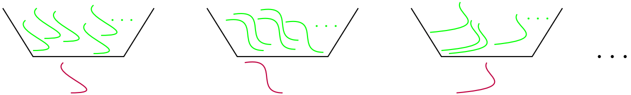
So we have a contradiction on our hands. The only way to resolve it is that no such function F exists.

I hope you appreciate just how very strange this is



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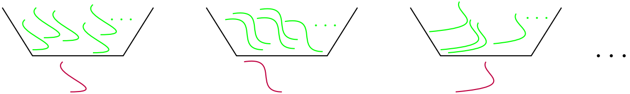

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Wait a minute, did we exactly show *more*?

Well, no, not quite. We showed that the number of forgotten-origin strings was, in this special sense, *not less than or equal to* the number of strings with origin:

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- In this context, one-to-one means that if s_1 is different from s_2, then $F(s_1)$ is not only different from $F(s_2)$; it's not even a shift of $F(s_2)$

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- Alternatively, you might first code the singly-infinite string as a real number, then try to encode the real number into the doubly-infinite string, in such a way that you can decode it after any shift

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Opportunities for research

- The example I have shown today is a small sample of a very active research field, usually called the study of *Borel equivalence relations*.
- If you are interested in mathematical logic, this is a newer subfield of set theory than many of the traditional ones, and may have more accessible open problems.

Partial chart of equivalence relations, ordered by definable cardinality of quotient