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A note on the notion of truth in fuzzy logic

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Dedicated to Petr Vopěnka on the occasion of his 65th birthday

Abstract

In fuzzy predicate logic, assignment of truth values may be partial, i.e. the truth value of a formula in an interpretation may be undefined (due to lack of some infinite suprema or infima in the underlying structure of truth values). A logic is *supersound* if each provable formula φ is true (has truth value 1) in each interpretation in which the truth value of φ is defined. It is shown that among the logics given by continuous t-norms, Gödel logic is the only one that is supersound; all others are (sound but) not supersound. This supports the view that the usual restriction of semantics to *safe* interpretations (in which the truth assignments is total) is very natural.

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1. Introduction

Fuzzy logic can be understood as a logic with a comparative notion of truth. Initiated by L. Zadeh, it remained for a long time ignored by most mathematical logicians (S. Gottwald and G. Takeuti being good counterexamples). The book [2] is an attempt to elaborate systems of fuzzy logic in the style of classical logic, stressing axiomatization, completeness, complexity, etc. The basic notion is a continuous t-norm $*$ on the real interval [0,1] as a truth function of conjunction & and its residuum $\Rightarrow$ as the corresponding truth function of implication $\rightarrow$; [0,1] with its standard lattice operations min, max and the operations $*, \Rightarrow$ is a t-algebra $[0,1]_*$ [1]. It follows from the results of [2] and [1] that t-algebras generate the variety of BL-algebras, i.e. that the class of BL-algebras (as defined in [2]) is exactly the variety of all algebras **L** (with arbitrary

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[1] Recall that min and max are definable from * and $\Rightarrow$, see below.

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domain) such that each identity valid in each t-algebra is valid in **L**. (Recall that a BL-algebra is a particular residuated lattice, i.e. an algebra $\mathbf{L} = (L, \cap, \cup, *. \Rightarrow, 0, 1)$ such that $(L, \cap, \cup, 0, 1)$ is a lattice with the top 1 and bottom 0, $(L, *, 1)$ is a commutative semigroup with a unit element 1 and the following hold for each $x, y, z, \in L$:

$$z \leqslant x \Rightarrow y \text{ iff } x * z \leqslant y,$$

$$x \cap y = x * (x \Rightarrow y),$$

$$(x \Rightarrow y) \cup (y \Rightarrow x) = 1.$$

It can be shown that the class of all BL-algebras is a variety.)

Each BL-algebra **L** determines corresponding interpretations of propositional calculus and predicate calculus. If $\mathscr{I}$ is a predicate language (consisting only of some predicates, no function symbols and no constants are necessary) then an **L** interpretation of $\mathscr{I}$ is a structure

$$\mathbf{M} = (M, (r_P)_{P_{\text{predicate}}}),$$

where $M \neq \emptyset$, $r_P : M^{\mathrm{ar}(P)} \to \mathbf{L}_{\mathrm{ar}(P)}$ is the arity of P; if ar $(P) = 0$ r_P is just an element of **L**. The Tarski-style definition of truth degree $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},e}$ of a formula φ in M under evaluation e of object variables, w.r.t. truth functions of **L** reads as follows:

$$\|P(x_1, \ldots, x_n)\|^{\mathbf{L}}_{\mathbf{M},e} = r_P(e(x_1), \ldots, e(x_n));$$

$$\|\varphi \& \psi\|^{\mathbf{L}}_{\mathbf{M},e} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e} * \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e}$$

$$\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},e} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},e}$$

$$\|\bar{0}\|^{\mathbf{L}}_{\mathbf{M},e} = 0, \|\bar{1}\|^{\mathbf{L}}_{\mathbf{M},e} = 1$$

$$\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},e} = \inf e_x \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e}$$

$$\|(\exists x)\varphi\|^{\mathbf{L}}_{\mathbf{M},e} = \sup e_x \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e},$$

where e_x runs over all evaluations differing from e at most in the value for the argument x.

Note the following defined connectives: $\neg\varphi$ stands for $\varphi \to \bar{0}$, $\varphi \wedge \psi$ stands for $\varphi \& (\varphi \to \psi)$, and $\varphi \vee \psi$ stands for $(\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. The truth function of $\wedge, \vee$ are $\cap, \cup$, respectively.

Needless to say, for each t-algebra, $\mathbf{L} = [0, 1]_*, \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e}$ is defined for each $\varphi, \mathbf{M}, e$ since the ordering of [0,1] is complete. But in general one has to deal with algebras whose ordering is not complete. An **L**-interpretation **M** is *safe* if all values $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}, e$ are defined (for all φ, e).

In [2] one can find an axiom system BL$\forall$ for the predicate logic over BL-algebras. The system is *sound* in the following sense: If BL$\forall$ $\vdash \varphi$ then φ is true in all safe **L**-interpretations **L** being any BL-chain). And the converse is also valid: this is completeness. More than that: BL$\forall$ proves φ iff φ is true in all safe **L**-interpretations for all BL-chains (linearly ordered BL-algebras).

There are important subvarieties of the variety of BL-algebras, notably MV-algebras, G-algebras and Π-algebras corresponding to logics stronger than BL∀ namely Łukasiewicz logic Ł∀, Gödel logic G∀ and product logic Π∀. Each of these varieties is generated by a particular continuous t-norm (t-algebra), namely Łukasiewicz, Gödel and product t-norm. More generally, each continuous t-norm generates a variety $\mathscr{V}(*)$ of BL-algebras: A BL-algebra $\mathbf{L}$ belongs to $\mathscr{V}(*)$ iff each identity (in the language of BL-algebras) valid in $[0,1]_*$ is valid in $\mathbf{L}$. This gives a corresponding logic $\mathscr{C}(*)\forall$. Its axioms are those of BL∀ plus all axioms given by identities valid in $\mathscr{V}(*)$—or a sufficient subset of them. (For example, Ł∀ is axiomatized by BL∀ plus the schema of $\neg\neg\varphi \to \varphi$ of double negation; $G\forall$ is axiomatized by BL∀ plus the axiom schema $\varphi \to (\varphi \& \varphi)$ of idempotence of conjunction; Π∀ is axiomatized by BL∀ plus two additional schemas, see [2].) Each logic $\mathscr{C}(*)\forall$ is sound and complete in the above sense, i.e. the following are equivalent:

(i) $\mathscr{C}(*)\forall$ proves φ;
(ii) for each linearly ordered $\mathbf{L} \in \mathscr{V}(*)$ and each safe $\mathbf{L}$-interpretation $\mathbf{M}$, φ is true in $\mathbf{M}$

(This follows from the strong completeness of BL∀, see [2].)

A logic is *supersound* if each provable formula φ is true in each $\mathbf{L}$-interpretation ($\mathbf{L}$ being any chain from the given variety) in which the truth value of φ is defined.

It was proved in [4] that Ł∀, Π∀ are not supersound whereas G∀ is. Here we show that Gödel logic is the only logic $\mathscr{C}(*)\forall$ given by a continuous t-norm which is supersound; the result also implies that BL∀ is not supersound. But, let us stress again, all the logics in question are sound and complete with respect to safe interpretations; safe interpretations appear to be the natural semantics for fuzzy predicate calculi.

2. The results

Theorem. *There is a formula φ such that $BL \vdash \varphi$ and for each continuous t-norm $*$ with non-idempotent element, there is an algebra $\mathbf{L} \in \mathscr{V}(*)$ and a (non-safe) interpretation $\mathbf{M}$ such that $\|\varphi\|^{\mathbf{L}}_{M} < 1$.*

The proof will be a generalization of the corresponding proof in [4]. To make the proof more readable we show that there is a theory T with a single axiom (denoted also by T) and a formula φ such that T proves φ over BL∀ and for each non-idempotent * there is an $\mathbf{L} \in \mathscr{V}(*)$ and a (non-safe) $\mathbf{L}$-interpretation $\mathbf{M}$ in which T is (meaningful and) true and φ is meaningful but not true (its truth value is <1). To get the theorem it suffices to apply the deduction theorem for BL∀ to get a k such that $\mathrm{BL}\forall \vdash T^k \to \varphi$ (T^k being $T\&\ldots\&T, k$ copies); in our $\mathbf{M}$, $(T^k \to \varphi)$ is meaningful but not true[2]. Before we start the proof we collect some preliminaries.

[2] A diligent reader may show that in this particular case one may take $k = 1$

First recall from [3] that a cut in a BL-chain $\mathbf{L}$ is a pair $X, Y \subseteq \mathbf{L}$ such that ($x \in X$ and $y \in Y$) implies ($x \leqslant y$ and $x * y = x$), $X \cup Y = \mathbf{L}$, and y is closed under $*$. Then $X \cap Y$ is either empty or contains a single idempotent. $\mathbf{L}$ is saturated if all cuts are of the latter kind. Each BL-chain has a saturation resulting by adding some idempotents. From the results of [1] it follows that each saturated BL-chain is an ordered sum of MV-chains, Π-chains and G-chains [3].

Lemma 1. (1) *Let* $\mathbf{L}$ *be a* BL-*chain and* c *an idempotent of* $\mathbf{L}$. *i.e.* $c * c = c$. *Then for each* $x \in \mathbf{L}$, $c * x = \min(x, c)$.

(2) *Let* $\mathbf{M}$ *be a safe* $\mathbf{L}$-*structure*; *let* $\mathbf{L}_1$ *be the saturation of* $\mathbf{L}$. *Then* $\mathbf{M}$ *is a safe* $\mathbf{L}_1$-*structure and* $\|\varphi\|^{\mathbf{L}}_{M,e} = \|\varphi\|^{\mathbf{L}}_{M,e}$, *for all* φ *and* e.

Proof. See [3]. For (2), the only thing to observe is that fact that if $X \in L$ has a supremum (infimum) u in $\mathbf{L}$ the u is the supremum (infimum) of X also in $\mathbf{L}_1$. This follows easily from the construction of $\mathbf{L}_1$. $\square$

Lemma 2. *Let* $\mathbf{L}$ *be an* MV-*chain or a* $\Pi -$ *chain.*

(1) *If* $0 < u \leq v < 1$ *then* $u * v < u$.

(2) *If* $0 < x < f < 1$ *and* $y = (f \Rightarrow x)$ *then* $f = (y \Rightarrow x)$, $x = y * f$ *and* $x < y < 1$.

Proof. Easy by representation by ordered Abelian groups (see [2]).

Definition. Our language has a single unary predicate P and a nullary predicate (propositional constant) C. The theory T has one single axiom

$$(\forall x)(\exists y)(P(x) \to (C \& P(y)));$$

φ is the formula $(\exists x)(P(x) \to (C \& P(x)))$.

Lemma 3. T *proves* φ *over* BL$\forall$.

Proof. We show that φ is true in each safe $\mathbf{L}$-model $\mathbf{M}$ of T, $\mathbf{L}$ being a saturated BL-chain. Observe that $T \vdash (\forall x)(P(x) \to C)$. Let $c = \|C\|_{\mathbf{M}}$ and for each $m \in \mathbf{M}$, let $P_m = \|P(m)\|_{\mathbf{M}}$. $\square$

If c is an idempotent, then since $P_m \leqslant c$ for each m, we have $P_m = c * P_m$, thus $\|(\forall x)(P(x)) \equiv (C \& P(x)))\|_{\mathbf{M}} = 1$. Thus let c be non-idempotent and let $[u, v]$ be the component of $\mathbf{L}$ containing c; by the result of [1], mentioned above, $[u, v]$ is an MV-chain or a Π-chain, Recall $P_m \leqslant c$ for all m. If $P_m \leqslant u$ for some m then $P_m * c = P_m$ which makes φ true.

[3] This shows how the famous characterization of continuous t-norms (Mostert and Shields) generalizes to BL-chains

The remaining case is that $u < P_m \leqslant c$ for all m; put $d = \|(\exists x)P(x)\|_M = \sup_m P_m$. We have $u < d \leqslant c < v$, thus $c * d < d$ (by Lemma 2 (1)). But observe that using properties of quantifiers provable in BL$\forall$ (see [2] 5.1.7, 5.1.14, 5.1.16,5.1.18) our axiom implies $(\forall x)(P(x) \to (C\&(\exists y)P(y))$, thus $(\exists x)P(x) \to (C\&(\exists y)P(y))$, which gives $d = c * d$, a contradiction. Lemma 3 is proved.

Lemma 4. *For each continuous t-norm $*$ with a non-idempotent element there is a linearly ordered $\mathscr{V}(*)$-algebra $\mathbf{L}$ and a non-safe $\mathbf{L}$-model $\mathbf{M}$ of T such that $\|\varphi\|_{\mathbf{M}} < 1$.*

Proof. Let F be a non-principal ultrafilter on ω and let $L = [0,1]_*^\omega/F$ be the F-ultrapower of the BL-algebra $[0,1]_*$ (on [0,1] given by $*$). Thus elements of $\mathbf{L}$ are mappings $h : \omega \to [0,1]$, operations are defined coordinatewise and $h_1 = h_2$ iff $\{ih_1(i) = h_2(i)\} \in F$; similarly for $<, \leqslant$.

Let b be non-idempotent in $*$ and let $u < b < v$ be the neighbour idempotents of b (thus on $[u,v], *$ is isomorphic either to the Łukasiewicz t-norm or to the product t-norm).

For each $x \in [0,1]$, let k_x be the constant function on ω with the value x. The mapping associating to $x \in [0,1]$ the element k_x of $\mathbf{L}$ is the elementary embedding of $[0,1]_*$ into $\mathbf{L}$. We use Łoś's theorem without mentioning it. Observe that k_b is non-idempotent in $\mathbf{L}$ and k_u, k_v are its idempotent neighbours; on $[k_u, k_v]$, the operation $*_{\mathbf{L}}$ is isomorphic to the $*$-operation of $[0,1]_{\mathrm{L}}^\omega/F$ or of $[0,1]_\Pi^\omega/F$ and the isomorphism also commutes with the $\Rightarrow$-operation for the case of first argument greater than the second and the second greater than k_u, e.g. by Lemma 2. Let $f \in [k_u, k_v]$ be an element less than k_v but infinitely close to it (e.g. $f(n) = v - (v-u)/(n+1)$ for each n; then for each $x \in [u,v], k_x < f$ in $[k_u, k_v]$).

Let $\mathbf{M} = \omega$; we define an $\mathbf{L}$-interpretation of P, C with the domain $\mathbf{M}$. Let $\|C\|_{\mathbf{M}} = f$, let $P_m = \|P(m)\|_{\mathbf{M}}$ be defined as follows: $P_0 = k_b$, $P_{m+1} = (f \Rightarrow P_m)$. ($P_{m+1}$ is infinitesimally greater than P_m, hence by induction P_m is infinitesimally greater than k_b, and f is infinitesimally close to k_v). Thus we also get $(P_m \Rightarrow (f * P_m)) = f$(for $m > 0$ directly from Lemma 2, for $m = 0$ in a similar way). Clearly, $\mathbf{M}$ is an $\mathbf{L}$-model of T; and $\|\varphi\|_{\mathbf{M}}^{\mathbf{L}} = f < 1_{\mathbf{L}}$. This completes the proof of the lemma and of the theorem. □

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