ELSEVIER

Annals of Pure and Applied Logic 109 (2001) 71–75

ANNALS OF PURE AND APPLIED LOGIC

www.elsevier.com/locate/apal

Three clouds may cover the plane ☆

Péter Komjáth

Department of Computer Science, Budapest, Kecskemeti u. 10–12, 1053 Hungary

In an interesting paper [2] Ginsburg and Linek considered the question whether the plane can be covered with 3 stars where a star consists of a point and a segment of finite length on every half-line from that point. We have not been able to prove this conjecture, but under the continuum hypothesis we show the stronger statement that the plane is the union of three clouds, where a cloud consists of a point plus finitely many points on every half-line emanating from that point. In fact, the continuum hypothesis is equivalent with the existence of such a covering. We also show that if the continuum is at most $\aleph_n$ (for some positive natural number n) then the plane is the union of $n+2$ clouds, but we are unable to show equivalence in this general case. Yet another variant is the notion of a circle, that is, a point plus one point on every half-line from that point. We show, in ZFC, that the plane is the union of countably many circles. (A simple deduction of this, from results on almost-disjoint sets, will also be given.) We conjecture that this is not possible with finitely many circles, but we can only prove this if the centers of the circles are distinct.

Definition. If a is a point on the plane, then a *cloud around a* is a set A which intersects every line e with $a \in e$ in a finite set.

Theorem 1. *If the continuum hypothesis holds then the plane is the union of three clouds.*

Proof. Assume that a, b, c are noncollinear points. We define the clouds A, B, C around a, b, c by transfinite recursion. Let $\mathscr{E}$ be the set of all lines going through one of a, b, or c. By closure we can find the increasing, continuous, countable decompositions

$$\mathscr{E} = \bigcup \{\mathscr{E}_\alpha : \alpha < \omega_1\}$$

☆ Research partially supported by Hungarian National Research Grant T 019476.
E-mail address: kope@cs.elte.hu (P. Komjáth).

0168-0072/01/$ - see front matter
PII: S0168-0072(01)00042-2

and

$$\mathbf{R}^2 = \bigcup\{X_\alpha : \alpha < \omega_1\},$$

such that

(a) $\mathscr{E}_0 = X_0 = \emptyset$;
(b) $a, b, c \in X_1$;
(c) if $e_1, e_2 \in \mathscr{E}_\alpha$ then their intersection is in X_α;
(d) if $p \in X_\alpha$, then the lines pa, pb, pc are in $\mathscr{E}_\alpha$.

We define the function $F(e) \in [e]^{<\omega}$ for every $e \in \mathscr{E}_{\alpha+1} - \mathscr{E}_\alpha$. If we have

$$\mathbf{R}^2 = \bigcup\{F(e) : e \cap \{a, b, c\} \neq \emptyset\},$$

then we can clearly define $A = \bigcup\{F(e) : a \in e\}$, $B = \bigcup\{F(e) : b \in e\}$, $C = \bigcup\{F(e) : c \in e\}$, and so A, B, C will be clouds around a, b, c, respectively, and $A \cup B \cup C = \mathbf{R}^2$.

Let g_1, g_2, g_3 denote the ab, ac, bc lines. Enumerate $\mathscr{E}_{\alpha+1} - \mathscr{E}_\alpha$ as

$$\mathscr{E}_{\alpha+1} - \mathscr{E}_\alpha = \{e_i : i < \omega\}.$$

Let $F(e_i)$ be a finite set containing the intersection points of $e_0 \cap e_i, \ldots, e_{i-1} \cap e_i$, $g_1 \cap e_i, g_2 \cap e_i, g_3 \cap e_i$. We show that $\bigcup\{F(e) : e \in \mathscr{E}\} = \mathbf{R}^2$. Pick an arbitrary $p \in \mathbf{R}^2$. There is a unique $\alpha < \omega_1$ such that $p \in X_{\alpha+1} - X_\alpha$. If $p \in g_1$ then the line $e = pc$ is in $\mathscr{E}_{\alpha+1} - \mathscr{E}_\alpha$ and $p \in F(e)$ by our construction. A similar argument works if $p \in g_2$ or $p \in g_3$. In the remaining case, the lines pa, pb, and pc are distinct and at most one of them is in $\mathscr{E}_\alpha$. So, there are two of them in $\mathscr{E}_{\alpha+1} - \mathscr{E}_\alpha$, and assume that they get the indices $j < i$ under the enumeration of this latter set. Now $\{p\} = e_j \cap e_i$ and so $p \in F(e_i)$, and we are done again. $\square$

Theorem 1′. If the continuum is at most $\aleph_n$ $(n = 1, 2, \ldots)$ then the plane is the union of $n + 2$ clouds.

Proof. Let $a_1, \ldots, a_{n+2}$ be distinct points on the plane, not all on a line. We show that the plane can be covered with $n + 2$ clouds with the respective centers $a_1, \ldots, a_{n+2}$.

First we notice that it suffices to cover those points p for which the half-lines $\overrightarrow{pa_1}, \ldots, \overrightarrow{pa_{n+2}}$ are different. Indeed the remaining points are those on the finitely many lines $g_1, \ldots, g_N$ joining some two of our points, and by condition, for every g_i there is some a_j not on it, so we can add g_i to the cloud around a_j.

Let $\mathscr{E}$ denote the set of all half-lines from one of the points a_i. We show the following statement.

Lemma. Assume that $0 \leqslant t \leqslant n$, X is a set of points, $\mathscr{F} \subseteq \mathscr{E}$, $|X| = |\mathscr{F}| = \aleph_t$, and if $p \in X$ then at least $t + 2$ half-lines of the form $\overrightarrow{a_i p}$ are in $\mathscr{F}$. Then there is a function F defined on $\mathscr{F}$ with $F(e) \in [e]^{<\omega}$ such that

$$X = \bigcup\{F(e) : e \in \mathscr{F}\}.$$

Proof. By induction on t. For $t = 0$ enumerate $\mathscr{F}$ as $\{e_0, e_1, \cdots\}$ and set $F(e_i) = (e_0 \cap e_i) \cup \cdots \cup (e_{i-1} \cap e_i) \cap X$. We have to show that X is covered by $\bigcup\{F(e) : e \in \mathscr{F}\}$. Let $p \in X$. By assumption there are $j < i < \omega$ that $p \in e_j$, $p \in e_i$, so $p \in F(e_i)$.

Assume now that $0 \leqslant t < n$, we have the result for t, and we are to prove it for $t+1$. We have, therefore, $|X| = |\mathscr{F}| = \aleph_{t+1}$.

Decompose X as well as $\mathscr{F}$ into the increasing, continuous union of subsets $X = \bigcup\{X_\alpha < \omega_{t+1}\}$, $\mathscr{F} = \bigcup\{\mathscr{F}_\alpha : \alpha < \omega_{t+1}\}$ with the following conditions:

1. if $p \in X_\alpha$, $\overrightarrow{a_i p} \in \mathscr{F}$ then $\overrightarrow{a_i p} \in \mathscr{F}_\alpha$;
2. if $e, e' \in \mathscr{F}_\alpha$, $e \cap e' = \{p\}$, $p \in X$, then $p \in X_\alpha$;
3. $X_0 = \mathscr{F}_0 = \emptyset$, $|X_{\alpha+1} - X_\alpha| = |\mathscr{F}_{\alpha+1} - \mathscr{F}_\alpha| = \aleph_t$.

This can be done by the usual closure arguments and, if needed, by padding, to ensure property 3.

We now observe that for $p \in X_{\alpha+1} - X_\alpha$ there are at least $t+2$ half-lines of the form $\overrightarrow{a_i p}$ in $\mathscr{F}_{\alpha+1} - \mathscr{F}_\alpha$ (as there are at least $t+3$ altogether, by property 2 above, only one can be in $\mathscr{F}_\alpha$, the rest will be included in $\mathscr{F}_{\alpha+1} - \mathscr{F}_\alpha$). So we can apply the previous case of the lemma, to get a function F_α on $\mathscr{F}_{\alpha+1} - \mathscr{F}_\alpha$ with range covering $X_{\alpha+1} - X_\alpha$, and then the union $\bigcup\{F_\alpha : \alpha < \omega_{t+1}\}$ works. $\square$

Clearly, the case $t = n$ of the lemma gives the theorem. $\square$

Theorem 2. *If the plane is the union of three clouds then the continuum hypothesis holds.*

Proof. Assume that the continuum is greater than $\aleph_1$ and the plane is the union of some clouds around a, b, and c. This means that there is a function $F(e) \in [e]^{<\omega}$ defined for every line e going through either a, or b, or c such that

$$\mathbf{R}^2 = \bigcup\{F(e) : e \cap \{a, b, c\} \neq \emptyset\}.$$

Clearly, a, b, and c are not collinear. With a line preserving transformation we can achieve that $a = (0,1)$, $b = (1,0)$, $c = (0,0)$. Let $d = (1,1)$. Assume that the point $p = (x, y)$ of the $abcd$ square is the intersection of the lines e (passing through a) and f (passing through b) where the dap angle is α and the dbp angle is β, $0 < \alpha, \beta < \pi/4$.

Then we have

$$\frac{y}{x} = \frac{1 - \tan\alpha}{1 - \tan\beta}.$$

Let $\{d_n : n < \omega\}$ be distinct real numbers between 0 and 1, and let $\{a_\xi : \xi < \omega_1\}$ be distinct real numbers between 0 and 1. Define the angles $\{\alpha_{\xi,n} : \xi < \omega_1, n < \omega\}$ with

$$1 - \tan\alpha_{\xi,n} = d_n a_\xi.$$

Let $e_{\xi,n}$ be the line going through a having an angle $\alpha_{\xi,n}$ with ad. The set

$$X = \bigcup\{F(e_{\xi,n}) : \xi < \omega_1, n < \omega\}$$

has cardinal at most $\aleph_1$. There is a real $0 < c < 1$ such that if $1 - \tan \beta_n = d_n c$ and f_n is the line going through b with an angle β_n with bd, then $X \cap f_n = \emptyset$ (as every point of X disqualifies only countably many c). Set

$$Y = \bigcup \{F(f_n) : n < \omega\},$$

a countable set. There is a $\xi < \omega_1$ such that $Y \cap e_{\xi,n} = \emptyset$ holds for every n (as every point of Y disqualifies only countably many ξ). Fix such a ξ. Set $\{p_n\} = e_{\xi,n} \cap f_n$. By our construction, p_n is not an element of $F(e_{\xi,n})$ or $F(f_n)$. Nevertheless, it is on the line g determined by the equation

$$\frac{y}{x} = \frac{a_\xi}{c}$$

and so $\{p_n : n < \omega\} \subseteq F(g)$, a contradiction. $\square$

Definition. A set $C \subseteq \mathbf{R}^2$ is a *circle around* $a \in \mathbf{R}^2$ if C has one point on every half-line from a.

Theorem 3. *If the points $a_0, a_1, \ldots$ are not all on a line then $\mathbf{R}^2 = C_0 \cup C_1 \cup \cdots$ where C_i is a circle around a_i.*

Proof. First we assume that finitely many lines cannot cover $\{a_0, a_1, \ldots\}$.

In this case, for every point $p \in \mathbf{R}^2$ the set of lines $E_p = \{a_i p : i < \omega\}$ is infinite. Notice that $|E_p \cap E_q| \leqslant 1$ for distinct points (only the connecting line pq can be in the intersection).

In [3] it is shown (see also [4]) that if $\mathscr{H}$ is a family of countably infinite sets with $|H \cap H'| \leqslant n$ for some natural number n whenever $H \neq H' \in \mathscr{H}$, then there is a one–one choice function on the family. If we apply this result to the family $\{E_p : p \in \mathbf{R}^2\}$ we get that to each p some $i = i(p) < \omega$ can be chosen that the half-lines (even the lines) $\overrightarrow{a_{i(p)}p}$ are different, and the theorem is proved (in the restricted case).

For the sake of completeness we show how to incorporate the proof in [3] to give a proof of the present theorem.

It suffices to show that there exists a well ordering $\prec$ of $\mathbf{R}^2$ such that

$$E_p \cap \left(\bigcup \{E_q : q \prec p\} \right)$$

is always finite (as in this case a simple transfinite recursion suffices for the selection of different elements from the sets $\{E_p : p \in \mathbf{R}^2\}$). We show the existence of such a well ordering for every subset $X \subseteq \mathbf{R}^2$ by transfinite induction on the cardinal $\kappa = |X|$.

This is obvious if $\kappa \leqslant \omega$ (take any well ordering of type κ). For $\kappa > \omega$ use Skolem-type closure arguments to decompose X into the increasing, continuous union $X = \bigcup \{X_\alpha : \alpha < \kappa\}$ such that $X_0 = \emptyset$, $|X_\alpha| < \kappa$ for $\alpha < \kappa$, and if $e \cap e' = \{p\}$ holds for some $e \in E_q$, $e' \in E_{q'}$, $q, q' \in X_\alpha$ then $p \in X_\alpha$, as well. Let $\prec_\alpha$ be a well ordering as stated on the set $X_{\alpha+1} - X_\alpha$, and put $q \prec p$ if $p \in X_{\alpha+1} - X_\alpha$ for some α and either $q \in X_\alpha$ or else $q \prec_\alpha p$.

As $|E_p \cap (\bigcup \{E_q : q \in X_\alpha\})| \leqslant 1$ we are done.

We are left with the case when finitely many lines cover $\{a_0, a_1, \ldots\}$. As we assumed that the points are not all collinear, we can assume, with no harm, that our set of points is $\{b, a_0, a_1, \ldots\}$ where $\{a_0, a_1, \ldots\}$ are on a line e while b is not. In this case, e is on a circle around b (yes, it is!), and using our earlier arguments on the two half-planes of $\mathbf{R}^2 - e$, we can cover $\mathbf{R}^2 - e$ with circles around $a_0, a_1, \ldots$. $\square$

There is a nice argument given by the participants of the Schweitzer competition (1999) for the weaker statement that some countably many circles cover the plane.

Let $B = \{b_\alpha : \alpha < 2^\omega\}$ be a transcendency base for $\mathbf{R}$. For every real number r there is a minimal finite subset $\mathrm{supp}(r) \subseteq B$ such that r is algebraic over $\mathrm{supp}(r)$. Given a point $(a, a') \in \mathbf{R}^2$ let $i < \omega$ be minimal that

$$\{b_{2i}, b_{2i+1}\} \cap (\mathrm{supp}(a) \cup \mathrm{supp}(a')) = \emptyset.$$

Put (a, a') into A_i. Now an easy argument shows that every A_i is a circle around the point (b_{2i}, b_{2i+1}).

References

[1] R. Davies, On a problem of Erdös concerning decompositions of the plane, Proc. Camb. Philos. Soc 59 (1963) 33–36.

[2] J. Ginsburg, V. Linek, A space-filling complete graph, Ars Combin. LVIII (2001), to appear.

[3] P. Komjáth, Families close to disjoint ones, Acta Math. Hungar 43 (1984) 199–204.

[4] E.W. Miller, On a property of families of sets, Comptes Rendus Varsovies 30 (1937) 31–38.

[5] J. Simms, Sierpiński's theorem, Simon Stevin 65 (1991) 69–163.

Note added in proof

In the meantime we proved that it is consistent that the plane is not the union of 3 stars.