Homework 7

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Contents

Remarks

Made a mistake while trying to convert markdown to tex. So there are some wrong feedlines for formulas. For example:

$$A \vdash A \vee B, B \vee A$$

(T1)

Means:

$$A \vdash A \vee B, B \vee A \qquad (T1)$$

1 Question

Prove that:

$$A \vdash A \vee B, B \vee A$$

(T1)

$$A \vee B \vdash\dashv B \vee A$$

(T2)

$$(A \vee B) \vee C \vdash\dashv A \vee (B \vee C)$$

(T3)

$$(A \vee B) \vdash\dashv \neg A \to B$$

(T4)

$$A \to B \vdash\dashv \neg A \vee B$$

(T5)

$$\neg(A \vee B) \vdash\dashv \neg A \wedge \neg B$$

(T6)

$$\neg(A \wedge B) \vdash\dashv \neg A \vee \neg B$$

(T7)

$$\emptyset \vdash A \vee \neg A$$

(T8)

2 Lemma

First we prove the following lemmas to simpilify the proof.

2.1 2.1. belong to ($\in$) (in)

If:

$$A \in \Sigma$$

Then:

$$\Sigma \vdash A$$

Proof:
Let Σ' be $\Sigma - A$, then:

$$A \vdash A$$

(Ref)

$$\Sigma', A \vdash A$$

(+)

$$\Sigma \vdash A$$

(1,2)

2.2 2.2. neg imply

$$\neg A \to B \vdash \neg B \to A$$

proof:

$$\neg A \to B, \neg A, \neg B \vdash \neg A \to B$$

(in)

$$\neg A \to B, \neg A, \neg B \vdash \neg A$$

(in)

From 1, 2, $\to -$:

$$\neg A \to B, \neg A, \vdash B$$

$$\neg A \to B, \neg A, \neg B \vdash B$$

(3)

$$\neg A \to B, \neg A, \neg B \vdash \neg B$$

(in)

From $\neg -$, 4 ,5 :

$$\neg A \to B, \neg B \vdash A$$

From 6, $rightarrow+$:

$$\neg A \to B \vdash \neg B \to A$$

2.3 2.3. transitivity1 (trans)

$$A \to B, B \to C \vdash A \to C$$

(trans)

Proof:

$$A \to B, B \to C, A \vdash A$$
$$A \to B, B \to C, A \vdash A \to B$$
$$A \to B, B \to C, A \vdash B$$
$$A \to B, B \to C, A \vdash B \to C$$
$$A \to B, B \to C, A \vdash C$$
$$A \to B, B \to C \vdash A \to C$$

2.4 2.4. transitivity2 (Tr)

IF:

$$\Sigma \vdash \Sigma'$$
$$\Sigma' \vdash A$$

Then:

$$\Sigma \vdash A$$

(Tr)

2.5 2.5. double neg

$$\neg\neg A \vdash A$$

(2neg)

$$A \vdash \neg\neg A$$

(2neg)

Proof1:

$$\neg\neg A, \neg A \vdash A$$
$$\neg\neg A, \neg A \vdash \neg A$$

From 1, 2, $\neg -$:

$$\neg\neg A \vdash A$$

Proof2:

$$\neg\neg\neg A \vdash \neg A$$

(2neg)

$$A, \neg\neg\neg A \vdash \neg A$$

(+)

$$A, \neg\neg\neg A \vdash A$$

(in)

From $\neg -$:

$$A \vdash \neg\neg A$$

2.6 2.6. reductio

If:

$$\Sigma, A \vdash B$$
$$\Sigma, A \vdash \neg B$$

Then:

$$\Sigma \vdash \neg A$$

(redct)

Proof:

$$\Sigma, A \vdash B$$
$$\neg\neg A \vdash A$$

From 1, 2, Tr:

$$\Sigma \neg\neg A \vdash B$$

Similarily:

$$\neg\neg A \vdash A$$
$$\Sigma, A \vdash \neg B$$
$$\Sigma \neg\neg A \vdash \neg B$$

Finally, from 3, 6, $\neg -$:

$$\Sigma \vdash \neg A$$

2.7 2.7. confliction

$$A, \neg A \vdash B$$

(conf)

$$A \vdash \neg A \to B$$

(conf)

$$\neg A \vdash A \to B$$

(conf)

Proof:

$$A, \neg A, \neg B \vdash A$$

(in)

$$A, \neg A, \neg B \vdash \neg A$$

(in)

$$A, \neg A \vdash B$$

(1,2)

From 3, $\to +$:

$$A \vdash \neg A \to B$$

$$\neg A \vdash A \to B$$

2.8 2.8. and

$$A \wedge B \vdash\dashv A, B$$

Proof:

$$A \wedge B \vdash A \wedge B$$

(in)

From 1, $\wedge -$:

$$A \wedge B \vdash A$$

$$A \wedge B \vdash B$$

$$A \wedge B \vdash A, B$$

And:

$$A, B \vdash A$$

$$A, B \vdash B$$

Then use $\wedge +$ rule;

$$A, B \vdash A \wedge B$$

2.9 2.9. switch

If:

$$A \dashv\vdash A'$$

$$\vdash A \to B$$

$$\vdash C \to A$$

$$\vdash A \vee D$$

$$\vdash A \wedge E$$

$$\vdash \neg A$$

Then it is trival that:

$$\vdash A' \to B$$

$$\vdash C \to A'$$

$$\vdash A' \vee D$$

$$\vdash A' \wedge E$$

$$\vdash \neg A'$$

Therefore, for A, A' in deduction formula, we can switch A and A'.

3 Proof

3.1 3.1. T1

$$A \vdash A$$

(Ref)

$$A \vdash A \vee B$$

(1,V+)

$$A \vdash B \vee A$$

(1,V+)

$$A \vdash A \vee B, B \vee A$$

(3,4)

3.2 3.2. T2

To prove:

$$A \lor B \vdash\dashv B \lor A$$

(T2)

We only need to prove one direction, for this formula is symmatric.

$$A \vdash B \lor A$$

(T1)

$$B \vdash B \lor A$$

(T1)

From 1,2,V-:

$$A \lor B \vdash B \lor A$$

Symmatricly:

$$B \lor A \vdash A \lor B$$

As a result:

$$A \lor B \vdash\dashv B \lor A$$

(4,5)

3.3 3.3. T3

$$C \vdash (B \lor C)$$

(T1)

$$(B \lor C) \vdash A \lor (B \lor C)$$

(T1)

$$C \vdash A \lor (B \lor C)$$

(1,2)

Similarily, we have:

$$A \vdash A \lor (B \lor C)$$

$$B \vdash A \lor (B \lor C)$$

Then from 4,5,V-:

$$A \lor B \vdash A \lor (B \lor C)$$

Finally:

$$(A \lor B) \lor C \vdash A \lor (B \lor C)$$

(3,6)

3.4 3.4. T4

Left to Right:

$$A, \neg A \vdash B$$

(conf)

$$A \vdash \neg A \to B$$

(conf)

$$B, \neg A \vdash B$$

(in)
From 3, $\to +$:

$$B \vdash \neg A \to B$$

From 3, 4, V-:

$$(A \vee B) \vdash \neg A \to B$$

Right to Left:

$$A \vdash A \vee B$$

(V+)
From 1, lemma neg imply:

$$\neg(A \vee B) \vdash \neg A$$

$$\neg A \to B, \neg(A \vee B) \vdash \neg A$$

(+)

$$\neg A \to B, \neg(A \vee B) \vdash \neg A \to B$$

(in)
From $\to -$:

$$\neg A \to B, \neg(A \vee B) \vdash B$$

Using V+:

$$\neg A \to B, \neg(A \vee B) \vdash (A \vee B)$$

Using in:

$$\neg A \to B, \neg(A \vee B) \vdash \neg(A \vee B)$$

(in)
From $\neg -$:

$$\neg A \to B \vdash (A \vee B)$$

To sum up:

$$(A \vee B) \vdash\dashv \neg A \to B$$

3.5 3.5. T5

$$(\neg A \vee B) \vdash\dashv \neg(\neg A) \to B$$

(T4)

$$\neg\neg A \vdash\dashv A$$

From 1, 2, switch:

$$\neg A \vee B \vdash\dashv A \to B$$

3.6 3.6. T6

Consider that:

$$\neg(\neg A \to B), A \vdash \neg(\neg A \to B)$$

(in)
From :

$$A, \neg A \vdash B$$

Using $\to +$, we gain:

$$\neg(\neg A \to B), A \vdash (\neg A \to B)$$

Then use $\neg-$:

$$\neg(\neg A \to B) \vdash \neg A$$

$$\neg(\neg A \to B) \vdash \neg B$$

Since:

$$\neg(\neg A \to B) \vdash \neg A$$

$$\neg(\neg A \to B) \vdash \neg B$$

Then:

$$\neg(\neg A \to B) \vdash \neg A \wedge \neg B$$

And:

$$\neg(\neg A \to B) \vdash \neg(\neg A \to B)$$

(in)
From T4, switch:

$$\neg(A \vee B) \vdash \neg(\neg A \to B)$$

$$\neg(A \vee B) \vdash \neg A \wedge \neg B$$

(Tr)
Another direction:

$$\neg A \wedge \neg B \vdash \neg A, \neg B$$

(and)

$$\neg A, \neg B, (A \vee B) \vdash \neg B$$

(in)

$$\neg A, \neg B, (A \vee B) \vdash \neg A$$

(in)

$$\neg A, \neg B, (A \vee B) \vdash \neg A \to B$$

(T4)
From $\to -$:

$$\neg A, \neg B, (A \vee B) \vdash B$$

Then from $\neg -$:

$$\neg A, \neg B \vdash \neg (A \vee B)$$

Using Tr:

$$\neg A \wedge \neg B \vdash \neg (A \vee B)$$

As a result:

$$\neg (A \vee B) \vdash\dashv \neg A \wedge \neg B$$

3.7 3.7. T7

$$\neg (A \wedge B) \vdash A \to \neg B$$

And:

$$\neg A \vee \neg B \vdash\dashv \neg\neg A \to \neg B$$

(T4)
From 2neg, switch:

$$\neg A \vee \neg B \to\vdash\dashv A \to \neg B$$

As a result (Tr):

$$\neg (A \wedge B) \vdash \neg A \vee \neg B$$

Another direction:
We have proved that:

$$\neg A \vee \neg B \to\vdash\dashv A \to \neg B$$

For (we have proved it many times):

$$A \to \neg B, A \vdash \neg B$$

Using Tr, and:

$$\neg A \vee \neg B, (A \wedge B) \vdash \neg B$$

From in, and:

$$\neg A \vee \neg B, (A \wedge B) \vdash B$$

Using $\neg -$:

$$\neg A \lor \neg B \vdash \neg(A \land B)$$

Both 2 directions have been proved:

$$\neg(A \land B) \vdash\dashv \neg A \lor \neg B$$

(T7)

3.8 3.8. T8

$$\neg A \vdash \neg A$$

(in)

From $\rightarrow +$:

$$\emptyset \vdash \neg A \rightarrow \neg A$$

From T4:

$$\neg A \rightarrow \neg A \vdash A \lor \neg A$$

As a result:

$$\emptyset \vdash A \lor \neg A$$

(1,3,Tr)