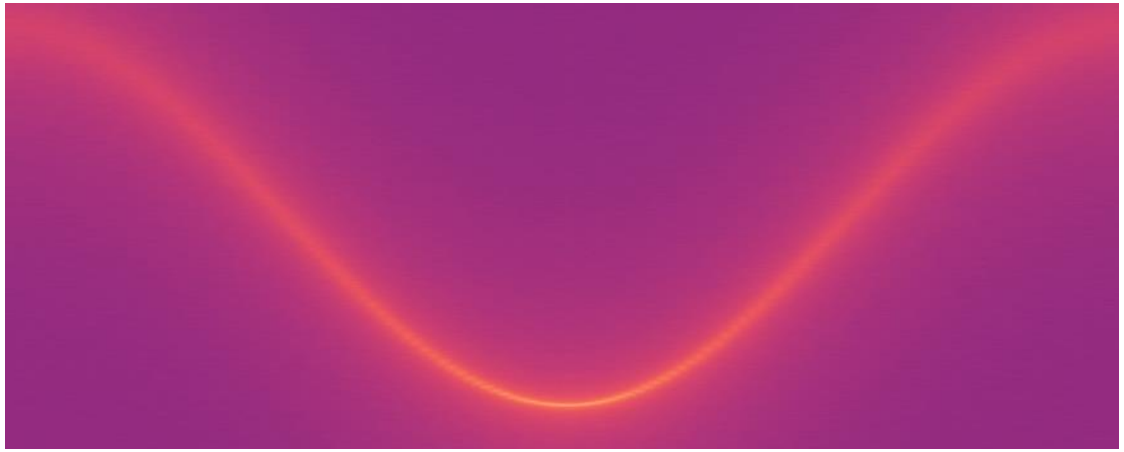
# Magnons in Ferromagnets



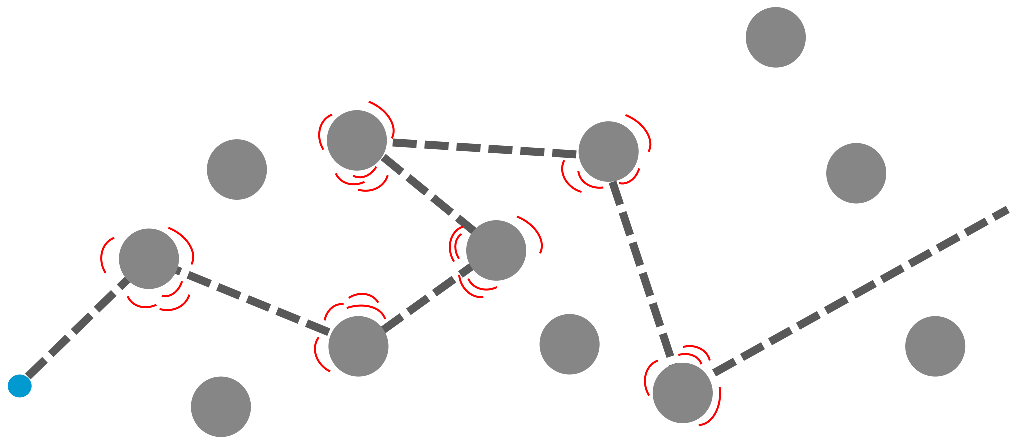
**Julian Beisch**
Konstanz, 17.12.2024

Universität Konstanz

# Motivation

## Current computing

- Electronics
- Information by moving electrons *(charge)*
  - But they scatter → Joule heating



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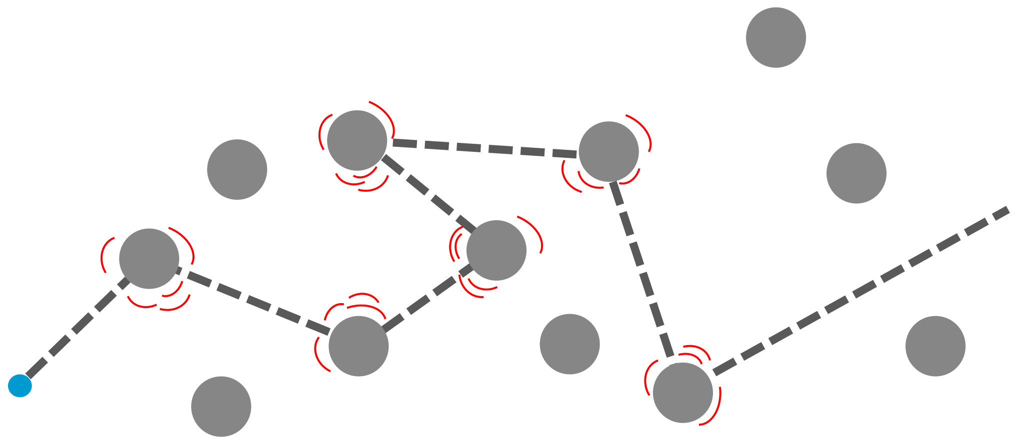
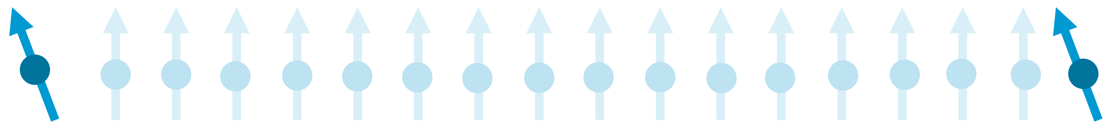


- Another property of electrons: spin
- Spintronics
- Make currents with spins, but how ?



**Caveat**

Spins depicted as arrows is just a semi-classical approximation of an otherwise quantum mechanical expectation distribution

# Classification

**Scope**

| | Bound $e^-$ | Quasi-free $e^-$ |
|---|---|---|
| Dia | Lamor Diamagnetism | Landau Diamagnetism |
| Para | Langevin Paramagnetism | Pauli Paramagnetism |
| IM | Cooperative Magnetism | Band Ferromagnetism |

# Spin-operators

## Spin-operators

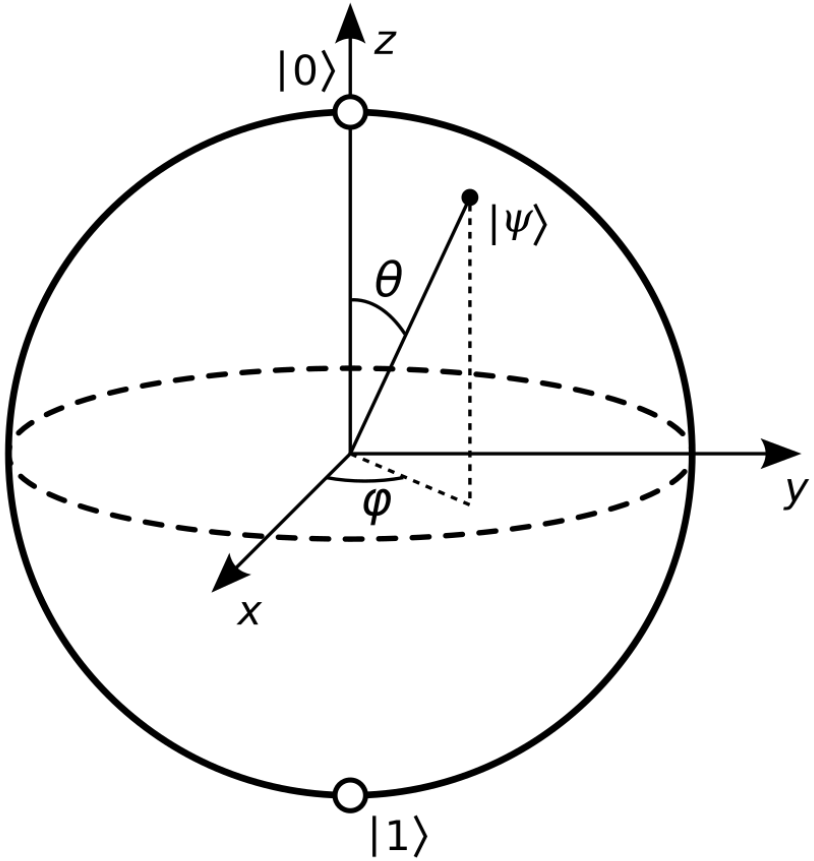
$$\left[\hat{S}_i^x, \hat{S}_j^y\right] = \mathrm{i}\hat{S}_i^z \delta_{i,j} \qquad + \text{cyclic permutation}$$

$$\hat{S}_i^+ = \hat{S}_i^x + \mathrm{i}\hat{S}_i^y$$

$$\hat{S}_i^- = \hat{S}_i^x - \mathrm{i}\hat{S}_i^y$$

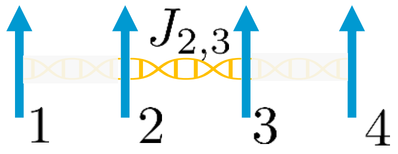
$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm\hat{S}_i^\pm \delta_{i,j}$$

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hat{S}_i^z \delta_{i,j}$$




Wikipedia.com

# Heisenberg Theory of Ferromagnetism




$$\mathcal{H} = -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta}$$

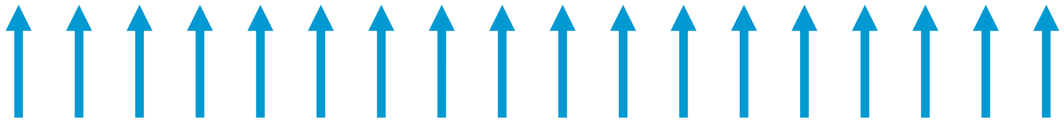
$$= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \left( \hat{S}_i^x \hat{S}_{i+\Delta}^x + \hat{S}_i^y \hat{S}_{i+\Delta}^y + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)$$

$$= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)$$

*W.Heisenberg* Z.Physik **49**, 619-636, (1928)

## Groundstate

$$|0\rangle = |S, S, S, \ldots, S\rangle$$

$|0\rangle$ 

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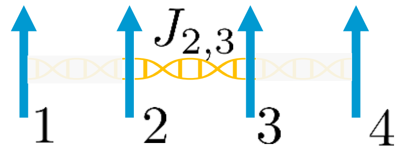


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## Groundstate

$$|0\rangle = |S, S, S, \ldots, S\rangle$$

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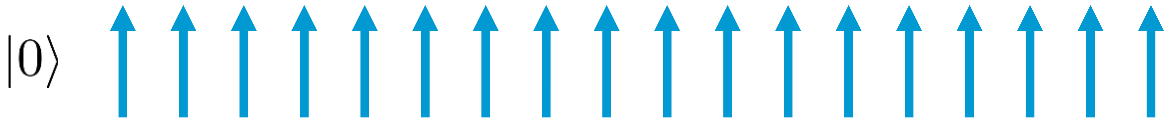
An eigenstate, with the eigenenergy $E_0$

$$\mathcal{H}|0\rangle = -\sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \underline{\hat{S}_i^+ \hat{S}_{i+\Delta}^-} + \underline{\hat{S}_i^- \hat{S}_{i+\Delta}^+} \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) |0\rangle$$

$$= \underline{0} - zJN \cdot S^2 |0\rangle = E_0 |0\rangle$$



# Bloch Theory of Ferromagnetism

## Excitations

Groundstate $|0\rangle = |S, S, S, \ldots, S\rangle$

How do excitations of this state look like?

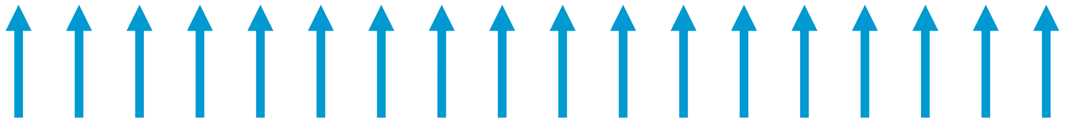
**Zur Theorie des Ferromagnetismus.**

Von **F. Bloch,** zurzeit in Utrecht.

(Eingegangen am 1. Februar 1930.)

*F.Bloch.* Z.Physik **61**, *206-219* (1930)

$|0\rangle$



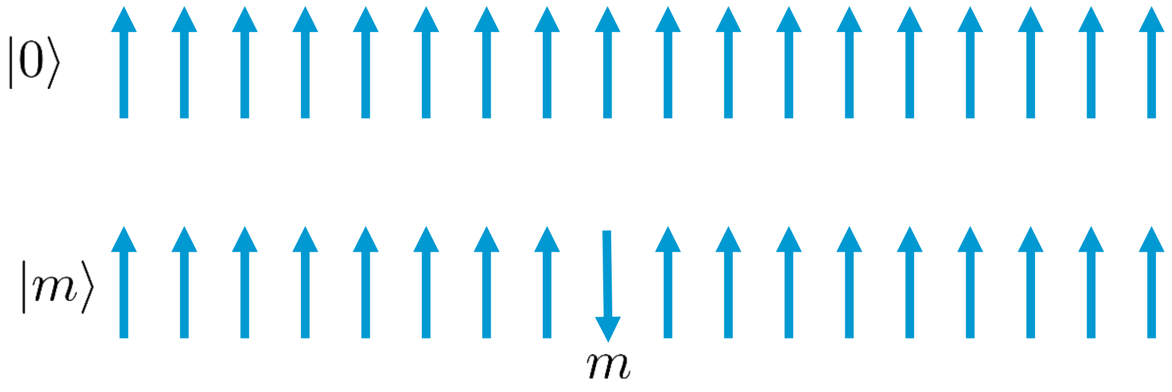
# Bloch Theory of Ferromagnetism

## Excitations

His approach was to consider one flipped spin

$$
\begin{aligned}
|m\rangle &= \frac{S_m^-}{\sqrt{2S}}|0\rangle \\
&= |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle
\end{aligned}
$$

Not an eigenstate anymore



# Bloch Theory of Ferromagnetism

$$\mathcal{H}|m\rangle = -\sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- |m\rangle + \hat{S}_i^- \hat{S}_{i+\Delta}^+ |m\rangle \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right)$$

## Excitations

His approach was to consider one flipped spin

$$|m\rangle = \frac{S_m^-}{\sqrt{2S}}|0\rangle = |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle$$

Not an eigenstate anymore

$$= -J \sum_{i,\Delta} \cdot \left( \frac{1}{2} \left\{ \delta_{i,m} \hat{S}_i^+ \hat{S}_{i+\Delta}^- |m\rangle + \delta_{i+\Delta,m} \hat{S}_i^- \hat{S}_{i+\Delta}^+ |m\rangle \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right)$$

$$= -J \cdot \left( \frac{1}{2} \left\{ \sum_{\Delta} \hat{S}_m^+ \hat{S}_{m+\Delta}^- |m\rangle + \sum_{\Delta} \hat{S}_{m-\Delta}^- \hat{S}_m^+ |m\rangle \right\} + \sum_{i,\Delta} \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right)$$

$$= -J \cdot \left( \frac{1}{2} \left\{ \sum_{\Delta} 2S|m+\Delta\rangle + \sum_{\Delta} 2S|m+\Delta\rangle \right\} + \sum_{i,\Delta} \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right)$$

$$= (E_0 + 2zJS)|m\rangle - 2JS \sum_{\Delta} |m+\Delta\rangle$$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

# Bloch Theory of Ferromagnetism

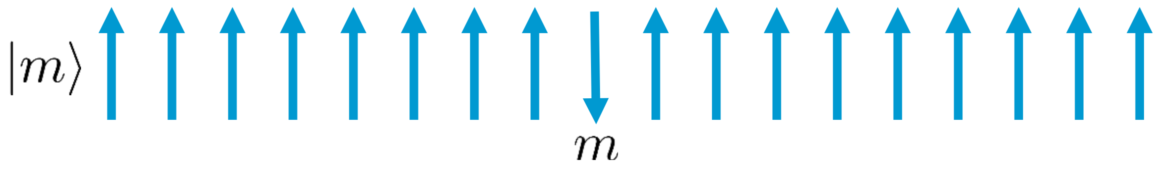
## Search for eigenstates

Not an eigenstate anymore.

The total magnetization was reduced by 1!

$$\begin{aligned}\mathcal{H}|m\rangle &= \mathcal{H}S_m^-|0\rangle \\ &= (E_0 + 2zJS)\,|m\rangle - 2JS\sum_{\Delta}|m+\Delta\rangle\end{aligned}$$

$$\hat{S}^z_{\mathrm{tot.}}|m\rangle = (NS-1)\,|m\rangle$$



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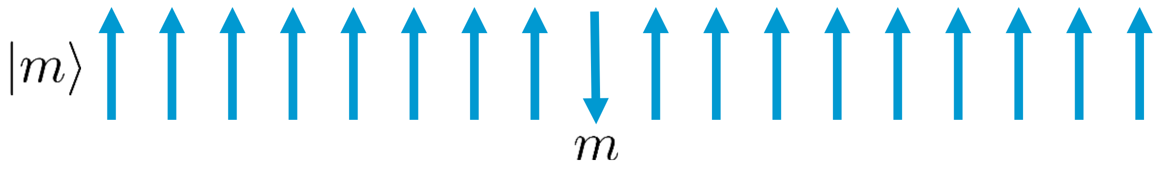
Hence we can guess an eigenstate



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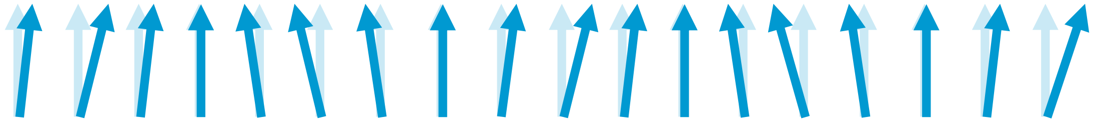
$$\hat{S}^-_{\vec{k}} \propto \frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^-_n$$

$|\vec{k}\rangle$



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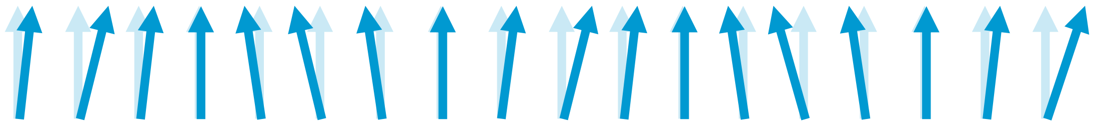
$|\vec{k}\rangle$ 

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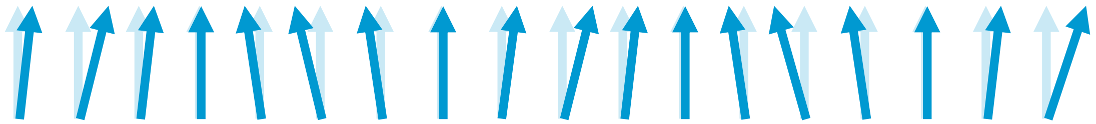

# Bloch Theory of Ferromagnetism

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle$$

## Properties of the eigenstates

The total **magnetization** is reduced

As well as an increase in energy

$$\begin{aligned}\hat{S}^z_{\text{tot.}}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^z_{\text{tot.}}|n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) (NS-1)|n\rangle \\ &= (NS-1)|\vec{k}\rangle\end{aligned}$$

But the average x and y component are still zero?

$$\langle\vec{k}|\hat{S}^x_i|\vec{k}\rangle = 0$$

$$\langle\vec{k}|\hat{S}^y_i|\vec{k}\rangle = 0$$

$$\langle\vec{k}|\hat{S}^z_i|\vec{k}\rangle = S - \frac{1}{N}$$

# Bloch Theory of Ferromagnetism

## Properties of the eigenstates

The total magnetization is reduced

As well as an increase in **energy**

$$\mathcal{H}|\vec{k}\rangle = \dots$$

# Ferromagnetic Dispersion relation

## Properties of the eigenstates

$$\mathcal{H}|\vec{k}\rangle = \left( (E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta>0} - \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) - \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right)$$

$$= \left( (E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta>0} -2\cos\left(\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right)$$

$$= \left( (E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta} -\cos\left(\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right)$$

$$= \left( E_0|\vec{k}\rangle + 2JS \sum_{\Delta} \left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right) |\vec{k}\rangle \right)$$

# Ferromagnetic Dispersion relation

## Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2\,J_1\left[1 - \cos\frac{2\,\pi\,k}{N}\right],$$

*F.Bloch.* Z.Physik **61**, *206-219* (1930)

$$\begin{aligned}\mathcal{H}|\vec{k}\rangle &= \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS\sum_{\Delta>0} - \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) - \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)|\vec{k}\rangle\right)\\ &= \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS\sum_{\Delta>0} -2\cos\left(\vec{k}\cdot\vec{r}_\Delta\right)|\vec{k}\rangle\right)\\ &= \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS\sum_{\Delta} -\cos\left(\vec{k}\cdot\vec{r}_\Delta\right)|\vec{k}\rangle\right)\\ &= \left(E_0|\vec{k}\rangle + 2JS\sum_{\Delta}\left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)|\vec{k}\rangle\right)\end{aligned}$$

Same result with semiclassical calculation

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*F.Bloch.* Z.Physik **61**, *206-219* (1930)

Same result with semiclassical calculation

$$\mathcal{H}|\vec{k}\rangle = \left(E_0|\vec{k}\rangle + 4JS\left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)|\vec{k}\rangle\right)$$

$$\Delta E = 4JS\left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)$$

For a linear chain with NN

S.Blundell, *Magnetism in Condensed matter* (2000)

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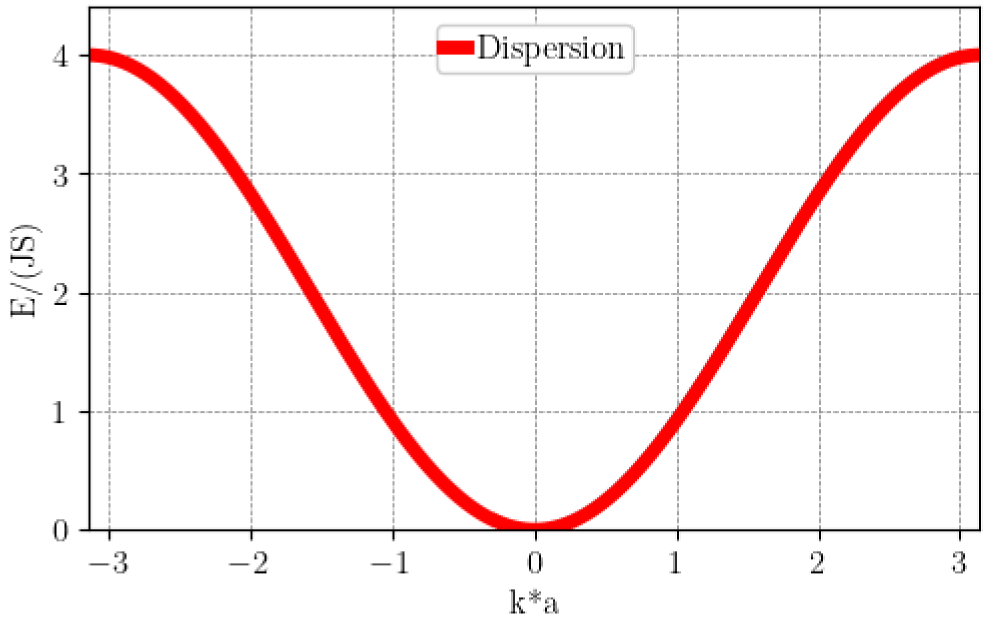

# Ferromagnetic Dispersion relation with a magnetic Field

**Addition of Zeeman term**

$$\mathcal{H}_Z = -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \vec{B} \cdot \hat{S}_i$$

$$= -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i B^z \cdot \hat{S}_i^z$$

$$= -\sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) + B^z \hat{S}_i^z$$

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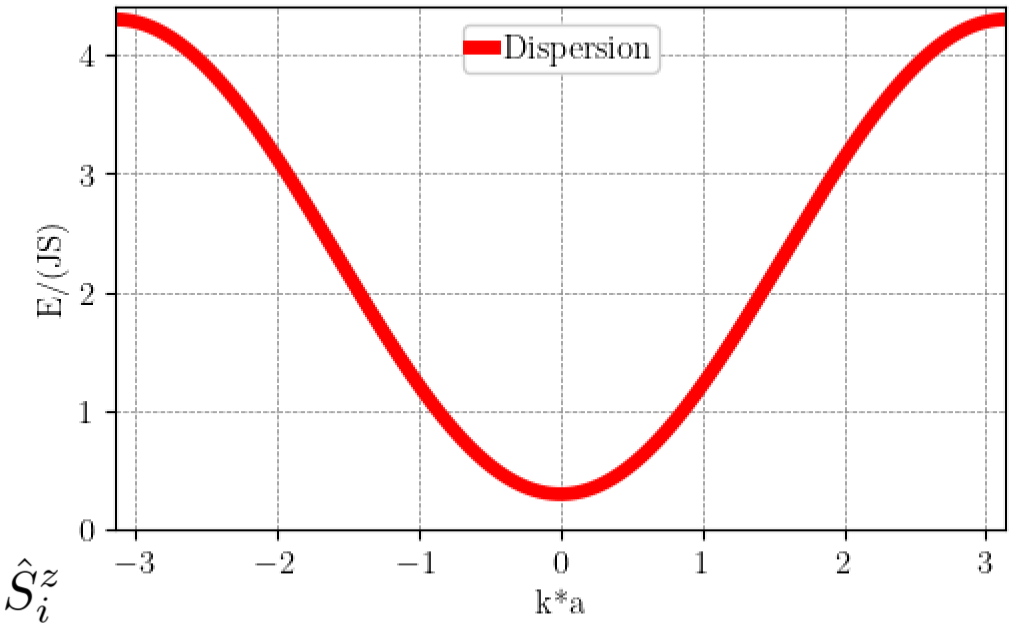


$$\mathcal{H}_Z |\vec{k}\rangle = \left( E_{0,Z} + B^z + 2JS \sum_{\Delta} \left( 1 - \cos(\vec{k}\vec{r}_\Delta) \right) \right) |k\rangle$$

$$E_{0,Z} = -zJN \cdot S^2 - NSB^z$$

# Recap

### First excited state

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) |n\rangle$$

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Collective excitation

$$|\langle m|\vec{k}\rangle|^2 = \left|\frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\delta_{m,n}\right|^2$$

$$= \frac{1}{N}\left|\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right)\right|^2 = \frac{1}{N}$$

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Collective excitation

By comparison to phonons:
-Well defined momentum $\hbar\vec{k}$
-Energy $\hbar E(\vec{k})$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ Quasiparticle

$$|\langle m|\vec{k}\rangle|^2 = \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2$$

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-Energy $\hbar E(\vec{k})$
} Quasiparticle

Reduces magnetization by 1 → Integer spin → Boson

# Recap

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Delocalization of a “flipped” spin over all sites

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$$|\langle m|\vec{k}\rangle|^2 = \left| \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \langle m|n\rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \delta_{m,n} \right|^2$$

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Collective excitation

By comparison to phonons:

$$\left.\begin{array}{l}\text{-Well defined momentum } \hbar\vec{k} \\ \text{-Energy } \hbar E(\vec{k})\end{array}\right\} \text{Quasiparticle}$$

Reduces magnetization by 1 → Integer spin → Boson

Magnon

# Magnons as Bosons

statistischen
Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik eines Einstein-Bose-Gases her bekannt ist;

*F.Bloch.* Z.Physik **61**, *206-219* (1930)

## Thermodynamical treatment

$$\hat{S}^z_{\mathrm{tot.}}|\vec{k}\rangle = (NS-1)|\vec{k}\rangle$$

Since it is a boson it must fulfill Bose-Einstein-statistic

$$n_{\mathrm{magnon}} \approx \int_0^\infty \frac{\mathrm{DOS}(\omega)\mathrm{d}\omega}{\exp(\hbar\omega/k_B T)-1} = \ldots \propto T^{3/2}$$

Bose factor

Number of
magnons @ T

# Magnons as Bosons

statistischen
Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik eines Einstein-Bose-Gases her bekannt ist;

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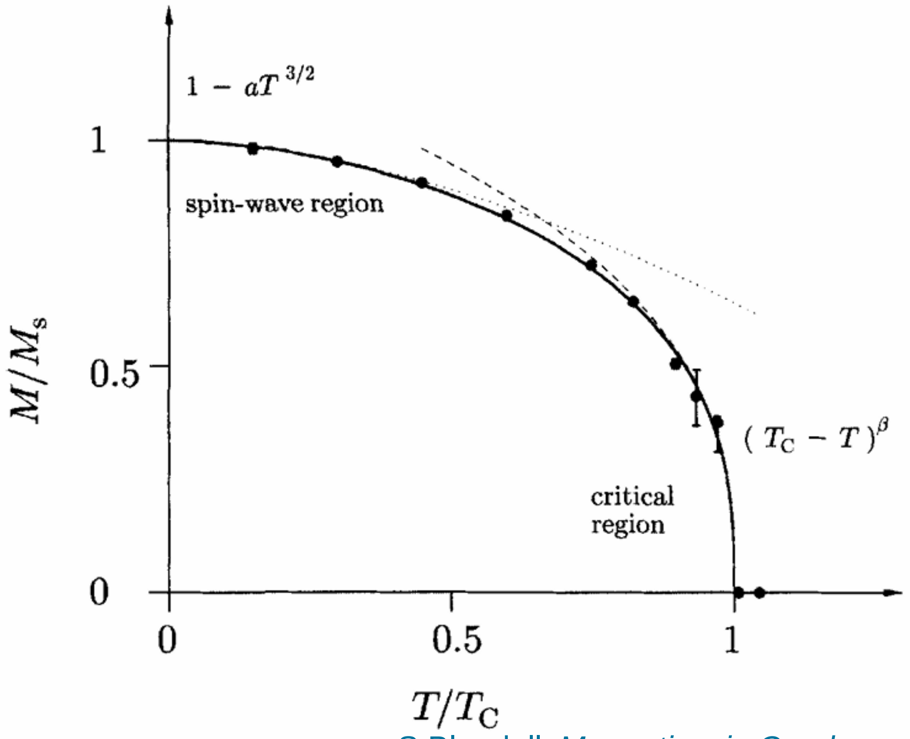


S.Blundell, *Magnetism in Condensed matter* (2000)

# Spin-Boson Transformation

### How to describe Magnons

The spin operators are not bosonic

Magnons are bosonic

Alternative approach:
LLG (analytically or numerically)
Or Schwinger representation
Or Dyson–Maleev representation

### Conditions

i. The **transformation** needs to be **Hermitian**, raising and lowering operators written as creation and annihilation boson operators need to be Hermitian conjugate of each other

ii. The **transformation** must be **unitary** to preserve the commutation relations between the spin operators.

iii. Must satisfy the **equality between the matrix elements** of the spin operators on $|0\rangle$ and the bosons on $|n\rangle$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

# A Spin-Boson Transformation

**Define higher magnon states**

$$|n_\sigma\rangle = |S, S, \dots, \underbrace{S - \sigma_n}_{n}, \dots, S\rangle$$

$$\hat{S}_n^+ |n_\sigma\rangle = \sqrt{2S}\sqrt{1 - \frac{\sigma_n - 1}{2S}}\sqrt{\sigma_n}|n_{\sigma-1}\rangle$$

$$\hat{S}_n^- |n_\sigma\rangle = \sqrt{2S}\sqrt{\sigma_n + 1}\sqrt{1 - \frac{\sigma_n}{2S}}|n_{\sigma+1}\rangle$$

DECEMBER 15, 1940 PHYSICAL REVIEW VOLUME 58

**Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet**

T. HOLSTEIN
*New York University, New York, New York*
AND
H. PRIMAKOFF*
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(Received July 31, 1940)

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Only for $\sigma \leq 2S$

$$\hat{S}_n^- |n_{2S}\rangle = \sqrt{2S}\sqrt{2S+1}\sqrt{1-\frac{2S}{2S}}|n_{2S+1}\rangle = 0$$

# A Spin-Boson Transformation

**Compare the "eigenvalues"**

$$|n_\sigma\rangle = |S, S, \ldots, \underbrace{S - \sigma_n}_{n}, \ldots, S\rangle$$

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$$\hat{S}_n^-|n_{2S}\rangle = \sqrt{2S}\sqrt{2S+1}\sqrt{1 - \frac{2S}{2S}}|n_{2S+1}\rangle = 0$$

Consider the states to be like 2nd quantized number states

$$\hat{a}_n|n_\sigma\rangle = \sqrt{\sigma_n}|n_{\sigma-1}\rangle$$

$$\hat{a}_n^\dagger|n_\sigma\rangle = \sqrt{\sigma_n + 1}|n_{\sigma+1}\rangle$$

# A Spin-Boson Transformation

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$$\hat{S}_n^- |n_\sigma\rangle = \sqrt{2S}\sqrt{\sigma_n + 1}\sqrt{1 - \frac{\sigma_n}{2S}}|n_{\sigma+1}\rangle$$

$$\hat{S}_n^+ = \sqrt{2S}\sqrt{1 - \frac{\boxed{\hat{a}_n^\dagger \hat{a}_n}}{2S}}\boxed{\hat{a}_n}$$

Consider the states to be like 2nd quantized number states

$$\boxed{\hat{a}_n^\dagger \hat{a}_n |n_{\sigma-1}\rangle = (\sigma_n - 1)|n_{\sigma-1}\rangle}$$

$$\boxed{\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n}|n_{\sigma-1}\rangle}$$

$$\hat{a}_n^\dagger |n_\sigma\rangle = \sqrt{\sigma_n + 1}|n_{\sigma+1}\rangle$$

# A Spin-Boson Transformation

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Consider the states to be like 2nd quantized number states

$$\boxed{\hat{a}_n^\dagger \hat{a}_n |n_{\sigma-1}\rangle = (\sigma_n - 1)|n_{\sigma-1}\rangle}$$

$$\boxed{\hat{a}_n^\dagger \hat{a}_n |n_\sigma\rangle = \sigma_n |n_\sigma\rangle}$$

$$\boxed{\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n}|n_{\sigma-1}\rangle}$$

$$\boxed{\hat{a}_n^\dagger |n_\sigma\rangle = \sqrt{\sigma_n + 1}|n_{\sigma+1}\rangle}$$

$$\hat{S}_n^- = \sqrt{2S}\,\boxed{\hat{a}_n^\dagger}\sqrt{1 - \frac{\boxed{\hat{a}_n^\dagger \hat{a}_n}}{2S}}$$

# The Holstein Primakoff Spin-Boson Transformation

## Combined

$$\hat{S}_n^- = \sqrt{2S}\hat{a}_n^\dagger\sqrt{1 - \frac{\hat{a}_n^\dagger\hat{a}_n}{2S}}$$

$$\hat{S}_n^+ = \sqrt{2S}\sqrt{1 - \frac{\hat{a}_n^\dagger\hat{a}_n}{2S}}\hat{a}_n$$

$$\hat{S}_n^z = S - \hat{a}_n^\dagger\hat{a}_n$$

$$\boxed{\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger\hat{a}_j \\ \hat{S}_j^+ &= \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger\hat{a}_j} \cdot \hat{a}_j \\ \hat{S}_j^- &= \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger\hat{a}_j}\end{aligned}}$$

*T-Holstein, H.Primakoff* PR **58**, 1098, (1940)

# The Holstein Primakoff Spin-Boson Transformation

## Checking the conditions

(i) Fulfilled

(ii) Does fulfill the commutation relations

(iii) Fulfilled by definition

$\Bigg\}$ for $\sigma \leq 2S$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

*T-Holstein, H.Primakoff* PR **58**, 1098, (1940)

$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm \hat{S}_i^\pm \delta_{i,j}$$ ✓

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = \ldots = 2\hat{S}_i^z \delta_{i,j}$$ ✓

# Holstein-Primakoff(HP)

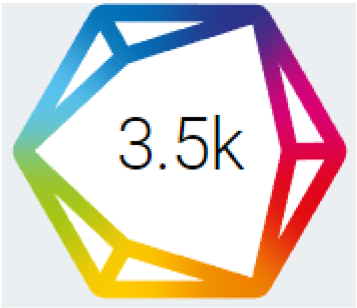
$$\begin{aligned}
\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\
\hat{S}_j^+ &= \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j \\
\hat{S}_j^- &= \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}
\end{aligned}$$

*T-Holstein, H.Primakoff* PR **58**, 1098, (1940)

$$\langle \hat{a}^\dagger \hat{a} \rangle < 2S$$

## Applications

HP framework is a powerful method for calculating dispersions and higher order interactions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# Linearized Holstein-Primakoff

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

*T-Holstein, H.Primakoff* PR **58**, 1098, (1940)

**Linearized HP**

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$
$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

## Sacrifices

Does not fulfill the boson commutation relations

Hence (ii) is violated

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$
$$[\hat{a}_i, \hat{a}_j] = 0$$

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hat{S}_i^z \delta_{i,j}$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \frac{1}{2S}\left[\hat{S}_i^+, \hat{S}_j^-\right] = \frac{1}{S}\hat{S}_i^z \delta_{i,j} \neq \delta_{i,j}$$
$$\approx \frac{S}{S}\delta_{i,j} = \delta_{i,j}$$

# Holstein-Primakoff

## Simplified application

Only linear Spin-Wave theory using the linearized HP

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

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$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$
$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

$$\mathcal{H} = -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta}$$
$$= -\sum_{i,\Delta} J \cdot \left(\frac{1}{2}\left\{\hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+\right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z\right)$$
$$= -\sum_{i,\Delta} J \cdot \left(S\left\{\hat{a}_i \hat{a}_{i+\Delta}^\dagger + \hat{a}_i^\dagger \hat{a}_{i+\Delta}\right\} + (S - \hat{n}_i)(S - \hat{n}_{i+\Delta})\right)$$
$$= -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{\hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta}\right\} + \underbrace{\sum_{i,\Delta} J \hat{n}_i \hat{n}_{i+\Delta}}_{\approx 0}$$
$$\approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{\hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta}\right\}$$

# Holstein-Primakoff

### Rewrite the Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator :

### Linearized HP

$$\hat{S}^z_j = S - \hat{a}^\dagger_j \hat{a}_j$$

$$\hat{S}^y_j = \frac{\hat{S}^+_j - \hat{S}^-_j}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left( \hat{a}_j - \hat{a}^\dagger_j \right)$$

$$\hat{S}^x_j = \frac{\hat{S}^+_j + \hat{S}^-_j}{2} \approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}^\dagger_j \right)$$

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# Holstein-Primakoff

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$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator :

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}_n = [\mathcal{H}, \hat{a}_n]$$

$$= 0 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \left[\hat{a}^\dagger_i \hat{a}_j, \hat{a}_n\right]$$

$$= -\sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \delta_{i,n}\hat{a}_j$$

$$= -\sum_{j} \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j$$

Problem: coupling between all sites!
Practically unsolvable for real systems

### Linearized HP

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# Holstein-Primakoff

## The new Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator :

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}_n = -\sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j$$

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## Going to Fourier-space

Lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \hat{a}^\dagger_n \exp(\mathrm{i}\vec{k}\vec{r}_n).$$

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Time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t}\hat{a}_n \right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

# Holstein-Primakoff

## The new Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

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Time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t}\hat{a}_n \right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Plug in

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( -\sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j \right) \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= \frac{1}{\sqrt{N}} \sum_n \sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= \dots$$

# Holstein-Primakoff

$$\mathcal{H} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

## Solving the differential equation

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \ldots = -\frac{\mathrm{i}}{\hbar}\mathscr{H}^{\mathrm{SW}}_{\vec{k}}\hat{a}(\vec{k}) \quad \Rightarrow \quad \hat{a}(\vec{k},t) = \hat{a}(\vec{k},0)\exp\left(-\mathrm{i}\omega_{\vec{k}}t\right)$$

With the spin-wave frequency

$$\begin{aligned}\hbar\omega_{\vec{k}} &= \mathscr{H}^{\mathrm{SW}}_{\vec{k}} \\ &= S\left(J_0 - J_{\vec{k}}\right)\end{aligned}$$

*L. Rósza,* Lecture Notes (2022)

$$J_{\vec{k}} \propto \sum_j J_{j,0}\exp(-\mathrm{i}\vec{k}(\vec{r}_j - \vec{r}_0)) \qquad J_0 = J_{\vec{k}=0}$$

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$$J_{\vec{k}} \propto \sum_j J_{j,0}\exp(-\mathrm{i}\vec{k}(\vec{r}_j - \vec{r}_0)) \qquad J_0 = J_{\vec{k}=0}$$

$$\hbar\omega_{\vec{k}} = \ldots \propto JS\sum_{\Delta}\left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)$$

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*L. Rósza,* Lecture Notes (2022)

$$\begin{aligned}\hat{a}_j(t) &= \frac{1}{\sqrt{N}}\sum_{\vec{k}}\exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k},t) \\ &= \frac{1}{\sqrt{N}}\sum_{\vec{k}}\exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)\end{aligned}$$

$$J_{\vec{k}} \propto \sum_j J_{j,0}\exp(-\mathrm{i}\vec{k}(\vec{r}_j - \vec{r}_0)) \qquad J_0 = J_{\vec{k}=0}$$

$$\hat{a}_j(\vec{k}_0,t) = \underbrace{\frac{1}{\sqrt{N}}\hat{a}(\vec{k}_0,0)}_{\hat{A}}\cdot\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)$$

$$\hbar\omega_{\vec{k}} = \ldots \propto JS\sum_{\Delta}\left(1-\cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)$$

$$\boxed{\hat{a}_j(\vec{k}_0,t) = \hat{A}\cdot\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)}$$

# Holstein-Primakoff

## Back to Real-space

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\boxed{\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)}$$

## Linearized HP

$$\boxed{\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right) \end{aligned}}$$

# Holstein-Primakoff

## Back to Real-space

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\boxed{\hat{a}_j(\vec{k}_0, t) = \hat{A}\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)}$$

S.Blundell, *Magnetism in Condensed matter* (2000)

## Linearized HP

$$\boxed{\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right) \end{aligned}}$$

$$\boxed{\begin{aligned} \hat{S}_j^z(t) &= S - |\hat{A}|^2 \\ \hat{S}_j^y(t) &= \sqrt{2S}\hat{A}\cdot\sin(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \\ \hat{S}_j^x(t) &= \sqrt{2S}\hat{A}\cdot\cos(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \end{aligned}}$$

This explains why the average we calculated earlier was 0

# Holstein-Primakoff & Bogoliubov transformation

**In-plane anisotropy**

**Considering more interactions**

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i$$

$$\mathscr{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov transformation

**Considering more interactions**

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$$\begin{aligned} \hat{S}_i \cdot \mathcal{K}_i \cdot \hat{S}_i &= K \hat{S}_i^x \cdot \hat{S}_i^x \\ &\approx K \frac{S}{2} \left(\hat{a}_i \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger\right) \\ &= K \frac{S}{2} \left(\hat{a}_i \hat{a}_i + 2\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger\right) + K \frac{S}{2} \end{aligned}$$

$$\mathcal{H}_{\text{iso}} \approx -E_0 - \sum_{i,j} \mathscr{H}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j$$

# Holstein-Primakoff & Bogoliubov transformation

**In-plane anisotropy**

$$\mathscr{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\mathcal{H} \approx -\tilde{E}_0 - \sum_{i,j} \tilde{\mathscr{H}}_{i,j}^{\mathrm{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2} \left( \underline{\hat{a}_i \hat{a}_i} + \underline{\hat{a}_i^\dagger \hat{a}_i^\dagger} \right)$$

How to diagonalize this ?

# Holstein-Primakoff & Bogoliubov transformation

In-plane anisotropy

$$\mathscr{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Considering more interactions

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Linearized HP

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$$\begin{aligned} \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i &= K \hat{S}_i^x \cdot \hat{S}_i^x \\ &\approx K\frac{S}{2}\left(\hat{a}_i\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger + \hat{a}_i^\dagger\hat{a}_i + \hat{a}_i^\dagger\hat{a}_i^\dagger\right) \\ &= K\frac{S}{2}\left(\hat{a}_i\hat{a}_i + 2\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i^\dagger\hat{a}_i^\dagger\right) + K\frac{S}{2} \end{aligned}$$

$$\mathcal{H} \approx -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2}\left(\hat{a}_i\hat{a}_i\right) + \mathrm{c.c}$$

How to diagonalize this ?

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger\right)^T$$

## How to diagonalize this

$$\begin{aligned}\mathcal{H} &= -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2} (\hat{a}_i \hat{a}_i) + \text{c.c} \\ &= -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$
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## How to diagonalize this

$$\begin{aligned}\mathcal{H} &= -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2} (\hat{a}_i \hat{a}_i) + \text{c.c} \\ &= -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

Seek a Bogoliubov transformation

$$\begin{aligned}\hat{a} &= U\hat{\alpha} + V^\star \hat{\alpha}^\dagger \\ \hat{a}^\dagger &= V\hat{\alpha} + U^\star \hat{\alpha}^\dagger\end{aligned}$$

That diagonalizes $\mathcal{H}$ to $$\mathcal{H} = -(\hat{\alpha}^\dagger, \hat{\alpha}) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$$

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$
$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

## How to diagonalize this

$$\mathcal{H} = -\left(\hat{\alpha}^\dagger, \hat{\alpha}\right)\begin{pmatrix}\omega & 0 \\ 0 & \omega\end{pmatrix}\begin{pmatrix}\hat{\alpha} \\ \hat{\alpha}^\dagger\end{pmatrix} + \text{c-number}$$

$$\hat{a} = U\hat{\alpha} + V^\star\hat{\alpha}^\dagger$$
$$\hat{a}^\dagger = V\hat{\alpha} + U^\star\hat{\alpha}^\dagger$$

Idea: Diagonalize with

$$\begin{pmatrix}H & \mathcal{K} \\ \mathcal{K}^\star & H^\star\end{pmatrix}\begin{pmatrix}u_n \\ v_n\end{pmatrix} = \omega_n \begin{pmatrix}u_n \\ v_n\end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

## How to diagonalize this

$$\mathcal{H} = -\left(\hat{\alpha}^\dagger, \hat{\alpha}\right) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \times$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger\right)^T$$

## How to diagonalize this

$$\mathcal{H} = -\left(\hat{\alpha}^\dagger, \hat{\alpha}\right)\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \times$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}}\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently $\mathcal{I} = \mathcal{T}\mathcal{I}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{I}\mathcal{T}$ $\quad \mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$

Thus it needs to be a pseudo unitary transformation

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

## How to diagonalize this

$$\mathcal{H} = -\left(\hat{\alpha}^\dagger, \hat{\alpha}\right) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

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Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \times$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently $\quad \mathcal{I} = \mathcal{T}\mathcal{I}\mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{I} \mathcal{T} \qquad \mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$

Thus it needs to be a pseudo unitary transformation

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger\right)^T$$

**How to diagonalize this**

$$\mathcal{H} = -\left(\hat{\alpha}^\dagger, \hat{\alpha}\right) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently

$$\mathcal{I} = \mathcal{T}\mathcal{I}\mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{I}\mathcal{T} \qquad \mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$\mathbb{I} = (\mathcal{I}\mathcal{T}) \cdot (\mathcal{I}\mathcal{T}^\dagger) = (\mathcal{I}\mathcal{T}^\dagger) \cdot (\mathcal{I}\mathcal{T})$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} = \mathcal{T}^{-1} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

## How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K} & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

What we need to solve

$$\underbrace{\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}}_{=:\mathscr{M}} = \begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}^\dagger \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} = \mathcal{T}^{-1} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Such that

$$\mathcal{H} = -\tilde{E}_0 - \sum_{i,j} \mathscr{M}_{i,j}^{\mathrm{diag}} \cdot \hat{\alpha}_i^\dagger \hat{\alpha}_j$$

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

# Holstein-Primakoff & Bogoliubov transformation

## How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - \sum_{i,j} \tilde{\mathscr{H}}^{\mathrm{SW}}_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2}\left(\underline{\hat{a}_i \hat{a}_i} + \underline{\hat{a}_i^\dagger \hat{a}_i^\dagger}\right)$$

$$= -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K} & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$= -\tilde{E}_0 - \sum_{i,j} \mathscr{M}^{\mathrm{diag}}_{i,j} \cdot \hat{\alpha}_i^\dagger \hat{\alpha}_j$$

Lattice Fourier transformation

$$\mathcal{H} = \sum_{\vec{k}} \mathscr{M}_{\vec{k}} \cdot \hat{\alpha}^\dagger(\vec{k})\hat{\alpha}(\vec{k})$$

# Visualising a Magnon

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = -\frac{\gamma}{(1+\alpha^2)\mu_S}\vec{S}_i \times \left(\vec{H}_i(t) + \alpha\vec{S}_i \times \vec{H}_i(t)\right)$$

$$\vec{H}_i(t) = -\frac{\partial \mathcal{H}}{\partial \vec{S}_i}$$

## Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

Time integration with Heun’s methods

Additional property is the damping $\alpha$

Engine_LLG\Test.ipynb

# Visualising a Magnon

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = -\frac{\gamma}{(1+\alpha^2)\mu_S}\vec{S}_i \times \left(\vec{H}_i(t) + \alpha\vec{S}_i \times \vec{H}_i(t)\right)$$

$$\vec{H}_i(t) = \vec{\xi}_i(t) - \frac{\partial\mathcal{H}}{\partial\vec{S}_i}$$

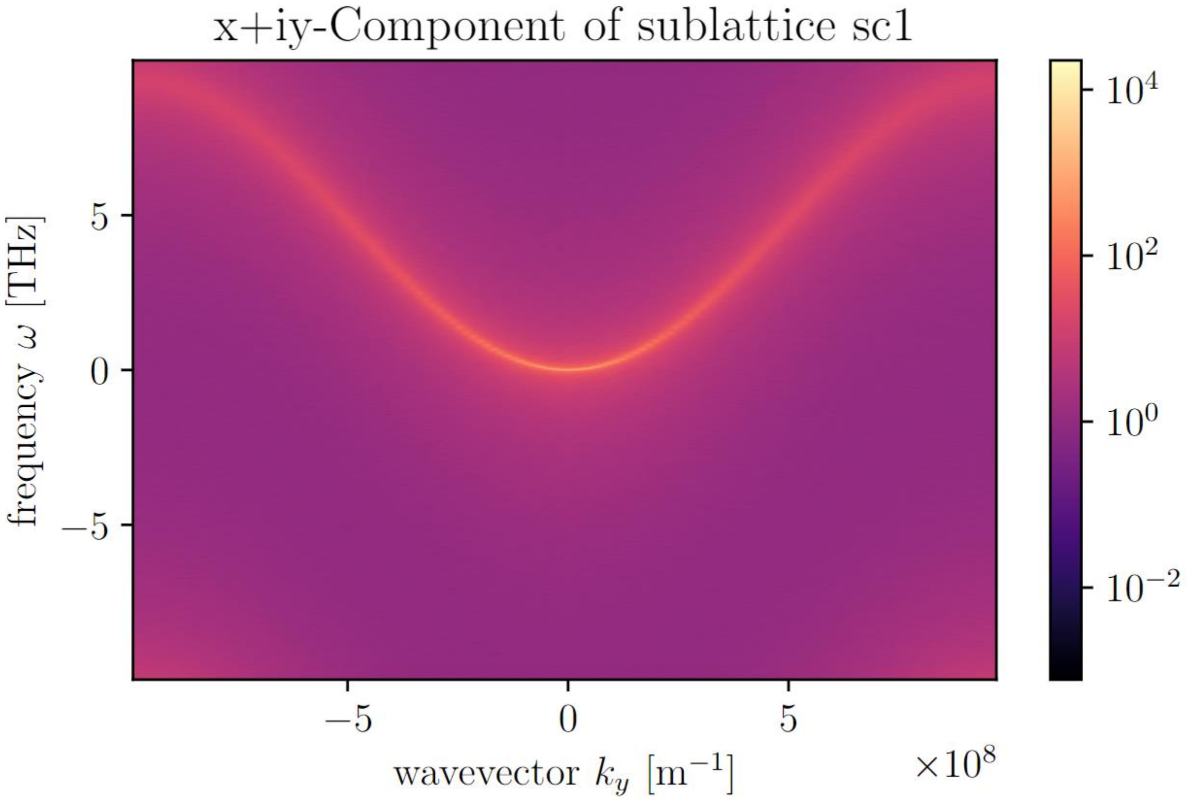
## Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

Time integration with Heun's methods

Additional property is the damping $\alpha$

Engine_LLG\Test.ipynb



# Outlook

## Current research

Easily obtain highly interesting dispersion relations.

Can be used for computing, without energy loss through Joule heating

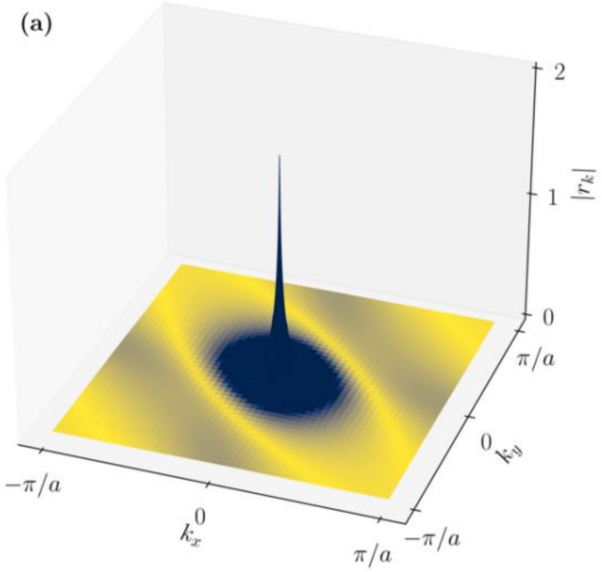
This is ongoing research in

- Spin wave diodes *J.Lan et.al.* PRX **5**, 041049, (2015)
- Spin wave transitors *A.Chumak et.al.* Nature C. **5**, 4700, (2014)

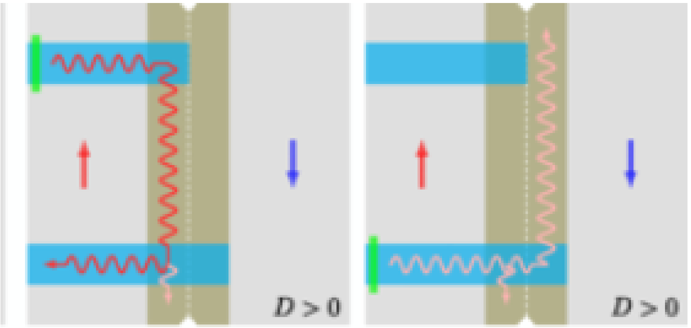
The most modern research even includes Altermagnets, which have distinct symmetry enforced properties regarding the dispersion in different directions

Topological magnons *P. McClarty* Annual Reviews **13**, 171-190, (2022)

Squeezed magnons *D. Wuhrer et.al. PR* **5**, 043124, (2023)

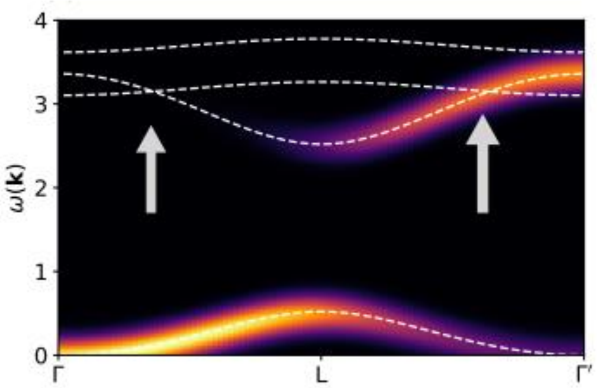



*D. Wuhrer et.al. PR* **5**, 043124, (2023)

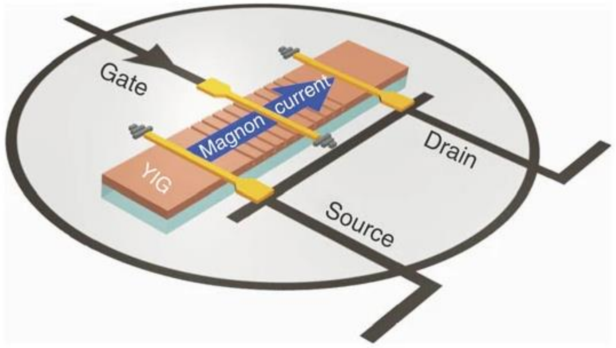



(c) Diode - forward direction (d) Diode - reverse direction

*J.Lan et.al.* PRX **5**, 041049, (2015)




Magnon transistor scheme




*A.Chumak et.al.* Nature C. **5**, 4700, (2014)