Fortschr. Phys. **45** (1997) 2, 145–156

Holstein-Primakoff/Bogoliubov Transformations and the Multiboson System

MICHAEL MARTIN NIETO [1])

Theoretical Division, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545, U.S.A. and Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany

D. RODNEY TRUAX [2])

Department of Chemistry, University of Calgary, Calgary, Alberta T2N 1N4, Canada

Abstract

As an aid to understanding the *displacement operator* definition of squeezed states for arbitrary systems, we investigate the properties of systems where there is a Holstein-Primakoff or Bogoliubov transformation. In these cases the *ladder-operator or minimum-uncertainty* definitions of squeezed states are equivalent to an extent displacement-operator definition. We exemplify this in a setting where there are operators satisfying $[A, A^\dagger] = 1$, but the A's are not necessarily the Fock space a's; the multiboson system. It has been previously observed that the ground state of a system often can be shown to to be a coherent state. We demonstrate why this must be so. We close with a discussion of an alternative, effective definition of displacement-operator squeezed states.

1. Introduction

As has now been known and studied for some time, there are three equivalent, widely-used definitions of the coherent states of the harmonic oscillator [1]–[7]. These are (1) the minimum-uncertainty, (2) annihilation- (or, more generally, ladder-) operator, and (3) displacement-operator methods. These methods have been extended to the squeezed states of the harmonic oscillator. Further, with one exception, general coherent and squeezed states have been obtained for general systems by these three methods. That exception is a general definition of squeezed states by the displacement-operator method.

With an aim towards understanding a general method, we can study systems where such a definition works. Specifically, after reviewing the coherent and sequeezed states for the harmonic oscillator and more general systems, we focus on why displacement-operator squeezed states often can not be obtained by a naive generalization of the harmonic-oscillator case. This is when there is, in general, no Bogoliubov transformation.

This problem does not exist in certain systems. In particular, we here study the multiboson formalism of BRANDT and GREENBERG [8], where the multi-boson operators obey canonical commutation relations, and hence one can proceed with calculations in the standard way. Elsewhere [9], we will study time-dependent systems which have isomorphic symmetry algebras.

We also explain the property of these definitions of squeezed and coherent states which is that the ground state is a member of the set of coherent states. In closing, we discuss an alternative, effective method for defining displacement-operator squeezed states.

[1]) Email: mmn@pion.lanl.gov
[2]) Email: truax@acs.ucalgary.ca

2. The Coherent and Squeezed States of the Harmonic Oscillator

2.1. Coherent states

Given the canonical commutation relations

$$[a, a^\dagger] = 1\,, \qquad [a, a] = 0\,, \tag{1}$$

where we adopt the realization

$$a = \frac{1}{\sqrt{2}}\,(x + ip)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}\,(x - ip)\,, \tag{2}$$

the definitions of displacement-operator and ladder-operator coherent states are well known. They are

$$D(\alpha)\,|0\rangle = |\alpha\rangle \tag{3}$$

and

$$a|\alpha = \alpha|\alpha\rangle\,, \tag{4}$$

where

$$D(\alpha) = \exp\left[\alpha a^\dagger - \bar{\alpha} a\right] = \exp\left[-\frac{1}{2}\,|\alpha|^2\right] \exp\left[\alpha a^\dagger\right] \exp\left[-\bar{\alpha} a\right] \tag{5}$$

and

$$|\alpha\rangle = \exp\left[-\frac{1}{2}\,|\alpha|^2\right] \sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{6}$$

The last equality in Eq. (5) comes from using a Baker-Campbell-Hausdorff relation. Observe that the definition (4) follows from the definition (3) by

$$[a, D(\alpha)] = \alpha D(\alpha)\,. \tag{7}$$

The coherent-state wave functions are $(m\omega/\hbar \to 1)$

$$\psi_{cs}(x) = \pi^{-1/4} \exp\left[-\frac{(x - x_0)^2}{2} + ip_0 x\right], \tag{8}$$

$$x_0 = \langle x\rangle\,, \qquad p_0 = \langle p\rangle\,, \tag{9}$$

$$\mathrm{Re}(\alpha) = x_0/2^{1/2}\,, \qquad \mathrm{Im}(\alpha) = p_0/2^{1/2}\,. \tag{10}$$

That is, the states are Gaussians with the width being that of the ground state.

2.2 Squeezed states

Squeezed states [10]–[14] can be defined by the displacement-operator method as the product of a unitary displacement operator and a unitary squeeze operator acting on the ground state:

$$D(\alpha)\,S(z)\,|0\rangle \equiv |\alpha, z\rangle\,, \qquad z = z_1 + iz_2 = re^{i\theta}\,. \tag{11}$$

θ is a phase which defines the starting time, $t_0 = (\theta/2\omega)$. $S(z)$ is given by

$$S(z) = \exp\left[\frac{1}{2}\,za^\dagger a^\dagger - \frac{1}{2}\,\bar{z}aa\right] \tag{12}$$

$$= \exp\left[\frac{1}{2}\,e^{i\theta}(\tanh r)\,a^\dagger a^\dagger\right]\left(\frac{1}{\cosh r}\right)^{\left(\frac{1}{2}+a^\dagger a\right)}\exp\left[-\frac{1}{2}\,e^{-i\theta}(\tanh r)\,aa\right] \tag{13}$$

$$= \exp\left[\frac{1}{2}\,e^{i\theta}(\tanh r)a^\dagger a^\dagger\right](\cosh r)^{-1/2}\sum_{n=0}^{\infty}\frac{(\operatorname{sech} r - 1)^n}{n!}\,(a^\dagger)^n\,(a)^n$$

$$\times \exp\left[-\frac{1}{2}\,e^{-i\theta}(\tanh r)\,aa\right], \tag{14}$$

where Eqs. (13) and (14) are obtained from BCH relations. Observe that

$$D(\alpha)\,S(z) = S(z)\,D(\gamma)\,, \qquad \gamma = \alpha\cosh r - \bar{\alpha}e^{i\theta}\sinh r\,. \tag{15}$$

Therefore, the ordering of D and S is only a convention.

The squeezed-state wave functions are given by a more complicated form of Eq. (8). Specifically, they are [15]

$$\psi_{ss} = D(\alpha)\,S(z)\,\psi_0$$
$$= \frac{1}{\pi^{1/4}}\,\frac{\exp\left[-ix_0p_0/2\right]}{\left[\mathcal{S}(1+i2\varkappa)\right]^{1/2}}\exp\left[-(x-x_0)^2\left(\frac{1}{2\mathcal{S}^2(1+i2\varkappa)} - i\varkappa\right) + ip_0x\right], \tag{16}$$

where

$$\mathcal{S} = \cosh r + \frac{z_1}{r}\,\sinh r = e^r\cos^2\frac{\varphi}{2} + e^{-r}\sin^2\frac{\varphi}{2} \tag{17}$$

and

$$\varkappa = \frac{z_2\sinh r}{2r\mathcal{S}}\,. \tag{18}$$

These wave functions are Gaussians which, in general, do not have the width of the ground state; i.e., they are squeezed by the squeeze parameters $\mathcal{S}$, $\varkappa$. The most commonly studied example is when z is real and positive, giving

$$\psi_{ss}(x) = [\pi s^2]^{-1/4}\exp\left[-\frac{(x-x_0)^2}{2s^2} + ip_0x\right], \qquad s = e^r\,. \tag{19}$$

The elements involved in S actually are an SU(1,1) group defined by

$$K_+ = \frac{1}{2}\,a^\dagger a^\dagger\,, \qquad K_- = \frac{1}{2}\,aa\,, \qquad K_0 = \frac{1}{2}\left(N + \frac{1}{2}\right), \tag{20}$$

where $N = a^\dagger a$. The operators K_0, $K_\pm$ satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm\,, \qquad [K_+, K_-] = -2K_0\,. \tag{21}$$

Therefore, S can be given by

$$S(z) = \exp\left[zK_+ - \bar{z}K_-\right] \tag{22}$$

$$= \exp\left[e^{i\theta}(\tanh r)\,K_+\right]\left(\frac{1}{\cosh r}\right)^{2K_0} \exp\left[-e^{-i\theta}(\tanh r)\,K_-\right]. \tag{23}$$

The commutation relations (1) and (21) close with

$$\begin{aligned}
&[K_+, a^\dagger] = 0\,, \qquad [K_-, a^\dagger] = a\,, \qquad [K_+, a] = -a^\dagger\,,\\
&[K_-, a] = 0\,, \qquad [K_0, a^\dagger] = \frac{1}{2}\,a^\dagger\,, \qquad [K_0, a] = -\frac{1}{2}\,a\,.
\end{aligned} \tag{24}$$

The ladder-operator definition of the squeezed states is

$$[\mu a - \nu a^\dagger]\,|\alpha, z\rangle = \beta|\alpha, z\rangle\,. \tag{25}$$

Again this follows from the displacement-operator definition because

$$\begin{aligned}
b &\equiv S(z)^{-1}\, aS(z) = (\cosh r)\,a + e^{i\theta}(\sinh r)\,a^\dagger\,,\\
b^\dagger &\equiv S(z)^{-1}\, a^\dagger S(z) = (\cosh r)\,a^\dagger + e^{-i\theta}(\sinh r)\,a\,.
\end{aligned} \tag{26}$$

where

$$[b, b^\dagger] = 1\,, \qquad b \equiv \mu a + \nu a^\dagger\,, \qquad |\mu|^2 - |\nu|^2 = 1\,. \tag{27}$$

Eq. (26) is a HOLSTEIN-PRIMAKOFF [16] or BOGOLIUBOV [17] transformation. When such a transformation exists, such as for the harmonic oscillator and for some other cases [23]–[25], there is no problem defining displacement-operator squeezed states. However, such a transformation does not always exist, and that is at the crux of the problem of finding a general definition for displacement-operator squeezed states.

Lastly, we note the time-dependent uncertainties in x and p. They are [18]

$$[\Delta x(t)]^2_{(\alpha,z)} = \frac{1}{2}\left[s^2\cos^2\omega t + \frac{1}{s^2}\sin^2\omega t\right], \tag{28}$$

$$[\Delta p(t)]^2_{(\alpha,z)} = \frac{1}{2}\left[\frac{1}{s^2}\cos^2\omega t + s^2\sin^2\omega t\right], \tag{29}$$

$$[\Delta x(t)]^2_{(\alpha,z)}\,[\Delta p(t)]^2_{(\alpha,z)} = \frac{1}{4}\left[1 + \frac{1}{4}\left(s^2 - \frac{1}{s^2}\right)^2 \sin^2[\omega t]\right]. \tag{30}$$

3. Generalized Coherent and Squeezed States

As discussed in Ref. [19], generalizations of the displacement-operator and ladder-operator coherent states have been widely discussed and studied [3, 20, 21, 22]. Also, a generalization of the minimum-uncertainty coherent states was found [26, 27], and this method turned out to also yield the generalized squeezed states as a byproduct.

Recently, we gave a generalized ladder-operator method to define squeezed states for arbitrary systems [19], and there we pointed out the problem which is at the crux of the present study. In general there is no Bogoliubov transformation and hence no connection between the ladder-operator and displacement-operator methods for defining squeezed states.

This can be exemplified by considering the ordinary squeeze operator acting on the ground state, with no displacement operator:

$$S(z)\,|0\rangle = |z\rangle\,. \tag{31}$$

In this form, $S(z)$ is the SU(1,1) displacement operator, and hence the states $|z\rangle$ are the SU(1,1) coherent states. Note that these coherent states have only even occupation numbers in the number basis. (Indeed, recall that one of early names for the squeezed states was "two-photon coherent states" [10].)

But if S is the displacement operator for SU(1,1), what is the SU(1,1) squeeze operator? A first guess would be to square the elements of S, i.e., to quare aa and $a^\dagger a^\dagger$ to yield operators exponentiated to the fourth power. But this leads to operators that are not well-defined [28, 29]; that is, all operators

$$U_j = \exp\left[\hat{z}_j(a^\dagger)^j - \hat{\bar{z}}_j(a)^j\right], \qquad j = 3,\,4,\,5,\,\ldots \tag{32}$$

So, there is no naive higher-order squeezing. Another way to state this is that there exist no simple operators which obey

$$\hat{S}(y)^{-1}\,aa\hat{S}(y) = \mu aa + \nu a^\dagger a^\dagger\,. \tag{33}$$

That is, there is no Bogoliubov transformation for the SU(1,1) elements. Hence, there is no obvious way to define the SU(1,1) squeezed states by the displacement-operator method.

4. Multiboson Operators

In a program to circumvent the problems with naive multiboson queezing, a productive collaboration [30]–[35] proposed using the generalized Bose operators of BRANDT and GREENBERG [8]. These latter two observed that if one defines the operators

$$A_j = \sum_{k=0}^{\infty} \alpha_{jk}(a^\dagger)^k\, a^{k+j}\,, \qquad j \geq 2\,, \tag{34}$$

$$\alpha_{jk} = \sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k-l)!}\left[\frac{1+[[l/j]]}{l!(l+j)!}\right]^{1/2} e^{i\varrho_l}\,, \tag{35}$$

where we denote the greatest-integer function by $[[y]]$, and the ϱ_l are arbitrary phases. Then, we have

$$[A_j^\dagger, A_j] = 1\,. \tag{36}$$

That is, these functions satisfy the canonical commutation relations even though they are not the ordinary boson operators. They also satisfy

$$[N, A_j] = [a^\dagger a, A_j] = -jA_j\,, \tag{37}$$

and

$$A_j|jn+k\rangle = \sqrt{n}|j(n-1)+k\rangle\,, \tag{38}$$

$$A_j^\dagger|jn+k\rangle = \sqrt{(n+1)}\,|j(n+1)+k\rangle\,, \qquad 0 \le k < j\,. \tag{39}$$

Note that for a given j we have j different sets of states. Each of them starts at a different lowest state $|k\rangle$, where $0 \le k < j$; i.e., $|0\rangle$, $|1\rangle$, $|2\rangle$, ..., $|j-1\rangle$.

If one acts on eigenstates of N, then from the normal-ordering theorems of Wilcox [36], a very useful form of A_j can be given [37]

$$A_j^\dagger = \left[[[\tilde{N}/j]]\,\frac{(\tilde{N}-j)!}{\tilde{N}!}\right]^{1/2} (a^\dagger)^j\,, \tag{40}$$

where $\tilde{N}$ is the eigenvalue of the operator N in the number operator basis.

The collaboration of Refs. [30]–[35] concentrated on investigating the properties of the states defined by

$$D(\alpha)\,V(z)\,|0\rangle = D(\alpha)\exp\left[zA_j^\dagger - \bar{z}A_j\right]|0\rangle = |\alpha, z_j\rangle\,. \tag{41}$$

In other words, they took an ordinary coherent state and then squeezed this state by the j-photon operators of A_j and $A_j^\dagger$. (Also, they studied [34] the properties of states obtained from a generalized set of Weyl-Heisenberg operators, A_j^η.)

5. Coherent and Squeezed States for the Multiboson Systems

5.1. Coherent states

Now, from our point of view, of finding general and consistent methods of obtaining coherent and squeezed states, another path suggests itself. Since the A_j's obey the canonical commutation relations of Eq. (36), which are identical in form to Eq. (1), this means one can use *these* operators in displacement operators. That is, we consider the operator V of equation (41) not to be a multiboson squeeze of a coherent state, but rather a multiboson displacement operator:

$$D_j(\alpha) = \exp\left[\alpha A_j^\dagger - \bar{\alpha}A_j\right] = \exp\left[-\frac{1}{2}\,|\alpha|^2\right]\exp\left[\alpha A_j^\dagger\right]\exp\left[-\bar{\alpha}A_j\right]. \tag{42}$$

Therefore, the multi-boson coherent states are

$$|\alpha(j,k)\rangle = D_j(\alpha)\,|k\rangle = \exp\left[-\frac{1}{2}\,|\alpha|^2\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|jn+k\rangle\,. \tag{43}$$

Again observe that for a given j we have j different sets of (coherent) states. Each of them again starts at a different lowest state $|k\rangle$, where $0 \le k < j$; i.e., $|0\rangle$, $|1\rangle$, $|2\rangle$, ..., $|j-1\rangle$.

That it why we label the states by the couple (j, k). [The states $|\alpha(j, 0)\rangle$ were studied in Ref. [33].]

These coherent states are, of course, consistent with the ladder-operator definition,

$$A_j|\alpha(j, k)\rangle = \alpha\,|\,\alpha(j, k)\rangle\,. \tag{44}$$

By using the number-state basis of the wave functions,

$$\psi_n = \left(\frac{a_0}{\pi^{1/2}2^n n!}\right)^{1/2} \exp\left[-\frac{1}{2}\,a_0^2x^2\right] H_n(a_0x)\,, \tag{45}$$

where $a_0^2 = (m\omega/\hbar)$ will now be set to 1 and the H are the Hermite polynomials, one can write the normalized coherent state wave functions as

$$\psi_{cs}(j, k)\,(x) = \pi^{-1/4} \exp\left[-\frac{1}{2}\,(|\alpha|^2 + x^2)\right] I_{(j,k)}(\alpha, x)\,, \tag{46}$$

where I is the sum

$$I_{(j,k)}(\alpha, x) = \sum_{n=0}^{\infty} \frac{\alpha^n H_{jn+k}(x)}{[n!(jn+k)!\,2^{jn+k}]^{1/2}}\,. \tag{47}$$

Note that for $(j, k) = (1, 0)$, we obtain the usual generating function result [38] for the ordinary coherent states,

$$I_{(1,0)}(x) = \exp\left[\sqrt{2}\,\alpha x - \alpha^2/2\right]. \tag{48}$$

The "natural quantum operators" for this system are [26, 27] (in dimensionless units)

$$X_j \equiv \frac{1}{\sqrt{2}}\,[A_j + A_j^\dagger]\,, \qquad P_j \equiv \frac{1}{i\sqrt{2}}\,[A_j - A_j^\dagger]\,. \tag{49}$$

But then, the Heisenberg-Weyl algebra tells us immediately that these are the operators directly connected to the minimum-uncertainty method. Therefore, we have that [19]

$$(\Delta X_j)^2_{(j,k)} = 1/2\,, \qquad (\Delta P_j)12_{j,k)} = 1/2\,. \tag{50}$$

We can also obtain information for the uncertainties of the physical position and momentum, x and p. We immediately observe that

$$\langle x\rangle_{(j,k)} = \langle p\rangle_{(j,k)} = 0\,, \qquad j > 1\,. \tag{51}$$

(For $j = 1$ we have the ordinary harmonic oscillator). For $j > 2$, we have, then, that

$$\begin{aligned}\langle x^2\rangle_{(j,k)} &= (\Delta x)^2_{(j,k)} = \langle p^2\rangle_{(j,k)} = (\Delta p)^2_{(j,k)}\\ &= \exp\left[-|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}\left[jn + k + \frac{1}{2}\right]\\ &= \frac{1}{2} + k + j|\alpha|^2\,, \qquad j > 2\,.\end{aligned} \tag{52}$$

The case $j = 2$ is slightly more complicated because the operators x^2 and p^2 connect different numbers states in the expectation values. In particular,

$$\langle x^2 \rangle_{(2,k)} = (\Delta x)^2_{(2,k)} = \frac{1}{2} + k + 2|\alpha|^2 + C_{(2,k)} \tag{53}$$

$$\langle p^2 \rangle_{(2,k)} = (\Delta p)^2_{(2,k)} = \frac{1}{2} + k + 2|\alpha|^2 - C_{(2,k)}\,, \tag{54}$$

where

$$C_{(2,k)} = \frac{1}{2}\left[\langle a^2 \rangle_{(2,k)} + \langle (a^\dagger)^2 \rangle_{(2,k)}\right], \tag{55}$$

which evaluates to

$$C_{(2,k)} = (\alpha + \bar{\alpha}) \exp\left[-|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \left[\frac{(n+1+k/2)\,(n+1/2+k/2)}{n+1}\right]^{1/2}. \tag{56}$$

5.2. Squeezed states

Because the A_j's define a Heisenberg-Weyl algebra, one can therefore define an SU(1,1) squeeze algebra in the normal way:

$$K_{j+} = \frac{1}{2} A_j^\dagger A_j^\dagger\,, \qquad K_{j-} = \frac{1}{2} A_j A_j\,, \qquad K_{j0} = \frac{1}{2}\left(A_j^\dagger A_j + \frac{1}{2}\right). \tag{57}$$

Then all these A_j's and K_j's again have the same commutation relations as before, and so all the results of the ordinary harmonic oscillator coherent and squeezed states goes through in the same manner, only with the a's being changed into the A_j's. The squeeze operators are therefore

$$S_j(z) = \exp\left[zK_{j+} - \bar{z}K_{j-}\right] \tag{58}$$

$$= \exp\left[e^{i\theta}(\tanh r)\, K_{j+}\right] \left(\frac{1}{\cosh}\, r\right)^{2K_{j0}} \exp\left[-e^{i\theta}(\tanh r)\, K_{j-}\right], \tag{59}$$

where

$$z = re^{i\theta}\,, \tag{60}$$

meaning the squeezed states are

$$D_j(\alpha)\, S_j(z)\, |k\rangle = |\alpha, z(j, k)\rangle\,. \tag{61}$$

Furthermore, all the mathematics of the ordinary squeezed states follows automatically, just changing notation. For example, there is a Bogoliubov transformation:

$$B_j \equiv S_j(z)^{-1} A_j S_j(z) = (\cosh r)\, A_j + e^{i\theta}(\sinh r)\, A_j^\dagger\,, \tag{62}$$

$$B_j^\dagger \equiv S_j(z)^{-1} A_j^\dagger S_j(z) = (\cosh r)\, A_j^\dagger + e^{-i\theta}(\sinh r)\, A_j\,. \tag{63}$$

where

$$[B_j, B_j^\dagger] = 1\,, \qquad B_j \equiv \mu A_j + \nu A_j^\dagger\,, \qquad |\mu|^2 - |\nu|^2 = 1\,. \tag{64}$$

This means, of course, that there is an equivalent ladder-operator definition of these squeezed states:

$$[\mu A_j - \nu A_j^\dagger]\,|\alpha, z(j, k)\rangle = \beta|\alpha, z(j, k)\rangle\,. \tag{65}$$

Again, from the the Heisenberg-Weyl algebra, it follows that

$$(\Delta X_j)^2_{ss}\,(\Delta P_j)^2_{ss} = 1/4\,. \tag{66}$$

Of course, being squeezed states the above equality holds at $t = 0$ and oscillates, and the uncertainty in each quadrature also oscillates.

6. The Ground State as a Coherent State

In finding the coherent and squeezed states for general systems, it has been noted that the ground state (or extremal state) is always a member for the set of coherent states [19, 26]. This is also true in the multi-boson case and we want to show that why it is true in general. Before continuing, however, note that this makes intuitive sense. The ground state is the closest quantum state to zero motion, which corresponds to a classical particle at rest. Therefore, the most-classical like states should include this state.

Starting from a minimum-uncertainty Hamiltonian system, the classical Hamiltonian is transformed to classical variables that vary as the sin and the cosine of the classical ωt. In these variables the Hamiltonian can be written as

$$H_{cl} = X^2/2 + P^2/2\,. \tag{67}$$

This is harmonic-oscillator like. Indeed, for the rest of this discussion keep the harmonic oscillator in mind for intuituve aid.

When the classical variables are changed to quantum operators, it is found that

$$X = \frac{\mathscr{A} + \mathscr{A}^\dagger}{\sqrt{2}}\,, \qquad P = \frac{\mathscr{A} - \mathscr{A}^\dagger}{i\sqrt{2}}\,, \tag{68}$$

where the $\mathscr{A}$'s are the lowering operators of the system. In general, these operators may be n-dependent or have to be made Hermitian with respect to the adjoint, but the statement holds.

Therefore, the states which minimize the uncertainty relation between X and P,

$$[X, P] = iG\,, \tag{69}$$

are those (squeezed) states which satisfy the eigenvalue equation

$$\left[X + \frac{i\Delta X}{\Delta P}\,P\right]\psi_{mus} = \left[\langle X\rangle + \frac{i\Delta X}{\Delta P}\,\langle P\rangle\right]\psi_{mus}\,. \tag{70}$$

[When dealing with symmetry, non-Hamiltonian systems, the starting point for the study is to simply consider the implications of the commutation relation (69)].

These states are, in general, squeezed states. This can be seen by writing X and P in terms of $\mathscr{A}$ and $\mathscr{A}^\dagger$. Then the left hand side of Eq. (70) is proportional to a linear combination of $\mathscr{A}$ and $\mathscr{A}^\dagger$, just like after any Bogoliubov transformation. To change to a coherent state, the relative uncertainties must be equal, i.e., $(\Delta X)/(\Delta P) = 1$, but then the left hand side of Eq. (70) is proportional simply to $\mathscr{A}$. Then taking the case corresponding to the smallest classical motion, $\langle X\rangle = \langle P\rangle = 0$, one is left with the equation

$$\mathscr{A}\psi_{mus} = 0\,. \tag{71}$$

But the state that is annihilated by the lowering operator is the ground state.

7. An Alternative, Effective Definition for Displacement-Operator Squeezed States

We close with a comment on how an alternative method can be used to define "displacement operator" squeezed states. This method can be used for the systems under discussion: systems with minimum-uncertainty, ladder-operator, and displacement operator coherent states, but only minimum-uncertainty or ladder-operator squeezed states. An example, where it has been used, suffices to explain the procedure.

The even and odd coherent states [39, 40] are defined in terms of the double destruction operator:

$$aa|\alpha\rangle_\pm = \alpha^2|\alpha\rangle_\pm\,. \tag{72}$$

They also can be defined in terms of an unusual displacement operator,

$$|\alpha\rangle_\pm = D_\pm(\alpha)\,|0\rangle = [2(1 \pm \exp[-2|\alpha|^2]^{-1/2}\,[D(\alpha)(-\alpha)]\,|0\rangle\,, \tag{73}$$

where D is the ordinary coherent-state displacement operator.

The even and odd squeezed states are generalized to those states satisfying the eigenvalue equation

$$\left[\left(\frac{1+q}{2}\right)aa + \left(\frac{1-q}{2}\right)a^\dagger a^\dagger\right]\psi_{ss} = \alpha^2\psi_{ss}\,. \tag{74}$$

The solutions are [19]

$$\begin{aligned}\psi_{Ess} &= N_E \exp\left[-\frac{x^2}{2}\,(q+\sqrt{q^2-1})\right]\\ &\quad\times \Phi\left(\left[\frac{1}{4}+\frac{\alpha^2}{2\sqrt{q^2-1}}\right], \frac{1}{2};\, x^2\sqrt{q^2-1}\right)\end{aligned} \tag{75}$$

$$\begin{aligned}\psi_{Oss} &= N_O x \exp\left[\frac{-x^2}{2}\,(q+\sqrt{q^2-1})\right]\\ &\quad\times \Phi\left(\left[\frac{3}{4}+\frac{\alpha^2}{2\sqrt{q^2-1}}\right], \frac{3}{2};\, x^2\sqrt{q^2-1}\right),\end{aligned} \tag{76}$$

where $\Phi(a, b; c)$ is the confluent hypergeometric function $\sum_{n=0}^{\infty} \frac{(a)_n c^n}{(b)_n n!}$. In the limit $q \to 1$, these become the even and odd coherent states.

But there are no displacement-operator squeezed states because there does not exist a unitary (Bogoliubov) transformation that can rotate aa into a linear combination of aa and $a^\dagger a^\dagger$. Therefore, an alternative idea is to simply use the ordinary squeeze operator, S, with the given displacement operator, and call these the displacement-operator squeezed states [41]. Here that would be

$$D_\pm(\alpha)\, S(z)\, |0\rangle = |\alpha, z\rangle_\pm \,. \tag{77}$$

In Ref. [42] these minimum-uncertainty and "displacement-operator" states were compared and found to be similar in nature. Since the "displacement-operator" states are more amenable to analytic calculations, they were then used for exploration of the time-dependence of the even and odd states of a trapped ion.

This, then, is a viable alternative to mathematically rigorous displacment-operator squeezed states.

Acknowledgements

We wish to thank G. M. D'Ariano and Wolfgang Schleich for helpful conversations on these topics. MMN acknowledges the support of the United States Department of Energy and the Alexander von Humboldt Foundation. DRT acknowledges a grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] E. Schrödinger, Naturwiss. **14**, 664 (1926).
[2] J. R. Klauder, Annals of Physics **11**, 123 (1960); J. Math. Phys. **4**, 1055 (1963).
[3] J. R. Klauder, J. Math. Phys. **4**, 1058 (1963).
[4] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[5] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 227 (1963).
[6] J. R. Klauder and B.-S. Skagerstam, *Coherent States – Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
[7] W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[8] R. A. Brandt and O. W. Greenberg, J. Math. Phys. **19**, 1168 (1969).
[9] M. M. Nieto and D. R. Truax, J. Math. Phys. to be published.
[10] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).
[11] J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).
[12] C. M. Caves, J. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmerman, Rev. Mod. Phys. **52**, 341 (1980).
[13] For a pedagogical review, see M. M. Nieto, in: *Frontiers of Nonequilibrium Statistical Physics*, eds. G. T. Moore and M. O. Scully (Plenum, New York, 1986) p. 287.
[14] D. Han, Y. S. Kim, and W. W. Zachary, eds., *Workshop on Squeezed States and Uncertainty Relations*, NASA Conference Publication 3135 (NASA, Washington, D.C., 1992).
[15] M. M. Nieto, Quant. Semiclass. Optics **8**, 1061 (1996).
[16] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[17] N. Bogolubov, J. Phys. USSR **11**, 292 (1947).
[18] See. Eq. (4.11) of Ref. [13]; M. M. Nieto, Quantum Opt. **6**, 9 (1994). The large round brackets in Eq. (5) should be squared.
[19] M. M. Nieto and D. R. Truax, Phys. Rev. Lett, **71**, 2843 (1993).
[20] A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971).
[21] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).

[22] A. M. PERELOMOV, Commun. Math. Phys. **26**, 222 (1972); *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).
[23] T. S. SANTHANAM, in: *Einstein Centennial Week Symposium on 'Symmetries in Science'*, ed. by B. Gruber and R. S. Millman (Plenum, NY, 1980), p. 337.
[24] G. S. AGARWAL and R. R. PURI, Phys. Rev. A **41**, 3782 (1990); A. T. FILIPPOV, D. GANGOPADHYAY, and A. P. ISAEV, J. Phys. A **24**, L63 (1991).
[25] J. KATRIEL and A. I. SOLOMON, J. Phys. A **24**, 2093 (1991).
[26] M. M. NIETO and L. M. SIMMONS, Jr., Phys. Rev. Lett. **41**, 207 (1978); Phys. Rev. D **20**, 1321 (1979), the first of a series concluding with M. M. NIETO, L. M. SIMMONS, Jr., and V. P. GUTSCHICK, Phys. Rev. D **23**, 927 (1981).
[27] M. M. NIETO, in Ref. [6], p. 429, gives a summary of this program.
[28] R. A. FISHER, M. M. NIETO, and V. D. SANDBERG, Phys. Rev. D **29**, 1107 (1984).
[29] D. R. TRUAX, Phys. Rev. D **31**, 1988 (1985).
[30] G. D'ARIANO, M. RASETTI, and M. VADACCHINO, Phys. Rev. D **32**, 1034 (1985).
[31] J. KATRIEL, A. I. SOLOMON, G. D'ARIANO, and M. RASETTI, Phys. Rev. D **34**, 2332 (1986).
[32] G. D'ARIANO and M. RASETTI, Phys. Rev, D **35,** 1239 (1987).
[33] J. KATRIEL, M. RASETTI, and A. I. SOLOMON, Phys. Rev. D **35**, 1248 (1987).
[34] J. KATRIEL, A. I. SOLOMON, G. D'ARIANO, and M. RASETTI, J. Opt. Soc. Am. B **4**, 1728 (1987).
[35] G. D'ARIANO, S. MOROSI, M. RASETTI, J. KATRIEL, and A. I. SOLOMON, Phys. Rev. D **36**, 2399 (1987).
[36] R. M. WILCOX, J. Math. Phys. **8**, 962 (1967).
[37] M. RASETTI, Int. J. Theor. Phys. **5**, 377 (1972).
[38] M. M. NIETO and D. R. TRUAX, Phys. Lett. A **208**, 8 (1995).
[39] V. V. DODONOV, I. A. MALKIN, and V. I. MAN'KO, Physica **72**, 597 (1974).
[40] M. M. NIETO, eprint atom-ph/9605010 (to be pulished).
[41] J. KLAUDER has suggested this possibility.
[42] M. M. NIETO, Phys. Lett. A **219**, 180 (1996).

(Manuscript received May 16, 1996)