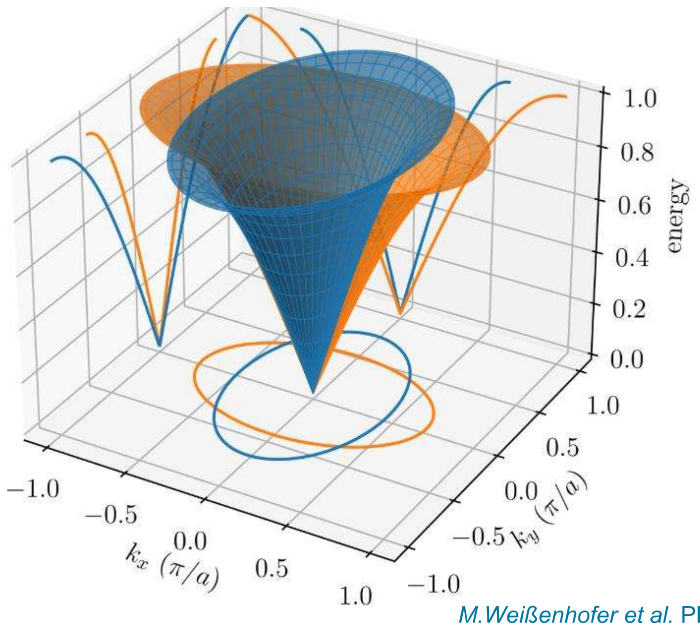
Universität Konstanz

Magnons in Ferromagnets



M.Weißenhofer et al. PRB **110**, *094427* (2024)

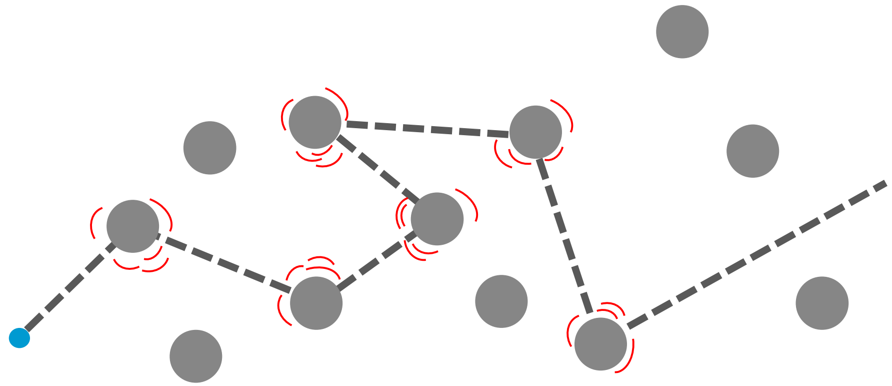
Julian Beisch

Konstanz, 17.12.2024

Motivation

Current computing

- Electronics
- Information by moving electrons *(charge)*
  - But they scatter ➡ Joule heating



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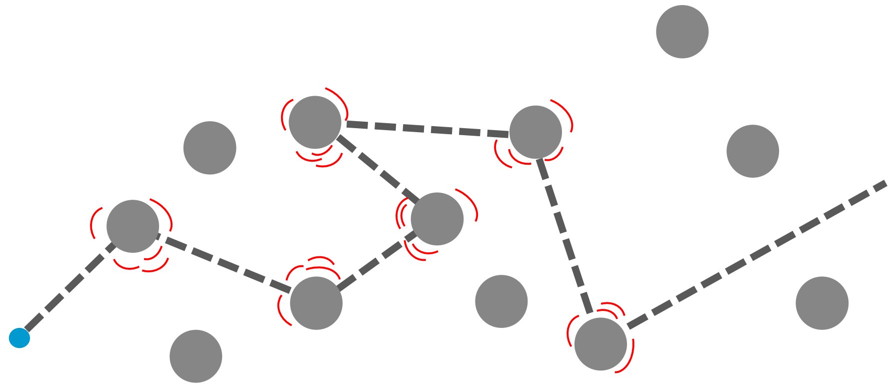
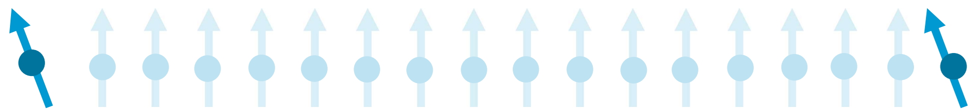


- Another property of electrons: spin
- spintronics
- Make currents with spins, but how ?



Classification

Scope

	Bound e^-	Quasi-free e^-
Dia	Lamor Diamagnetism	Landau Diamagnetism
Para	Langevin Paramagnetism	Pauli Paramagnetism
IM	Cooperative Magnetism	Band Ferromagnetism

Spin-operators

Spin-operators

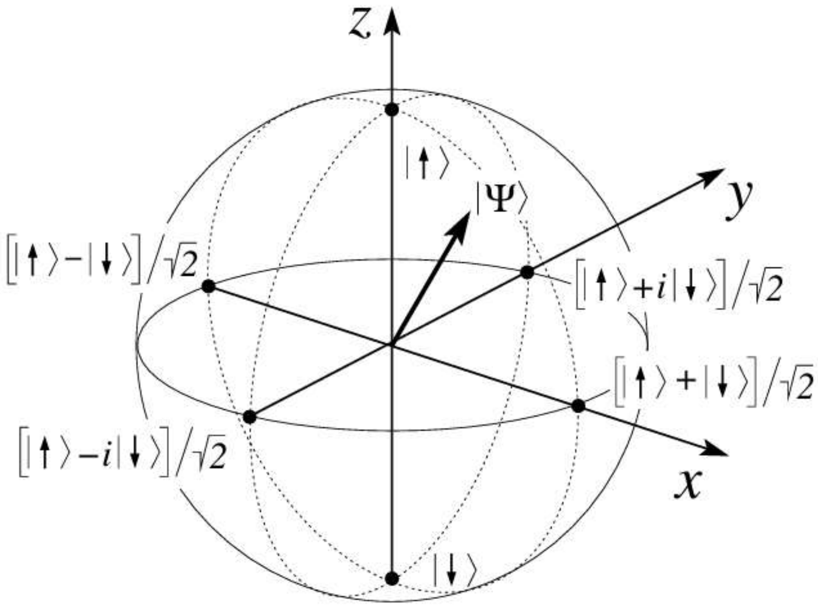
$$\left[\hat{S}_i^x, \hat{S}_j^y\right] = \mathrm{i}\hat{S}_i^y \delta_{i,j} \qquad + \text{cyclic permutation}$$

$$\hat{S}_i^+ = \hat{S}_i^x + \mathrm{i}\hat{S}_i^y$$

$$\hat{S}_i^- = \hat{S}_i^x - \mathrm{i}\hat{S}_i^y$$

$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm\hat{S}_i^\pm \delta_{i,j}$$

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hat{S}_i^z \delta_{i,j}$$



Heisenberg Theory of Ferromagnetism

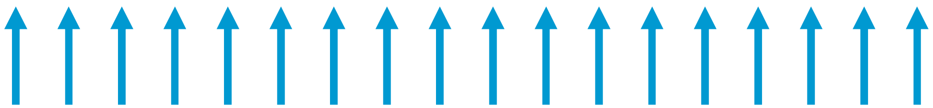
$$\mathcal{H} = -\sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j$$
$$= -\sum_{i \neq j} J_{i,j} \cdot \left(\hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right)$$
$$= -\sum_{i \neq j} J_{i,j} \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right)$$

W.Heisenberg Z.Physik **49**, 619-636, (1928)

Ferromagnetic groundstate

Groundstate $|0\rangle = |S, S, S, \ldots, S\rangle$

$|0\rangle$



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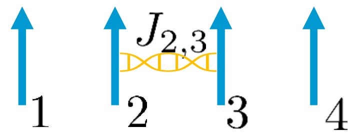

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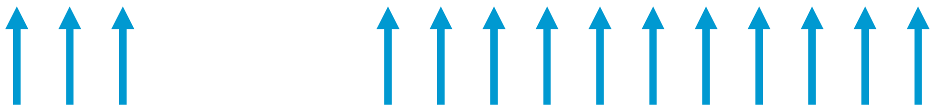
Ferromagnetic groundstate

Groundstate $|0\rangle = |S, S, S, \ldots, S\rangle$

An eigenstate, with the eigenenergy E_0

$$\mathcal{H}|0\rangle = -\sum_{i \neq j} J \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right) |0\rangle$$
$$= 0 - JN \cdot S^2 |0\rangle = E_0 |0\rangle$$

$|0\rangle$ 

Heisenberg Theory of Ferromagnetism

Excitations

Groundstate $|0\rangle = |S, S, S, \ldots, S\rangle$

How do excitations of this state look like?

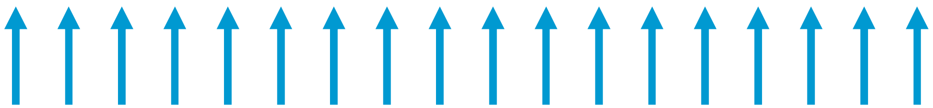
Zur Theorie des Ferromagnetismus.

Von **F. Bloch,** zurzeit in Utrecht.

(Eingegangen am 1. Februar 1930.)

F.Bloch. Z.Physik **61**, *206-219* (1930)

$|0\rangle$

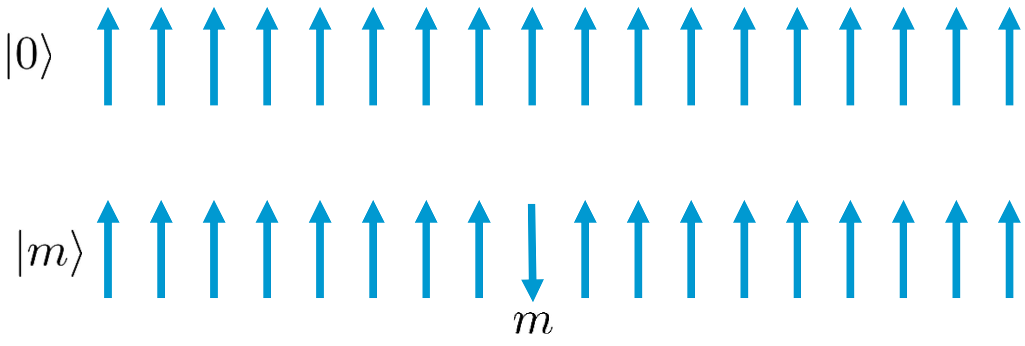


Heisenberg Theory of Ferromagnetism

Excitations

His approach was to consider one flipped spin

$$
\begin{aligned}
|m\rangle &= S_m^-|0\rangle \\
&= |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle
\end{aligned}
$$



Not an eigenstate anymore

Heisenberg Theory of Ferromagnetism

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Not an eigenstate anymore

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle$$

$$
\begin{aligned}
\mathcal{H}|m\rangle &= -\sum_{i\neq j} J \cdot \left(\frac{1}{2}\left\{\hat{S}_i^+\hat{S}_j^-|m\rangle + \hat{S}_i^-\hat{S}_j^+|m\rangle\right\} + \hat{S}_i^z\hat{S}_j^z|m\rangle\right) \\
&= -\sum_{i\neq j} J \cdot \left(\frac{1}{2}\left\{\delta_{i,m}\hat{S}_i^+\hat{S}_j^-|m\rangle + \delta_{j,m}\hat{S}_i^-\hat{S}_j^+|m\rangle\right\} + \hat{S}_i^z\hat{S}_j^z|m\rangle\right) \\
&= -J \cdot \left(\frac{1}{2}\left\{\sum_j \hat{S}_m^+\hat{S}_j^-|m\rangle + \sum_i \hat{S}_i^-\hat{S}_m^+|m\rangle\right\} + \sum_{i\neq j}\hat{S}_i^z\hat{S}_j^z|m\rangle\right) \\
&= -J \cdot \left(\frac{1}{2}\left\{\sum_j S|j\rangle + \sum_i S|i\rangle\right\} + \sum_{i\neq j}\hat{S}_i^z\hat{S}_j^z|m\rangle\right) \\
&= (E_0 + 2JS)\,|m\rangle - JS\sum_n |n\rangle
\end{aligned}
$$

Heisenberg Theory of Ferromagnetism

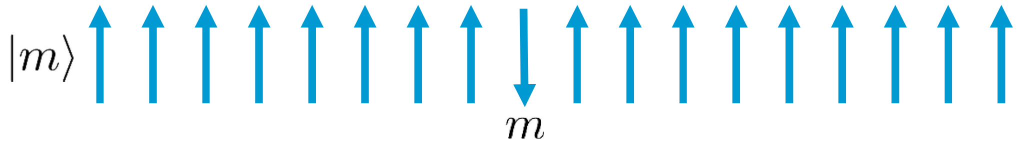
Search for Eigenstates

Not an eigenstate anymore.

The total magnetization was reduced by 1!

$$\begin{aligned}\mathcal{H}|m\rangle &= \mathcal{H}S_m^-|0\rangle \\ &= (E_0 + 2JS)\,|m\rangle - JS\sum_n |n\rangle\end{aligned}$$

$$\hat{S}^z_{\mathrm{tot.}}|m\rangle = (NS-1)\,|m\rangle$$



Heisenberg Theory of Ferromagnetism

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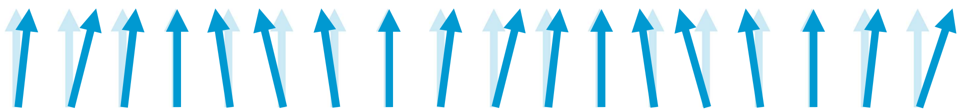
Hence we can guess an eigenstate

$$|\vec{k}\rangle \propto \sum_n |n\rangle$$

$|\vec{k}\rangle$



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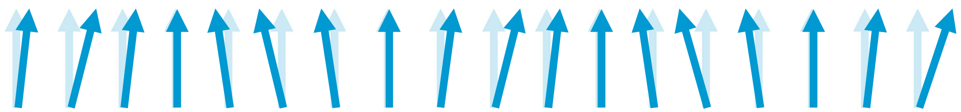
The total magnetization was reduced by 1!

$$\hat{S}^z_{\mathrm{tot.}}|m\rangle = (NS - 1)\,|m\rangle$$

Hence we can guess an eigenstate

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

$|\vec{k}\rangle$ 

Heisenberg Theory of Ferromagnetism

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle$$

Properties of the eigenstates

The total **magnetization** is reduced

As well as an increase in energy

$$\begin{aligned}\hat{S}^z_{\mathrm{tot.}}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^z_{\mathrm{tot.}}|n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) (NS-1)|n\rangle \\ &= (NS-1)|\vec{k}\rangle\end{aligned}$$

But the average x and y component are still zero?

$$\langle\vec{k}|\hat{S}^x_i|\vec{k}\rangle = 0$$
$$\langle\vec{k}|\hat{S}^y_i|\vec{k}\rangle = 0$$

Heisenberg Theory of Ferromagnetism

Properties of the eigenstates

The total magnetization is reduced

As well as an increase in **energy**

$$\mathcal{H}|\vec{k}\rangle = \ldots$$

Ferromagnetic Dispersion relation

$$\mathcal{H}|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \mathcal{H}|n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \left((E_0 + 2JS)\,|n\rangle - JS \sum_j |j\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2JS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |j\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2JS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp^{\mathrm{i}\vec{k}\cdot\vec{r}_n} \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_j\right) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right) |j\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2JS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n - \mathrm{i}\vec{k}\cdot\vec{r}_j\right) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right) |j\rangle \right)$$

Ferromagnetic Dispersion relation

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$$= \left((E_0+2JS)|k\rangle - \frac{1}{\sqrt{N}}JS\sum_\Delta\sum_j \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right)|j\rangle\right)$$

$$= \left((E_0+2JS)|k\rangle - JS\sum_\Delta \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)|k\rangle\right)$$

$$= \left(E_0|k\rangle - JS\sum_\Delta\left(2-\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)\right)|k\rangle\right)$$

$$= \left(E_0|k\rangle - JS\sum_{\Delta>0}\left(2-\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)-\exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)\right)|k\rangle\right)$$

$$= \left(E_0|k\rangle - JS\sum_{\Delta>0}\left(2-2\cdot\cos(\vec{k}\vec{r}_\Delta)\right)|k\rangle\right)$$

$$= \left(E_0 - 2JS\sum_{\Delta>0}\left(1-\cos(\vec{k}\vec{r}_\Delta)\right)\right)|k\rangle$$

For a linear chain with NN this simplifies to removal of sum

Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2J_1\left[1-\cos\frac{2\pi k}{N}\right],$$

F.Bloch. Z.Physik **61**, *206-219* (1930)

Same result with semiclassical calculation

Ferromagnetic Dispersion relation

Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2\,J_1\left[1 - \cos\frac{2\,\pi\,k}{N}\right],$$

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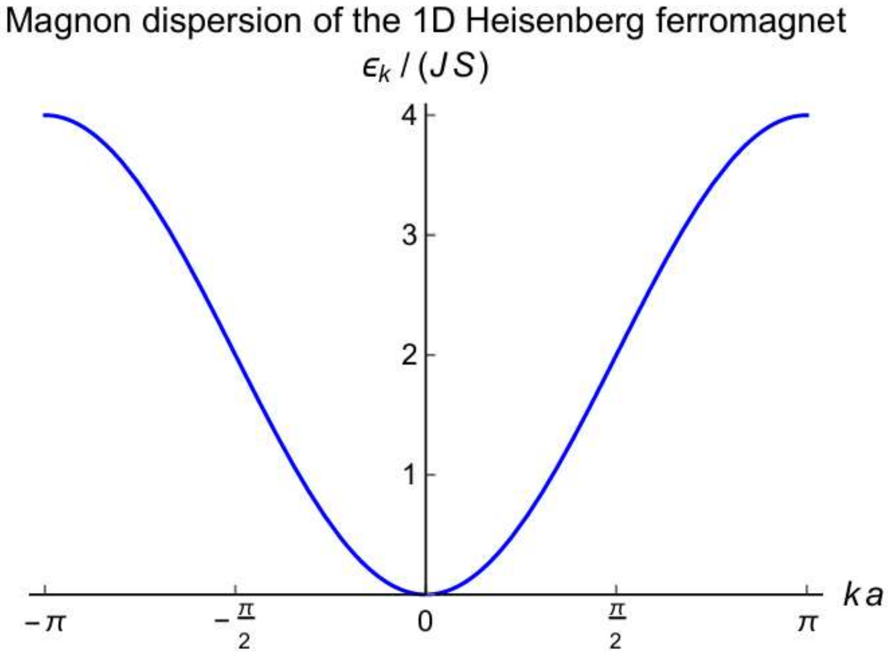
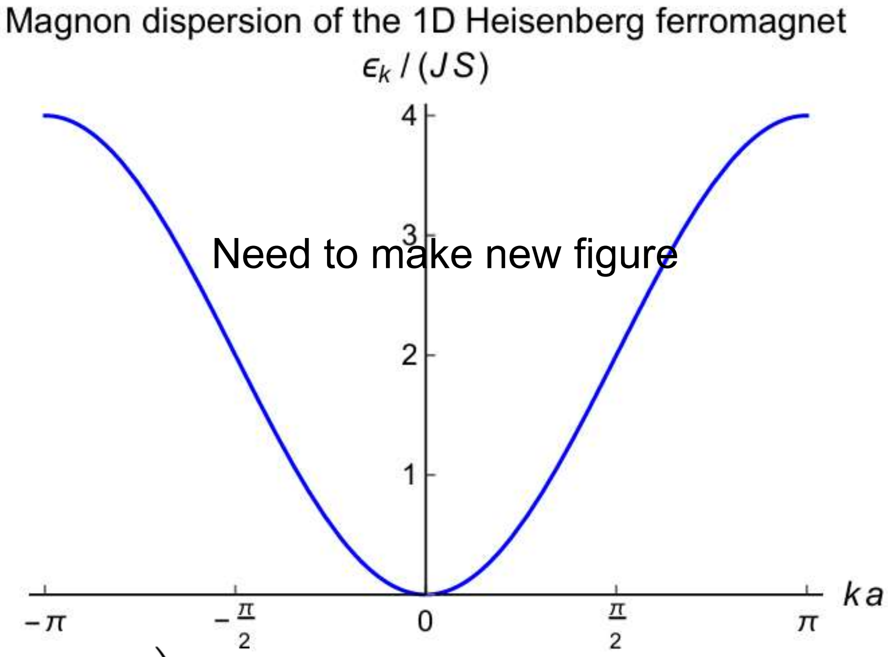


Ferromagnetic Dispersion relation with a magnetic Field

Addition of Zeeman term

$$\begin{aligned}\mathcal{H} &= -\sum_{i\neq j} J_{i,j}\cdot \hat{S}_i\cdot\hat{S}_j - \sum_i \vec{B}_i\cdot\hat{S}_i\\ &= -\sum_{i\neq j} J_{i,j}\cdot \hat{S}_i\cdot\hat{S}_j - \sum_i B_i^z\cdot\hat{S}_i^z\\ &= -\sum_{i\neq j} J_{i,j}\cdot\left(\frac{1}{2}\left\{\hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+\right\} + \hat{S}_i^z\hat{S}_j^z + B_i^z\hat{S}_i^z\right)\end{aligned}$$



$$\mathcal{H}|\vec{k}\rangle = \left(E_0 - B^z((N-1)S - (S-1)) - 2JS\sum_{\Delta>0}\left(1-\cos(\vec{k}\vec{r}_\Delta)\right)\right)|k\rangle$$

Recap

What is this eigenstate

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) |n\rangle$$

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Collective excitation

By comparison to phonons:
-Well defined momentum $\hbar\vec{k}$
-Energy $\hbar\epsilon(\vec{k})$

$\Bigg\}$ Quasiparticle

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$$|\langle m|\vec{k}\rangle|^2 = \left|\frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\delta_{m,n}\right|^2$$

$$= \frac{1}{N}\left|\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right)\right|^2 = \frac{1}{N}$$

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By comparison to phonons:

-Well defined momentum $\hbar\vec{k}$

-Energy $\hbar\epsilon(\vec{k})$

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Reduces magnetization by 1 → integer spin → Boson

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$$\left.\begin{array}{l}\text{-Well defined momentum } \hbar\vec{k} \\ \text{-Energy } \hbar\epsilon(\vec{k})\end{array}\right\} \text{Quasiparticle}$$

Reduces magnetization by 1 → integer spin → Boson

Magnon

Magnons as Bosons

statistischen
Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik eines Einstein-Bose-Gases her bekannt ist;

F.Bloch. Z.Physik **61**, *206-219* (1930)

Thermodynamical treatment

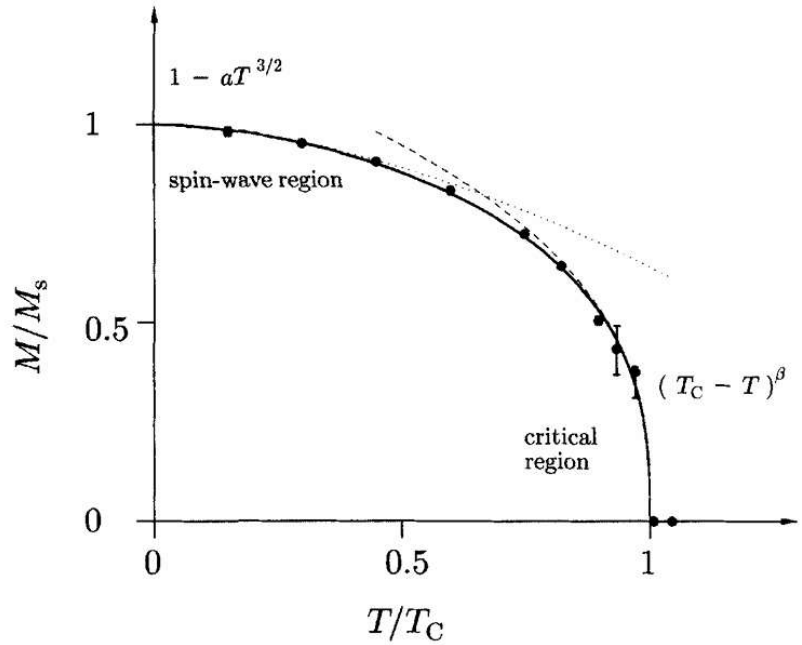
$$\hat{S}^z_{\text{tot.}}|\vec{k}\rangle = (NS-1)|\vec{k}\rangle$$

Since it is a boson it must fulfill Bose-Einstein-statistic

$$n_{\text{magnon}} \approx \int_0^\infty \frac{\text{DOS}(\omega)\text{d}\omega}{\exp(\hbar\omega/k_B T)-1} \Longleftarrow \text{Bose factor}$$

$$= \ldots \propto T^{3/2}$$

Number of magnons @ T



S.Blundell, *Magnetism in Condensed matter* (2000)

Spin-Boson Transformation

How to describe Magnons

The spin operators are not bosonic

Magnons are bosonic

Alternative approach:
LLG (analytically or numerically)
Or Schwinger representation
Or Dyson–Maleev representation,

Conditions

i. The **transformation** needs to be **Hermitian**, raising and lowering operators written as creation and annihilation boson operators need to be Hermitian conjugate of each other

ii. The **transformation** must be **unitary** to preserve the commutation relations between the spin operators.

iii. Must satisfy the **equality between the matrix elements** of the spin operators on $|0\rangle$ and the bosons on $|n\rangle$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

Simple Spin-Boson Transformation

Direct attempt

Operator $\hat{b}^\dagger$ can form the one magnon state

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

$$\Downarrow$$

$$\hat{b}^\dagger_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^-_n$$

$$|\vec{k}\rangle = \hat{b}^\dagger_{\vec{k}}|0\rangle$$

Simple Spin-Boson Transformation

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Operator $\hat{b}^\dagger$ can form the one magnon state

$$\hat{b}_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^+_n$$

Does not fulfill the Boson commutation relations

Close for small excitation

$$[\hat{b}^\dagger_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = 0$$
$$[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}] = 0$$

$$\left[\hat{S}^+_i, \hat{S}^-_j\right] = 2\hat{S}^z_i\delta_{i,j}$$

$$[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = \frac{1}{2SN}\sum_n \exp\left(-\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{r}_n\right)2\hat{S}^z_n \neq \delta_{\vec{k},\vec{k}'}$$

$$\approx \frac{1}{N}\sum_n \exp\left(-\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{r}_n\right) = \delta_{k,k'}$$

Simple Spin-Boson Transformation

$$\hat{b}^\dagger_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^-_n$$

$$\hat{b}_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^+_n$$

Direct attempt

Operator $\hat{b}^\dagger$ can form the one magnon state

Does not fulfill the Boson commutation relations

Close for small excitation

Fourier transform back to lattice operators

$$\hat{b}^\dagger_j = \frac{1}{\sqrt{2S}} \hat{S}^-_j \qquad \hat{b}_j = \frac{1}{\sqrt{2S}} \hat{S}^+_j$$

$$\hat{S}^-_j = \sqrt{2S}\hat{b}^\dagger_j \qquad \hat{S}^+_j = \sqrt{2S}\hat{b}_j$$

$$\hat{n}_j = \hat{b}^\dagger_j \hat{b}_j$$

The number of bosons/magnons.

$$\hat{S}^z_j = S - \hat{b}^\dagger_j \hat{b}_j$$

A magnon reduces the z-comp of the magnetization

Holstein-Primakoff(HP)

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$

$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

More sophisticated attempt

HP framework is a powerful method for calculating dispersions and higher order interactions

$$\left[\hat{S}_j^+, \hat{S}_k^-\right] = \ldots = \delta_{j,k} \cdot 2 \cdot \hat{S}_j^z$$

Holstein-Primakoff

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

Not so sophisticated anymore

Only linear Spin-Wave theory using the linearized HP

Linearized form

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$
$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

$$\mathcal{H} = -\sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j$$
$$= -\sum_{i \neq j} J_{i,j} \cdot \left(\frac{1}{2}\left\{\hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+\right\} + \hat{S}_i^z \hat{S}_j^z\right)$$
$$= -\sum_{i \neq j} J_{i,j} \cdot \left(S\left\{\hat{a}_i \hat{a}_j^\dagger + \hat{a}_i^\dagger \hat{a}_j\right\} + (S - \hat{n}_i)(S - \hat{n}_j)\right)$$

Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} = -\sum_{i\neq j} J \cdot \hat{S}_i \cdot \hat{S}_j$$

$$= -\sum_{i\neq j} J \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right)$$

$$= -\sum_{i\neq j} J \cdot \left(S \left\{ \hat{a}_i \hat{a}_j^\dagger + \hat{a}_i^\dagger \hat{a}_j \right\} + (S - \hat{n}_i)(S - \hat{n}_j) \right)$$

$$= -\frac{NJS^2}{2} - \sum_{i\neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i + \underbrace{\sum_{ij} J \hat{n}_i \hat{n}_j}_{\approx 0}$$

$$\approx -\frac{NJS^2}{2} - \sum_{i\neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

$$\begin{aligned}
\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\
\hat{S}_j^y &= \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger \right) \\
\hat{S}_j^x &= \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger \right) \\
\hat{S}_j^+ &\approx \sqrt{2S} \cdot \hat{a}_j \\
\hat{S}_j^- &\approx \sqrt{2S} \cdot \hat{a}_j^\dagger
\end{aligned}$$

These would be magnon-magnon interactions

Holstein-Primakoff

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{\hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j\right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \hat{a}_n = [\mathcal{H}, \hat{a}_n]$$

$$= 0 - \sum_{i \neq j} JS \left\{\left[\hat{a}_j^\dagger \hat{a}_i, \hat{a}_n\right] + \left[\hat{a}_i^\dagger \hat{a}_j, \hat{a}_n\right]\right\} + JS \sum_i [\hat{n}_i, \hat{a}_n]$$

$$= 0 - \sum_{i \neq j} JS \{-\delta_{jn} \hat{a}_i - \delta_{in} \hat{a}_j\} + JS \sum_i -\delta_{in} \hat{a}_i$$

$$= -JS \left\{-\sum_{i \neq j} \delta_{jn} \hat{a}_i - \sum_{i \neq j} \delta_{in} \hat{a}_j\right\} + JS \sum_i -\delta_{in} \hat{a}_i$$

Problem: coupling between all sites practically unsolvable for real systems

Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \hat{a}_n = \ldots$$

$$= -JS \left\{ -\sum_i \hat{a}_i - \sum_j \hat{a}_j \right\} - JS\hat{a}_n$$

$$= 2JS \sum_i \hat{a}_i - JS\hat{a}_n$$

$$= JS \sum_i \hat{a}_i \left(2 - \delta_{in}\right)$$

Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i\neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$i\hbar \frac{\partial}{\partial t} \hat{a}_n = \dots$$

$$= -JS \left\{ -\sum_i \hat{a}_i - \sum_j \hat{a}_j \right\} - JS\hat{a}_n$$

$$= 2JS \sum_i \hat{a}_i - JS\hat{a}_n$$

$$= JS \sum_i \hat{a}_i \left(2 - \delta_{in}\right)$$

Going to Fourier-space

Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_j),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(\mathrm{i}\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t} \hat{a}_n \right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i\neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \hat{a}_n = \dots$$

$$= -JS \left\{ -\sum_i \hat{a}_i - \sum_j \hat{a}_j \right\} - JS\hat{a}_n$$

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$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t} \hat{a}_n \right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Now we can plug in

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(JS \sum_i \hat{a}_i \left(2 - \delta_{in}\right) \right) \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= \frac{JS}{\sqrt{N}} \sum_n \sum_i \hat{a}_i \left(2 - \delta_{in}\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= \dots$$

Holstein-Primakoff

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Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_j),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(\mathrm{i}\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{a}(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_n \left(JS\sum_i \hat{a}_i\left(2-\delta_{in}\right)\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n) \\ &= JS\sum_n \left(2-\delta_{0n}\right) \exp\left(\mathrm{i}\vec{k}(\vec{r}_0-\vec{r}_n)\right) \hat{a}(\vec{k}) \end{aligned}$$

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = -\frac{\mathrm{i}}{\hbar}\mathcal{W}_{\vec{k}}\hat{a}(\vec{k})$$

L. Rósza, Lecture Notes (2022)

$$\hbar\omega_{\vec{k}} = S\left(J_0 - 2J_{\vec{k}}\right) \qquad \hat{a}(\vec{k},t) = \hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)$$

Holstein-Primakoff

Going to Fourier-space

Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_j),$$

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$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(JS\sum_i \hat{a}_i\left(2-\delta_{in}\right)\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= JS\sum_n \left(2-\delta_{0n}\right) \exp\left(\mathrm{i}\vec{k}(\vec{r}_0-\vec{r}_n)\right)\hat{a}(\vec{k})$$

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = -\frac{\mathrm{i}}{\hbar}\mathcal{W}_{\vec{k}}\hat{a}(\vec{k})$$

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$$\hbar\omega_{\vec{k}} = S\left(J_0 - 2J_{\vec{k}}\right) \qquad \hat{a}(\vec{k},t) = \hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)$$

$$\hat{a}_j(\vec{k},t) = \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k},t)$$

$$= \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)$$

$$\hat{a}_j(\vec{k}_0,t) = \frac{1}{\sqrt{N}}\hat{a}(\vec{k}_0,0)\cdot\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)$$

$$\boxed{\hat{a}_j(\vec{k}_0,t) = \hat{A}\cdot\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)}$$

Holstein-Primakoff

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And the time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Now we can plug in

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\boxed{\hat{a}_j(\vec{k}_0, t) = \hat{A}\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)}$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^x \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

Holstein-Primakoff

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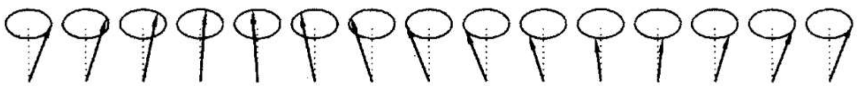
And the time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Now we can plug in



S.Blundell, *Magnetism in Condensed matter* (2000)

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\boxed{\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)}$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^x \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

$$\boxed{\begin{aligned} \hat{S}_j^x(t) &= \sqrt{2S}\hat{A} \cdot \cos(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \\ \hat{S}_j^y(t) &= \sqrt{2S}\hat{A} \cdot \sin(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \\ \hat{S}_j^z(t) &= S - |\hat{A}|^2 \end{aligned}}$$

Now we can understand why the average we calculated earlier was 0

Holstein-Primakoff & Bogolibuv transformation

Considering more interactions

Adding on site anisotropy

$$\mathcal{H} = -\sum_{i \neq j} J \cdot \hat{S}_i \cdot \hat{S}_j - \sum_i \mathcal{K}_i \cdot \hat{S}_i \cdot \hat{S}_i$$

$$= \ldots \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i - \sum_i \mathcal{K}_i \left(\underline{2\hat{a}_i \hat{a}_i} + \underline{2\hat{a}_i^\dagger \hat{a}_i^\dagger} + S^2 - 2\hat{n}_i + \underbrace{\left(\hat{a}_i^\dagger \hat{a}_i \right)^2}_{\approx 0} \right)$$

Let's simplify this

$$\mathcal{H} = \sum_{i \neq j} \hat{a}_i^\dagger H \hat{a}_i + \sum_i \hat{a}_i^\dagger \mathcal{K} \hat{a}_i^\dagger + \hat{a}_i \mathcal{K} \hat{a}_i$$

$$= \frac{1}{2} \left(\hat{a}, \hat{a}^\dagger \right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K} & H \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger \right)^T$$

This slide is still work in progress

Visualising a Magnon

Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = -\frac{\gamma}{(1+\alpha^2)\mu_S}\vec{S}_i \times \left(\vec{H}_i(t) + \alpha\vec{S}_i \times \vec{H}_i(t)\right)$$

$$\vec{H}_i(t) = -\frac{\partial\mathcal{H}}{\partial\vec{S}_i}$$

Time integration with Heun's methods

Additional property is the damping α

Engine_LLG\Test.ipynb

Outlook

Current research

Easily obtain highly interesting dispersion relations.

Can be used for computing, without energy loss through Joule heating

This is ongoing research in

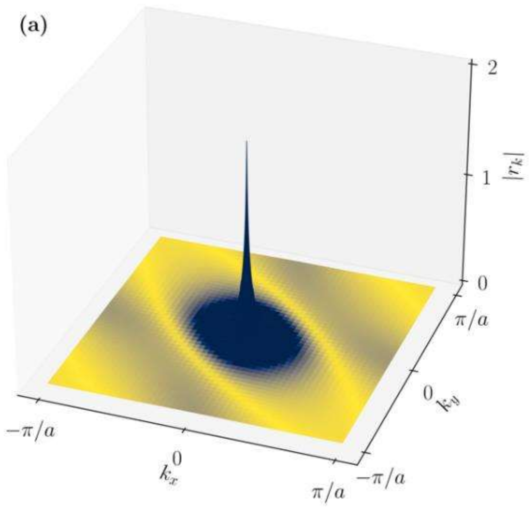
Spin wave diodes *J.Lan et.al.* PRX **5**, 041049, (2015)

Spin wave transitors *A.Chumak et.al.* Nature C. **5**, 4700, (2014)

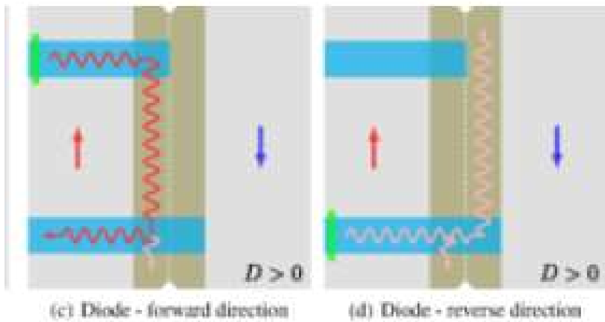
The most modern research even includes Altermagnets, which have distinct symmetry enforced properties regarding the dispersion in different directions

Topological magnons *P. McClarty* Annual Reviews **13**, 171-190, (2022)

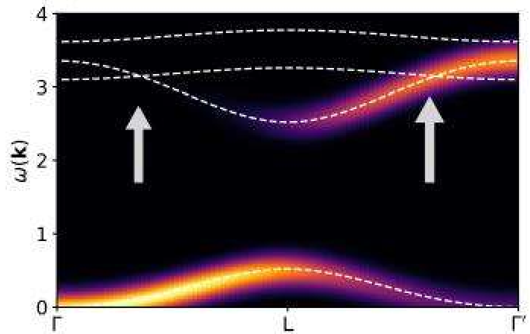
Squeezed magnons *D. Wuhrer et.al. PR* **5**, 043124, (2023)



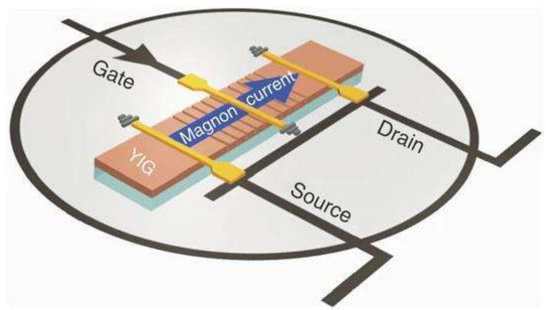
D. Wuhrer et.al. PR **5**, 043124, (2023)



J.Lan et.al. PRX **5**, 041049, (2015)



Magnon transistor scheme



A.Chumak et.al. Nature C. **5**, 4700, (2014)