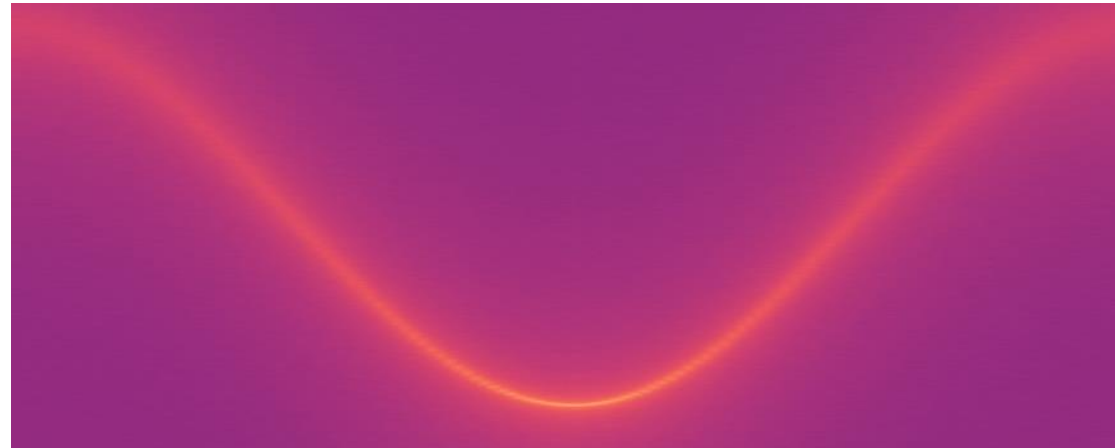


# Magnons in Ferromagnets



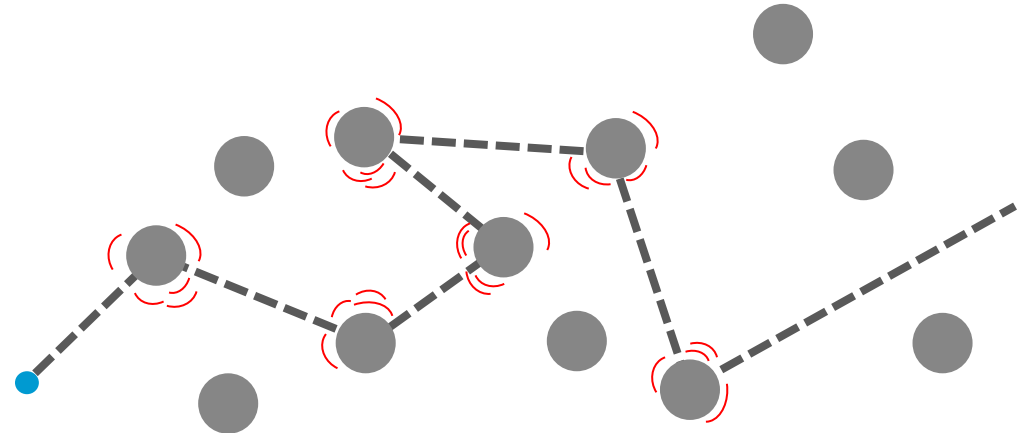
Julian Beisch

Konstanz, 17.12.2024

# Motivation

## Current computing

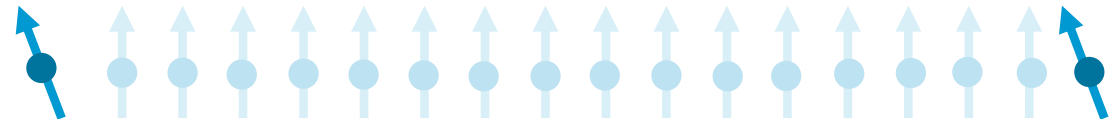
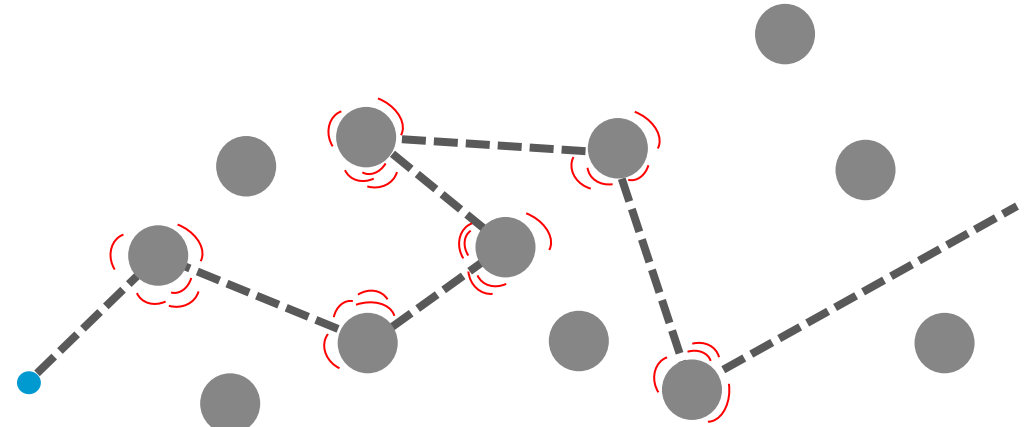
- Electronics
- Information by moving electrons (*charge*)
  - But they scatter → Joule heating



# Motivation

## Current computing

- Electronics
- Information by moving electrons (*charge*)
  - But they scatter → Joule heating
- Another property of electrons: Spin
- Spintronics
- Make currents with spins, but how ?



### Caveat

Spins depicted as arrows is just a semi-classical approximation of an otherwise quantum mechanical expectation distribution

# Classification

## Scope

	Bound $e^-$	Quasi-free $e^-$
Dia	Lamor diamagnetism	Landau diamagnetism
Para	Langevin paramagnetism	Pauli paramagnetism
IM	Cooperative magnetism	Band ferromagnetism

# Spin-operators

## Spin-operators

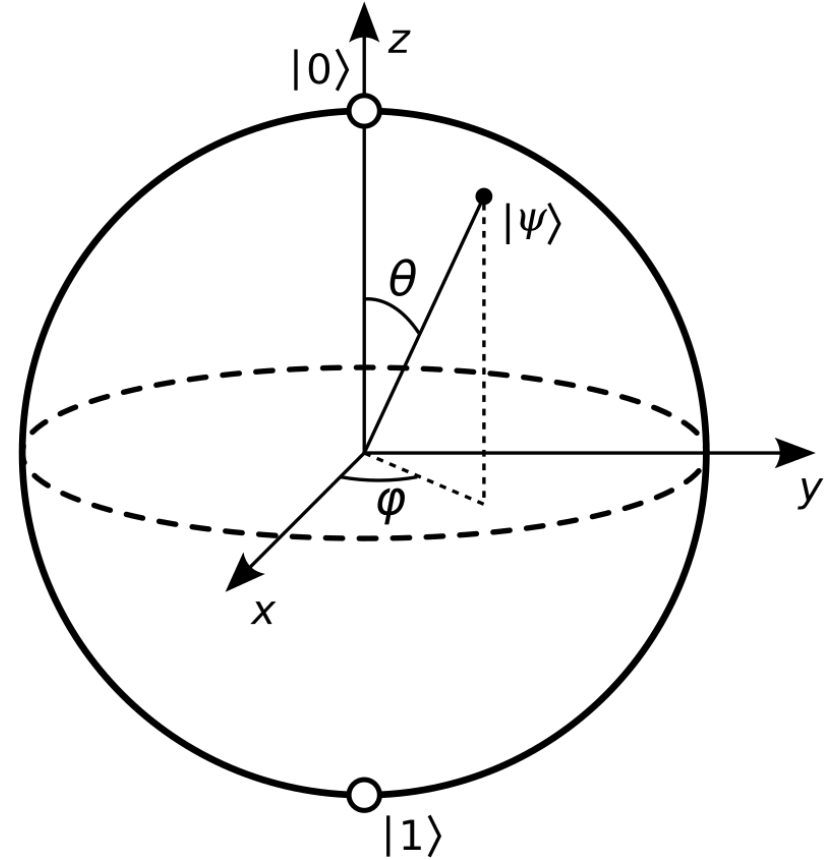
$$[\hat{S}_i^x, \hat{S}_j^y] = i\hat{S}_i^z \delta_{i,j} \quad + \text{cyclic permutation}$$

$$\hat{S}_i^+ = \hat{S}_i^x + i\hat{S}_i^y$$

$$\hat{S}_i^- = \hat{S}_i^x - i\hat{S}_i^y$$

$$[\hat{S}_i^z, \hat{S}_j^\pm] = \pm \hat{S}_i^\pm \delta_{i,j}$$

$$[\hat{S}_i^+, \hat{S}_j^-] = 2\hat{S}_i^z \delta_{i,j}$$

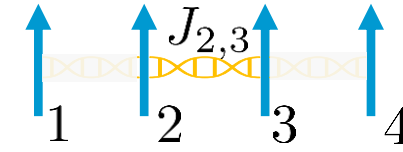


[Wikipedia.com](https://en.wikipedia.org/wiki/Bloch_sphere)

# Heisenberg Theory of Ferromagnetism

## Groundstate

$$|0\rangle = |S, S, S, \dots, S\rangle$$



$$\begin{aligned}\mathcal{H} &= - \sum_{i,\Delta} J_{i,i+\Delta} \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} \\ &= - \sum_{i,\Delta} J_{i,i+\Delta} \cdot \left( \hat{S}_i^x \hat{S}_{i+\Delta}^x + \hat{S}_i^y \hat{S}_{i+\Delta}^y + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) \\ &= - \sum_{i,\Delta} J_{i,i+\Delta} \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)\end{aligned}$$

*W.Heisenberg Z.Physik* **49**, 619-636, (1928)



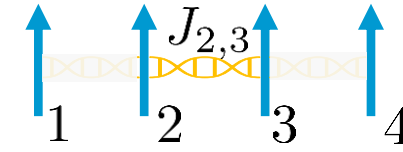
# Heisenberg Theory of Ferromagnetism

## Groundstate

$$|0\rangle = |S, S, S, \dots, S\rangle$$

An eigenstate, with the eigenenergy  $E_0$

$$\begin{aligned}\mathcal{H}|0\rangle &= - \sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \underline{\hat{S}_i^+ \hat{S}_{i+\Delta}^-} + \underline{\hat{S}_i^- \hat{S}_{i+\Delta}^+} \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) |0\rangle \\ &= \underline{0} - zJN \cdot S^2 |0\rangle = E_0 |0\rangle\end{aligned}$$



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*W.Heisenberg Z.Physik* **49**, 619-636, (1928)



# Bloch Theory of Ferromagnetism

## Excitations

Groundstate  $|0\rangle = |S, S, S, \dots, S\rangle$

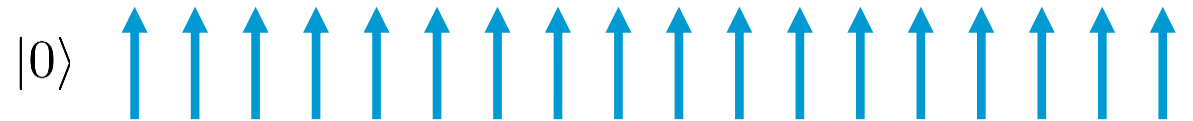
How do excitations of this state look like?

## Zur Theorie des Ferromagnetismus.

Von **F. Bloch**, zurzeit in Utrecht.

(Eingegangen am 1. Februar 1930.)

*F.Bloch. Z.Physik* **61**, 206-219 (1930)





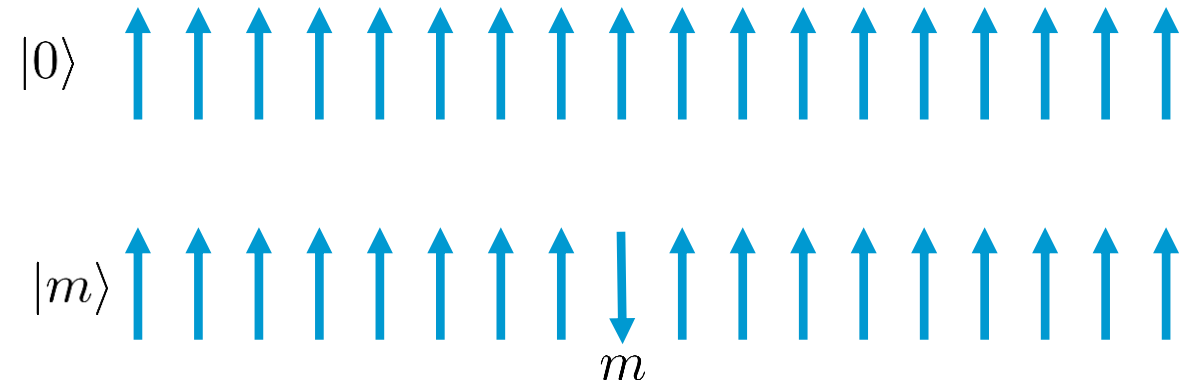
# Bloch Theory of Ferromagnetism

## Excitations

His approach was to consider one flipped spin

$$\begin{aligned}|m\rangle &= \frac{S_m^-}{\sqrt{2S}}|0\rangle \\ &= |S, S, \dots, \underbrace{S-1}_m, \dots, S\rangle\end{aligned}$$

Not an eigenstate anymore



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His approach was to consider one flipped spin

$$|m\rangle = \frac{S_m^-}{\sqrt{2S}}|0\rangle$$

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$$\begin{aligned}\mathcal{H}|m\rangle &= - \sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- |m\rangle + \hat{S}_i^- \hat{S}_{i+\Delta}^+ |m\rangle \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right) \\ &= -J \sum_{i,\Delta} \cdot \left( \frac{1}{2} \left\{ \delta_{i,m} \hat{S}_i^+ \hat{S}_{i+\Delta}^- |m\rangle + \delta_{i+\Delta,m} \hat{S}_i^- \hat{S}_{i+\Delta}^+ |m\rangle \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right) \\ &= -J \cdot \left( \frac{1}{2} \left\{ \sum_{\Delta} \hat{S}_m^+ \hat{S}_{m+\Delta}^- |m\rangle + \sum_{\Delta} \hat{S}_{m-\Delta}^- \hat{S}_m^+ |m\rangle \right\} + \sum_{i,\Delta} \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right) \\ &= -J \cdot \left( \frac{1}{2} \left\{ \sum_{\Delta} 2S |m+\Delta\rangle + \sum_{\Delta} 2S |m-\Delta\rangle \right\} + \sum_{i,\Delta} \hat{S}_i^z \hat{S}_{i+\Delta}^z |m\rangle \right) \\ &= (E_0 + 2zJS) |m\rangle - 2JS \sum_{\Delta} |m+\Delta\rangle\end{aligned}$$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

# Bloch Theory of Ferromagnetism

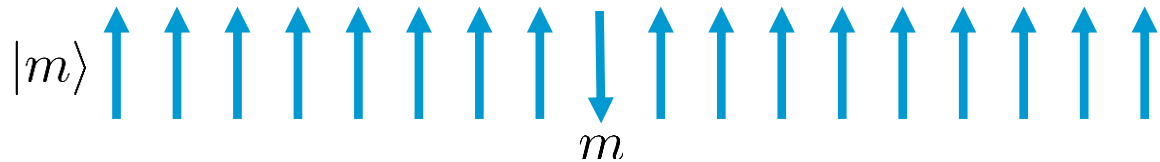
## Search for eigenstates

Not an eigenstate anymore.

The total magnetization was reduced by 1!

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$$\hat{S}_{\text{tot.}}^z |m\rangle = (NS - 1) |m\rangle$$



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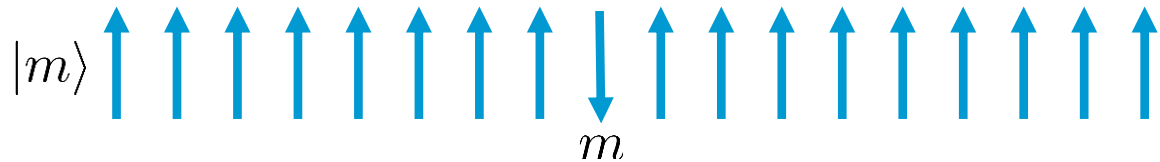
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$$|m\rangle \propto S_m^-|0\rangle$$



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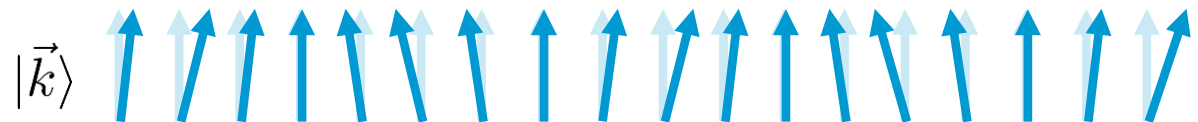
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$$\hat{S}_{\vec{k}}^- \propto \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \hat{S}_n^-$$



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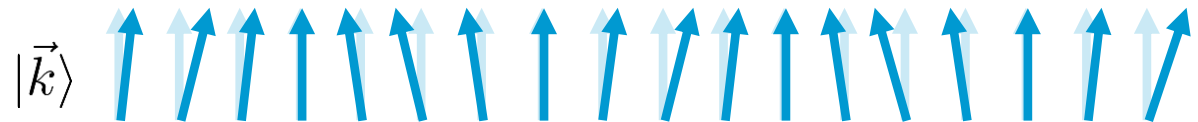
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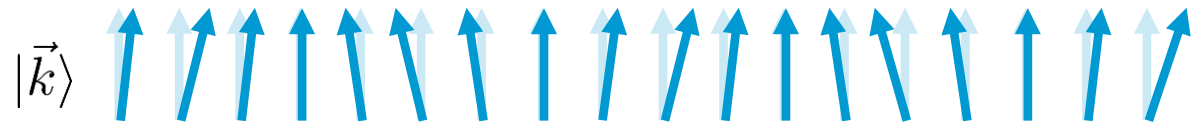
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# Bloch Theory of Ferromagnetism

## Properties of the eigenstates

The total **magnetization** is reduced

As well as an increase in energy

But the average x and y components are still zero?

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) |n\rangle$$

$$\begin{aligned} \hat{S}_{\text{tot.}}^z |\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \hat{S}_{\text{tot.}}^z |n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) (NS - 1) |n\rangle \\ &= (NS - 1) |\vec{k}\rangle \end{aligned}$$

$$\langle \vec{k} | \hat{S}_i^x | \vec{k} \rangle = 0$$

$$\langle \vec{k} | \hat{S}_i^y | \vec{k} \rangle = 0$$

$$\langle \vec{k} | \hat{S}_i^z | \vec{k} \rangle = S - \frac{1}{N}$$



# Bloch Theory of Ferromagnetism

## Properties of the eigenstates

The total magnetization is reduced

As well as an increase in **energy**

$$\mathcal{H}|\vec{k}\rangle = \dots$$

# Ferromagnetic Dispersion Relation

## Properties of the eigenstates

$$\begin{aligned}\mathcal{H}|\vec{k}\rangle &= \left( (E_0 + 2zJS) |\vec{k}\rangle + 2JS \sum_{\Delta>0} -\exp(-i\vec{k} \cdot \vec{r}_\Delta) - \exp(i\vec{k} \cdot \vec{r}_\Delta) |\vec{k}\rangle \right) \\ &= \left( (E_0 + 2zJS) |\vec{k}\rangle + 2JS \sum_{\Delta>0} -2 \cos(\vec{k} \cdot \vec{r}_\Delta) |\vec{k}\rangle \right) \\ &= \left( (E_0 + 2zJS) |\vec{k}\rangle + 2JS \sum_{\Delta} -\cos(\vec{k} \cdot \vec{r}_\Delta) |\vec{k}\rangle \right) \\ &= \left( E_0 |\vec{k}\rangle + 2JS \sum_{\Delta} \left( 1 - \cos(\vec{k} \cdot \vec{r}_\Delta) \right) |\vec{k}\rangle \right)\end{aligned}$$

# Ferromagnetic Dispersion Relation

## Properties of the eigenstates

$$\epsilon_{\vec{k}} \stackrel{r=1.}{=} 2 J_1 \left[ 1 - \cos \frac{2 \pi \vec{k}}{N} \right],$$

*F.Bloch. Z.Physik 61, 206-219 (1930)*

$$\begin{aligned} \mathcal{H}|\vec{k}\rangle &= \left( (E_0 + 2zJS) |\vec{k}\rangle + 2JS \sum_{\Delta>0} -\exp(-i\vec{k} \cdot \vec{r}_{\Delta}) - \exp(i\vec{k} \cdot \vec{r}_{\Delta}) |\vec{k}\rangle \right) \\ &= \left( (E_0 + 2zJS) |\vec{k}\rangle + 2JS \sum_{\Delta>0} -2 \cos(\vec{k} \cdot \vec{r}_{\Delta}) |\vec{k}\rangle \right) \\ &= \left( (E_0 + 2zJS) |\vec{k}\rangle + 2JS \sum_{\Delta} -\cos(\vec{k} \cdot \vec{r}_{\Delta}) |\vec{k}\rangle \right) \\ &= \left( E_0 |\vec{k}\rangle + 2JS \sum_{\Delta} \left( 1 - \cos(\vec{k} \cdot \vec{r}_{\Delta}) \right) |\vec{k}\rangle \right) \end{aligned}$$

Same result with semiclassical calculation

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Same result with semiclassical calculation

$$\mathcal{H}|\vec{k}\rangle = \left( E_0 |\vec{k}\rangle + 4JS \left( 1 - \cos(\vec{k} \cdot \vec{r}_{\Delta}) \right) |\vec{k}\rangle \right) \quad \text{For a linear chain with NN}$$

$$\Delta E = 4JS \left( 1 - \cos(\vec{k} \cdot \vec{r}_{\Delta}) \right)$$

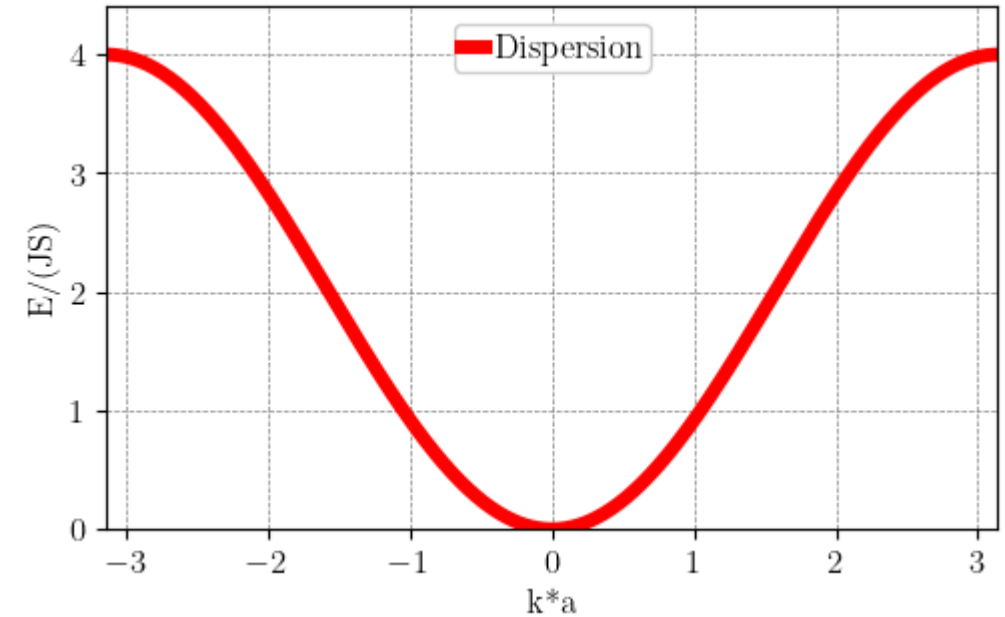
*S.Blundell, Magnetism in Condensed matter (2000)*

# Ferromagnetic Dispersion Relation

## Properties of the eigenstates

$$\epsilon_k \stackrel{r=1.}{=} 2 J_1 \left[ 1 - \cos \frac{2 \pi k}{N} \right],$$

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# Ferromagnetic Dispersion Relation with a Magnetic Field

## Addition of Zeeman term

$$\begin{aligned}\mathcal{H}_Z &= - \sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \vec{B} \cdot \hat{S}_i \\ &= - \sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i B^z \cdot \hat{S}_i^z \\ &= - \sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) + B^z \hat{S}_i^z\end{aligned}$$

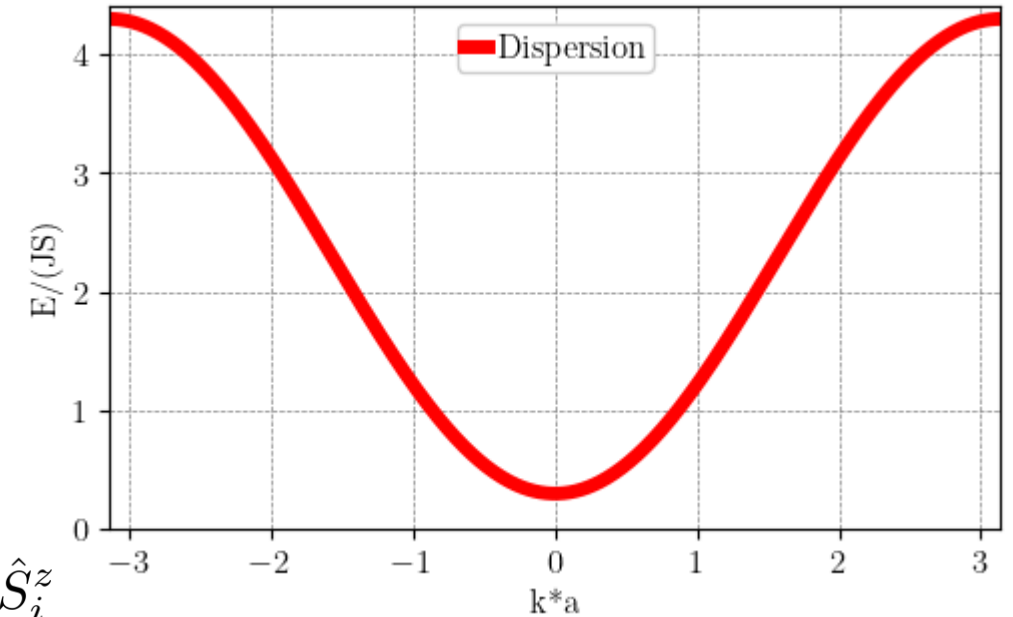
# Ferromagnetic Dispersion Relation with a Magnetic Field

## Addition of Zeeman term

$$\begin{aligned}
 \mathcal{H}_Z &= - \sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \vec{B} \cdot \hat{S}_i \\
 &= - \sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i B^z \cdot \hat{S}_i^z \\
 &= - \sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) + B^z \hat{S}_i^z
 \end{aligned}$$

$$\mathcal{H}_Z |\vec{k}\rangle = \left( E_{0,Z} + B^z + 2JS \sum_{\Delta} \left( 1 - \cos(\vec{k} \vec{r}_{\Delta}) \right) \right) |k\rangle$$

$$E_{0,Z} = -zJN \cdot S^2 - NSB^z$$



# Recap

## First excited state

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) |n\rangle$$



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Collective excitation

$$\begin{aligned} |\langle m | \vec{k} \rangle|^2 &= \left| \frac{1}{\sqrt{N}} \sum_n \exp(\mathrm{i}\vec{k} \cdot \vec{r}_n) \langle m | n \rangle \right|^2 \\ &= \left| \frac{1}{\sqrt{N}} \sum_n \exp(\mathrm{i}\vec{k} \cdot \vec{r}_n) \delta_{m,n} \right|^2 \\ &= \frac{1}{N} \left| \exp(\mathrm{i}\vec{k} \cdot \vec{r}_m) \right|^2 = \frac{1}{N} \end{aligned}$$

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## Collective excitation

By comparison to phonons:

-Well defined momentum  $\hbar\vec{k}$   
-Energy  $\hbar E(\vec{k})$

} Quasiparticle

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Reduces magnetization by 1 → Integer spin → Boson

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Magnon

$$\begin{aligned} |\langle m|\vec{k}\rangle|^2 &= \left| \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \langle m|n\rangle \right|^2 \\ &= \left| \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \delta_{m,n} \right|^2 \\ &= \frac{1}{N} \left| \exp(i\vec{k} \cdot \vec{r}_m) \right|^2 = \frac{1}{N} \end{aligned}$$

# Magnons as Bosons

## Thermodynamical treatment

$$\hat{S}_{\text{tot.}}^z |\vec{k}\rangle = (NS - 1) |\vec{k}\rangle$$

statistischen  
Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik  
eines Einstein-Bose-Gases her bekannt ist;

*F.Bloch. Z.Physik 61, 206-219 (1930)*

Since it is a boson it must fulfill the Bose-Einstein statistic

$$n_{\text{magnon}} \approx \int_0^\infty \frac{\text{DOS}(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1} \leftarrow \text{Bose factor}$$
$$= \dots \propto T^{3/2}$$

Number of  
magnons @ T

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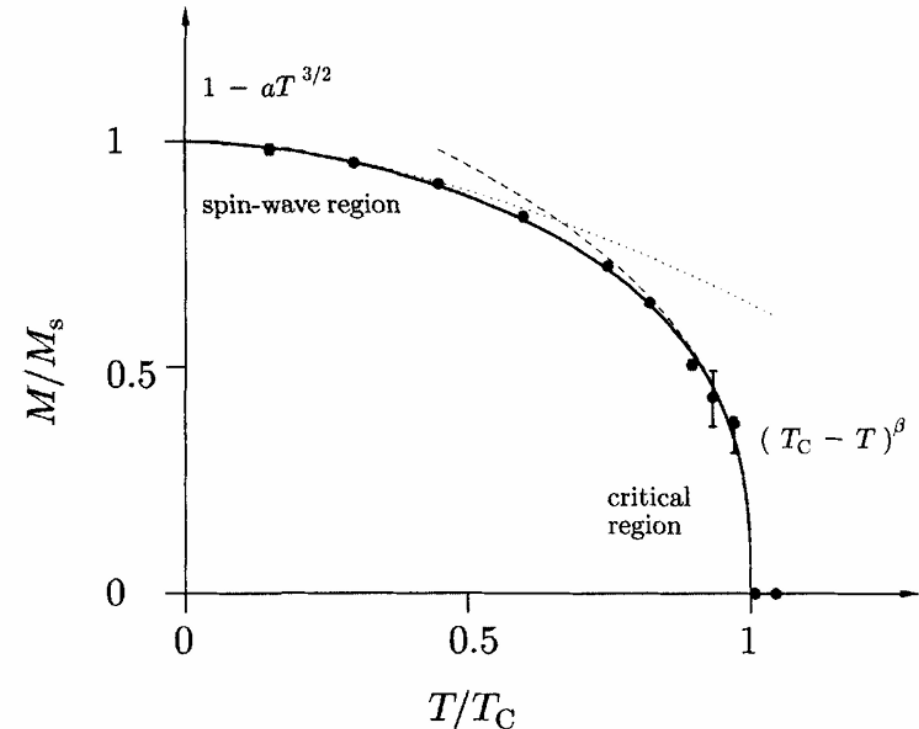
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# Spin-Boson Transformation

## How to describe magnons

The spin operators are not bosonic

Magnons are bosonic

Alternative approach:

LLG (analytically or numerically)

Or Schwinger representation

Or Dyson–Maleev representation

## Conditions

- i. The **transformation** needs to be **Hermitian**, raising and lowering operators written as creation and annihilation boson operators need to be Hermitian conjugate of each other
- ii. The **transformation** must be **unitary** to preserve the commutation relations between the spin operators.
- iii. Must satisfy the **equality between the matrix elements** of the spin operators on  $|0\rangle$  and the bosons on  $|n\rangle$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

# A Spin-Boson Transformation

DECEMBER 15, 1940

PHYSICAL REVIEW

VOLUME 58

## Define higher magnon states

$$|n_\sigma\rangle = |S, S, \dots, \underbrace{S - \sigma_n}_{n}, \dots, S\rangle$$



$$\hat{S}_n^+ |n_\sigma\rangle = \sqrt{2S} \sqrt{1 - \frac{\sigma_n - 1}{2S}} \sqrt{\sigma_n} |n_{\sigma-1}\rangle$$

$$\hat{S}_n^- |n_\sigma\rangle = \sqrt{2S} \sqrt{\sigma_n + 1} \sqrt{1 - \frac{\sigma_n}{2S}} |n_{\sigma+1}\rangle$$

## Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet

T. HOLSTEIN

*New York University, New York, New York*

AND

H. PRIMAKOFF\*

*Polytechnic Institute of Brooklyn, Brooklyn, New York*

(Received July 31, 1940)



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Only for  $\sigma \leq 2S$

$$\hat{S}_n^- |n_{2S}\rangle = \sqrt{2S} \sqrt{2S + 1} \sqrt{1 - \frac{2S}{2S}} |n_{2S+1}\rangle = 0$$

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Consider the states to be like 2<sup>nd</sup> quantized number states



$$\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n} |n_{\sigma-1}\rangle$$

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# A Spin-Boson Transformation

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$$\hat{S}_n^+ = \sqrt{2S} \sqrt{1 - \frac{\hat{a}_n^\dagger \hat{a}_n}{2S}} \hat{a}_n$$

Consider the states to be like 2<sup>nd</sup> quantized number states



$$\hat{a}_n^\dagger \hat{a}_n |n_{\sigma-1}\rangle = (\sigma_n - 1) |n_{\sigma-1}\rangle$$

$$\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n} |n_{\sigma-1}\rangle$$

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# A Spin-Boson Transformation

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Consider the states to be like 2<sup>nd</sup> quantized number states



$$\hat{a}_n^\dagger \hat{a}_n |n_{\sigma-1}\rangle = (\sigma_n - 1) |n_{\sigma-1}\rangle$$

$$\hat{a}_n^\dagger \hat{a}_n |n_\sigma\rangle = \sigma_n |n_\sigma\rangle$$

$$\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n} |n_{\sigma-1}\rangle$$

$$\hat{a}_n^\dagger |n_\sigma\rangle = \sqrt{\sigma_n + 1} |n_{\sigma+1}\rangle$$

$$\hat{S}_n^- = \sqrt{2S} \hat{a}_n^\dagger \sqrt{1 - \frac{\hat{a}_n^\dagger \hat{a}_n}{2S}}$$

# The Holstein Primakoff Spin-Boson Transformation

Combined

$$\hat{S}_n^- = \sqrt{2S} \hat{a}_n^\dagger \sqrt{1 - \frac{\hat{a}_n^\dagger \hat{a}_n}{2S}}$$

$$\hat{S}_n^+ = \sqrt{2S} \sqrt{1 - \frac{\hat{a}_n^\dagger \hat{a}_n}{2S}} \hat{a}_n$$

$$\hat{S}_n^z = S - \hat{a}_n^\dagger \hat{a}_n \quad \leftarrow$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^+ = \hat{S}_j^x + i\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$

$$\hat{S}_j^- = \hat{S}_j^x - i\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

*T-Holstein, H.Primakoff PR 58, 1098, (1940)*

# The Holstein Primakoff Spin-Boson Transformation

## Checking the conditions

- (i) Fulfilled
  - (ii) Does fulfill the commutation relations
  - (iii) Fulfilled by definition
- } for  $\sigma \leq 2S$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^+ &= \hat{S}_j^x + i\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j \\ \hat{S}_j^- &= \hat{S}_j^x - i\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\end{aligned}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

$$[\hat{S}_i^z, \hat{S}_j^\pm] = \pm \hat{S}_i^\pm \delta_{i,j}$$



$$[\hat{S}_i^+, \hat{S}_j^-] = \dots = 2\hat{S}_i^z \delta_{i,j}$$



# The Holstein Primakoff Spin-Boson Transformation

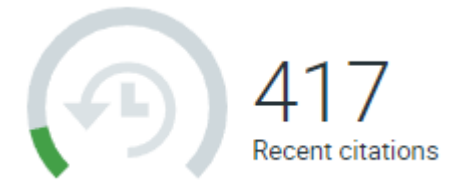
## Applications

HP framework is a powerful method for calculating dispersions and higher order interactions

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^+ &= \hat{S}_j^x + i\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j \\ \hat{S}_j^- &= \hat{S}_j^x - i\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\end{aligned}$$

$$\langle \hat{a}^\dagger \hat{a} \rangle < 2S$$

*T-Holstein, H.Primakoff* PR **58**, 1098, (1940)



[Journals.aps.org](https://journals.aps.org)

# Linearized Holstein-Primakoff

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^+ &= \hat{S}_j^x + i\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j \\ \hat{S}_j^- &= \hat{S}_j^x - i\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\end{aligned}$$

*T-Holstein, H.Primakoff PR 58, 1098, (1940)*

## Sacrifices

Does not fulfill the boson commutation relations

Hence (ii) is violated

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$

$$[\hat{a}_i, \hat{a}_j] = 0$$



$$[\hat{S}_i^+, \hat{S}_j^-] = 2\hat{S}_i^z \delta_{i,j}$$

$$\begin{aligned}[\hat{a}_i, \hat{a}_j^\dagger] &= \frac{1}{2S} [\hat{S}_i^+, \hat{S}_j^-] = \frac{1}{S} \hat{S}_i^z \delta_{i,j} \neq \delta_{i,j} \\ &\approx \frac{S}{S} \delta_{i,j} = \delta_{i,j}\end{aligned}$$



## Linearized HP



$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &= \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger) \\ \hat{S}_j^x &= \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} (\hat{a}_j + \hat{a}_j^\dagger) \\ \hat{S}_j^+ &\approx \sqrt{2S} \cdot \hat{a}_j \\ \hat{S}_j^- &\approx \sqrt{2S} \cdot \hat{a}_j^\dagger\end{aligned}$$



# Holstein-Primakoff

## Simplified application

Only linear Spin-Wave theory using the linearized HP

$$\mathcal{H} = - \sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta}$$

$$= - \sum_{i,\Delta} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)$$

$$= - \sum_{i,\Delta} J \cdot \left( S \left\{ \hat{a}_i \hat{a}_{i+\Delta}^\dagger + \hat{a}_i^\dagger \hat{a}_{i+\Delta} \right\} + (S - \hat{n}_i) (S - \hat{n}_{i+\Delta}) \right)$$

$$= -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\} + \underbrace{\sum_{i,\Delta} J \hat{n}_i \hat{n}_{i+\Delta}}_{\approx 0}$$

$$\approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^+ = \hat{S}_j^x + i\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$

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T-Holstein, H.Primakoff PR **58**, 1098, (1940)

## Linearized HP

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger)$$

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$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

# Holstein-Primakoff

## Rewrite the Hamiltonian

$$\begin{aligned}\mathcal{H} &\approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\} \\ &\approx -NJS^2 - \sum_{i,j} \mathcal{H}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j\end{aligned}$$

Time evolution of an operator:

## Linearized HP

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &= \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right) \\ \hat{S}_j^x &= \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right) \\ \hat{S}_j^+ &\approx \sqrt{2S} \cdot \hat{a}_j \\ \hat{S}_j^- &\approx \sqrt{2S} \cdot \hat{a}_j^\dagger\end{aligned}$$

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Time evolution of an operator:

$$\begin{aligned}i\hbar \frac{\partial}{\partial t} \hat{a}_n &= [\mathcal{H}, \hat{a}_n] \\ &= 0 - \sum_{i,j} \mathcal{H}_{i,j}^{\text{SW}} \cdot [\hat{a}_i^\dagger \hat{a}_j, \hat{a}_n] \\ &= - \sum_{i,j} \mathcal{H}_{i,j}^{\text{SW}} \cdot \delta_{i,n} \hat{a}_j \\ &= - \sum_j \mathcal{H}_{n,j}^{\text{SW}} \cdot \hat{a}_j\end{aligned}$$

**Problem:** Coupling between all sites!  
Practically unsolvable for real systems

## Linearized HP

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &= \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger) \\ \hat{S}_j^x &= \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} (\hat{a}_j + \hat{a}_j^\dagger) \\ \hat{S}_j^+ &\approx \sqrt{2S} \cdot \hat{a}_j \\ \hat{S}_j^- &\approx \sqrt{2S} \cdot \hat{a}_j^\dagger\end{aligned}$$

# Holstein-Primakoff

## The new Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$
$$\approx -NJS^2 - \sum_{i,j} \mathcal{H}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j$$

Time evolution of an operator:

$$i\hbar \frac{\partial}{\partial t} \hat{a}_n = - \sum_j \mathcal{H}_{n,j}^{\text{SW}} \cdot \hat{a}_j$$

**Problem:** Coupling between all sites!  
Practically unsolvable for real systems

## Going to Fourier-space

Lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n \exp(-i\vec{k}\vec{r}_n),$$
$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n^\dagger \exp(i\vec{k}\vec{r}_n).$$

# Holstein-Primakoff

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Time evolution:

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

# Holstein-Primakoff

## The new Hamiltonian

$$\begin{aligned}\mathcal{H} &\approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\} \\ &\approx -NJS^2 - \sum_{i,j} \mathcal{H}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j\end{aligned}$$

Time evolution of an operator:

$$i\hbar \frac{\partial}{\partial t} \hat{a}_n = - \sum_j \mathcal{H}_{n,j}^{\text{SW}} \cdot \hat{a}_j$$

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Time evolution:

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

Plug in

$$\begin{aligned}i\hbar \frac{\partial}{\partial t} \hat{a}(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_n \left( - \sum_j \mathcal{H}_{n,j}^{\text{SW}} \cdot \hat{a}_j \right) \exp(-i\vec{k}\vec{r}_n) \\ &= \frac{1}{\sqrt{N}} \sum_n \sum_j \mathcal{H}_{n,j}^{\text{SW}} \cdot \hat{a}_j \exp(-i\vec{k}\vec{r}_n) \\ &= \dots\end{aligned}$$

# Holstein-Primakoff

## Solving the differential equation

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \dots = -\frac{i}{\hbar} \mathcal{H}_{\vec{k}}^{\text{SW}} \hat{a}(\vec{k}) \quad \longrightarrow \quad \hat{a}(\vec{k}, t) = \hat{a}(\vec{k}, 0) \exp(-i \omega_{\vec{k}} t)$$

With the spin-wave frequency

$$\begin{aligned} \hbar \omega_{\vec{k}} &= \mathcal{H}_{\vec{k}}^{\text{SW}} \\ &= S (J_0 - J_{\vec{k}}) \end{aligned}$$

*L. Rósa, Lecture Notes (2022)*

$$J_{\vec{k}} \propto \sum_j J_{j,0} \exp(-i \vec{k}(\vec{r}_j - \vec{r}_0)) \quad J_0 = J_{\vec{k}=0}$$

# Holstein-Primakoff

## Solving the differential equation

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$$\hbar \omega_{\vec{k}} = \dots \propto JS \sum_{\Delta} \left( 1 - \cos(\vec{k} \cdot \vec{r}_{\Delta}) \right)$$



# Holstein-Primakoff

## Solving the differential equation

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \dots = -\frac{i}{\hbar} \mathcal{H}_{\vec{k}}^{\text{SW}} \hat{a}(\vec{k}) \quad \longrightarrow \quad \hat{a}(\vec{k}, t) = \hat{a}(\vec{k}, 0) \exp(-i \omega_{\vec{k}} t)$$

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*L. Rósz, Lecture Notes (2022)*

$$\begin{aligned} \hat{a}_j(t) &= \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(i \vec{k} \vec{r}_j) \hat{a}(\vec{k}, t) \\ &= \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(i \vec{k} \vec{r}_j) \hat{a}(\vec{k}, 0) \cdot \exp(-i \omega_{\vec{k}} t) \end{aligned}$$

$$J_{\vec{k}} \propto \sum_j J_{j,0} \exp(-i \vec{k} (\vec{r}_j - \vec{r}_0)) \quad J_0 = J_{\vec{k}=0}$$

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \hat{a}(\vec{k}_0, 0) \cdot \exp(i \vec{k}_0 \vec{r}_j - i \omega_{\vec{k}_0} t)$$

$$\hbar \omega_{\vec{k}} = \dots \propto JS \sum_{\Delta} \left( 1 - \cos(\vec{k} \cdot \vec{r}_{\Delta}) \right)$$

$$\hat{a}_j(\vec{k}_0, t) = \hat{A} \cdot \exp(i \vec{k}_0 \vec{r}_j - i \omega_{\vec{k}_0} t)$$

# Holstein-Primakoff

## Back to Real-space

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(i\vec{k}_0 \vec{r}_j - i\omega_{\vec{k}_0} t) \hat{a}(\vec{k}_0, 0)$$

$$\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(i\vec{k}_0 \vec{r}_j - i\omega_{\vec{k}_0} t)$$

## Linearized HP

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger)$$

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S.Blundell, *Magnetism in Condensed matter* (2000)

## Linearized HP

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} (\hat{a}_j + \hat{a}_j^\dagger)\end{aligned}$$

$$\begin{aligned}\hat{S}_j^z(t) &= S - |\hat{A}|^2 \\ \hat{S}_j^y(t) &= \sqrt{2S} \hat{A} \cdot \sin(\vec{k}_0 \vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \\ \hat{S}_j^x(t) &= \sqrt{2S} \hat{A} \cdot \cos(\vec{k}_0 \vec{r}_j + \omega_{\vec{k}_0} t + \phi_A)\end{aligned}$$

This explains why the average we calculated earlier was 0

# Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

$$\mathcal{H} = - \sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathcal{K}_i \cdot \hat{S}_i$$

In-plane anisotropy

$$\mathcal{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov Transformation

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$$\begin{aligned} \hat{S}_i \cdot \mathcal{K}_i \cdot \hat{S}_i &= K \hat{S}_i^x \cdot \hat{S}_i^x \\ &\approx K \frac{S}{2} \left( \hat{a}_i \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger \right) \\ &= K \frac{S}{2} \left( \hat{a}_i \hat{a}_i + 2 \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger \right) + K \frac{S}{2} \end{aligned}$$

Linearized HP

$$\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right) \end{aligned}$$

$$\mathcal{H}_{\text{iso}} \approx -E_0 - \sum_{i,j} \mathcal{H}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j$$

# Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

$$\mathcal{H} = - \sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathcal{K}_i \cdot \hat{S}_i$$

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Linearized HP

$$\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right) \end{aligned}$$

$$\mathcal{H} \approx -\tilde{E}_0 - \sum_{i,j} \tilde{\mathcal{H}}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2} \left( \underline{\hat{a}_i \hat{a}_i} + \underline{\hat{a}_i^\dagger \hat{a}_i^\dagger} \right)$$

How to diagonalize this?

# Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

$$\mathcal{H} = - \sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathcal{K}_i \cdot \hat{S}_i$$

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Linearized HP

$$\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right) \end{aligned}$$

$$\mathcal{H} \approx -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K} (\hat{a}_i \hat{a}_i) + \text{c.c}$$

How to diagonalize this ?

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\begin{aligned}\mathcal{H} &= -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K} (\hat{a}_i \hat{a}_i) + \text{c.c} \\ &= -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\hat{a} &= (\hat{a}_1, \dots, \hat{a}_N)^T \\ \hat{a}^\dagger &= (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T\end{aligned}$$



# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\begin{aligned}\mathcal{H} &= -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K} (\hat{a}_i \hat{a}_i) + \text{c.c} \\ &= -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^* & H^* \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

Seek a Bogoliubov transformation

$$\begin{aligned}\hat{a} &= U \hat{\alpha} + V^* \hat{\alpha}^\dagger \\ \hat{a}^\dagger &= V \hat{\alpha} + U^* \hat{\alpha}^\dagger\end{aligned}$$

That diagonalizes  $\mathcal{H}$  to  $\mathcal{H} = -(\hat{\alpha}^\dagger, \hat{\alpha}) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$

$$\begin{aligned}\hat{a} &= (\hat{a}_1, \dots, \hat{a}_N)^T \\ \hat{a}^\dagger &= (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T\end{aligned}$$

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$
$$\hat{a}^\dagger = (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T$$

$$\hat{a} = U\hat{\alpha} + V^\star\hat{\alpha}^\dagger$$
$$\hat{a}^\dagger = V\hat{\alpha} + U^\star\hat{\alpha}^\dagger$$

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

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$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \text{X}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T$$

$$\hat{a} = U \hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V \hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \text{X}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently  $\mathcal{I} = \mathcal{T} \mathcal{I} \mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{I} \mathcal{T}$

$$\mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

Thus it needs to be a pseudo unitary transformation

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T$$

$$\hat{a} = U \hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V \hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently

$$\mathcal{I} = \mathcal{T} \mathcal{I} \mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{I} \mathcal{T} \quad \mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$\mathbb{I} = (\mathcal{I} \mathcal{T}) \cdot (\mathcal{I} \mathcal{T}^\dagger) = (\mathcal{I} \mathcal{T}^\dagger) \cdot (\mathcal{I} \mathcal{T})$$

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T$$

$$\hat{a} = U \hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V \hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} = \mathcal{T}^{-1} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

What we need to solve

$$\underbrace{\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}}_{=:\mathcal{M}} = \begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}^\dagger \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}$$

Such that

$$\mathcal{H} = -\tilde{E}_0 - \sum_{i,j} \mathcal{M}_{i,j}^{\text{diag}} \cdot \hat{a}_i^\dagger \hat{a}_j$$

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T$$

$$\hat{a} = U\hat{\alpha} + V^\star\hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star\hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} = \mathcal{T}^{-1} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

# Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K} (\hat{a}_i \hat{a}_i) + \text{c.c}$$

$$= -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$= -\tilde{E}_0 - \sum_{i,j} \mathcal{M}_{i,j}^{\text{diag}} \cdot \hat{\alpha}_i^\dagger \hat{\alpha}_j$$

Lattice Fourier transformation

$$\mathcal{H} = \sum_{\vec{k}} \mathcal{M}_{\vec{k}} \cdot \hat{\alpha}^\dagger(\vec{k}) \hat{\alpha}(\vec{k})$$

# Visualising a Magnon

## Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

Time integration with Heun's methods

Additional property is the damping  $\alpha$

Engine\_LLG\Test.ipynb

$$\frac{d\vec{S}_i}{dt} = -\frac{\gamma}{(1 + \alpha^2)\mu_S} \vec{S}_i \times \left( \vec{H}_i(t) + \alpha \vec{S}_i \times \vec{H}_i(t) \right)$$
$$\vec{H}_i(t) = -\frac{\partial \mathcal{H}}{\partial \vec{S}_i}$$



# Visualising a Magnon

## Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

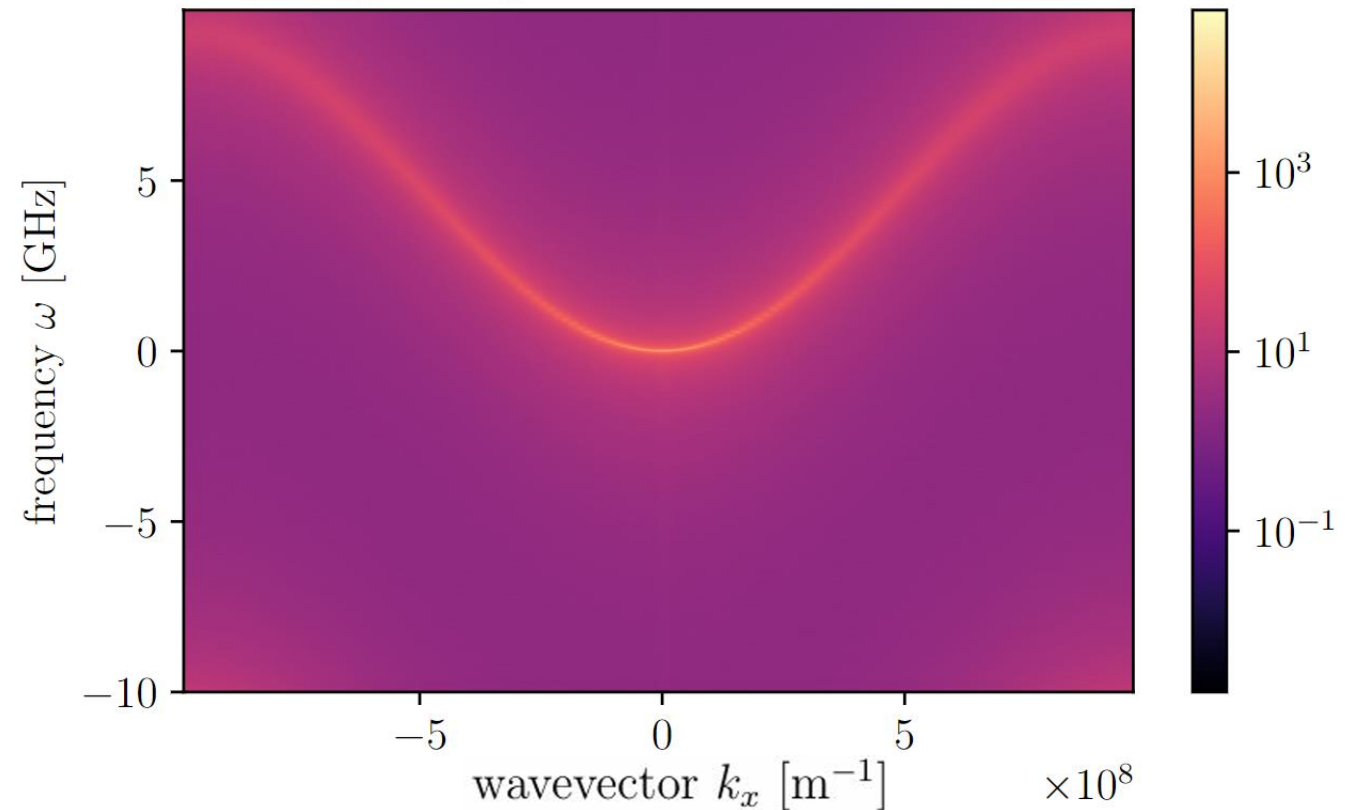
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Engine\_LLG\Test.ipynb

$$\frac{d\vec{S}_i}{dt} = -\frac{\gamma}{(1+\alpha^2)\mu_S} \vec{S}_i \times \left( \vec{H}_i(t) + \alpha \vec{S}_i \times \vec{H}_i(t) \right)$$
$$\vec{H}_i(t) = \vec{\xi}_i(T, t) - \frac{\partial \mathcal{H}}{\partial \vec{S}_i}$$

x+iy-Component of sublattice sc1



# Outlook

## Current research

Easily obtain highly interesting dispersion relations.

Can be used for computing, without energy loss through Joule heating

This is ongoing research in

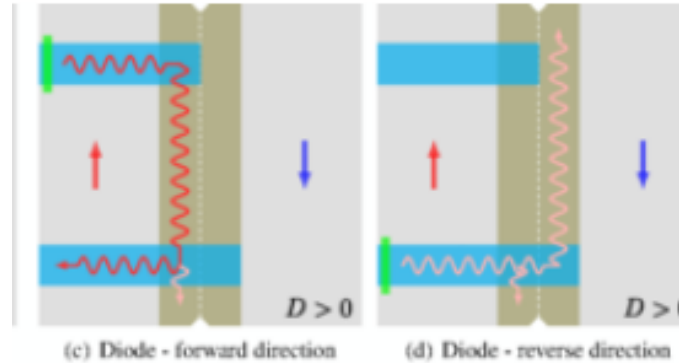
Spin wave diodes *J.Lan et.al. PRX 5, 041049, (2015)*

Spin wave transistors *A.Chumak et.al. Nature C. 5, 4700, (2014)*

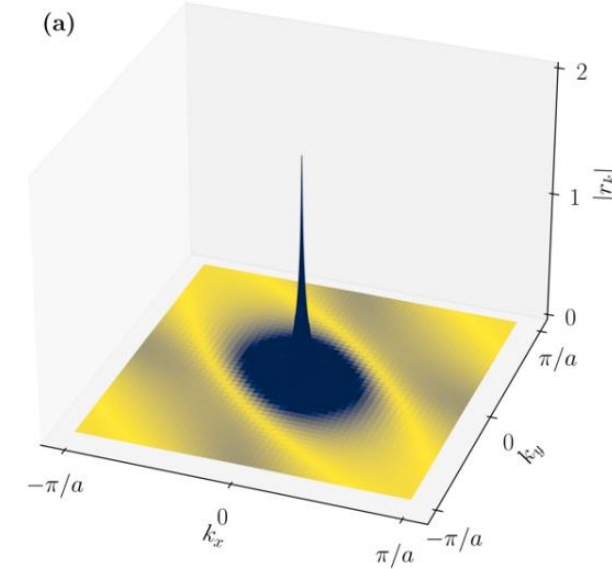
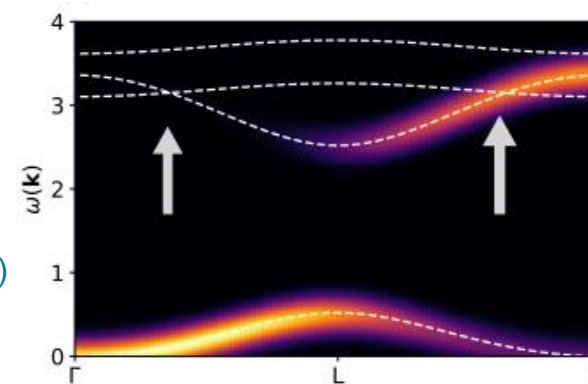
The most modern research even includes Altermagnets, which have distinct symmetry enforced properties regarding the dispersion in different directions

Topological magnons *P. McClarty Annual Reviews 13, 171-190, (2022)*

Squeezed magnons *D. Wuhler et.al. PR 5, 043124, (2023)*

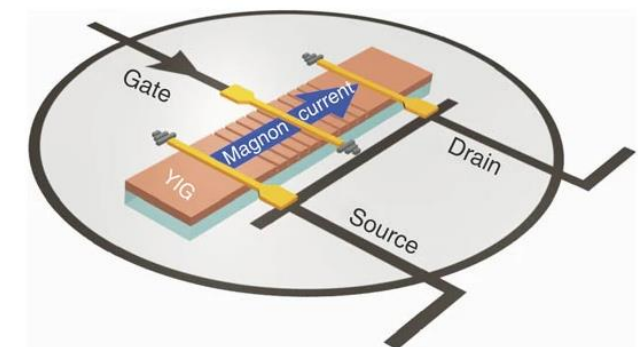


*J.Lan et.al. PRX 5, 041049, (2015)*



*D. Wuhler et.al. PR 5, 043124, (2023)*

Magnon transistor scheme



*A.Chumak et.al. Nature C. 5, 4700, (2014)*