Chapter 6

PERTURBATION THEORY IN PLANAR FERROMAGNETS

6.1. Bogoliubov Transformation

Until now we have studied isotropic or uniaxial ferromagnetic systems for which the ground state is known exactly. In this chapter, we will study the *planar* ferromagnet in which the single-ion easy-plane anisotropy changes dramatically the scenario. First of all, the ground state of the planar ferromagnet is not known exactly, then the bilinear boson Hamiltonian is no longer diagonal but a Bogoliubov transformation has to be used to diagonalize it. Even worse, the DM spin-boson transformation[15,16] leads to a bilinear Hamiltonian that is non-hermitian violating one of the postulates of the quantum mechanics. This fact also occurred in the isotropic ferromagnet, but only when the interaction between spin waves was accounted for, and in Section 3.4, we showed that this drawback does not affect the physical quantities at low temperature. Moreover, two other inconsistencies occur: the *kinematical consistency*, that is, the disappearance of any anisotropy dependence of the spin wave spectrum for $S = \frac{1}{2}$ is lost as well as the existence of a Goldstone mode at $\boldsymbol{k} = 0$ as required by the invariance property of the Hamiltonian under rotation about the z-axis. This intriguing puzzle was tackled[46] some time ago: only a very careful treatment of the terms coming from the perturbation expansion in $\frac{1}{S}$ allows to overcome any physical inconsistence. The detailed procedure to arrive at a correct spin wave spectrum may be useful to make the reader more familiar with the many-body theory in unorthodox systems. The many-body approach to this problem is similar to that used to treat the ^{4}He case[47] even though the absence of the Bose–Einstein condensation makes the problem more difficult.

Let us consider the exchange Hamiltonian

$$\mathcal{H} = -J \sum_{i,\boldsymbol{\delta}} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+\boldsymbol{\delta}} + D \sum_i (S_i^z)^2, \tag{6.1.1}$$

where i runs over all sites of a Bravais lattice; $J > 0$ is the ferromagnetic exchange integral between a spin located on the lattice site i and its NN. For $D > 0$, the single-ion anisotropy favours the xy-plane and the Hamiltonian (6.1.1) is invariant under rotation about the z-axis. This symmetry implies the existence of a Goldstone

mode[48] or a "soft mode" at $\boldsymbol{k} = 0$ in the spin wave excitation spectrum. Moreover, for $S = \frac{1}{2}$ the single-ion anisotropy reduces to a constant so that any dependence of the spin wave spectrum on the anisotropy D has to disappear (kinematical consistency). Without loss of generality, we assume that x is the quantization axis in the xy-plane. By using the DM spin-boson transformation

$$\begin{aligned} S_i^x &= S - a_i^+ a_i, \\ S_i^y &= \frac{\sqrt{2S}}{2}\left(a_i + a_i^+ - \frac{1}{2S} a_i^+ a_i a_i\right), \\ S_i^z &= \frac{\sqrt{2S}}{2i}\left(a_i - a_i^+ - \frac{1}{2S} a_i^+ a_i a_i\right) \end{aligned} \tag{6.1.2}$$

and the Fourier transforms (2.2.14) together with a normal ordering of the Bose operators that is putting all creation Bose operators $a_{\boldsymbol{k}}^+$ on the left of the Bose destruction operators $a_{\boldsymbol{k}}$, one obtains

$$\mathcal{H} = E_0 + \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 \tag{6.1.3}$$

where

$$E_0 = -zJS^2N + \frac{1}{2}DSN. \tag{6.1.4}$$

The bilinear Hamiltonian is

$$\mathcal{H}_2 = \mathcal{H}_0 + \mathcal{H}_2^{\mathrm{NH}} \tag{6.1.5}$$

where

$$\mathcal{H}_0 = 2zJS \sum_{\boldsymbol{q}} \left[(\omega_{\boldsymbol{q}} + \delta) a_{\boldsymbol{q}}^+ a_{\boldsymbol{q}} - \frac{1}{2}\delta(a_{\boldsymbol{q}} a_{-\boldsymbol{q}} + a_{\boldsymbol{q}}^+ a_{-\boldsymbol{q}}^+)\right] \tag{6.1.6}$$

with

$$\omega_{\boldsymbol{q}} = 1 - \gamma_{\boldsymbol{q}}, \quad d = \frac{D}{2zJ}, \quad \delta = d\left(1 - \frac{1}{2S}\right) \tag{6.1.7}$$

is the hermitian bilinear Hamiltonian and

$$\mathcal{H}_2^{\mathrm{NH}} = -\frac{zJd}{2} \sum_{\boldsymbol{q}} a_{\boldsymbol{q}}^+ a_{-\boldsymbol{q}}^+ \tag{6.1.8}$$

is the "non-hermitian" bilinear Hamiltonian. The four-operator Hamiltonian is

$$\mathcal{H}_4 = \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4} \delta_{\boldsymbol{q}_1 + \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4} [V_{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4}\ a_{\boldsymbol{q}_1}^+ a_{\boldsymbol{q}_2}^+ a_{\boldsymbol{q}_3} a_{\boldsymbol{q}_4} + D^{(1)} a_{\boldsymbol{q}_1}^+ a_{-\boldsymbol{q}_2} a_{\boldsymbol{q}_3} a_{\boldsymbol{q}_4}] \tag{6.1.9}$$

with

$$V_{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4} = -\frac{zJ}{4N}[\gamma_{\boldsymbol{q}_1 - \boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_1 - \boldsymbol{q}_4} + \gamma_{\boldsymbol{q}_2 - \boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_2 - \boldsymbol{q}_4} - 2(\gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2}) + 4d] \tag{6.1.10}$$

and

$$D^{(1)} = \frac{zJ\delta}{N}. \tag{6.1.11}$$

The six-operator Hamiltonian is

$$\mathcal{H}_6 = \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} D^{(2)}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3,\boldsymbol{q}_4+\boldsymbol{q}_5+\boldsymbol{q}_6}\ a^+_{\boldsymbol{q}_1}a^+_{\boldsymbol{q}_2}a_{-\boldsymbol{q}_3}a_{\boldsymbol{q}_4}a_{\boldsymbol{q}_5}a_{\boldsymbol{q}_6} \tag{6.1.12}$$

with

$$D^{(2)} = -\frac{zJd}{4SN^2}. \tag{6.1.13}$$

Unlike the isotropic ferromagnet, the bilinear Hamiltonian (6.1.5) of the planar ferromagnet is no longer diagonal. The merit of the DM transformation is in generating only a finite number of interaction potentials even though the boson Hamiltonian so obtained is no longer hermitian. The non-hermiticity of the bilinear Hamiltonian (6.1.5) prevents its diagonalization using the Bogoliubov transformation. To overcome this unpleasant feature we limit ourselves to diagonalize the hermitian bilinear Hamiltonian (6.1.6) and treating the non-hermitian bilinear Hamiltonian (6.1.8) as a perturbation. We introduce the Bogoliubov transformation

$$a_{\boldsymbol{q}} = l_{\boldsymbol{q}}(\alpha_{\boldsymbol{q}} - x_{\boldsymbol{q}}\alpha^+_{-\boldsymbol{q}}), \quad a^+_{\boldsymbol{q}} = l_{\boldsymbol{q}}(\alpha^+_{\boldsymbol{q}} - x_{\boldsymbol{q}}\alpha_{-\boldsymbol{q}}) \tag{6.1.14}$$

where $l_{\boldsymbol{q}}$ and $x_{\boldsymbol{q}}$ are scalar functions of the wavevector $\boldsymbol{q}$. The requirement that $\alpha_{\boldsymbol{q}}$ and $\alpha^+_{\boldsymbol{q}}$ are Bose operators satisfying the same commutation rules of $a_{\boldsymbol{q}}$ and $a^+_{\boldsymbol{q}}$ leads to the relationships

$$l_{\boldsymbol{q}} = l_{-\boldsymbol{q}} = l^*_{\boldsymbol{q}}, \quad x_{\boldsymbol{q}} = x_{-\boldsymbol{q}} = x^*_{\boldsymbol{q}}, \quad l^2_{\boldsymbol{q}}(1 - x^2_{\boldsymbol{q}}) = 1. \tag{6.1.15}$$

Equations in (6.1.15) are simultaneously satisfied assuming

$$l_{\boldsymbol{q}} = \cosh\phi_{\boldsymbol{q}}, \quad x_{\boldsymbol{q}} = \tanh\phi_{\boldsymbol{q}}. \tag{6.1.16}$$

By means of Eq. (6.1.14), the Hamiltonian (6.1.6) becomes

$$\begin{aligned}\mathcal{H}_0 = 2zJS\sum_{\boldsymbol{q}}\Big\{&[l^2_{\boldsymbol{q}}(1+x^2_{\boldsymbol{q}})(\omega_{\boldsymbol{q}}+\delta) + 2\delta l^2_{\boldsymbol{q}}x_{\boldsymbol{q}}]\alpha^+_{\boldsymbol{q}}\alpha_{\boldsymbol{q}}\\ &- [2l^2_{\boldsymbol{q}}x_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+\delta) + \delta l^2_{\boldsymbol{q}}(1+x^2_{\boldsymbol{q}})]\frac{1}{2}(\alpha_{\boldsymbol{q}}\alpha_{-\boldsymbol{q}} + \alpha^+_{\boldsymbol{q}}\alpha^+_{-\boldsymbol{q}})\\ &+ l^2_{\boldsymbol{q}}x^2_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+\delta) + \delta l^2_{\boldsymbol{q}}x_{\boldsymbol{q}}\Big\}.\end{aligned} \tag{6.1.17}$$

Obviously, the Hamiltonian (6.1.17) can be diagonalized choosing

$$2l^2_{\boldsymbol{q}}x_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+\delta) + \delta l^2_{\boldsymbol{q}}(1+x^2_{\boldsymbol{q}}) = 0 \tag{6.1.18}$$

or, using Eq. (6.1.16),

$$\tanh 2\phi_{\boldsymbol{q}} = -\frac{\delta}{\omega_{\boldsymbol{q}} + \delta}. \tag{6.1.19}$$

Using the relationships between the hyperbolic functions

$$l_{\boldsymbol{q}}^2 = \cosh^2 \phi_{\boldsymbol{q}} = \frac{1}{2}\left(\frac{1}{\sqrt{1 - \tanh^2 2\phi_{\boldsymbol{q}}}} + 1\right), \tag{6.1.20}$$

$$l_{\boldsymbol{k}}^2 x_{\boldsymbol{q}}^2 = \sinh^2 \phi_{\boldsymbol{q}} = \frac{1}{2}\left(\frac{1}{\sqrt{1 - \tanh^2 2\phi_{\boldsymbol{q}}}} - 1\right), \tag{6.1.21}$$

$$2l_{\boldsymbol{q}}^2 x_{\boldsymbol{q}} = \sinh 2\phi_{\boldsymbol{q}} = \frac{\tanh 2\phi_{\boldsymbol{q}}}{\sqrt{1 - \tanh^2 2\phi_{\boldsymbol{q}}}} \tag{6.1.22}$$

and Eq. (6.1.19), the Hamiltonian (6.1.6) becomes

$$\mathcal{H}_0 = 2zJS \sum_{\boldsymbol{q}} \sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}} + 2\delta)}\alpha_{\boldsymbol{q}}^+ \alpha_{\boldsymbol{q}} + zJSN[X - Y + \delta(X + Z)], \tag{6.1.23}$$

where

$$X = \frac{1}{N}\sum_{\boldsymbol{q}} 2l_{\boldsymbol{q}}^2 x_{\boldsymbol{q}}^2 = \frac{1}{N}\sum_{\boldsymbol{q}} \frac{\omega_{\boldsymbol{q}} + \delta}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}} + 2\delta)}} - 1, \tag{6.1.24}$$

$$Y = \frac{1}{N}\sum_{\boldsymbol{q}} 2l_{\boldsymbol{q}}^2 x_{\boldsymbol{q}}^2 \gamma_{\boldsymbol{q}} = \frac{1}{N}\sum_{\boldsymbol{q}} \frac{\omega_{\boldsymbol{q}} + \delta}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}} + 2\delta)}}\gamma_{\boldsymbol{q}}, \tag{6.1.25}$$

$$Z = \frac{1}{N}\sum_{\boldsymbol{q}} 2l_{\boldsymbol{q}}^2 x_{\boldsymbol{q}} = -\delta\frac{1}{N}\sum_{\boldsymbol{q}} \frac{1}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}} + 2\delta)}} \tag{6.1.26}$$

and

$$T = \frac{1}{N}\sum_{\boldsymbol{q}} 2l_{\boldsymbol{q}}^2 x_{\boldsymbol{q}} \gamma_{\boldsymbol{q}} = -\delta\frac{1}{N}\sum_{\boldsymbol{q}} \frac{\gamma_{\boldsymbol{q}}}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}} + 2\delta)}}. \tag{6.1.27}$$

As one can see from Eq. (6.1.23), the choice of $\mathcal{H}_0$ as "unperturbed" Hamiltonian leads to a spin wave spectrum kinematically consistent with a soft mode at $\boldsymbol{k} = 0$. Note that the Bogoliubov transformation (6.1.14) enters a first correction to the "classical" ground state E_0 given by Eq. (6.1.4). Replacing Eq. (6.1.14) into Eqs. (6.1.9) and (6.1.12), after normal ordering the Hamiltonian (6.1.3) becomes

$$\mathcal{H} = E_0 + \Delta E_0 + \mathcal{H}_0 + \mathcal{H}_2' + \mathcal{H}_4' + \mathcal{H}_6', \tag{6.1.28}$$

where E_0 is given by Eq. (6.1.4) and

$$\Delta E_0 = zJSN\Big\{X - Y + \delta(X+Z) - \frac{1}{4S}[-dZ + (X-Y)^2 + T(T-Z) + d(Z^2+2X^2) + 3\delta XZ] + \frac{3d}{32S^2}Z(Z^2+4X^2)\Big\}. \tag{6.1.29}$$

In Eq. (6.1.29), the terms linear, quadratic and cubic in X, Y, Z and T come from the normal ordering of $\mathcal{H}_2$, $\mathcal{H}_4$ and $\mathcal{H}_6$, respectively. The unperturbed Hamiltonian is

$$\mathcal{H}_0 = 2zJS\sum_{\boldsymbol{q}}\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+2\delta)}\alpha^+_{\boldsymbol{q}}\alpha_{\boldsymbol{q}}, \tag{6.1.30}$$

and the remaining part of the bilinear Hamiltonian coming from the Hamiltonian (6.1.8) and from the normal ordering of Hamiltonians (6.1.9) and (6.1.12) becomes

$$\mathcal{H}'_2 = zJ\sum_{\boldsymbol{k}}\left(-P_{\boldsymbol{q}}\alpha^+_{\boldsymbol{q}}\alpha_{\boldsymbol{q}} + \frac{1}{2}Q_{\boldsymbol{q}}\alpha^+_{\boldsymbol{q}}\alpha^+_{-\boldsymbol{q}} + \frac{1}{2}R_{\boldsymbol{q}}\alpha_{\boldsymbol{q}}\alpha_{-\boldsymbol{q}}\right) \tag{6.1.31}$$

where

$$P_{\boldsymbol{q}} = \frac{1}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+2\delta)}}[A_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+\delta) - \delta B_{\boldsymbol{q}}], \tag{6.1.32}$$

$$Q_{\boldsymbol{q}} = \frac{1}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+2\delta)}}[-\delta A_{\boldsymbol{q}} + B_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+\delta)] - C_{\boldsymbol{q}}, \tag{6.1.33}$$

$$R_{\boldsymbol{q}} = \frac{1}{\sqrt{\omega_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+2\delta)}}[-\delta A_{\boldsymbol{q}} + B_{\boldsymbol{q}}(\omega_{\boldsymbol{q}}+\delta)] + C_{\boldsymbol{q}} \tag{6.1.34}$$

with

$$A_{\boldsymbol{q}} = (X-Y)\omega_{\boldsymbol{q}} + \frac{1}{2}\delta(3Z+4X) + \frac{d}{2S}X(2-3Z), \tag{6.1.35}$$

$$B_{\boldsymbol{q}} = -\frac{1}{2}(Z-T)\gamma_{\boldsymbol{q}} - \frac{1}{2}T\omega_{\boldsymbol{q}} - \frac{\delta}{2}(1-2Z-3X) - \frac{d}{16S}(4-8Z+9Z^2+12X^2) \tag{6.1.36}$$

and

$$C_{\boldsymbol{k}} = \frac{1}{2}(Z-T)\gamma_{\boldsymbol{q}} - \frac{1}{2}T\omega_{\boldsymbol{q}} + \frac{\delta}{2}(1+3X) + \frac{d}{16S}(4-3Z^2-12X^2). \tag{6.1.37}$$

The four-operator Hamiltonian $\mathcal{H}'_4$ is given by

$$\mathcal{H}'_4 = \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} l_{\boldsymbol{q}_1}l_{\boldsymbol{q}_2}l_{\boldsymbol{q}_3}l_{\boldsymbol{q}_4}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{q}_4}(\Phi^{(1)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha_{\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4} + \Phi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha^+_{-\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4} + \Phi^{(3)}_{\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4}\alpha^+_{\boldsymbol{q}_1}\alpha_{-\boldsymbol{q}_2}\alpha_{\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4} + \Phi^{(4)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4}\alpha_{-\boldsymbol{q}_1}\alpha_{-\boldsymbol{q}_2}\alpha_{\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4} + \Phi^{(5)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_4}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha^+_{-\boldsymbol{q}_3}\alpha^+_{-\boldsymbol{q}_4}), \tag{6.1.38}$$

where

$$\begin{aligned}\Phi^{(1)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} &= V_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_2,-\boldsymbol{q}_4,-\boldsymbol{q}_1,\boldsymbol{q}_3}\\ &+ x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_1,-\boldsymbol{q}_3,-\boldsymbol{q}_2,\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_1,-\boldsymbol{q}_4,-\boldsymbol{q}_2,\boldsymbol{q}_3}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}\\ &- \frac{3}{2}D^{(1)}[x_{\boldsymbol{q}_1} + x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}(x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_4})]\\ &+ 3D^{(3)}Z(1 + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4})\\ &+ 6D^{(3)}X[x_{\boldsymbol{q}_1} + x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}(x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_4})], \end{aligned} \tag{6.1.39}$$

$$\begin{aligned}\Phi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4} &= -\frac{2}{3}(x_{\boldsymbol{q}_1}V_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}V_{\boldsymbol{q}_1,-\boldsymbol{q}_3,-\boldsymbol{q}_2,\boldsymbol{q}_4} + x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_1}\\ &+ x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,-\boldsymbol{q}_1,-\boldsymbol{q}_3,\boldsymbol{q}_2})\\ &+ D^{(1)}(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4})\\ &- 2D^{(3)}Z(x_{\boldsymbol{q}_1} + x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4})\\ &- 4D^{(3)}X(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}), \end{aligned} \tag{6.1.40}$$

$$\begin{aligned}\Phi^{(3)}_{\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} &= -\frac{2}{3}(x_{\boldsymbol{q}_2}V_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} + x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_1,-\boldsymbol{q}_3,-\boldsymbol{q}_2,\boldsymbol{q}_4} + x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_1,-\boldsymbol{q}_4,-\boldsymbol{q}_2,\boldsymbol{q}_3}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_3,-\boldsymbol{q}_2,-\boldsymbol{q}_4,\boldsymbol{q}_1} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_1}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1})\\ &+ D^{(1)}(1 + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4})\\ &- 2D^{(3)}Z(x_{\boldsymbol{q}_1} + x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}\\ &+ x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4})\\ &- 4D^{(3)}X(1 + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}), \end{aligned} \tag{6.1.41}$$

$$\begin{aligned}\Phi^{(4)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} &= \frac{1}{6}(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}V_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_1,-\boldsymbol{q}_3,-\boldsymbol{q}_2,\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_1,-\boldsymbol{q}_4,-\boldsymbol{q}_2,\boldsymbol{q}_3}\\ &+ x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_2,-\boldsymbol{q}_4,-\boldsymbol{q}_1,\boldsymbol{q}_3} + x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_4,\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1})\\ &- \frac{1}{4}D^{(1)}(x_{\boldsymbol{q}_1} + x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_4})\\ &+ \frac{1}{2}D^{(3)}Z(1 + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}\\ &+ x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}) + D^{(3)}X(x_{\boldsymbol{q}_1} + x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_4}) \end{aligned} \tag{6.1.42}$$

and

$$\begin{aligned}\Phi^{(5)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_4} = \frac{1}{6}(&x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}V_{\boldsymbol{q}_4,\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_4,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_1} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_3,-\boldsymbol{q}_2,-\boldsymbol{q}_4,\boldsymbol{q}_1}\\ &+ x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}V_{\boldsymbol{q}_4,-\boldsymbol{q}_1,-\boldsymbol{q}_3,\boldsymbol{q}_2} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_3,-\boldsymbol{q}_1,-\boldsymbol{q}_4,\boldsymbol{q}_2} + x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}V_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4})\\ &- \frac{1}{4}D^{(1)}[x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}(x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_4}) + x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}(x_{\boldsymbol{q}_1}+x_{\boldsymbol{q}_2})]\\ &+ \frac{1}{2}D^{(3)}Z(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}\\ &+ x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}) + D^{(3)}X[x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}(x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_4})\\ &+ x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}(x_{\boldsymbol{q}_1}+x_{\boldsymbol{q}_2})] \end{aligned} \tag{6.1.43}$$

with

$$D^{(3)} = \frac{zJd}{4SN}. \tag{6.1.44}$$

The potentials Φ are written in a symmetric even though heavy form which is convenient in the perturbation expansion since it allows to group many terms in a single Feynman diagram with an appropriated multiplicity. For instance, all perturbation terms containing the potentials $\Phi^{(1)}$ that differ only by the exchange of the first and second or the third and fourth subscripts give the same contribution so that a multiplicity 4 can be introduced retaining only one term. Analogously, the potentials $\Phi^{(2)}$ and $\Phi^{(3)}$ are invariant under any permutation of the first, second and third or second, third and fourth subscripts, respectively. The potentials $\Phi^{(4)}$ and $\Phi^{(5)}$ are invariant under any of the 24 permutations of their subscripts. Note that $D^{(1)}$ given by Eq. (6.1.11) is kinematically consistent whereas $D^{(3)}$ given by Eq. (6.1.44) is not kinematically consistent.

The six-operator Hamiltonian $\mathcal{H}'_6$ is given by

$$\begin{aligned}\mathcal{H}'_6 = \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} & l_{\boldsymbol{q}_1}l_{\boldsymbol{q}_2}l_{\boldsymbol{q}_3}l_{\boldsymbol{q}_4}l_{\boldsymbol{q}_5}l_{\boldsymbol{q}_6}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3,\boldsymbol{q}_4+\boldsymbol{q}_5+\boldsymbol{q}_6}\\ &\times(\Psi^{(1)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha^+_{\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4}\alpha_{\boldsymbol{q}_5}\alpha_{\boldsymbol{q}_6}\\ &+\Psi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,-\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha^+_{\boldsymbol{q}_3}\alpha^+_{-\boldsymbol{q}_4}\alpha_{\boldsymbol{q}_5}\alpha_{\boldsymbol{q}_6}\\ &+\Psi^{(3)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha_{-\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4}\alpha_{\boldsymbol{q}_5}\alpha_{\boldsymbol{q}_6}\\ &+\Psi^{(4)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,-\boldsymbol{q}_4,-\boldsymbol{q}_5,\boldsymbol{q}_6}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha^+_{\boldsymbol{q}_3}\alpha^+_{-\boldsymbol{q}_4}\alpha^+_{-\boldsymbol{q}_5}\alpha_{\boldsymbol{q}_6}\\ &+\Psi^{(5)}_{\boldsymbol{q}_1,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6}\alpha^+_{\boldsymbol{q}_1}\alpha_{-\boldsymbol{q}_2}\alpha_{-\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4}\alpha_{\boldsymbol{q}_5}\alpha_{\boldsymbol{q}_6}\\ &+\Psi^{(6)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6}\alpha_{-\boldsymbol{q}_1}\alpha_{-\boldsymbol{q}_2}\alpha_{-\boldsymbol{q}_3}\alpha_{\boldsymbol{q}_4}\alpha_{\boldsymbol{q}_5}\alpha_{\boldsymbol{q}_6}\\ &+\Psi^{(7)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,-\boldsymbol{q}_4,-\boldsymbol{q}_5,-\boldsymbol{q}_6}\alpha^+_{\boldsymbol{q}_1}\alpha^+_{\boldsymbol{q}_2}\alpha^+_{\boldsymbol{q}_3}\alpha^+_{-\boldsymbol{q}_4}\alpha^+_{-\boldsymbol{q}_5}\alpha^+_{-\boldsymbol{q}_6}) \end{aligned} \tag{6.1.45}$$

where

$$\Psi^{(1)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} = -\frac{4}{3}D^{(2)}[x_{\boldsymbol{q}_1}+x_{\boldsymbol{q}_2}+x_{\boldsymbol{q}_3}+(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}) \times (x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_6})+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}(x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6})], \tag{6.1.46}$$

$$\Psi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,-\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} = D^{(2)}[x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4} + (x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}) \times (x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_6})+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6}], \tag{6.1.47}$$

$$\Psi^{(3)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} = D^{(2)}[1+(x_{\boldsymbol{q}_1}+x_{\boldsymbol{q}_2})(x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_6}) + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}(x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6})], \tag{6.1.48}$$

$$\Psi^{(4)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,-\boldsymbol{q}_4,-\boldsymbol{q}_5,\boldsymbol{q}_6} = -\frac{2}{5}D^{(2)}[x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5})x_{\boldsymbol{q}_6}], \tag{6.1.49}$$

$$\Psi^{(5)}_{\boldsymbol{q}_1,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} = -\frac{2}{5}D^{(2)}[x_{\boldsymbol{q}_2}+x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_1}(x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6})], \tag{6.1.50}$$

$$\Psi^{(6)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_4,\boldsymbol{q}_5,\boldsymbol{q}_6} = \frac{1}{15}D^{(2)}(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_5} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4} + x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6}) \tag{6.1.51}$$

and

$$\Psi^{(7)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,-\boldsymbol{q}_4,-\boldsymbol{q}_5,-\boldsymbol{q}_6} = \frac{1}{15}D^{(2)}(x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_1}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_6} + x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_2}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6}+x_{\boldsymbol{q}_3}x_{\boldsymbol{q}_4}x_{\boldsymbol{q}_5}x_{\boldsymbol{q}_6}). \tag{6.1.52}$$

Like the potentials Φ, the potentials Ψ also have been written in a symmetric form convenient to group them in a reduced number of Feynman diagrams with

appropriate multiplicity. For instance, $\Psi^{(6)}$ and $\Psi^{(7)}$ are invariant under all the $6! = 720$ permutations of their subscripts.

6.2. The Dyson Matrix Equation

Analogously to what we have done for the isotropic ferromagnet in Chapter 4, we introduce the temperature Green function in order to perform a perturbation expansion in the interaction Hamiltonian. For the planar ferromagnet Hamiltonian, however, it is convenient to introduce the temperature *matrix* Green function, the elements of which are given by

$$\mathcal{G}^{11}_{\boldsymbol{k}}(\tau) = -\langle T[\alpha^{\mathrm{H}}_{\boldsymbol{k}}(\tau)\alpha^{+}_{\boldsymbol{k}}]\rangle, \tag{6.2.1}$$

$$\mathcal{G}^{12}_{\boldsymbol{k}}(\tau) = -\langle T[\alpha^{\mathrm{H}}_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle, \tag{6.2.2}$$

$$\mathcal{G}^{21}_{\boldsymbol{k}}(\tau) = -\langle T[\alpha^{+\mathrm{H}}_{-\boldsymbol{k}}(\tau)\alpha^{+}_{\boldsymbol{k}}]\rangle \tag{6.2.3}$$

and

$$\mathcal{G}^{22}_{\boldsymbol{k}}(\tau) = -\langle T[\alpha^{+\mathrm{H}}_{-\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle = \mathcal{G}^{11}_{-\boldsymbol{k}}(-\tau), \tag{6.2.4}$$

where $\tau = it$ is an imaginary time and T is the ordering time operator that puts the operators within the square brackets in such a way that the time decreases going from the first to the last. The superscript H stays for "Heisenberg picture". The introduction of the matrix Green function dates back to the paper by Beliaev[47] for the liquid helium and of Nambu[49] for the superconductivity and applied to the antiferromagnet by Harris, Kumar, Halpering and Hohenberg.[50] According to Eq. (4.1.18), we can write a perturbation expansion for each of the four temperature Green functions (6.2.1)–(6.2.4)

$$\mathcal{G}^{11}_{\boldsymbol{k}}(\tau) = -\sum_{n=0}^{\infty}\left(-\frac{1}{\hbar}\right)^{n}\frac{1}{n!}\int_0^{\beta\hbar} d\tau_1\cdots \times\int_0^{\beta\hbar} d\tau_n\langle T[\mathcal{H}'(\tau_1)\cdots\mathcal{H}'(\tau_n)\alpha_{\boldsymbol{k}}(\tau)\alpha^{+}_{\boldsymbol{k}}]\rangle^{\mathrm{c}}_0, \tag{6.2.5}$$

$$\mathcal{G}^{12}_{\boldsymbol{k}}(\tau) = -\sum_{n=0}^{\infty}\left(-\frac{1}{\hbar}\right)^{n}\frac{1}{n!}\int_0^{\beta\hbar} d\tau_1\cdots \times\int_0^{\beta\hbar} d\tau_n\langle T[\mathcal{H}'(\tau_1)\cdots\mathcal{H}'(\tau_n)\alpha_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle^{\mathrm{c}}_0, \tag{6.2.6}$$

$$\mathcal{G}^{21}_{\boldsymbol{k}}(\tau) = -\sum_{n=0}^{\infty}\left(-\frac{1}{\hbar}\right)^{n}\frac{1}{n!}\int_0^{\beta\hbar} d\tau_1\cdots \times\int_0^{\beta\hbar} d\tau_n\langle T[\mathcal{H}'(\tau_1)\cdots\mathcal{H}'(\tau_n)\alpha^{+}_{-\boldsymbol{k}}(\tau)\alpha^{+}_{\boldsymbol{k}}]\rangle^{\mathrm{c}}_0 \tag{6.2.7}$$

and

$$\mathcal{G}^{22}_{\boldsymbol{k}}(\tau) = \mathcal{G}^{11}_{-\boldsymbol{k}}(-\tau), \tag{6.2.8}$$

where the subscript "0" means that the average is performed over the statistical ensemble with the unperturbed Hamiltonian $\mathcal{H}_0$ given by (6.1.30) and the superscript "c" means that only contributions corresponding to connected diagrams have to be retained in the sum. The interaction Hamiltonian $\mathcal{H}'$ is given by the sum

$$\mathcal{H}' = \mathcal{H}'_2 + \mathcal{H}'_4 + \mathcal{H}'_6, \tag{6.2.9}$$

where $\mathcal{H}'_2$, $\mathcal{H}'_4$ and $\mathcal{H}'_6$ are given by Eqs. (6.1.31), (6.1.38) and (6.1.45), respectively. The unperturbed temperature Green functions are given by

$$\mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau) = -\langle T[\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle = \begin{cases} -(1+n_{\boldsymbol{k}})e^{-\epsilon_{\boldsymbol{k}}\tau} & \text{for } \tau > 0, \\ -n_{\boldsymbol{k}}e^{-\epsilon_{\boldsymbol{k}}\tau} & \text{for } \tau < 0, \end{cases} \tag{6.2.10}$$

$$-\langle T[\alpha_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle = -\langle T[\alpha^+_{-\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle = 0, \tag{6.2.11}$$

$$-\langle T[\alpha^+_{-\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle = \begin{cases} -n_{\boldsymbol{k}}e^{\epsilon_{\boldsymbol{k}}\tau} & \text{for } \tau > 0, \\ -(1+n_{\boldsymbol{k}})e^{\epsilon_{\boldsymbol{k}}\tau} & \text{for } \tau < 0, \end{cases} \tag{6.2.12}$$

where

$$\hbar\epsilon_{\boldsymbol{k}} = 2zJS\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}} + 2\delta)} \tag{6.2.13}$$

is the unperturbed spin wave spectrum. Note that from Eqs. (6.2.10) and (6.2.12), one obtains

$$-\langle T[\alpha^+_{-\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle = \mathcal{G}^{(0)}_{-\boldsymbol{k}}(-\tau) = \mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau) \tag{6.2.14}$$

and the generalized Wick's theorem applied to the thermal averages occurring in Eqs. (6.2.5)–(6.2.7) leads to contractions containing only the unperturbed propagator given by Eq. (6.2.10) similar to the unperturbed propagator of the isotropic ferromagnet given by Eq. (4.1.20). The Fourier transforms (4.1.24) and (4.1.22) become

$$\mathcal{G}^{\mathrm{ij}}_{\boldsymbol{k}}(i\omega_n) = \int_0^{\beta\hbar} d\tau e^{i\omega_n\tau}\mathcal{G}^{\mathrm{ij}}_{\boldsymbol{k}}(\tau) \tag{6.2.15}$$

and

$$\mathcal{G}^{\mathrm{ij}}_{\boldsymbol{k}}(\tau) = \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{\mathrm{ij}}_{\boldsymbol{k}}(i\omega_n), \tag{6.2.16}$$

where i,j = 1,2 and $\omega_n = \frac{2\pi n}{\beta\hbar}$. The Fourier transform of the unperturbed temperature Green function coincides with Eq. (4.1.25) replacing the unperturbed spin

wave spectrum of the isotropic ferromagnet by the spin wave spectrum of the planar ferromagnet (6.2.13) that is

$$\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\boldsymbol{k}}}. \tag{6.2.17}$$

The Dyson equation (3.5.2) becomes the *matrix* Dyson equation

$$\begin{aligned}\begin{pmatrix} \mathcal{G}^{11}_{\boldsymbol{k}}(i\omega_n) & \mathcal{G}^{12}_{\boldsymbol{k}}(i\omega_n) \\ \mathcal{G}^{21}_{\boldsymbol{k}}(i\omega_n) & \mathcal{G}^{22}_{\boldsymbol{k}}(i\omega_n) \end{pmatrix} &= \begin{pmatrix} \mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n) & 0 \\ 0 & \mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n) \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n) & 0 \\ 0 & \mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n) \end{pmatrix} \begin{pmatrix} \Sigma^*_{11}(\boldsymbol{k}, i\omega_n) & \Sigma^*_{12}(\boldsymbol{k}, i\omega_n) \\ \Sigma^*_{21}(\boldsymbol{k}, i\omega_n) & \Sigma^*_{22}(\boldsymbol{k}, i\omega_n) \end{pmatrix} \\ &\times \begin{pmatrix} \mathcal{G}^{11}_{\boldsymbol{k}}(i\omega_n) & \mathcal{G}^{12}_{\boldsymbol{k}}(i\omega_n) \\ \mathcal{G}^{21}_{\boldsymbol{k}}(i\omega_n) & \mathcal{G}^{22}_{\boldsymbol{k}}(i\omega_n) \end{pmatrix}\end{aligned} \tag{6.2.18}$$

where $\Sigma^*_{\mathrm{ij}}(\boldsymbol{k}, i\omega_n)$ are the proper self-energies with i,j = 1,2 and

$$\Sigma^*_{22}(\boldsymbol{k}, i\omega_n) = \Sigma^*_{11}(-\boldsymbol{k}, -i\omega_n) = \Sigma^*_{11}(\boldsymbol{k}, -i\omega_n), \tag{6.2.19}$$

where the last equality holds for Bravais lattices with a centre of symmetry. The solution of the matrix equation (6.2.18) is then given by

$$\mathcal{G}^{11}_{\boldsymbol{k}}(i\omega_n) = \frac{i\omega_n + \epsilon_{\boldsymbol{k}} + \Sigma^*_{22}(\boldsymbol{k}, i\omega_n)}{F(\boldsymbol{k}, i\omega_n)}, \tag{6.2.20}$$

$$\mathcal{G}^{12}_{\boldsymbol{k}}(i\omega_n) = -\frac{\Sigma^*_{12}(\boldsymbol{k}, i\omega_n)}{F(\boldsymbol{k}, i\omega_n)}, \tag{6.2.21}$$

$$\mathcal{G}^{21}_{\boldsymbol{k}}(i\omega_n) = -\frac{\Sigma^*_{21}(\boldsymbol{k}, i\omega_n)}{F(\boldsymbol{k}, i\omega_n)}, \tag{6.2.22}$$

$$\mathcal{G}^{22}_{\boldsymbol{k}}(i\omega_n) = \frac{-i\omega_n + \epsilon_{\boldsymbol{k}} + \Sigma^*_{11}(\boldsymbol{k}, i\omega_n)}{F(\boldsymbol{k}, i\omega_n)}, \tag{6.2.23}$$

where

$$\begin{aligned} F(\boldsymbol{k}, i\omega_n) &= [i\omega_n - \epsilon_{\boldsymbol{k}} - \Sigma^*_{11}(\boldsymbol{k}, i\omega_n)][i\omega_n + \epsilon_{\boldsymbol{k}} + \Sigma^*_{11}(\boldsymbol{k}, -i\omega_n)] \\ &\quad + \Sigma^*_{12}(\boldsymbol{k}, i\omega_n)\Sigma^*_{21}(\boldsymbol{k}, i\omega_n) \\ &= [i\omega_n - \Sigma^*_{\mathrm{A}}(\boldsymbol{k}, i\omega_n)]^2 - [\epsilon_{\boldsymbol{k}} + \Sigma^*_{\mathrm{S}}(\boldsymbol{k}, i\omega_n)]^2 \\ &\quad + \Sigma^*_{12}(\boldsymbol{k}, i\omega_n)\Sigma^*_{21}(\boldsymbol{k}, i\omega_n) \end{aligned} \tag{6.2.24}$$

with

$$\Sigma^*_{\mathrm{A}}(\boldsymbol{k}, i\omega_n) = \frac{1}{2}[\Sigma^*_{11}(\boldsymbol{k}, i\omega_n) - \Sigma^*_{22}(\boldsymbol{k}, i\omega_n)] \tag{6.2.25}$$

and

$$\Sigma^*_{\rm S}(\boldsymbol{k}, i\omega_n) = \frac{1}{2}[\Sigma^*_{11}(\boldsymbol{k}, i\omega_n) + \Sigma^*_{22}(\boldsymbol{k}, i\omega_n)]. \tag{6.2.26}$$

The poles of the temperature Green functions (6.2.20)–(6.2.23) are the roots of the equation $F(\boldsymbol{k}, i\omega) = 0$. The perturbation theory is meaningful only if the correction to the unperturbed spin wave spectrum is small compared to the unperturbed spectrum $\epsilon_{\boldsymbol{k}}$ so that the argument $i\omega_n$ inside the self-energy contribution may be replaced by $\epsilon_{\boldsymbol{k}}$ and the poles of the temperature Green functions (6.2.20)–(6.2.23) are given by the roots of the equation

$$[\omega - \Sigma^*_{\rm A}(\boldsymbol{k}, \epsilon_{\boldsymbol{k}})]^2 - [\epsilon_{\boldsymbol{k}} + \Sigma^*_{\rm S}(\boldsymbol{k}, \epsilon_{\boldsymbol{k}})]^2 + \Sigma^*_{12}(\boldsymbol{k}, \epsilon_{\boldsymbol{k}})\Sigma^*_{21}(\boldsymbol{k}, \epsilon_{\boldsymbol{k}}) = 0. \tag{6.2.27}$$

6.3. First-order Perturbation Theory

In this section, we perform explicitly the calculation of the first-order self-energies $\Sigma_{\rm ij}(\boldsymbol{k})$. As for the isotropic ferromagnet, the first-order self-energies of the planar ferromagnet are independent of the frequency $i\omega_n$. In order to simplify the notations, we neglect everywhere the superscript $^{(1)}$ on both the Green functions and the self-energies indicating the first-order approximation and suppress the asterisk on the *proper* self-energy since at the first-order there is no difference between the self-energy and the proper self-energy. The way followed in Chapter 4 for the isotropic ferromagnet is the same we follow here. The main difference between the isotropic case and the planar one is that in the planar ferromagnet the interaction Hamiltonian is made up of more terms and four temperature Green functions are to be accounted for. The first-order perturbation theory corresponds to the first term of the series expansions (6.2.5)–(6.2.8) that is

$$\mathcal{G}^{11}_{\boldsymbol{k}}(\tau) = \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0, \tag{6.3.1}$$

$$\mathcal{G}^{12}_{\boldsymbol{k}}(\tau) = \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle^{\rm c}_0, \tag{6.3.2}$$

$$\mathcal{G}^{21}_{\boldsymbol{k}}(\tau) = \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'(\tau_1)\alpha^+_{-\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0 \tag{6.3.3}$$

and

$$\mathcal{G}^{22}_{\boldsymbol{k}}(\tau) = \mathcal{G}^{11}_{\boldsymbol{k}}(-\tau), \tag{6.3.4}$$

where $\mathcal{H}'$ is given by Eq. (6.2.9). Each of the Eqs. (6.3.1)–(6.3.3) consists of three contributions corresponding to the three terms occurring in the interaction Hamiltonian $\mathcal{H}'$ given by Eq. (6.2.9). The generalized Wick's theorem illustrated in Chapter 4 and the Fourier transform (6.2.16) enable us to write the contribution of

Eq. (6.3.1) coming from $\mathcal{H}'_2$ in the form

$$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_2(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0 = -\frac{1}{\hbar} zJP_{\boldsymbol{k}} \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau_1)$$
$$= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}[\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)]^2[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_2}, \quad (6.3.5)$$

where

$$[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_2} = -\frac{1}{\hbar} zJP_{\boldsymbol{k}}. \quad (6.3.6)$$

The Feynman diagram corresponding to the first-order self-energy given by Eq. (6.3.6) is represented by the first term of the first line of Fig. 6.1.

The contribution to Eq. (6.3.1) coming from of $\mathcal{H}'_4$ becomes

$$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_4(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0$$
$$= -\frac{1}{\hbar}\sum_{\boldsymbol{p}} 4l^2_{\boldsymbol{k}}l^2_{\boldsymbol{p}}\Phi^{(1)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{k},\boldsymbol{p}} \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{p}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau_1)$$
$$= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}[\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)]^2[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_4}, \quad (6.3.7)$$

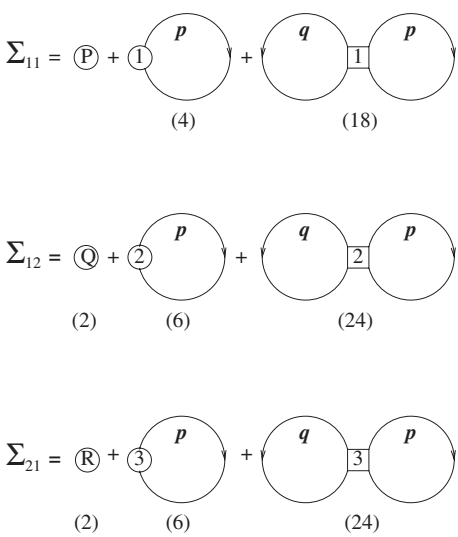

Fig. 6.1. First-order self-energy. The circles labelled by P, Q and R represent the potential interactions of $\mathcal{H}'_2$. The circles containing 1, 2 and 3 represent the potential interactions $\Phi^{(1)}$, $\Phi^{(2)}$ and $\Phi^{(3)}$ of $\mathcal{H}'_4$, respectively. The squares containing 1, 2 and 3 represent the potential interactions $\Psi^{(1)}$, $\Psi^{(2)}$ and $\Psi^{(3)}$ of $\mathcal{H}'_6$, respectively. The numbers in brackets under the diagrams indicate their multiplicity.

where

$$[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_4} = \frac{4}{\hbar}\sum_{\boldsymbol{p}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 \Phi^{(1)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{k},\boldsymbol{p}} n_{\boldsymbol{p}}$$

$$= -\frac{2zJ}{\hbar}\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}[A'_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+\delta) - B'_{\boldsymbol{k}}\delta] \tag{6.3.8}$$

with

$$A'_{\boldsymbol{k}} = (\lambda_2 + \delta\lambda_1)\omega_{\boldsymbol{k}} + 2d(\lambda_1+\delta\lambda_0)\left(1-\frac{3Z}{4S}\right) - \frac{3}{2}\delta^2\lambda_0 + \frac{3d\delta}{2S}\lambda_0 X, \tag{6.3.9}$$

$$B'_{\boldsymbol{k}} = \frac{1}{2}\delta\lambda_1\gamma_{\boldsymbol{k}} + \frac{1}{2}\delta(\lambda_0-\lambda_1)\omega_{\boldsymbol{k}} + \frac{3}{2}\delta(\lambda_1+\delta\lambda_0)$$
$$- d\delta\lambda_0\left(1-\frac{9Z}{8S}\right) - \frac{3d}{2S}X(\lambda_1+\delta\lambda_0), \tag{6.3.10}$$

$$\lambda_r = \frac{1}{N}\sum_{\boldsymbol{p}}\frac{(\omega_{\boldsymbol{p}})^r n_{\boldsymbol{p}}}{\sqrt{\omega_{\boldsymbol{p}}(\omega_{\boldsymbol{p}}+2\delta)}} \tag{6.3.11}$$

and

$$n_{\boldsymbol{p}} = \frac{1}{e^{\beta\hbar\epsilon_{\boldsymbol{p}}}-1}. \tag{6.3.12}$$

To obtain the self-energy (6.3.8) whose Feynman diagram is represented by the second term of the first line of Fig. 6.1, the relationship (4.2.16) is used in which the isotropic unperturbed spectrum $\omega_{\boldsymbol{k}}$ has been replaced by the planar unperturbed spectrum given by Eq. (6.2.13).

The contribution to Eq. (6.3.1) coming from of $\mathcal{H}'_6$ is given by

$$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_6(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0$$

$$= \frac{18}{\hbar}\sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^2 \Psi^{(1)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{p}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{q}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau_1)$$

$$= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}[\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)]^2[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_6} \tag{6.3.13}$$

where

$$[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_6} = \frac{18}{\hbar}\sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^2 \Psi^{(1)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} n_{\boldsymbol{p}} n_{\boldsymbol{q}}$$

$$= -\frac{6zJ}{\hbar S}\frac{d}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}[A''_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+\delta) - B''_{\boldsymbol{k}}\delta] \tag{6.3.14}$$

with

$$A''_{\boldsymbol{k}} = \delta\lambda_0(\lambda_1+\delta\lambda_0) \tag{6.3.15}$$

and

$$B''_{\boldsymbol{k}} = -\frac{1}{2}(\lambda_1+\delta\lambda_0)^2 - \frac{3}{8}\delta^2\lambda_0^2. \tag{6.3.16}$$

The self-energy (6.3.14) is represented by the third diagram of the first line of Fig. 6.1.

The same procedure is followed to evaluate the three contributions generated from Eq. (6.3.2). The contribution coming from $\mathcal{H}'_2$ is

$$\begin{aligned}\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_2(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle_0^{\rm c} \\ &= \frac{1}{2\hbar} zJ(Q_{\boldsymbol{k}}+Q_{-\boldsymbol{k}})\int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau_1) \\ &= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-i\omega_n)[\Sigma_{12}(\boldsymbol{k})]_{\mathcal{H}'_2}, \end{aligned} \tag{6.3.17}$$

where

$$[\Sigma_{12}(\boldsymbol{k})]_{\mathcal{H}'_2} = \frac{1}{\hbar} zJQ_{\boldsymbol{k}}. \tag{6.3.18}$$

The corresponding Feynman diagram is given by the first term of the second line of Fig. 6.1.

The contribution to Eq. (6.3.2) coming from of $\mathcal{H}'_4$ is

$$\begin{aligned}\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_4(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle_0^{\rm c} \\ &= -\frac{6}{\hbar}\sum_{\boldsymbol{p}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} \Phi^{(2)}_{\boldsymbol{k},\boldsymbol{p},-\boldsymbol{k},\boldsymbol{p}}\int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{p}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau_1) \\ &= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-i\omega_n)[\Sigma_{12}(\boldsymbol{k})]_{\mathcal{H}'_4}, \end{aligned} \tag{6.3.19}$$

where

$$\begin{aligned}[\Sigma_{12}(\boldsymbol{k})]_{\mathcal{H}'_4} &= \frac{6}{\hbar}\sum_{\boldsymbol{p}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} \Phi^{(2)}_{\boldsymbol{k},\boldsymbol{p},-\boldsymbol{k},\boldsymbol{p}} n_{\boldsymbol{p}} \\ &= \frac{2zJ}{\hbar}\left\{\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}[-A'_{\boldsymbol{k}}\delta + B'_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+\delta)] - C'_{\boldsymbol{k}}\right\} \end{aligned} \tag{6.3.20}$$

with

$$\begin{aligned}C'_{\boldsymbol{k}} = &-\frac{1}{2}\delta\lambda_1\gamma_{\boldsymbol{k}} + \frac{1}{2}\delta(\lambda_0-\lambda_1)\omega_{\boldsymbol{k}} + \frac{3}{2}\delta(\lambda_1+\delta\lambda_0) \\ &-\frac{3d}{2S}X(\lambda_1+\delta\lambda_0) + \frac{3d\delta}{8S}Z\lambda_0. \end{aligned} \tag{6.3.21}$$

The Feynman diagram corresponding to Eq. (6.3.20) is given by the second term of the second line of Fig. 6.1.

The contribution to Eq. (6.3.2) coming from of $\mathcal{H}'_6$ is

$$\begin{aligned}
&\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_6(\tau_1)\alpha_{\boldsymbol{k}}(\tau)\alpha_{-\boldsymbol{k}}]\rangle_0^{\mathrm{c}} \\
&\quad = \frac{24}{\hbar}\sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^2 \Psi^{(2)}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{p}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{q}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau_1) \\
&\quad = \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-i\omega_n)[\Sigma_{12}(\boldsymbol{k})]_{\mathcal{H}'_6},
\end{aligned} \tag{6.3.22}$$

where

$$\begin{aligned}
[\Sigma_{12}(\boldsymbol{k})]_{\mathcal{H}'_6} &= \frac{24}{\hbar}\sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^2 \Psi^{(2)}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\ n_{\boldsymbol{p}}\ n_{\boldsymbol{q}} \\
&= \frac{6zJ}{\hbar S} d\left\{\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}[-A''_{\boldsymbol{k}}\delta + B''_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+\delta)] - C''_{\boldsymbol{k}}\right\}
\end{aligned} \tag{6.3.23}$$

with

$$C''_{\boldsymbol{k}} = -\frac{1}{2}(\lambda_1+\delta\lambda_0)^2 - \frac{1}{8}\delta^2\lambda_0^2. \tag{6.3.24}$$

The Feynman diagram corresponding to Eq. (6.3.23) is given by the third term of the second line of Fig. 6.1.

As for Eq. (6.3.3), the contribution coming from $\mathcal{H}'_2$ is

$$\begin{aligned}
\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_2(\tau_1)\alpha^+_{-\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle_0^{\mathrm{c}} &= \frac{1}{2\hbar} zJ(R_{\boldsymbol{k}}+R_{-\boldsymbol{k}}) \\
&\quad \times \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau_1) \\
&= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-i\omega_n)[\Sigma_{21}(\boldsymbol{k})]_{\mathcal{H}'_2},
\end{aligned} \tag{6.3.25}$$

where

$$[\Sigma_{21}(\boldsymbol{k})]_{\mathcal{H}'_2} = \frac{1}{\hbar} zJR_{\boldsymbol{k}}. \tag{6.3.26}$$

The Feynman diagram corresponding to the self-energy (6.3.26) is shown by the first term of the third line of Fig. 6.1.

The contribution coming from of $\mathcal{H}'_4$ is given by

$$\begin{aligned}\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_4(\tau_1)\alpha^+_{-\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0 \\ = -\frac{6}{\hbar}\sum_{\boldsymbol{p}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} \Phi^{(3)}_{\boldsymbol{p},-\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}} \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{p}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau_1) \\ = \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-i\omega_n)[\Sigma_{21}(\boldsymbol{k})]_{\mathcal{H}'_4}, \qquad (6.3.27)\end{aligned}$$

where

$$\begin{aligned}[\Sigma_{21}(\boldsymbol{k})]_{\mathcal{H}'_4} = \frac{6}{\hbar}\sum_{\boldsymbol{p}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} \Phi^{(3)}_{\boldsymbol{p},-\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}} n_{\boldsymbol{p}} \\ = \frac{2zJ}{\hbar}\left\{\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}[-A'_{\boldsymbol{k}}\delta + B'_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+\delta)] + C'_{\boldsymbol{k}}\right\}. \qquad (6.3.28)\end{aligned}$$

The corresponding Feynman diagram is given by the second term of the third line of Fig. 6.1.

The contribution to Eq. (6.3.3) coming from of $\mathcal{H}'_6$ is

$$\begin{aligned}\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau_1 \langle T[\mathcal{H}'_6(\tau_1)\alpha^+_{-\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\rm c}_0 \\ = \frac{24}{\hbar}\sum_{\boldsymbol{p},\boldsymbol{q}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} l^2_{\boldsymbol{q}} \Psi^{(3)}_{\boldsymbol{p},\boldsymbol{q},-\boldsymbol{k},\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \int_0^{\beta\hbar} d\tau_1 \mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau-\tau_1)\mathcal{G}^{(0)}_{\boldsymbol{p}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{q}}(0^-)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau_1) \\ = \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-i\omega_n)[\Sigma_{21}(\boldsymbol{k})]_{\mathcal{H}'_6}, \qquad (6.3.29)\end{aligned}$$

where

$$\begin{aligned}[\Sigma_{21}(\boldsymbol{k})]_{\mathcal{H}'_6} = \frac{24}{\hbar}\sum_{\boldsymbol{p},\boldsymbol{q}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} l^2_{\boldsymbol{q}} \Psi^{(3)}_{\boldsymbol{p},\boldsymbol{q},-\boldsymbol{k},\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\, n_{\boldsymbol{p}}\, n_{\boldsymbol{q}} \\ = \frac{6zJ}{\hbar S}d\left\{\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}[-A''_{\boldsymbol{k}}\delta + B''_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+\delta)] + C''_{\boldsymbol{k}}\right\}. \qquad (6.3.30)\end{aligned}$$

The corresponding Feynman diagram is given by the third term of the third line of Fig. 6.1. In conclusion the first-order self-energies are

$$\begin{aligned}\Sigma_{11}(\boldsymbol{k}) = -\frac{zJ}{\hbar}\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}\left[\left(A_{\boldsymbol{k}} + 2A'_{\boldsymbol{k}} + \frac{6d}{S}A''_{\boldsymbol{k}}\right)\right. \\ \left.\times\ (\omega_{\boldsymbol{k}}+\delta) - \left(B_{\boldsymbol{k}} + 2B'_{\boldsymbol{k}} + \frac{6d}{S}B''_{\boldsymbol{k}}\right)\delta\right], \qquad (6.3.31)\end{aligned}$$

$$\Sigma_{12}(\boldsymbol{k}) = \frac{zJ}{\hbar}\left\{\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}\left[-\left(A_{\boldsymbol{k}}+2A'_{\boldsymbol{k}}+\frac{6d}{S}A''_{\boldsymbol{k}}\right)\delta\right.\right.$$
$$\left.\left.+\left(B_{\boldsymbol{k}}+2B'_{\boldsymbol{k}}+\frac{6d}{S}B''_{\boldsymbol{k}}\right)(\omega_{\boldsymbol{k}}+\delta)\right]-\left(C_{\boldsymbol{k}}+2C'_{\boldsymbol{k}}+\frac{6d}{S}C''_{\boldsymbol{k}}\right)\right\}, \quad (6.3.32)$$

$$\Sigma_{21}(\boldsymbol{k}) = \frac{zJ}{\hbar}\left\{\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}\left[-\left(A_{\boldsymbol{k}}+2A'_{\boldsymbol{k}}+\frac{6d}{S}A''_{\boldsymbol{k}}\right)\delta\right.\right.$$
$$\left.\left.+\left(B_{\boldsymbol{k}}+2B'_{\boldsymbol{k}}+\frac{6d}{S}B''_{\boldsymbol{k}}\right)(\omega_{\boldsymbol{k}}+\delta)\right]+\left(C_{\boldsymbol{k}}+2C'_{\boldsymbol{k}}+\frac{6d}{S}C''_{\boldsymbol{k}}\right)\right\} \quad (6.3.33)$$

and

$$\Sigma_{22}(\boldsymbol{k}) = \Sigma_{11}(\boldsymbol{k}), \quad (6.3.34)$$

where $A_{\boldsymbol{k}}$, $B_{\boldsymbol{k}}$, $C_{\boldsymbol{k}}$, $A'_{\boldsymbol{k}}$, $B'_{\boldsymbol{k}}$, $C'_{\boldsymbol{k}}$, $A''_{\boldsymbol{k}}$, $B''_{\boldsymbol{k}}$ and $C''_{\boldsymbol{k}}$ are given by Eqs. (6.1.35), (6.1.36), (6.1.37), (6.3.9), (6.3.11), (6.3.21), (6.3.15), (6.3.16) and (6.3.24), respectively. From Eqs. (6.3.34), (6.2.25) and (6.2.26), one obtains $\Sigma_{\rm A}=0$ and $\Sigma_{\rm S}=\Sigma_{11}$ so that the first-order spin wave spectrum that can be deduced from Eq. (6.2.27) becomes

$$\omega^2 = [\epsilon_{\boldsymbol{k}}+\Sigma_{11}(\boldsymbol{k})]^2-\Sigma_{12}(\boldsymbol{k})\Sigma_{21}(\boldsymbol{k})$$
$$= \left(\frac{2zJS}{\hbar}\right)^2\left\{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)-\frac{1}{S}\left[\left(A_{\boldsymbol{k}}+2A'_{\boldsymbol{k}}+\frac{6d}{S}A''_{\boldsymbol{k}}\right)(\omega_{\boldsymbol{k}}+\delta)\right.\right.$$
$$\left.-\left(B_{\boldsymbol{k}}+2B'_{\boldsymbol{k}}+\frac{6d}{S}B''_{\boldsymbol{k}}\right)\delta\right]$$
$$+\frac{1}{4S^2}\left[\left(A_{\boldsymbol{k}}+2A'_{\boldsymbol{k}}+\frac{6d}{S}A''_{\boldsymbol{k}}\right)^2-\left(B_{\boldsymbol{k}}+2B'_{\boldsymbol{k}}+\frac{6d}{S}B''_{\boldsymbol{k}}\right)^2\right.$$
$$\left.\left.+\left(C_{\boldsymbol{k}}+2C'_{\boldsymbol{k}}+\frac{6d}{S}C''_{\boldsymbol{k}}\right)^2\right]\right\}. \quad (6.3.35)$$

As one can see from Eq. (6.3.35), contributions of order 1 (unperturbed spectrum), $\frac{1}{S}$, $\frac{1}{S^2}$ and $\frac{1}{S^3}$ appear inside the curl brackets. Notice that *all* contributions within $\frac{1}{S}$ are contained in Eq. (6.3.35) while not all contributions of the order $\frac{1}{S^2}$ and $\frac{1}{S^3}$ appear because terms of the same order are generated by higher orders of the perturbation expansion. For this reason, Eq. (6.3.35) is consistent only within terms of order $\frac{1}{S}$ so that the first-order spectrum becomes

$$\omega^2 = \left(\frac{2zJS}{\hbar}\right)^2\left\{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)-\frac{\omega_{\boldsymbol{k}}}{S}[(X-Y)\omega_{\boldsymbol{k}}+\delta(3X+Z-Y+T)\right.$$
$$\left.+2(\lambda_2+\delta\lambda_1)\omega_{\boldsymbol{k}}+2\delta\lambda_2+4\delta\lambda_1(1+\delta)]\right\}. \quad (6.3.36)$$

In writing Eq. (6.3.36) the "sum rule" $Z-T+\delta(1+X+Z)=0$ is used. As one can see from Eq. (6.3.36), the first-order spin wave spectrum is kinematically consistent

and shows a soft mode at $\boldsymbol{k} = 0$. This result was obtained by Rastelli and Lindgård[18] at $T = 0$ and Balucani *et al.*[51] at $T \neq 0$. These authors did not use the DM spin-boson transformation because in presence of easy-plane anisotropy it leads to a non-hermitian bilinear Hamiltonian as shown by Eq. (6.1.5) that cannot be diagonalized using the Bogoliubov transformation. In particular, these authors used a modified version of the HP spin-boson transformation introduced by Lindgård and Danielsen[52] in 1974: the matching of matrix element (MME) method. The MME method consists on writing the spin operators and their powers in a normal ordered boson expansion where any expansion in $\frac{1}{S}$ is avoided. Rastelli and Lindgård[18] showed that the MME method leads to the same results as those obtained from the normal ordered HP transformation if a "generalized" Bogoliubov transformation[53] is performed. Indeed, the generalized Bogoliubov transformation consists on diagonalizing the bilinear boson Hamiltonian only *after* having normal ordered the higher-order terms of the boson Hamiltonian. As shown in Sec. 6.1, the terms $\mathcal{H}_4$ and $\mathcal{H}_6$ give contributions to both the ground-state energy (6.1.19) and the bilinear Hamiltonian (6.1.31). The generation of higher-order terms in $\frac{1}{S}$ is a general feature of the easy-plane amsotropy independently of which spin-boson transformation is assumed. However, the great advantage of the DM transformation used in this section with respect to the normal ordered HP transformation and the MME method is that the number of interaction potentials is *finite*. Balucani *et al.*,[51] using the MME method, obtained a spectrum similar to that of Eq. (6.3.36) by means of the Green function equation of motion method pushed to second order. Subsequently, Rastelli and Tassi[46] showed that the DM transformation led to the same result performing a first-order perturbation expansion by means of the many-body technique as illustrated in the present section.

Now we perform an explicit calculation of the temperature-independent terms X, Y, Z, T and of the temperature-dependent terms λ_1, λ_2 occurring in the first-order spectrum (6.3.36). We will transform the sums over the wavevector into an integral containing the isotropic ferromagnet density of states by $D(\omega)$ that satisfies the properties

$$\int_0^{\omega_m} d\omega D(\omega) = \frac{1}{N} \sum_{\boldsymbol{q}} 1 = 1, \tag{6.3.37}$$

$$\int_0^{\omega_m} d\omega \; \omega D(\omega) = \frac{1}{N} \sum_{\boldsymbol{q}} (1 - \gamma_{\boldsymbol{q}}) = 1 \tag{6.3.38}$$

and

$$\int_0^{\omega_m} d\omega \frac{D(\omega)}{\omega} = \frac{1}{N} \sum_{\boldsymbol{q}} \frac{1}{1 - \gamma_{\boldsymbol{q}}} = I_1 \tag{6.3.39}$$

where ω_m is the maximum spin wave frequency of the isotropic ferromagnet and I_1 is the Watson's integral: for a SC lattice, $\omega_m = 2$ and $I_1 = 1.51637$.

Let us begin evaluating Z given by Eq. (6.1.26). Taking advantage from the fact that in the actual compounds the easy-plane anisotropy is generally much smaller

than the exchange interaction, we assume that $\delta \ll 1$, so that we may expand Z in ascending powers of δ limiting ourselves to the first few terms. By the use of the density of states $D(\omega)$ defined by Eqs. (6.3.37)–(6.3.39), Eq. (6.1.26) becomes

$$Z = -\delta \int_0^{\omega_m} \frac{D(\omega) d\omega}{\sqrt{\omega(\omega + 2\delta)}}. \tag{6.3.40}$$

Obviously, we do not know the analytic expression of $D(\omega)$ but we know that for small frequencies the density of states behaves like $D(\omega) = c\sqrt{\omega}$ as a consequence of the quadratic dependence of the dispersion relation of the isotropic ferromagnet on the wavevector. Then we break the integration over the frequency in two parts: in the first integral the frequency range is assumed to be $0 < \omega < \omega_0$, while in the second integral the frequency range is $\omega_0 < \omega < \omega_m$. Assuming

$$\delta \ll \omega_0 \ll \omega_m, \tag{6.3.41}$$

in the first integral we may replace $D(\omega)$ with $c\sqrt{\omega}$ so that the integration can be easily performed and in the second integral we may expand the function in powers of δ obtaining

$$Z = -\delta \left\{ 2c[\sqrt{\omega_0 + 2\delta} - \sqrt{2\delta}] + \int_{\omega_0}^{\omega_m} d\omega \, \frac{D(\omega)}{\omega} \left(1 - \frac{\delta}{\omega} + \frac{3}{2}\frac{\delta^2}{\omega^2} + \cdots \right) \right\}. \tag{6.3.42}$$

Under the assumption (6.3.41), we may expand the function inside the square brackets in powers of δ obtaining

$$Z = -I_1 \delta + c(2\delta)^{\frac{3}{2}} - I_2 \delta^2 + O(\delta^3), \tag{6.3.43}$$

where

$$I_1 = 2c\sqrt{\omega_0} + \int_{\omega_0}^{\omega_m} d\omega \frac{D(\omega)}{\omega} = \int_0^{\omega_0} d\omega \frac{c\sqrt{\omega}}{\omega} + \int_{\omega_0}^{\omega_m} d\omega \frac{D(\omega)}{\omega} = \int_0^{\omega_m} d\omega \frac{D(\omega)}{\omega} \tag{6.3.44}$$

in agreement with Eq. (6.3.39) and

$$I_2 = \frac{2c}{\sqrt{\omega_0}} - \int_{\omega_0}^{\omega_m} d\omega \frac{D(\omega)}{\omega^2}. \tag{6.3.45}$$

Even though we do not know the numerical value of I_2, we have been able to write an expansion of Z in powers of δ picking the non-analytic term coming from the low frequency square-root dependence of the density of state of the isotropic ferromagnet. In a similar way, we obtain

$$X = \frac{1}{3} c(2\delta)^{\frac{3}{2}} - \frac{1}{2} I_2 \delta^2 + O(\delta^3), \tag{6.3.46}$$

$$T = (1 - I_1)\delta + c(2\delta)^{\frac{3}{2}} - (I_1 + I_2)\delta^2 + \frac{2}{3} c(2\delta)^{\frac{5}{2}} + O(\delta^3) \tag{6.3.47}$$

and

$$Y = \frac{1}{3}c(2\delta)^{\frac{3}{2}} - \frac{1}{2}(I_1 + I_2)\delta^2 + \frac{2}{5}c(2\delta)^{\frac{5}{2}} + O(\delta^3). \tag{6.3.48}$$

As for the temperature-dependent terms λ_1 and λ_2 occurring in Eq. (6.3.36), we proceed in a way similar to that used for Z. First of all, we transform the sums over the wavevectors into integrals over the frequency making use of the density of states $D(\omega)$. Then we break the integrals in two parts: in the first integral ($0 < \omega < \omega_0$), we replace the density of states by $D(\omega) = c\sqrt{\omega}$ and the occupation number by its expansion

$$n = \frac{1}{e^{\beta\hbar\epsilon} - 1} \simeq \frac{1}{\beta\hbar\epsilon} = \frac{\theta}{\omega(\omega + 2\delta)}, \tag{6.3.49}$$

where

$$\theta = \frac{k_B T}{2zJS}. \tag{6.3.50}$$

The approximation (6.3.49) is justified for $\omega_0 \ll \theta$, that is for temperatures such that $D(2S-1) \ll k_B T \ll 2zJS$. In the second integral ($\omega_0 < \omega < \omega_m$), we may expand the integrand function in powers of $\frac{\delta}{\omega}$ since $\frac{\delta}{\omega_0} \ll 1$. By doing so, we obtain

$$\lambda_1 = c\theta \int_0^{\omega_0} d\omega \frac{\sqrt{\omega}}{\omega + 2\delta} + \int_{\omega_0}^{\omega_m} d\omega \frac{D(\omega)}{e^{\frac{\omega}{\theta}} - 1} \left(1 - \frac{\delta}{\omega} - \frac{\delta}{\theta} \frac{e^{\frac{\omega}{\theta}}}{e^{\frac{\omega}{\theta}} - 1} + O(\delta^2)\right). \tag{6.3.51}$$

Using the relationships

$$\int_0^{\omega_0} d\omega \frac{\sqrt{\omega}}{\omega + 2\delta} = 2\sqrt{\omega_0} - 2\sqrt{2\delta} \arctan \sqrt{\frac{\omega_0}{2\delta}} = 2\sqrt{\omega_0} - \pi\sqrt{2\delta} + O(\delta) \tag{6.3.52}$$

and

$$2\theta c\sqrt{\omega_0} = \int_0^{\omega_0} d\omega c\sqrt{\omega} \frac{\theta}{\omega} \simeq \int_0^{\omega_0} d\omega \frac{D(\omega)}{e^{\frac{\omega}{\theta}} - 1}, \tag{6.3.53}$$

Equation (6.3.51) becomes

$$\lambda_1 = \int_0^{\omega_m} d\omega D(\omega) n^{(0)}(\omega) - \pi\theta c\sqrt{2\delta} = \sigma_0 - \pi\theta c\sqrt{2\delta} + O(\delta), \tag{6.3.54}$$

where

$$\sigma_0 = \frac{1}{N} \sum_{\boldsymbol{q}} n_{\boldsymbol{q}}^{(0)} = \frac{1}{N} \sum_{\boldsymbol{q}} \frac{1}{e^{\frac{\omega_{\boldsymbol{q}}}{\theta}} - 1}. \tag{6.3.55}$$

In Eqs. (6.3.54) and (6.3.55), $n^{(0)}$ is the Bose factor of the isotropic ferromagnet. Following the same procedure, we obtain for λ_2

$$\lambda_2 = \sigma_1 - 2\delta\sigma_0 \tag{6.3.56}$$

where

$$\sigma_1 = \frac{1}{N}\sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} n^{(0)}_{\boldsymbol{q}}. \tag{6.3.57}$$

Replacing Eqs. (6.3.43), (6.3.46)–(6.3.48), (6.3.54) and (6.3.56) into Eq. (6.3.35), one obtains for the renormalized spin wave spectrum

$$\omega^2 = \left(\frac{2zJS}{\hbar}\right)^2 \left\{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}} + 2\delta) + \frac{\delta^2}{2S}\omega_{\boldsymbol{k}}(4I_1 - 2 - I_1\omega_{\boldsymbol{k}}) - 2\frac{\omega_{\boldsymbol{k}}}{S}[(\omega_{\boldsymbol{k}} + \delta)\sigma_1 + \delta(2 - \omega_{\boldsymbol{k}})\sigma_0] + \frac{\delta^2}{4S^2}[-1 + (I_1 - 1)\omega^2_{\boldsymbol{k}}] - \frac{2}{S^2}\delta\omega_{\boldsymbol{k}}\sigma_0 + O\left(\delta^{\frac{5}{2}}, \theta\delta^{\frac{3}{2}}\right)\right\}. \tag{6.3.58}$$

Equation (6.3.58) reproduces the first-order spectrum given by Eq. (55) of Rastelli and Tassi[46] except for a minor change in the term proportional to σ_0. A factor 3 should be replaced by a factor 2 in the last line of Eq. (55). Moreover, the contribution of the spectrum (6.3.58) proportional to $\frac{1}{S}$ recovers Eq. (25) of Balucani *et al.*[51] for the temperature-dependent contribution once the thermal averages $\langle a^+_{\boldsymbol{p}} a_{\boldsymbol{p}}\rangle$ are expressed in terms of the Bose factors $\langle \alpha^+_{\boldsymbol{p}} \alpha_{\boldsymbol{p}}\rangle$. Notice that the $\frac{1}{S}$ contribution is correct while the contribution of order $\frac{1}{S^2}$ is manifestly incorrect: indeed, the Goldstone theorem is violated because of the presence of the term $-\frac{\delta^2}{4S^2}$ that remains finite and negative at $\boldsymbol{k} = 0$. As announced, terms of the same order in $\frac{1}{S}$ come from the second-order perturbation theory as we will see in the next section.

6.4. Second-Order Perturbation Theory

The only contribution of order $\frac{1}{S^2}$ to the squared renormalized spin wave spectrum of Eq. (6.3.35) coming from the second-order perturbation theory is given by $2\epsilon_{\boldsymbol{k}}\Sigma^*_{11}(\boldsymbol{k}, \epsilon_{\boldsymbol{k}})$ where Σ^*_{11} is the second-order proper self-energy evaluated for $i\omega_n = \epsilon_{\boldsymbol{k}}$ coming from the perturbation expansion (6.2.5), that is,

$$\mathcal{G}^{11}_{\boldsymbol{k}}(\tau) = -\frac{1}{2!\hbar^2}\int_0^{\beta\hbar} d\tau_1 \int_0^{\beta\hbar} d\tau_2 \langle T[\mathcal{H}'(\tau_1)\mathcal{H}'(\tau_2)\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\mathrm{c}}_0, \tag{6.4.1}$$

where $\mathcal{H}'$ is restricted to the sum $\mathcal{H}'_2 + \mathcal{H}'_4$, since any product involving $\mathcal{H}'_6$ which is of order $\frac{1}{S^2}$ leads to terms of order at least $\frac{1}{S^3}$ neglected in the present approximation. The term $\mathcal{H}'_2(\tau_1)\mathcal{H}'_2(\tau_2)$ leads to a reducible self-energy so that the terms contributing to the proper self-energy entering the matrix Dyson equation are restricted to the

products $\mathcal{H}'_2(\tau_1)\mathcal{H}'_4(\tau_2)$, $\mathcal{H}'_4(\tau_1)\mathcal{H}'_2(\tau_2)$ and $\mathcal{H}'_4(\tau_1)\mathcal{H}'_4(\tau_2)$. The terms $\mathcal{H}'_2(\tau_1)\mathcal{H}'_4(\tau_2)$ and $\mathcal{H}'_4(\tau_1)\mathcal{H}'_2(\tau_2)$ give the same contribution since they are transformed into one another changing the time variables τ_1 and τ_2. This invariance under time permutation is accounted for keeping only one term, for instance $\mathcal{H}'_4(\tau_1)\mathcal{H}'_2(\tau_2)$ and neglecting the factor $\frac{1}{2!}$ in the expansion (6.4.1). Moreover, within the present approximation, the potentials P, Q, R given by Eqs. (6.1.32)–(6.1.37) and $\Phi^{(i)}$ given by Eqs. (6.1.38)–(6.1.43) have to be restricted to their parts independent of S and $D^{(3)} = \frac{zJd}{4SN}$, respectively.

Let us begin evaluating the contribution to the proper self-energy coming from $\mathcal{H}'_4\mathcal{H}'_2$. From Eq. (6.4.1), one obtains

$$\begin{aligned}\mathcal{G}^{11}_{\boldsymbol{k}}(\tau) &= -\frac{1}{\hbar^2}\int_0^{\beta\hbar} d\tau_1 \int_0^{\beta\hbar} d\tau_2 \langle T[\mathcal{H}'_4(\tau_1)\mathcal{H}'_2(\tau_2)\alpha_{\boldsymbol{k}}(\tau)\alpha^+_{\boldsymbol{k}}]\rangle^{\mathrm{c}}_0 \\ &= \frac{1}{\beta\hbar}\sum_n e^{-i\omega_n\tau}[\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega_n)]^2[\Sigma^*_{11}(\boldsymbol{k})]_{\mathcal{H}'_4\mathcal{H}'_2}, \end{aligned} \tag{6.4.2}$$

where

$$\begin{aligned}[\Sigma^*_{11}(\boldsymbol{k})]_{\mathcal{H}'_2\mathcal{H}'_4} &= 4\beta\frac{zJ}{\hbar}\sum_{\boldsymbol{q}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{q}}\ \Phi^{(1)}_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{k},\boldsymbol{q}}\ P_{\boldsymbol{q}} n_{\boldsymbol{q}}(1+n_{\boldsymbol{q}}) \\ &\quad - 3\frac{zJ}{\hbar^2}\sum_{\boldsymbol{q}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{q}}\ \Phi^{(2)}_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}}\ R_{\boldsymbol{q}}\ \frac{1+2n_{\boldsymbol{q}}}{2\epsilon_{\boldsymbol{q}}} \\ &\quad - 3\frac{zJ}{\hbar^2}\sum_{\boldsymbol{q}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{q}}\ \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q},\boldsymbol{q},\boldsymbol{k}}\ Q_{\boldsymbol{q}}\ \frac{1+2n_{\boldsymbol{q}}}{2\epsilon_{\boldsymbol{q}}}. \end{aligned} \tag{6.4.3}$$

The three contributions to the second-order proper self-energy (6.4.3) are represented by the Feynman diagrams of the first line of Fig. 6.2. With the help of such diagrams it is easier to write their analytic translation using the same rules given in Chapter 4 for the isotropic ferromagnet. For instance, the third diagram of the first line of Fig. 6.2 corresponds to a product between the potentials $-\frac{zJ}{2}Q_{\boldsymbol{q}}$ and $l^2_{\boldsymbol{k}} l^2_{\boldsymbol{q}}\Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q},\boldsymbol{q},\boldsymbol{k}}$ times a factor 6 coming from the number of ways of contracting the boson operators that lead to six equivalent contributions since the potential $\Phi^{(3)}$ is invariant under the permutation of the last three subscripts ($3! = 6$ is the multiplicity of the diagram). The sums over internal frequencies are reduced to a single sum over the only frequency remaining after the use of the "frequency conservation" rule applied to the internal lines of the diagram. Finally, using Eqs. (6.2.17) and (4.2.16) one obtains

$$\begin{aligned}\frac{1}{\beta\hbar}\sum_r \mathcal{G}^{(0)}_{\boldsymbol{q}}(i\omega_r)\mathcal{G}^{(0)}_{\boldsymbol{q}}(-i\omega_r) &= \frac{1}{\beta\hbar}\sum_r \frac{1}{i\omega_r - \epsilon_{\boldsymbol{q}}}\ \frac{1}{-i\omega_r - \epsilon_{\boldsymbol{q}}} \\ &= -\frac{1}{\beta\hbar}\sum_r \frac{1}{2\epsilon_{\boldsymbol{q}}}\left(\frac{1}{i\omega_r - \epsilon_{\boldsymbol{q}}} - \frac{1}{i\omega_r + \epsilon_{\boldsymbol{q}}}\right) \\ &= \frac{1}{2\epsilon_{\boldsymbol{q}}}[n_{\boldsymbol{q}} - n(-\epsilon_{\boldsymbol{q}})] = \frac{1+2n_{\boldsymbol{q}}}{2\epsilon_{\boldsymbol{q}}}. \end{aligned} \tag{6.4.4}$$

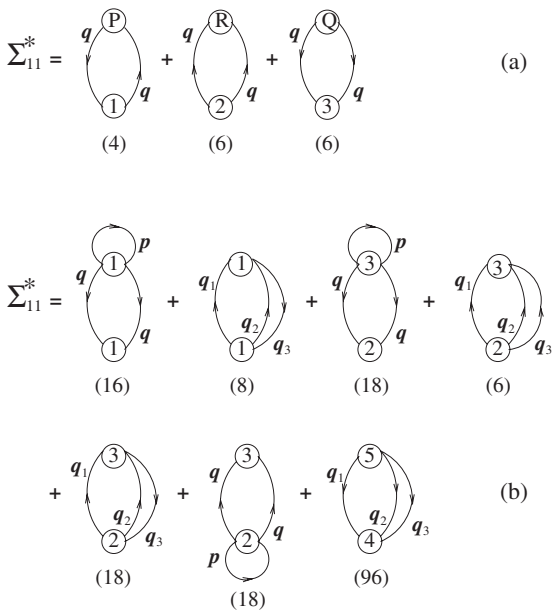

Fig. 6.2. Second-order proper self-energy $[\Sigma^*_{11}]_{\mathcal{H}'_4\mathcal{H}'_2}$ (a) and $[\Sigma^*_{11}]_{\mathcal{H}'_4\mathcal{H}'_4}$ (b). The circles labelled by P, Q and R of (a) represent the potentials occurring in $\mathcal{H}'_2$. The circles containing 1, 2, 3, 4 and 5 in (a) and (b) represent the potentials $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$, $\Phi^{(4)}$ and $\Phi^{(5)}$ occurring in $\mathcal{H}'_4$, respectively. The numbers in brackets under the diagrams indicate their multiplicity.

Neglecting terms of order higher than $\frac{\delta^{5/2}}{S^2}$, the proper self-energy (6.4.3) reduces to

$$[\Sigma^*_{11}(\boldsymbol{k})]_{\mathcal{H}'_2\mathcal{H}'_4} = \frac{2zJS}{\hbar}\frac{\delta^2}{4S^2}\frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\delta)}}\left\{\frac{1}{2}[1+2\omega_{\boldsymbol{k}}-(2-I_1)\omega^2_{\boldsymbol{k}}]\right.$$
$$+\ \frac{\omega^2_{\boldsymbol{k}}}{2\theta}[2(1-2I_1)\rho_1+I_1\ \rho_2]+\sigma_0\omega_{\boldsymbol{k}}[4-I_1-(2-I_1)\omega_{\boldsymbol{k}}]$$
$$\left.+\ \sigma_1[1-(2-I_1)\omega_{\boldsymbol{k}}]\right\}, \tag{6.4.5}$$

with

$$\rho_1 = \frac{1}{N}\sum_{\boldsymbol{q}}\omega_{\boldsymbol{q}}n^{(0)}_{\boldsymbol{q}}(1+n^{(0)}_{\boldsymbol{q}}) \simeq \frac{3}{2}\theta\ \sigma_0, \tag{6.4.6}$$

$$\rho_2 = \frac{1}{N}\sum_{\boldsymbol{q}}\omega^2_{\boldsymbol{q}}n^{(0)}_{\boldsymbol{q}}(1+n^{(0)}_{\boldsymbol{q}}) \simeq \frac{5}{2}\theta\ \sigma_1 \tag{6.4.7}$$

where $n^{(0)}_{\boldsymbol{q}}$ is the Bose factor of the isotropic model. The relationship between ρ_r and σ_r is obtained as follows: in terms of the density of states, the sum over wavevectors is transformed into a sum over frequencies like

$$\rho_r = \int_0^{\omega_m} d\omega D(\omega)\omega^r\frac{e^{\frac{\omega}{\theta}}}{(e^{\frac{\omega}{\theta}}-1)^2} = \theta\int_0^{\omega_m} d\omega[r\omega^{r-1}D(\omega)+\omega^r D'(\omega)]\frac{1}{e^{\frac{\omega}{\theta}}-1}, \tag{6.4.8}$$

where an integration by parts has been performed and the integrated contribution vanishes because the density of states is zero for $\omega = 0$ and $\omega = \omega_m$. At low temperature, the main contribution to the integral comes from the low frequency region where the density of states may be replaced by the function $D(\omega) = c\sqrt{\omega}$. Using the change of variable $\omega \to x = \frac{\omega}{\theta}$ and assuming that the upper limit of the integral $\frac{\omega_m}{\theta}$ goes to infinity, Eq. (6.4.8) becomes

$$\rho_r = c\left(r+\frac{1}{2}\right)\theta^{r+\frac{3}{2}}\int_0^\infty dx\frac{x^{r-\frac{1}{2}}}{e^x-1} = a_{r+\frac{1}{2}}\ c\ \left(r+\frac{1}{2}\right)\theta^{r+\frac{3}{2}} \tag{6.4.9}$$

with[3]

$$a_r = \Gamma(r)\zeta(r) \tag{6.4.10}$$

where Γ and ζ are the Gamma-function and the Riemann function, respectively. By a similar procedure, we obtain for σ_r

$$\sigma_r = \int_0^{\omega_m} d\omega D(\omega)\omega^r\frac{1}{e^{\frac{\omega}{\theta}}-1} = c\ \theta^{r+\frac{3}{2}}\int_0^\infty dx\frac{x^{r+\frac{1}{2}}}{e^x-1} = a_{r+\frac{3}{2}}\ c\ \theta^{r+\frac{3}{2}}. \tag{6.4.11}$$

From Eqs. (6.4.9) and (6.4.11), we obtain

$$\frac{\rho_{r+1}}{\sigma_r} = \left(r+\frac{3}{2}\right)\theta \tag{6.4.12}$$

so that the temperature-dependent contribution to the self-energy of Eq. (6.4.5) is of the order $\delta^2\theta^{\frac{3}{2}}/S^2$, and consistently with Eq. (6.3.58) has to be neglected. Then the contribution to the renormalized spin wave spectrum coming from the self-energy (6.4.5) becomes

$$2\epsilon_{\boldsymbol{k}}[\Sigma_{11}(\boldsymbol{k})]_{\mathcal{H}'_2\mathcal{H}'_4} = \left(\frac{2zJS}{\hbar}\right)^2\left\{\frac{\delta^2}{4S^2}[1+2\omega_{\boldsymbol{k}}-(2-I_1)\omega_{\boldsymbol{k}}^2]+O\left(\delta^{\frac{5}{2}},\delta^2\theta^{\frac{3}{2}}\right)\right\}. \tag{6.4.13}$$

Note that the term $\frac{\delta^2}{4S^2}$ of Eq. (6.4.13) cancels out the analogous term of the first-order spectrum (6.3.58) restoring the soft mode at $\boldsymbol{k} = 0$.

Let us define $[\Sigma^*_{11}(\boldsymbol{k}, i\omega_n)]_{\mathcal{H}'_4\mathcal{H}'_4}$ as the proper self-energy obtained from the perturbation expansion (6.4.1) with $\mathcal{H}' = \mathcal{H}'_4$. It consists of seven contributions represented by the Feynman diagrams classified as (b) in Fig. 6.2. Their anlaytic translation is given by

$$\begin{aligned}[\Sigma^*_{11}(\boldsymbol{k}, i\omega_n)]_{\mathcal{H}'_4\mathcal{H}'_4} = &-\frac{16}{\hbar^2}\sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^4\ \Phi^{(1)}_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{k},\boldsymbol{q}}\Phi^{(1)}_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{q},\boldsymbol{p}}\ \beta\hbar\ n_{\boldsymbol{p}}n_{\boldsymbol{q}}(1+n_{\boldsymbol{q}})\\ &+\frac{8}{\hbar^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}}\ \Phi^{(1)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}}\Phi^{(1)}_{\boldsymbol{k},\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}\\ &\times\frac{n_{\boldsymbol{q}_3}(1+n_{\boldsymbol{q}_1})(1+n_{\boldsymbol{q}_2})-n_{\boldsymbol{q}_1}n_{\boldsymbol{q}_2}(1+n_{\boldsymbol{q}_3})}{i\omega_n-\epsilon_{\boldsymbol{q}_1}-\epsilon_{\boldsymbol{q}_2}+\epsilon_{\boldsymbol{q}_3}}\end{aligned}$$

$$
\begin{aligned}
&- \frac{18}{\hbar^2} \sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^4 \; \Phi^{(2)}_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}} \Phi^{(3)}_{\boldsymbol{p},-\boldsymbol{q},\boldsymbol{q},\boldsymbol{p}} \; \frac{n_{\boldsymbol{p}}(1+2n_{\boldsymbol{q}})}{2\epsilon_{\boldsymbol{q}}} \\
&+ \frac{6}{\hbar^2} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2 \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}} \; \Phi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1} \\
&\times \frac{(1+n_{\boldsymbol{q}_1})(1+n_{\boldsymbol{q}_2})(1+n_{\boldsymbol{q}_3}) - n_{\boldsymbol{q}_1} n_{\boldsymbol{q}_2} n_{\boldsymbol{q}_3}}{i\omega_n - \epsilon_{\boldsymbol{q}_1} - \epsilon_{\boldsymbol{q}_2} - \epsilon_{\boldsymbol{q}_3}} \\
&- \frac{18}{\hbar^2} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2 \delta_{\boldsymbol{q}_1+\boldsymbol{k},\boldsymbol{q}_2+\boldsymbol{q}_3} \; \Phi^{(2)}_{\boldsymbol{k},\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3} \Phi^{(3)}_{\boldsymbol{q}_3,-\boldsymbol{q}_2,\boldsymbol{q}_1,\boldsymbol{k}} \\
&\times \frac{n_{\boldsymbol{q}_3}(1+n_{\boldsymbol{q}_1})(1+n_{\boldsymbol{q}_2}) - n_{\boldsymbol{q}_1} n_{\boldsymbol{q}_2}(1+n_{\boldsymbol{q}_3})}{i\omega_n + \epsilon_{\boldsymbol{q}_1} + \epsilon_{\boldsymbol{q}_2} - \epsilon_{\boldsymbol{q}_3}} \\
&- \frac{18}{\hbar^2} \sum_{\boldsymbol{p},\boldsymbol{q}} l_{\boldsymbol{k}}^2 l_{\boldsymbol{p}}^2 l_{\boldsymbol{q}}^4 \; \Phi^{(2)}_{\boldsymbol{p},\boldsymbol{q},-\boldsymbol{q},\boldsymbol{p}} \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q},\boldsymbol{q},\boldsymbol{k}} \; \frac{n_{\boldsymbol{p}}(1+2n_{\boldsymbol{q}})}{2\epsilon_{\boldsymbol{q}}} \\
&- \frac{96}{\hbar^2} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2 \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}} \\
&\times \Phi^{(4)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(5)}_{\boldsymbol{k},\boldsymbol{q}_3,-\boldsymbol{q}_2,-\boldsymbol{q}_1} \\
&\times \frac{(1+n_{\boldsymbol{q}_1})(1+n_{\boldsymbol{q}_2})(1+n_{\boldsymbol{q}_3}) - n_{\boldsymbol{q}_1} n_{\boldsymbol{q}_2} n_{\boldsymbol{q}_3}}{i\omega_n + \epsilon_{\boldsymbol{q}_1} + \epsilon_{\boldsymbol{q}_2} + \epsilon_{\boldsymbol{q}_3}}. \qquad (6.4.14)
\end{aligned}
$$

The lowest order in temperature is obtained by neglecting in Eq. (6.4.14) all terms involving products of two or more Bose factors labelled by different wavevectors. By doing so, we obtain

$$
\begin{aligned}
[\Sigma^*_{11}(\boldsymbol{k}, i\omega_n)]_{\mathcal{H}'_4\mathcal{H}'_4} &= \frac{6}{\hbar^2} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2 \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}} \\
&\times \left[\frac{\Phi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}}{i\omega_n - \epsilon_{\boldsymbol{q}_1} - \epsilon_{\boldsymbol{q}_2} - \epsilon_{\boldsymbol{q}_3}} \right. \\
&\left. - \; 16 \; \frac{\Phi^{(4)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(5)}_{\boldsymbol{k},\boldsymbol{q}_3,-\boldsymbol{q}_2,-\boldsymbol{q}_1}}{i\omega_n + \epsilon_{\boldsymbol{q}_1} + \epsilon_{\boldsymbol{q}_2} + \epsilon_{\boldsymbol{q}_3}} \right] \\
&+ \frac{8}{\hbar^2} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2 \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}} \\
&\times \frac{\Phi^{(1)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(1)}_{\boldsymbol{k},\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}}{i\omega_n - \epsilon_{\boldsymbol{q}_1} - \epsilon_{\boldsymbol{q}_2} + \epsilon_{\boldsymbol{q}_3}} \; n_{\boldsymbol{q}_3} \\
&+ \frac{18}{\hbar^2} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3} l_{\boldsymbol{k}}^2 l_{\boldsymbol{q}_1}^2 l_{\boldsymbol{q}_2}^2 l_{\boldsymbol{q}_3}^2 \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}}
\end{aligned}
$$

$$\times \left[\frac{\Phi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}}{i\omega_n - \epsilon_{\boldsymbol{q}_1} - \epsilon_{\boldsymbol{q}_2} - \epsilon_{\boldsymbol{q}_3}} - \frac{\Phi^{(2)}_{\boldsymbol{k},\boldsymbol{q}_3,-\boldsymbol{q}_2,\boldsymbol{q}_1} \Phi^{(3)}_{\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}}}{i\omega_n - \epsilon_{\boldsymbol{q}_1} + \epsilon_{\boldsymbol{q}_2} + \epsilon_{\boldsymbol{q}_3}} \right.$$

$$\left. - 16 \frac{\Phi^{(4)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(5)}_{\boldsymbol{k},\boldsymbol{q}_3,-\boldsymbol{q}_2,-\boldsymbol{q}_1}}{i\omega_n + \epsilon_{\boldsymbol{q}_1} + \epsilon_{\boldsymbol{q}_2} + \epsilon_{\boldsymbol{q}_3}} \right] n_{\boldsymbol{q}_1}$$

$$- \frac{9}{\hbar^2} \sum_{\boldsymbol{p},\boldsymbol{q}} l^2_{\boldsymbol{k}} l^2_{\boldsymbol{p}} l^4_{\boldsymbol{q}} \big[\Phi^{(2)}_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}} \Phi^{(3)}_{\boldsymbol{p},-\boldsymbol{q},\boldsymbol{q},\boldsymbol{p}}$$

$$+ \Phi^{(2)}_{\boldsymbol{p},\boldsymbol{q},-\boldsymbol{q},\boldsymbol{p}} \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q},\boldsymbol{q},\boldsymbol{k}} \big] \frac{n_{\boldsymbol{p}}}{\epsilon_{\boldsymbol{q}}}. \tag{6.4.15}$$

Now, we select the terms of Eq. (6.4.15) consistent with the spectrum (6.3.58) and (6.4.13), that is, we neglect all contributions higher than δ^2/S^2 in the temperature-independent terms and all contributions higher than $\theta^{\frac{5}{2}}$ or $\delta\theta^{\frac{3}{2}}$ in the temperature-dependent terms. Accordingly, in Eqs. (6.3.43) and (6.3.46)–(6.3.48) we may claim that

$$\frac{1}{N} \sum_{\boldsymbol{q}} l^2_{\boldsymbol{q}} \sim \frac{1}{N} \sum_{\boldsymbol{q}} l^2_{\boldsymbol{q}} \omega_{\boldsymbol{q}} = O(1), \tag{6.4.16}$$

$$\frac{1}{N} \sum_{\boldsymbol{q}} l^2_{\boldsymbol{q}} x_{\boldsymbol{q}} \sim \frac{1}{N} \sum_{\boldsymbol{q}} l^2_{\boldsymbol{q}} x_{\boldsymbol{q}} \omega_{\boldsymbol{q}} = O(\delta) \tag{6.4.17}$$

and

$$\sum_{\boldsymbol{q}} l^2_{\boldsymbol{k}} x^2_{\boldsymbol{q}} = O\left(\delta^{\frac{3}{2}}\right), \quad \sum_{\boldsymbol{q}} l^2_{\boldsymbol{k}} x^2_{\boldsymbol{q}} \omega_{\boldsymbol{q}} = O(\delta^2). \tag{6.4.18}$$

Equations (6.4.16)–(6.4.18) allow us to establish the general rule that the powers of δ increase according to the number of factors $x_{\boldsymbol{q}}$ associated to $l^2_{\boldsymbol{q}}$. This remark leads to the conclusion that the main anisotropy contributions come from the terms generated by the products $\Phi^{(i)}\Phi^{(j)}$ having the minimum number of $x_{\boldsymbol{q}}$. Consistently with the order $\frac{1}{S^2}$, we can write Eq. (6.1.10) in the form

$$V_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} = -\frac{zJ}{2N}(v_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} + 2\delta), \tag{6.4.19}$$

where

$$v_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} = \frac{1}{2}(\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_4} + \gamma_{\boldsymbol{q}_2-\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_2-\boldsymbol{q}_4}) - (\gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2}). \tag{6.4.20}$$

Using Eqs. (6.1.38)–(6.1.43), the product of potentials occurring under the sum of the temperature-independent term of Eq. (6.4.15) may be replaced by

$$\Phi^{(2)}_{\boldsymbol{q}_1,\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{k}} \Phi^{(3)}_{\boldsymbol{k},-\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1} = \left(\frac{zJ}{2N}\right)^2 \left\{ \frac{4}{3} [x^2_{\boldsymbol{q}_1}\ v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}}\ v_{\boldsymbol{k},-\boldsymbol{q}_1,-\boldsymbol{q}_3,\boldsymbol{q}_2} \right.$$

$$+ 2x_{\boldsymbol{q}_1} x_{\boldsymbol{q}_2}\ v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}}\ v_{\boldsymbol{k},-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_1}]$$

$$\left. + 4\delta x_{\boldsymbol{q}_1} v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}} + O\left(\delta^{\frac{5}{2}}\right) \right\} \tag{6.4.21}$$

and

$$\Phi^{(4)}_{-\boldsymbol{q}_1,-\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}}\Phi^{(5)}_{\boldsymbol{k},\boldsymbol{q}_3,-\boldsymbol{q}_2,-\boldsymbol{q}_1} = \left(\frac{zJ}{N}\right)^2 O(\delta^3). \tag{6.4.22}$$

Replacing Eq. (6.4.21) in the temperature-independent part of the proper self-energy (6.4.15), we obtain

$$\begin{aligned}[\Sigma^*_{11}(\boldsymbol{k},\epsilon_{\boldsymbol{k}},T=0)]_{\mathcal{H}'_4\mathcal{H}'_4} = \frac{2zJS}{\hbar}\frac{\delta^2}{8S^2}\frac{1}{N^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}}\\ \times\left[\frac{1}{\omega^2_{\boldsymbol{q}_1}}v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}}\ v_{\boldsymbol{k},-\boldsymbol{q}_1,-\boldsymbol{q}_3,\boldsymbol{q}_2}\right.\\ +\ \frac{2}{\omega_{\boldsymbol{q}_1}\omega_{\boldsymbol{q}_2}}v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}}\ v_{\boldsymbol{k},-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_1}\\ -\ \left.\frac{6}{\omega_{\boldsymbol{q}_1}}v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}}\right]\frac{1}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_3}}.\end{aligned} \tag{6.4.23}$$

As for the temperature-dependent part of the proper self-energy (6.4.15), we can see that the terms of order $\theta^{\frac{5}{2}}$ and $\delta\theta^{\frac{3}{2}}$ come from the contribution containing $\Phi^{(1)}$ that gives

$$\begin{aligned}[\Sigma^*_{11}(\boldsymbol{k},\epsilon_{\boldsymbol{k}},T\neq 0)]_{\mathcal{H}'_4\mathcal{H}'_4} = \frac{2zJS}{\hbar}\frac{1}{2S^2}\left[2\delta\sigma_0+\frac{1}{N^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}}\right.\\ \left.\times\ \frac{v_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}}\ v_{\boldsymbol{k},\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_3}}n^{(0)}_{\boldsymbol{q}_3}+O(\delta^2\theta)\right].\end{aligned} \tag{6.4.24}$$

Equation (6.4.24) reduces to the isotropic result (3.4.25) for $\delta = 0$ and it is of order $\theta^{\frac{5}{2}}$. All other terms in Eq. (6.4.15) give higher-order contributions. The renormalized second-order spin wave spectrum is the sum of (6.3.58), (6.4.13), (6.4.23) and (6.4.24), where the last two equations have to be multiplied by $2\epsilon_{\boldsymbol{k}}$. The result is

$$\begin{aligned}\omega^2 = \left(\frac{2zJS}{\hbar}\right)^2\omega_{\boldsymbol{k}}\left\{\omega_{\boldsymbol{k}}+2\delta+\frac{\delta^2}{2S}(4I_1-2-I_1\omega_{\boldsymbol{k}})+\frac{\delta^2}{4S^2}[2+(2I_1-3)\omega_{\boldsymbol{k}}]\right.\\ +\ \frac{\delta^2}{4S^2}\frac{1}{N^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}}\frac{v_{\boldsymbol{q}_2,-\boldsymbol{q}_3,-\boldsymbol{q}_1,\boldsymbol{k}}}{\omega_{\boldsymbol{q}_1}}\left(\frac{v_{\boldsymbol{k},-\boldsymbol{q}_1,-\boldsymbol{q}_3,\boldsymbol{q}_2}}{\omega_{\boldsymbol{q}_1}}\right.\\ +\ \left.2\frac{v_{\boldsymbol{k},-\boldsymbol{q}_2,-\boldsymbol{q}_3,\boldsymbol{q}_1}}{\omega_{\boldsymbol{q}_2}}-6\right)\frac{1}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_3}}\\ -\ \frac{2}{S}[(\omega_{\boldsymbol{k}}+\delta)\sigma_1+\delta(2-\omega_{\boldsymbol{k}})\sigma_0]+\frac{2\delta}{S^2}\sigma_0\\ +\ \left.\frac{1}{S^2N^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{k}}\frac{v_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{k}}\ v_{\boldsymbol{k},\boldsymbol{q}_3,\boldsymbol{q}_2,\boldsymbol{q}_1}}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_3}}n^{(0)}_{\boldsymbol{q}_3}\right\}.\end{aligned} \tag{6.4.25}$$

Equation (6.4.25) coincides with Eq. (60) of Rastelli and Tassi[46] except for minor changes that amend Eq. (60). As one can see, the renormalized spectrum (6.4.25) shows kinematical consistency and satisfies the Goldstone theorem.

In conclusion, we have shown that the DM spin-boson transformation leads to the same results obtained from the HP transformation[18] and MME method[51] if all contributions of the same order in $\frac{1}{S}$ are carefully taken into account: the apparent discrepencies are simply due to the treatment of terms not homogeneous in the perturbation expansion. The spectrum (6.4.25) was obtained after a systematic use of the many-body theory. This approach, even though very heavy, can be worked out. On the contrary, the method of the Green function equation of motion illustrated in Chapter 3 much more hardly could be pushed to the same order of approximation.