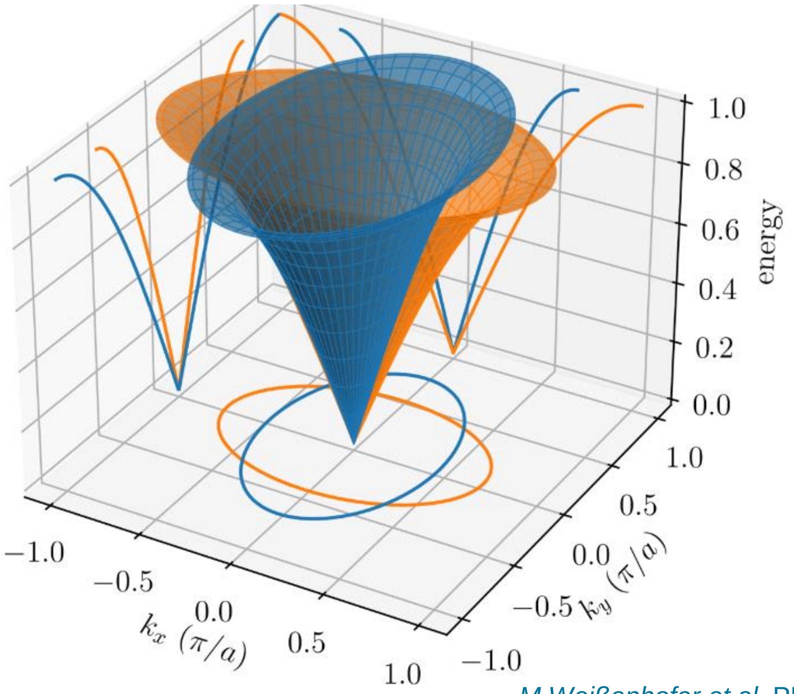
Universität Konstanz

Seminartalk: Magnons in Ferromagnets



M.Weißenhofer et al. PRB **110**, *094427* (2024)

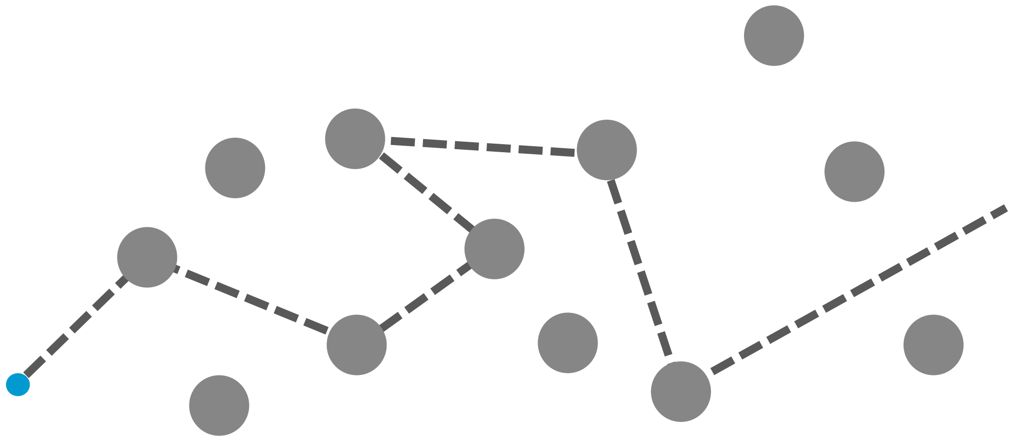
Julian Beisch

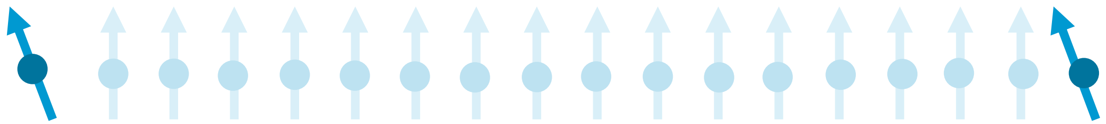
Konstanz, 17.12.2024

Motivation

Modern life

- Most applications are based on electronics
- Information is transferred by moving electrons (charge)
  - But they scatter ➔ Joule heat
- Maybe we can go around this motion
- Use another property of electrons: spin
- Make currents with spins, but how ?





Classification

Scope

- Not the quantum mechanical deviation for a spin
- Consider only bound electrons
  - Insulators

	Bound e^-	Quasi-free e^-
Dia	Lamor Diamagnetism	Landau Diamagnetism
Para	Langevin Paramagnetism	Pauli Paramagnetism
IM	Cooperative Magnetism	Band Ferromagnetism

Spin-operators

Spin-operators

We repeat them here since they are quite important for this talk

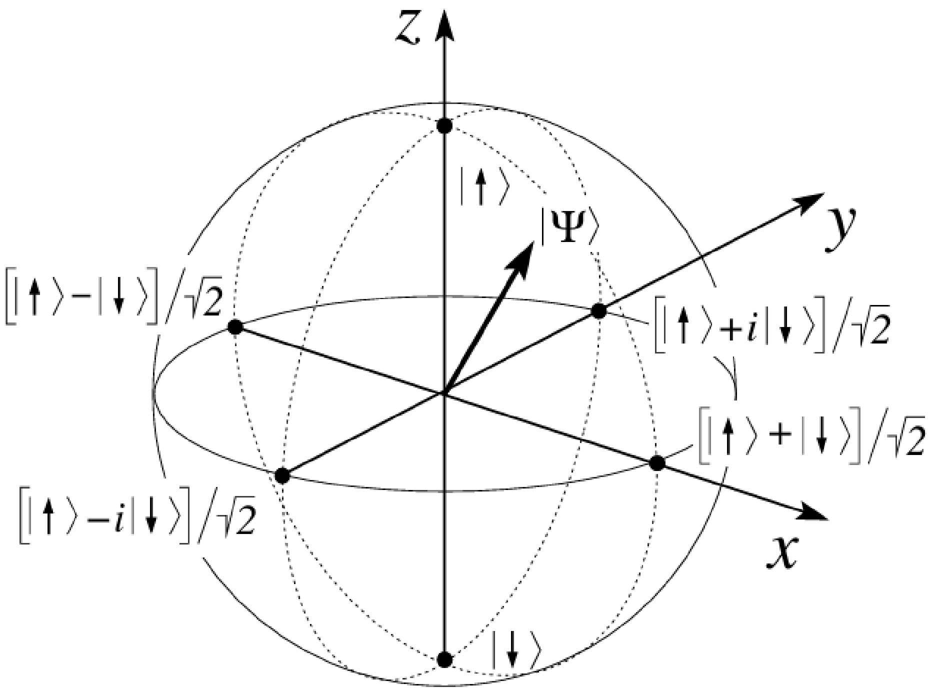
$$\left[\hat{S}_i^x, \hat{S}_j^y\right] = \mathrm{i}\hat{S}_i^y \delta_{i,j} \qquad + \text{cyclic permutation}$$

$$\hat{S}_i^+ = \hat{S}_i^x + \mathrm{i}\hat{S}_i^y$$

$$\hat{S}_i^- = \hat{S}_i^x - \mathrm{i}\hat{S}_i^y$$

$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm\hat{S}_i^\pm \delta_{i,j}$$

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hat{S}_i^z \delta_{i,j}$$



Heisenberg Theory of Ferromagnetism

$$\mathcal{H} = -\sum_{i\neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j$$
$$= -\sum_{i\neq j} J_{i,j} \cdot \left(\hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right)$$
$$= -\sum_{i\neq j} J_{i,j} \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right)$$

W.Heisenberg Z.Physik **49**, 619-636, (1928)

Ferromagnetic groundstate

Derivation of magnons for ferromagnets -> positive J

Concepts can also be applied to antiferromagnets and altermagnets (not part of this talk)

The groundstate is

$$|0\rangle = |S, S, S, \ldots, S\rangle$$

An eigenstate, with the eigenenergy E_0

$$\mathcal{H}|0\rangle = -\sum_{i\neq j} \underline{J} \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right) |0\rangle$$
$$= 0 - JN \cdot S^2 |0\rangle = E_0 |0\rangle$$

Heisenberg Theory of Ferromagnetism

Excitations

How do excitations of this state look like?

The same question was asked by Felix Block

He found several low-lying excited states

We will now obtain the same result as him, but we will use a more modern path

Zur Theorie des Ferromagnetismus.

Von **F. Bloch**, zurzeit in Utrecht.

(Eingegangen am 1. Februar 1930.)

F.Bloch. Z.Physik **61**, *206-219* (1930)

Heisenberg Theory of Ferromagnetism

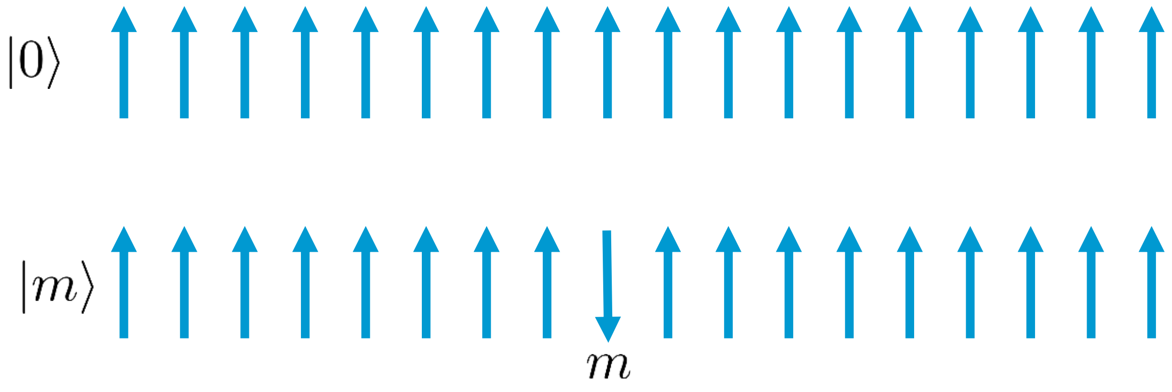
Excitations

Consider one flipped spin

the state flipped/tilted first

$$
\begin{aligned}
|m\rangle &= S_m^-|0\rangle \\
&= |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle
\end{aligned}
$$

Not an eigenstate anymore, which we can quickly calculate



Heisenberg Theory of Ferromagnetism

Excitations

His approach was to consider one flipped spin

the state flipped/tilted first

$$|m\rangle = S_m^-|0\rangle$$
$$= |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle$$

Not an eigenstate anymore, which we can quickly calculate

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle$$

$$\mathcal{H}|m\rangle = -\sum_{i\neq j} J \cdot \left(\frac{1}{2}\left\{\underline{\hat{S}_i^+\hat{S}_j^-}|m\rangle + \underline{\hat{S}_i^-\hat{S}_j^+}|m\rangle\right\} + \hat{S}_i^z\hat{S}_j^z|m\rangle\right)$$

$$= -\sum_{i\neq j} J \cdot \left(\frac{1}{2}\left\{\delta_{i,m}\hat{S}_i^+\hat{S}_j^-|m\rangle + \delta_{j,m}\hat{S}_i^-\hat{S}_j^+|m\rangle\right\} + \hat{S}_i^z\hat{S}_j^z|m\rangle\right)$$

$$= -J \cdot \left(\frac{1}{2}\left\{\sum_j \hat{S}_m^+\hat{S}_j^-|m\rangle + \sum_i \hat{S}_i^-\hat{S}_m^+|m\rangle\right\} + \sum_{i\neq j}\hat{S}_i^z\hat{S}_j^z|m\rangle\right)$$

$$= -J \cdot \left(\frac{1}{2}\left\{\sum_j S|j\rangle + \sum_i S|i\rangle\right\} + \sum_{i\neq j}\hat{S}_i^z\hat{S}_j^z|m\rangle\right)$$

$$= (E_0 + 2JS)\,|m\rangle - \underline{JS\sum_n}|n\rangle$$

Heisenberg Theory of Ferromagnetism

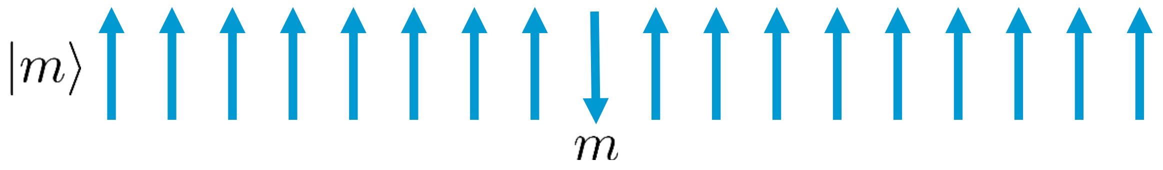
Search for Eigenstates

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle$$
$$= (E_0 + 2JS)\,|m\rangle - JS\sum_n |n\rangle$$

Not an eigenstate anymore.

$$\hat{S}^z_{\text{tot.}}|m\rangle = (NS-1)\,|m\rangle$$

The total magnetization was reduced by 1



Heisenberg Theory of Ferromagnetism

Search for Eigenstates

Not an eigenstate anymore.

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle = (E_0 + 2JS)\,|m\rangle - JS\sum_n |n\rangle$$

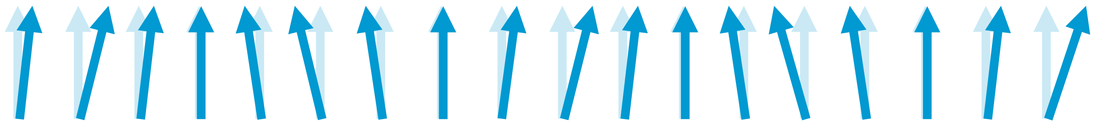
The total magnetization was reduced by 1

$$\hat{S}^z_{\text{tot.}}|m\rangle = (NS-1)\,|m\rangle$$

But if we can see a linear combination of states

Hence we can find an eigenstate

$$|\vec{k}\rangle \propto \sum_n \quad |n\rangle$$

$|\vec{k}\rangle$ 

Heisenberg Theory of Ferromagnetism

Search for eigenstates

Not an eigenstate anymore.

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle$$
$$= (E_0 + 2JS)\,|m\rangle - JS\sum_n |n\rangle$$

The total magnetization was reduced by 1

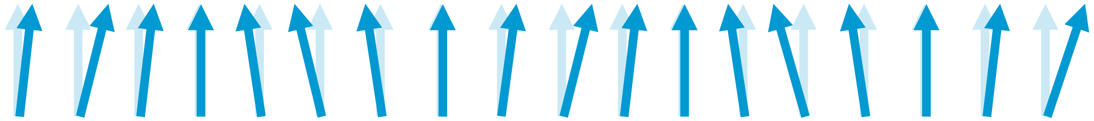
$$\hat{S}^z_{\mathrm{tot.}}|m\rangle = (NS-1)\,|m\rangle$$

But if we can see a linear combination of states

Hence we can find an eigenstate

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

$|\vec{k}\rangle$



Heisenberg Theory of Ferromagnetism

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

Properties of the eigenstates

With this new state we can show that the total **magnetization** compared to before reduced itself

$$\begin{aligned} \hat{S}^z_{\mathrm{tot.}}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^z_{\mathrm{tot.}}|n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)(NS-1)|n\rangle \\ &= (NS-1)|\vec{k}\rangle \end{aligned}$$

Aswell as an increase in energy

The total magnetization in z is reduced, but the average x and y component are still zero? How can that be?

$$\langle\vec{k}|\hat{S}^x_i|\vec{k}\rangle = 0$$

$$\langle\vec{k}|\hat{S}^y_i|\vec{k}\rangle = 0$$

Heisenberg Theory of Ferromagnetism

Properties of the eigenstates

With this new state we can show that the total magnetization compared to before reduced itself

Aswell as an increase in **energy**

$$\mathcal{H}|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) \mathcal{H}|n\rangle$$

Ferromagnetic Dispersion relation

$$\mathcal{H}|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \mathcal{H}|n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \left((E_0 + 2JS)\,|n\rangle - JS \sum_j |j\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2JS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |j\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2JS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp^{\mathrm{i}\vec{k}\cdot\vec{r}_n} \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_j\right) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right) |j\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2JS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n - \mathrm{i}\vec{k}\cdot\vec{r}_j\right) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right) |j\rangle \right)$$

Ferromagnetic Dispersion relation

$$
\begin{aligned}
\mathcal{H}|\vec{k}\rangle &= \frac{1}{\sqrt{N}}\left(\sum_n (E_0+2JS)\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle - JS\sum_{n,j}\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n-\mathrm{i}\vec{k}\cdot\vec{r}_j\right)\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right)|j\rangle\right)\\
&= \left((E_0+2JS)\,|k\rangle - \frac{1}{\sqrt{N}}JS\sum_{\Delta}\sum_{j}\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_j\right)|j\rangle\right)\\
&= \left((E_0+2JS)\,|k\rangle - JS\sum_{\Delta}\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)|k\rangle\right)\\
&= \left(E_0|k\rangle - JS\sum_{\Delta}\left(2-\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)\right)|k\rangle\right)\\
&= \left(E_0|k\rangle - JS\sum_{\Delta>0}\left(2-\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)-\exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)\right)|k\rangle\right)\\
&= \left(E_0|k\rangle - JS\sum_{\Delta>0}\left(2-2\cdot\cos(\vec{k}\vec{r}_\Delta)\right)|k\rangle\right)\\
&= \left(E_0 - 2JS\sum_{\Delta>0}\left(1-\cos(\vec{k}\vec{r}_\Delta)\right)\right)|k\rangle
\end{aligned}
$$

Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2\,J_1\left[1-\cos\frac{2\pi k}{N}\right],$$

F.Bloch. Z.Physik **61**, *206-219* (1930)

We can also find this using a semiclassical calculation

What we just calculated is the Dispersion relation

For a linear chain with NN this simplifies to removal of sum

Ferromagnetic Dispersion relation

Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2\,J_1\left[1 - \cos\frac{2\,\pi\,k}{N}\right],$$

F.Bloch. Z.Physik **61**, *206-219* (1930)

We can also find this using a semiclassical
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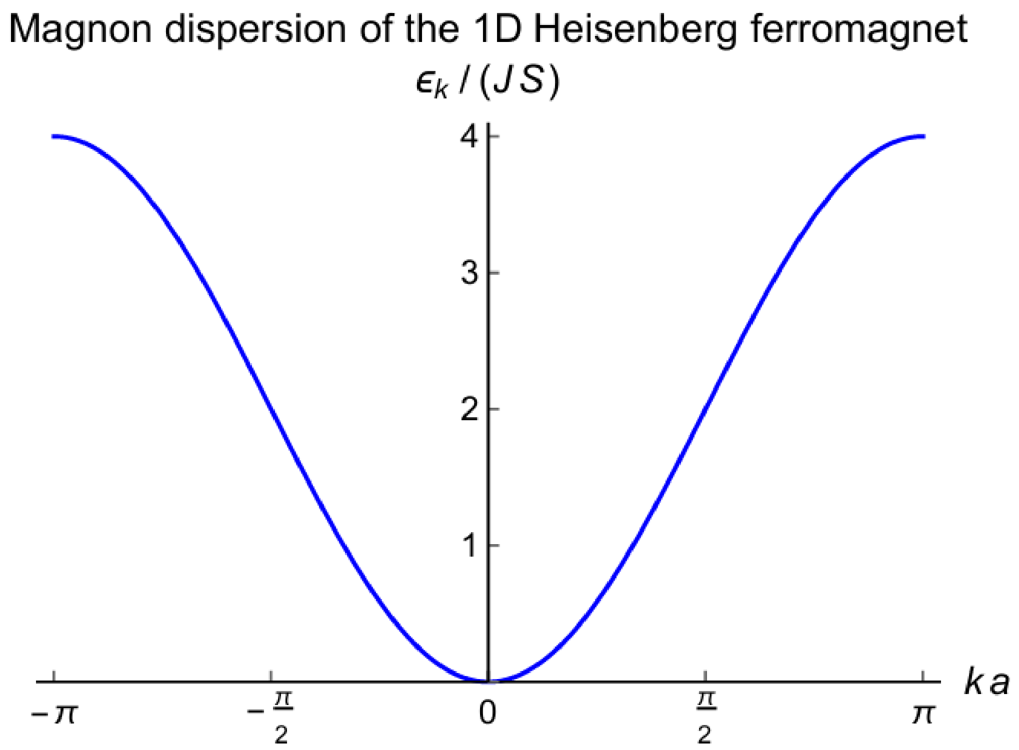
What we just calculated is the Dispersion relation



For a linear chain with NN this simplifies to removal of sum

Recap

What is this eigenstate

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

Hence a collective excitation of the system

By comparison to phonons:

$$\left.\begin{array}{l}\text{-Well defined momentum } \hbar\vec{k} \\ \text{-Energy } \hbar\epsilon(\vec{k})\end{array}\right\} \text{Quasiparticle}$$

The probability to find the excitation at any site m is 1/N

One magnon reduces magnetization by 1 S -> integer spin -> Boson

$$|\langle m|\vec{k}\rangle|^2 = \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\delta_{m,n}\right|^2$$

$$= \frac{1}{N}\left|\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right)\right|^2 = \frac{1}{N}$$

Magnons as Bosons

Nach dem in § 1 Gesagten gehören alle Zahlenanordnungen $(k_1 \ldots k_j \ldots k_r)$, die auseinander durch Vertauschung der k_j entstehen, zu demselben stationären Zustand und sind also nur mit dem statistischen Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik eines Einstein-Bose-Gases her bekannt ist; wie dort kann man einen

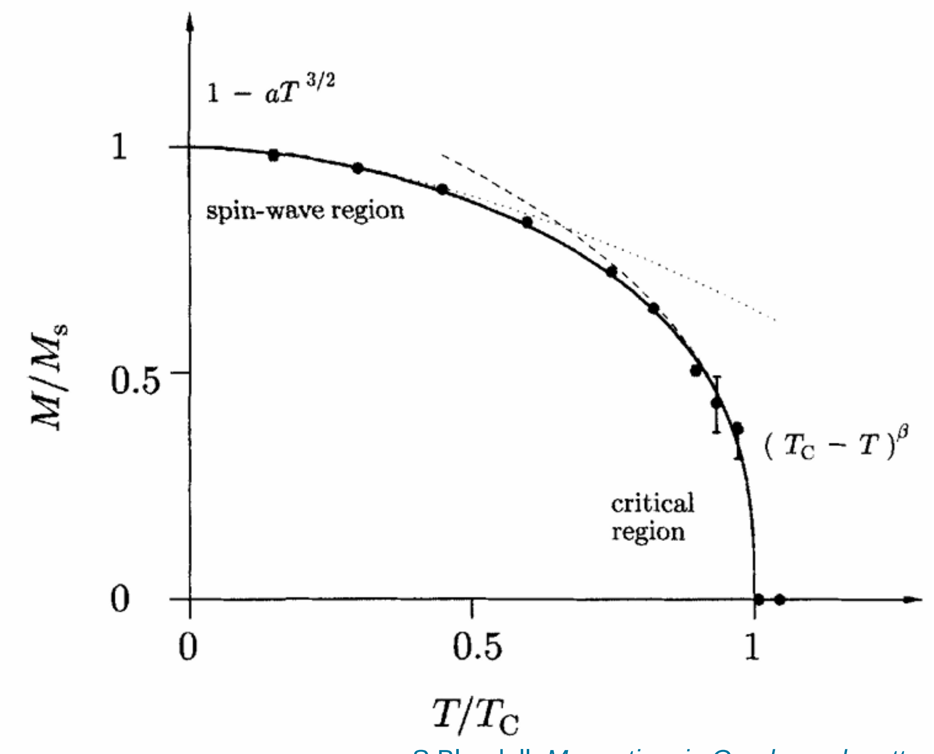
F.Bloch. Z.Physik **61**, *206-219* (1930)

Thermodynamical treatment

$$\hat{S}^z_{\text{tot.}}|\vec{k}\rangle = (NS-1)|\vec{k}\rangle$$

Since it is a boson it must fulfill Bose-Einstein-statistic

$$n_{\text{magnon}} \approx \int_0^\infty \frac{g(\omega)\mathrm{d}\omega}{\exp(\hbar\omega/k_B T)-1}$$
$$= \ldots \propto T^{3/2}$$



S.Blundell, *Magnetism in Condensed matter* (2000)

Spin-Boson Transformation

How to describe Magnons

The spin operators are fermionic

Magnons are bosonic

Search transformation that writes the ang. Moment.
Operators as bosonic creation and annihilation operators

Alternative approach would be to solve the LLG (analytically or numerically)
Holstein-Primakoff representation
Or Schwinger representation
Or Dyson–Maleev representation,

Conditions

i. The transformation needs to be Hermitian, raising and lowering operators written as creation and annihilation boson operators need to be Hermitian conjugate from each other
ii. The transformation must be unitary to preserve the commutation relations between the spin operators.
iii. Must satisfy the equality between the matrix elements of the spin operators on $|0\rangle$ and the bosons on $|n\rangle$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

Simple Spin-Boson Transformation

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

$$\hat{b}^\dagger_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^-_n$$

$$|\vec{k}\rangle = \hat{b}^\dagger_{\vec{k}}|0\rangle$$

$$\hat{b}_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^+_n$$

Direct attempt

For this we will look at an operator $\hat{b}$ that can form our one magnon state

This operator does not fulfill the Boson commutation relations, but is close for small excitation

$$\left[\hat{S}^+_i, \hat{S}^-_j\right] = 2\hat{S}^z_i \delta_{i,j}$$

$$[\hat{b}^\dagger_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = 0$$

$$[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}] = 0$$

$$[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = \frac{1}{2SN}\sum_n \exp\left(-\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{r}_n\right)2\hat{S}^z_n \neq \delta_{\vec{k},\vec{k}'}$$

$$\approx \frac{1}{N}\sum_n \exp\left(-\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{r}_n\right) = \delta_{k,k'}$$

Simple Spin-Boson Transformation

$$\hat{b}^\dagger_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^-_n$$

$$\hat{b}_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^+_n$$

Direct attempt

For this we will look at an operator $\hat{b}$ that can form our one magnon state

This operator does not fulfill the Boson commutation relations, but is close for small excitation

Fourier transform back to lattice operators

$$\hat{b}^\dagger_j = \frac{1}{\sqrt{2S}} \hat{S}^-_j \qquad \hat{b}_j = \frac{1}{\sqrt{2S}} \hat{S}^+_j$$

$$\hat{S}^-_j = \sqrt{2S}\hat{b}^\dagger_j \qquad \hat{S}^+_j = \sqrt{2S}\hat{b}_j$$

$$\hat{n}_j = \hat{b}^\dagger_j \hat{b}_j$$

$$\hat{S}^z_j = S - \hat{b}^\dagger_j \hat{b}_j$$

This we know as the number of bosons/magnons.
We also know a magnon
Reduces the z-comp of
The magnetization
Hence we can conclude

Simple Spin-Boson Transformation

$$\hat{b}^{\dagger}_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^-_n$$

$$\hat{b}_{\vec{k}} = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \hat{S}^+_n$$

Direct attempt

For this we will look at an operator $\hat{b}$ that can form our one magnon state

This operator does not fulfill the Boson commutation relations, but is close for small excitation

This means we can search for a correction of those terms

Fourier transform back to lattice operators

$$\hat{S}^-_j = \sqrt{2S}\hat{b}^{\dagger}_j \qquad \hat{S}^+_j = \sqrt{2S}\hat{b}_j$$

$$\hat{S}^z_j = S - \hat{b}^{\dagger}_j\hat{b}_j$$

using $[\hat{b}_i, \hat{b}^{\dagger}_j] = \delta_{ij}$

$$[\hat{n}_i, \hat{b}_i] = -\hat{b}_i$$

$$[\hat{n}_i, \hat{b}^{\dagger}_i] = \hat{b}^{\dagger}_i$$

we find $[\hat{S}^+_j, \hat{S}^-_i] = 2S\delta_{ij} \neq 2\delta_{ij}\hat{S}^z_j$

Holstein-Primakoff(HP)

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

$$\left[\hat{S}_j^+, \hat{S}_k^-\right] = \ldots = \delta_{j,k} \cdot 2 \cdot \hat{S}_j^z$$

More sophisticated attempt

HP framework is a powerful method for calculating dispersions and higher order interactions

We will do it for a ferromagnetic system.

$$\hat{S}_j^z |S\rangle_j = S|S\rangle_j$$
$$\hat{S}_j^- |S\rangle_j = \sqrt{2S}|S-1\rangle_j$$
$$\hat{S}_j^+ |S-1\rangle_j = \sqrt{2S}|S\rangle_j$$

Holstein-Primakoff

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

Not so sophisticated anymore

Only do linear Spin-Wave theory using only the linearized HP

Want to express the Hamiltonian

$$\begin{aligned}\mathcal{H} &= -\sum_{i\neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j \\ &= -\sum_{i\neq j} J_{i,j} \cdot \left(\frac{1}{2}\left\{\hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+\right\} + \hat{S}_i^z\hat{S}_j^z\right) \\ &= -\sum_{i\neq j} J_{i,j} \cdot \left(S\left\{\hat{a}_i\hat{a}_j^\dagger + \hat{a}_i^\dagger\hat{a}_j\right\} + (S - \hat{n}_i)(S - \hat{n}_j)\right)\end{aligned}$$

Linearized form

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$
$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

Holstein-Primakoff

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^+ \approx \sqrt{2S}\cdot\hat{a}_j$$
$$\hat{S}_j^- \approx \sqrt{2S}\cdot\hat{a}_j^\dagger$$

We want to express the Hamiltonian

$$\begin{aligned}
\mathcal{H} &= -\sum_{i\neq j} J\cdot\hat{S}_i\cdot\hat{S}_j \\
&= -\sum_{i\neq j} J\cdot\left(\frac{1}{2}\left\{\hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+\right\} + \hat{S}_i^z\hat{S}_j^z\right) \\
&= -\sum_{i\neq j} J\cdot\left(S\left\{\hat{a}_i\hat{a}_j^\dagger + \hat{a}_i^\dagger\hat{a}_j\right\} + (S-\hat{n}_i)(S-\hat{n}_j)\right) \\
&= -\frac{NJS^2}{2} - \sum_{i\neq j} JS\cdot\left\{\hat{a}_j^\dagger\hat{a}_i + \hat{a}_i^\dagger\hat{a}_j\right\} + JS\sum_i \hat{n}_i + \underbrace{\sum_{ij} J\hat{n}_i\hat{n}_j}_{\approx 0} \\
&\approx -\frac{NJS^2}{2} - \sum_{i\neq j} JS\cdot\left\{\hat{a}_j^\dagger\hat{a}_i + \hat{a}_i^\dagger\hat{a}_j\right\} + JS\sum_i \hat{n}_i
\end{aligned}$$

These would be magnon-magnon interactions

Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i\neq j} JS \cdot \left\{\hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j\right\} + JS \sum_i \hat{n}_i$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

What can we do with this expression?
We can look at the time evolution of an operator

$$\begin{aligned}
\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}_n &= [\mathcal{H}, \hat{a}_n] \\
&= 0 - \sum_{i\neq j} JS \left\{\left[\hat{a}_j^\dagger \hat{a}_i, \hat{a}_n\right] + \left[\hat{a}_i^\dagger \hat{a}_j, \hat{a}_n\right]\right\} + JS \sum_i [\hat{n}_i, \hat{a}_n] \\
&= 0 - \sum_{i\neq j} JS \left\{-\delta_{jn}\hat{a}_i - \delta_{in}\hat{a}_j\right\} + JS \sum_i -\delta_{in}\hat{a}_i \\
&= -JS \left\{-\sum_{i\neq j}\delta_{jn}\hat{a}_i - \sum_{i\neq j}\delta_{in}\hat{a}_j\right\} + JS \sum_i -\delta_{in}\hat{a}_i
\end{aligned}$$

Problem: coupling between all sites
practically unsolvable for real systems

Holstein-Primakoff

We want to express the Hamiltonian

Want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

What can we do with this expression.

We can look at the time evolution of an operator

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{a}_n &= \dots \\ &= -JS \left\{ -\sum_i \hat{a}_i - \sum_j \hat{a}_j \right\} - JS\hat{a}_n \\ &= 2JS \sum_i \hat{a}_i - JS\hat{a}_n \\ &= JS \sum_i \hat{a}_i \left(2 - \delta_{in}\right) \end{aligned}$$

Going to Fourierspace

Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-i\vec{k}\vec{r}_j),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(i\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

Now we can plug in

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{a}(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_n \left(JS \sum_i \hat{a}_i \left(2 - \delta_{in}\right) \right) \exp(-i\vec{k}\vec{r}_n) \\ &= \frac{JS}{\sqrt{N}} \sum_n \sum_i \hat{a}_i \left(2 - \delta_{in}\right) \exp(-i\vec{k}\vec{r}_n) \end{aligned}$$

Holstein-Primakoff

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And the time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{a}(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_n \left(JS\sum_i \hat{a}_i\,(2-\delta_{in})\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n) \\
&= \frac{JS}{\sqrt{N}} \sum_n \sum_i \hat{a}_i\,(2-\delta_{in}) \exp(-\mathrm{i}\vec{k}\vec{r}_n) \\
&= \frac{JS}{\sqrt{N}} \sum_n \sum_i \frac{1}{\sqrt{N}} \sum_{\vec{k}'} \exp(\mathrm{i}\vec{k}'\vec{r}_i)\hat{a}(\vec{k}')\,(2-\delta_{in}) \exp(-\mathrm{i}\vec{k}\vec{r}_n) \\
&= \frac{JS}{N} \sum_n \sum_i \sum_{\vec{k}'} (2-\delta_{in}) \exp(\mathrm{i}\vec{k}'\vec{r}_i + 0 - \mathrm{i}\vec{k}\vec{r}_n)\hat{a}(\vec{k}') \\
&= \frac{JS}{N} \sum_{\vec{k}',i,n} (2-\delta_{in}) \exp(\mathrm{i}\vec{k}'\vec{r}_i - \mathrm{i}\vec{k}\vec{r}_i + \mathrm{i}\vec{k}\vec{r}_i - \mathrm{i}\vec{k}\vec{r}_n)\hat{a}(\vec{k}') \\
&= \frac{JS}{N} \sum_{\vec{k}',i,n} (2-\delta_{in}) \exp\left(\mathrm{i}(\vec{k}'-\vec{k})\vec{r}_i + \mathrm{i}\vec{k}(\vec{r}_i-\vec{r}_n)\right) \hat{a}(\vec{k}') \\
&= JS \sum_{\vec{k}'} \sum_i \frac{1}{N} \exp\left(\mathrm{i}(\vec{k}'-\vec{k})\vec{r}_i\right) \cdot \sum_n (2-\delta_{0n}) \exp\left(\mathrm{i}\vec{k}(\vec{r}_0-\vec{r}_n)\right) \hat{a}(\vec{k}') \\
&= JS \sum_{\vec{k}'} \delta_{\vec{k}',\vec{k}} \sum_n (2-\delta_{0n}) \exp\left(\mathrm{i}\vec{k}(\vec{r}_0-\vec{r}_n)\right) \hat{a}(\vec{k}') \\
&= JS \sum_n (2-\delta_{0n}) \exp\left(\mathrm{i}\vec{k}(\vec{r}_0-\vec{r}_n)\right) \hat{a}(\vec{k})
\end{aligned}$$

Holstein-Primakoff

Going to Fourierspace

Using the lattice Fourier trafo

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_j),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(\mathrm{i}\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(JS\sum_i \hat{a}_i\,(2-\delta_{in})\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= JS\sum_n (2-\delta_{0n}) \exp\left(\mathrm{i}\vec{k}(\vec{r}_0-\vec{r}_n)\right) \hat{a}(\vec{k})$$

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = -\frac{\mathrm{i}}{\hbar}\mathcal{W}_{\vec{k}}\hat{a}(\vec{k})$$

$$\hbar\omega_{\vec{k}} = S\left(J_0 - 2J_{\vec{k}}\right) \qquad \hat{a}(\vec{k},t) = \hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)$$

$$\hat{a}_j(\vec{k},t) = \frac{1}{\sqrt{N}}\sum_{\vec{k}} \exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k},t)$$

$$= \frac{1}{\sqrt{N}}\sum_{\vec{k}} \exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)$$

$$\hat{a}_j(\vec{k}_0,t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)\hat{a}(\vec{k}_0,0)$$

$$\hat{a}_j(\vec{k}_0,t) = \hat{A}\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)$$

Holstein-Primakoff

Going to Fourierspace

Using the lattice Fourier trafo

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_j),$$

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And the time evolution

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Now we can plug in

S.Blundell, *Magnetism in Condensed matter* (2000)

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^x \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

$$\boxed{\begin{aligned} \hat{S}_j^x(t) &= \sqrt{2S}\hat{A} \cdot \cos(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \\ \hat{S}_j^y(t) &= \sqrt{2S}\hat{A} \cdot \sin(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A) \\ \hat{S}_j^z(t) &= S - |\hat{A}|^2 \end{aligned}}$$

Now we can understand why the average we calculated earlier was 0

Visualising a Magnon

Numerical methods

Based on the Lamor-frequency we can numerically integrate a system of spins using the Landau Lifshitz Gilbert equation.

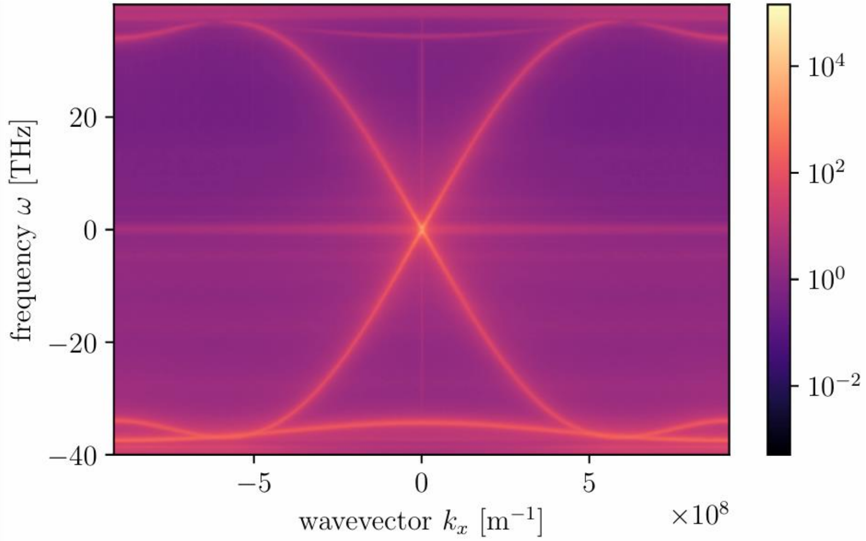
Time integration done with Heun's methods

With this method one could also determine the dispersion relation

Additional property is the damping α

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = -\frac{\gamma}{(1+\alpha^2)\mu_S}\vec{S}_i \times \left(\vec{H}_i(t) + \alpha \vec{S}_i \times \vec{H}_i(t)\right)$$

$$\vec{H}_i(t) = -\frac{\partial \mathcal{H}}{\partial \vec{S}_i}$$



Outlook

What is the research doing

With this we can easily obtain highly interesting dispersion relations.

Can be used for computing, without energy loss through Joule heating

This is ongoing research in

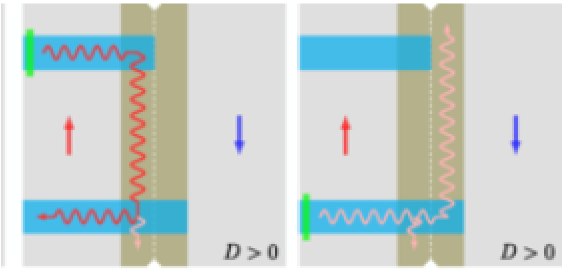
Spin wave diodes

Spin wave transitors *J.Lan et.al.* PRX **5**, 041049, (2015)

The most modern research even includes Altermagnets, which have distinct symmetry enforced properties regarding the dispersion in different directions

Topological magnons

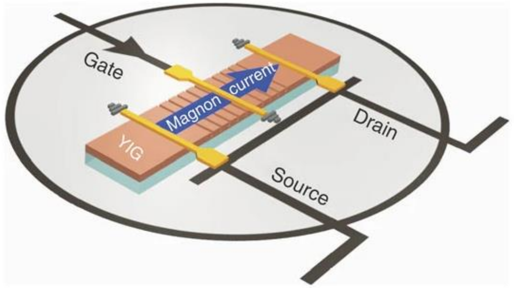
Squezzed magnons *Add cool sources here et.al.* Nature C. **5**, 4700, (2014)



(c) Diode - forward direction (d) Diode - reverse direction

J.Lan et.al. PRX **5**, 041049, (2015)

Magnon transistor scheme



A.Chumak et.al. Nature C. **5**, 4700, (2014)