Chapter 8

SPIN WAVES IN MULTILAYERS

8.1. Spin Green Functions and Random Phase Approximation

In the previous chapters, we have studied the spin waves in *infinite* systems that is in perfect periodic lattices. On the contrary, in this chapter we will study the spin waves in *finite* systems describing magnetic multilayers or spin waves in a semi-infinite system describing the effect of a surface on a magnetic system. The main point is the loss of periodicity in the direction perpendicular to the magnetic layers. According to Chapter 3, we define the retarded Green function[24]

$$G_{l,m}(t) \equiv \langle\langle S_l^+; S_m^- \rangle\rangle = -i\theta(t)\langle [S_l^+(t), S_m^-]\rangle \tag{8.1.1}$$

and its time Fourier transform

$$G_{l,m}(\omega + i\epsilon) \equiv \langle\langle S_l^+; S_m^- \rangle\rangle_{\omega+i\epsilon} = \int_{-\infty}^{+\infty} dt e^{i\omega t} G_{l,m}(t). \tag{8.1.2}$$

The equation of motion of the time Green function (8.1.1) is

$$\frac{d}{dt}\langle\langle S_l^+; S_m^- \rangle\rangle = -2\ i\delta(t)\delta_{l,m}\langle S_l^z\rangle + \left\langle\left\langle -\frac{i}{\hbar}[S_l^+, \mathcal{H}]; S_m^- \right\rangle\right\rangle, \tag{8.1.3}$$

where the commutation rule $[S_l^+, S_m^-] = 2S_l^z\delta_{l,m}$ has been used and $\mathcal{H}$ is the Heisenberg Hamiltonian

$$\mathcal{H} = -\sum_{j,\boldsymbol{\delta}} J_\delta \left[S_j^z S_{j+\boldsymbol{\delta}}^z + \frac{1}{2}(S_j^+ S_{j+\boldsymbol{\delta}}^- + S_j^- S_{j+\boldsymbol{\delta}}^+)\right] \tag{8.1.4}$$

with $J_{\boldsymbol{\delta}} > 0$. The equation of motion of the Green function (8.1.2) is then

$$\hbar\omega\langle\langle S_l^+; S_m^- \rangle\rangle_\omega = 2\hbar\delta_{l,m}\langle S_l^z\rangle + \sum_{\boldsymbol{\delta}} 2J_{\boldsymbol{\delta}}\langle\langle S_l^+ S_{l+\boldsymbol{\delta}}^z - S_{l+\boldsymbol{\delta}}^+ S_l^z; S_m^- \rangle\rangle_\omega. \tag{8.1.5}$$

The infinite chain of equations for the various Green functions is made finite by a convenient decoupling. In particular, the random phase approximation[28] (RPA) consists of decoupling the Green function occurring in the last term of Eq. (8.1.5) as

$$\langle\langle S_l^+ S_{l+\boldsymbol{\delta}}^z - S_{l+\boldsymbol{\delta}}^+ S_l^z ; S_m^- \rangle\rangle_\omega \simeq \langle S_l^z \rangle [G_{l,m}(\omega) - G_{l+\boldsymbol{\delta},m}(\omega)] \tag{8.1.6}$$

where one neglects the correlations between S^+ on one lattice site and S^z on another lattice site. This approximation becomes exact at zero temperature because $S_l^z|0\rangle = S|0\rangle$ where S is a c-number. The RPA allows to write Eq. (8.1.5) in the form

$$\left(\hbar\omega - \langle S_l^z \rangle \sum_{\boldsymbol{\delta}} 2J_{\boldsymbol{\delta}}\right) G_{l,m}(\omega) + \langle S_l^z \rangle \sum_{\boldsymbol{\delta}} 2J_{\boldsymbol{\delta}}\; G_{l+\boldsymbol{\delta},m}(\omega) = 2\hbar\delta_{l,m}\langle S_l^z \rangle. \tag{8.1.7}$$

8.2. Multilayers

Starting from Eq. (8.1.7), we will obtain the spin wave spectrum for a stacking of ferromagnetic planes (multilayer). Suppose, we pile up the layers along the z-axis. Assuming PBC in the xy-planes, we may define a spatial Fourier transform as

$$G_{l,m}(\omega) = \frac{1}{\sqrt{N_s}} \sum_{\boldsymbol{k}_\parallel} e^{i\boldsymbol{k}_\parallel \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j)} G_{n,n'}(\boldsymbol{k}_\parallel, \omega) \tag{8.2.1}$$

where $l = \boldsymbol{r}_i + n\mathbf{u}_z$ and $m = \boldsymbol{r}_j + n'\mathbf{u}_z$ in which n, n' label the planes and $\boldsymbol{r}_i$, $\boldsymbol{r}_j$ are 2D lattice vectors within the xy-planes parallel to the lower ($n = 1$) and upper ($n = N$) surface of the multilayer; $\boldsymbol{k}_\parallel$ is a wavevector in the $k_x k_y$-plane and N_s is the number of the spins belonging to a plane of the multilayer. Suppose that the exchange interaction between NN spins in the xy-planes is $2J_\parallel$ and the exchange interaction between NN spins belonging to adjacent planes is $2J_\perp$. Such a model is called[76] "free surface model" since there is no difference between the exchange interactions between the NN spins on the surfaces and the exchange interactions between NN spins in the layers of the bulk. This model could be appropriated to describe a film of N layers. The Heisenberg Hamiltonian for such a film is

$$\mathcal{H} = -\sum_{s=1}^{N} \sum_{j\boldsymbol{\delta}_\parallel} J_\parallel \boldsymbol{S}_j^{(s)} \cdot \boldsymbol{S}_{j+\boldsymbol{\delta}_\parallel}^{(s)} - \sum_{s=1}^{N-1} \sum_{j\boldsymbol{\delta}_\perp} 2J_\perp \boldsymbol{S}_j^{(s)} \cdot \boldsymbol{S}_{j+\boldsymbol{\delta}_\perp}^{(s+1)}, \tag{8.2.2}$$

where $\boldsymbol{S}_j^{(s)}$ is the spin located at the site $\boldsymbol{r}_j + s\mathbf{u}_z$. The absence of a factor 2 before the exchange integral $J_\parallel$ is due to a double counting coming from the sum over j and $\delta_\parallel$. In the limit $T \to 0$, the equation of motion (8.1.7) splits into N equations of motion for the Green functions $G_{n,n'}(\boldsymbol{k}_\parallel, \omega)$ with $n = 1, 2, \ldots, N$ given by

$$[\hbar(\omega - \omega_\parallel) - 2z_\perp J_\perp S] G_{1,n'}(\boldsymbol{k}_\parallel, \omega) + 2z_\perp J_\perp S \gamma_\perp G_{2,n'}(\boldsymbol{k}_\parallel, \omega) = 2\hbar S \delta_{1,n'} \tag{8.2.3}$$

for $n = 1$ corresponding to the lower surface of the film,

$$\begin{aligned} &[\hbar(\omega - \omega_\parallel) - 4z_\perp J_\perp S] G_{n,n'}(\boldsymbol{k}_\parallel, \omega) + 2z_\perp J_\perp S \gamma_\perp [G_{n-1,n'}(\boldsymbol{k}_\parallel, \omega) + G_{n+1,n'}(\boldsymbol{k}_\parallel, \omega)] \\ &\quad = 2\hbar S \delta_{n,n'} \end{aligned} \tag{8.2.4}$$

for $1 < n < N-1$ corresponding to the internal layers (bulk) and

$$[\hbar(\omega-\omega_{\|}) - 2z_\perp J_\perp S]G_{N,n'}(\boldsymbol{k}_{\|},\omega) + 2z_\perp J_\perp S\gamma_\perp G_{N-1,n'}(\boldsymbol{k}_{\|},\omega) = 2\hbar S\delta_{N,n'} \tag{8.2.5}$$

for $n = N$ corresponding to the upper surface of the film. The frequency $\omega_{\|}$ and the structure factors occurring in Eqs. (8.2.3)–(8.2.5) are given by

$$\hbar\omega_{\|} = 2z_{\|}J_{\|}S(1-\gamma_{\|}) \tag{8.2.6}$$

with

$$\gamma_{\|} = \frac{1}{z_{\|}}\sum_{\boldsymbol{\delta}_{\|}} e^{i\boldsymbol{k}_{\|}\cdot\boldsymbol{\delta}_{\|}} \tag{8.2.7}$$

and

$$\gamma_\perp = \frac{1}{z_\perp}\sum_{\boldsymbol{\delta}_\perp} e^{i\boldsymbol{k}_{\|}\cdot\boldsymbol{\delta}_\perp}, \tag{8.2.8}$$

respectively. In Eqs. (8.2.3)–(8.2.8), $z_{\|}$ and $z_\perp$ are the numbers of the NN spins in and out of xy-planes, respectively. Notice that for planes stacked in such a way that the spins along the z-direction lie on a straight line one has $z_\perp = 1$ and $\boldsymbol{k}_{\|}\cdot\boldsymbol{\delta}_\perp = 0$. This is the case for SC and H lattices, for instance. Defining

$$\hbar\omega = \hbar\omega_{\|} + 4z_\perp J_\perp S(1-\gamma_\perp\cos\theta) \tag{8.2.9}$$

one may write the N equations (8.2.3)–(8.2.5) in a matrix form as

$$\mathbf{A}\times\mathbf{G} = \frac{\hbar}{z_\perp J_\perp \gamma_\perp}\mathbf{1} \tag{8.2.10}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\gamma_\perp} - 2\cos\theta & 1 & 0 & \dots & 0 & 0 \\ 1 & -2\cos\theta & 1 & \dots & 0 & 0 \\ 0 & 1 & -2\cos\theta & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2\cos\theta & 1 \\ 0 & 0 & 0 & \dots & 1 & \frac{1}{\gamma_\perp} - 2\cos\theta \end{pmatrix}. \tag{8.2.11}$$

The matrix elements of $\mathbf{G}$ are $G_{n,n'}(\boldsymbol{k}_{\|},\omega)$ with $n, n' = 1,\dots,N$ and $\mathbf{1}$ is the unit matrix of dimension N. As one can see, the matrix $\mathbf{A}$ is a tridiagonal matrix symmetric with respect to its diagonals. The inverse matrix $\mathbf{A}^{-1}$ is no longer tridiagonal but it is yet symmetric with respect to its two diagonals so that the number of distinct $G_{n,n'}(\boldsymbol{k}_{\|},\omega)$ is drastically reduced. The solution of the matrix equation (8.2.10) is

$$G_{n,n'}(\boldsymbol{k}_{\|},\omega) = \frac{\hbar}{z_\perp J_\perp \gamma_\perp}(-1)^{n+n'}\frac{\det\mathbf{A}_{n'n}}{\det\mathbf{A}}, \tag{8.2.12}$$

where $\mathbf{A}_{n'n}$ is the matrix left after the n'th row and the nth column have been suppressed. Notice that the poles of the Green functions of Eq. (8.2.12) occur at frequencies ω_l (or θ_l according to Eq. (8.2.9)) for which the determinant of $\mathbf{A}$ vanishes. In order to evaluate such a determinant, we split the matrix $\mathbf{A}$ into two matrices like

$$\mathbf{A} = \mathbf{\Delta} - \mathbf{A}_0, \tag{8.2.13}$$

where $\mathbf{\Delta}$ is the square matrix of dimension N whose elements are all zero except $\Delta_{1,1} = \Delta_{N,N} = \frac{1}{\gamma_\perp}$ and

$$\mathbf{A}_0 = \begin{pmatrix} 2\cos\theta & -1 & 0 & \dots & 0 & 0 \\ -1 & 2\cos\theta & -1 & \dots & 0 & 0 \\ 0 & -1 & 2\cos\theta & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2\cos\theta & -1 \\ 0 & 0 & 0 & \dots & -1 & 2\cos\theta \end{pmatrix}. \tag{8.2.14}$$

In this way, the determinant of $\mathbf{A}$ becomes

$$\det\mathbf{A} = (-1)^N D_N \ \det(\mathbf{1} - \mathbf{A}_0^{-1}\mathbf{\Delta}) \tag{8.2.15}$$

where $D_N = \det\mathbf{A}_0$, $\mathbf{A}_0^{-1}$ is the inverse matrix of $\mathbf{A}_0$ and $\mathbf{1}$ is the unit matrix of dimension N. A recursion formula for the determinant D_N is easily obtained expanding the determinant itself along the first row. One obtains

$$D_N = 2\cos\theta D_{N-1} - D_{N-2}, \tag{8.2.16}$$

where D_{N-1} and D_{N-2} are determinants with the same structure of D_N but with one and two rows and columns suppressed, respectively. It is direct to verify that the solution of Eq. (8.2.16) is

$$D_N = \frac{\sin(N+1)\theta}{\sin\theta} \tag{8.2.17}$$

and to obtain

$$\mathbf{A}_0^{-1}\mathbf{\Delta} = \begin{pmatrix} \frac{D_{N-1}}{\gamma_\perp D_N} & 0 & \dots & 0 & \frac{1}{\gamma_\perp D_N} \\ \frac{D_{N-2}}{\gamma_\perp D_N} & 0 & \dots & 0 & \frac{D_1}{\gamma_\perp D_N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{D_1}{\gamma_\perp D_N} & 0 & \dots & 0 & \frac{D_{N-2}}{\gamma_\perp D_N} \\ \frac{1}{\gamma_\perp D_N} & 0 & \dots & 0 & \frac{D_{N-1}}{\gamma_\perp D_N} \end{pmatrix} \tag{8.2.18}$$

so that

$$\det(\mathbf{1} - \mathbf{A}_0^{-1}\boldsymbol{\Delta}) = \left(1 - \frac{D_{N-1}}{\gamma_\perp D_N}\right)^2 - \left(\frac{1}{\gamma_\perp D_N}\right)^2. \tag{8.2.19}$$

Replacing Eqs. (8.2.19) and (8.2.17) into Eq. (8.2.15), one obtains

$$\det \mathbf{A} = \frac{(-1)^N}{\gamma_\perp^2 D_N}[(\gamma_\perp D_N - D_{N-1})^2 - 1] \tag{8.2.20}$$

with D_N given by Eq. (8.2.17).

The neutron scattering cross-section for a film can be deduced from Eq. (3.1.4). In order to simplify the formulas, we consider the neutron cross-section for a scattering wavevector directed perpendicularly to the surface of the film that is $\boldsymbol{K} = K_z\mathbf{u}_z$. For such a geometry, Eq. (3.1.4) reduces to

$$\frac{d^2\sigma}{d\Omega dE'} = r_0^2\frac{k'}{k}\left[\frac{1}{2}gF(\boldsymbol{K})\right]^2 e^{-2W(\boldsymbol{K})}\ 2S_\perp(\boldsymbol{K},\omega) \tag{8.2.21}$$

with

$$S_\perp(K_z,\omega) = -\frac{1}{4\pi\hbar}\sum_{n,n'} e^{-iK_z(n-n')}\Im\{[1+n(\omega)]G_{n,n'}(\omega) + n(-\omega)G_{n,n'}(-\omega)\}, \tag{8.2.22}$$

where $G_{n,n'}(\omega) = G_{n,n'}(\boldsymbol{K}_\parallel = 0,\omega)$. For $\boldsymbol{K}_\parallel = 0$, the solutions of the equation $\det \mathbf{A} = 0$ with $\det \mathbf{A}$ given by Eq. (8.2.20) are

$$\theta = \frac{\pi}{N}l, \quad l = 0,1,2,\ldots,N-1 \tag{8.2.23}$$

so that the Green functions $G_{n,n'}(\omega)$ appearing in Eq. (8.2.12) can be written in the form

$$G_{n,n'}(\omega) = \sum_{l=0}^{N-1}\frac{c_{n,n'}^{(l)}}{\omega - \omega_l} \tag{8.2.24}$$

with

$$\omega_l = \frac{4z_\perp J_\perp S}{\hbar}\left(1 - \cos\frac{\pi}{N}l\right). \tag{8.2.25}$$

By using the identity (3.2.12), one has

$$\Im G_{n,n'}(\pm\omega) = -\pi\sum_{l=0}^{N-1} c_{n,n'}^{(l)}\delta(\omega \mp \omega_l) \tag{8.2.26}$$

so that

$$S_\perp(K_z,\omega) = \sum_{l=0}^{N-1} I_l(K_z)\{[1+n(\omega_l)]\delta(\omega-\omega_l) + n(\omega_l)\delta(\omega+\omega_l)\} \tag{8.2.27}$$

with

$$I_l(K_z) = \frac{1}{4\hbar}\sum_{n,n'} c_{n,n'}^{(l)}e^{-iK_z(n-n')}. \tag{8.2.28}$$

The neutron cross-section (8.2.21) is a sum of δ-peaks with intensity $I_l(K_z)$ whose number coincides with the number of layers of the film. Note that the existence of δ-peaks in the neutron cross-section is directly connected to the RPA approximation that becomes exact as $T \to 0$. Higher-order decoupling has to be used at finite temperature and the related Green functions change the δ-peaks into Lorentzian peaks, leading to the renormalization and damping of the excitations.

8.3. Bilayer

In order to show in detail the results illustrated in the previous section, we perform explicitly the calculations for a bilayer (two layers). For a bilayer, the matrix (8.2.11) becomes

$$\mathbf{A}_2 = \begin{pmatrix} \frac{1}{\gamma_\perp} - 2\cos\theta & 1 \\ 1 & \frac{1}{\gamma_\perp} - 2\cos\theta \end{pmatrix}. \tag{8.3.1}$$

The distinct Green functions given by Eq. (8.2.12) are

$$G_{1,1}(\boldsymbol{k}_\parallel, \omega) = \frac{\hbar}{z_\perp J_\perp \gamma_\perp} \frac{\frac{1}{\gamma_\perp} - 2\cos\theta}{\det \mathbf{A}_2} \tag{8.3.2}$$

and

$$G_{1,2}(\boldsymbol{k}_\parallel, \omega) = -\frac{\hbar}{z_\perp J_\perp \gamma_\perp} \frac{1}{\det \mathbf{A}_2} \tag{8.3.3}$$

where

$$\det \mathbf{A}_2 = \left(\frac{1}{\gamma_\perp} - 2\cos\theta\right)^2 - 1. \tag{8.3.4}$$

Owing to the properties of the matrix $\mathbf{A}_2$, one has $G_{2,1} = G_{1,2}$ (symmetry about the principal diagonal of the matrix) and $G_{2,2} = G_{1,1}$ (symmetry about the other diagonal of the matrix). Using the relationship (8.2.9), one can transform Eqs. (8.3.2) and (8.3.3) into

$$G_{1,1}(\boldsymbol{k}_\parallel, \omega) = S\left(\frac{1}{\omega - \omega_0} + \frac{1}{\omega - \omega_1}\right) \tag{8.3.5}$$

and

$$G_{1,2}(\boldsymbol{k}_\parallel, \omega) = S\left(\frac{1}{\omega - \omega_0} - \frac{1}{\omega - \omega_1}\right), \tag{8.3.6}$$

where

$$\omega_l = \omega_\parallel + \frac{2 z_\perp J_\perp S}{\hbar}(1 - \gamma_\perp \cos \pi l) \tag{8.3.7}$$

with $l = 0, 1$. Using the relationship (3.2.12), the imaginary parts of the Green functions (8.3.5) and (8.3.6) are given by

$$\Im G_{1,1}(\boldsymbol{k}_\parallel, \pm\omega) = -\pi\ S[\delta(\omega \mp \omega_0) + \delta(\omega \mp \omega_1)] \tag{8.3.8}$$

and

$$\Im G_{1,2}(\boldsymbol{k}_{\|}, \pm\omega) = -\pi\ S[\delta(\omega \mp \omega_0) - \delta(\omega \mp \omega_1)]. \tag{8.3.9}$$

The neutron cross-section is given by (8.2.27) with $N = 2$ and

$$I_l(K_z) = \frac{S}{2\hbar}(1 + \cos K_z \cos \pi l) \tag{8.3.10}$$

with $l = 0, 1$.

8.4. Trilayer

In this section, we perform explicitly the calculations for a trilayer (three layers) for which the matrix (8.2.11) becomes

$$\mathbf{A}_3 = \begin{pmatrix} \dfrac{1}{\gamma_\perp} - 2\cos\theta & 1 & 0 \\ 1 & -2\cos\theta & 1 \\ 0 & 1 & \dfrac{1}{\gamma_\perp} - 2\cos\theta \end{pmatrix}. \tag{8.4.1}$$

The distinct Green functions given by Eq. (8.2.12) are

$$G_{1,1}(\boldsymbol{k}_{\|}, \omega) = \frac{\hbar}{z_\perp J_\perp \gamma_\perp} \frac{4\cos^2\theta - \frac{2}{\gamma_\perp}\cos\theta - 1}{\det \mathbf{A}_3}, \tag{8.4.2}$$

$$G_{1,2}(\boldsymbol{k}_{\|}, \omega) = -\frac{\hbar}{z_\perp J_\perp \gamma_\perp}\ \frac{\frac{1}{\gamma_\perp} - 2\cos\theta}{\det \mathbf{A}_3}, \tag{8.4.3}$$

$$G_{1,3}(\boldsymbol{k}_{\|}, \omega) = \frac{\hbar}{z_\perp J_\perp \gamma_\perp}\ \frac{1}{\det \mathbf{A}_3} \tag{8.4.4}$$

and

$$G_{2,2}(\boldsymbol{k}_{\|}, \omega) = \frac{\hbar}{z_\perp J_\perp \gamma_\perp}\ \frac{\left(\frac{1}{\gamma_\perp} - 2\cos\theta\right)^2}{\det \mathbf{A}_3}, \tag{8.4.5}$$

where

$$\det \mathbf{A}_3 = \left(\frac{1}{\gamma_\perp} - 2\cos\theta\right)\left(2\cos\theta - \frac{1 - \sqrt{1 + 8\gamma_\perp^2}}{2\gamma_\perp}\right)\left(2\cos\theta - \frac{1 + \sqrt{1 + 8\gamma_\perp^2}}{2\gamma_\perp}\right). \tag{8.4.6}$$

Owing to the properties of the matrix $\mathbf{A}_3$, one has $G_{2,1} = G_{1,2}$, $G_{1,3} = G_{3,1}$ (symmetry about the principal diagonal of the matrix) and $G_{3,3} = G_{1,1}$, $G_{2,3} = G_{3,2}$ (symmetry about the other diagonal of the matrix). Using the relationship (8.2.9), one can transform Eqs. (8.4.2)–(8.4.5) into

$$\begin{aligned} G_{1,1}(\boldsymbol{k}_{\|}, \omega) = \frac{S}{4}\Bigg(& \frac{\sqrt{1 + 8\gamma_\perp^2} + 1}{2\sqrt{1 + 8\gamma_\perp^2}}\ \frac{1}{\omega - \omega_0} + \frac{1}{\omega - \omega_1} \\ & + \frac{\sqrt{1 + 8\gamma_\perp^2} - 1}{2\sqrt{1 + 8\gamma_\perp^2}}\ \frac{1}{\omega - \omega_2}\Bigg), \end{aligned} \tag{8.4.7}$$

$$G_{1,2}(\boldsymbol{k}_{\|},\omega)=\frac{S}{2}\frac{\gamma_{\perp}}{\sqrt{1+8\gamma_{\perp}^{2}}}\left(\frac{1}{\omega-\omega_{0}}-\frac{1}{\omega-\omega_{2}}\right), \tag{8.4.8}$$

$$\begin{aligned}G_{1,3}(\boldsymbol{k}_{\|},\omega)=\frac{S}{4}\Bigg(&\frac{\sqrt{1+8\gamma_{\perp}^{2}}+1}{2\sqrt{1+8\gamma_{\perp}^{2}}}\;\frac{1}{\omega-\omega_{0}}-\frac{1}{\omega-\omega_{1}}\\&+\frac{\sqrt{1+8\gamma_{\perp}^{2}}-1}{2\sqrt{1+8\gamma_{\perp}^{2}}}\;\frac{1}{\omega-\omega_{2}}\Bigg),\end{aligned} \tag{8.4.9}$$

$$G_{2,2}(\boldsymbol{k}_{\|},\omega)=\frac{S}{4}\left(\frac{\sqrt{1+8\gamma_{\perp}^{2}}-1}{\sqrt{1+8\gamma_{\perp}^{2}}}\;\frac{1}{\omega-\omega_{0}}+\frac{\sqrt{1+8\gamma_{\perp}^{2}}+1}{\sqrt{1+8\gamma_{\perp}^{2}}}\;\frac{1}{\omega-\omega_{2}}\right), \tag{8.4.10}$$

where

$$\omega_{0}=\omega_{\|}+\frac{z_{\perp}J_{\perp}S}{\hbar}\left(3-\sqrt{1+8\gamma_{\perp}^{2}}\right), \tag{8.4.11}$$

$$\omega_{1}=\omega_{\|}+\frac{2z_{\perp}J_{\perp}S}{\hbar} \tag{8.4.12}$$

and

$$\omega_{2}=\omega_{\|}+\frac{z_{\perp}J_{\perp}S}{\hbar}\left(3+\sqrt{1+8\gamma_{\perp}^{2}}\right). \tag{8.4.13}$$

Using the relationship (3.2.12), the imaginary parts of the Green functions (8.4.7)–(8.4.10) are given by

$$\begin{aligned}\Im G_{1,1}(\boldsymbol{k}_{\|},\pm\omega)=-\frac{\pi S}{4}\Bigg[&\frac{\sqrt{1+8\gamma_{\perp}^{2}}+1}{2\sqrt{1+8\gamma_{\perp}^{2}}}\delta(\omega\mp\omega_{0})+\delta(\omega\mp\omega_{1})\\&+\frac{\sqrt{1+8\gamma_{\perp}^{2}}-1}{2\sqrt{1+8\gamma_{\perp}^{2}}}\delta(\omega\mp\omega_{2})\Bigg],\end{aligned} \tag{8.4.14}$$

$$\Im G_{1,2}(\boldsymbol{k}_{\|},\pm\omega)=-\frac{\pi S}{2}\;\frac{\gamma_{\perp}}{\sqrt{1+8\gamma_{\perp}^{2}}}[\delta(\omega\mp\omega_{0})-\delta(\omega\mp\omega_{2})], \tag{8.4.15}$$

$$\begin{aligned}\Im G_{1,3}(\boldsymbol{k}_{\|},\pm\omega)=-\frac{\pi S}{4}\Bigg[&\frac{\sqrt{1+8\gamma_{\perp}^{2}}+1}{2\sqrt{1+8\gamma_{\perp}^{2}}}\delta(\omega\mp\omega_{0})-\delta(\omega\mp\omega_{1})\\&+\frac{\sqrt{1+8\gamma_{\perp}^{2}}-1}{2\sqrt{1+8\gamma_{\perp}^{2}}}\delta(\omega\mp\omega_{2})\Bigg],\end{aligned} \tag{8.4.16}$$

$$\Im G_{2,2}(\boldsymbol{k}_{\|},\pm\omega)=-\frac{\pi S}{4}\left[\frac{\sqrt{1+8\gamma_{\perp}^{2}}-1}{\sqrt{1+8\gamma_{\perp}^{2}}}\delta(\omega\mp\omega_{0})+\frac{\sqrt{1+8\gamma_{\perp}^{2}}+1}{\sqrt{1+8\gamma_{\perp}^{2}}}\delta(\omega\mp\omega_{2})\right]. \tag{8.4.17}$$

The neutron cross-section is given by (8.2.27) with $N = 3$ and

$$I_0(K_z) = \frac{S}{3\hbar}\cos^4\frac{K_z}{2}, \tag{8.4.18}$$

$$I_1(K_z) = \frac{S}{4\hbar}\sin^2 K_z, \tag{8.4.19}$$

$$I_2(K_z) = \frac{S}{3\hbar}\sin^4\frac{K_z}{2} \tag{8.4.20}$$

and

$$\omega_l = \frac{4z_\perp J_\perp S}{\hbar}\left(1 - \cos\frac{\pi l}{3}\right) \tag{8.4.21}$$

with $l = 0, 1, 2$.

8.5. Classical Spin Waves in Multilayers

In order to get information about the spin dynamics in multilayers, we make use of the classical mechanics assuming the spins to be classical vectors. According to Section 1.6, the classical equations of motion for the spin components are given by the torque equations (1.6.5)–(1.6.7) the solution of which may be written

$$S^x_{j,n} = u_n \cos(\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_j - \omega t), \tag{8.5.1}$$

$$S^y_{j,n} = u_n \sin(\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_j - \omega t) \tag{8.5.2}$$

and

$$S^z_{j,n} = \sqrt{S^2 - u_n^2}, \tag{8.5.3}$$

where $\boldsymbol{r}_j$ is a 2D lattice vector in the layer and n labels the N layers. As one can see from Eqs. (8.5.1)–(8.5.3), each spin of the film rotates around the z-axis sweeping the surface of a cone of apex angle $\alpha_n = \arcsin\frac{u_n}{S} \simeq \frac{u_n}{S}$. While in the ferromagnetic "bulk" of Section 1.6, the apex angle α was the same at each lattice site, in the ferromagnetic "multilayer" of the present section the apex angle is a function of the number n of the layer. In this way, the torque equation (1.6.4) leads to N torque equations, one for each layer, and a homogeneous linear system of equations for the amplitudes u_n is obtained. The matrix form of such a system is

$$\mathbf{A}_N \times \mathbf{U}_N = 0 \tag{8.5.4}$$

where the square matrix $\mathbf{A}_N$ of dimension N is given by Eq. (8.2.11) and $\mathbf{U}_N$ is a vector whose components are the amplitude u_n with $n = 1, 2, \ldots, N$, proportional to the apex angles of the cones in each plane. The frequency occurring in Eqs. (8.5.1) and (8.5.2) is the same that appears in Eq. (8.2.9). The values of θ that satisfy the determinant equation $\det \mathbf{A}_N = 0$ give the frequencies of the normal modes of the film. Because of the nature of the matrix $\mathbf{A}_N$, the vector $\mathbf{U}_N$ in Eq. (8.5.4) has a defined parity: its components are symmetric or antisymmetric with respect the centre of the multilayer. This fact allows us to reduce the number of independent equations of the linear system. Let us distinguish between the multilayers with an even or odd number of layers.

For N *even*, the symmetric vector $\mathbf{U}_N^{(\mathrm{s})}$ has components satisfying the relationship $u_{\frac{N}{2}+l} = u_{\frac{N}{2}-l+1}$ with $l = 1, 2, \ldots, \frac{N}{2}$. Then Eq. (8.5.4) reduces to

$$\left[\frac{1}{\gamma_\perp} - 2\cos\theta^{(\mathrm{s})} + 1\right] u_1 = 0 \tag{8.5.5}$$

for $N = 2$ and

$$\mathbf{A}_{\frac{N}{2}}^{(\mathrm{s})} \times \mathbf{V}_{\frac{N}{2}} = 0 \tag{8.5.6}$$

for $N = 4$, 6, 8, etc. where

$$\mathbf{A}_{\frac{N}{2}}^{(\mathrm{s})} = \begin{pmatrix} \frac{1}{\gamma_\perp} - 2\cos\theta & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2\cos\theta & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2\cos\theta & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & (1 - 2\cos\theta) \end{pmatrix} \tag{8.5.7}$$

is a square matrix of dimension $\frac{N}{2}$ and $\mathbf{V}_{\frac{N}{2}}$ is a vector of components u_1, u_2, $\ldots$, $u_{\frac{N}{2}}$. The determinant of the matrix (8.5.7) can be evaluated in a way similar to that of Section 8.2. One obtains

$$\begin{aligned} \det \mathbf{A}_{\frac{N}{2}}^{(\mathrm{s})} &= \frac{(-1)^{\frac{N}{2}}}{\gamma_\perp} \left[\left(\gamma_\perp - \frac{D_{\frac{N}{2}-1}}{D_{\frac{N}{2}}}\right)\left(D_{\frac{N}{2}} - D_{\frac{N}{2}-1}\right) - \frac{1}{D_{\frac{N}{2}}}\right] \\ &= \frac{(-1)^{\frac{N}{2}}}{\gamma_\perp \cos\frac{\theta^{(\mathrm{s})}}{2}} \left(\gamma_\perp \cos\frac{N+1}{2}\theta^{(\mathrm{s})} - \cos\frac{N-1}{2}\theta^{(\mathrm{s})}\right). \end{aligned} \tag{8.5.8}$$

The antisymmetric solution of Eq. (8.5.4) corresponds to a vector $\mathbf{U}_N^{(\mathrm{a})}$ whose components satisfy the relationship $u_{\frac{N}{2}+l} = -u_{\frac{N}{2}-l+1}$ with $l = 1, 2, \ldots, \frac{N}{2}$. Then Eq. (8.5.4) reduces to

$$\left[\frac{1}{\gamma_\perp} - 2\cos\theta^{(\mathrm{a})} - 1\right] u_1 = 0 \tag{8.5.9}$$

for $N = 2$ and

$$\mathbf{A}_{\frac{N}{2}}^{(\mathrm{a})} \times \mathbf{V}_{\frac{N}{2}} = 0 \tag{8.5.10}$$

for $N = 4$, 6, 8, etc. where

$$\mathbf{A}_{\frac{N}{2}}^{(\mathrm{a})} = \begin{pmatrix} \frac{1}{\gamma_\perp} - 2\cos\theta & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2\cos\theta & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2\cos\theta & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -(1 + 2\cos\theta) \end{pmatrix}. \tag{8.5.11}$$

The determinant of the matrix (8.5.11) is given by

$$\det \mathbf{A}^{(a)}_{\frac{N}{2}} = \frac{(-1)^{\frac{N}{2}}}{\gamma_\perp}\left[\left(\gamma_\perp - \frac{D_{\frac{N}{2}-1}}{D_{\frac{N}{2}}}\right)\left(D_{\frac{N}{2}} - D_{\frac{N}{2}-1}\right) + \frac{1}{D_{\frac{N}{2}}}\right]$$
$$= \frac{(-1)^{\frac{N}{2}}}{\gamma_\perp \sin\frac{\theta^{(a)}}{2}}\left(\gamma_\perp \sin\frac{N+1}{2}\theta^{(a)} - \sin\frac{N-1}{2}\theta^{(a)}\right). \qquad (8.5.12)$$

For N *odd*, the symmetric vector $\mathbf{U}^{(s)}_N$ has components satisfying the relationship $u_{\frac{N+1}{2}+l} = u_{\frac{N+1}{2}-l}$ with $l = 1, 2, \ldots, \frac{N-1}{2}$. Then, Eq. (8.5.4) reduces to

$$\mathbf{A}^{(s)}_{\frac{N+1}{2}} \times \mathbf{V}_{\frac{N+1}{2}} = 0. \qquad (8.5.13)$$

where

$$\mathbf{A}^{(s)}_{\frac{N+1}{2}} = \begin{pmatrix} \frac{1}{\gamma_\perp} - 2\cos\theta & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2\cos\theta & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2\cos\theta & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -2\cos\theta \end{pmatrix} \qquad (8.5.14)$$

is a square matrix of dimension $\frac{N+1}{2}$ and $\mathbf{V}_{\frac{N+1}{2}}$ is a vector of components u_1, $u_2, \ldots, u_{\frac{N+1}{2}}$. The determinant of the matrix (8.5.14) is given by

$$\det \mathbf{A}^{(s)}_{\frac{N+1}{2}} = \frac{(-1)^{\frac{N+1}{2}}}{\gamma_\perp}\left[\left(\gamma_\perp - \frac{D_{\frac{N-1}{2}}}{D_{\frac{N+1}{2}}}\right)\left(D_{\frac{N+1}{2}} - D_{\frac{N-3}{2}}\right) - \frac{2\cos\theta^{(s)}}{D_{\frac{N+1}{2}}}\right]$$
$$= (-1)^{\frac{N+1}{2}}\frac{2}{\gamma_\perp}\left(\gamma_\perp \cos\frac{N+1}{2}\theta^{(s)} - \cos\frac{N-1}{2}\theta^{(s)}\right). \qquad (8.5.15)$$

The antisymmetric vector $\mathbf{U}^{(a)}_N$ has components satisfying the relationship $u_{\frac{N+1}{2}+l} = -u_{\frac{N+1}{2}-l}$ with $l = 1, 2, \ldots, \frac{N-1}{2}$ and $u_{\frac{N+1}{2}} = 0$. Then Eq. (8.5.4) reduces to Eq. (8.5.9) for $N = 3$ and to

$$\mathbf{A}^{(a)}_{\frac{N-1}{2}} \times \mathbf{V}_{\frac{N-1}{2}} = 0 \qquad (8.5.16)$$

for $N = 5$, 7, 9, etc., where

$$\mathbf{A}^{(a)}_{\frac{N-1}{2}} = \begin{pmatrix} \frac{1}{\gamma_\perp} - 2\cos\theta & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2\cos\theta & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2\cos\theta & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -2\cos\theta \end{pmatrix} \qquad (8.5.17)$$

is a square matrix of dimension $\frac{N-1}{2}$ and $\mathbf{V}_{\frac{N-1}{2}}$ is a vector of components u_1, $u_2, \ldots, u_{\frac{N-1}{2}}$. The determinant of the matrix (8.5.17) is given by

$$\det \mathbf{A}^{(\mathrm{a})}_{\frac{N-1}{2}} = \frac{(-1)^{\frac{N-1}{2}}}{\gamma_\perp}\left(\gamma_\perp D_{\frac{N-1}{2}} - D_{\frac{N-3}{2}}\right)$$
$$= \frac{(-1)^{\frac{N}{2}}}{\gamma_\perp \sin\theta^{(\mathrm{a})}}\left(\gamma_\perp \sin\frac{N+1}{2}\theta^{(\mathrm{a})} - \sin\frac{N-1}{2}\theta^{(\mathrm{a})}\right). \tag{8.5.18}$$

The frequencies of the normal modes of the film are obtained from Eq. (8.2.9) where θ are the solutions of the determinant equation $\det \mathbf{A} = 0$. Note that Eqs. (8.5.8) and (8.5.15) lead to the same symmetric spin wave frequencies. The same occurs for the antisymmetric spin wave frequencies of Eqs. (8.5.12) and (8.5.18).

Let us perform explicit calculations for small N. For $N = 2$ (bilayer), Eqs. (8.5.5) and (8.2.9) give

$$\omega^{(\mathrm{s})} = \omega_\parallel + \frac{2z_\perp J_\perp S}{\hbar}(1-\gamma_\perp), \quad u_1 = u_2 = u \tag{8.5.19}$$

for the symmetric normal mode while Eqs. (8.5.9) and (8.2.9) give

$$\omega^{(\mathrm{a})} = \omega_\parallel + \frac{2z_\perp J_\perp S}{\hbar}(1+\gamma_\perp), \quad u_1 = -u_2 = u \tag{8.5.20}$$

for the antisymmetric normal mode. The arbitrary constant u appearing in Eqs. (8.5.19) and (8.5.20) can be fixed only giving the initial conditions. A snapshot of the spin configurations for the bilayer is shown in Fig. 8.1 where the index $n = 1, 2$ labels the two layers of the film and $\omega^{(\mathrm{s})}$ and $\omega^{(\mathrm{a})}$ indicate the normal mode frequencies corresponding to the symmetric and antisymmetric configurations. The apex angle of the cone is given by $\alpha = \arccos\frac{u}{S}$ and it is the same for each spin. The spins rotate about the z-axis with frequency $\omega^{(\mathrm{s})}$ or $\omega^{(\mathrm{a})}$ according to Eqs. (8.5.1)–(8.5.3). When the symmetric normal mode is excited, the spins in the two planes move in phase. When the antisymmetric normal mode is excited, the spins in the two planes move out of phase of an angle π. No qualitative difference exists between the spin waves in a bilayer and the ferromagnetic spin waves studied in Section 1.6 for a 3D lattice with PBC (see Eqs. (1.6.8)–(1.6.11)) except that in the bilayer, the allowed values of k_z, where the z-axis is chosen perpendicular to the film, have to be restricted to $k_z = 0$ and $k_z = \pi$.

For $N = 3$ (trilayer), the frequencies of the symmetric normal modes can be obtained from Eq. (8.5.15) that gives

$$2\gamma_\perp \cos^2\theta^{(\mathrm{s})} - \cos\theta^{(\mathrm{s})} - \gamma_\perp = 0. \tag{8.5.21}$$

The two solutions of Eq. (8.5.21) lead to the frequencies

$$\omega_1^{(\mathrm{s})} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 - \sqrt{1+8\gamma_\perp^2}\right) \tag{8.5.22}$$

with

$$u_1 = u_3 = u, \quad u_2 = \frac{\sqrt{1+8\gamma_\perp^2}-1}{2\gamma_\perp}u \tag{8.5.23}$$

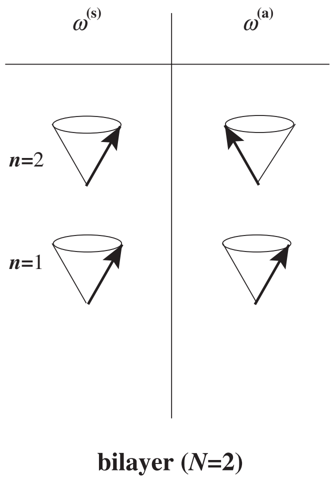

Fig. 8.1. Spin configuration of a bilayer.

and

$$\omega_2^{(\mathrm{s})} = \omega_{\|} + \frac{z_{\perp} J_{\perp} S}{\hbar}\left(3 + \sqrt{1 + 8\gamma_{\perp}^2}\right) \tag{8.5.24}$$

with

$$u_1 = u_3 = u, \quad u_2 = -\frac{\sqrt{1 + 8\gamma_{\perp}^2} + 1}{2\gamma_{\perp}} u. \tag{8.5.25}$$

Since $-1 \leq \gamma_{\perp} \leq 1$, in Eq. (8.5.23) $|u_2/u| \leq 1$ so that for this mode the apex angle of the cone $\alpha_n = \arccos \frac{u_n}{S}$ *decreases* going from the surface $n = 1$ (or $n = 3$) to the internal layer $n = 2$: for this reason the corresponding normal mode is called[77] "monotonic surface magnon". On the contrary, in Eq. (8.5.25) $|u_2/u| \geq 1$ so that in this case the apex angle *increases* going from the surface to the centre of the film and the corresponding mode is called "bulk magnon". Equations (8.5.9) and (8.2.9) give the frequency of the antisymmetric normal mode

$$\omega^{(\mathrm{a})} = \omega_{\|} + \frac{2z_{\perp} J_{\perp} S}{\hbar}(1 + \gamma_{\perp}), \quad u_1 = -u_3 = u, \quad u_2 = 0. \tag{8.5.26}$$

The apex angle of the cone is the same on the two surfaces and is zero on the layer at the centre of the film: however, the spins on the two surfaces are out of phase of π. A snapshot of the spin configurations for the trilayer is shown in Fig. 8.2 where $n = 1, 2, 3$ labels the three layers of the film and $\omega_1^{(\mathrm{s})}$, $\omega_2^{(\mathrm{s})}$ and $\omega^{(\mathrm{a})}$ refer to the two symmetric and to the antisymmetric normal modes, respectively. In the trilayer, we have two monotonic surface magnons (one symmetric and one antisymmetric) and one (symmetric) bulk magnon. Note that the spin waves differ qualitatively from those of the 3D systems since the apex angle α_n of the cones swept by the spins changes from one layer to the other. For $\boldsymbol{k}_{\|} = 0$ ($\gamma_{\perp} = 1$), Eqs. (8.5.22) and (8.5.24)

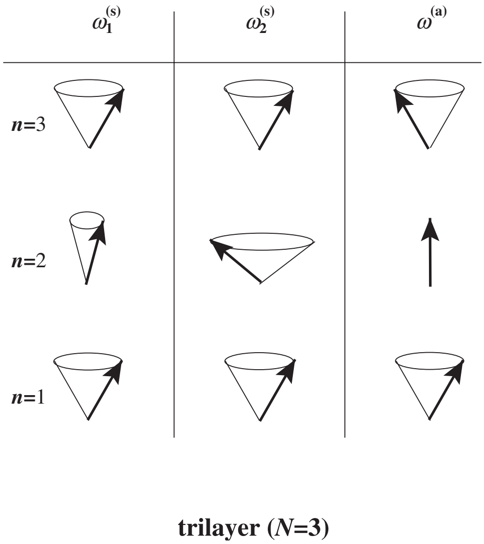

Fig. 8.2. Spin configuration of a trilayer.

and (8.5.26) reduce to

$$\omega_1^{(s)}(0) = 0, \quad u_1 = u_2 = u_3 = u, \tag{8.5.27}$$

$$\omega_2^{(s)}(0) = \frac{6z_\perp J_\perp S}{\hbar}, \quad u_1 = u_3 = u, \quad u_2 = -2u \tag{8.5.28}$$

and

$$\omega_1^{(a)}(0) = \frac{4z_\perp J_\perp S}{\hbar}, \quad u_1 = -u_3 = u, \quad u_2 = 0. \tag{8.5.29}$$

For $N = 4$ (four-layer), the frequencies of the symmetric normal modes can be obtained from Eq. (8.5.8) that gives

$$4\gamma_\perp \cos^2\theta^{(s)} - 2(1+\gamma_\perp)\cos\theta^{(s)} + 1 - \gamma_\perp = 0. \tag{8.5.30}$$

The two solutions of Eqs. (8.5.30) and (8.2.9) give the frequencies of the two symmetric spin waves

$$\omega_1^{(s)} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 - \gamma_\perp - \sqrt{(1-\gamma_\perp)^2 + 4\gamma_\perp^2}\right) \tag{8.5.31}$$

with

$$u_1 = u_4 = u, \quad u_2 = u_3 = \frac{\sqrt{(1-\gamma_\perp)^2 + 4\gamma_\perp^2} - (1-\gamma_\perp)}{2\gamma_\perp} u \tag{8.5.32}$$

and

$$\omega_2^{(s)} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 - \gamma_\perp + \sqrt{(1-\gamma_\perp)^2 + 4\gamma_\perp^2}\right) \tag{8.5.33}$$

with

$$u_1 = u_4 = u, \quad u_2 = u_3 = -\frac{\sqrt{(1-\gamma_\perp)^2 + 4\gamma_\perp^2} + (1-\gamma_\perp)}{2\gamma_\perp} u. \tag{8.5.34}$$

The frequencies of the antisymmetric normal modes are obtained from Eq. (8.5.12) that becomes

$$4\gamma_\perp \cos^2\theta^{(\mathrm{a})} - 2(1-\gamma_\perp)\cos\theta^{(\mathrm{a})} - (1+\gamma_\perp) = 0 \tag{8.5.35}$$

so that the two frequencies of the antisymmetric normal modes are

$$\omega_1^{(\mathrm{a})} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 + \gamma_\perp - \sqrt{(1+\gamma_\perp)^2 + 4\gamma_\perp^2}\right) \tag{8.5.36}$$

with

$$u_1 = -u_4 = u, \quad u_2 = -u_3 = \frac{\sqrt{(1+\gamma_\perp)^2 + 4\gamma_\perp^2} - (1+\gamma_\perp)}{2\gamma_\perp} u \tag{8.5.37}$$

and

$$\omega_2^{(\mathrm{a})} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 + \gamma_\perp + \sqrt{(1+\gamma_\perp)^2 + 4\gamma_\perp^2}\right) \tag{8.5.38}$$

with

$$u_1 = -u_4 = u, \quad u_2 = -u_3 = -\frac{\sqrt{(1+\gamma_\perp)^2 + 4\gamma_\perp^2} + (1+\gamma_\perp)}{2\gamma_\perp} u. \tag{8.5.39}$$

The spin configuration of the four-layer is shown in Fig. 8.3. Since $-1 \le \gamma_\perp \le 1$, in Eqs. (8.5.32) and (8.5.37) one has $|u_2/u| \le 1$ and in Eqs. (8.5.34) and (8.5.39) one has $|u_2/u| \ge 1$ so that also for $N = 4$ we have two monotonic surface modes

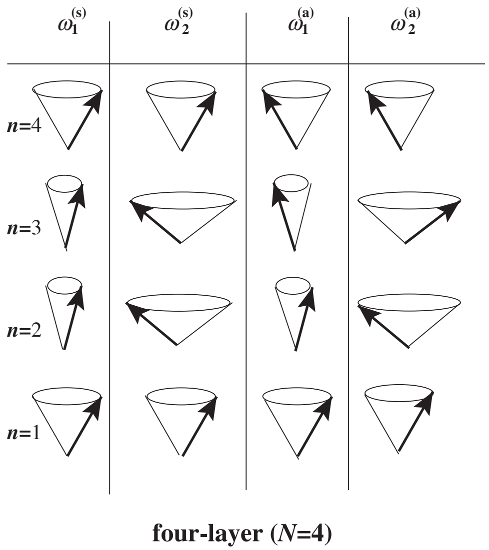

Fig. 8.3. Spin configuration of a four-layer.

(one symmetric and one antisymmetric) and two bulk magnons. Note that the frequencies of the surface magnons $\omega_1^{(s)}$ and $\omega_1^{(a)}$ are lower than the frequencies of the bulk magnons $\omega_2^{(s)}$ and $\omega_2^{(a)}$. For $\boldsymbol{k}_\parallel = 0$, Eqs. (8.5.31)–(8.5.34) and (8.5.36)–(8.5.39) reduce to

$$\omega_1^{(s)}(0) = 0, \quad u_1 = u_2 = u_3 = u_4 = u, \tag{8.5.40}$$

$$\omega_2^{(s)}(0) = \frac{4z_\perp J_\perp S}{\hbar}, \quad u_1 = u_4 = u, \quad u_2 = u_3 = -u, \tag{8.5.41}$$

$$\omega_1^{(a)}(0) = \frac{2z_\perp J_\perp S}{\hbar}(2 - \sqrt{2}), \quad u_1 = -u_4 = u, \quad u_2 = -u_3 = (\sqrt{2} - 1)u \tag{8.5.42}$$

and

$$\omega_2^{(a)}(0) = \frac{2z_\perp J_\perp S}{\hbar}(2 + \sqrt{2}), \quad u_1 = -u_4 = u, \quad u_2 = -u_3 = -(\sqrt{2} + 1)u. \tag{8.5.43}$$

For $N = 5$ (five-layer), the frequencies of the symmetric normal modes can be obtained from Eq. (8.5.15) that becomes

$$4\gamma_\perp \cos^3\theta^{(s)} - 2\cos^2\theta^{(s)} - 3\gamma_\perp \cos\theta^{(s)} + 1 = 0. \tag{8.5.44}$$

The solutions of the cubic equation (8.5.44) cannot be given in analytic form for a generic $\boldsymbol{k}_\parallel$ wavevector. Anyway, the numerical solution for several values of $\boldsymbol{k}_\parallel$ shows that the lowest frequency, say $\omega_1^{(s)}$, leads to a monotonic surface mode. The other two higher frequencies $\omega_2^{(s)}$ and $\omega_3^{(s)}$ correspond to bulk magnons. The frequencies of the antisymmetric normal modes are obtained from Eq. (8.5.18) that leads to

$$4\gamma_\perp \cos^2\theta^{(a)} - 2\cos\theta^{(a)} - \gamma_\perp = 0 \tag{8.5.45}$$

so that the two antisymmetric spin wave frequencies are

$$\omega_1^{(a)} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 - \sqrt{1 + 4\gamma_\perp^2}\right) \tag{8.5.46}$$

with

$$u_1 = -u_5 = u, \quad u_2 = -u_4 = \frac{\sqrt{1 + 4\gamma_\perp^2} - 1}{2\gamma_\perp}u, \quad u_3 = 0 \tag{8.5.47}$$

and

$$\omega_2^{(a)} = \omega_\parallel + \frac{z_\perp J_\perp S}{\hbar}\left(3 + \sqrt{1 + 4\gamma_\perp^2}\right) \tag{8.5.48}$$

with

$$u_1 = -u_5 = u, \quad u_2 = -u_4 = -\frac{\sqrt{1 + 4\gamma_\perp^2} + 1}{2\gamma_\perp}u, \quad u_3 = 0. \tag{8.5.49}$$

Since $-1 \le \gamma_\perp \le 1$, in Eq. (8.5.47) one has $|u_2/u| \le 1$ so that the frequency $\omega_1^{(a)}$ corresponds to a monotonic surface magnon while in Eq. (8.5.49) one has $|u_2/u| \ge 1$

so that the frequency $\omega_2^{(a)}$ corresponds to a bulk magnon. For $\boldsymbol{k}_\parallel = 0$, Eq. (8.5.44) gives the solutions

$$\omega_1^{(s)}(0) = 0, \quad u_1 = u_2 = u_3 = u, \tag{8.5.50}$$

$$\omega_2^{(s)}(0) = \frac{z_\perp J_\perp S}{\hbar}(5-\sqrt{5}), \quad u_1 = u, \quad u_2 = -\frac{3-\sqrt{5}}{2}u, \quad u_3 = -(\sqrt{5}-1)u \tag{8.5.51}$$

and

$$\omega_3^{(s)}(0) = \frac{z_\perp J_\perp S}{\hbar}(5+\sqrt{5}), \quad u_1 = u, \quad u_2 = -\frac{3+\sqrt{5}}{2}u, \quad u_3 = (\sqrt{5}+1)u, \tag{8.5.52}$$

while Eqs. (8.5.46)–(8.5.49) reduce to

$$\omega_1^{(a)}(0) = \frac{z_\perp J_\perp S}{\hbar}(3-\sqrt{5}), \quad u_1 = u, \quad u_2 = \frac{\sqrt{5}-1}{2}u, \quad u_3 = 0 \tag{8.5.53}$$

and

$$\omega_2^{(a)} = \frac{z_\perp J_\perp S}{\hbar}(3+\sqrt{5}), \quad u_1 = u, \quad u_2 = -\frac{\sqrt{5}+1}{2}u, \quad u_3 = 0. \tag{8.5.54}$$

For a BCC lattice with the wavevector $\boldsymbol{k}_\parallel$ along the $(1,1,0)$ direction, one has $\gamma_\parallel = \cos k_\parallel$ and $\gamma_\perp = \cos^2 \frac{k_\parallel}{2}$. The spin wave frequencies are shown in the left panel of Fig. 8.4, assuming $J_\perp = J_\parallel$. There are three symmetric (s) spin waves frequencies (continuous curves) and two antisymmetric (a) spin waves frequencies (dashed curves). The lowest energy branches (s,1) and (a,1) are monotonic surface magnons. All other branches correspond to bulk spin waves. For $N = 6$ we obtain similar results. The spin wave frequencies are obtained from Eq. (8.5.8) (symmetric normal modes) and from Eq. (8.5.12) (antisymmetric normal modes). For a BCC lattice with the wavevector $\boldsymbol{k}_\parallel$ along the direction (1,1,0), the spin wave frequencies are shown in the right panel of Fig. 8.4 for $J_\perp = J_\parallel$. The symmetric normal mode frequencies are given by continuous curves and the antisymmetric normal mode frequencies are given by dashed curves. The lowest two branches (s,1) and (a,1) are monotonic surface magnons. For $\boldsymbol{k}_\parallel = 0$, the symmetric spin wave frequencies become

$$\omega_1^{(s)}(0) = 0, \quad u_1 = u_2 = u_3 = u, \tag{8.5.55}$$

$$\omega_2^{(s)}(0) = \frac{2z_\perp J_\perp S}{\hbar}, \quad u_1 = -u_3 = u, \quad u_2 = 0 \tag{8.5.56}$$

and

$$\omega_3^{(s)}(0) = \frac{3z_\perp J_\perp S}{\hbar}, \quad u_1 = u_3 = u, \quad u_2 = -2u. \tag{8.5.57}$$

The antisymmetric spin wave frequencies are

$$\omega_1^{(a)} = \frac{2z_\perp J_\perp S}{\hbar}(2-\sqrt{3}), \quad u_1 = u, \quad u_2 = (\sqrt{3}-1)u, \quad u_3 = (2-\sqrt{3})u, \tag{8.5.58}$$

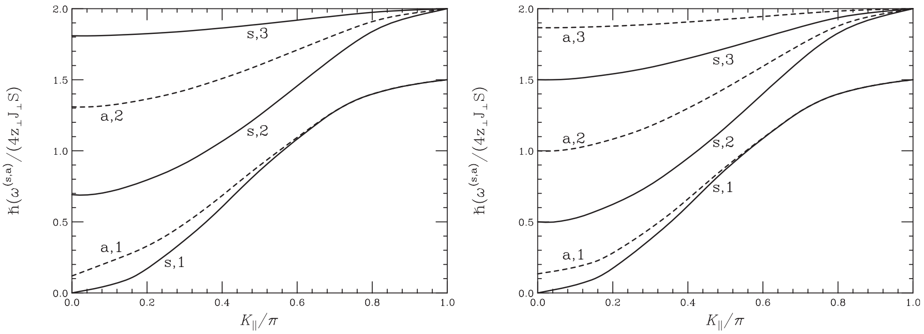

Fig. 8.4. Spin wave frequency in a film made up of five layers (left panel) and six layers (right panel) of a BCC structure with $J_\perp = J_\parallel$. The wavevector $\boldsymbol{k}_\parallel$ is along the (1,1,0) direction. The labels (s) and (a) indicate the symmetric (continuous curves) and antisymmetric (dashed curves) normal modes, respectively.

$$\omega_2^{(a)}(0) = \frac{4z_\perp J_\perp S}{\hbar}, \quad u_1 = -u_2 = -u_3 = u \tag{8.5.59}$$

and

$$\omega_3^{(a)}(0) = \frac{2z_\perp J_\perp S}{\hbar}(2+\sqrt{3}), \quad u_1 = u, \quad u_2 = -(\sqrt{3}+1)u, \quad u_3 = (2+\sqrt{3})u. \tag{8.5.60}$$

The existence of a soft mode at $\boldsymbol{k}_\parallel = 0$ for any film thickness reflects the rotation invariance of the Heisenberg Hamiltonian. This soft mode disappears, introducing a surface anisotropy as could be the case when the film is grown over a substrate of another material.

Now, we are able to investigate the spin wave frequencies for a film made up of an arbitrary number of layers. For N even the spin wave frequencies are given from Eq. (8.2.9) where θ is obtained from Eq. (8.5.8) in the symmetric case and from Eq. (8.5.12) in the antisymmetric case. In particular, the lowest symmetric spin wave frequency is obtained from Eq. (8.2.9) where $\theta_1^{(s)}$ is the solution of the equation

$$\frac{\gamma_\perp \cosh \frac{N+1}{2}\theta_1^{(s)} - \cosh \frac{N-1}{2}\theta_1^{(s)}}{\cosh \frac{\theta_1^{(s)}}{2}} = 0. \tag{8.5.61}$$

The remaining $\frac{N}{2} - 1$ symmetric spin wave frequencies are still obtained from Eq. (8.2.9) where $\theta_l^{(s)}$ are the solutions of the equation

$$\frac{\gamma_\perp \cos \frac{N+1}{2}\theta_l^{(s)} - \cos \frac{N-1}{2}\theta_l^{(s)}}{\cos \frac{\theta_l^{(s)}}{2}} = 0. \tag{8.5.62}$$

In the antisymmetric case, one has to distinguish between the case

i) $\frac{N-1}{N+1} < \gamma_\perp < 1$ corresponding to magnons whose wavevectors belong to a region around the zone centre ($0 < k_\parallel < 2\arccos\sqrt{\frac{N-1}{N+1}}$ for a BCC lattice along (1,1,0) direction) and the case

ii) $\gamma_\perp < \frac{N-1}{N+1}$ corresponding to magnons whose wavevectors are in a region around the ZC ($2\arccos\sqrt{\frac{N-1}{N+1}} < k_\parallel < \pi$ for a BCC lattice along (1,1,0) direction).

In the case i), the $\frac{N}{2}$ antisymmetric spin wave frequencies are obtained from Eq. (8.2.9) where $\theta_l^{(\mathrm{a})}$ are the solutions of the equation

$$\frac{\gamma_\perp \sin\frac{N+1}{2}\theta_l^{(\mathrm{a})} - \sin\frac{N-1}{2}\theta_l^{(\mathrm{a})}}{\sin\frac{\theta_l^{(\mathrm{a})}}{2}} = 0. \tag{8.5.63}$$

In the case ii), $\frac{N}{2}-1$ spin wave frequencies are still given by Eq. (8.5.63); however, the remaining (lowest) frequency is obtained from Eq. (8.2.9) where $\theta_1^{(\mathrm{a})}$ is the solution of the equation

$$\frac{\gamma_\perp \sinh\frac{N+1}{2}\theta_1^{(\mathrm{a})} - \sinh\frac{N-1}{2}\theta_1^{(\mathrm{a})}}{\sinh\frac{\theta_1^{(\mathrm{a})}}{2}} = 0. \tag{8.5.64}$$

The existence of two distinct cases originates from the nature of the function in the left-hand side of Eq. (8.5.64). Indeed, it is an increasing monotonic function of θ whose value at $\theta = 0$ is positive in the case i) and negative in the case ii) implying that Eq. (8.5.64) has no solution in the former case and one solution in the latter case. It is worthwhile noticing that the eigenvectors belonging to $\theta_1^{(\mathrm{s})}$ and $\theta_1^{(\mathrm{a})}$ have components whose magnitude decreases going from $n = 1$ (surface) to the centre of the film $n = \frac{N}{2}$ so that these two modes are referred as[77] "monotonic surface magnons" in contrast with the remaining $N-2$ symmetric and antisymmetric modes that are classified as "bulk magnons".

For N odd, the symmetric and antisymmetric spin wave frequencies are obtained from Eqs. (8.5.15) and (8.5.18), respectively. As for the symmetric case, $\theta_1^{(\mathrm{s})}$ is the solution of the equation

$$\gamma_\perp \cosh\frac{N+1}{2}\theta_1^{(\mathrm{s})} - \cosh\frac{N-1}{2}\theta_1^{(\mathrm{s})} = 0 \tag{8.5.65}$$

and the remaining $\frac{N-1}{2}$ spin wave frequencies are given by Eq. (8.2.9) where $\theta_l^{(\mathrm{s})}$ are the solutions of the equation

$$\gamma_\perp \cos\frac{N+1}{2}\theta_l^{(\mathrm{s})} - \cos\frac{N-1}{2}\theta_l^{(\mathrm{s})} = 0. \tag{8.5.66}$$

As before, in the antisymmetric case one has to distinguish between the two cases

i) $\frac{N-1}{N+1} < \gamma_\perp < 1$ and

ii) $\gamma_\perp < \frac{N-1}{N+1}$.

In the case i), all the $\frac{N-1}{2}$ antisymmetric spin wave frequencies are obtained from the $\theta_l^{(a)}$ that satisfy the equation

$$\frac{\gamma_\perp \sin \frac{N+1}{2}\theta_l^{(a)} - \sin \frac{N-1}{2}\theta_l^{(a)}}{\sin \theta_l^{(a)}} = 0. \tag{8.5.67}$$

In the case ii), $\frac{N-3}{2}$ spin wave frequencies are still obtained from Eq. (8.5.67) while the remaining (lowest) frequency $\theta_1^{(a)}$ is obtained from the equation

$$\frac{\gamma_\perp \sinh \frac{N+1}{2}\theta_1^{(a)} - \sinh \frac{N-1}{2}\theta_1^{(a)}}{\sinh \theta_1^{(a)}} = 0. \tag{8.5.68}$$

Also, for N odd, the eigenvectors belonging to $\theta_1^{(s)}$ and $\theta_1^{(a)}$ have components decreasing with n going from $n = 1$ (surface) to the centre of the film $n = \frac{N+1}{2}$ so that the existence of two "monotonic surface magnons" is a peculiarity of a BCC film with an arbitrary number of layers. For $\boldsymbol{k}_\parallel = 0$ ($\gamma_\perp = \gamma_\parallel = 1$), the spin wave frequencies are given by

$$\omega_l(0) = \frac{4z_\perp J_\perp S}{\hbar}(1 - \cos\theta_l) \tag{8.5.69}$$

with $\theta_l = \theta_l^{(s)} = \frac{2\pi l}{N}$ for the symmetric modes and $\theta_l = \theta_l^{(a)} = \frac{(2l+1)\pi}{N}$ for the antisymmetric modes. The soft mode corresponding to $l = 0$ in Eq. (8.5.69) leads to an eigenvector with components given by $u_n = u$: in this particular case, the monotonic surface magnon reduces to a conventional ferromagnetic spin wave with all spins parallel along a direction forming an angle $\alpha = \arccos \frac{u}{S}$ with the z-axis.

For SC and H lattices ($z_\perp = \gamma_\perp = 1$), an analytic result can be achieved. The symmetric spin wave frequencies are given by

$$\omega_l^{(s)} = \omega_\parallel + \frac{4J_\perp S}{\hbar}\left(1 - \cos\frac{2\pi l}{N}\right) \tag{8.5.70}$$

and the corresponding eigenvectors components u_n^l are given by

$$u_n^l = u\ \frac{\cos\left[\frac{\pi l}{N}(N + 1 - 2n)\right]}{\cos\frac{\pi l}{N}} \tag{8.5.71}$$

for N even, where $l = 0, 1, \ldots, \frac{N}{2} - 1$ labels the $\frac{N}{2}$ symmetric normal modes and $u_{\frac{N}{2}+n} = u_{\frac{N}{2}-n+1}$. For N odd, the frequencies are still given by Eq. (8.5.70) while the eigenvectors components are given by

$$u_n^l = u\ \cos\left[\frac{\pi l}{N}(N + 1 - 2n)\right] \tag{8.5.72}$$

with $l = 0, 1, \ldots, \frac{N-1}{2}$ and $n = 1, 2, \ldots, \frac{N+1}{2}$.

The antisymmetric spin wave frequencies are given by

$$\omega_l^{(\mathrm{a})} = \omega_{\|} + \frac{4J_\perp S}{\hbar}\left[1 - \cos\frac{(2l+1)\pi}{N}\right] \tag{8.5.73}$$

and the corresponding eigenvectors components are given by

$$u_n^l = u\,\frac{\sin\left[\left(l+\frac{1}{2}\right)\frac{\pi}{N}(N+1-2n)\right]}{\sin\left(l+\frac{1}{2}\right)\frac{\pi}{N}} \tag{8.5.74}$$

for N even, with $l = 0, 1, \ldots, \frac{N}{2} - 1$, $n = 1, 2, \ldots, \frac{N}{2}$ and $u_{\frac{N}{2}+n} = -u_{\frac{N}{2}-n+1}$. For N odd, the antisymmetric spin wave frequencies are still given by Eq. (8.5.73) while the eigenvectors components are given by

$$u_n^l = u\,\frac{\sin\left[\left(l+\frac{1}{2}\right)\frac{\pi}{N}(N+1-2n)\right]}{\sin\frac{(2l+1)\pi}{N}} \tag{8.5.75}$$

with $n = 1, \ldots, \frac{N-1}{2}$ and $l = 0, 1, \ldots, \frac{N-3}{2}$. Obviously, the remaining components are given by $u_{\frac{N+1}{2}+n} = -u_{\frac{N+1}{2}-n}$ and $u_{\frac{N+1}{2}} = 0$.

8.6. Classical Spin Waves in a Semi-Infinite Medium

In order to investigate how the presence of a surface modifies the spin waves of a 3D lattice, we study a model made up of a semi-infinite stacking of layers.[78] For a semi-infinite medium we can write a matrix equation like (8.5.4) where **A** and **U** are now an *infinite* square matrix and a vector with an infinite number of components, respectively. In particular, the linear system for the amplitudes u_n becomes

$$(1 - 2\lambda\gamma_\perp)u_1 + \gamma_\perp u_2 = 0 \tag{8.6.1}$$

for the surface ($n = 1$) with

$$\omega = 2z_{\|}J_{\|}S(1-\gamma_{\|}) + 4z_\perp J_\perp S(1 - \lambda\gamma_\perp) \tag{8.6.2}$$

and

$$-2\lambda\gamma_\perp u_n + u_{n-1} + u_{n+1} = 0 \tag{8.6.3}$$

for the bulk ($n > 1$) where $\gamma_{\|}$ and $\gamma_\perp$ are given by Eqs. (8.2.7) and (8.2.8), respectively. We point out that Eq. (8.6.3) represents a set of *infinite* linear equations at variance with the multilayer case where only a *finite* number of linear equations $(N-2)$ for the bulk and one further equation for the second surface were required. The complication entered by the infinite number of equations, however, is balanced by the absence of the second surface equation so that a solution can be easily found. Indeed, for the surface mode we assume

$$u_n = u\,e^{-(n-1)\phi} \tag{8.6.4}$$

where ϕ is a parameter to be determined imposing that Eq. (8.6.4) is solution of the system (8.6.1)–(8.6.3). Replacing Eq. (8.6.4) into Eqs. (8.6.3) and (8.6.1), we find

$$\lambda = \frac{1}{2}(e^{\phi} + e^{-\phi}), \quad e^{\phi} = \frac{1}{\gamma_\perp} \tag{8.6.5}$$

so that the surface magnon frequency becomes

$$\omega = 2z_{\|} J_{\|} S(1 - \gamma_{\|}) + 2z_\perp J_\perp S(1 - \gamma_\perp^2) \tag{8.6.6}$$

and the amplitudes become $u_n = u(\gamma_\perp)^{n-1}$. Since $|\gamma_\perp| \leq 1$, the amplitudes u_n decrease moving away from the surface, confirming that the spin wave with frequency (8.6.6) is really the "surface magnon" of the semi-infinite medium.

To recover the "bulk magnons", we look for the other solutions of the system (8.6.1)–(8.6.3). To do this, let us define $\lambda = \cos\theta$ and look for the zeros of the determinant of the infinite matrix $\mathbf{A}_\infty$ given by

$$\mathbf{A}_\infty = \begin{pmatrix} 2\cos\theta - \dfrac{1}{\gamma_\perp} & -1 & 0 & \dots \\ -1 & 2\cos\theta & -1 & \dots \\ 0 & -1 & -2\cos\theta & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{8.6.7}$$

For finite N, the determinant of the square matrix $\mathbf{A}_N$ is obtained by expanding along the first row. One has

$$\det \mathbf{A}_N = \left(2\cos\theta - \frac{1}{\gamma_\perp}\right) D_{N-1} - D_{N-2} \tag{8.6.8}$$

with D_N given by Eq. (8.2.17). Then the zeros of the determinant of $\mathbf{A}_N$ are given by the solutions of the equation

$$\tan N\theta = \frac{\gamma_\perp \sin\theta}{1 - \gamma_\perp \cos\theta} \tag{8.6.9}$$

with $0 < \theta < \pi$. The left-hand function of Eq. (8.6.9) is a function of θ with a series of asymptotes at $\theta = (l + \frac{1}{2})\frac{\pi}{N}$ while the right-hand function is a positive function with a maximum at $\theta_m = \arccos(\gamma_\perp)$ going to zero at $\theta = 0$ and π. Each solution of Eq. (8.6.9) is restricted to the interval $(l - \frac{1}{2})\frac{\pi}{N} < \theta_l < (l + \frac{1}{2})\frac{\pi}{N}$. For $N \to \infty$, the solutions become $\theta_l \to \frac{\pi}{N} l$ so that the bulk spin wave excitation energies of a semi-infinite system are given by

$$\hbar\omega = 2z_{\|} J_{\|} S(1 - \gamma_{\|}) + 4z_\perp J_\perp S\left(1 - \gamma_\perp \cos\frac{\pi l}{N}\right) \tag{8.6.10}$$

with l integer and $N \to \infty$. The amplitudes can be obtained directly by the system (8.6.1)–(8.6.3), that is,

$$\begin{aligned} u_1 &= u, \quad u_2 = \frac{u}{\gamma_\perp}(2\gamma_\perp \cos\theta - 1), \\ u_3 &= \frac{u}{\gamma_\perp}(4\gamma_\perp \cos^2\theta - 2\cos\theta - \gamma_\perp), \\ u_4 &= \frac{u}{\gamma_\perp}(8\gamma_\perp \cos^3\theta - 4\cos^2\theta - 4\gamma_\perp \cos\theta + 1), \end{aligned} \tag{8.6.11}$$

and so on.

For the SC and H lattices ($\gamma_\perp = z_\perp = 1$), one has

$$u_1 = u, \quad u_n = \frac{\cos \frac{2n-1}{2}\theta}{\cos \frac{\theta}{2}}\, u \tag{8.6.12}$$

with $n > 1$. Notice that the well-known spin wave frequencies of the corresponding 3D lattice with PBC along all the three directions is not recovered. Indeed, the 3D spin waves for SC and H lattices with PBC are characterized by energies

$$\hbar\omega = 2z_\parallel J_\parallel S(1 - \gamma_\parallel) + 4J_\perp S(1 - \cos k_z) \tag{8.6.13}$$

with $k_z = \frac{2\pi l}{N}$ and constant amplitudes $u_n = u$. This result should have been expected since the presence of a surface (semi-infinite system) or two surfaces (film) is incompatible with the PBC that assure that each spin interacts in the same way with its NN. Indeed, the spins belonging to a surface interact only with their NN spins on the same surface and the NN spins of the bulk so that the equations of motion of the spins on the surfaces differ from the equation of motion of the spins in the bulk. Consequently, solutions like "surface magnons" appear and the "bulk magnons" are no longer characterized by uniform amplitudes. It is not surprising that the dynamics is deeply affected by the boundary conditions even though the thermal properties of macroscopic samples are not.