

# Magnons in Ferromagnets

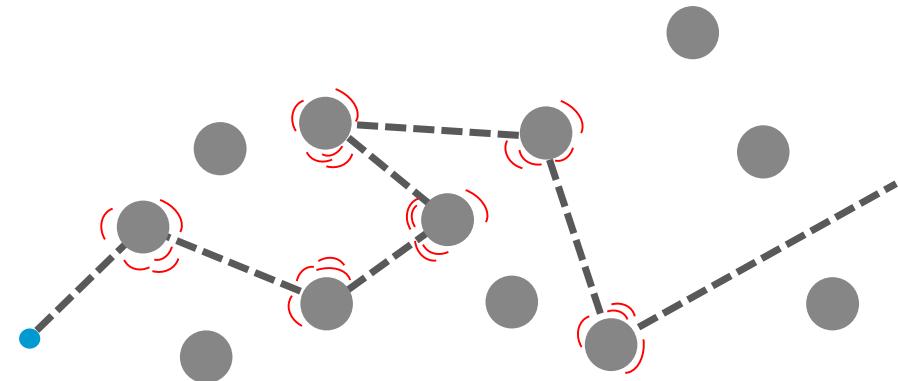
Julian Beisch

Konstanz, 17.12.2024

# Motivation

## Current computing

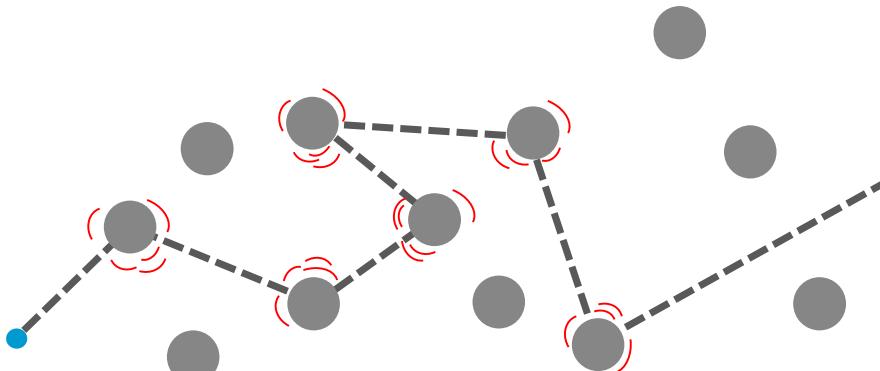
- Electronics
- Information by moving electrons (*charge*)
  - But they scatter → Joule heating



# Motivation

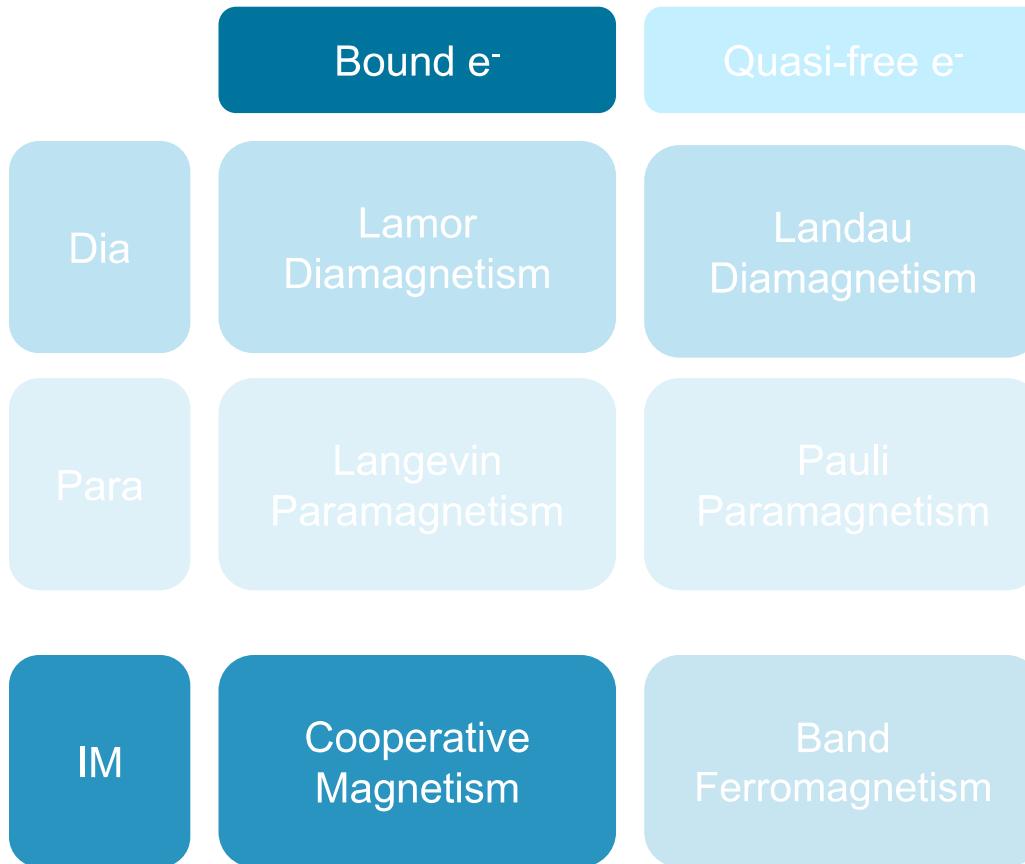
## Current computing

- Electronics
- Information by moving electrons (*charge*)
  - But they scatter → Joule heating
- Another property of electrons: spin
- spintronics
- Make currents with spins, but how ?



# Classification

Scope



# Spin-operators

## Spin-operators

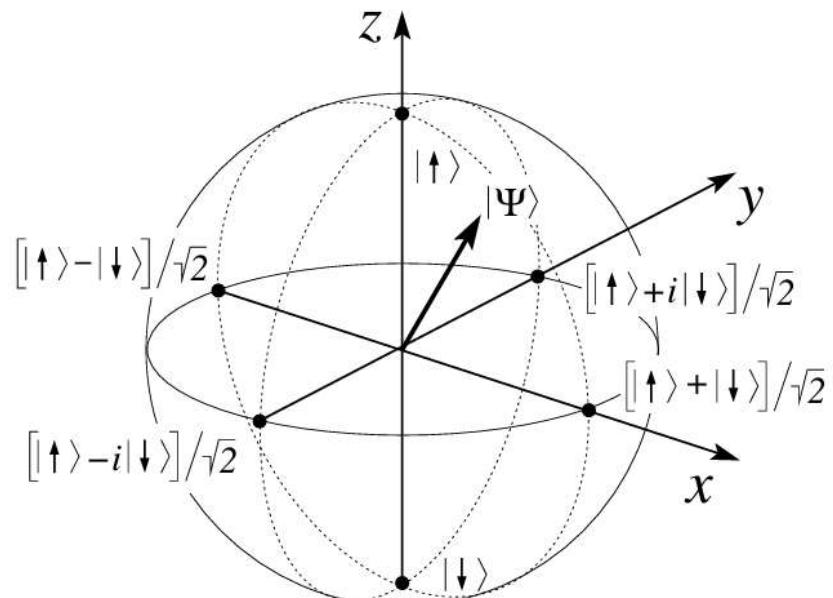
$$[\hat{S}_i^x, \hat{S}_j^y] = i\hat{S}_i^y \delta_{i,j} \quad + \text{cyclic permutation}$$

$$\hat{S}_i^+ = \hat{S}_i^x + i\hat{S}_i^y$$

$$\hat{S}_i^- = \hat{S}_i^x - i\hat{S}_i^y$$

$$[\hat{S}_i^z, \hat{S}_j^\pm] = \pm \hat{S}_i^\pm \delta_{i,j}$$

$$[\hat{S}_i^+, \hat{S}_j^-] = 2\hat{S}_i^z \delta_{i,j}$$



# Heisenberg Theory of Ferromagnetism

Ferromagnetic groundstate

Groundstate  $|0\rangle = |S, S, S, \dots, S\rangle$

$$\begin{aligned}\mathcal{H} &= - \sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j \\ &= - \sum_{i \neq j} J_{i,j} \cdot \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right) \\ &= - \sum_{i \neq j} J_{i,j} \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right)\end{aligned}$$

*W.Heisenberg Z.Physik 49, 619-636, (1928)*



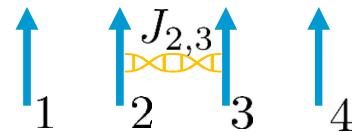
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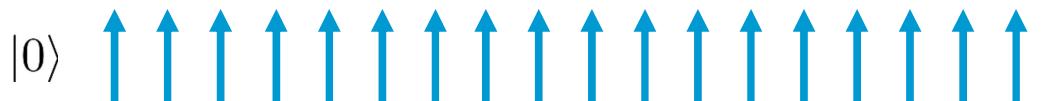
An eigenstate, with the eigenenergy  $E_0$

$$\begin{aligned}\mathcal{H}|0\rangle &= -\sum_{i \neq j} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right) |0\rangle \\ &= 0 - JN \cdot S^2 |0\rangle = E_0 |0\rangle\end{aligned}$$



$$\begin{aligned}\mathcal{H} &= -\sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j \\ &= -\sum_{i \neq j} J_{i,j} \cdot \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right) \\ &= -\sum_{i \neq j} J_{i,j} \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right)\end{aligned}$$

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# Heisenberg Theory of Ferromagnetism

## Excitations

Groundstate  $|0\rangle = |S, S, S, \dots, S\rangle$

How do excitations of this state look like?

## Zur Theorie des Ferromagnetismus.

Von **F. Bloch**, zurzeit in Utrecht.

(Eingegangen am 1. Februar 1930.)

*F.Bloch. Z.Physik 61, 206-219 (1930)*

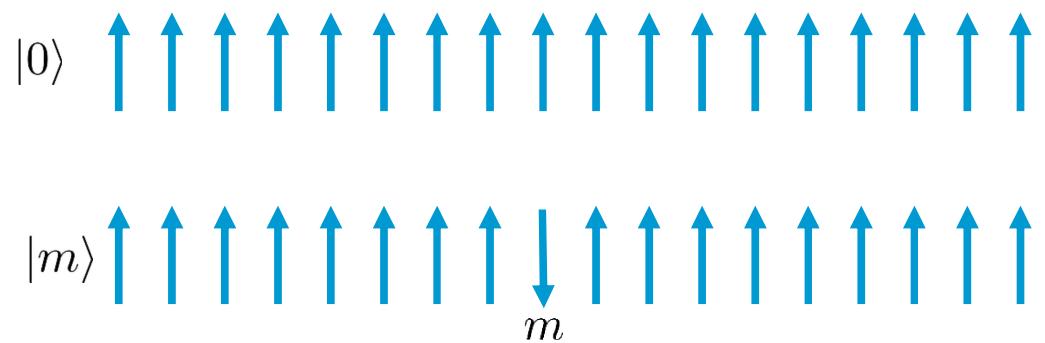


# Heisenberg Theory of Ferromagnetism

## Excitations

His approach was to consider one flipped spin

$$\begin{aligned}|m\rangle &= S_m^- |0\rangle \\&= |S, S, \dots, \underbrace{S-1}_m, \dots, S\rangle\end{aligned}$$



Not an eigenstate anymore

# Heisenberg Theory of Ferromagnetism

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Not an eigenstate anymore

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle$$

$$\mathcal{H}|m\rangle = - \sum_{i \neq j} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- |m\rangle + \hat{S}_i^- \hat{S}_j^+ |m\rangle \right\} + \hat{S}_i^z \hat{S}_j^z |m\rangle \right)$$

$$= - \sum_{i \neq j} J \cdot \left( \frac{1}{2} \left\{ \delta_{i,m} \hat{S}_i^+ \hat{S}_j^- |m\rangle + \delta_{j,m} \hat{S}_i^- \hat{S}_j^+ |m\rangle \right\} + \hat{S}_i^z \hat{S}_j^z |m\rangle \right)$$

$$= -J \cdot \left( \frac{1}{2} \left\{ \sum_j \hat{S}_m^+ \hat{S}_j^- |m\rangle + \sum_i \hat{S}_i^- \hat{S}_m^+ |m\rangle \right\} + \sum_{i \neq j} \hat{S}_i^z \hat{S}_j^z |m\rangle \right)$$

$$= -J \cdot \left( \frac{1}{2} \left\{ \sum_j S|j\rangle + \sum_i S|i\rangle \right\} + \sum_{i \neq j} \hat{S}_i^z \hat{S}_j^z |m\rangle \right)$$

$$= (E_0 + 2JS) |m\rangle - JS \sum_n |n\rangle$$

# Heisenberg Theory of Ferromagnetism

## Search for Eigenstates

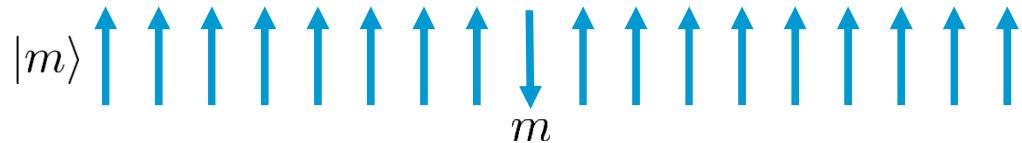
Not an eigenstate anymore.

The total magnetization was reduced by 1!

$$\mathcal{H}|m\rangle = \mathcal{H}S_m^-|0\rangle$$

$$= (E_0 + 2JS) |m\rangle - JS \sum_n |n\rangle$$

$$\hat{S}_{\text{tot.}}^z|m\rangle = (NS - 1) |m\rangle$$



# Heisenberg Theory of Ferromagnetism

## Search for eigenstates

Not an eigenstate anymore.

The total magnetization was reduced by 1!

Hence we can guess an eigenstate

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$$|\vec{k}\rangle \propto \sum_n |n\rangle$$



# Heisenberg Theory of Ferromagnetism

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$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) |n\rangle$$



# Heisenberg Theory of Ferromagnetism

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) |n\rangle$$

## Properties of the eigenstates

The total **magnetization** is reduced

As well as an increase in **energy**

$$\begin{aligned}\hat{S}_{\text{tot.}}^z |\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \hat{S}_{\text{tot.}}^z |n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) (NS - 1) |n\rangle \\ &= (NS - 1) |\vec{k}\rangle\end{aligned}$$

But the average x and y component are still zero?

$$\begin{aligned}\langle \vec{k} | \hat{S}_i^x | \vec{k} \rangle &= 0 \\ \langle \vec{k} | \hat{S}_i^y | \vec{k} \rangle &= 0\end{aligned}$$

# Heisenberg Theory of Ferromagnetism

## Properties of the eigenstates

The total magnetization is reduced

As well as an increase in **energy**

$$\mathcal{H}|\vec{k}\rangle = \dots$$

# Ferromagnetic Dispersion relation

$$\begin{aligned}\mathcal{H}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) \mathcal{H}|n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) \left( (E_0 + 2JS)|n\rangle - JS \sum_j |j\rangle \right) \\ &= \frac{1}{\sqrt{N}} \left( \sum_n (E_0 + 2JS) \exp\left(i\vec{k} \cdot \vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp\left(i\vec{k} \cdot \vec{r}_n\right) |j\rangle \right) \\ &= \frac{1}{\sqrt{N}} \left( \sum_n (E_0 + 2JS) \exp\left(i\vec{k} \cdot \vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp^{i\vec{k} \cdot \vec{r}_n} \exp\left(-i\vec{k} \cdot \vec{r}_j\right) \exp\left(i\vec{k} \cdot \vec{r}_j\right) |j\rangle \right) \\ &= \frac{1}{\sqrt{N}} \left( \sum_n (E_0 + 2JS) \exp\left(i\vec{k} \cdot \vec{r}_n\right) |n\rangle - JS \sum_{n,j} \exp\left(i\vec{k} \cdot \vec{r}_n - i\vec{k} \cdot \vec{r}_j\right) \exp\left(i\vec{k} \cdot \vec{r}_j\right) |j\rangle \right)\end{aligned}$$

# Ferromagnetic Dispersion relation

$$\begin{aligned}
 \mathcal{H}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \left( \sum_n (E_0 + 2JS) \exp(i\vec{k} \cdot \vec{r}_n) |n\rangle - JS \sum_{n,j} \exp(i\vec{k} \cdot \vec{r}_n - i\vec{k} \cdot \vec{r}_j) \exp(i\vec{k} \cdot \vec{r}_j) |j\rangle \right) \\
 &= \left( (E_0 + 2JS) |k\rangle - \frac{1}{\sqrt{N}} JS \sum_{\Delta} \sum_j \exp(i\vec{k} \cdot \vec{r}_{\Delta}) \exp(i\vec{k} \cdot \vec{r}_j) |j\rangle \right) \\
 &= \left( (E_0 + 2JS) |k\rangle - JS \sum_{\Delta} \exp(i\vec{k} \cdot \vec{r}_{\Delta}) |k\rangle \right) \\
 &= \left( E_0 |k\rangle - JS \sum_{\Delta} (2 - \exp(i\vec{k} \cdot \vec{r}_{\Delta})) |k\rangle \right) \\
 &= \left( E_0 |k\rangle - JS \sum_{\Delta>0} (2 - \exp(i\vec{k} \cdot \vec{r}_{\Delta}) - \exp(-i\vec{k} \cdot \vec{r}_{\Delta})) |k\rangle \right) \\
 &= \left( E_0 |k\rangle - JS \sum_{\Delta>0} (2 - 2 \cdot \cos(\vec{k} \cdot \vec{r}_{\Delta})) |k\rangle \right) \\
 &= \left( E_0 - 2JS \sum_{\Delta>0} (1 - \cos(\vec{k} \cdot \vec{r}_{\Delta})) \right) |k\rangle
 \end{aligned}$$

For a linear chain with NN this simplifies to removal of sum

$$\varepsilon_k = 2J_1 \left[ 1 - \cos \frac{2\pi k}{N} \right],$$

*F.Bloch. Z.Physik 61, 206-219 (1930)*

Same result with semiclassical calculation

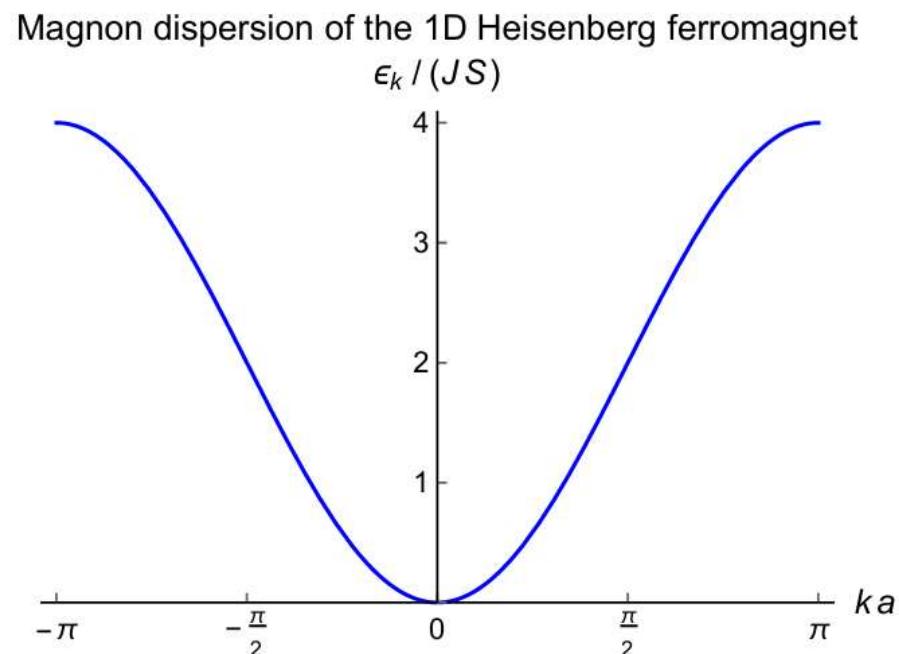
# Ferromagnetic Dispersion relation

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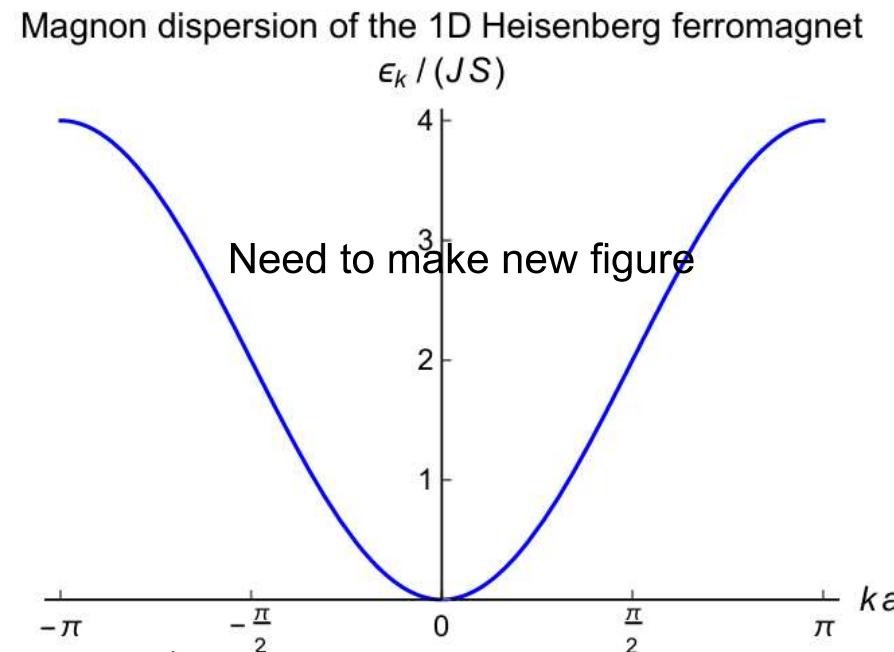


# Ferromagnetic Dispersion relation with a magnetic Field

Addition of Zeeman term

$$\begin{aligned}\mathcal{H} &= - \sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j - \sum_i \vec{B}_i \cdot \hat{S}_i \\ &= - \sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j - \sum_i B_i^z \cdot \hat{S}_i^z \\ &= - \sum_{i \neq j} J_{i,j} \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z + B_i^z \hat{S}_i^z \right)\end{aligned}$$

$$\mathcal{H}|\vec{k}\rangle = \left( E_0 - B^z((N-1)S - (S-1)) - 2JS \sum_{\Delta>0} \left( 1 - \cos(\vec{k}\vec{r}_\Delta) \right) \right) |k\rangle$$



# Recap

## **What is this eigenstate**

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) |n\rangle$$

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## Collective excitation

By comparison to phonons:

- Well defined momentum  $\hbar\vec{k}$
- Energy  $\hbar\epsilon(\vec{k})$

} Quasiparticle

# Recap

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$$\begin{aligned} |\langle m | \vec{k} \rangle|^2 &= \left| \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \langle m | n \rangle \right|^2 \\ &= \left| \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \delta_{m,n} \right|^2 \\ &= \frac{1}{N} \left| \exp(i\vec{k} \cdot \vec{r}_m) \right|^2 = \frac{1}{N} \end{aligned}$$

Reduces magnetization by 1 → integer spin → Boson

# Recap

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Reduces magnetization by 1 → integer spin → Boson

**Magnon**

# Magnons as Bosons

## Thermodynamical treatment

$$\hat{S}_{\text{tot}}^z |\vec{k}\rangle = (NS - 1) |\vec{k}\rangle$$

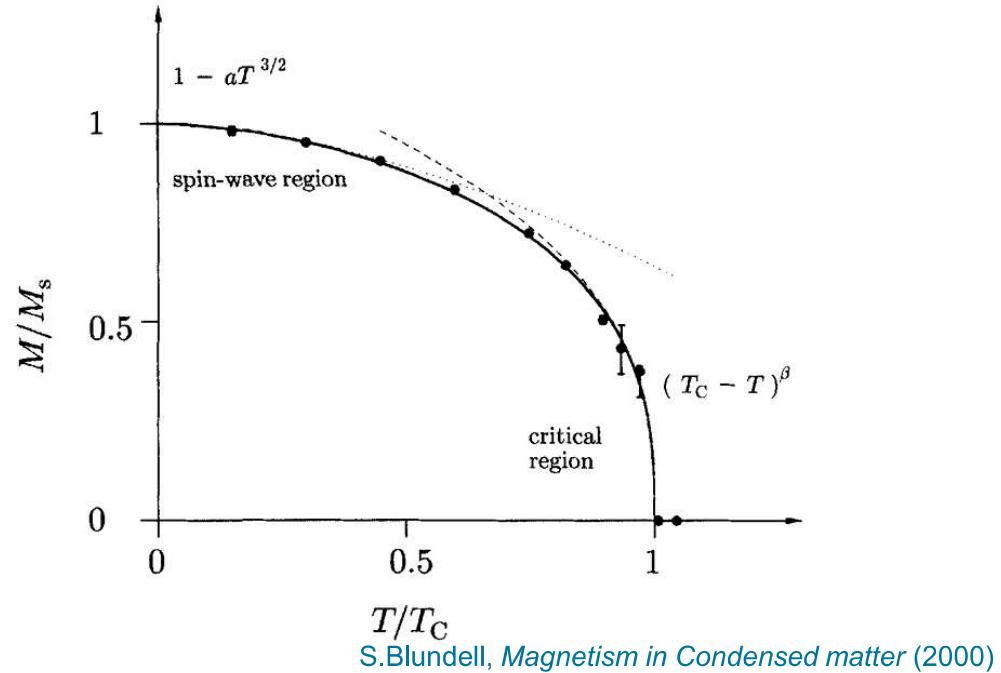
Since it is a boson it must fulfill Bose-Einstein-statistic

$$n_{\text{magnon}} \approx \int_0^\infty \frac{\text{DOS}(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1} \xleftarrow{\text{Bose factor}} \dots \propto T^{3/2}$$

Number of magnons @ T

statistischen Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik eines Einstein-Bose-Gases her bekannt ist;

F.Bloch, Z.Physik **61**, 206-219 (1930)



# Spin-Boson Transformation

## How to describe Magnons

The spin operators are not bosonic

Magnons are bosonic

Alternative approach:

LLG (analytically or numerically)

Or Schwinger representation

Or Dyson–Maleev representation,

## Conditions

- i. The **transformation** needs to be **Hermitian**, raising and lowering operators written as creation and annihilation boson operators need to be Hermitian conjugate of each other
- ii. The **transformation** must be **unitary** to preserve the commutation relations between the spin operators.
- iii. Must satisfy the **equality between the matrix elements** of the spin operators on  $|0\rangle$  and the bosons on  $|n\rangle$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

# Simple Spin-Boson Transformation

Direct attempt

Operator  $\hat{b}^\dagger$  can form the one magnon state

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) |n\rangle$$



$$\hat{b}_{\vec{k}}^\dagger = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) \hat{S}_n^-$$

$$|\vec{k}\rangle = \hat{b}_{\vec{k}}^\dagger |0\rangle$$

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# Simple Spin-Boson Transformation

Direct attempt

Operator  $\hat{b}^\dagger$  can form the one magnon state

Does not fulfill the Boson commutation relations

Close for small excitation

$$[\hat{b}_{\vec{k}}^\dagger, \hat{b}_{\vec{k}'}^\dagger] = 0$$

$$[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}] = 0$$



$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) |n\rangle$$



$$\hat{b}_{\vec{k}}^\dagger = \frac{1}{\sqrt{2SN}} \sum_n \exp(i\vec{k} \cdot \vec{r}_n) \hat{S}_n^-$$

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$$[\hat{S}_i^+, \hat{S}_j^-] = 2\hat{S}_i^z \delta_{i,j}$$



$$[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}^\dagger] = \frac{1}{2SN} \sum_n \exp(-i(\vec{k} - \vec{k}') \cdot \vec{r}_n) 2\hat{S}_n^z \neq \delta_{\vec{k}, \vec{k}'}$$

$$\approx \frac{1}{N} \sum_n \exp(-i(\vec{k} - \vec{k}') \cdot \vec{r}_n) = \delta_{k, k'}$$

# Simple Spin-Boson Transformation

$$\hat{b}_{\vec{k}}^\dagger = \frac{1}{\sqrt{2SN}} \sum_n \exp\left(i\vec{k} \cdot \vec{r}_n\right) \hat{S}_n^-$$

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## Direct attempt

Operator  $\hat{b}^\dagger$  can form the one magnon state

Does not fulfill the Boson commutation relations

Close for small excitation

## Fourier transform back to lattice operators

$$\hat{b}_j^\dagger = \frac{1}{\sqrt{2S}} \hat{S}_j^- \quad \hat{b}_j = \frac{1}{\sqrt{2S}} \hat{S}_j^+$$

$$\hat{S}_j^- = \sqrt{2S} \hat{b}_j^\dagger \quad \hat{S}_j^+ = \sqrt{2S} \hat{b}_j$$

$$\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$$

The number of  
bosons/magnons.

$$\hat{S}_j^z = S - \hat{b}_j^\dagger \hat{b}_j$$

A magnon reduces the z-comp  
of the magnetization

# Holstein-Primakoff(HP)

$$\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^+ &= \hat{S}_j^x + i\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j \\ \hat{S}_j^- &= \hat{S}_j^x - i\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\end{aligned}$$

T-Holstein, H.Primakoff PR 58, 1098, (1940)

## More sophisticated attempt

HP framework is a powerful method for calculating dispersions and higher order interactions

$$[\hat{S}_j^+, \hat{S}_k^-] = \dots = \delta_{j,k} \cdot 2 \cdot \hat{S}_j^z$$

# Holstein-Primakoff

Not so sophisticated anymore

Only linear Spin-Wave theory using the linearized HP

$$\begin{aligned}
 \mathcal{H} &= - \sum_{i \neq j} J_{i,j} \cdot \hat{S}_i \cdot \hat{S}_j \\
 &= - \sum_{i \neq j} J_{i,j} \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right) \\
 &= - \sum_{i \neq j} J_{i,j} \cdot \left( S \left\{ \hat{a}_i \hat{a}_j^\dagger + \hat{a}_i^\dagger \hat{a}_j \right\} + (S - \hat{n}_i)(S - \hat{n}_j) \right)
 \end{aligned}$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^+ = \hat{S}_j^x + i \hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$

$$\hat{S}_j^- = \hat{S}_j^x - i \hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

T-Holstein, H.Primakoff PR 58, 1098, (1940)

Linearized form

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

# Holstein-Primakoff

We want to express the Hamiltonian

$$\begin{aligned}
 \mathcal{H} &= - \sum_{i \neq j} J \cdot \hat{S}_i \cdot \hat{S}_j \\
 &= - \sum_{i \neq j} J \cdot \left( \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} + \hat{S}_i^z \hat{S}_j^z \right) \\
 &= - \sum_{i \neq j} J \cdot \left( S \left\{ \hat{a}_i \hat{a}_j^\dagger + \hat{a}_i^\dagger \hat{a}_j \right\} + (S - \hat{n}_i)(S - \hat{n}_j) \right) \\
 &= - \frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i + \underbrace{\sum_{ij} J \hat{n}_i \hat{n}_j}_{\approx 0} \\
 &\approx - \frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i
 \end{aligned}$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger)$$

$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} (\hat{a}_j + \hat{a}_j^\dagger)$$

$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

These would be magnon-magnon interactions

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We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{a}_n &= [\mathcal{H}, \hat{a}_n] \\ &= 0 - \sum_{i \neq j} JS \left\{ [\hat{a}_j^\dagger \hat{a}_i, \hat{a}_n] + [\hat{a}_i^\dagger \hat{a}_j, \hat{a}_n] \right\} + JS \sum_i [\hat{n}_i, \hat{a}_n] \\ &= 0 - \sum_{i \neq j} JS \{-\delta_{jn} \hat{a}_i - \delta_{in} \hat{a}_j\} + JS \sum_i -\delta_{in} \hat{a}_i \\ &= -JS \left\{ -\sum_{i \neq j} \delta_{jn} \hat{a}_i - \sum_{i \neq j} \delta_{in} \hat{a}_j \right\} + JS \sum_i -\delta_{in} \hat{a}_i \end{aligned}$$

$$\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &= \frac{\hat{S}_j^+ - \hat{S}_j^-}{2i} \approx \frac{\sqrt{2S}}{2i} (\hat{a}_j - \hat{a}_j^\dagger) \\ \hat{S}_j^x &= \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} (\hat{a}_j + \hat{a}_j^\dagger) \\ \hat{S}_j^+ &\approx \sqrt{2S} \cdot \hat{a}_j \\ \hat{S}_j^- &\approx \sqrt{2S} \cdot \hat{a}_j^\dagger \end{aligned}$$

**Problem:** coupling between all sites practically unsolvable for real systems

# Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$i\hbar \frac{\partial}{\partial t} \hat{a}_n = \dots$$

$$\begin{aligned} &= -JS \left\{ -\sum_i \hat{a}_i - \sum_j \hat{a}_j \right\} - JS \hat{a}_n \\ &= 2JS \sum_i \hat{a}_i - JS \hat{a}_n \end{aligned}$$

$$= JS \sum_i \hat{a}_i (2 - \delta_{in})$$

# Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

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Going to Fourier-space

Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-i\vec{k}\vec{r}_j),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(i\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

# Holstein-Primakoff

We want to express the Hamiltonian

$$\mathcal{H} \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i$$

Time evolution of an operator :

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{a}_n &= \dots \\ &= -JS \left\{ -\sum_i \hat{a}_i - \sum_j \hat{a}_j \right\} - JS \hat{a}_n \\ &= 2JS \sum_i \hat{a}_i - JS \hat{a}_n \\ &= JS \sum_i \hat{a}_i (2 - \delta_{in}) \end{aligned}$$

Going to Fourier-space

Using the lattice Fourier transformation

$$\begin{aligned} \hat{a}(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-i\vec{k}\vec{r}_j), \\ \hat{a}^\dagger(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(i\vec{k}\vec{r}_j). \end{aligned}$$

And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

Now we can plug in

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{a}(\vec{k}) &= \frac{1}{\sqrt{N}} \sum_n \left( JS \sum_i \hat{a}_i (2 - \delta_{in}) \right) \exp(-i\vec{k}\vec{r}_n) \\ &= \frac{JS}{\sqrt{N}} \sum_n \sum_i \hat{a}_i (2 - \delta_{in}) \exp(-i\vec{k}\vec{r}_n) \\ &= \dots \end{aligned}$$

# Holstein-Primakoff

## Going to Fourier-space

Using the lattice Fourier transformation

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And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

$$i\hbar \frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( JS \sum_i \hat{a}_i (2 - \delta_{in}) \right) \exp(-i\vec{k}\vec{r}_n)$$

$$= JS \sum_n (2 - \delta_{0n}) \exp(i\vec{k}(\vec{r}_0 - \vec{r}_n)) \hat{a}(\vec{k})$$

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = -\frac{i}{\hbar} \mathcal{W}_{\vec{k}} \hat{a}(\vec{k})$$

*L. Rósza, Lecture Notes (2022)*

$$\hbar\omega_{\vec{k}} = S(J_0 - 2J_{\vec{k}})$$

$$\hat{a}(\vec{k}, t) = \hat{a}(\vec{k}, 0) \cdot \exp(-i\omega_{\vec{k}}t)$$

# Holstein-Primakoff

## Going to Fourier-space

Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-i\vec{k}\vec{r}_j),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(i\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

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*L. Rósza, Lecture Notes (2022)*

$$\hbar\omega_{\vec{k}} = S(J_0 - 2J_{\vec{k}}) \quad \hat{a}(\vec{k}, t) = \hat{a}(\vec{k}, 0) \cdot \exp(-i\omega_{\vec{k}}t)$$

$$\hat{a}_j(\vec{k}, t) = \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(i\vec{k}\vec{r}_j) \hat{a}(\vec{k}, t)$$

$$= \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(i\vec{k}\vec{r}_j) \hat{a}(\vec{k}, 0) \cdot \exp(-i\omega_{\vec{k}}t)$$

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \hat{a}(\vec{k}_0, 0) \cdot \exp(i\vec{k}_0\vec{r}_j - i\omega_{\vec{k}_0}t)$$

$$\hat{a}_j(\vec{k}_0, t) = \hat{A} \cdot \exp(i\vec{k}_0\vec{r}_j - i\omega_{\vec{k}_0}t)$$

# Holstein-Primakoff

## Going to Fourier-space

Using the lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j \exp(-i\vec{k}\vec{r}_j),$$
$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_j \hat{a}_j^\dagger \exp(i\vec{k}\vec{r}_j).$$

And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

Now we can plug in

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(i\vec{k}_0 \vec{r}_j - i\omega_{\vec{k}_0} t) \hat{a}(\vec{k}_0, 0)$$

$$\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(i\vec{k}_0 \vec{r}_j - i\omega_{\vec{k}_0} t)$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^x \approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right)$$

# Holstein-Primakoff

## Going to Fourier-space

Using the lattice Fourier transformation

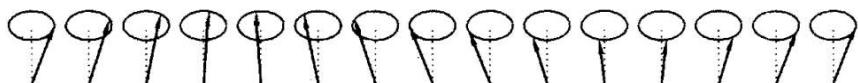
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And the time evolution

$$\frac{\partial}{\partial t} \hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left( \frac{\partial}{\partial t} \hat{a}_n \right) \exp(-i\vec{k}\vec{r}_n),$$

Now we can plug in



S.Blundell, *Magnetism in Condensed matter* (2000)

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(i\vec{k}_0 \vec{r}_j - i\omega_{\vec{k}_0} t) \hat{a}(\vec{k}_0, 0)$$

$$\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(i\vec{k}_0 \vec{r}_j - i\omega_{\vec{k}_0} t)$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2i} \left( \hat{a}_j - \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^x \approx \frac{\sqrt{2S}}{2} \left( \hat{a}_j + \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^x(t) = \sqrt{2S} \hat{A} \cdot \cos(\vec{k}_0 \vec{r}_j + \omega_{\vec{k}_0} t + \phi_A)$$

$$\hat{S}_j^y(t) = \sqrt{2S} \hat{A} \cdot \sin(\vec{k}_0 \vec{r}_j + \omega_{\vec{k}_0} t + \phi_A)$$

$$\hat{S}_j^z(t) = S - |\hat{A}|^2$$

Now we can understand why the average we calculated earlier was 0

# Holstein-Primakoff & Bogoliubov transformation

## Considering more interactions

Adding on site anisotropy

$$\begin{aligned}\mathcal{H} &= - \sum_{i \neq j} J \cdot \hat{S}_i \cdot \hat{S}_j - \sum_i \mathcal{K}_i \cdot \hat{S}_i \cdot \hat{S}_i \\ &= \dots \approx -\frac{NJS^2}{2} - \sum_{i \neq j} JS \cdot \left\{ \hat{a}_j^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_j \right\} + JS \sum_i \hat{n}_i - \sum_i \mathcal{K}_i \left( \underline{2\hat{a}_i^\dagger \hat{a}_i} + \underline{2\hat{a}_i^\dagger \hat{a}_i^\dagger} + S^2 - 2\hat{n}_i + \underbrace{\left( \hat{a}_i^\dagger \hat{a}_i \right)^2}_{\approx 0} \right)\end{aligned}$$

Let's simplify this

$$\begin{aligned}\mathcal{H} &= \sum_{i \neq j} \hat{a}_i^\dagger H \hat{a}_i + \sum_i \hat{a}_i^\dagger \mathcal{K} \hat{a}_i^\dagger + \hat{a}_i \mathcal{K} \hat{a}_i \\ &= \frac{1}{2} (\hat{a}, \hat{a}^\dagger) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K} & H \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\hat{a} &= (\hat{a}_1, \dots, \hat{a}_N)^T \\ \hat{a}^\dagger &= (\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^T\end{aligned}$$

This slide is still work in progress

# Visualising a Magnon

## Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

Time integration with Heun's methods

Additional property is the damping  $\alpha$

$$\frac{d\vec{S}_i}{dt} = -\frac{\gamma}{(1 + \alpha^2)\mu_S} \vec{S}_i \times (\vec{H}_i(t) + \alpha \vec{S}_i \times \vec{H}_i(t))$$
$$\vec{H}_i(t) = -\frac{\partial \mathcal{H}}{\partial \vec{S}_i}$$

[Engine\\_LLG\Test.ipynb](#)

# Outlook

## Current research

Easily obtain highly interesting dispersion relations.

Can be used for computing, without energy loss through  
Joule heating

This is ongoing research in

Spin wave diodes

*J.Lan et.al. PRX 5, 041049, (2015)*

Spin wave transitors

*A.Chumak et.al. Nature C. 5, 4700, (2014)*

The most modern research even includes Altermagnets,  
which have distinct symmetry enforced properties regarding  
the dispersion in different directions

Topological magnons

*P. McClarty Annual Reviews 13, 171-190, (2022)*

Squeezed magnons

*D. Wührer et.al. PR 5, 043124, (2023)*

