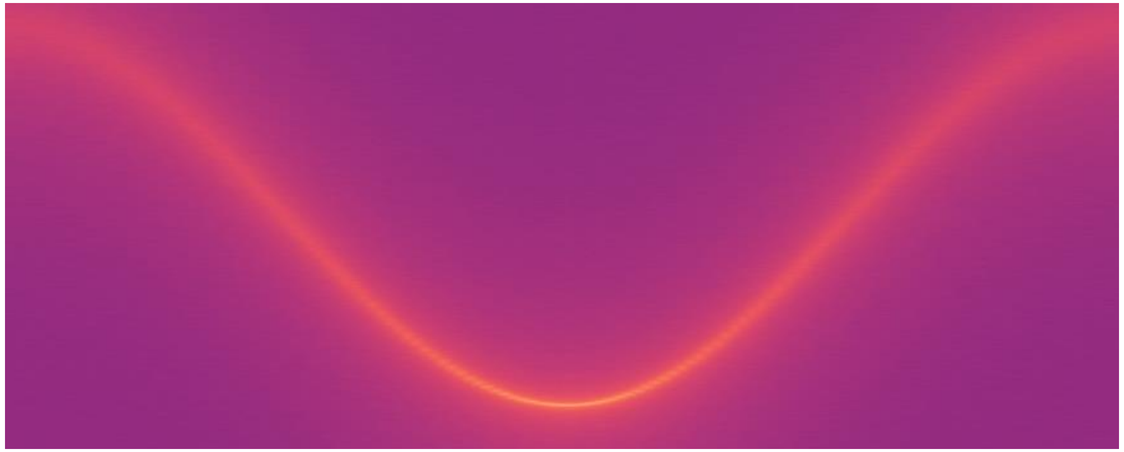
Universität Konstanz



Magnons in Ferromagnets

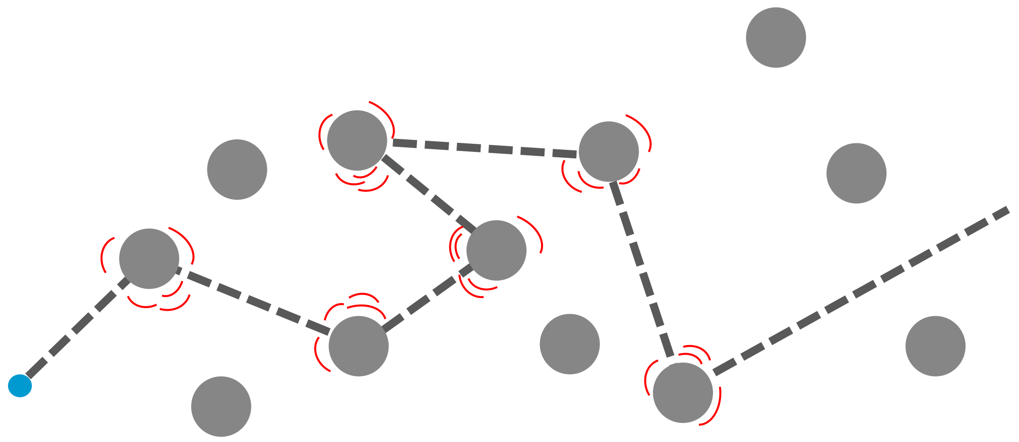
Julian Beisch

Konstanz, 17.12.2024

Motivation

Current computing

- Electronics
- Information by moving electrons *(charge)*
  - But they scatter → Joule heating



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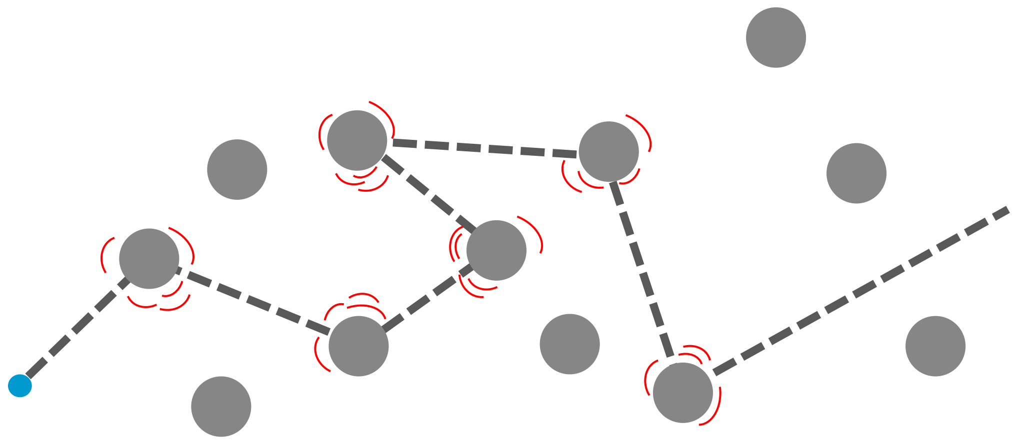
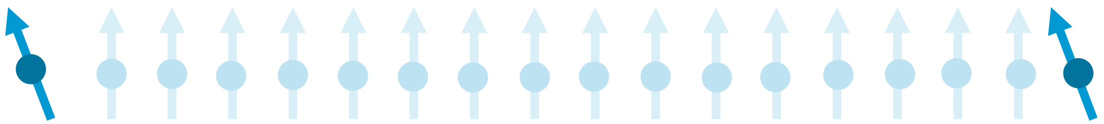


- Another property of electrons: Spin
- Spintronics
- Make currents with spins, but how ?



Caveat

Spins depicted as arrows is just a semi-classical approximation of an otherwise quantum mechanical expectation distribution

Classification

Scope

	Bound e^-	Quasi-free e^-
Dia	Lamor diamagnetism	Landau diamagnetism
Para	Langevin paramagnetism	Pauli paramagnetism
IM	Cooperative magnetism	Band ferromagnetism

Spin-operators

Spin-operators

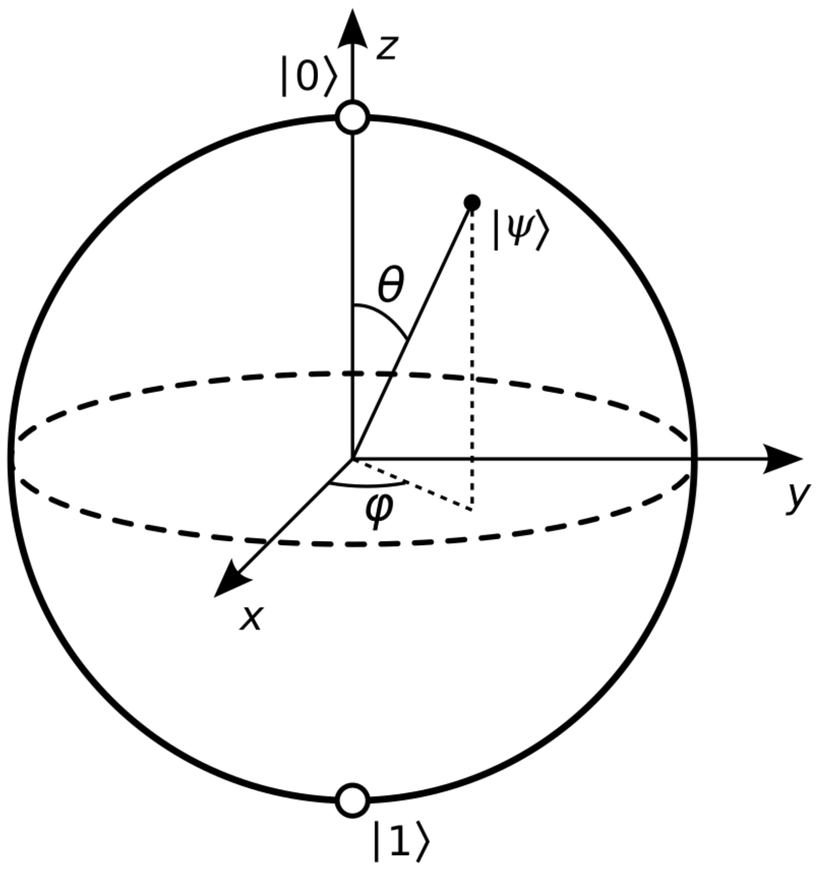
$$\left[\hat{S}_i^x, \hat{S}_j^y\right] = \mathrm{i}\hat{S}_i^z \delta_{i,j} \qquad + \text{cyclic permutation}$$

$$\hat{S}_i^+ = \hat{S}_i^x + \mathrm{i}\hat{S}_i^y$$

$$\hat{S}_i^- = \hat{S}_i^x - \mathrm{i}\hat{S}_i^y$$

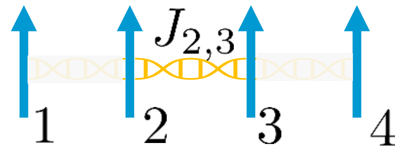
$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm\hat{S}_i^\pm \delta_{i,j}$$

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hat{S}_i^z \delta_{i,j}$$



Wikipedia.com

Heisenberg Theory of Ferromagnetism



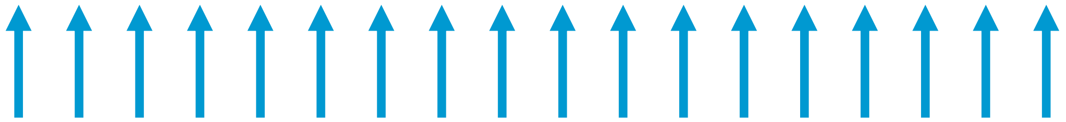
$$\begin{aligned}\mathcal{H} &= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} \\ &= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \left(\hat{S}_i^x \hat{S}_{i+\Delta}^x + \hat{S}_i^y \hat{S}_{i+\Delta}^y + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) \\ &= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)\end{aligned}$$

W.Heisenberg Z.Physik **49**, 619-636, (1928)

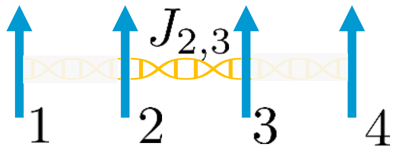
Groundstate

$$|0\rangle = |S, S, S, \ldots, S\rangle$$

$|0\rangle$



Heisenberg Theory of Ferromagnetism



$$\mathcal{H} = -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta}$$

$$= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \left(\hat{S}_i^x \hat{S}_{i+\Delta}^x + \hat{S}_i^y \hat{S}_{i+\Delta}^y + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)$$

$$= -\sum_{i,\Delta} J_{i,i+\Delta} \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)$$

Groundstate

$$|0\rangle = |S, S, S, \ldots, S\rangle$$

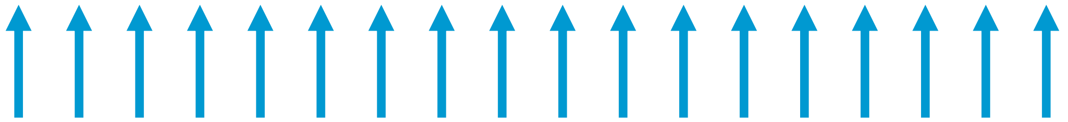
W.Heisenberg Z.Physik **49**, 619-636, (1928)

An eigenstate, with the eigenenergy E_0

$$\mathcal{H}|0\rangle = -\sum_{i,\Delta} J \cdot \left(\frac{1}{2} \left\{ \underline{\hat{S}_i^+ \hat{S}_{i+\Delta}^-} + \underline{\hat{S}_i^- \hat{S}_{i+\Delta}^+} \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) |0\rangle$$

$$= \underline{0} - zJN \cdot S^2 |0\rangle = E_0 |0\rangle$$

$|0\rangle$



Bloch Theory of Ferromagnetism

Excitations

Groundstate $|0\rangle = |S, S, S, \ldots, S\rangle$

How do excitations of this state look like?

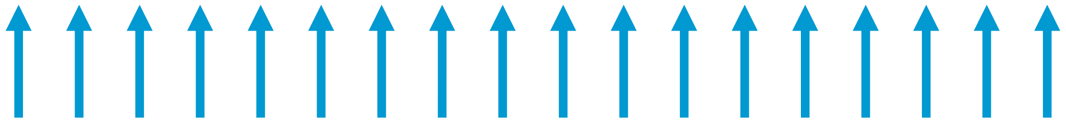
Zur Theorie des Ferromagnetismus.

Von **F. Bloch**, zurzeit in Utrecht.

(Eingegangen am 1. Februar 1930.)

F.Bloch. Z.Physik **61**, *206-219* (1930)

$|0\rangle$



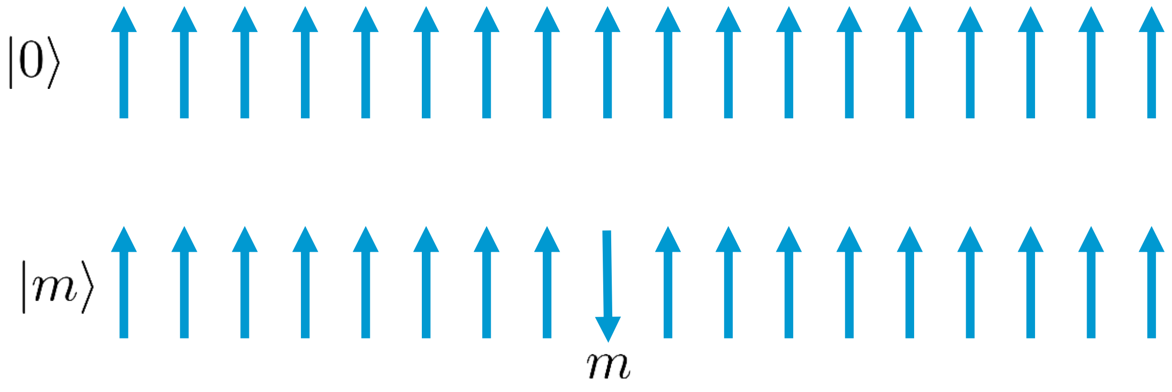
Bloch Theory of Ferromagnetism

Excitations

His approach was to consider one flipped spin

$$
\begin{aligned}
|m\rangle &= \frac{S_m^-}{\sqrt{2S}}|0\rangle \\
&= |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle
\end{aligned}
$$

Not an eigenstate anymore



Bloch Theory of Ferromagnetism

$$\mathcal{H}|m\rangle = -\sum_{i,\Delta} J \cdot \left(\frac{1}{2}\left\{\hat{S}_i^+\hat{S}_{i+\Delta}^-|m\rangle + \hat{S}_i^-\hat{S}_{i+\Delta}^+|m\rangle\right\} + \hat{S}_i^z\hat{S}_{i+\Delta}^z|m\rangle\right)$$

Excitations

His approach was to consider one flipped spin

$$|m\rangle = \frac{S_m^-}{\sqrt{2S}}|0\rangle = |S, S, \ldots, \underbrace{S-1}_{m}, \ldots, S\rangle$$

Not an eigenstate anymore

$$= -J\sum_{i,\Delta} \cdot \left(\frac{1}{2}\left\{\delta_{i,m}\hat{S}_i^+\hat{S}_{i+\Delta}^-|m\rangle + \delta_{i+\Delta,m}\hat{S}_i^-\hat{S}_{i+\Delta}^+|m\rangle\right\} + \hat{S}_i^z\hat{S}_{i+\Delta}^z|m\rangle\right)$$

$$= -J \cdot \left(\frac{1}{2}\left\{\sum_{\Delta}\hat{S}_m^+\hat{S}_{m+\Delta}^-|m\rangle + \sum_{\Delta}\hat{S}_{m-\Delta}^-\hat{S}_m^+|m\rangle\right\} + \sum_{i,\Delta}\hat{S}_i^z\hat{S}_{i+\Delta}^z|m\rangle\right)$$

$$= -J \cdot \left(\frac{1}{2}\left\{\sum_{\Delta}2S|m+\Delta\rangle + \sum_{\Delta}2S|m+\Delta\rangle\right\} + \sum_{i,\Delta}\hat{S}_i^z\hat{S}_{i+\Delta}^z|m\rangle\right)$$

$$= (E_0 + 2zJS)\,|m\rangle - 2JS\sum_{\Delta}|m+\Delta\rangle$$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

Bloch Theory of Ferromagnetism

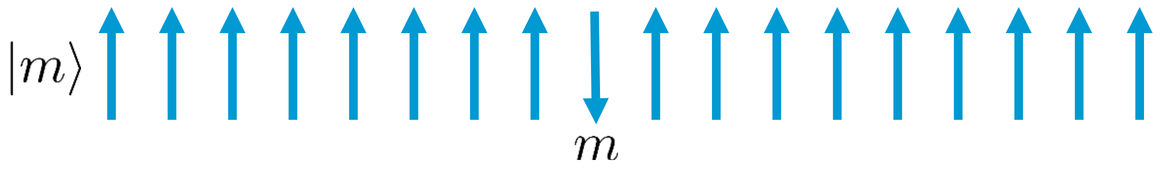
Search for eigenstates

Not an eigenstate anymore.

The total magnetization was reduced by 1!

$$\begin{aligned}\mathcal{H}|m\rangle &= \mathcal{H}S_m^-|0\rangle \\ &= (E_0 + 2zJS)\,|m\rangle - 2JS\sum_{\Delta}|m+\Delta\rangle\end{aligned}$$

$$\hat{S}^z_{\mathrm{tot.}}|m\rangle = (NS-1)\,|m\rangle$$



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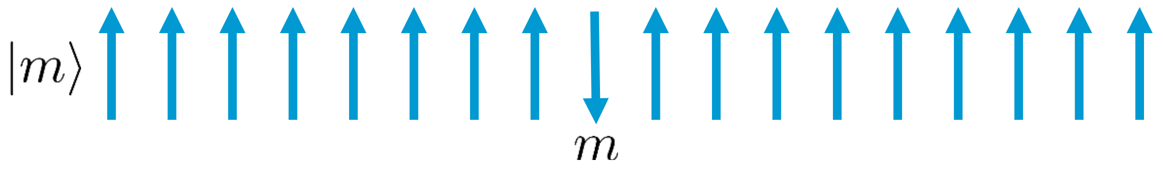
$$\hat{S}^z_{\text{tot.}}|m\rangle = (NS-1)\,|m\rangle$$

$$|m\rangle \propto S_m^-|0\rangle$$

Hence we can guess an eigenstate



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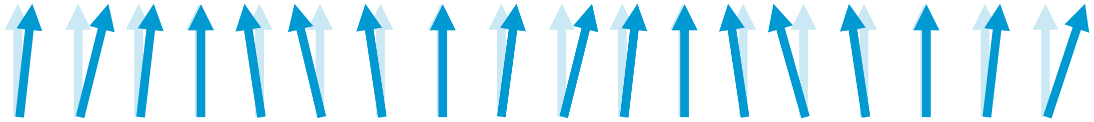
$$\hat{S}^-_{\vec{k}} \propto \frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\hat{S}^-_n$$

$|\vec{k}\rangle$



Bloch Theory of Ferromagnetism

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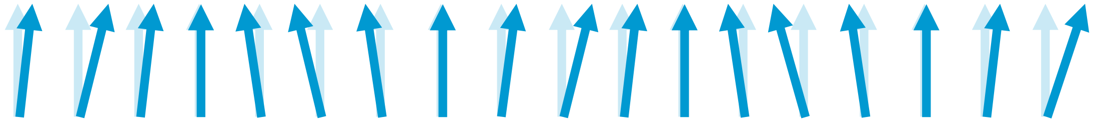

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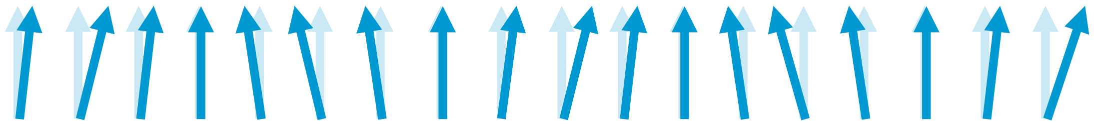
$|\vec{k}\rangle$ 

Bloch Theory of Ferromagnetism

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) |n\rangle$$

Properties of the eigenstates

The total **magnetization** is reduced

As well as an increase in energy

$$\begin{aligned}\hat{S}^z_{\text{tot.}}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) \hat{S}^z_{\text{tot.}}|n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) (NS-1)|n\rangle \\ &= (NS-1)|\vec{k}\rangle\end{aligned}$$

But the average x and y components are still zero?

$$\begin{aligned}\langle \vec{k}|\hat{S}^x_i|\vec{k}\rangle &= 0 \\ \langle \vec{k}|\hat{S}^y_i|\vec{k}\rangle &= 0 \\ \langle \vec{k}|\hat{S}^z_i|\vec{k}\rangle &= S - \frac{1}{N}\end{aligned}$$

Bloch Theory of Ferromagnetism

Properties of the eigenstates

The total magnetization is reduced

As well as an increase in **energy**

$$\mathcal{H}|\vec{k}\rangle = \dots$$

Ferromagnetic Dispersion Relation

$$\begin{aligned}
\mathcal{H}|\vec{k}\rangle &= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \mathcal{H}|n\rangle \\
&= \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \left((E_0 + 2zJS)\,|n\rangle - 2JS \sum_\Delta |n+\Delta\rangle \right) \\
&= \frac{1}{\sqrt{N}} \left(\sum_n (E_0 + 2zJS) \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle - 2JS \sum_{n,\Delta} \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n+\Delta\rangle \right) \\
&= \left((E_0 + 2zJS)\,|\vec{k}\rangle - \frac{1}{\sqrt{N}} 2JS \sum_{\Delta,m} \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_{m-\Delta}\right) |m\rangle \right) \\
&= \left((E_0 + 2zJS)\,|\vec{k}\rangle - 2JS \sum_\Delta \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right)
\end{aligned}$$

Ferromagnetic Dispersion Relation

Properties of the eigenstates

$$
\begin{aligned}
\mathcal{H}|\vec{k}\rangle &= \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta>0} - \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) - \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right) \\
&= \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta>0} -2\cos\left(\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right) \\
&= \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta} -\cos\left(\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right) \\
&= \left(E_0|\vec{k}\rangle + 2JS \sum_{\Delta} \left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right) |\vec{k}\rangle \right)
\end{aligned}
$$

Ferromagnetic Dispersion Relation

$$\mathcal{H}|\vec{k}\rangle = \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta>0} - \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) - \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right)|\vec{k}\rangle \right)$$

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Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2\,J_1\left[1 - \cos\frac{2\,\pi\,k}{N}\right],$$

F.Bloch. Z.Physik **61**, *206-219* (1930)

Same result with semiclassical calculation

Ferromagnetic Dispersion Relation

$$\mathcal{H}|\vec{k}\rangle = \left((E_0 + 2zJS)\,|\vec{k}\rangle + 2JS \sum_{\Delta>0} - \exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) - \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) |\vec{k}\rangle \right)$$

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Same result with semiclassical calculation

$$\mathcal{H}|\vec{k}\rangle = \left(E_0|\vec{k}\rangle + 4JS \left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right) |\vec{k}\rangle \right)$$

$$\Delta E = 4JS\left(1 - \cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)$$

For a linear chain with NN

S.Blundell, *Magnetism in Condensed matter* (2000)

Ferromagnetic Dispersion Relation

Properties of the eigenstates

$$\varepsilon_k \overset{r=1.}{=} 2\,J_1\left[1-\cos\frac{2\,\pi\,k}{N}\right],$$

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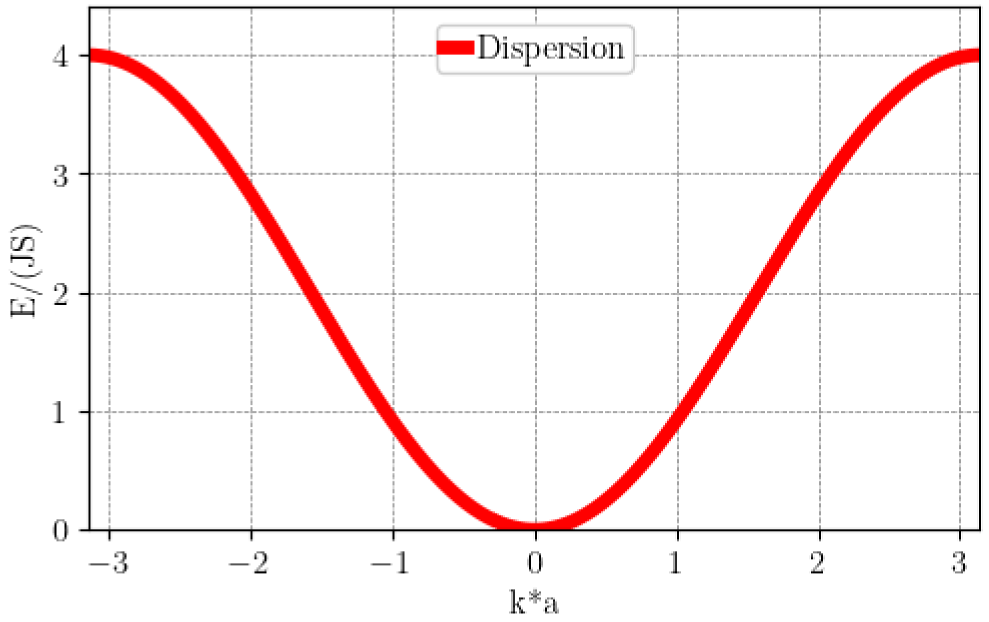
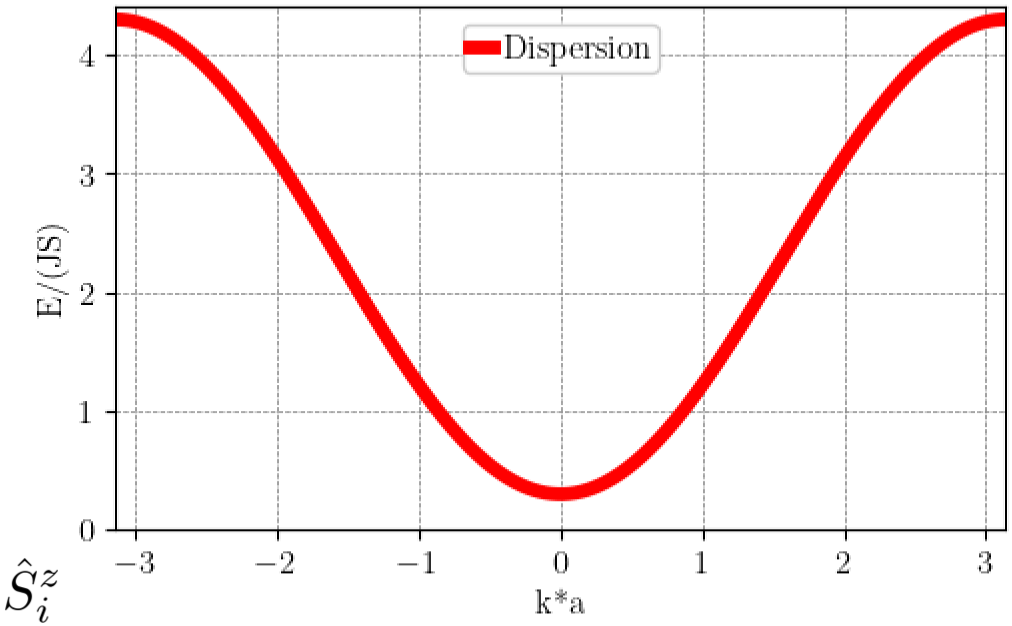


Ferromagnetic Dispersion Relation with a Magnetic Field

Addition of Zeeman term

$$\begin{aligned}
\mathcal{H}_Z &= -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \vec{B} \cdot \hat{S}_i \\
&= -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i B^z \cdot \hat{S}_i^z \\
&= -\sum_{i,\Delta} J \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right) + B^z \hat{S}_i^z
\end{aligned}$$

Ferromagnetic Dispersion Relation with a Magnetic Field



Addition of Zeeman term

$$\mathcal{H}_Z = -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \vec{B} \cdot \hat{S}_i$$

$$= -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i B^z \cdot \hat{S}_i^z$$

$$= -\sum_{i,\Delta} J \cdot \left(\frac{1}{2}\left\{\hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+\right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z\right) + B^z \hat{S}_i^z$$

$$\mathcal{H}_Z|\vec{k}\rangle = \left(E_{0,Z} + B^z + 2JS\sum_{\Delta}\left(1 - \cos(\vec{k}\vec{r}_\Delta)\right)\right)|k\rangle$$

$$E_{0,Z} = -zJN \cdot S^2 - NSB^z$$

Recap

First excited state

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k} \cdot \vec{r}_n\right) |n\rangle$$

Recap

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$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

Collective excitation

$$\begin{aligned}|\langle m|\vec{k}\rangle|^2 &= \left|\frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2 \\ &= \left|\frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\delta_{m,n}\right|^2 \\ &= \frac{1}{N}\left|\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right)\right|^2 = \frac{1}{N}\end{aligned}$$

Recap

First excited state

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

Collective excitation

By comparison to phonons:
-Well defined momentum $\hbar\vec{k}$
-Energy $\hbar E(\vec{k})$

Quasiparticle

$$|\langle m|\vec{k}\rangle|^2 = \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\delta_{m,n}\right|^2$$

$$= \frac{1}{N}\left|\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right)\right|^2 = \frac{1}{N}$$

Recap

First excited state

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) |n\rangle$$

$$|\langle m|\vec{k}\rangle|^2 = \left| \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \langle m|n\rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right) \delta_{m,n} \right|^2$$

$$= \frac{1}{N} \left| \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right) \right|^2 = \frac{1}{N}$$

Collective excitation

By comparison to phonons:

-Well defined momentum $\hbar\vec{k}$

-Energy $\hbar E(\vec{k})$

Quasiparticle

Reduces magnetization by 1 → Integer spin → Boson

Recap

First excited state

Delocalization of a “flipped” spin over all sites

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)|n\rangle$$

$$|\langle m|\vec{k}\rangle|^2 = \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\langle m|n\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{N}}\sum_n \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_n\right)\delta_{m,n}\right|^2$$

$$= \frac{1}{N}\left|\exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_m\right)\right|^2 = \frac{1}{N}$$

Collective excitation

By comparison to phonons:
-Well defined momentum $\hbar\vec{k}$
-Energy $\hbar E(\vec{k})$

$\Big\}$ Quasiparticle

Reduces magnetization by 1 → Integer spin → Boson

Magnon

Magnons as Bosons

statistischen
Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik eines Einstein-Bose-Gases her bekannt ist;

F.Bloch. Z.Physik **61**, *206-219* (1930)

Thermodynamical treatment

$$\hat{S}^z_{\mathrm{tot.}}|\vec{k}\rangle = (NS-1)|\vec{k}\rangle$$

Since it is a boson it must fulfill the Bose-Einstein statistic

$$n_{\mathrm{magnon}} \approx \int_0^\infty \frac{\mathrm{DOS}(\omega)\mathrm{d}\omega}{\exp(\hbar\omega/k_B T)-1}$$
$$= \ldots \propto T^{3/2}$$

Bose factor

Number of
magnons @ T

Magnons as Bosons

statistischen
Gewicht 1 zu zählen. Der Sachverhalt ist derselbe, wie er von der Statistik
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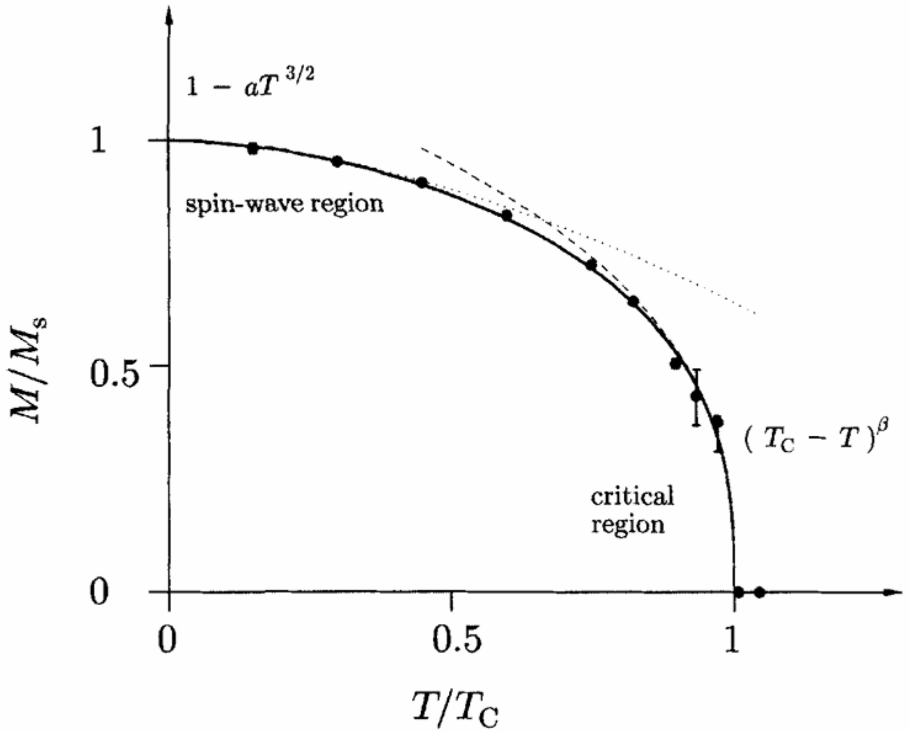
$$= \ldots \propto T^{3/2}$$

Number of magnons @ T



S.Blundell, *Magnetism in Condensed matter* (2000)

Magnons as Bosons

$$\hbar\omega \approx 2JSk^2a^2$$
$$\omega \approx k^2 DOS(k)\mathrm{d}k \propto q^2\mathrm{d}q$$
$$DOS\mathrm{d}\omega \propto \sqrt{\omega}\mathrm{d}\omega$$

Thermodynamical treatment

$$\hat{S}^z_{\mathrm{tot.}}|\vec{k}\rangle = (NS-1)|\vec{k}\rangle$$

Since it is a boson it must fulfill Bose-Einstein-statistic

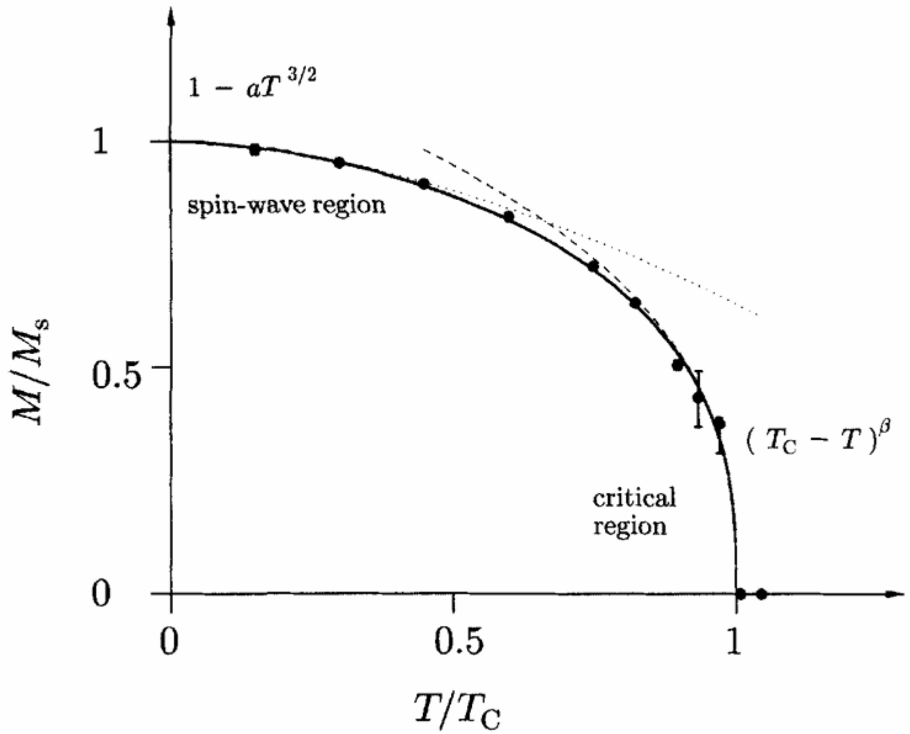
$$n_{\mathrm{magnon}} \approx \int_0^\infty \frac{\mathrm{DOS}(\omega)\mathrm{d}\omega}{\exp(\hbar\omega/k_BT)-1}$$

Bose factor

Number of magnons @ T

$$= \left(\frac{k_BT}{\hbar}\right)^{\frac{3}{2}} \int_0^\infty \frac{\sqrt{\frac{\hbar\omega}{k_BT}}}{\exp(\hbar\omega/k_BT)-1}\mathrm{d}\left(\frac{\hbar\omega}{k_BT}\right) \propto T^{3/2}$$

Each thermally excited magnon reduces the magnetization by 1



S.Blundell, *Magnetism in Condensed matter* (2000)

Spin-Boson Transformation

How to describe magnons

The spin operators are not bosonic

Magnons are bosonic

Alternative approach:
LLG (analytically or numerically)
Or Schwinger representation
Or Dyson–Maleev representation

Conditions

i. The **transformation** needs to be **Hermitian**, raising and lowering operators written as creation and annihilation boson operators need to be Hermitian conjugate of each other

ii. The **transformation** must be **unitary** to preserve the commutation relations between the spin operators.

iii. Must satisfy the **equality between the matrix elements** of the spin operators on $|0\rangle$ and the bosons on $|n\rangle$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

A Spin-Boson Transformation

Define higher magnon states

$$|n_\sigma\rangle = |S, S, \ldots, \underbrace{S - \sigma_n}_{n}, \ldots, S\rangle$$

$$\hat{S}_n^+ |n_\sigma\rangle = \sqrt{2S}\sqrt{1 - \frac{\sigma_n - 1}{2S}}\sqrt{\sigma_n}|n_{\sigma-1}\rangle$$

$$\hat{S}_n^- |n_\sigma\rangle = \sqrt{2S}\sqrt{\sigma_n + 1}\sqrt{1 - \frac{\sigma_n}{2S}}|n_{\sigma+1}\rangle$$

DECEMBER 15, 1940 PHYSICAL REVIEW VOLUME 58

Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet

T. HOLSTEIN
New York University, New York, New York
AND
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Polytechnic Institute of Brooklyn, Brooklyn, New York

(Received July 31, 1940)

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Only for $\sigma \leq 2S$

$$\hat{S}_n^-|n_{2S}\rangle = \sqrt{2S}\sqrt{2S+1}\sqrt{1 - \frac{2S}{2S}}|n_{2S+1}\rangle = 0$$

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A Spin-Boson Transformation

Compare the "eigenvalues"

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Only for $\sigma \leq 2S$

$$\hat{S}_n^- |n_{2S}\rangle = \sqrt{2S}\sqrt{2S+1}\sqrt{1 - \frac{2S}{2S}}|n_{2S+1}\rangle = 0$$

Consider the states to be like 2nd quantized number states

$$\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n}|n_{\sigma-1}\rangle$$

$$\hat{a}_n^\dagger |n_\sigma\rangle = \sqrt{\sigma_n + 1}|n_{\sigma+1}\rangle$$

A Spin-Boson Transformation

Consider the states to be like 2nd quantized number states

$$\boxed{\hat{a}_n^\dagger \hat{a}_n |n_{\sigma-1}\rangle = (\sigma_n - 1)|n_{\sigma-1}\rangle}$$

$$\boxed{\hat{a}_n |n_\sigma\rangle = \sqrt{\sigma_n}|n_{\sigma-1}\rangle}$$

$$\hat{a}_n^\dagger |n_\sigma\rangle = \sqrt{\sigma_n + 1}|n_{\sigma+1}\rangle$$

Compare the "eigenvalues"

$$|n_\sigma\rangle = |S, S, \ldots, \underbrace{S - \sigma_n}_{n}, \ldots, S\rangle$$

$$\hat{S}_n^+ |n_\sigma\rangle = \sqrt{2S}\sqrt{1 - \frac{\boxed{\sigma_n - 1}}{2S}}\boxed{\sqrt{\sigma_n}}|n_{\sigma-1}\rangle$$

$$\hat{S}_n^- |n_\sigma\rangle = \sqrt{2S}\sqrt{\sigma_n + 1}\sqrt{1 - \frac{\sigma_n}{2S}}|n_{\sigma+1}\rangle$$

$$\hat{S}_n^+ = \sqrt{2S}\sqrt{1 - \frac{\boxed{\hat{a}_n^\dagger \hat{a}_n}}{2S}}\boxed{\hat{a}_n}$$

A Spin-Boson Transformation

Compare the “eigenvalues”

$$|n_\sigma\rangle = |S, S, \dots, \underbrace{S - \sigma_n}_{n}, \dots, S\rangle$$

$$\hat{S}_n^+|n_\sigma\rangle = \sqrt{2S}\sqrt{1 - \frac{\boxed{\sigma_n - 1}}{2S}}\boxed{\sqrt{\sigma_n}}|n_{\sigma-1}\rangle$$

$$\hat{S}_n^-|n_\sigma\rangle = \sqrt{2S}\boxed{\sqrt{\sigma_n + 1}}\sqrt{1 - \frac{\boxed{\sigma_n}}{2S}}|n_{\sigma+1}\rangle$$

Consider the states to be like 2nd quantized number states

$$\boxed{\hat{a}_n^\dagger \hat{a}_n|n_{\sigma-1}\rangle = (\sigma_n - 1)|n_{\sigma-1}\rangle}$$

$$\boxed{\hat{a}_n^\dagger \hat{a}_n|n_\sigma\rangle = \sigma_n|n_\sigma\rangle}$$

$$\boxed{\hat{a}_n|n_\sigma\rangle = \sqrt{\sigma_n}|n_{\sigma-1}\rangle}$$

$$\boxed{\hat{a}_n^\dagger|n_\sigma\rangle = \sqrt{\sigma_n + 1}|n_{\sigma+1}\rangle}$$

$$\hat{S}_n^+ = \sqrt{2S}\sqrt{1 - \frac{\boxed{\hat{a}_n^\dagger \hat{a}_n}}{2S}}\boxed{\hat{a}_n}$$

$$\hat{S}_n^- = \sqrt{2S}\boxed{\hat{a}_n^\dagger}\sqrt{1 - \frac{\boxed{\hat{a}_n^\dagger \hat{a}_n}}{2S}}$$

The Holstein Primakoff Spin-Boson Transformation

Combined

$$\hat{S}_n^- = \sqrt{2S}\hat{a}_n^\dagger\sqrt{1-\frac{\hat{a}_n^\dagger\hat{a}_n}{2S}}$$

$$\hat{S}_n^+ = \sqrt{2S}\sqrt{1-\frac{\hat{a}_n^\dagger\hat{a}_n}{2S}}\hat{a}_n$$

$$\hat{S}_n^z = S - \hat{a}_n^\dagger\hat{a}_n$$

$$\boxed{\begin{aligned}\hat{S}_j^z &= S - \hat{a}_j^\dagger\hat{a}_j\\ \hat{S}_j^+ &= \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S-\hat{a}_j^\dagger\hat{a}_j}\cdot\hat{a}_j\\ \hat{S}_j^- &= \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger\cdot\sqrt{2S-\hat{a}_j^\dagger\hat{a}_j}\end{aligned}}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940) is the expectation value of the spin-deviation operator when the temperature of the specimen is T, and involves, first an average over Ψ_E, and then an average over the Boltzmann distribution of the eigenstates of the specimen.

The operators of (2) have the following properties,

$$S^+{}_l\Psi_{n_l}=(2S)^{\frac{1}{2}}(1-(n_l-1)/2S)^{\frac{1}{2}}(n_l)^{\frac{1}{2}}\Psi_{n_l-1},$$
$$S^-{}_l\Psi_{n_l}=(2S)^{\frac{1}{2}}(n_l+1)^{\frac{1}{2}}(1-n_l/2S)^{\frac{1}{2}}\Psi_{n_l+1},\quad \mathfrak{n}_l\Psi_{n_l}=n_l\Psi_{n_l}. \tag{3}$$

Introducing the well-known creation and destruction operators defined by[13]

$$a^*{}_l\Psi_{n_l}=(n_l+1)^{\frac{1}{2}}\Psi_{n_l+1},\quad a_l\Psi_{n_l}=(n_l)^{\frac{1}{2}}\Psi_{n_l-1}, \tag{4}$$

one obtains, upon comparing (3) and (4)

$$S^+{}_l=(2S)^{\frac{1}{2}}(1-a^*{}_la_l/2S)^{\frac{1}{2}}a_l,\quad S^-{}_l=(2S)^{\frac{1}{2}}a^*{}_l(1-a^*{}_la_l/2S)^{\frac{1}{2}},\quad \mathfrak{n}_l=a^*{}_la_l. \tag{5}$$

---

[12] It is to be noted that $\sum_{l=1}^{N}S^{(z)}{}_l$ does not commute with the magnetic interaction portion of the Hamiltonian (1).

[13] In Eq. (4), n_l is allowed to run from 0 to ∞ rather than from 0 to $2S$ as in Eq. (3). The discrepancy is only apparent, since the transition from states with $n_l\leqq 2S$ to states with $n_l>2S$ will never occur. e.g.

$$S^-{}_l\Psi_{2S}=(2S)^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}(1-2S/2S)^{\frac{1}{2}}\Psi_{2S+1}=0.$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

The Holstein Primakoff Spin-Boson Transformation

Checking the conditions

(i) Fulfilled

(ii) Does fulfill the commutation relations

(iii) Fulfilled by definition

$\Bigg\}$ for $\sigma \leq 2S$

E.Rastelli, *Statistical Mechanics of Magnetic Excitations* (2013)

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm \hat{S}_i^\pm \delta_{i,j}$$ ✓

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = \ldots = 2\hat{S}_i^z \delta_{i,j}$$ ✓

Holstein-Primakoff(HP)

$$
\begin{aligned}
\left[\hat{S}_j^+, \hat{S}_k^-\right] &= \left[\sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j, \hat{a}_k^\dagger \cdot \sqrt{2S - \hat{a}_k^\dagger \hat{a}_k}\right] \\
&\stackrel{k\neq j}{=} \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\left[\hat{a}_j, \hat{a}_k^\dagger\right]\sqrt{2S - \hat{a}_k^\dagger \hat{a}_k} + \left[\sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}, \hat{a}_k^\dagger\right]\hat{a}_j\sqrt{2S - \hat{a}_k^\dagger \hat{a}_k} \\
&\quad + \hat{a}_k^\dagger\sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\left[\hat{a}_j, \sqrt{2S - \hat{a}_k^\dagger \hat{a}_k}\right] + \hat{a}_k^\dagger\left[\sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}, \sqrt{2S - \hat{a}_k^\dagger \hat{a}_k}\right]\hat{a}_j \\
&= 0 \\
&\stackrel{k=j}{=} \left(\sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j \cdot \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\right) - \left(\hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j\right) \\
&= \left(\sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \left(\hat{a}_j^\dagger \hat{a}_j + 1\right) \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\right) - \left(\hat{a}_j^\dagger \cdot \left(2S - \hat{a}_j^\dagger \hat{a}_j\right) \cdot \hat{a}_j\right) \\
&= \left(\hat{a}_j^\dagger \hat{a}_j + 1\right) \cdot \left(2S - \hat{a}_j^\dagger \hat{a}_j\right) - \left(\hat{a}_j^\dagger \hat{a}_j \cdot \left(2S - (\hat{a}_j^\dagger \hat{a}_j - 1)\right)\right) \\
&= \left(\hat{a}_j^\dagger \hat{a}_j + 1\right) \cdot \left(2S - \hat{a}_j^\dagger \hat{a}_j\right) - \hat{a}_j^\dagger \hat{a}_j \cdot \left(2S - \hat{a}_j^\dagger \hat{a}_j + 1\right) \\
&= \left(\hat{a}_j^\dagger \hat{a}_j + 1\right) \cdot \left(2S - \hat{a}_j^\dagger \hat{a}_j\right) = \left(2S - \hat{a}_j^\dagger \hat{a}_j\right) - \hat{a}_j^\dagger \hat{a}_j = 2 \cdot \left(S - \hat{a}_j^\dagger \hat{a}_j\right) = \delta_{j,k} \cdot 2 \cdot \hat{S}_j^z
\end{aligned}
$$

The Holstein Primakoff Spin-Boson Transformation

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

T-Holstein, H.Primakoff PR **58**, 1098, (1940)

Applications

$$\langle \hat{a}^\dagger \hat{a} \rangle < 2S$$

HP framework is a powerful method for calculating dispersions and higher order interactions

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Linearized Holstein-Primakoff

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$
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T-Holstein, H.Primakoff PR **58**, 1098, (1940)

Linearized HP

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$
$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right)$$
$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$
$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

Sacrifices

Does not fulfill the boson commutation relations

Hence (ii) is violated

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$
$$[\hat{a}_i, \hat{a}_j] = 0$$

✓

$$\left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hat{S}_i^z \delta_{i,j}$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \frac{1}{2S}\left[\hat{S}_i^+, \hat{S}_j^-\right] = \frac{1}{S}\hat{S}_i^z \delta_{i,j} \neq \delta_{i,j}$$
$$\approx \frac{S}{S}\delta_{i,j} = \delta_{i,j}$$

✗

Holstein-Primakoff

Simplified application

Only linear Spin-Wave theory using the linearized HP

$$\mathcal{H} = -\sum_{i,\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta}$$

$$= -\sum_{i,\Delta} J \cdot \left(\frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_{i+\Delta}^- + \hat{S}_i^- \hat{S}_{i+\Delta}^+ \right\} + \hat{S}_i^z \hat{S}_{i+\Delta}^z \right)$$

$$= -\sum_{i,\Delta} J \cdot \left(S \left\{ \hat{a}_i \hat{a}_{i+\Delta}^\dagger + \hat{a}_i^\dagger \hat{a}_{i+\Delta} \right\} + (S - \hat{n}_i)(S - \hat{n}_{i+\Delta}) \right)$$

$$= -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\} + \underbrace{\sum_{i,\Delta} J \hat{n}_i \hat{n}_{i+\Delta}}_{\approx 0}$$

$$\approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}_{i+\Delta}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^+ = \hat{S}_j^x + \mathrm{i}\hat{S}_j^y = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j} \cdot \hat{a}_j$$

$$\hat{S}_j^- = \hat{S}_j^x - \mathrm{i}\hat{S}_j^y = \hat{a}_j^\dagger \cdot \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}$$

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Linearized HP

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y = \frac{\hat{S}_j^+ - \hat{S}_j^-}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^x = \frac{\hat{S}_j^+ + \hat{S}_j^-}{2} \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger \right)$$

$$\hat{S}_j^+ \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}_j^- \approx \sqrt{2S} \cdot \hat{a}_j^\dagger$$

Holstein-Primakoff

Rewrite the Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator:

Linearized HP

$$\hat{S}^z_j = S - \hat{a}^\dagger_j \hat{a}_j$$

$$\hat{S}^y_j = \frac{\hat{S}^+_j - \hat{S}^-_j}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}^\dagger_j\right)$$

$$\hat{S}^x_j = \frac{\hat{S}^+_j + \hat{S}^-_j}{2} \approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}^\dagger_j\right)$$

$$\hat{S}^+_j \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}^-_j \approx \sqrt{2S} \cdot \hat{a}^\dagger_j$$

Holstein-Primakoff

Rewrite the Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}_n = [\mathcal{H}, \hat{a}_n]$$

$$= 0 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \left[\hat{a}^\dagger_i \hat{a}_j, \hat{a}_n\right]$$

$$= -\sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \delta_{i,n}\hat{a}_j$$

$$= -\sum_{j} \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j$$

Problem: Coupling between all sites!
Practically unsolvable for real systems

Linearized HP

$$\hat{S}^z_j = S - \hat{a}^\dagger_j \hat{a}_j$$

$$\hat{S}^y_j = \frac{\hat{S}^+_j - \hat{S}^-_j}{2\mathrm{i}} \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}^\dagger_j\right)$$

$$\hat{S}^x_j = \frac{\hat{S}^+_j + \hat{S}^-_j}{2} \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}^\dagger_j\right)$$

$$\hat{S}^+_j \approx \sqrt{2S} \cdot \hat{a}_j$$

$$\hat{S}^-_j \approx \sqrt{2S} \cdot \hat{a}^\dagger_j$$

Holstein-Primakoff

The new Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}_n = -\sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j$$

Problem: Coupling between all sites!
Practically unsolvable for real systems

Going to Fourier-space

Lattice Fourier transformation

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

$$\hat{a}^\dagger(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \hat{a}^\dagger_n \exp(\mathrm{i}\vec{k}\vec{r}_n).$$

Holstein-Primakoff

The new Hamiltonian

$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

$$\approx -NJS^2 - \sum_{i,j} \mathscr{H}^{\mathrm{SW}}_{i,j} \cdot \hat{a}^\dagger_i \hat{a}_j$$

Time evolution of an operator:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{a}_n = -\sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j$$

Problem: Coupling between all sites!
Practically unsolvable for real systems

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Time evolution:

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(\frac{\partial}{\partial t}\hat{a}_n\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n),$$

Holstein-Primakoff

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$$\mathcal{H} \approx -NJS^2 - \sum_{i,\Delta} JS \cdot \left\{ \hat{a}^\dagger_{i+\Delta}\hat{a}_i + \hat{a}^\dagger_i \hat{a}_{i+\Delta} - \hat{n}_i - \hat{n}_{i+\Delta} \right\}$$

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Plug in

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}} \sum_n \left(-\sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j\right) \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= \frac{1}{\sqrt{N}} \sum_n \sum_j \mathscr{H}^{\mathrm{SW}}_{n,j} \cdot \hat{a}_j \exp(-\mathrm{i}\vec{k}\vec{r}_n)$$

$$= \dots$$

Holstein-Primakoff

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \frac{1}{\sqrt{N}}\sum_{n}\exp(-\mathrm{i}\vec{k}\vec{r}_n)\left(-\frac{\mathrm{i}}{\hbar}\sum_{j}\mathscr{H}^{\mathrm{SW}}_{n,j}\overbrace{\frac{1}{\sqrt{N}}\sum_{\vec{k}'}\exp(\mathrm{i}\vec{k}'\vec{r}_j)\hat{a}(\vec{k}')}^{\hat{a}_j = \frac{1}{\sqrt{N}}\sum_{\vec{k}}\exp(\mathrm{i}\vec{k}\vec{r}_j)\hat{a}(\vec{k})}\right)$$

$$= -\frac{\mathrm{i}}{\hbar\cdot N}\sum_{\vec{k}'}\sum_{n,j}\mathscr{H}^{\mathrm{SW}}_{n,j}\exp(-\mathrm{i}\vec{k}\vec{r}_n + 0 + \mathrm{i}\vec{k}'\vec{r}_j)\hat{a}(\vec{k}')$$

$$= -\frac{\mathrm{i}}{\hbar\cdot N}\sum_{\vec{k}'}\sum_{n,j}\mathscr{H}^{\mathrm{SW}}_{n,j}\exp(-\mathrm{i}\vec{k}\vec{r}_n + \mathrm{i}\vec{k}\vec{r}_j - \mathrm{i}\vec{k}\vec{r}_j + \mathrm{i}\vec{k}'\vec{r}_j)\hat{a}(\vec{k}')$$

$$= -\frac{\mathrm{i}}{\hbar\cdot N}\sum_{\vec{k}'}\sum_{n,j}\mathscr{H}^{\mathrm{SW}}_{n,\underline{j}}\exp(\mathrm{i}\vec{k}(-\vec{r}_n + \vec{r}_{\underline{j}}))\exp(\mathrm{i}\vec{r}_j(-\vec{k}+\vec{k}'))\hat{a}(\vec{k}')$$

$$= -\frac{\mathrm{i}}{\hbar}\sum_{\vec{k}'}\underbrace{\left(\frac{1}{N}\sum_{j}\exp(\mathrm{i}\vec{r}_j(\vec{k}'-\vec{k}))\right)}_{=\delta_{\vec{k},\vec{k}'}}\underbrace{\left(\sum_{n}\exp(\mathrm{i}\vec{k}(\vec{r}_{\underline{0}}-\vec{r}_n))\mathscr{H}^{\mathrm{SW}}_{n,\underline{0}}\right)}_{=\mathscr{H}^{SW}_{\vec{k}}}\hat{a}(\vec{k}')$$

Using translational invariance

$$= -\frac{\mathrm{i}}{\hbar}\mathscr{H}^{\mathrm{SW}}_{\vec{k}}\hat{a}(\vec{k})$$

L. Rósza, Lecture Notes (2022)

Holstein-Primakoff

$$\boxed{\mathcal{H} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})}$$

Solving the differential equation

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \ldots = -\frac{\mathrm{i}}{\hbar}\mathscr{H}^{\mathrm{SW}}_{\vec{k}}\hat{a}(\vec{k}) \quad \Rightarrow \quad \hat{a}(\vec{k},t) = \hat{a}(\vec{k},0)\exp\left(-\mathrm{i}\omega_{\vec{k}}t\right)$$

With the spin-wave frequency

$$\begin{aligned}\hbar\omega_{\vec{k}} &= \mathscr{H}^{\mathrm{SW}}_{\vec{k}} \\ &= S\left(J_0 - J_{\vec{k}}\right)\end{aligned}$$

L. Rósza, Lecture Notes (2022)

$$J_{\vec{k}} \propto \sum_j J_{j,0}\exp(-\mathrm{i}\vec{k}(\vec{r}_j - \vec{r}_0)) \qquad J_0 = J_{\vec{k}=0}$$

Holstein-Primakoff

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L. Rósza, Lecture Notes (2022)

$$\underline{J_{\vec{k}}} \propto \sum_j J_{j,0}\exp(-\mathrm{i}\vec{k}(\vec{r}_j - \vec{r}_0)) \qquad J_0 = J_{\vec{k}=0}$$

$$\hbar\omega_{\vec{k}} = \ldots \propto JS\sum_{\Delta}\left(1 - \cos\left(\vec{k}\cdot\vec{r}_{\Delta}\right)\right)$$

Holstein-Primakoff

$$\boxed{\mathcal{H} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})}$$

Solving the differential equation

$$\frac{\partial}{\partial t}\hat{a}(\vec{k}) = \ldots = -\frac{\mathrm{i}}{\hbar}\mathscr{H}_{\vec{k}}^{\mathrm{SW}}\hat{a}(\vec{k}) \quad \Rightarrow \quad \hat{a}(\vec{k},t) = \hat{a}(\vec{k},0)\exp\left(-\mathrm{i}\omega_{\vec{k}}t\right)$$

With the spin-wave frequency

$$\hbar\omega_{\vec{k}} = \mathscr{H}_{\vec{k}}^{\mathrm{SW}}$$
$$= S\left(J_0 - J_{\vec{k}}\right)$$

L. Rósza, Lecture Notes (2022)

$$\hat{a}_j(t) = \frac{1}{\sqrt{N}}\sum_{\vec{k}}\exp(\mathrm{i}\vec{k}\vec{r}_j)\overbrace{\hat{a}(\vec{k},t)}$$
$$= \frac{1}{\sqrt{N}}\sum_{\vec{k}}\exp(\mathrm{i}\vec{k}\vec{r}_j)\underbrace{\hat{a}(\vec{k},0)\cdot\exp(-\mathrm{i}\omega_{\vec{k}}t)}$$

$$J_{\vec{k}} \propto \sum_j J_{j,0}\exp(-\mathrm{i}\vec{k}(\vec{r}_j - \vec{r}_0)) \qquad J_0 = J_{\vec{k}=0}$$

$$\hat{a}_j(\vec{k}_0,t) = \underbrace{\frac{1}{\sqrt{N}}\hat{a}(\vec{k}_0,0)}\cdot\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)$$

$$\hbar\omega_{\vec{k}} = \ldots \propto JS\sum_{\Delta}\left(1-\cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)$$

$$\boxed{\hat{a}_j(\vec{k}_0,t) = \hat{A}\cdot\exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0}t)}$$

Holstein-Primakoff Dispersion Relation

$$
\begin{aligned}
\hbar\omega_{\vec{k}} &= S\left(J_0 - J_{\vec{k}}\right) \\
&= S\sum_{\Delta} J\exp(0) - \sum_{\Delta} J\exp\left(-\mathrm{i}\vec{k}(r_\Delta)\right) \\
&\propto zJS + JS\sum_{\Delta>0} -\exp\left(-\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) - \exp\left(\mathrm{i}\vec{k}\cdot\vec{r}_\Delta\right) \\
&\propto zJS + JS\sum_{\Delta>0} -2\cos\left(\vec{k}\cdot\vec{r}_\Delta\right) \\
&\propto zJS + JS\sum_{\Delta} -\cos\left(\vec{k}\cdot\vec{r}_\Delta\right) \\
&\propto JS\sum_{\Delta}\left(1-\cos\left(\vec{k}\cdot\vec{r}_\Delta\right)\right)
\end{aligned}
$$

Holstein-Primakoff

Back to Real-space

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\boxed{\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)}$$

Linearized HP

$$\boxed{\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right) \end{aligned}}$$

Holstein-Primakoff

Back to Real-space

$$\hat{a}_j(\vec{k}_0, t) = \frac{1}{\sqrt{N}} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)\hat{a}(\vec{k}_0, 0)$$

$$\boxed{\hat{a}_j(\vec{k}_0, t) = \hat{A} \exp(\mathrm{i}\vec{k}_0\vec{r}_j - \mathrm{i}\omega_{\vec{k}_0} t)}$$

S.Blundell, *Magnetism in Condensed matter* (2000)

Linearized HP

$$\hat{S}_j^z = S - \hat{a}_j^\dagger \hat{a}_j$$

$$\hat{S}_j^y \approx \frac{\sqrt{2S}}{2\mathrm{i}}\left(\hat{a}_j - \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^x \approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

$$\hat{S}_j^z(t) = S - |\hat{A}|^2$$

$$\hat{S}_j^y(t) = \sqrt{2S}\hat{A} \cdot \sin(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A)$$

$$\hat{S}_j^x(t) = \sqrt{2S}\hat{A} \cdot \cos(\vec{k}_0\vec{r}_j + \omega_{\vec{k}_0} t + \phi_A)$$

This explains why the average we calculated earlier was 0

Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i$$

In-plane anisotropy

$$\mathscr{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Holstein-Primakoff & Bogoliubov Transformation

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$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i$$

Linearized HP

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$$\begin{aligned} \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i &= K \hat{S}_i^x \cdot \hat{S}_i^x \\ &\approx K \frac{S}{2} \left(\hat{a}_i \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger \right) \\ &= K \frac{S}{2} \left(\hat{a}_i \hat{a}_i + 2\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger \right) + K \frac{S}{2} \end{aligned}$$

$$\mathcal{H}_{\text{iso}} \approx -E_0 - \sum_{i,j} \mathscr{H}_{i,j}^{\text{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j$$

Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i$$

In-plane anisotropy

$$\mathscr{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Linearized HP

$$\boxed{\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2\mathrm{i}} \left(\hat{a}_j - \hat{a}_j^\dagger\right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2} \left(\hat{a}_j + \hat{a}_j^\dagger\right) \end{aligned}}$$

$$\begin{aligned} \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i &= K \hat{S}_i^x \cdot \hat{S}_i^x \\ &\approx K \frac{S}{2} \left(\hat{a}_i \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger\right) \\ &= K \frac{S}{2} \left(\hat{a}_i \hat{a}_i + 2\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger\right) + K \frac{S}{2} \end{aligned}$$

$$\mathcal{H} \approx -\tilde{E}_0 - \sum_{i,j} \tilde{\mathscr{H}}_{i,j}^{\mathrm{SW}} \cdot \hat{a}_i^\dagger \hat{a}_j + \frac{KS}{2} \left(\underline{\hat{a}_i \hat{a}_i} + \underline{\hat{a}_i^\dagger \hat{a}_i^\dagger}\right)$$

How to diagonalize this?

Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_i \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i$$

In-plane anisotropy

$$\mathscr{K}_i = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Linearized HP

$$\boxed{\begin{aligned} \hat{S}_j^z &= S - \hat{a}_j^\dagger \hat{a}_j \\ \hat{S}_j^y &\approx \frac{\sqrt{2S}}{2i}\left(\hat{a}_j - \hat{a}_j^\dagger\right) \\ \hat{S}_j^x &\approx \frac{\sqrt{2S}}{2}\left(\hat{a}_j + \hat{a}_j^\dagger\right) \end{aligned}}$$

$$\begin{aligned} \hat{S}_i \cdot \mathscr{K}_i \cdot \hat{S}_i &= K \hat{S}_i^x \cdot \hat{S}_i^x \\ &\approx K\frac{S}{2}\left(\hat{a}_i\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger + \hat{a}_i^\dagger\hat{a}_i + \hat{a}_i^\dagger\hat{a}_i^\dagger\right) \\ &= K\frac{S}{2}\left(\hat{a}_i\hat{a}_i + 2\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i^\dagger\hat{a}_i^\dagger\right) + K\frac{S}{2} \end{aligned}$$

$$\mathcal{H} \approx -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K}\left(\hat{a}_i\hat{a}_i\right) + \mathrm{c.c}$$

How to diagonalize this ?

Holstein-Primakoff & Bogoliubov Transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

How to diagonalize this

$$\begin{aligned}\mathcal{H} &= -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K}\left(\hat{a}_i \hat{a}_i\right) + \text{c.c} \\ &= -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

Holstein-Primakoff & Bogoliubov Transformation

$$\hat{a} = (\hat{a}_1, \dots, \hat{a}_N)^T$$
$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger\right)^T$$

How to diagonalize this

$$\begin{aligned}\mathcal{H} &= -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K}\left(\hat{a}_i \hat{a}_i\right) + \text{c.c} \\ &= -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\end{aligned}$$

Seek a Bogoliubov transformation

$$\begin{aligned}\hat{a} &= U\hat{\alpha} + V^\star \hat{\alpha}^\dagger \\ \hat{a}^\dagger &= V\hat{\alpha} + U^\star \hat{\alpha}^\dagger\end{aligned}$$

That diagonalizes $\mathcal{H}$ to $$\mathcal{H} = -\left(\hat{\alpha}^\dagger, \hat{\alpha}\right) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} + \text{c-number}$$

Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$\hat{a} = \left(\hat{a}_1, \ldots, \hat{a}_N\right)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

Holstein-Primakoff & Bogoliubov Transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently

Holstein-Primakoff & Bogoliubov Transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - (\hat{a}^\dagger, \hat{a}) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Idea: Diagonalize with

$$\begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad ✗$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Since we need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently $\quad \mathcal{I} = \mathcal{T}\mathcal{I}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{I}\mathcal{T} \qquad \mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$

Thus it needs to be a pseudo unitary transformation

Holstein-Primakoff & Bogoliubov Transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$\hat{a} = U\hat{\alpha} + V^\star\hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star\hat{\alpha}^\dagger$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

Need to preserve the commutation relations:

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Or equivalently

$$\mathcal{I} = \mathcal{T}\mathcal{I}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{I}\mathcal{T} \qquad \mathcal{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$\mathbb{I} = (\mathcal{I}\mathcal{T}) \cdot (\mathcal{I}\mathcal{T}^\dagger) = (\mathcal{I}\mathcal{T}^\dagger) \cdot (\mathcal{I}\mathcal{T})$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} = \mathcal{T}^{-1} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Holstein-Primakoff & Bogoliubov Transformation

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)^T$$

$$\hat{a}^\dagger = \left(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger\right)^T$$

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$\hat{a} = U\hat{\alpha} + V^\star \hat{\alpha}^\dagger$$

$$\hat{a}^\dagger = V\hat{\alpha} + U^\star \hat{\alpha}^\dagger$$

What we need to solve

$$\underbrace{\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}}_{=:\mathscr{M}} = \begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}^\dagger \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U & V^\star \\ V & U^\star \end{pmatrix}}_{=\mathcal{T}} \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix}$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}^\dagger \end{pmatrix} = \mathcal{T}^{-1} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Such that

$$\mathcal{H} = -\tilde{E}_0 - \sum_{i,j} \mathscr{M}_{i,j}^{\text{diag}} \cdot \hat{\alpha}_i^\dagger \hat{\alpha}_j$$

$$U^\dagger U - V^\dagger V = \mathbb{I}$$

$$V^\dagger U^\star - U^\dagger V^\star = 0$$

Holstein-Primakoff & Bogoliubov Transformation

How to diagonalize this

$$\mathcal{H} = -\tilde{E}_0 - \sum_{i,j} H_{i,j} \cdot \hat{a}_i^\dagger \hat{a}_j + \mathcal{K}\left(\underline{\hat{a}_i \hat{a}_i}\right) + \text{c.c}$$

$$= -\tilde{E}_0 - \left(\hat{a}^\dagger, \hat{a}\right) \begin{pmatrix} H & \mathcal{K} \\ \mathcal{K}^\star & H^\star \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

$$= -\tilde{E}_0 - \sum_{i,j} \mathscr{M}_{i,j}^{\text{diag}} \cdot \hat{\alpha}_i^\dagger \hat{\alpha}_j$$

Lattice Fourier transformation

$$\mathcal{H} = \sum_{\vec{k}} \mathscr{M}_{\vec{k}} \cdot \hat{\alpha}^\dagger(\vec{k}) \hat{\alpha}(\vec{k})$$

Holstein-Primakoff & Bogoliubov Transformation

Considering more interactions

Adding DMI

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_{i,j} \vec{D}_{i,j} \cdot \hat{S}_i \times \hat{S}_j$$

$$\mathcal{H}_{\mathrm{iso}} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

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$$\mathcal{H} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \mu^\star_{\vec{k}} \underline{\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(-\vec{k})} + \mu_{\vec{k}} \underline{\hat{a}(\vec{k})\hat{a}(-\vec{k})}$$

$$\mu_{\vec{k}} = \mu_{\vec{k}}(J_0, J_{\vec{k}}, D_{\vec{k}}, \theta)$$

Holstein-Primakoff & Bogoliubov Transformation

Bogoliubov transformation

$$\boxed{\begin{aligned}\hat{\alpha}(\vec{k}) &= u_{\vec{k}}\hat{a}(\vec{k}) - v^{\star}_{\vec{k}}\hat{a}^{\dagger}(-\vec{k})\\ \hat{\alpha}^{\dagger}(\vec{k}) &= u^{\star}_{\vec{k}}\hat{a}^{\dagger}(\vec{k}) - v_{\vec{k}}\hat{a}(\vec{k})\end{aligned}}$$

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_{i,j} \vec{D}_{i,j} \cdot \hat{S}_i \times \hat{S}_j$$

$$\mathcal{H}_{\text{iso}} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}}\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})$$

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$$\mathcal{H} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}}\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k}) + \mu^{\star}_{\vec{k}}\underline{\hat{a}^{\dagger}(\vec{k})\hat{a}^{\dagger}(-\vec{k})} + \mu_{\vec{k}}\underline{\hat{a}(\vec{k})\hat{a}(-\vec{k})}$$

$$\begin{aligned}\left[\hat{\alpha}_{\vec{k}}, \hat{\alpha}^{\dagger}_{\vec{k}'}\right] &= \left[u_{\vec{k}}\hat{a}_{\vec{k}} + v_{\vec{k}}\hat{a}^{\dagger}_{\vec{k}}, u_{\vec{k}'}\hat{a}^{\dagger}_{\vec{k}'} + v_{\vec{k}'}\hat{a}_{\vec{k}'}\right]\\ &= u_{\vec{k}}u_{\vec{k}'}\underbrace{\left[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}\right]}_{=\delta_{\vec{k},\vec{k}'}} + v_{\vec{k}}v_{\vec{k}'}\underbrace{\left[\hat{a}^{\dagger}_{\vec{k}}, \hat{a}_{\vec{k}'}\right]}_{=-\delta_{\vec{k},\vec{k}'}}\\ &= \left(u^2_{\vec{k}} - v^2_{\vec{k}}\right)\delta_{\vec{k},\vec{k}'}\end{aligned}$$

$$\mu_{\vec{k}} = \mu_{\vec{k}}(J_0, J_{\vec{k}}, D_{\vec{k}}, \theta)$$

Holstein-Primakoff & Bogoliubov Transformation

$$\mathcal{H} = -\sum_{i\Delta} J \cdot \hat{S}_i \cdot \hat{S}_{i+\Delta} - \sum_{i,j} \vec{D}_{i,j} \cdot \hat{S}_i \times \hat{S}_j$$

Bogoliubov transformation

$$\mathcal{H}_{\text{iso}} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

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$$\boxed{\begin{aligned}\hat{\alpha}(\vec{k}) &= u_{\vec{k}}\hat{a}(\vec{k}) - v^\star_{\vec{k}}\hat{a}^\dagger(-\vec{k}) \\ \hat{\alpha}^\dagger(\vec{k}) &= u^\star_{\vec{k}}\hat{a}^\dagger(\vec{k}) - v_{\vec{k}}\hat{a}(\vec{k})\end{aligned}}$$

$$\mathcal{H} \approx E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \mu^\star_{\vec{k}} \underline{\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(-\vec{k})} + \mu_{\vec{k}} \underline{\hat{a}(\vec{k})\hat{a}(-\vec{k})}$$

$$\left|u_{\vec{k}}\right|^2 - \left|v_{\vec{k}}\right|^2 = 1$$

$$\mu_{\vec{k}} = \mu_{\vec{k}}(J_0, J_{\vec{k}}, D_{\vec{k}}, \theta)$$

$$\mathcal{H} = \sum_{\vec{k}} \Upsilon_{\vec{k}} \cdot \hat{\alpha}^\dagger(\vec{k})\hat{\alpha}(\vec{k})$$

Visualising a Magnon

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = -\frac{\gamma}{(1+\alpha^2)\mu_S}\vec{S}_i \times \left(\vec{H}_i(t) + \alpha\vec{S}_i \times \vec{H}_i(t)\right)$$

$$\vec{H}_i(t) = -\frac{\partial\mathcal{H}}{\partial\vec{S}_i}$$

Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

Time integration with Heun’s methods

Additional property is the damping α

Engine_LLG\Test.ipynb

Visualising a Magnon

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = -\frac{\gamma}{(1+\alpha^2)\mu_S}\vec{S}_i \times \left(\vec{H}_i(t) + \alpha\vec{S}_i \times \vec{H}_i(t)\right)$$

$$\vec{H}_i(t) = \vec{\xi}_i(T,t) - \frac{\partial \mathcal{H}}{\partial \vec{S}_i}$$

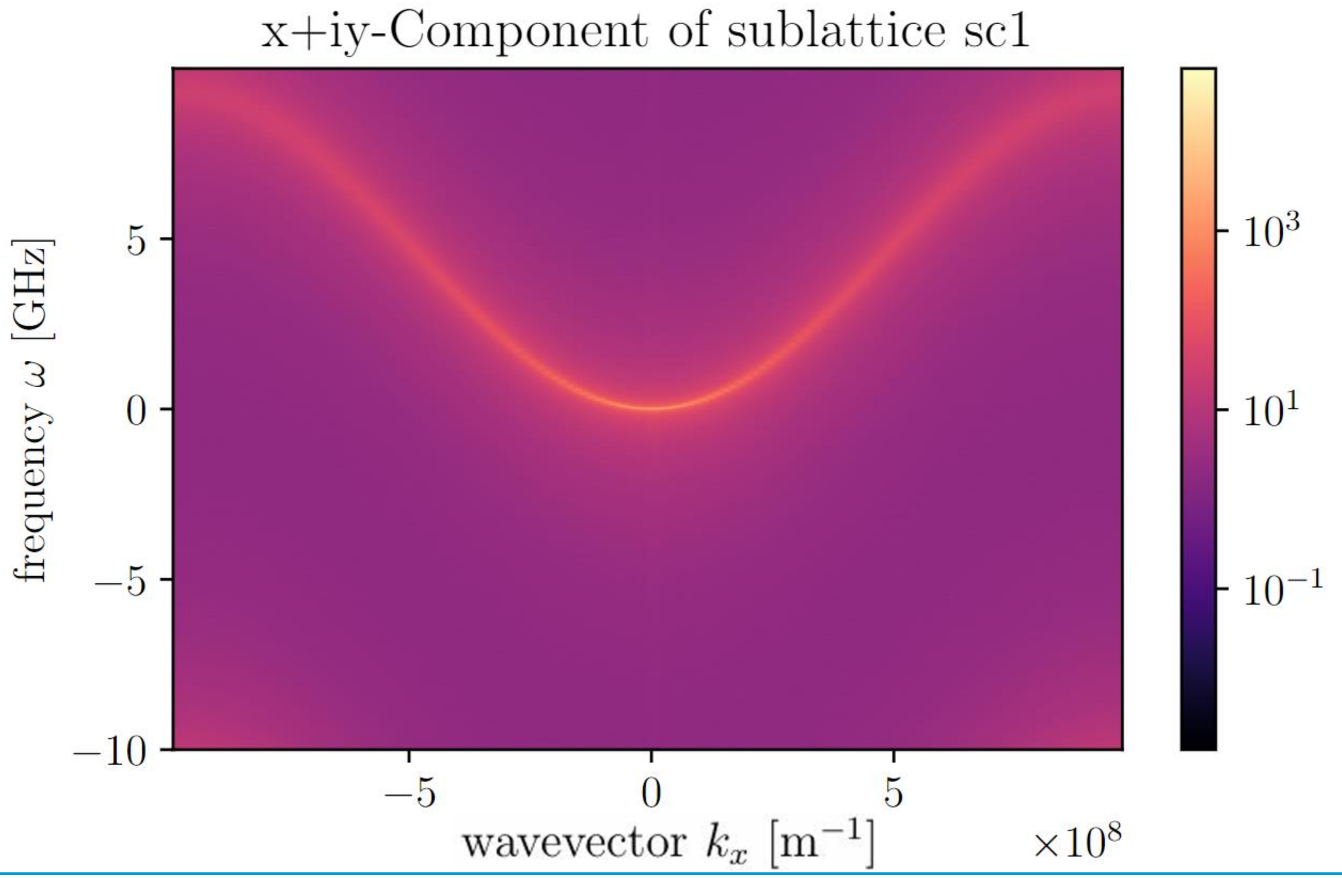
Numerical methods

Integrate using the Landau Lifshitz Gilbert equation.

Time integration with Heun’s methods

Additional property is the damping α

Engine_LLG\Test.ipynb



Outlook

Current research

Easily obtain highly interesting dispersion relations.

Can be used for computing, without energy loss through Joule heating

This is ongoing research in

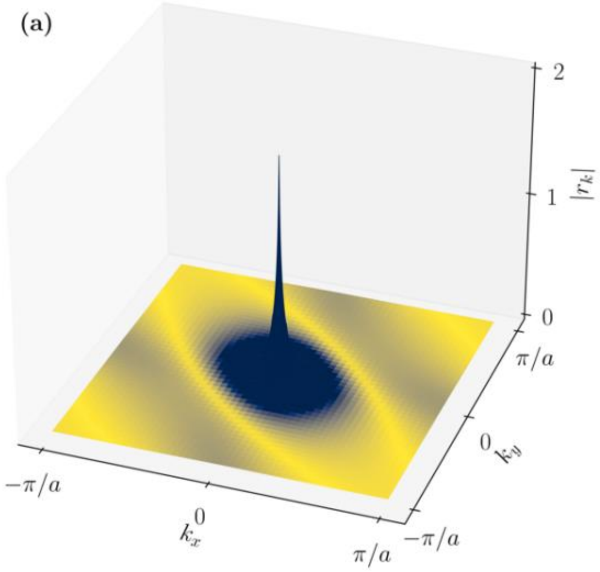
Spin wave diodes *J.Lan et.al.* PRX **5**, 041049, (2015)

Spin wave transitors *A.Chumak et.al.* Nature C. **5**, 4700, (2014)

The most modern research even includes Altermagnets, which have distinct symmetry enforced properties regarding the dispersion in different directions

Topological magnons *P. McClarty* Annual Reviews **13**, 171-190, (2022)

Squeezed magnons *D. Wuhrer et.al. PR* **5**, 043124, (2023)



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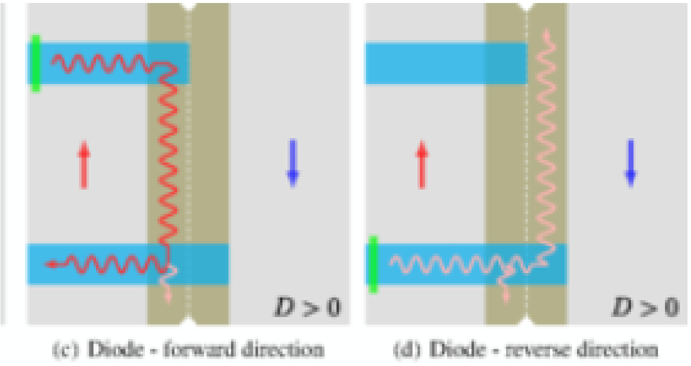
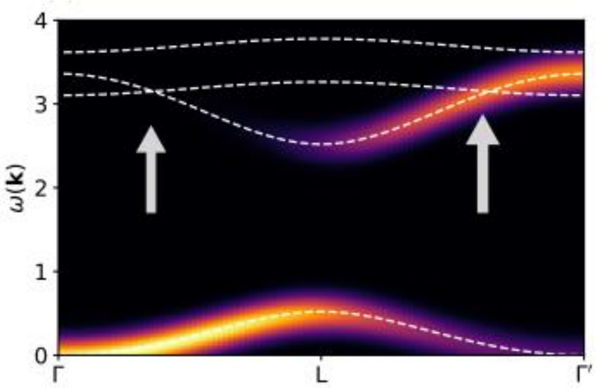
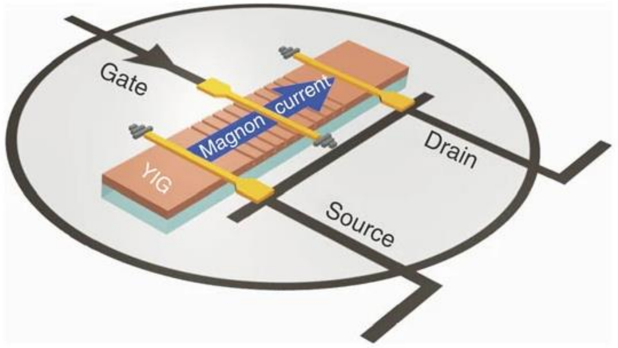


J.Lan et.al. PRX **5**, 041049, (2015)



Magnon transistor scheme



A.Chumak et.al. Nature C. **5**, 4700, (2014)