Chapter 3

INTERACTING SPIN WAVES IN FERROMAGNETS

3.1. Neutron Scattering Cross-Section

Thermal neutrons used in the experiments have a typical wavelength $\lambda \sim 1.81\,\text{Å}$ (wavevector $k_i = \frac{2\pi}{\lambda} \sim 3.47\,\text{Å}^{-1}$) and energy $E_i = \frac{\hbar^2 k_i^2}{2M_n} \sim 25\,\text{meV} = 290\,\text{K}$. The wavelength and the energy of the neutrons of the incident beam are comparable with the lattice spacing and with the energy of the elementary excitations of the sample, respectively, so that they are suitable probes to investigate the spin wave dispersion relation. Neutrons interact with matter through nuclear forces (neutron-ion interaction) and magnetic forces (neutron-electron interaction) due to the interaction between the neutron spin $\sigma = 1/2$ and the orbital and spin momentum of the electrons of an ion. The partial differential magnetic cross-section for unpolarized neutrons of incident energy E_i scattered into an element of solid angle Ω with energy between E_f and $E_f + dE_f$ is given by[23]

$$\frac{d^2\sigma}{d\Omega dE_f} = r_0^2 \frac{k_f}{k_i}\left[\frac{1}{2} g F(\boldsymbol{K})\right]^2 e^{-2W(\boldsymbol{K})} \sum_{\alpha,\beta}\left(\delta_{\alpha,\beta} - \frac{K_\alpha K_\beta}{K^2}\right) S^{\alpha\beta}(\boldsymbol{K},\omega) \qquad (3.1.1)$$

where

$$S^{\alpha\beta}(\boldsymbol{K},\omega) = \sum_{i,j} e^{-i\boldsymbol{K}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt e^{-i\omega t}\langle S_i^\alpha S_j^\beta(t)\rangle \qquad (3.1.2)$$

is called the *dynamical structure factor*; $r_0 = \frac{\gamma e^2}{m_e c^2} = -0.54\times 10^{-12}\,\text{cm}$; g is the gyromagnetic ratio ($g = 2$ for spin-only scattering); E_f is the final energy of the neutron, $\boldsymbol{k}_f$ and $\boldsymbol{k}_i$ are the wavevectors of the outgoing and incoming neutron, respectively: $\boldsymbol{K} = \boldsymbol{k}_i - \boldsymbol{k}_f$ is the scattering wavevector; $F(\boldsymbol{K})$ is the atomic form factor and $W(\boldsymbol{K}) = \frac{1}{2}\langle(\boldsymbol{K}\cdot\boldsymbol{u})^2\rangle$ is the Debye-Waller factor, in which $\langle \boldsymbol{u}^2\rangle$ is the mean square displacement of the ion. Labels α and β refer to the cartesian components x, y, z; $\hbar\omega = E_i - E_f$ is the transferred energy between the neutron and the sample: if $\hbar\omega > 0$, the neutron *gives* energy to the sample while for $\hbar\omega < 0$, the neutron *gets* energy from the sample. The cross-section (3.1.1) is evaluated by treating the interaction between the magnetic moment of the neutron and the magnetic moment of the ion ($g\mu_B \boldsymbol{S}_i$) in the first Born approximation and assuming that the orbital angular momentum of the ion is zero (ion with half-filled shells) or quenched by the crystal field. If the z component of the total spin of the system $\sum_i S_i^z$ is a constant of the motion, as it occurs for the isotropic or uniaxial ferromagnet, one has

$$\langle S_i^+ S_j^+(t)\rangle = \langle S_i^- S_j^-(t)\rangle = \langle S_i^+ S_j^z(t)\rangle = \langle S_i^- S_j^z(t)\rangle = 0 \tag{3.1.3}$$

since the raising and lowering spin operators S_i^+ and S_i^- change the z-component of the total momentum by a unit. Equation (3.1.3) implies that all non-diagonal terms in Eq. (3.1.1) vanish and the cross-section (3.1.1) reduces to

$$\frac{d^2\sigma}{d\Omega dE_f} = r_0^2 \frac{k_f}{k_i}\left[\frac{1}{2} gF(\boldsymbol{K})\right]^2 e^{-2W(\boldsymbol{K})}\left[\left(1 - \frac{K_z^2}{K^2}\right) S_\parallel(\boldsymbol{K},\omega) + \left(1 + \frac{K_z^2}{K^2}\right) S_\perp(\boldsymbol{K},\omega)\right] \tag{3.1.4}$$

where

$$S_\parallel(\boldsymbol{K},\omega) \equiv S^{zz}(\boldsymbol{K},\omega) = \sum_{i,j} e^{-i\boldsymbol{K}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt e^{-i\omega t}\langle S_i^z S_j^z(t)\rangle \tag{3.1.5}$$

and

$$\begin{aligned} S_\perp(\boldsymbol{K},\omega) &= S^{xx}(\boldsymbol{K},\omega) = S^{yy}(\boldsymbol{K},\omega) \\ &= \frac{1}{4}\sum_{i,j} e^{-i\boldsymbol{K}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt e^{-i\omega t}\langle S_i^+ S_j^-(t) \\ &\quad + S_i^- S_j^+(t)\rangle. \end{aligned} \tag{3.1.6}$$

Using the HP or DM spin-boson transformations (2.1.6) or (2.1.7), keeping only the higher order terms and using the Fourier transforms (2.2.14), one can write the correlation functions occurring in Eqs. (3.1.5) and (3.1.6) in terms of magnon creation and destruction operators:

$$\langle S_i^z S_j^z(t)\rangle \simeq S^2 - \frac{2S}{N}\sum_{\boldsymbol{q}}\langle a_{\boldsymbol{q}} a_{\boldsymbol{q}}^+\rangle \simeq \left\langle \frac{1}{N}\sum_i S_i^z\right\rangle^2 = \langle S_i^z\rangle^2 \tag{3.1.7}$$

and

$$\langle S_i^+ S_j^-(t) + S_i^- S_j^+(t)\rangle \simeq \frac{2S}{N}\sum_{\boldsymbol{q}}[e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}\langle a_{\boldsymbol{q}} a_{\boldsymbol{q}}^+(t)\rangle + e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}\langle a_{\boldsymbol{q}}^+ a_{\boldsymbol{q}}(t)\rangle]. \tag{3.1.8}$$

Replacing Eqs. (3.1.7) and (3.1.8) into Eqs. (3.1.5) and (3.1.6), for a lattice with one atom per unit cell, one obtains

$$S_\parallel(\boldsymbol{K},\omega) = (N\langle S_i^z\rangle)^2 \sum_{\boldsymbol{G}} \delta_{\boldsymbol{K},\boldsymbol{G}}\delta(\hbar\omega) \tag{3.1.9}$$

and

$$S_\perp(\boldsymbol{K},\omega) = 2SN\sum_{\boldsymbol{q},\boldsymbol{G}}\delta_{\boldsymbol{K}+\boldsymbol{q},\boldsymbol{G}}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dte^{-i\omega t}\langle a_{\boldsymbol{q}}a^+_{\boldsymbol{q}}(t)\rangle$$
$$+\,2SN\sum_{\boldsymbol{q},\boldsymbol{G}}\delta_{\boldsymbol{K}-\boldsymbol{q},\boldsymbol{G}}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dte^{-i\omega t}\langle a^+_{\boldsymbol{q}}a_{\boldsymbol{q}}(t)\rangle \qquad (3.1.10)$$

where $\boldsymbol{G}$ is a reciprocal lattice vector. To obtain Eqs. (3.1.9) and (3.1.10), the following relations have been used:

$$\sum_{i,j}e^{-i\boldsymbol{K}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} = N^2\sum_{\boldsymbol{G}}\delta_{\boldsymbol{K},\boldsymbol{G}} \qquad (3.1.11)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}dte^{-i\omega t} = \delta(\omega). \qquad (3.1.12)$$

Equation (3.1.11) takes into account that the vectors $\boldsymbol{q}$ of the Fourier transform (2.2.14) belong to the first Brilloin zone (BZ) while the scattering wavevector $\boldsymbol{K}$ is not restricted to the first BZ. Equation (3.1.12) is the well known representation of the Dirac δ-function. For ferromagnets with more than one atom per unit cell (for instance, the BCC and FCC lattices in which the cubic cell is chosen as unit cell or the honeycomb lattice in 2D), the relationship (3.1.11) should be replaced by

$$\sum_{i,j}e^{-i\boldsymbol{K}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} = N^2_{\text{cell}}\sum_{\boldsymbol{G}}\delta_{\boldsymbol{K},\boldsymbol{G}}\sum_{l,l'=1,n}e^{-i\boldsymbol{K}\cdot(\boldsymbol{\rho}_l-\boldsymbol{\rho}_{l'})} \qquad (3.1.13)$$

where $\boldsymbol{\rho}_l$ is the position of the l-th atom in the unit cell. N_{cell} is the number of unit cells and $N = nN_{\text{cell}}$ is the total number of atoms of the sample, n being the number of atoms in the unit cell. Obviously, for $n = 1$ and $\boldsymbol{\rho}_1 = 0$, Eq. (3.1.13) reduces to Eq. (3.1.11).

In the harmonic approximation, the time evolution of the boson operators occurring in Eq. (3.1.10) is given by

$$a^+_{\boldsymbol{q}}(t) = a^+_{\boldsymbol{q}}e^{i\omega_{\boldsymbol{q}}t}, \quad a_{\boldsymbol{q}}(t) = a_{\boldsymbol{q}}e^{-i\omega_{\boldsymbol{q}}t} \qquad (3.1.14)$$

so that Eq. (3.1.10) becomes

$$S_\perp(\boldsymbol{K},\omega) = 2SN\sum_{\boldsymbol{q},\boldsymbol{G}}[(1+n_{\boldsymbol{q}})\delta_{\boldsymbol{K}+\boldsymbol{q},\boldsymbol{G}}\delta(\hbar\omega-\hbar\omega_{\boldsymbol{q}}) + n_{\boldsymbol{q}}\delta_{\boldsymbol{K}-\boldsymbol{q},\boldsymbol{G}}\delta(\hbar\omega+\hbar\omega_{\boldsymbol{q}})]. \qquad (3.1.15)$$

Then the first term of (3.1.4) gives the elastic contribution ($k_i = k_f$, $E_i = E_f$)

$$\left(\frac{d^2\sigma}{d\Omega dE_f}\right)_{\text{el}} = r_0^2\left[\frac{1}{2}gF(\boldsymbol{K})\right]^2 e^{-2W(\boldsymbol{K})}\left(1-\frac{K_z^2}{K^2}\right)(N\langle S_i^z\rangle)^2\delta(\hbar\omega)\sum_{\boldsymbol{G}}\delta_{\boldsymbol{K},\boldsymbol{G}} \qquad (3.1.16)$$

and the second term give the one-magnon inelastic contribution

$$\left(\frac{d^2\sigma}{d\Omega dE_f}\right)_{\text{inel}} = r_0^2 \frac{k_f}{k_i}\left[\frac{1}{2}gF(\boldsymbol{K})\right]^2 e^{-2W(\boldsymbol{K})}\left(1+\frac{K_z^2}{K^2}\right) \times \frac{NS}{2}\sum_{\boldsymbol{q},\boldsymbol{G}}[(1+n_{\boldsymbol{q}})\delta_{\boldsymbol{K}+\boldsymbol{q},\boldsymbol{G}}\delta(\hbar\omega-\hbar\omega_{\boldsymbol{q}}) + n_{\boldsymbol{q}}\delta_{\boldsymbol{K}-\boldsymbol{q},\boldsymbol{G}}\delta(\hbar\omega+\hbar\omega_{\boldsymbol{q}})]. \tag{3.1.17}$$

The elastic scattering cross-section (3.1.16) is proportional to the square of the magnetization leading to "Bragg peaks" located at each reciprocal lattice vector $\boldsymbol{G}$. The presence of the factor $\delta(\hbar\omega)$ implies that there is no energy transfer between the neutron and the sample so that the scattering is *elastic*. The elastic neutron scattering geometry is shown in Fig. 3.1 by the circumference that represents the locus of points such that $k'_f = k_i$. When the wavevctor $\boldsymbol{k'}_f$ falls on the reciprocal lattice point $\boldsymbol{G}'$, a Bragg peak occurs since $\boldsymbol{K} = \boldsymbol{G}'$. However, the Bragg peaks corresponding to scattering wavevectors $\boldsymbol{K}$ parallel to the magnetization direction disappear because the geometrical factor $1-(K_z/K)^2$ in Eq. (3.1.16) vanishes. This peculiarity can be used to establish the orientation of the magnetic moment in the ferromagnetic sample.

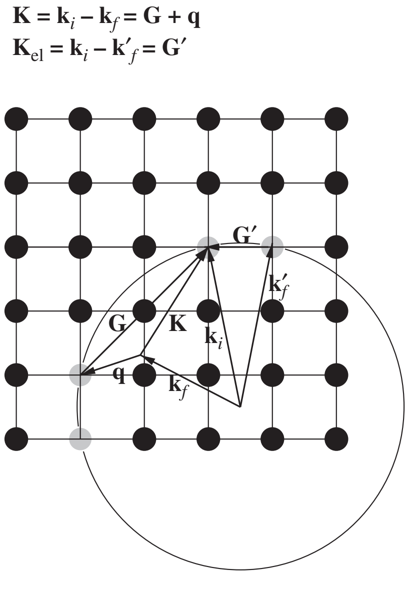

Fig. 3.1. Geometrical representation of a neutron scattering experiment: the grey and black circles are the reciprocal lattice points of the sample under investigation; $\boldsymbol{k}_i$ is the wavevector of the incident neutron, $\boldsymbol{k}_f$ and $\boldsymbol{k'}_f$ are the wavectors of two outgoing neutrons. The wavevctor $\boldsymbol{k'}_f$ lying on the circumference of radius $k'_f = k_i$ gives rise to an elastic Bragg peak since it falls on a reciprocal lattice point (grey circle). For $k_f < k_i$, the inelastic scattering corresponds to the creation of a magnon of wavevector $\boldsymbol{q} = \boldsymbol{K} - \boldsymbol{G}$ and frequency $\omega_{\boldsymbol{q}}$. $\boldsymbol{K}$ and $\boldsymbol{K}_{\text{el}} = \boldsymbol{G}'$ are the inelastic and elastic scattering wavevectors, respectively.

The inelastic cross-section (3.1.17) corresponds to a process in which *one* magnon is created or destroyed. In Fig. 3.1, the inelastic scattering corresponds to the creation of a magnon of wavevector $\boldsymbol{q} = \boldsymbol{K} - \boldsymbol{G}$ where $\boldsymbol{q}$ is restricted to the first BZ centered at the reciprocal lattice vector $\boldsymbol{G}$ and energy $\hbar\omega_{\boldsymbol{q}} = \frac{\hbar^2}{2M_n}(k_i^2 - k_f^2)$. As one can see, the choice of the vector $\boldsymbol{G}$ would not be unique as for the momentum conservation but it becomes unique when the energy conservation is satisfied. In general, a magnon of momentum $\hbar\boldsymbol{q}$ and energy $\hbar\omega_{\boldsymbol{q}}$ is created or destroyed at the expense of the incident neutron: the argument of the Dirac δ-function appearing in the first term of (3.1.17) implies that $\hbar\omega > 0$ so that the incident neutron gives a fraction of its energy to the sample creating a magnon of energy $\hbar\omega_{\boldsymbol{q}}$: such an event can occur even at $T = 0$ explaining the factor $1 + n_{\boldsymbol{q}}$ in front of this term. On the contrary, the argument of the Dirac δ-function appearing in the second term of (3.1.17) implies that $\hbar\omega < 0$ so that the neutron gets an amount of energy from the sample corresponding to the destruction of a magnon of energy $\hbar\omega_{\boldsymbol{q}}$: this event cannot occur at $T = 0$ since the magnons exist only at finite temperature, explaining the factor $n_{\boldsymbol{q}}$ in front of this term.

3.2. Boson Green Function

As shown in the previous section, the one-magnon inelastic neutron scattering (3.1.10) is an important probe to obtain the dispersion relation of the spin wave spectrum in a ferromagnet. Note that the cross-section (3.1.10) reduces to (3.1.17) in the harmonic approximation: this is a good approximation at very low temperatures where the magnon-magnon interaction can be ignored. As the temperature increases, however, the magnon-magnon interaction cannot be ignored and some systematic perturbative expansion has to be introduced in order to evaluate the correlation functions occurring in Eq. (3.1.10).

In this chapter, we illustrate the method of the Green function equation of motion introduced by Zubarev[24] in 1960. The correlation functions occurring in Eq. (3.1.10) may be written in terms of a boson Green function that satisfies a well defined equation of motion. Let us follow Zubarev[24] and define the *retarded* (R) and the *advanced* (A) Green functions as

$$G^{(\mathrm{R,A})}_{\boldsymbol{k},\boldsymbol{k}'}(t) = \mp i\theta(\pm t)\langle[a_{\boldsymbol{k}}(t), a^+_{\boldsymbol{k}'}]\rangle \equiv \langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle^{(\mathrm{R,A})} \tag{3.2.1}$$

where the upper (lower) sign refers to R(A), respectively; $\theta(t)$ is the step-function which is 1 if its argument is positive and 0 if its argument is negative. The equation of motion of the retarded or advanced Green function (3.2.1) is

$$i\hbar\frac{d}{dt}\langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle^{(\mathrm{R,A})} = \hbar\delta(t)\langle[a_{\boldsymbol{k}}, a^+_{\boldsymbol{k}'}]\rangle + \langle\langle[a_{\boldsymbol{k}}, \mathcal{H}]; a^+_{\boldsymbol{k}'}\rangle\rangle^{(\mathrm{R,A})}. \tag{3.2.2}$$

The first term of the right-hand side comes from the derivative of the step-function, the second term comes from the equation of motion of the operator $a_{\boldsymbol{k}}$ in the

Heisenberg representation

$$i\hbar \frac{da_{\boldsymbol{k}}}{dt} = [a_{\boldsymbol{k}}, \mathcal{H}] \tag{3.2.3}$$

where $\mathcal{H}$ is the boson Hamiltonian containing the dynamical interaction between the magnons. The correlation functions are given by

$$\begin{aligned}\langle a_{\boldsymbol{k}}(t) a^+_{\boldsymbol{k}'}\rangle &= \frac{1}{\mathcal{Q}} \sum_{\mu,\nu} \langle \mu | e^{i\mathcal{H}t/\hbar} a_{\boldsymbol{k}} e^{-i\mathcal{H}t/\hbar} |\nu\rangle \langle \nu | a^+_{\boldsymbol{k}'} |\mu\rangle e^{-\beta E_\mu} \\ &= \int_{-\infty}^{+\infty} d\omega' e^{-i\omega' t}\, e^{\beta\hbar\omega'} J(\omega') \end{aligned} \tag{3.2.4}$$

and

$$\langle a^+_{\boldsymbol{k}'} a_{\boldsymbol{k}}(t)\rangle = \int_{-\infty}^{+\infty} d\omega' e^{-i\omega' t}\, J(\omega') \tag{3.2.5}$$

where $|\mu\rangle$ and $|\nu\rangle$ are the eigenstates of the Hamiltonian $\mathcal{H}$ with eigenvalues E_μ and E_ν, respectively; the time evolution of the boson operator $a_{\boldsymbol{k}}(t)$ in the Heisenberg representation has been used and the spectral intensity of the correlation function $J(\omega')$ is defined as

$$J(\omega') = \frac{1}{\mathcal{Q}} \sum_{\mu,\nu} \langle \mu | a_{\boldsymbol{k}} |\nu\rangle \langle \nu | a^+_{\boldsymbol{k}'} |\mu\rangle e^{-\beta E_\nu} \delta\left(\omega' - \frac{E_\nu - E_\mu}{\hbar}\right). \tag{3.2.6}$$

Then the commutator occurring in Eq. (3.2.1) becomes

$$\langle [a_{\boldsymbol{k}}(t), a^+_{\boldsymbol{k}'}]\rangle = \int_{-\infty}^{+\infty} d\omega' e^{-i\omega' t} (e^{\beta\hbar\omega'} - 1) J(\omega'). \tag{3.2.7}$$

Using the integral representation of the step-function[24]

$$\theta(t) = \frac{i}{2\pi} \lim_{\epsilon\to 0} \int_{-\infty}^{+\infty} dx \frac{e^{-ixt}}{x + i\epsilon} \tag{3.2.8}$$

and the δ-function representation (3.1.12), the frequency Fourier transforms of the retarded and advanced Green functions (3.2.1) become

$$G^{(\mathrm{R,A})}_{\boldsymbol{k},\boldsymbol{k}'}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} G^{(\mathrm{R,A})}_{\boldsymbol{k},\boldsymbol{k}'}(t) = \lim_{\epsilon\to 0} \int_{-\infty}^{+\infty} d\omega' (e^{\beta\hbar\omega'} - 1) \frac{J(\omega')}{\omega - \omega' \pm i\epsilon}. \tag{3.2.9}$$

If we define a generalized frequency Green function in the complex z-plane

$$G_{\boldsymbol{k},\boldsymbol{k}'}(z) = \int_{-\infty}^{+\infty} d\omega' (e^{\beta\hbar\omega'} - 1) \frac{J(\omega')}{z - \omega'}, \tag{3.2.10}$$

the retarded and advanced frequency Green functions are given by

$$G^{(\mathrm{R,A})}_{\boldsymbol{k},\boldsymbol{k}'}(\omega) = \lim_{\epsilon\to 0} G_{\boldsymbol{k},\boldsymbol{k}'}(\omega \pm i\epsilon). \tag{3.2.11}$$

Using the operator identity

$$\lim_{\epsilon\to 0}\frac{1}{\omega-\omega_0\pm i\epsilon}=\frac{\mathrm{P}}{\omega-\omega_0}\mp i\pi\delta(\omega-\omega_0) \tag{3.2.12}$$

where P means "principal value", from Eq. (3.2.11) one obtains

$$\lim_{\epsilon\to 0}[G_{\boldsymbol{k},\boldsymbol{k}'}(\omega+i\epsilon)-G_{\boldsymbol{k},\boldsymbol{k}'}(\omega-i\epsilon)]=-2\pi i(e^{\beta\hbar\omega}-1)J(\omega). \tag{3.2.13}$$

Equation (3.2.13) establishes the link between the spectral intensity $J(\omega)$ and the generalized frequency Green function. By means of Eq. (3.2.13), the correlation functions (3.2.4) and (3.2.5) become

$$\langle a_{\boldsymbol{k}}(t)a^{+}_{\boldsymbol{k}'}\rangle=\lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}[1+n(\omega)][G_{\boldsymbol{k},\boldsymbol{k}'}(\omega+i\epsilon)-G_{\boldsymbol{k},\boldsymbol{k}'}(\omega-i\epsilon)] \tag{3.2.14}$$

and

$$\langle a^{+}_{\boldsymbol{k}'}a_{\boldsymbol{k}}(t)\rangle=\lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}n(\omega)[G_{\boldsymbol{k},\boldsymbol{k}'}(\omega+i\epsilon)-G_{\boldsymbol{k},\boldsymbol{k}'}(\omega-i\epsilon)] \tag{3.2.15}$$

where $n(\omega)=(e^{\beta\hbar\omega}-1)^{-1}$. Due to the time-inedependence of the Hamiltonian $\mathcal{H}$, one has

$$\langle a_{\boldsymbol{q}}a^{+}_{\boldsymbol{q}}(t)\rangle=\langle a_{\boldsymbol{q}}(-t)a^{+}_{\boldsymbol{q}}\rangle \tag{3.2.16}$$

and the dynamical structure factor (3.1.10) becomes

$$\begin{aligned}S_{\perp}(\boldsymbol{K},\omega)&=i\frac{SN}{\hbar\pi}\sum_{\boldsymbol{q},\boldsymbol{G}}\delta_{\boldsymbol{K}+\boldsymbol{q},\boldsymbol{G}}\lim_{\epsilon\to 0}[1+n(\omega)][G_{\boldsymbol{q}}(\omega+i\epsilon)-G_{\boldsymbol{q}}(\omega-i\epsilon)]\\&\quad+i\frac{SN}{\hbar\pi}\sum_{\boldsymbol{q},\boldsymbol{G}}\delta_{\boldsymbol{K}-\boldsymbol{q},\boldsymbol{G}}n(-\omega)\lim_{\epsilon\to 0}[G_{\boldsymbol{q}}(-\omega+i\epsilon)-G_{\boldsymbol{q}}(-\omega-i\epsilon)]\end{aligned} \tag{3.2.17}$$

where $G_{\boldsymbol{q}}(\omega)\equiv G_{\boldsymbol{q},\boldsymbol{q}}(\omega)$. Finally, using the spectral representation of the retarded and advanced Green function

$$G^{(\mathrm{R,A})}_{\boldsymbol{k},\boldsymbol{k}'}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}G_{\boldsymbol{k},\boldsymbol{k}'}(\omega\pm i\epsilon) \tag{3.2.18}$$

from Eq. (3.2.2), one obtains the equation of motion for the generalized frequency Green function $G_{\boldsymbol{k},\boldsymbol{k}'}(\omega)\equiv\langle\langle a_{\boldsymbol{k}};a_{\boldsymbol{k}'}\rangle\rangle_{\omega}$

$$\hbar\omega\langle\langle a_{\boldsymbol{k}};a^{+}_{\boldsymbol{k}'}\rangle\rangle_{\omega}=\hbar\langle[a_{\boldsymbol{k}},a^{+}_{\boldsymbol{k}'}]\rangle+\langle\langle[a_{\boldsymbol{k}},\mathcal{H}];a^{+}_{\boldsymbol{k}'}\rangle\rangle_{\omega}. \tag{3.2.19}$$

In conclusion, if we are able to solve the equation of motion of the generalized Green function (3.2.19) to some degree of approximation, we can evaluate directly the dynamical structure factor (3.2.17) that enters the one-magnon inelastic neutron cross-section (3.1.4).

3.3. First-Order Approximation

In order to solve the equation of motion (3.2.19), we begin evaluating the commutator $[a_{\boldsymbol{k}}, \mathcal{H}]$: to do this, we split the boson Hamiltonian $\mathcal{H}$ in two parts

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int} \tag{3.3.1}$$

where the $\mathcal{H}_0$ is the *unperturbed* (harmonic) Hamiltonian given by

$$\mathcal{H}_0 = \sum_{\boldsymbol{q}} \hbar\omega_{\boldsymbol{q}} a^+_{\boldsymbol{q}} a_{\boldsymbol{q}} \tag{3.3.2}$$

and $\mathcal{H}_{\rm int}$ is the interaction Hamiltonian given by

$$\mathcal{H}_{\rm int} = -\frac{zJ}{N} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{q}_4} v_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} a^+_{\boldsymbol{q}_1} a^+_{\boldsymbol{q}_2} a_{\boldsymbol{q}_3} a_{\boldsymbol{q}_4}. \tag{3.3.3}$$

The interaction potential v occurring in Eq. (3.3.3) is

$$v^{DM}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} = \frac{1}{4}(\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_4} + \gamma_{\boldsymbol{q}_2-\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_2-\boldsymbol{q}_4}) - \frac{1}{2}(\gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2}) \tag{3.3.4}$$

or

$$\begin{aligned} v^{HP}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} &= \frac{1}{4}(\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_4} + \gamma_{\boldsymbol{q}_2-\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_2-\boldsymbol{q}_4}) - S\left(1 - \sqrt{1-\frac{1}{2S}}\right) \\ &\quad \times (\gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_4}) = v^{DM}_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4} + \frac{1}{4}(\gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} - \gamma_{\boldsymbol{q}_3} - \gamma_{\boldsymbol{q}_4}) \\ &\quad + \frac{1}{4}\left[1 - 4S\left(1 - \sqrt{1-\frac{1}{2S}}\right)\right](\gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_3} + \gamma_{\boldsymbol{q}_4}) \end{aligned} \tag{3.3.5}$$

for the DM or HP spin-boson transformation, respectively. Note that both potentials (3.3.4) and (3.3.5) are invariant under the exchange $\boldsymbol{q}_1 \Leftrightarrow \boldsymbol{q}_2$ and/or $\boldsymbol{q}_3 \Leftrightarrow \boldsymbol{q}_4$. As for the HP spin-boson transformation, the NO expansion of Eq. (2.2.3) allows us to neglect all terms except $\mathcal{H}_4^{HP}$ of Eq. (2.2.18) with v^{HP} given by Eq. (2.2.19) since all other terms give contributions of higher order in temperature. From Eqs. (3.3.1)–(3.3.3), the commutator in Eq. (3.2.19) gives

$$[a_{\boldsymbol{k}}, \mathcal{H}] = \hbar\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} - \frac{2zJ}{N} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} v_{\boldsymbol{k},\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}. \tag{3.3.6}$$

Replacing Eq. (3.3.6) into the equation of motion (3.2.19), one has

$$(\omega - \omega_{\boldsymbol{k}})\langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega = \delta_{\boldsymbol{k},\boldsymbol{k}'} - \frac{2zJ}{\hbar N}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} v_{\boldsymbol{k},\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}\langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega. \tag{3.3.7}$$

Equation (3.3.7), which is exact, shows that the evaluation of the two-operator (one particle) Green function needs the knowledge of the four-operator (two particle) Green function entered by the commutator of $a_{\boldsymbol{k}}$ with the interaction Hamiltonian $\mathcal{H}_{\text{int}}$. Analogously, the equation of motion of the four-operator Green function involves the six-operator (three-particle) Green function as we will see in the next section. In this way, an *infinite hierarchy* of equations of motion is generated so that an appropriate *decoupling scheme* at some order has to be assigned in order to truncate the infinite hierarchy. We will follow a *systematic* criterion of truncation that allows us to control the order of the perturbation at each step: in particular, we will consider a series expansion in the interaction potential v. To show the procedure, let us begin with a simple exercise: let us evaluate the zero-order ($v = 0$) perturbation theory corresponding to the harmonic approximation. For $v = 0$, the solution of Eq. (3.3.7) is

$$\langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle^{(0)}_\omega = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) \tag{3.3.8}$$

with

$$G^{(0)}_{\boldsymbol{k}}(\omega) = \frac{1}{\omega - \omega_{\boldsymbol{k}}}. \tag{3.3.9}$$

Using the operator identity (3.2.12), the zero-order Green function (3.3.9) becomes

$$G^{(0)}_{\boldsymbol{k}}(\omega \pm i\epsilon) = \frac{\mathrm{P}}{\omega - \omega_{\boldsymbol{k}}} \mp i\pi\delta(\omega - \omega_{\boldsymbol{k}}) \tag{3.3.10}$$

and

$$G^{(0)}_{\boldsymbol{k}}(-\omega \pm i\epsilon) = -\frac{\mathrm{P}}{\omega + \omega_{\boldsymbol{k}}} \mp i\pi\delta(\omega + \omega_{\boldsymbol{k}}). \tag{3.3.11}$$

Replacing Eqs. (3.3.10) and (3.3.11) into Eq. (3.2.17), one obtains

$$S_\perp(\boldsymbol{K},\omega) = 2\frac{NS}{\hbar}\sum_{\boldsymbol{q},\boldsymbol{G}}[(1 + n^{(0)}_{\boldsymbol{q}})\ \delta_{\boldsymbol{K}+\boldsymbol{q},\boldsymbol{G}}\ \delta(\omega - \omega_{\boldsymbol{q}}) + n^{(0)}_{\boldsymbol{q}}\ \delta_{\boldsymbol{K}-\boldsymbol{q},\boldsymbol{G}}\ \delta(\omega + \omega_{\boldsymbol{q}})] \tag{3.3.12}$$

where $n^{(0)}_{\boldsymbol{k}} = (e^{\beta\hbar\omega_{\boldsymbol{k}}} - 1)^{-1}$. The result (3.3.12) obviously coincides with the harmonic result (3.1.17).

To go beyond the zero-order approximation, we need to keep the second term in the right-hand side of Eq. (3.3.7) into account. Let us write explicitly the frequency Fourier transform of the four-operator frequency Green function appearing in Eq. (3.3.7)

$$\langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a^+_{\boldsymbol{k}'}\rangle\rangle_{\omega\pm i\epsilon} = \mp i \int_{-\infty}^{+\infty} dt\theta(\pm t) e^{i\omega t} \langle [a^+_{\boldsymbol{q}_1}(t) a_{\boldsymbol{q}_2}(t) a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}(t), a^+_{\boldsymbol{k}'}]\rangle. \tag{3.3.13}$$

A systematic perturbation expansion in the interaction potential is established assuming that in the higher-order Green functions, the time evolution of the boson operators is ruled by the harmonic Hamiltonian. In this way, the time dependent boson operators appearing in Eq. (3.3.13) may be approximated as follows

$$a_{\boldsymbol{k}}(t) = e^{i\mathcal{H}t/\hbar} a_{\boldsymbol{k}} e^{-i\mathcal{H}t/\hbar} = e^{i\mathcal{H}_0 t/\hbar} a_{\boldsymbol{k}} e^{-i\mathcal{H}_0 t/\hbar} + O(\mathcal{H}_{\rm int}) = a_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t} + O(v) \tag{3.3.14}$$

where v is the magnon-magnon interaction potential. As one can see from Eq. (3.3.14), the neglected terms are at least of order v. Then the first-order calculation gives

$$\begin{aligned}\langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a^+_{\boldsymbol{k}'}\rangle\rangle_{\omega\pm i\epsilon} &= \mp i \int_{-\infty}^{+\infty} dt\theta(\pm t) e^{i(\omega+\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2}-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2})t} \\ &\quad \times \langle [a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}, a^+_{\boldsymbol{k}'}]\rangle + O(v) \\ &= \frac{\langle [a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}, a^+_{\boldsymbol{k}'}]\rangle}{\omega+\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2}-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} \pm i\epsilon} + O(v)\end{aligned} \tag{3.3.15}$$

where the last step of Eq. (3.3.15) is obtained from the integral representation of the step-function (3.2.8). In Eq. (3.3.15), all the boson operators within the commutator are at $t = 0$ so that the commutator itself can be evaluated using the commutation rules (2.2.15) leading to

$$\langle [a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}, a^+_{\boldsymbol{k}'}]\rangle = \langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_1}\rangle \delta_{\boldsymbol{k},\boldsymbol{k}'}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{k},\boldsymbol{q}_2}). \tag{3.3.16}$$

Replacing Eq. (3.3.16) into Eq. (3.3.15), one has

$$\begin{aligned}&\langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a^+_{\boldsymbol{k}'}\rangle\rangle_{\omega\pm i\epsilon} \\ &\quad = \langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_1}\rangle \delta_{\boldsymbol{k},\boldsymbol{k}'}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{k},\boldsymbol{q}_2}) \frac{1}{\omega-\omega_{\boldsymbol{k}} \pm i\epsilon} + O(v) \\ &\quad = n^{(0)}_{\boldsymbol{q}_1} \delta_{\boldsymbol{k},\boldsymbol{k}'}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{k},\boldsymbol{q}_2}) G^{(0)}_{\boldsymbol{k}}(\omega \pm i\epsilon) + O(v).\end{aligned} \tag{3.3.17}$$

Consistent with the first-order perturbation theory, the average $\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_1}\rangle = n^{(0)}_{\boldsymbol{k}}$ is obtained from Eq. (3.2.15) by putting $t = 0$ and replacing $G_{\boldsymbol{k}}(\omega \pm i\epsilon)$ by $G^{(0)}_{\boldsymbol{k}}(\omega \pm i\epsilon)$ given in Eq. (3.3.9). Using Eq. (3.3.17) and from Eq. (3.3.7), one obtains

$$(\omega - \omega_{\boldsymbol{k}})\langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle^{(1)}_{\omega} = \delta_{\boldsymbol{k},\boldsymbol{k}'} \left[1 - \frac{4zJ}{\hbar N} \sum_{\boldsymbol{q}_1} v_{\boldsymbol{k},\boldsymbol{q}_1,\boldsymbol{k},\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{q}_1} G^{(0)}_{\boldsymbol{k}}(\omega)\right]. \tag{3.3.18}$$

Equation (3.3.18) has been obtained using the invariance of the interaction potential v under the exchange of the first two and/or the last two labels. The first-order two-operator Green function is then given by

$$\langle\langle a_{\boldsymbol{k}}; a^{+}_{\boldsymbol{k}'}\rangle\rangle^{(1)}_{\omega} \equiv \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(1)}_{\boldsymbol{k}}(\omega) \tag{3.3.19}$$

where

$$G^{(1)}_{\boldsymbol{k}}(\omega) = G^{(0)}_{\boldsymbol{k}}(\omega) + \Sigma^{(1)}(\boldsymbol{k})\big[G^{(0)}_{\boldsymbol{k}}(\omega)\big]^2 \tag{3.3.20}$$

where the first-order *self-energy* $\Sigma^{(1)}(\boldsymbol{k})$ is given by

$$\Sigma^{(1)}(\boldsymbol{k}) = -\frac{4zJ}{\hbar N}\sum_{\boldsymbol{q}} v_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{k},\boldsymbol{q}} n^{(0)}_{\boldsymbol{q}}. \tag{3.3.21}$$

Using the DM potential (3.3.4), the self-energy (3.3.21) becomes

$$\Sigma^{(1)}_{DM}(\boldsymbol{k}) = -\frac{4zJ}{\hbar N}\sum_{\boldsymbol{q}} v^{DM}_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{k},\boldsymbol{q}} n^{(0)}_{\boldsymbol{q}} = -\frac{2zJ}{\hbar N}\sum_{\boldsymbol{q}}(1+\gamma_{\boldsymbol{k}-\boldsymbol{q}}-\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{q}})n^{(0)}_{\boldsymbol{q}} \tag{3.3.22}$$

while the HP potential (3.3.5) gives

$$\begin{aligned}\Sigma^{(1)}_{HP}(\boldsymbol{k}) &= -\frac{4zJ}{\hbar N}\sum_{\boldsymbol{q}}\left\{v^{DM}_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{k},\boldsymbol{q}} + \frac{1}{2}\left[1-4S\left(1-\sqrt{1-\frac{1}{2S}}\right)\right](\gamma_{\boldsymbol{k}}+\gamma_{\boldsymbol{q}})\right\} n^{(0)}_{\boldsymbol{q}}\\ &= \Sigma^{(1)}_{DM}(\boldsymbol{k}) - \frac{2zJ}{\hbar N}\sum_{\boldsymbol{q}}\left[1-4S\left(1-\sqrt{1-\frac{1}{2S}}\right)\right](\gamma_{\boldsymbol{k}}+\gamma_{\boldsymbol{q}})n^{(0)}_{\boldsymbol{q}}\\ &= \Sigma^{(1)}_{DM}(\boldsymbol{k}) + \left[\frac{1}{8S}+O\left(\frac{1}{S^2}\right)\right]\frac{2zJ}{\hbar N}\sum_{\boldsymbol{q}}(\gamma_{\boldsymbol{k}}+\gamma_{\boldsymbol{q}})\, n^{(0)}_{\boldsymbol{q}}. \end{aligned} \tag{3.3.23}$$

As one can see, the DM and HP first-order self-energies differ at the order $\frac{1}{S}$. Note that the unperturbed spectrum $\omega_{\boldsymbol{k}}$ is of order $O(S)$ and $\Sigma^{(1)}_{DM}(\boldsymbol{k})$ is of order $O(1)$. Moreover, the HP self-energy contains an infinite number of terms when expanded in powers of $\frac{1}{S}$ and it violates the Goldstone theorem since Eq. (3.3.23) does not vanish for $k = 0$, giving a spurious contribution to the Goldstone mode of the order $T^{\frac{3}{2}}$. This puzzle will be solved in the next section. Finally, note that both the DM and HP first-order self-energies are real functions independent of the frequency.

3.4. Second-Order Approximation

In this section, we perform the second-order perturbation expansion and we show that the second-order self-energy is no longer a real function. Instead of treating the four-operator Green function appearing in Eq. (3.3.7) in a perturbative way, we write the exact equation of the motion of the four-operator Green function and we apply the approximation (3.1.14) to the six-operator Green function. The equation of motion of the four-operator Green function is

$$\begin{aligned}\hbar\omega\langle\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega &= \hbar\langle[a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2},a^+_{\boldsymbol{k}'}]\rangle\\ &\quad+\langle\langle[a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2},\mathcal{H}_0];a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\quad+\langle\langle[a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2},\mathcal{H}_{\text{int}}];a^+_{\boldsymbol{k}'}\rangle\rangle_\omega. \qquad (3.4.1)\end{aligned}$$

The evaluation of the three terms on the right-hand side of Eq. (3.4.1) gives

$$\langle[a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2},a^+_{\boldsymbol{k}'}]\rangle = \langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_1}\rangle\delta_{\boldsymbol{k},\boldsymbol{k}'}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}+\delta_{\boldsymbol{k},\boldsymbol{q}_2}), \qquad (3.4.2)$$

$$\begin{aligned}&\langle\langle[a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2},\mathcal{H}_0];a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\qquad=\hbar(\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_1})\langle\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega \qquad (3.4.3)\end{aligned}$$

and

$$\begin{aligned}&\langle\langle[a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2},\mathcal{H}_{\text{int}}];a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\quad=-\frac{2zJ}{N}\left\{\sum_{\boldsymbol{p}}v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}}\langle\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{p}}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\right.\\ &\qquad+\sum_{\boldsymbol{p},\boldsymbol{q}}v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}\langle\langle a^+_{\boldsymbol{q}_1}a^+_{\boldsymbol{p}}a_{\boldsymbol{q}_2}a_{\boldsymbol{q}}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\qquad+\sum_{\boldsymbol{p},\boldsymbol{q}}v_{\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}\langle\langle a^+_{\boldsymbol{q}_1}a^+_{\boldsymbol{p}}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}a_{\boldsymbol{q}}a_{\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\qquad\left.-\sum_{\boldsymbol{p},\boldsymbol{q}}v_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}_1,\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1}\langle\langle a^+_{\boldsymbol{p}}a^+_{\boldsymbol{q}}a_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\right\}. \qquad (3.4.4)\end{aligned}$$

By means of Eqs. (3.4.2)–(3.4.4), the equation of motion of the four-operator Green function (3.4.1) becomes

$$\begin{aligned}&(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_1})\langle\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\quad=\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_1}\rangle\delta_{\boldsymbol{k},\boldsymbol{k}'}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}+\delta_{\boldsymbol{k},\boldsymbol{q}_2})\\ &\qquad-\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p}}v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}}\langle\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{p}}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\qquad-\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p},\boldsymbol{q}}v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}\langle\langle a^+_{\boldsymbol{q}_1}a^+_{\boldsymbol{p}}a_{\boldsymbol{q}_2}a_{\boldsymbol{q}}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\qquad-\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p},\boldsymbol{q}}v_{\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}\langle\langle a^+_{\boldsymbol{q}_1}a^+_{\boldsymbol{p}}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}a_{\boldsymbol{q}}a_{\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega\\ &\qquad+\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p},\boldsymbol{q}}v_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}_1,\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1}\langle\langle a^+_{\boldsymbol{p}}a^+_{\boldsymbol{q}}a_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega. \qquad (3.4.5)\end{aligned}$$

Note that Eq. (3.4.5) is exact but it is of no use since it involves the six-operator Green functions. Following the systematic approach introduced in the previous section, we replace the time evolution of the boson operators in the Green functions of the right-hand side of Eq. (3.4.5) by the time evolution given by Eq. (3.3.14). In so doing, we neglect terms of order $O(v^2)$ in Eq. (3.4.5) and terms of order $O(v^3)$ in Eq. (3.3.7). Within the second-order perturbation expansion, the terms occurring in Eq. (3.4.5) are given by

$$\langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{p}} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) n^{(0)}_{\boldsymbol{q}_1}(\delta_{\boldsymbol{q}_1,\boldsymbol{p}} + \delta_{\boldsymbol{k},\boldsymbol{p}}) + O(v), \tag{3.4.6}$$

$$\begin{aligned}
&\langle\langle a^+_{\boldsymbol{q}_1} a^+_{\boldsymbol{p}} a_{\boldsymbol{q}_2} a_{\boldsymbol{q}} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega \\
&\quad = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) n^{(0)}_{\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{p}} (\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}\delta_{\boldsymbol{p},\boldsymbol{q}} + \delta_{\boldsymbol{q},\boldsymbol{q}_1}\delta_{\boldsymbol{p},\boldsymbol{q}_2} + \delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}\delta_{\boldsymbol{k},\boldsymbol{q}} \\
&\qquad + \delta_{\boldsymbol{p},\boldsymbol{q}_2}\delta_{\boldsymbol{k},\boldsymbol{q}} + \delta_{\boldsymbol{q},\boldsymbol{q}_1}\delta_{\boldsymbol{k},\boldsymbol{q}_2} + \delta_{\boldsymbol{p},\boldsymbol{q}}\delta_{\boldsymbol{k},\boldsymbol{q}_2}) + O(v),
\end{aligned} \tag{3.4.7}$$

$$\begin{aligned}
&\langle\langle a^+_{\boldsymbol{q}_1} a^+_{\boldsymbol{p}} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} a_{\boldsymbol{q}} a_{\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) n^{(0)}_{\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{p}} (\delta_{\boldsymbol{k},\boldsymbol{q}_2}\delta_{\boldsymbol{p},\boldsymbol{q}} \\
&\qquad + \delta_{\boldsymbol{q},\boldsymbol{q}_1}\delta_{\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} + \delta_{\boldsymbol{k},\boldsymbol{q}_2}\delta_{\boldsymbol{k},\boldsymbol{q}} + \delta_{\boldsymbol{k},\boldsymbol{q}}\delta_{\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} \\
&\qquad + \delta_{\boldsymbol{q},\boldsymbol{q}_1}\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{p},\boldsymbol{q}}\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}) + O(v)
\end{aligned} \tag{3.4.8}$$

and

$$\begin{aligned}
&\langle\langle a^+_{\boldsymbol{p}} a^+_{\boldsymbol{q}} a_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega \\
&\quad = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) n^{(0)}_{\boldsymbol{p}} n^{(0)}_{\boldsymbol{q}} (\delta_{\boldsymbol{q},\boldsymbol{q}_1}\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{p},\boldsymbol{q}_2}\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{q},\boldsymbol{q}_1}\delta_{\boldsymbol{k},\boldsymbol{q}_2} \\
&\qquad + \delta_{\boldsymbol{p},\boldsymbol{q}_1}\delta_{\boldsymbol{k},\boldsymbol{q}_2} + \delta_{\boldsymbol{p},\boldsymbol{q}_2}\delta_{\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} + \delta_{\boldsymbol{q},\boldsymbol{q}_2}\delta_{\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}) + O(v).
\end{aligned} \tag{3.4.9}$$

By means of Eqs. (3.4.6)–(3.4.9), the sums in Eq. (3.4.5) become

$$\begin{aligned}
&-\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p}} v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}} \langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{p}} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{p}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega \\
&\quad = -\frac{4zJ}{\hbar N}\delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{q}_1,\boldsymbol{k}} n^{(0)}_{\boldsymbol{q}_1} + O(v^2),
\end{aligned} \tag{3.4.10}$$

$$\begin{aligned}
&-\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p},\boldsymbol{q}} v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}} \langle\langle a^+_{\boldsymbol{q}_1} a^+_{\boldsymbol{p}} a_{\boldsymbol{q}_2} a_{\boldsymbol{q}} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega \\
&\quad = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) \\
&\qquad \times \left\{-\frac{4zJ}{\hbar N} v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{k},\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{q}_2} + [\Sigma^{(1)}(\boldsymbol{k})\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \Sigma^{(1)}(\boldsymbol{q}_1)\delta_{\boldsymbol{k},\boldsymbol{q}_2}] n^{(0)}_{\boldsymbol{q}_1}\right\} \\
&\qquad + O(v^2),
\end{aligned} \tag{3.4.11}$$

$$\begin{aligned}
&-\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p},\boldsymbol{q}} v_{\boldsymbol{q}_2,\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}} \langle\langle a^+_{\boldsymbol{q}_1} a^+_{\boldsymbol{p}} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} a_{\boldsymbol{q}} a_{\boldsymbol{q}_2+\boldsymbol{p}-\boldsymbol{q}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega = \delta_{\boldsymbol{k},\boldsymbol{k}'} G^{(0)}_{\boldsymbol{k}}(\omega) \\
&\qquad \times \left\{-\frac{4zJ}{\hbar N} v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{k},\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{q}_1} n^{(0)}_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} + [\Sigma^{(1)}(\boldsymbol{k})\delta_{\boldsymbol{k},\boldsymbol{q}_2} + \Sigma^{(1)}(\boldsymbol{q}_1)\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}] n^{(0)}_{\boldsymbol{q}_1}\right\} \\
&\qquad + O(v^2)
\end{aligned} \tag{3.4.12}$$

and

$$\frac{2zJ}{\hbar N}\sum_{\boldsymbol{p},\boldsymbol{q}} v_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}_1,\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1}\langle\langle a^+_{\boldsymbol{p}}a^+_{\boldsymbol{q}}a_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega$$
$$= \delta_{\boldsymbol{k},\boldsymbol{k}'}G^{(0)}_{\boldsymbol{k}}(\omega)\left[\frac{4zJ}{\hbar N}v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{q}_1,\boldsymbol{k}}n^{(0)}_{\boldsymbol{q}_2}n^{(0)}_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}\right.$$
$$\left. -\Sigma^{(1)}(\boldsymbol{q}_1)n^{(0)}_{\boldsymbol{q}_1}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}+\delta_{\boldsymbol{k},\boldsymbol{q}_2})\right]+O(v^2) \tag{3.4.13}$$

where $\Sigma^{(1)}(\boldsymbol{q})$ is the first-order self-energy (3.3.21). Replacing Eqs. (3.4.10)–(3.4.13) into the equation of motion (3.4.5), one obtains

$$(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_1})\langle\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2};a^+_{\boldsymbol{k}'}\rangle\rangle_\omega$$
$$=\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_1}\rangle\delta_{\boldsymbol{k},\boldsymbol{k}'}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}+\delta_{\boldsymbol{k},\boldsymbol{q}_2})$$
$$+\delta_{\boldsymbol{k},\boldsymbol{k}'}G^{(0)}_{\boldsymbol{k}}(\omega)\left\{(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}+\delta_{\boldsymbol{k},\boldsymbol{q}_2})\Sigma^{(1)}(\boldsymbol{k})n^{(0)}_{\boldsymbol{q}_1}-\frac{4zJ}{\hbar N}v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{q}_1,\boldsymbol{k}}\right.$$
$$\left.\times\left[n^{(0)}_{\boldsymbol{q}_1}(1+n^{(0)}_{\boldsymbol{q}_2}+n^{(0)}_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2})-n^{(0)}_{\boldsymbol{q}_2}n^{(0)}_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}\right]\right\}+O(v^2). \tag{3.4.14}$$

It remains to evaluate the average $\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_1}\rangle$ at the consistent order in v. To do this, we replace in the correlation function (3.2.15) $G_{\boldsymbol{k}}(\omega)$ by $G^{(1)}_{\boldsymbol{k}}(\omega)$ given by Eq. (3.3.20), thus obtaining

$$\langle a^+_{\boldsymbol{q}_1}a_{\boldsymbol{q}_1}\rangle=\lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{-\infty}^{+\infty}d\omega n(\omega)[G^{(1)}_{\boldsymbol{q}_1}(\omega+i\epsilon)-G^{(1)}_{\boldsymbol{q}_1}(\omega-i\epsilon)]$$
$$=\lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{-\infty}^{+\infty}d\omega n(\omega)[G^{(0)}_{\boldsymbol{q}_1}(\omega+i\epsilon)-G^{(0)}_{\boldsymbol{q}_1}(\omega-i\epsilon)]$$
$$+\Sigma^{(1)}(\boldsymbol{q}_1)\lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{-\infty}^{+\infty}d\omega n(\omega)\{[G^{(0)}_{\boldsymbol{q}_1}(\omega+i\epsilon)]^2-[G^{(0)}_{\boldsymbol{q}_1}(\omega-i\epsilon)]^2\}$$
$$=n^{(0)}_{\boldsymbol{q}_1}+\Sigma^{(1)}(\boldsymbol{q}_1)\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega n(\omega)\frac{2\epsilon(\omega-\omega_{\boldsymbol{q}_1})}{[(\omega-\omega_{\boldsymbol{q}_1})^2+\epsilon^2]^2}+O(v^2). \tag{3.4.15}$$

The integral occurring in Eq. (3.4.15) may be evaluated by parts leading to

$$\int_{-\infty}^{+\infty}d\omega n(\omega)\frac{2\epsilon(\omega-\omega_{\boldsymbol{q}_1})}{[(\omega-\omega_{\boldsymbol{q}_1})^2+\epsilon^2]^2}=\int_{-\infty}^{+\infty}d\omega\left[\frac{dn(\omega)}{d\omega}\right]\frac{\epsilon}{(\omega-\omega_{\boldsymbol{q}_1})^2+\epsilon^2} \tag{3.4.16}$$

since the first term vanishes when evaluated for $\omega=\pm\infty$ and

$$\frac{dn(\omega)}{d\omega}=-\beta\hbar\frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}=-\beta\hbar n(\omega)[1+n(\omega)].$$

Using the definition of the δ-function

$$\delta(\omega-\omega_0)=\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{(\omega-\omega_0)^2+\epsilon^2},$$

one obtains

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega n(\omega)\frac{2\epsilon(\omega-\omega_{\boldsymbol{q}_1})}{[(\omega-\omega_{\boldsymbol{q}_1})^2+\epsilon^2]^2} = -\beta\hbar n_{\boldsymbol{q}_1}^{(0)}(1+n_{\boldsymbol{q}_1}^{(0)}) \tag{3.4.17}$$

so that

$$\langle a_{\boldsymbol{q}_1}^+ a_{\boldsymbol{q}_1}\rangle \equiv n_{\boldsymbol{q}_1}^{(1)} = n_{\boldsymbol{q}_1}^{(0)}[1-\beta\hbar(1+n_{\boldsymbol{q}_1}^{(0)})\Sigma^{(1)}(\boldsymbol{q}_1)]. \tag{3.4.18}$$

Replacing Eq. (3.4.18) into Eq. (3.4.14), one obtains the two-particle Green function

$$\begin{aligned}
&\langle\langle a_{\boldsymbol{q}_1}^+ a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a_{\boldsymbol{k}'}^+\rangle\rangle_\omega \\
&\quad = \delta_{\boldsymbol{k},\boldsymbol{k}'} G_{\boldsymbol{k}}^{(0)}(\omega)\Bigg\{ n_{\boldsymbol{q}_1}^{(0)}(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2}+\delta_{\boldsymbol{k},\boldsymbol{q}_2})[1-\beta\hbar(1+n_{\boldsymbol{q}_1}^{(0)})\Sigma^{(1)}(\boldsymbol{q}_1)+G_{\boldsymbol{k}}^{(0)}(\omega)\Sigma^{(1)}(\boldsymbol{k})] \\
&\qquad - \frac{4zJ}{\hbar N} v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{q}_1,\boldsymbol{k}} \frac{n_{\boldsymbol{q}_1}^{(0)}(1+n_{\boldsymbol{q}_2}^{(0)}+n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}) - n_{\boldsymbol{q}_2}^{(0)} n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}}{\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_1}}\Bigg\} \\
&\qquad + O(v^2).
\end{aligned} \tag{3.4.19}$$

Substituting Eq. (3.4.19) into Eq. (3.3.7), the second-order one-particle Green function becomes

$$\begin{aligned}
(\omega-\omega_{\boldsymbol{k}})\langle\langle a_{\boldsymbol{k}}; a_{\boldsymbol{k}'}^+\rangle\rangle_\omega^{(2)} = \delta_{\boldsymbol{k},\boldsymbol{k}'}\{1+\Sigma^{(1)}(\boldsymbol{k})G_{\boldsymbol{k}}^{(0)}(\omega) + [\Sigma_a^{(2)}(\boldsymbol{k})+\Sigma_b^{(2)}(\boldsymbol{k},\omega) \\
+ \Sigma_c^{(2)}(\boldsymbol{k},\omega)]G_{\boldsymbol{k}}^{(0)}(\omega)\}
\end{aligned} \tag{3.4.20}$$

where

$$\Sigma_a^{(2)}(\boldsymbol{k},\omega) = G_{\boldsymbol{k}}^{(0)}(\omega)[\Sigma^{(1)}(\boldsymbol{k})]^2, \tag{3.4.21}$$

$$\Sigma_b^{(2)}(\boldsymbol{k}) = 4\beta\hbar\left(\frac{zJ}{\hbar N}\right)\sum_{\boldsymbol{q}} v_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{k},\boldsymbol{q}} n_{\boldsymbol{q}}^{(0)}(1+n_{\boldsymbol{q}}^{(0)})\Sigma^{(1)}(\boldsymbol{q}) \tag{3.4.22}$$

and

$$\Sigma_c^{(2)}(\boldsymbol{k},\omega) = 2\left(\frac{2zJ}{\hbar N}\right)^2 \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} A_{\boldsymbol{k}}(\boldsymbol{q}_1,\boldsymbol{q}_2)\frac{n_{\boldsymbol{q}_1}^{(0)}(1+n_{\boldsymbol{q}_2}^{(0)}+n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}) - n_{\boldsymbol{q}_2}^{(0)} n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}}{\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_1}} \tag{3.4.23}$$

with

$$A_{\boldsymbol{k}}(\boldsymbol{q}_1,\boldsymbol{q}_2) = v_{\boldsymbol{k},\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} v_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{q}_1,\boldsymbol{k}}. \tag{3.4.23a}$$

Within the perturbation theory, in Eq. (3.4.23) ω has to be replaced by $\omega_{\boldsymbol{k}} \pm i\epsilon$ where the infinitesimal imaginary part is reminiscent of the choice of the argument of the generalized Green function $G_{\boldsymbol{k}}(\omega \pm i\epsilon)$ corresponding to the retarded or advanced Green function, respectively. Using Eq. (3.2.12), the self-energy (3.4.23) becomes

$$\Sigma_c^{(2)}(\boldsymbol{k}, \omega_{\boldsymbol{k}} \pm i\epsilon) = \Sigma_c^{(2)'}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \mp i\Sigma_c^{(2)''}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \tag{3.4.24}$$

where the real part (renormalization) and the imaginary part (damping) of the self-energy (3.4.24) are given by

$$\Sigma_c^{(2)'}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) = 2\left(\frac{2zJ}{\hbar N}\right)^2 \mathrm{P}\sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} A_{\boldsymbol{k}}(\boldsymbol{q}_1, \boldsymbol{q}_2) \frac{n_{\boldsymbol{q}_1}^{(0)}(1 + n_{\boldsymbol{q}_2}^{(0)} + n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}) - n_{\boldsymbol{q}_2}^{(0)} n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}}{\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} - \omega_{\boldsymbol{q}_2} + \omega_{\boldsymbol{q}_1}} \tag{3.4.25}$$

and

$$\begin{aligned}\Sigma_c^{(2)''}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) = 2\pi \left(\frac{2zJ}{\hbar N}\right)^2 \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} &[n_{\boldsymbol{q}_1}^{(0)}(1 + n_{\boldsymbol{q}_2}^{(0)} + n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}) - n_{\boldsymbol{q}_2}^{(0)} n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}] \\ &\times A_{\boldsymbol{k}}(\boldsymbol{q}_1, \boldsymbol{q}_2)\ \delta(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} - \omega_{\boldsymbol{q}_2} + \omega_{\boldsymbol{q}_1}),\end{aligned} \tag{3.4.26}$$

respectively. Defining

$$\langle\langle a_{\boldsymbol{k}}; a_{\boldsymbol{k}'}^{+}\rangle\rangle_{\omega}^{(2)} = \delta_{\boldsymbol{k},\boldsymbol{k}'} G_{\boldsymbol{k}}^{(2)}(\omega) \tag{3.4.27}$$

from Eq. (3.4.20), we obtain the second-order one-particle Green function

$$\begin{aligned}G_{\boldsymbol{k}}^{(2)}(\omega \pm i\epsilon) = G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon)\{1 &+ [\Sigma^{(1)}(\boldsymbol{k}) + \Sigma_b^{(2)}(\boldsymbol{k}) + \Sigma_c^{(2)}(\boldsymbol{k}, \omega_{\boldsymbol{k}} \pm i\epsilon)]G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon) \\ &+ [\Sigma^{(1)}(\boldsymbol{k}) G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon)]^2\}\end{aligned} \tag{3.4.28}$$

where the self-energy $\Sigma_a^{(2)}$ has been replaced by Eq. (3.4.21). Note that the second-order self-energy $\Sigma_a^{(2)}$ defined in Eq. (3.4.21) is a *reducible* self-energy since it is the square of the first-order self-energy $\Sigma^{(1)}$ times an unperturbed one-particle Green function. The self-energy $\Sigma_b^{(2)}$ defined in Eq. (3.4.22) is a real function of the wavevector $\boldsymbol{k}$ and of the temperature T and contributes only to the renormalization but not to the damping while the self-energy $\Sigma_c^{(2)}$ defined in Eq. (3.4.24) is characterized by both a real contribution (renormalization) given by Eq. (3.4.25) and an imaginary contribution (damping) given by Eq. (3.4.26).

Before evaluating explicitly the second-order self-energies $\Sigma_b^{(2)}$ and $\Sigma_c^{(2)}$, we wish to solve the puzzle about the serious discrepancy between the DM and HP first-order self-energies pointed out at the end of Section 3.3. First of all, we note that the main contribution in temperature to the second-order self-energy renormalization comes from $\Sigma_c^{(2)'}$ given by Eq. (3.4.25) where a term with a single Bose factor $n_{\boldsymbol{q}_1}^{(0)}$ occurs. All other contributions contain products of two Bose factors leading to higher powers in temperature. Using the DM potential (3.3.4) in Eq. (3.4.23a), one has

$$A_{\boldsymbol{k}}^{DM}(\boldsymbol{q}_1,\boldsymbol{q}_2) = \frac{1}{4}(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1})(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{q}_2}) \tag{3.4.29}$$

while taking the HP potential (3.3.5), one has

$$\begin{aligned}
&A_{\boldsymbol{k}}^{HP}(\boldsymbol{q}_1,\boldsymbol{q}_2)\\
&\quad = A_{\boldsymbol{k}}^{DM}(\boldsymbol{q}_1,\boldsymbol{q}_2) + \Bigg\{ \frac{1}{16}(\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}_1} - \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{q}_2})^2\\
&\qquad + \frac{1}{4}\left[1 - 4S\left(1 - \sqrt{1 - \frac{1}{2S}}\right)\right](\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2})(\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} + \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2})\\
&\qquad - \frac{1}{16}\left[1 - 16S^2\left(1 - \sqrt{1 - \frac{1}{2S}}\right)^2\right](\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} + \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2})^2 \Bigg\}.
\end{aligned} \tag{3.4.30}$$

Neglecting terms with more than one Bose factor (this is consistent with the hypothesis of neglecting the potential w^{HP} with respect to v^{HP} as discussed in Section 3.3) and expanding the terms in the square brackets in powers of $\frac{1}{S}$, the real part of the second-order HP self-energy becomes

$$\begin{aligned}
\Sigma_{HP}^{(2)}(\boldsymbol{k},\omega_{\boldsymbol{k}}) &= \Sigma_{DM}^{(2)}(\boldsymbol{k},\omega_{\boldsymbol{k}}) + \left(\frac{2zJ}{\hbar N}\right)^2\\
&\quad \times \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} \Bigg\{ \frac{1}{8}(\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}_1} - \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{q}_2})^2\\
&\quad - \frac{1}{16S}\left(1 + \frac{1}{4S} + \cdots\right)(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2})(\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} + \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2})\\
&\quad + \frac{1}{32S}\left(1 + \frac{5}{16S} + \cdots\right)(\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}_1} + \gamma_{\boldsymbol{q}_2} + \gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2})^2 \Bigg\}\\
&\quad \times \frac{n_{\boldsymbol{q}_1}^{(0)}}{\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} - \omega_{\boldsymbol{q}_2} + \omega_{\boldsymbol{q}_1}}.
\end{aligned} \tag{3.4.31}$$

Replacing $\omega_{\boldsymbol{k}}$ with $\frac{2zJS}{\hbar}(1-\gamma_{\boldsymbol{k}})$ in the denominator of Eq. (3.4.31), one obtains

$$\Sigma_{HP}^{(2)}(\boldsymbol{k},\omega_{\boldsymbol{k}}) = \Sigma_{DM}^{(2)}(\boldsymbol{k},\omega_{\boldsymbol{k}}) - \left[\frac{1}{8S} + O\left(\frac{1}{S^2}\right)\right]\left(\frac{2zJ}{\hbar}\right)\frac{1}{N}\sum_{\boldsymbol{q}}(\gamma_{\boldsymbol{k}} + \gamma_{\boldsymbol{q}})n_{\boldsymbol{q}}^{(0)}. \tag{3.4.32}$$

As well as the first-order self-energies (3.3.22) and (3.3.23), the second-order HP and DM self-energies also differ from each other. However, summing the first-order result (3.3.23) to the second-order result (3.4.32), we obtain

$$\Sigma_{HP}^{(1)}(\boldsymbol{k}) + \Sigma_{HP}^{(2)}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) = \Sigma_{DM}^{(1)}(\boldsymbol{k}) + \Sigma_{DM}^{(2)}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) + O\left(\frac{1}{S^2}\right). \tag{3.4.33}$$

Even though the DM and HP spin-boson transformations lead to different results in a perturbation expansion in powers of the interaction potential v, they agree when all terms of the same order $\frac{1}{S}$ and temperature are taken into account. The perturbation expansion in v^{DM} coincides with the expansion in $\frac{1}{S}$ since v^{DM} does not depend on S. On the contrary, a further expansion in $\frac{1}{S}$ is required when the perturbation expansion in v^{HP} is used since v^{HP} does depend on S. Incidentally, an agreement between DM and HP perturbation expansion was also proven for terms of order $\frac{1}{S^2}$ when the third-order terms of the DM and HP self-energies are taken into account.[18] This result confirms that both DM and HP spin-boson transformations lead to the same low temperature self-energy when the perturbation parameter is assumed to be $\frac{1}{S}$ instead of the interaction potential v. Moreover, since at low temperature one finds $\Sigma_{HP} = \Sigma_{DM}$, the conclusion is that the effect of the unphysical states (kinematical interaction) is negligible for both the HP and the DM transformation even though a rigorous proof exists only for the DM transformation. Obviously, the perturbation calculation is greatly simplified by the use of the DM spin-boson transformation.

3.5. Dyson's Equation

Equation (3.4.28) is a good starting point to infer the Dyson's equation of the many-body theory. Indeed, pushing the perturbation expansion to the infinity, one expects to obtain

$$\begin{aligned} G_{\boldsymbol{k}}(\omega \pm i\epsilon) &= G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon) \sum_{n=0}^{\infty} \{[\Sigma^{(1)}(\boldsymbol{k}) + \Sigma_b^{(2)}(\boldsymbol{k}) \\ &\quad + \Sigma_c^{(2)}(\boldsymbol{k}, \omega_{\boldsymbol{k}} \pm i\epsilon) + \cdots] G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon)\}^n \\ &= \frac{G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon)}{1 - [\Sigma^{(1)}(\boldsymbol{k}) + \Sigma_b^{(2)}(\boldsymbol{k}) + \Sigma_c^{(2)}(\boldsymbol{k}, \omega_{\boldsymbol{k}} \pm i\epsilon) + \cdots] G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon)}. \end{aligned} \tag{3.5.1}$$

In the geometric series (3.5.1), the common ratio is given by the sum of all *irreducible* contributions to the *proper* self-energy[13] $\Sigma^*(\boldsymbol{k}, \omega_{\boldsymbol{k}} \pm i\epsilon)$ times the unperturbed generalized Green function $G_{\boldsymbol{k}}^{(0)}(\omega \pm i\epsilon)$ while all the powers in the geometric series with $n > 1$ come from the *reducible* contributions to the self-energy. Using Eq. (3.3.9), the generalized Green function of Eq. (3.5.1) becomes

$$G_{\boldsymbol{k}}(\omega \pm i\epsilon) = \frac{1}{\omega - \omega_{\boldsymbol{k}} - \Sigma^*(\boldsymbol{k}, \omega_{\boldsymbol{k}} \pm i\epsilon)} \tag{3.5.2}$$

where

$$\Sigma^*(\boldsymbol{k}, \omega \pm i\epsilon) = \Sigma^{*'}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \mp i\Sigma^{*''}(\boldsymbol{k}, \omega_{\boldsymbol{k}}). \tag{3.5.3}$$

As one can see, the generalized Green function (3.5.2) has a simple pole at a complex frequency

$$\omega = \omega_{\boldsymbol{k}} + \Sigma^{*'}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \mp i\Sigma^{*''}(\boldsymbol{k}, \omega_{\boldsymbol{k}}). \tag{3.5.4}$$

Note that only after the sum over the infinite (reducible) terms of the geometric series (3.5.1), the pole of the one-particle Green function is shifted from its (real) unperturbed value $\omega = \omega_{\boldsymbol{k}}$. Equation (3.5.1) may also be written as

$$G_{\boldsymbol{k}}(\omega) = G_{\boldsymbol{k}}^{(0)}(\omega) + G_{\boldsymbol{k}}^{(0)}(\omega)\Sigma^*(\boldsymbol{k}, \omega)G_{\boldsymbol{k}}(\omega), \tag{3.5.5}$$

that is the celebrated Dyson's equation of the many-body theory.[13]

Since the first-order self-energy $\Sigma^{(1)}(\boldsymbol{k})$ is frequency-independent, the first-order perturbation theory leads to a simple renormalization of the magnon frequency

$$\Delta\omega_{\boldsymbol{k}}^{(1)} = \Sigma^{(1)}(\boldsymbol{k}). \tag{3.5.6}$$

On the contrary, the second-order self-energy leads to either a renormalization given by

$$\Delta\omega_{\boldsymbol{k}}^{(2)} = \Sigma_b^{(2)}(\boldsymbol{k}) + \Sigma_c^{(2)'}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \tag{3.5.7}$$

and a damping given by

$$\Gamma_{\boldsymbol{k}}^{(2)} = |\Sigma_c^{(2)''}(\boldsymbol{k}, \omega_{\boldsymbol{k}})|. \tag{3.5.8}$$

Both renormalization and damping of the magnon frequency are due to the interaction with all other magnons.

Using Eq. (3.5.2), the dynamical structure factor (3.2.17) becomes

$$\begin{aligned} S_\perp(\boldsymbol{K}, \omega) = {} & 2\frac{SN}{\hbar\pi}\sum_{\boldsymbol{q},\boldsymbol{G}} \delta_{\boldsymbol{K}+\boldsymbol{q},\boldsymbol{G}}[1 + n(\omega)]\frac{\Sigma^{*''}(\boldsymbol{q}, \omega_{\boldsymbol{q}})}{[\omega - \omega_{\boldsymbol{q}} - \Sigma^{*'}(\boldsymbol{q}, \omega_{\boldsymbol{q}})]^2 + [\Sigma^{*''}(\boldsymbol{q}, \omega_{\boldsymbol{q}})]^2} \\ & + 2\frac{SN}{\hbar\pi}\sum_{\boldsymbol{q},\boldsymbol{G}} \delta_{\boldsymbol{K}-\boldsymbol{q},\boldsymbol{G}}\ n(-\omega)\frac{\Sigma^{*''}(\boldsymbol{q}, \omega_{\boldsymbol{q}})}{[\omega + \omega_{\boldsymbol{q}} + \Sigma^{*'}(\boldsymbol{q}, \omega_{\boldsymbol{q}})]^2 + [\Sigma^{*''}(\boldsymbol{q}, \omega_{\boldsymbol{q}})]^2}. \end{aligned} \tag{3.5.9}$$

As one can see by comparing Eqs. (3.5.9) and (3.3.12), the existence of a damping $\Sigma^{*''}$ changes the δ-like peaks of the unperturbed system into Lorentzian peaks located at the renormalized magnon frequency

$$\omega = \pm[\omega_{\boldsymbol{q}} + \Sigma^{*'}(\boldsymbol{q}, \omega_{\boldsymbol{q}})]. \tag{3.5.10}$$

The peak width is proportional to the magnon damping $\Sigma^{*''}$ and the height of the peak is proportional to $[\Sigma^{*''}]^{-1}$. Only for $T \to 0$, the poles of the Green function lead to δ-like peaks located at the unperturbed magnon frequencies wich are the lowest eigenvalues of the Heisenberg Hamiltonian. For $T \neq 0$, the poles of the Green function do not correspond to eigenvalues of the Heisenberg Hamiltonian but their shift and broadening correspond to the effect on the correlation function of the interaction between the spin waves. Note that for $\Sigma^{*''} \to 0$, the Lorentian peaks reduce to δ-peaks located at the renormalized frequencies (3.5.10).

Several truncations of the Green function hierarchy were proposed in literature. Some of them[26,27] give specific rules similar to (3.3.14) in order to write the many-particle Green functions in terms of the one-particle Green function. However, many other uncontrolled decouplings have been proposed to write the two-particle Green function of Eq. (3.3.7) in terms of the one-particle Green function. Among them, the most well known is perhaps that suggested by Tahir-Kheli and Ter Haar[28]

$$\langle\langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2} a_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega \simeq \langle a^+_{\boldsymbol{q}_1} a_{\boldsymbol{q}_1}\rangle(\delta_{\boldsymbol{q}_1,\boldsymbol{q}_2} + \delta_{\boldsymbol{k},\boldsymbol{q}_2})\langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle_\omega, \tag{3.5.11}$$

leading to the one-particle Green function

$$\langle\langle a_{\boldsymbol{k}}; a^+_{\boldsymbol{k}'}\rangle\rangle_{\omega\pm i\epsilon} = \frac{\delta_{\boldsymbol{k},\boldsymbol{k}'}}{\omega - \omega_{\boldsymbol{k}} - \widetilde{\Sigma}(\boldsymbol{k}) \pm i\epsilon} \tag{3.5.12}$$

where the self-energy $\widetilde{\Sigma}(\boldsymbol{k})$ is given by the first-order self-energy (3.3.22) in which the unperturbed Bose factor $n^{(0)}_{\boldsymbol{k}}$ is replaced by a Bose factor with a renormalized frequency $\omega_{\boldsymbol{k}} + \widetilde{\Sigma}(\boldsymbol{k})$ that is

$$\widetilde{\Sigma}(\boldsymbol{k}) = -\omega_{\boldsymbol{k}}\frac{\widetilde{\alpha}}{S} \tag{3.5.13}$$

with

$$\widetilde{\alpha} = \frac{1}{N}\sum_{\boldsymbol{p}}(1-\gamma_{\boldsymbol{p}})\frac{1}{e^{\beta\hbar\omega_{\boldsymbol{p}}(1-\widetilde{\alpha}/S)} - 1}. \tag{3.5.14}$$

Equation (3.5.14) is a self-consistent equation for $\widetilde{\alpha}$. The self-energy (3.5.13) reduces to the first-order result (3.3.22) putting $\widetilde{\alpha} = 0$ in the right-hand side of Eq. (3.5.14). The decoupling (3.5.11) allows us to get some (uncontrolled) informations about the behaviour of a ferromagnet at higher temperatures. Following a procedure very similar to that illustrated in Section 2.4, Eq. (3.5.14) for a SC lattice becomes

$$\begin{aligned}\widetilde{\alpha} = \sum_{n=1}^{\infty} e^{-12\beta JSn(1-\widetilde{\alpha}/S)}\left\{I_0\left[4\beta JSn\left(1-\frac{\widetilde{\alpha}}{S}\right)\right]\right\}^2 \\ \times\left\{I_0\left[4\beta JSn\left(1-\frac{\widetilde{\alpha}}{S}\right)\right] - I_1\left[4\beta JSn\left(1-\frac{\widetilde{\alpha}}{S}\right)\right]\right\}\end{aligned} \tag{3.5.15}$$

where I_0 and I_1 are the modified Bessel functions of order 0 and 1, respectively.[4] For $T \to 0$, Equation (3.5.15) has two solutions: $\widetilde{\alpha}_1 = 0$ and $\widetilde{\alpha}_2 = S$. In the low temperature limit, $\widetilde{\alpha}_1$ can be obtained using the asymptotic expansions of I_0 given by Eq. (2.4.16) and of I_1 given by

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 - \frac{3}{8z} - \frac{15}{128z^2} + \cdots\right), \tag{3.5.16}$$

Table 3.1. Physical ($\widetilde{\alpha}_1$) and unphysical ($\widetilde{\alpha}_2$) solutions of Eq. (3.5.15) with the corresponding magnetizations m_1 and m_2 obtained from Eq. (3.5.19) for a SC lattice with $S = 1$ for several temperatures. In the numerical calculation, the sums occurring in Eqs. (3.5.15) and (3.5.19) are truncated at $n = 30$, consistent with the precision quoted in the Table.

$\frac{k_BT}{2JS}$	$\widetilde{\alpha}_1$	$\widetilde{\alpha}_2$	m_1	m_2
0	0	1	1	−0.5286
0.1	2.439×10^{-5}	0.9883	0.9984	−0.5130
0.5	1.574×10^{-3}	0.9394	0.9809	−0.4479
1	1.144×10^{-2}	0.8721	0.9397	−0.3578
1.5	3.784×10^{-2}	0.7951	0.8738	−0.2541
2	8.851×10^{-2}	0.7025	0.7761	−0.1282
2.5	0.1814	0.5753	0.6215	0.0472
2.7	0.2496	0.4952	0.5166	0.1591
2.8	0.3151	0.4241	0.4182	0.2601
2.82426	0.3689	0.3689	0.3395	0.3395

leading to

$$\widetilde{\alpha}_1 = \pi\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}}. \tag{3.5.17}$$

In the same limit $\widetilde{\alpha}_2$ is given by

$$\widetilde{\alpha}_2 = S - A\left(\frac{k_BT}{8\pi JS}\right) \tag{3.5.18}$$

where A cannot be obtained by the asymptotic expansion since the argument of the Bessel functions is finite for $T \to 0$ and $\widetilde{\alpha}_2 \to S$. For $S = 1$, the numerical calculation gives $A = 1.456$. The two solutions of Eq. (3.5.15) are given in Table 3.1 for $S = 1$ and several temperatures. The smaller solution $\widetilde{\alpha}_1$ corresponds to a minimum of the free energy and represents the *physical* solution of Eq. (3.5.15) while the solution $\widetilde{\alpha}_2$ is unphysical. As shown in Table 3.1, Eq. (3.5.15) has two solutions for $T < T_{\max} = 2.8243(2JS/k_B)$, one solution for $T = T_{\max}$ and no solution for $T > T_{\max}$. From Eq. (2.4.21), one obtains the spontaneous magnetization corresponding to the solutions of Eq. (3.5.15):

$$m_i = \langle S_i^z\rangle = S - \sum_{n=1}^{\infty} e^{-12\beta JSn(1-\widetilde{\alpha}_i/S)}\left\{I_0\left[4\beta JSn\left(1-\frac{\widetilde{\alpha}_i}{S}\right)\right]\right\}^3 \tag{3.5.19}$$

where $i = 1, 2$ corresponds to the physical and unphysical solution of Eq. (3.5.15), respectively. In the last two columns of Table 3.1, the values of m_1 and m_2 are given for a SC lattice with $S = 1$ at various temperatures. As one can see, the magnetization m_1 decreses from 1 to 0.3395 as the temperature increase from 0 to $T_{\max}$. The unphysical solution m_2 decreases in a re-entrant way from $T_{\max}$ to 0, crossing the temperature axis at $T \simeq 2.5(2JS/k_B)$. The self-consistent Eq. (3.5.14) was obtained independently using a variational theorem[29] by Micheline Bloch in 1962. The author pointed out the surprising good agreement between $T_{\max}$ and the critical temperature of the SC Heisenberg model obtained by a high temperature series expansion[30]

$$\frac{k_B T_c}{2J} = \frac{5}{2}[0.579\ S(S+1) - 0.072] \tag{3.5.20}$$

that gives $T_c \simeq 2.715(2J/k_B)$ for $S = 1$. This agreement was not to be taken too seriously since at $T_{\max}$, a discontinuity in the magnetization ($\Delta m = 0.3395$) is found in the present calculation pointing out the occurrence of a wrong first-order phase transition. On the other hand, it is well known that the critical phenomena are driven by excitations that cannot be described in terms of spin waves only. However, in the low temperature limit, where the spin waves are the only excitations of the system, the physical solution (3.5.17) coincides with the first term of the low temperature series expansion. Moreover, using the asymptotic expansion (2.4.16), the low temperature expansion of the magnetization m_1 becomes

$$\begin{aligned} m_1 &= S - \zeta\left(\frac{3}{2}\right)\left(\frac{k_B T}{8\pi JS}\right)^{\frac{3}{2}} \frac{1}{(1-\widetilde{\alpha}_1/S)} \\ &= S - \zeta\left(\frac{3}{2}\right)\left(\frac{k_B T}{8\pi JS}\right)^{\frac{3}{2}} - \frac{3\pi}{2S}\zeta\left(\frac{3}{2}\right)\zeta\left(\frac{5}{2}\right)\left(\frac{k_B T}{8\pi JS}\right)^{4}, \end{aligned} \tag{3.5.21}$$

leading to the correct $T^{3/2}$ and T^4 contributions obtained by Bloch[21] and by Dyson,[15] respectively.

3.6. Renormalization and Damping

In Section 3.5, we have seen how the spin wave spectrum is modified by the interaction between the spin waves. Within the second-order perturbation theory, the real part of the renormalized spectrum reads

$$\omega_{\boldsymbol{k}}^{(2)} = \omega_{\boldsymbol{k}} + \Sigma^{(1)}(\boldsymbol{k}) + \Sigma_b^{(2)}(\boldsymbol{k}) + \Sigma_c^{(2)'}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \tag{3.6.1}$$

where $\Sigma^{(1)}(\boldsymbol{k})$, $\Sigma_b^{(2)}(\boldsymbol{k})$ and $\Sigma_c^{(2)'}(\boldsymbol{k}, \omega_{\boldsymbol{k}})$ are given by Eqs. (3.3.21), (3.4.22) and (3.4.31), respectively. The magnon damping is given by Eq. (3.4.32): note that the inverse of the damping is the lifetime of the magnon. In this section, we evaluate the low temperature behaviour of such contributions assuming that the interaction potential is the DM potential v^{DM} given by Eq. (3.3.4). The first-order self-energy becomes

$$\Sigma^{(1)}(\boldsymbol{k}) = -\frac{2zJ}{\hbar N}\sum_{\boldsymbol{q}}(1 + \gamma_{\boldsymbol{k}-\boldsymbol{q}} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}}) n_{\boldsymbol{q}}^{(0)} = -\alpha\frac{\omega_{\boldsymbol{k}}}{S} \tag{3.6.2}$$

where

$$\alpha = \frac{1}{N}\sum_{\boldsymbol{q}}(1 - \gamma_{\boldsymbol{q}}) n_{\boldsymbol{q}}^{(0)}. \tag{3.6.3}$$

To obtain the last step of Eq. (3.6.2), the relationships $\sum_{\boldsymbol{q}} \gamma_{\boldsymbol{k}-\boldsymbol{q}} n_{\boldsymbol{q}}^{(0)} = \sum_{\boldsymbol{q}} \gamma_{\boldsymbol{k}} \gamma_{\boldsymbol{q}} n_{\boldsymbol{q}}^{(0)}$ and $\hbar\omega_{\boldsymbol{k}} = 2zJS(1-\gamma_{\boldsymbol{k}})$ have to be used. The low temperature limit of α for a SC lattice was obtained in Eq. (3.5.17), making use of the asymptotic expansions of the Bessel functions occurring in Eq. (3.5.15). A more direct way to arrive at the same result consists of taking the long wavelength limit in the factor $(1-\gamma_{\boldsymbol{q}})$ as well as in the magnon frequency occurring in the Bose factor and replacing the cubic cell by a sphere of infinite radius. In doing so, Eq. (3.6.3) becomes

$$\begin{aligned}\alpha &\simeq \frac{1}{(2\pi)^3} \int_0^\infty dq q^2 \frac{1}{6} q^2 \frac{1}{e^{2\beta JSq^2}-1} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \\ &= \frac{1}{24\pi^2} \left(\frac{k_BT}{2JS}\right)^{\frac{5}{2}} \int_0^\infty dx \frac{x^{\frac{3}{2}}}{e^x-1}. \end{aligned} \tag{3.6.4}$$

Using the relationship[3]

$$\int_0^\infty dx \frac{x^{\frac{3}{2}}}{e^x-1} = \frac{3}{4}\sqrt{\pi}\zeta\left(\frac{5}{2}\right), \tag{3.6.5}$$

Eq. (3.6.4) becomes

$$\alpha = \pi\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}} = 7.53\times 10^{-3}\left(\frac{k_BT}{2JS}\right)^{\frac{5}{2}} \tag{3.6.6}$$

and the first-order renormalized spin wave spectrum reduces to

$$\omega_{\boldsymbol{k}}^{(1)} = \omega_{\boldsymbol{k}}\left(1-\frac{\alpha}{S}\right) \simeq \omega_{\boldsymbol{k}}\left[1-\frac{\pi}{S}\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}}\right]. \tag{3.6.7}$$

Equation (3.6.7) shows that the renormalized magnon frequency is lower than the unperturbed one leading to the conclusion that the effect of the magnon-magnon interaction is to lower the spin wave energy. The fact that the renormalization is the same for every wavevector $\boldsymbol{k}$ is peculiar of the first-order self-energy. The first-order spectrum (3.6.7) leads to the inelastic cross-section (3.1.17) with $\omega_{\boldsymbol{q}}$ replaced by $\omega_{\boldsymbol{q}}^{(1)}$. We stress that no damping occurs in the first-order perturbation theory. The spontaneous magnetization is obtained from Eq. (2.4.10) replacing $\omega_{\boldsymbol{q}}$ by $\omega_{\boldsymbol{q}}^{(1)}$. In the long wavelength limit, Eq. (3.5.21) is recovered.

From Eq. (3.4.22), one can see that the second-order self-energy $\Sigma_b^{(2)}$ is a real function independent of the frequency that can be written

$$\Sigma_b^{(2)}(\boldsymbol{k}) = -\frac{\omega_{\boldsymbol{k}}}{S}\left(\frac{2zJ}{k_BT}\right)\alpha\rho \tag{3.6.8}$$

where α is given by Eq. (3.6.3) and ρ is given by

$$\rho = \frac{1}{N}\sum_{\boldsymbol{p}}(1-\gamma_{\boldsymbol{p}})^2 n_{\boldsymbol{p}}^{(0)}(1+n_{\boldsymbol{p}}^{(0)}). \tag{3.6.8a}$$

At low temperature for a SC lattice, α is given by Eq. (3.6.6) and ρ becomes

$$\rho \simeq \frac{1}{2\pi^2}\int_0^\infty p^2 dp \left(\frac{1}{6}p^2\right)^2 \frac{e^{2JS\beta p^2}}{(e^{2JS\beta p^2}-1)^2}$$
$$= \frac{1}{144\pi^2}\left(\frac{k_BT}{2JS}\right)^{\frac{7}{2}}\int_0^\infty dx \frac{x^{5/2}e^x}{(e^x-1)^2}. \tag{3.6.8b}$$

Integrating by part and using Eq. (3.6.5), one obtains

$$\rho = \frac{5}{3}\pi^2\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{7}{2}} = 3.14\times 10^{-3}\left(\frac{k_BT}{2JS}\right)^{\frac{7}{2}} \tag{3.6.8c}$$

so that Eq. (3.6.8) becomes

$$\Sigma_b^{(2)} = -\frac{5}{2}\,\frac{\omega_{\boldsymbol{k}}}{S^2}\left[\pi\zeta\left(\frac{5}{2}\right)\right]^2\left(\frac{k_BT}{8\pi JS}\right)^5. \tag{3.6.9}$$

From Eq. (3.6.9), one can see that $\Sigma_b^{(2)}$ is of the order T^5, therefore it is negligible with respect to the first-order result given by Eq. (3.6.7). On the contrary, the main term coming from the temperature expansion of $\Sigma_c^{(2)'}$ is of the same order as $\Sigma^{(1)}$ ($T^{5/2}$). Indeed, replacing v^{DM} given by Eq. (3.3.4) in Eq. (3.4.25), the real part (renormalization) of the self-energy $\Sigma_c^{(2)}$ becomes

$$\Sigma_c^{(2)'}(\boldsymbol{k},\omega_{\boldsymbol{k}}) = \frac{1}{2}\left(\frac{2zJ}{\hbar N}\right)^2\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}[n_{\boldsymbol{q}_1}^{(0)}(1+n_{\boldsymbol{q}_2}^{(0)}+n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}) - n_{\boldsymbol{q}_2}^{(0)}n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}]$$
$$\times\frac{(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{q}_1})(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{q}_2})}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{q}_1}}. \tag{3.6.10}$$

The main temperature contribution of Eq. (3.6.10) is obtained, neglecting all terms containing products of two Bose factors: this is correct when the Bose factors refer to "independent" frequencies like in this case. However, this is no longer correct when the presence of a δ-function of the frequencies, like in Eq. (3.4.26), makes them no more independent.[31] Using the relationship $\hbar\omega_{\boldsymbol{k}} = 2zJS(1-\gamma_{\boldsymbol{k}})$, adding and subtracting the term $\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_2}$ in the first factor of the numerator, the function within the sum of Eq. (3.6.10) becomes

$$\left[\frac{(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{q}_2})^2}{\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{q}_1}}+\gamma_{\boldsymbol{k}-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{q}_2}\right]n_{\boldsymbol{q}_1}^{(0)}. \tag{3.6.10a}$$

Summing over $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, the last term of Eq. (3.6.10a) becomes

$$\begin{aligned}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}&(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{q}_2})n^{(0)}_{\boldsymbol{q}_1}\\ &=\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}(\gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1}\gamma_{\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1}\gamma_{\boldsymbol{q}_2}-\gamma_{\boldsymbol{q}_2})n^{(0)}_{\boldsymbol{q}_1}\\ &=\sum_{\boldsymbol{q}_1}(\gamma_{\boldsymbol{k}}+\gamma_{\boldsymbol{q}_1}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1}-1)n^{(0)}_{\boldsymbol{q}_1}\sum_{\boldsymbol{q}_2}\gamma_{\boldsymbol{q}_2}=0\end{aligned}\tag{3.6.10b}$$

since $\sum_{\boldsymbol{q}_2}\gamma_{\boldsymbol{q}_2}=0$. Then the main contribution of the real part of the second-order self-energy $\Sigma^{(2)}_c$ becomes

$$\Sigma^{(2)'}_c(\boldsymbol{k},\omega_{\boldsymbol{k}})=\frac{1}{2S^2}\left(\frac{2zJS}{\hbar}\right)\frac{1}{N^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}\frac{(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}-\gamma_{\boldsymbol{q}_2})^2}{\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}+\gamma_{\boldsymbol{q}_2}-\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{q}_1}}n^{(0)}_{\boldsymbol{q}_1}.\tag{3.6.11}$$

The explicit calculation of (3.6.11) will be performed for a SC lattice for which $\gamma_{\boldsymbol{k}}=\frac{1}{3}\sum_\alpha\cos k_\alpha$ where $\alpha=x,y,z$ and the lattice constant is assumed to be $a=1$. Replacing $\boldsymbol{q}_1$ by $\boldsymbol{p}$ and $\boldsymbol{q}_2$ by $\frac{1}{2}(\boldsymbol{k}+\boldsymbol{p})-\boldsymbol{\rho}$, Equation (3.6.11) becomes

$$\Sigma^{(2)'}_c(\boldsymbol{k},\omega_{\boldsymbol{k}})=\frac{16J}{\hbar S}\frac{1}{N^2}\sum_{\boldsymbol{p},\boldsymbol{\rho}}\frac{\sum_{\alpha,\beta}\sin\frac{k_\alpha}{2}\sin\frac{k_\beta}{2}\sin\frac{p_\alpha}{2}\sin\frac{p_\beta}{2}\cos\rho_\alpha\cos\rho_\beta}{\sum_\gamma\cos\frac{1}{2}(k_\gamma+p_\gamma)[\cos\rho_\gamma-\cos\frac{1}{2}(k_\gamma-p_\gamma)]}\tag{3.6.12}$$

where $\alpha,\beta,\gamma=x,y,z$. The main tepmerature contribution to the self-energy (3.6.12) is obtained by expanding the function under the sum in powers of $\boldsymbol{p}$ and retaining the first term of such expansion. In view of the sum over $\boldsymbol{p}$, only the terms with $\alpha=\beta$ in Eq. (3.6.12) give non zero contributions so that the self-energy (3.6.12) becomes

$$\Sigma^{(2)'}_c(\boldsymbol{k},\omega_{\boldsymbol{k}})=\frac{4J}{\hbar S}\sum_\alpha\sin^2\frac{k_\alpha}{2}\frac{1}{N}\sum_{\boldsymbol{p}}p_\alpha^2n^{(0)}_{\boldsymbol{p}}\frac{1}{N}\sum_{\boldsymbol{\rho}}\frac{\cos^2\rho_\alpha}{\sum_\gamma\cos\frac{k_\gamma}{2}[\cos\rho_\gamma-\cos\frac{k_\gamma}{2}]}.\tag{3.6.13}$$

For symmetry reasons, the sum over $\boldsymbol{p}$ in Eq. (3.6.13) is the same for $\alpha=x,y$ and z so that one has

$$\frac{1}{N}\sum_{\boldsymbol{p}}p_\alpha^2n^{(0)}_{\boldsymbol{p}}=\frac{1}{3}\,\frac{1}{N}\sum_{\boldsymbol{p}}p^2n^{(0)}_{\boldsymbol{p}}=2\pi\,\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}}\tag{3.6.14}$$

where we have used Eqs. (3.6.3) and (3.6.6). The sum over $\boldsymbol{\rho}$ in Eq. (3.6.13) may be written as

$$I_\alpha(\boldsymbol{k})=-\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi d\rho_x\,d\rho_y\,d\rho_z\frac{\cos^2\rho_\alpha}{\sum_\gamma\cos\frac{k_\gamma}{2}[\cos\frac{k_\gamma}{2}-\cos\rho_\gamma]}\tag{3.6.15}$$

so that the second-order renormalization (3.6.13) becomes

$$\Sigma_c^{(2)'}(\boldsymbol{k},\omega_{\boldsymbol{k}}) = \frac{8J}{\hbar S}\,\pi\,\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}}\sum_{\alpha}\,I_\alpha(\boldsymbol{k})\sin^2\frac{k_\alpha}{2}. \tag{3.6.16}$$

As announced, the main temperature contribution of the real part of the second-order self-energy $\Sigma_c^{(2)'}$ given by Eq. (3.6.16) is of the same order of the main temperature contribution to the first-order self-energy $\Sigma^{(1)}$ given by Eqs. (3.6.2)–(3.6.6). In the long wavelength limit ($\boldsymbol{k}\to 0$), Eq. (3.6.16) reduces to

$$\Sigma_c^{(2)'}(\boldsymbol{k}\to 0,\omega_{\boldsymbol{k}}) = -\frac{2J}{3\hbar S}I_c^{(2)}|\boldsymbol{k}|^2\pi\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}} \tag{3.6.17}$$

where

$$I_c^{(2)} = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi dx\;dy\;dz\;\frac{\cos^2 x}{1-\frac{1}{3}(\cos x+\cos y+\cos z)} = 0.88686. \tag{3.6.17a}$$

and the second-order magnon spectrum reads

$$\omega_{\boldsymbol{k}}^{(2)} \simeq \frac{2JS}{\hbar}k^2\left[1-\frac{\pi}{S}\,\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}}\left(1+\frac{I_c^{(2)}}{3S}\right)\right]. \tag{3.6.18}$$

The evaluation of the main temperature contribution of the damping coming from the second-order self-energy $\Sigma_c^{(2)''}$ is more subtle. As mentioned previously[31] in the calculation of the imaginary part of the self-energy, we cannot neglect terms containing two Bose factors since the energies occurring in the Bose factors are not independent due to the δ-function condition on the frequencies appearing in Eq. (3.4.32). Indeed, using the identities

$$n(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{p}}) = \frac{n_{\boldsymbol{q}}^{(0)}n_{\boldsymbol{p}}^{(0)}}{1+n_{\boldsymbol{q}}^{(0)}+n_{\boldsymbol{p}}^{(0)}}, \tag{3.6.19}$$

$$\begin{aligned}1-\frac{n_{\boldsymbol{p}}^{(0)}}{1+n_{\boldsymbol{q}}^{(0)}+n_{\boldsymbol{p}}^{(0)}} &= \frac{1+n_{\boldsymbol{q}}^{(0)}}{1+n_{\boldsymbol{q}}^{(0)}+n_{\boldsymbol{p}}^{(0)}} = [1+n(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{p}})]\frac{1}{1+n_{\boldsymbol{p}}^{(0)}}\\ &= [1+n(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{p}})](1-e^{-\beta\hbar\omega_{\boldsymbol{p}}})\end{aligned} \tag{3.6.20}$$

and the relationship between the frequencies occurring in the argument of the δ-function ($\omega_{\boldsymbol{q}_2}+\omega_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}=\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{q}_1}$), one has

$$\begin{aligned}&n_{\boldsymbol{q}_1}^{(0)}\big(1+n_{\boldsymbol{q}_2}^{(0)}+n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}\big)-n_{\boldsymbol{q}_2}^{(0)}n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}\\ &\quad=\big(1+n_{\boldsymbol{q}_2}^{(0)}+n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}\big)\big[n_{\boldsymbol{q}_1}^{(0)}-n(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{q}_1})\big]\\ &\quad=n_{\boldsymbol{q}_1}^{(0)}\big(1+n_{\boldsymbol{q}_2}^{(0)}+n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}\big)[1+n(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{q}_1})](1-e^{-\beta\hbar\omega_{\boldsymbol{k}}}).\end{aligned} \tag{3.6.21}$$

Replacing Eq. (3.6.21) into Eq. (3.4.26), the imaginary part (damping) of the self-energy $\Sigma_c^{(2)}$ becomes

$$\begin{aligned}\Sigma_c^{(2)''}&(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \\ &= (1 - e^{-\beta\hbar\omega_{\boldsymbol{k}}})\frac{\pi}{2S^2}\left(\frac{2zJS}{\hbar N^2}\right)\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1})^2 \\ &\quad \times \delta(\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1}) n_{\boldsymbol{q}_1}^{(0)}(1 + n_{\boldsymbol{q}_2}^{(0)} + n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)})[1 + n(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{q}_1})].\end{aligned} \tag{3.6.22}$$

For magnons of small wavevectors such that $\beta\hbar\omega_{\boldsymbol{k}} \ll 1$, the presence of the Bose factor $n_{\boldsymbol{q}_1}^{(0)}$ implies that $\boldsymbol{q}_2$ is also small and all the γ's in the argument of the δ-function can be expanded in powers of their wavevectors so that

$$\delta(\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1}) \simeq \frac{z}{2}\delta[\boldsymbol{k}\cdot\boldsymbol{q}_1 - \boldsymbol{q}_2\cdot(\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2)] \tag{3.6.23}$$

where the lattice constant $a = 1$ has been assumed. In the same way, the interaction potential becomes

$$(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1})^2 = \frac{4}{z^2}[\boldsymbol{q}_2\cdot(\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2)]^2 \tag{3.6.24}$$

so that Eq. (3.6.22) reduces to

$$\begin{aligned}\Sigma_c^{(2)''}(\boldsymbol{k}\to 0, \omega_{\boldsymbol{k}}) &= \pi\frac{2J}{\hbar S}\left(\frac{2JS}{k_BT}\right)k^2\frac{1}{N^2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}(\boldsymbol{k}\cdot\boldsymbol{q}_1)^2 \\ &\quad \times \delta[\boldsymbol{k}\cdot\boldsymbol{q}_1 - \boldsymbol{q}_2\cdot(\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2)] n_{\boldsymbol{q}_1}^{(0)}(1 + 2n_{\boldsymbol{q}_2}^{(0)})[1 + n(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{q}_1})]\end{aligned} \tag{3.6.25}$$

where the Bose factor $n_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2}^{(0)}$ in Eq. (3.6.22) has been replaced by $n_{\boldsymbol{q}_2}^{(0)}$ due to the symmetry of the interaction potential and of the argument of the δ-function under the exchange of $\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2$ with $\boldsymbol{q}_2$. Moreover, in the low temperature limit, the spin wave spectra occurring in the Bose factors of Eq. (3.6.25) may be replaced by their expressions in the long wavelength limit. In order to evaluate the sums over $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ in Eq. (3.6.25), we choose spherical coordinates in such a way that θ_1 is the polar angle between $\boldsymbol{k}$ and $\boldsymbol{q}_1$ and θ_2 is the polar angle between $\boldsymbol{q}_2$ and $\boldsymbol{k}+\boldsymbol{q}_1$. This choice implies that

$$(\boldsymbol{k}+\boldsymbol{q}_1)^2 = k^2 + q_1^2 + 2kq_1\cos\theta_1 \tag{3.6.26a}$$

and

$$\boldsymbol{q}_2\cdot(\boldsymbol{k}+\boldsymbol{q}_1) = q_2\cos\theta_2\sqrt{k^2 + q_1^2 + 2kq_1\cos\theta_1} \tag{3.6.26b}$$

so that

$$\begin{aligned}(\boldsymbol{k}\cdot\boldsymbol{q}_1)^2&\delta[\boldsymbol{k}\cdot\boldsymbol{q}_1 - \boldsymbol{q}_2\cdot(\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2)] \\ &= k^2q_1^2\cos^2\theta_1\,\delta(kq_1\cos\theta_1 - q_2\cos\theta_2\sqrt{k^2+q_1^2+2kq_1\cos\theta_1} + q_2^2).\end{aligned} \tag{3.6.26c}$$

From Eq. (3.6.26c), one can see that all the functions occurring in Eq. (3.6.25) do not depend explicitly on the azimuthal angles ϕ_1 and ϕ_2 so that a direct integration can be performed over these two angles leading to a factor $(2\pi)^2$. Putting $\boldsymbol{q}_1 = \boldsymbol{p}$, $\boldsymbol{q}_2 = \boldsymbol{q}$, $\cos\theta_1 = x$ and $\cos\theta_2 = y$, Eq. (3.6.25) becomes

$$\Sigma_c^{(2)''}(\boldsymbol{k} \to 0, \omega_{\boldsymbol{k}}) = \frac{J}{\hbar S}\left(\frac{2JS}{k_BT}\right) k^4 \, \frac{1}{(2\pi)^3}\int_0^\infty dp\, p^4 n_p^{(0)}[1 + n(\omega_k + \omega_p)] \times \int_{-1}^{1} dx \, \frac{x^2}{\sqrt{k^2+p^2+2kpx}} \int_0^\infty dq\; q(1+2n_q^{(0)}) \times \int_{-1}^{1} dy \, \delta\left(\frac{q^2+kpx}{q\sqrt{k^2+p^2+2kpx}} - y\right). \tag{3.6.27}$$

Due to the argument of the δ-function, the integral over y is 1 if

$$-1 < \frac{q^2+kpx}{q\sqrt{k^2+p^2+2kpx}} < 1 \tag{3.6.28}$$

and zero otherwise. The condition (3.6.28) implies that

$$-q^{(-)}(x) < q < q^{(+)}(x) \tag{3.6.29a}$$

for $x < 0$ ($\frac{\pi}{2} < \theta_1 < \pi$) and

$$q^{(-)}(x) < q < q^{(+)}(x) \tag{3.6.29b}$$

for $x > 0$ ($0 < \theta_1 < \frac{\pi}{2}$) where

$$q^{(\pm)}(x) = \frac{1}{2}\left[\sqrt{k^2+p^2+2kpx} \pm \sqrt{k^2+p^2-2kpx}\right]. \tag{3.6.29c}$$

Noticing that $q^{(-)}(-x) = -q^{(-)}(x)$ and $q^{(+)}(-x) = q^{(+)}(x)$, the last three integrals of Eq. (3.6.27) give

$$\int_{-1}^{1} dx \, \frac{x^2}{\sqrt{k^2+p^2+2kpx}} \int_0^\infty dq\, q(1+2n_q^{(0)}) \int_{-1}^{1} dy\, \delta\left(\frac{q^2+kpx}{q\sqrt{k^2+p^2+2kpx}} - y\right) = \int_0^1 dx\; x^2\left(\frac{1}{\sqrt{k^2+p^2+2kpx}} + \frac{1}{\sqrt{k^2+p^2-2kpx}}\right)\int_{q^{(-)}(x)}^{q^{(+)}(x)} dq\, q(1+2n_q^{(0)}). \tag{3.6.30}$$

The last integration of Eq. (3.6.30) can be easily performed if we make the variable change $q^2 = \rho$. Indeed,

$$\int_{q^{(-)}(x)}^{q^{(+)}(x)} dq\, q(1+2n_q^{(0)}) = \int_{\rho^{(-)}(x)}^{\rho^{(+)}(x)} d\rho \, \left(\frac{1}{2} + \frac{1}{e^{2\beta JS\rho}-1}\right) = \frac{1}{2}[\rho^{(+)}(x) - [\rho^{(-)}(x)] + \left(\frac{k_BT}{2JS}\right)\ln\frac{1-e^{-2\beta JS\rho^{(+)}(x)}}{1-e^{-2\beta JS\rho^{(-)}(x)}} \tag{3.6.31}$$

where

$$\rho^{(\pm)}(x) = \frac{1}{2}[k^2 + p^2 \pm \sqrt{(k^2+p^2)^2 - 4k^2p^2x^2}]. \tag{3.6.31a}$$

By means of Eq. (3.6.31), the damping (3.6.27) becomes

$$\Sigma_c^{(2)''}(\boldsymbol{k} \to 0, \omega_{\boldsymbol{k}}) = \frac{J}{\hbar S}\left(\frac{2JS}{k_BT}\right) k^4 \frac{1}{8\pi^3}[I_1(k) + I_2(k)] \tag{3.6.32}$$

where

$$I_1(k) = \frac{1}{2}\int_0^\infty dp\ p^4 n_p^{(0)}[1 + n(\omega_k + \omega_p)] \int_{-1}^1 dx\ x^2\sqrt{k^2+p^2-2kpx} \tag{3.6.33}$$

and

$$\begin{aligned} I_2(k) = \left(\frac{k_BT}{2JS}\right)\int_0^\infty dp\ p^4 n_p^{(0)}[1 + n(\omega_k+\omega_p)] \\ \times \int_{-1}^1 dx \frac{x^2}{\sqrt{k^2+p^2+2kpx}} \ln\frac{1-e^{-2\beta JS\rho^{(+)}(x)}}{1-e^{-2\beta JS\rho^{(-)}(x)}}. \end{aligned} \tag{3.6.34}$$

In the long wavelength limit the integral in Eq. (3.6.33) may be evaluated at $k = 0$ since it remains finite in that limit. One has[3]

$$\begin{aligned} I_1(0) &= \frac{1}{3}\int_0^\infty dp\ p^5 n_p^{(0)}(1 + n_p^{(0)}) = \frac{1}{6}\left(\frac{k_BT}{2JS}\right)^3 \int_0^\infty dx\, x^2 \frac{e^x}{(e^x-1)^2} \\ &= \frac{1}{6}\left(\frac{k_BT}{2JS}\right)^3 \left[\left|-\frac{x^2}{e^x-1}\right|_0^\infty + 2\int_0^\infty dx\ \frac{x}{e^x-1}\right] = \frac{\pi^2}{18}\left(\frac{k_BT}{2JS}\right)^3. \end{aligned} \tag{3.6.35}$$

The evaluation of the integral (3.6.34) is more difficult since it diverges for $k \to 0$. In order to evaluate this singular contribution,[31] the integral $I_2(k)$ may be broken in two parts:

$$I_2(k) = I_2^{(1)} + I_2^{(2)} \tag{3.6.36}$$

where

$$\begin{aligned} I_2^{(1)} = \left(\frac{k_BT}{2JS}\right)\int_0^{\lambda k} dp\, p^4 n_p^{(0)}[1 + n(\omega_k+\omega_p)] \\ \times \int_{-1}^1 dx \frac{x^2}{\sqrt{k^2+p^2+2kpx}} \ln\frac{1-e^{-2\beta JS\rho^{(+)}(x)}}{1-e^{-2\beta JS\rho^{(-)}(x)}} \end{aligned} \tag{3.6.37}$$

and

$$\begin{aligned} I_2^{(2)} = \left(\frac{k_BT}{2JS}\right)\int_{\lambda k}^\infty dp\, p^4 n_p^{(0)}[1 + n(\omega_k+\omega_p)] \\ \times \int_{-1}^1 dx \frac{x^2}{\sqrt{k^2+p^2+2kpx}} \ln\frac{1-e^{-2\beta JS\rho^{(+)}(x)}}{1-e^{-2\beta JS\rho^{(-)}(x)}} \end{aligned} \tag{3.6.38}$$

with the assumption that $\lambda \gg 1$ even though $\lambda k \ll 1$. For instance, this can be achieved by choosing[31] $\lambda = (\frac{k_BT}{2JSk^2})^{1/4}$. Since $\lambda k \ll 1$, the Bose factors in Eq. (3.6.37) may by replaced by

$$n_p^{(0)}[1 + n(\omega_k + \omega_p)] \simeq \left(\frac{k_BT}{2JS}\right)^2 \frac{1}{p^2(k^2 + p^2)} \tag{3.6.39}$$

and the exponential functions may be expanded in powers of $2\beta JS\rho^{(\pm)}$. By means of the variable change $p = \xi k$, Eq. (3.6.37) becomes

$$I_2^{(1)} = \left(\frac{k_BT}{2JS}\right)^3 \int_0^\lambda d\xi \frac{\xi^2}{1+\xi^2} \int_{-1}^1 dx \frac{x^2}{\sqrt{1+\xi^2+2x\xi}} \times \ln \frac{1+\xi^2+\sqrt{(1+\xi^2)^2 - 4\xi^2x^2}}{1+\xi^2-\sqrt{(1+\xi^2)^2-4\xi^2x^2}}. \tag{3.6.40}$$

Since $\lambda \gg 1$, we can split the integral over ξ in Eq. (3.6.40) in two integrals: the former going from 0 to 1 and the latter going from 1 to λ. In the second integral, a variable change from ξ to $\frac{1}{\xi}$ may be performed so that Eq. (3.6.40) becomes

$$I_2^{(1)} = \left(\frac{k_BT}{2JS}\right)^3 \left[\int_0^1 d\xi\ \frac{\xi^2}{1+\xi^2} S(\xi) + \int_{1/\lambda}^1 d\xi \frac{1}{\xi(1+\xi^2)} S(\xi)\right] \tag{3.6.41}$$

where

$$S(\xi) = \int_{-1}^1 dx \frac{x^2}{\sqrt{1+\xi^2+2x\xi}} \ln \frac{1+\xi^2+\sqrt{(1+\xi^2)^2-4x^2\xi^2}}{1+\xi^2-\sqrt{(1+\xi^2)^2-4x^2\xi^2}}. \tag{3.6.42}$$

The first integral of Eq. (3.6.41) may be split in two parts: one going from 0 to $1/\lambda$ and the other from $1/\lambda$ to 1. Then Eq. (3.6.41) becomes

$$I_2^{(1)} = \left(\frac{k_BT}{2JS}\right)^3 \left[I_2^{(1a)} + I_2^{(1b)}\right] \tag{3.6.43}$$

where

$$I_2^{(1a)} = \int_{1/\lambda}^1 d\xi\ \frac{1+\xi^3}{\xi(1+\xi^2)} S(\xi) \tag{3.6.44}$$

and

$$I_2^{(1b)} = \int_0^{1/\lambda} d\xi\ \frac{\xi^2}{1+\xi^2} S(\xi). \tag{3.6.45}$$

Making the variable change $x = \frac{1+\xi^2}{2\xi} y$ in the integral of Eq. (3.6.42), one obtains[3]

$$S(\xi) = \frac{(1+\xi^2)^{5/2}}{8\xi^3} \int_{-\frac{2\xi}{1+\xi^2}}^{\frac{2\xi}{1+\xi^2}} dy \frac{y^2}{\sqrt{1+y}} \ln \frac{1+\sqrt{1-y^2}}{1-\sqrt{1-y^2}} = -\frac{4}{15}(5+2\xi^2)\ln\xi - \frac{4}{15}\frac{(2+\xi^2)(1+\xi^2)}{\xi^2} + \frac{8}{15}\frac{(1+\xi^2)^{\frac{5}{2}}}{\xi^3}\ln(\sqrt{1+\xi^2}+\xi). \tag{3.6.46}$$

Replacing Eq. (3.6.46) into Eqs. (3.6.44) and (3.6.45) and expanding for large values of λ, one obtains

$$I_2^{(1a)} = \frac{2}{3}\ln^2\lambda + \frac{4}{9}\ln\lambda + C + O\left(\frac{1}{\lambda}\ln\lambda\right) \tag{3.6.47}$$

and

$$I_2^{(1b)} = \frac{2}{3\lambda^2} + O\left(\frac{1}{\lambda^3}\ln\lambda\right) \tag{3.6.48}$$

with

$$\begin{aligned} C &= \frac{56}{135} + \frac{7}{60}\pi^2 - \frac{4}{15}\ln^2(\sqrt{2}+1) - \frac{4}{5}G \\ &\quad - \frac{8}{15}[\mathrm{Li}_2(\sqrt{2}-1) - \mathrm{Li}_2(-\sqrt{2}+1)] = 0.17552256 \end{aligned} \tag{3.6.49}$$

where $G = 0.91596559$ is the Catalan's constant and $\mathrm{Li}_2(x)$ the polylogarithm function given by

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^s}. \tag{3.6.50}$$

The numerical value of the constant C in Eq. (3.6.49) differs from that given by Harris[31] in Eq. (B8) where the loss of a factor $\frac{4}{15}$ in Eq. (B7) leads to a negative value of C. Replacing Eqs. (3.6.47) and (3.6.48) into Eq. (3.6.43), one obtains

$$I_2^{(1)} = \left(\frac{k_BT}{2JS}\right)^3\left[\frac{2}{3}\ln^2\lambda + \frac{4}{9}\ln\lambda + C + O\left(\frac{1}{\lambda}\ln\lambda\right)\right]. \tag{3.6.51}$$

The integration range over p in Eq. (3.6.38) allows us to neglect k in comparison to p in the square root and ω_k with respect to ω_p in the Bose factor. Recalling that $\rho^{(\pm)}(x)$ are even functions of x and making the variable change $y = x^2$, Eq. (3.6.38) becomes

$$I_2^{(2)} = \left(\frac{k_BT}{2JS}\right)\int_{\lambda k}^{\infty} dp\, p^3 n_p^{(0)}(1+n_p^{(0)})\int_0^1 dy\,\sqrt{y}\;\ln\frac{1-e^{-2\beta JS\rho^{(+)}(y)}}{1-e^{-2\beta JS\rho^{(-)}(y)}} \tag{3.6.52}$$

where

$$\rho^{(\pm)}(y) = \frac{1}{2}[k^2+p^2 \pm \sqrt{(k^2+p^2)^2 - 4k^2p^2y}]. \tag{3.6.52a}$$

Integrating by parts the last integral of Eq. (3.6.52), one obtains

$$\begin{aligned} &\int_0^1 dy\;\sqrt{y}\;\ln\frac{1-e^{-2\beta JS\rho^{(+)}(y)}}{1-e^{-2\beta JS\rho^{(-)}(y)}} \\ &\quad = \frac{2}{3}\ln\frac{1-e^{-2\beta JS\rho^{(+)}(1)}}{1-e^{-2\beta JS\rho^{(-)}(1)}} \\ &\qquad - \frac{2}{3}\left(\frac{2JS}{k_BT}\right)\int_0^1 dy\;y^{3/2}\left[\frac{1}{e^{2\beta JS\rho^{(+)}(y)}-1}\;\frac{d\rho^{(+)}}{dy} - \frac{1}{e^{2\beta JS\rho^{(-)}(y)}-1}\;\frac{d\rho^{(-)}}{dy}\right] \end{aligned}$$

$$= \frac{2}{3} \ln \frac{1 - e^{-2\beta JS\rho^{(+)}(1)}}{1 - e^{-2\beta JS\rho^{(-)}(1)}} - \frac{2}{3} \left(\frac{2JS}{k_B T} \right) \int_{\rho^{(+)}(0)}^{\rho^{(+)}(1)} d\rho^{(+)} \frac{y[\rho^{(+)}]^{\frac{3}{2}}}{e^{2\beta JS\rho^{(+)}} - 1}$$

$$+ \frac{2}{3} \left(\frac{2JS}{k_B T} \right) \int_{\rho^{(-)}(0)}^{\rho^{(-)}(1)} d\rho^{(-)} \frac{y[\rho^{(-)}]^{\frac{3}{2}}}{e^{2\beta JS\rho^{(-)}} - 1} \tag{3.6.53}$$

where

$$y(\rho) = \frac{\rho \left(k^2 + p^2 - \rho \right)}{(kp)^2} \tag{3.6.54}$$

is the inverse function of Eq. (3.6.52a). Using the relationships $\rho^{(+)}(1) = p^2$, $\rho^{(+)}(0) = k^2 + p^2$, $\rho^{(-)}(1) = k^2$ and $\rho^{(-)}(0) = 0$, obtained from Eq. (3.6.52a), Eq. (3.6.53) becomes

$$I_2^{(2)} = I_2^{(2a)} + I_2^{(2b)} + I_2^{(2c)} \tag{3.6.55}$$

where

$$I_2^{(2a)} = \frac{2}{3k^3} \int_{\lambda k}^{\infty} dp\, n_p^{(0)} (1 + n_p^{(0)}) \int_0^{k^2} d\rho\, \frac{[\rho(k^2 + p^2 - \rho)]^{\frac{3}{2}}}{e^{2\beta JS\rho} - 1}, \tag{3.6.56}$$

$$I_2^{(2b)} = \frac{2}{3k^3} \int_{\lambda k}^{\infty} dp\, n_p^{(0)} (1 + n_p^{(0)}) \int_{p^2}^{p^2 + k^2} d\rho\, \frac{[\rho(k^2 + p^2 - \rho)]^{\frac{3}{2}}}{e^{2\beta JS\rho} - 1} \tag{3.6.57}$$

and

$$I_2^{(2c)} = \frac{2}{3} \left(\frac{k_B T}{2JS} \right) \int_{\lambda k}^{\infty} dp\; p^3\; n_p^{(0)} (1 + n_p^{(0)}) \ln \frac{1 - e^{-2\beta JSp^2}}{1 - e^{-2\beta JSk^2}}. \tag{3.6.58}$$

Because of the chain of inequalities

$$0 < \rho < k^2 \ll (\lambda k)^2 < p^2, \tag{3.6.59}$$

the integral of Eq. (3.6.56) may be approximated by

$$\begin{aligned} I_2^{(2a)} &\simeq \frac{2}{3k^3} \left(\frac{k_B T}{2JS} \right) \int_{\lambda k}^{\infty} dp\; p^3\; n_p^{(0)} (1 + n_p^{(0)}) \int_0^{k^2} d\rho \sqrt{\rho} \\ &= \frac{2}{9} \left(\frac{k_B T}{2JS} \right)^3 \int_{2\beta JS(\lambda k)^2}^{\infty} dx \frac{x e^x}{(e^x - 1)^2} \\ &= \frac{2}{9} \left(\frac{k_B T}{2JS} \right)^3 \left[\frac{2\beta JS(\lambda k)^2}{e^{2\beta JS(\lambda k)^2} - 1} - \ln(1 - e^{-2\beta JS\lambda^2 k^2}) \right]. \end{aligned} \tag{3.6.60}$$

The second row of Eq. (3.6.60) is obtained by the replacement $x = 2\beta JSp^2$ and the third row is obtained by an integration by parts. Expanding Eq. (3.6.60) for

$2\beta JS(\lambda k)^2 \ll 1$, one obtains

$$I_2^{(2a)} = \frac{2}{9}\left(\frac{k_B T}{2JS}\right)^3 \left\{\ln\frac{k_B T}{2JSk^2} - 2\ln\lambda + 1 + O\left[\left(\frac{2JS\lambda^2 k^2}{k_B T}\right)^2\right]\right\}. \tag{3.6.61}$$

From the chain of inequalities given in Eq. (3.6.59), the integral over p of Eq. (3.6.57) may be evaluated by means of the mean value theorem leading to

$$\int_{p^2}^{p^2+k^2} d\rho\, \frac{[\rho(k^2+p^2-\rho)]^{\frac{3}{2}}}{e^{2\beta JS\rho}-1} \simeq (1-\xi)^{\frac{3}{2}} k^5 p^3 \frac{1}{e^{2\beta JSp^2}-1} \tag{3.6.62}$$

where $0 < \xi < 1$ is due to the mean value theorem. Then Eq. (3.6.57) reduces to

$$\begin{aligned} I_2^{(2b)} &= \frac{2}{3}(1-\xi)^{\frac{3}{2}} k^2 \int_{\lambda k}^{\infty} dp\, p^3\, n_p^{(0)2}(1+n_p^{(0)}) \\ &= \frac{(1-\xi)^{3/2}}{3}\left(\frac{k_B T}{2JS}\right)^2 k^2 \int_{2\beta JS(\lambda k)^2}^{\infty} dx \frac{xe^x}{(e^x-1)^3}. \end{aligned} \tag{3.6.63}$$

After some integrations by parts, Eq. (3.6.63) becomes

$$\begin{aligned} I_2^{(2b)} = \frac{(1-\xi)^{\frac{3}{2}}}{6}\left(\frac{k_B T}{2JS}\right)^2 k^2 &\left[\frac{2\beta JS(\lambda k)^2}{(e^{2\beta JS\lambda^2 k^2}-1)^2}\right. \\ &\left. + \frac{1}{e^{2\beta JS(\lambda k)^2}-1} + 2\ln(1-e^{-2\beta JS\lambda^2 k^2})\right]. \end{aligned} \tag{3.6.64}$$

Expanding Eq. (3.6.64) in powers of $2\beta JS(\lambda k)^2$, one obtains

$$I_2^{(2b)} = \frac{(1-\xi)^{\frac{3}{2}}}{3}\left(\frac{k_B T}{2JS}\right)^3 \left[\frac{1}{\lambda^2} + \frac{2JSk^2}{k_B T}\left(\frac{1}{2}\ln\frac{2JSk^2}{k_B T} + \ln\lambda - \frac{3}{4} + \cdots\right)\right] \tag{3.6.65}$$

By the variable change $x = 2\beta JSp^2$, the integral of Eq. (3.6.58) becomes

$$\begin{aligned} I_2^{(2c)} = &-\frac{1}{3}\left(\frac{k_B T}{2JS}\right)^3 \ln(1-e^{-2\beta JS\lambda^2 k^2}) \int_{2\beta JS(\lambda k)^2}^{\infty} dx \frac{xe^x}{(e^x-1)^2} \\ &+ \frac{1}{3}\left(\frac{k_B T}{2JS}\right)^3 \int_{2\beta JS(\lambda k)^2}^{\infty} dx \frac{xe^x}{(e^x-1)^2}\ln(1-e^x). \end{aligned} \tag{3.6.66}$$

After some integrations by parts, one obtains

$$\begin{aligned} I_2^{(2c)} = &-\frac{1}{3}\left(\frac{k_B T}{2JS}\right)^3 \ln(1-e^{-2\beta JSk^2})\left[\frac{2\beta JS(\lambda k)^2}{e^{2\beta JS(\lambda k)^2}-1} - \ln(1-e^{-2\beta JS\lambda^2 k^2})\right] \\ &+ \frac{1}{3}\left(\frac{k_B T}{2JS}\right)^3 \left[\left(\frac{2\beta JS\lambda^2 k^2}{e^{2\beta JS\lambda^2 k^2}-1} - 1 + 2\beta JS\lambda^2 k^2\right)\ln(1-e^{-2\beta JS\lambda^2 k^2})\right. \\ &\left. + \frac{2\beta JS(\lambda k)^2}{e^{2\beta JS(\lambda k)^2}-1} - \frac{1}{2}\ln^2(1-e^{-2\beta JS\lambda^2 k^2}) - \mathrm{Li}_2(e^{-2\beta JS\lambda^2 k^2})\right] \end{aligned} \tag{3.6.67}$$

where the polylogarithm function $\mathrm{Li}_2(x)$ is given by Eq. (3.6.50). Expanding Eq. (3.6.67) in powers of $2\beta JS(\lambda k)^2$, one obtains

$$I_2^{(2c)} = \frac{1}{3}\left(\frac{k_BT}{2JS}\right)^3\left[\frac{1}{2}\ln^2\frac{k_BT}{2JSk^2} + \ln\frac{k_BT}{2JSk^2} - 2\ln^2\lambda + 1 - \frac{\pi^2}{6} + O\left(2\beta JS\lambda^2k^2\right)\right]. \tag{3.6.68}$$

Replacing Eqs. (3.6.61), (3.6.65) and (3.6.68) into Eq. (3.6.55), one obtains

$$I_2^{(2)} = \left(\frac{k_BT}{2JS}\right)^3\left[\frac{1}{6}\ln^2\frac{k_BT}{2JSk^2} + \frac{5}{9}\ln\frac{k_BT}{2JSk^2} - \frac{2}{3}\ln^2\lambda - \frac{4}{9}\ln\lambda + \frac{5}{9} - \frac{\pi^2}{18} + O\left(\frac{2JS\lambda^2k^2}{k_BT}\right)\right]. \tag{3.6.69}$$

From Eqs. (3.6.36), (3.6.51) and (3.6.69), one obtains

$$I_2(k\to 0) = \left(\frac{k_BT}{2JS}\right)^3\left[\frac{1}{6}\ln^2\frac{k_BT}{2JSk^2} + \frac{5}{9}\ln\frac{k_BT}{2JSk^2} + C + \frac{5}{9} - \frac{\pi^2}{18}\right] \tag{3.6.70}$$

where C is given by Eq. (3.6.49). Finally, from Eqs. (3.6.32), (3.6.35), and (3.6.70), one obtains

$$\Sigma_c^{(2)''}(\boldsymbol{k}\to 0,\omega_{\boldsymbol{k}}) = \frac{Jk^4}{8\pi^3\hbar S}\left(\frac{k_BT}{2JS}\right)^2\left[\frac{1}{6}\ln^2\left(\frac{k_BT}{2JSk^2}\right) + \frac{5}{9}\ln\left(\frac{k_BT}{2JSk^2}\right) + K\right] \tag{3.6.71}$$

where

$$K = \frac{1}{3}C + \frac{5}{9} = 0.73107812. \tag{3.6.72}$$

The damping given by Eq. (3.6.71) was obtained by A. B. Harris.[31] As one can see from Eq. (3.6.18) and (3.6.71), in the long wavelength and low temperature limit, the renormalization and damping of a spin wave of momentum $\boldsymbol{k}$ are of order $k^2T^{5/2}$ and k^4T^2, respectively. Moreover, singular terms appear in the damping.

In the short wavelength limit, that is for magnons with $\beta\hbar\omega_{\boldsymbol{k}} \gg 1$, in Eq. (3.6.22) one has $1 - e^{\beta\hbar\omega_{\boldsymbol{k}}} \simeq 1$, $n(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{q}_1}) \simeq n_{\boldsymbol{k}}^{(0)} \simeq 0$, $n_{\boldsymbol{q}_2}^{(0)} \simeq 0$ since $\boldsymbol{q}_2 \simeq \boldsymbol{k}$ due to the δ-function argument so that the imaginary part of the second-order self-energy becomes

$$\Sigma_c^{(2)''}(\boldsymbol{k},\omega_{\boldsymbol{k}}) = \frac{\pi}{2S^2}\left(\frac{2zJS}{\hbar N^2}\right)\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}(\gamma_{\boldsymbol{k}-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_1-\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1})^2 \times \delta(\gamma_{\boldsymbol{k}+\boldsymbol{q}_1-\boldsymbol{q}_2} + \gamma_{\boldsymbol{q}_2} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{q}_1})n_{\boldsymbol{q}_1}^{(0)}. \tag{3.6.73}$$

Putting $\boldsymbol{q}_1 = \boldsymbol{p}$ and $\boldsymbol{q}_2 = \frac{1}{2}(\boldsymbol{k}+\boldsymbol{p}) - \boldsymbol{\rho}$, for a SC lattice the main temperature contribution of Eq. (3.6.73) becomes

$$\Sigma_c^{(2)''}(\boldsymbol{k},\omega_{\boldsymbol{k}}) = \frac{4\pi J}{\hbar S}\sum_\alpha \sin^2\frac{k_\alpha}{2}\frac{1}{N}\sum_{\boldsymbol{p}} p_\alpha^2 n_{\boldsymbol{p}}^{(0)}\frac{1}{N}\sum_{\boldsymbol{\rho}}$$
$$\times \cos^2\rho_\alpha \delta\left[\sum_\gamma \cos\frac{k_\gamma}{2}\left(\cos\rho_\gamma - \cos\frac{k_\gamma}{2}\right)\right]$$
$$= \frac{8J}{\hbar S}\pi^2\,\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}}\sum_\alpha \sin^2\frac{k_\alpha}{2}L_\alpha(\boldsymbol{k}) \qquad (3.6.74)$$

where $\alpha,\gamma = x,y,z$ and Eq. (3.6.14) has been used. Moreover,

$$L_\alpha(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\rho_x d\rho_y d\rho_z \cos^2\rho_\alpha\,\delta\left[\sum_\gamma \cos\frac{k_\gamma}{2}\left(\cos\rho_\gamma - \cos\frac{k_\gamma}{2}\right)\right]. \qquad (3.6.75)$$

As one can see, the damping is proportional to $T^{5/2}$ for $\hbar\omega_{\boldsymbol{k}} \gg k_BT$. For simplicity, we evaluate the integral in Eq. (3.6.75) for $\boldsymbol{k} = (k,k,k)$. One has

$$L_\alpha(\boldsymbol{k}) = L(k) = \frac{1}{4\pi^3}\frac{1}{\cos\frac{k}{2}}\iint_D d\rho_x d\rho_y \frac{(3\cos\frac{k}{2} - \cos\rho_x - \cos\rho_y)^2}{\sqrt{1-(3\cos\frac{k}{2} - \cos\rho_x - \cos\rho_y)^2}} \qquad (3.6.76)$$

where the domain D is given by

$$\cos\rho_x + \cos\rho_y > 3\cos\frac{k}{2} - 1 \qquad (3.6.77)$$

for $k < k_0 = \arccos\frac{1}{3} = 0.78365\pi$ and

$$3\cos\frac{k}{2} + 1 > \cos\rho_x + \cos\rho_y > 3\cos\frac{k}{2} - 1 \qquad (3.6.78)$$

for $k > k_0$. For $k \lesssim \frac{\pi}{2}$, the domain (3.6.77) reduces to

$$\rho_x^2 + \rho_y^2 < 6\left(1 - \cos\frac{k}{2}\right) \qquad (3.6.79)$$

and the integral of Eq. (3.6.76) becomes

$$L(k) \simeq \frac{1}{2\pi^2}\,\frac{1}{\cos\frac{k}{2}}\int_0^{\sqrt{6(1-\cos k/2)}} \rho\, d\rho \frac{\left(3\cos\frac{k}{2} - 2 + \frac{1}{2}\rho^2\right)^2}{\sqrt{1-\left(3\cos\frac{k}{2} - 2 + \frac{1}{2}\rho^2\right)^2}}$$
$$= \frac{1}{2\pi^2}\,\frac{1}{\cos\frac{k}{2}}\int_{3\cos(k/2)-2}^{1} d\xi\,\frac{\xi^2}{\sqrt{1-\xi^2}}$$

$$= \frac{1}{2\pi^2}\,\frac{1}{\cos\frac{k}{2}}\left[\frac{\pi}{4}+\frac{1}{2}\left(3\cos\frac{k}{2}-2\right)\sqrt{1-\left(3\cos\frac{k}{2}-2\right)^2}\right.$$
$$\left.-\frac{1}{2}\arcsin\left(3\cos\frac{k}{2}-2\right)\right]. \tag{3.6.80}$$

Expanding Eq. (3.6.80) in powers of k, one has

$$L(k) \simeq \frac{1}{4\pi^2}\sqrt{3}k\left(1-\frac{5}{48}k^2+\cdots\right). \tag{3.6.81}$$

Replacing Eq. (3.6.81) into Eq. (3.6.74), the imaginary part of the second order self-energy for $\hbar\omega_{\boldsymbol{k}} \gg k_BT$ and $k \lesssim \frac{\pi}{2}$ becomes

$$\Sigma_c^{(2)''}(\boldsymbol{k},\omega_{\boldsymbol{k}}) = \frac{J}{2\hbar S}|\boldsymbol{k}|^3\zeta\left(\frac{5}{2}\right)\left(\frac{k_BT}{8\pi JS}\right)^{\frac{5}{2}} \tag{3.6.82}$$

where $|\boldsymbol{k}| = \sqrt{3}k$. The damping (3.6.82) was obtained by Lovesey.[23] Note that in the Lovesey's damping formula, a factor 2 in the denominator is missing.

In conclusion, for $\hbar\omega_{\boldsymbol{k}} \ll k_BT$, the damping is given by Eq. (3.6.71) while for $\hbar\omega_{\boldsymbol{k}} \gg k_BT$, it is given by Eq. (3.6.74) that reduces to Eq. (3.6.82) for $\boldsymbol{k}$ neither too small nor too near the ZB.