Exercises for Mathematical Programming
186.835 VU 3.0 – SS 2021
Fabian Holzberger 11921655

Exercise 1 *Cycle-Elimination Cuts (2* ***points****)*

Consider the polyhedra of the following two ILP formulations for the minimum spanning tree problem:

- cycle elimination formulation (CEC)

$$\min \sum_{e \in E} w_e x_e \tag{1}$$

$$\text{s.t.} \sum_{e \in E} x_e = n - 1 \tag{2}$$

$$\sum_{e \in C} x_e \leq |C| - 1 \qquad \forall C \subseteq E,\ |C| \geq 2,\ C \text{ forms a cycle} \tag{3}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E \tag{4}$$

- subtour elimination formulation (SUB)

$$\min \sum_{e \in E} w_e x_e \tag{5}$$

$$\text{s.t.} \sum_{e \in E} x_e = n - 1 \tag{6}$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subseteq V, S \neq \emptyset \tag{7}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E \tag{8}$$

Prove or disprove:

$$P_{\text{cec}} = P_{\text{sub}}$$

Proposition: $P_{\text{cec}} = P_{\text{sub}}$ for $x_e \in \mathbb{Q} \cap [0, 1]$.
Proof of the proposition:
The graphs are considered loop free. To show $P_{sub} \subseteq P_{cec}$ we assume $\exists x \in P_{sub}(x \notin P_{cec})$ for a contradiction. Therefore, we select an x such that $x \in P_{sub}$ and $x \notin P_{cec}$. Since $x \in P_{sub}$ we know $\sum_{e \in E} x_e = n - 1$ is true, implying together with $x \notin P_{cec}$ that we must have $\sum_{e \in C_1} x_e > |C_1| - 1$ for some cycle $C_1 \subseteq E$ and $|C_1| \geq 2$. W.l.o.g. let $C_1 = ((0, 1), ..., (m, 0))$. Note that C_1 defines a subtour $S_1 := \{0, 1, ..., m\}$ with edges $C_1 \subseteq E(S_1)$ and $|C_1| = |S_1|$ since it is a cycle. Therefore $S_1 \neq \emptyset$ and $S_1 \subseteq V$ implying since $x \in P_{sub}$ that $\sum_{e \in C_1} x_e \leq \sum_{e \in E(S_1)} x_e \leq |S_1| - 1 = |C_1| - 1$ which is a contradiction.
Next show $P_{cec} \subseteq P_{sub}$. First select an arbitrary $x \in P_{cec}$ which implies $\sum_{e \in C} x_e \leq |C| - 1 \quad \forall |C| \geq 2$ and $C \subseteq E$.
Consider an arbitrary cycle C_1. If we were to remove any edge t from C_1 we end up with a path $C_1 \setminus \{t\}$ that fulfills $|C_1| - 1 = |C_1 \setminus \{t\}|$. Therefore if C_1 satisfies the CEC $\sum_{e \in C_1} x_e \leq |C_1| - 1$ we have $\max_{x_e} \sum_{x_e \in C_1} x_e = |C_1| - 1 = |C_1 \setminus \{t\}| = \max_{x_e} \sum_{x_e \in C_1 \setminus \{t\}} x_e$.
Now take an arbitrary subtour S. Any cycle of S can be replaced by the path that we obtain by removing an edge from the corresponding cycle. By this we obtain a new acyclic graph $\hat{S}$ with $\max_{x_e} \sum_{x_e \in E(S)} x_e = \max_{x_e} \sum_{x_e \in E(\hat{S})} x_e$. In any acyclic graph the numver of vertices is at least one more than the number of edges. Therfore we have $\max_{x_e} \sum_{x_e \in E(\hat{S})} x_e = |E(\hat{S})| \leq |\hat{S}| - 1$. Since $|S| = |\hat{S}|$ we further have $\max_{x_e} \sum_{x_e \in E(S)} x_e \leq |S| - 1$ which shows that $\sum_{x_e \in E(S)} x_e \leq |S| - 1$ and we follow $x \in P_{sub}$. ■

Exercise 2 *(Prize Collecting) Steiner Tree Problem (2* ***points****)*

Consider the *Steiner tree problem on a graph (STP)* and the *Prize Collecting Steiner tree problem on a graph (PCSTP)*, which are defined as follows:

- STP: Given an undirected graph $G = (V, E)$ with edge weights $w_e \geq 0$, $\forall e \in E$, whose node set is partitioned into terminal nodes T and potential Steiner nodes S, i.e. $S \cup T = V$, $S \cap T = \emptyset$. The problem is to find a minimum weight subtree of G that contains all terminal nodes.
- PCSTP: Given an undirected graph $G = (V, E)$ with edge weights $w_e \geq 0$, $\forall e \in E$ and node profits $p_i \geq 0$, $\forall i \in V$. The problem is to find a subtree $G' = (V', E')$ of G that yields maximum profit, i.e. $\max \sum_{i \in V'} p_i - \sum_{e \in E'} w_e$.

Provide ILP formulations for each problem using *directed cutset constraints.*
Answer:
ILP for STP:

$$\begin{align}
\min \sum_{e \in E} w_e x_e \tag{9}\\
\sum_{(i,j) \in \delta^+(S)} y_{ij} \geq 1 \quad \forall S \subset V, r \in S, S^c \cap T \neq \emptyset \tag{10}\\
y_{ij} + y_{ji} = x_e \quad \forall e = \{i, j\} \in E \tag{11}\\
y_{ij}, x_e \in \mathbb{B} \quad \forall e = \{i, j\} \in E \tag{12}
\end{align}$$

We select arcs in the graph by the binary variables y_{ij}, that model the arc with tail at vertex i and head at vertex j. The arcs y_{ij}, y_{ji} correspond to the edge $e = \{i, j\}$, that is modeled by their binary variable x_e. By constraint (11) we know that only one arc can be chosen and also that whenever we choose an arc we also choose the corresponding edge. For the directed graph we choose the root as $r \in T$ arbitrarily. Then by the cut-set constrained (10) we state that every subset S of V that includes the root r must have an outgoing arc into the complementary set S^c if the complementary set contains a terminal.
ILP for PCSTP:

$$\begin{align}
\max \left(\sum_{i \in V} p_i z_i - \sum_{e \in E} w_e x_e \right) \tag{13}\\
\sum_{(i,j) \in \delta^+(S)} y_{ij} \geq z_k \quad \forall S \subset V, r \in S, k \in S^c \tag{14}\\
\sum_{(i,j) \in A} y_{ji} = z_i \quad \forall i \in V \setminus \{r\} \tag{15}\\
y_{ij} + y_{ji} = x_e \quad \forall e = \{i, j\} \in E \tag{16}\\
y_{ij}, x_e \in \mathbb{B} \quad \forall e = \{i, j\} \in E \tag{17}\\
z_i \in \mathbb{B} \quad \forall i \in V \tag{18}\\
\tag{19}
\end{align}$$

The ILP is similar to the STP. We now introduce the binary variable z_i that is true if we choose a vertex i when constructing the subgraph G'. By that we can easily formulate the objective function in (13) by summing up the profits of the vertices in G' and subtracting the weights of the edges in E'. To select vertcies when we have an ingoing edge we use constraint (16), that states that if there exists any ingoing edge into vertex i we must set z_i true. Further since z_i is binary we can only have one ingoing edge. The cutset constraint (14) ensures that all selected vertices are connected to the root r.
