ex3

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1.1 Exercise 6 Chvátal-Gomory Cutting Planes

Draw initial Polyhedra:

[1]:
```
import numpy as np
from matplotlib import pyplot as plt
import sympy as sym
```

[2]:
```
plt.rcParams['figure.dpi'] = 500
plt.rcParams['savefig.dpi'] = 1000
```

[3]:
```
# use this function to return integer solutions
def get_sols(x,y):
    mask = np.logical_and(x>=0,y>=0)
    mask = np.logical_and((2*x+3*y)<=10,mask)
    mask = np.logical_and((-4*x+6*y)<=9,mask)
    return mask
```

[4]:
```
# get our integer solutions from condidates
x    = np.arange(6)
X,Y  = np.meshgrid(x,x)
mask = get_sols(X,Y)
```

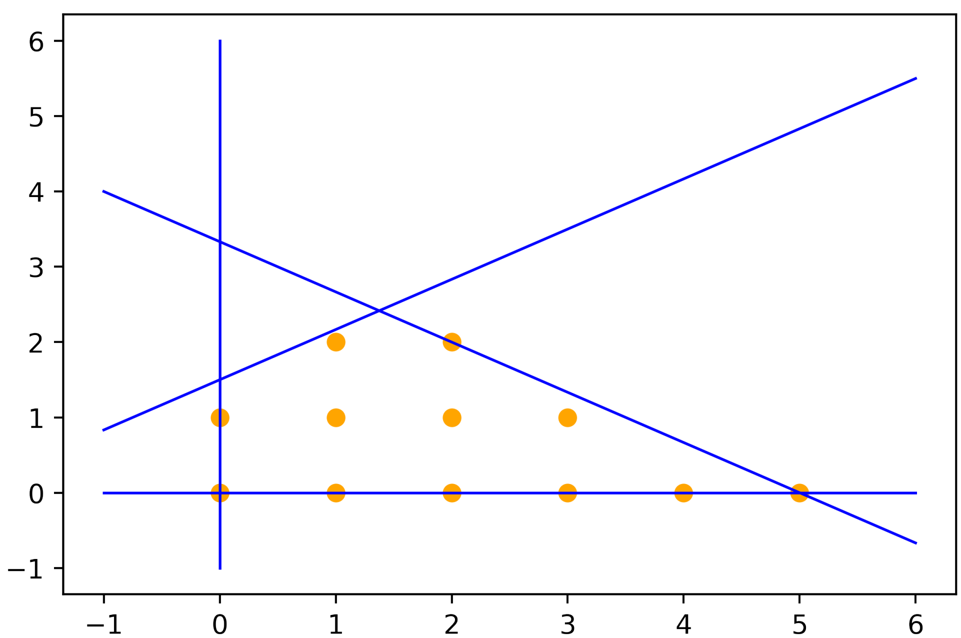
[5]:
```
# plot integer solutions and all cutting planes
def plot_polyhed(X_i, Y_i, x_curve, y_curve, col):
    plt.plot(X_i,Y_i,"o",color="orange")
    for val in zip(x_curve,y_curve, col):
        plt.plot(val[0], val[1],color=val[2],linewidth=1)
```

[6]:
```
# cuttiong planes and color
x_curve = []
y_curve = []
col     = []
```

[7]:
```
# add initial planes
x = np.linspace(-1,6,10)
```

```
# x2>=0
x_curve.append(x)
y_curve.append(x*0)
col.append("blue")
# x1>=0
x_curve.append(x*0)
y_curve.append(x)
col.append("blue")
# 2x1+ 3x2 <=10
x_curve.append(x)
y_curve.append(10/3-2/3*x)
col.append("blue")
# -4x1+ 6x2 <=9
x_curve.append(x)
y_curve.append(9/6+4/6*x)
col.append("blue")
```

[8]:
```
plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
```



Assume simplex method finds a fractional solution. Convert equations to standard form by adding slack variables x_3, x_4:

$$-4x_1 + 6x_2 + x_3 = 9 \tag{1}$$
$$2x_1 + 3x_2 + x_4 = 10 \tag{2}$$

We have for example $x_B = (x_1, x_2)$ and $x_N = (x_3, x_4)$ for a fractional solution.

Find next cutting plane:

[9]:
```
# A = (B, AN)
B  = sym.Matrix(2, 2, [-4, 6, 2, 3])
AN = sym.Matrix(2, 2, [1, 0, 0, 1])
b  = sym.Matrix(2, 1, [9, 10])
```

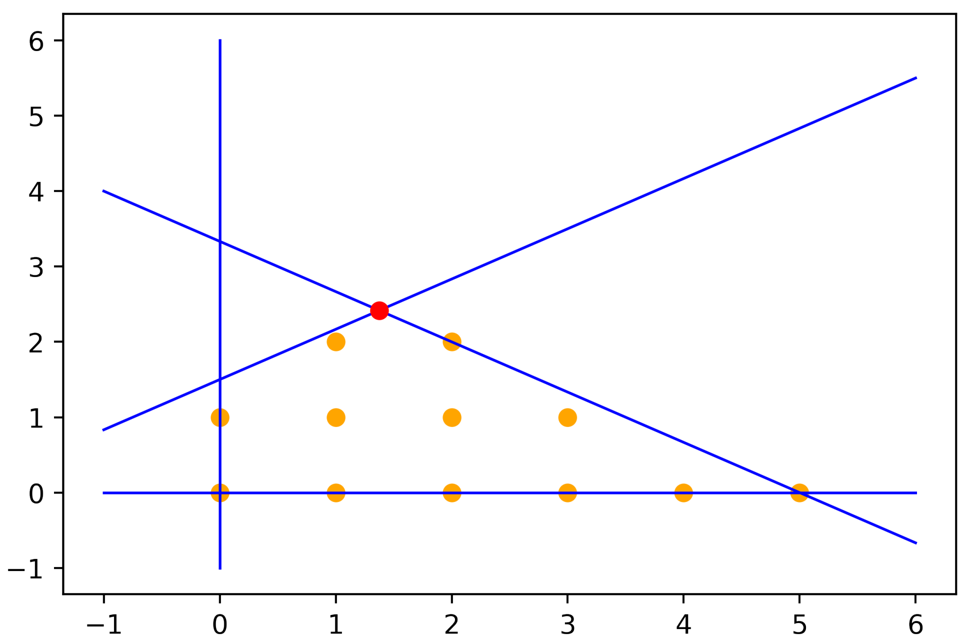
[10]:
```
# check for fractional solutions and generate inequalities by them
BIb = B.inv()*b
BIb
```

[10]: $\begin{bmatrix} \frac{11}{8} \\ \frac{29}{12} \end{bmatrix}$

This fractional solution is drawn as red dot in the plot below:

[11]:
```
plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
plt.plot(11/8,29/12,"o",color="red")
```

[11]: `[<matplotlib.lines.Line2D at 0x7ff1bf64c340>]`



Apply rounding:

[12]:
```
BIb.applyfunc(lambda x: sym.floor(x))
```

[12]: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

[13]:

```
BI = B.inv()*AN
BI.applyfunc(lambda x: sym.floor(x))
```

[13]:

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

We have then the following inequalities:

$$x_1 - x_3 \leq 1 \quad (3)$$
$$x_2 \leq 2 \quad (4)$$
$$(5)$$

If we substitute the non-basic variables by above's definition, we get:

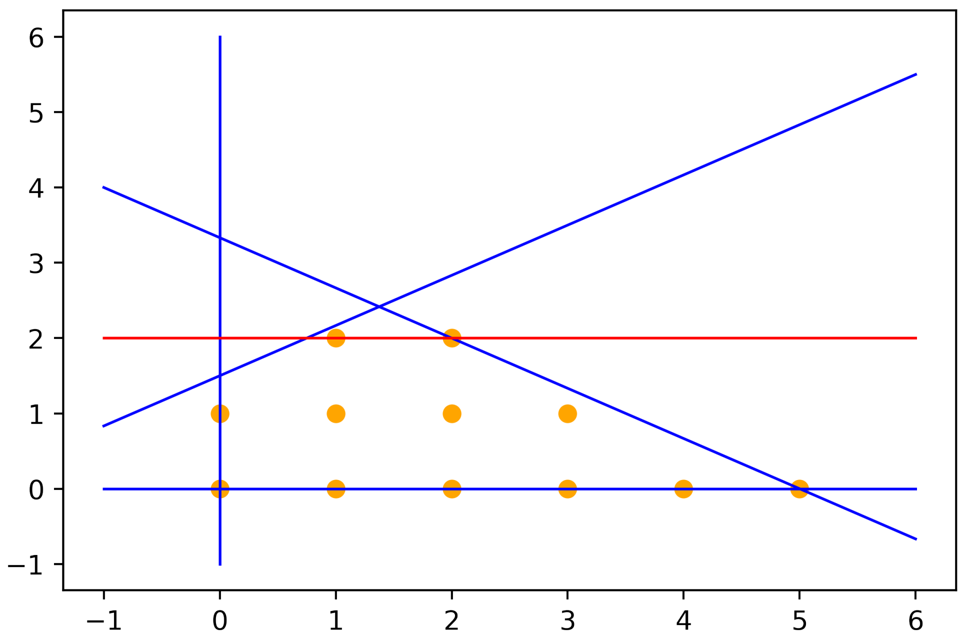
$$-3x_1 + 6x_2 \leq 1 \quad (6)$$
$$x_2 \leq 2 \quad (7)$$
$$(8)$$

Where we take the second inequality since the first one is redundant.

[14]:

```
# Add new inequality to plot
x_curve.append(x)
y_curve.append(2+x*0)
col.append("red")

plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
```



Find next cutting plane:

The standard formulation is now:

$$-4x_1 + 6x_2 + x_3 = 9 \tag{9}$$
$$2x_1 + 3x_2 + x_4 = 10 \tag{10}$$
$$x_2 + x_5 = 2 \tag{11}$$

Again assume the simplex method finds a fractional solution. Since we have three equations, we need three basic variables. This time we take x_1, x_2, x_4 as basic variables.

[15]:
```
B  = sym.Matrix(3, 3, [-4, 6, 0, 2, 3, 1, 0, 1, 0])
AN = sym.Matrix(3, 2, [1,0,0,0,0,1])
b  = sym.Matrix(3, 1, [9, 10, 2])
B
```

[15]:

$$\begin{bmatrix} -4 & 6 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

[16]:
```
# check for fractional solutions and generate inequalities by them
BIb = B.inv()*b
BIb
```

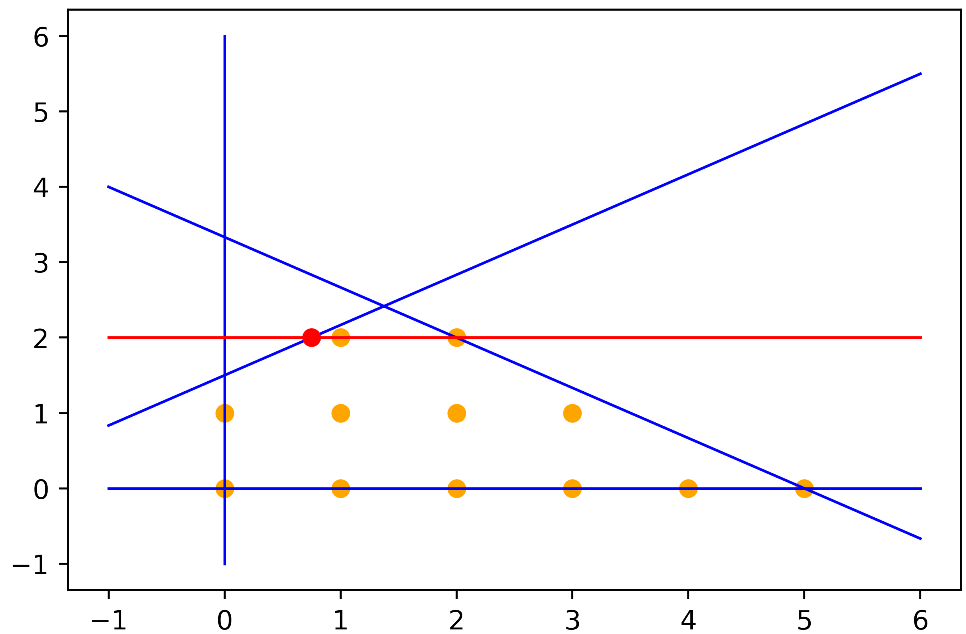
[16]:

$$\begin{bmatrix} \frac{3}{4} \\ 2 \\ \frac{5}{2} \end{bmatrix}$$

The fractional solution is shown as red dot in the plot below:

[17]:
```
plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
plt.plot(3/4, 2,"o",color="red")
```

[17]:
```
[<matplotlib.lines.Line2D at 0x7ff1bf536df0>]
```



Apply rounding:

[18]:
```
BI = B.inv()*AN
BI.applyfunc(lambda x: sym.floor(x))
```

[18]:
$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & -6 \end{bmatrix}$$

[19]:
```
BIb.applyfunc(lambda x: sym.floor(x))
```

[19]:
$$\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

We have then the following inequalities:

$$x_1 - x_3 \leq 0 \tag{12}$$
$$x_4 - 6x_5 \leq 2 \tag{13}$$
$$\tag{14}$$

If we substitute the non-basic variables x_3, x_5 by above's definition we get:

$$-3x_1 + 5x_2 \leq 7 \tag{15}$$
$$-2x_1 + 3x_2 \leq 4 \tag{16}$$

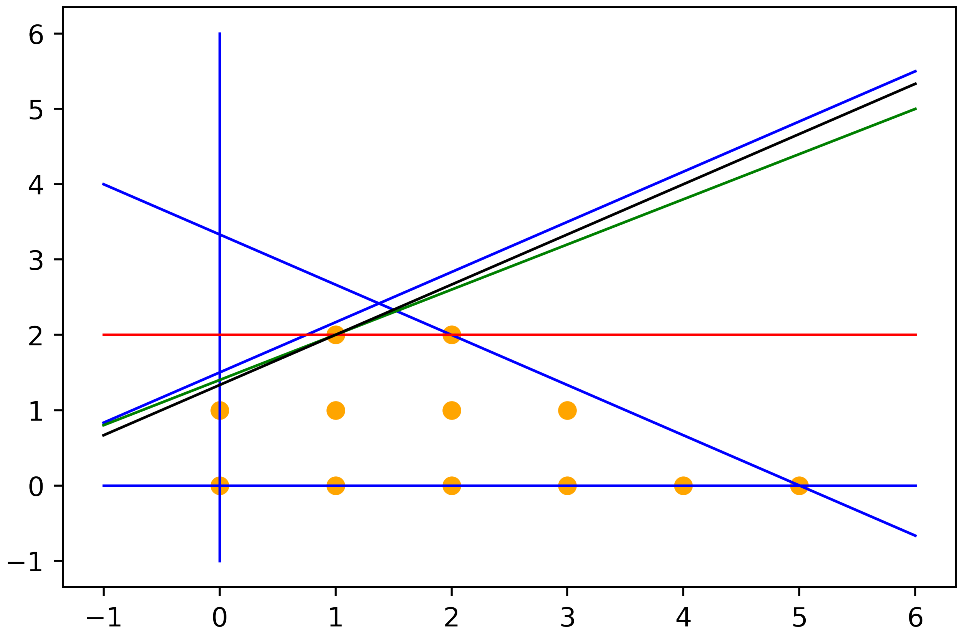
Next we show the impact of the new inequalities in green and black:

[20]:

```
x_curve.append(x)
y_curve.append(7/5+3/5*x)
col.append("green")

x_curve.append(x)
y_curve.append(4/3+2/3*x)
col.append("black")

plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
```



Since the second inequality cuts of a bigger piece off, we take it and obtain the standard formulation:

$$2x_1 + 3x_2 + x_3 = 10 \tag{17}$$

$$x_2 + x_4 = 2 \tag{18}$$

$$-2x_1 + 3x_2 + x_5 = 4 \tag{19}$$

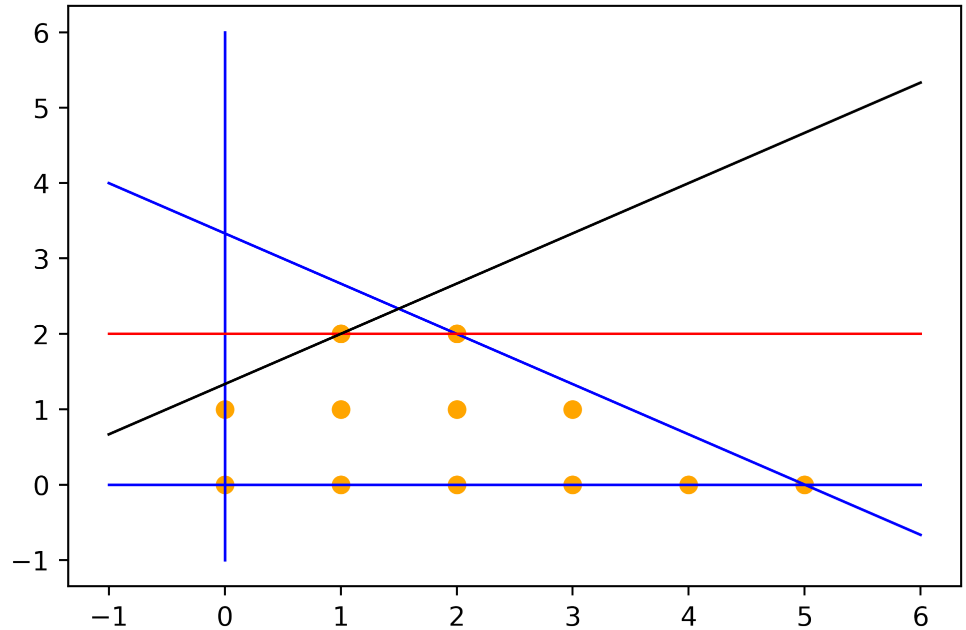
where the first equation from before became redundant and therefore we don't consider it anymore.

[21]:

```
# delete planes that we dont consider anymore
del x_curve[5]
del y_curve[5]
del col[5]
del x_curve[3]
del y_curve[3]
del col[3]
```

The current polyhedra is therefore:

[22]:
```
plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
```



Find next cutting plane:

The simplex method can still find a fractional solution. We get a fractional solution if we take x_2, x_3, x_5 as basic variables:

[23]:
```
B  = sym.Matrix(3, 3, [3, 1, 0,
                       1, 0, 1,
                       3, 0, 0])
AN = sym.Matrix(3, 2, [2,0,
                       0,0,
                      -2,1])

b  = sym.Matrix(3, 1, [10, 2, 4])
B
```

[23]:

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

[24]:
```
# check for fractional solutions and generate inequalities by them
BIb = B.inv()*b
BIb
```

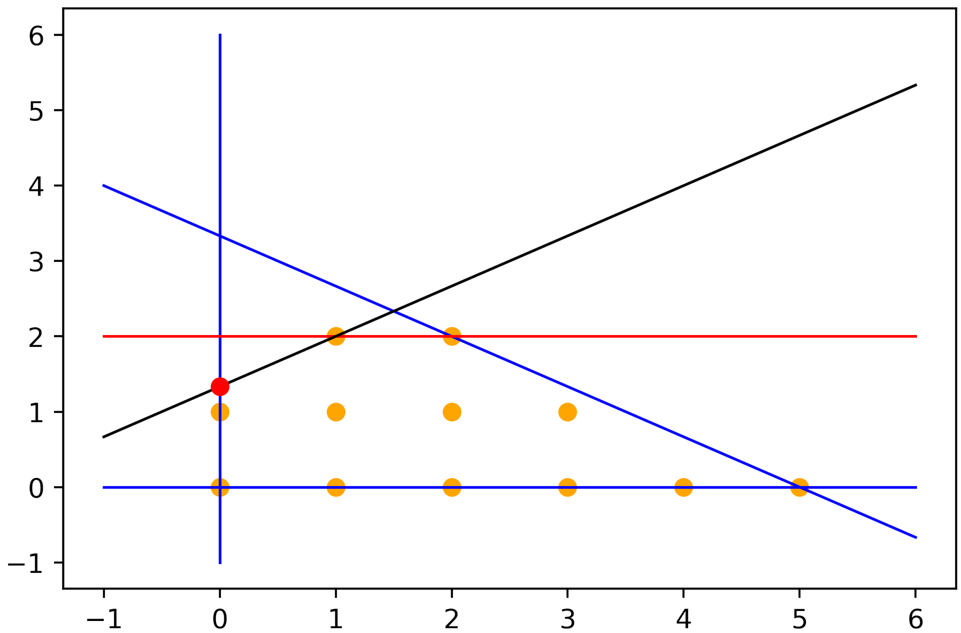
[24]:

$$\begin{bmatrix} \frac{4}{3} \\ 6 \\ \frac{2}{3} \end{bmatrix}$$

This solution corresponds to the red point drawn below (note $x_1 = 0$) :

[25]:
```
plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
plt.plot(0, 4/3,"o",color="red")
```

[25]: [<matplotlib.lines.Line2D at 0x7ff1bf40c430>]



Apply rounding:

[26]:
```
BI = B.inv()*AN
BI.applyfunc(lambda x: sym.floor(x))
```

[26]:
$$\begin{bmatrix} -1 & 0 \\ 4 & -1 \\ 0 & -1 \end{bmatrix}$$

[27]:
```
BIb.applyfunc(lambda x: sym.floor(x))
```

[27]:
$$\begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

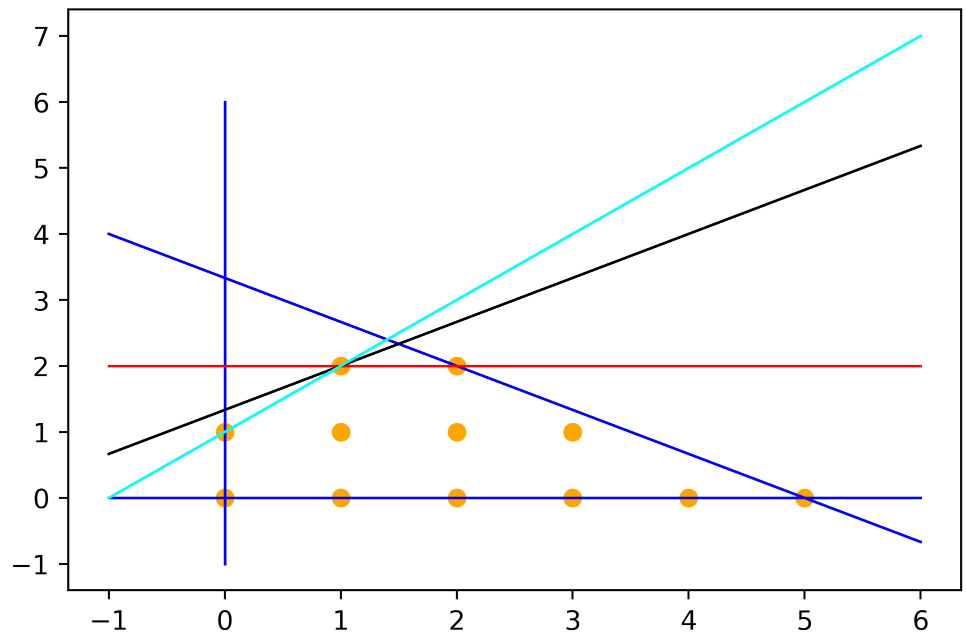
We just need to consider the first inequality that gives:

$$x_2 - x_1 \leq 1 \tag{20}$$

If we draw the cutting plane that corresponds to this equation in cyan, we get:

[28]:
```
x_curve.append(x)
y_curve.append(1+x)
col.append("cyan")

plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
```



The cutting plane in black from before is now redundant, and we see that the convex hull of our polyhedra is given by:

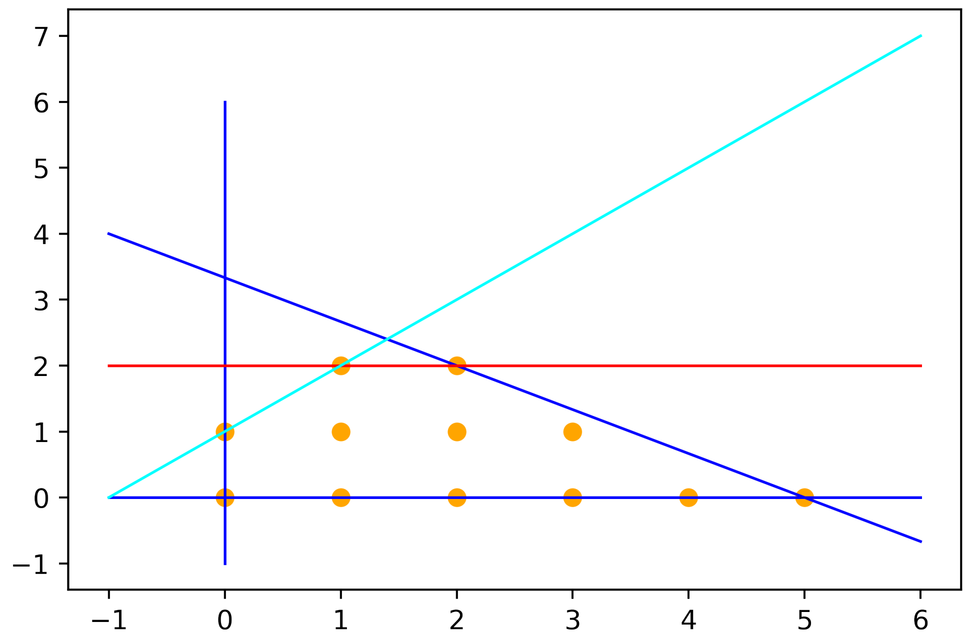
$$2x_1 + 3x_2 \leq 10 \tag{21}$$

$$x_2 \leq 2 \tag{22}$$

$$x_2 - x_1 \leq 1 \tag{23}$$

$$x_1, x_2 \geq 0 \tag{24}$$

[29]:
```
del x_curve[-2]
del y_curve[-2]
del col[-2]
plot_polyhed(X[mask], Y[mask], x_curve, y_curve, col)
```



1.2 Exercise 7 Cover Inequality

[30]:
```
# plot list of variables with constraint value lower than 15
weights = np.array([12,9,7,5,5,3],dtype="int")
values  = np.unpackbits( np.arange(2**6, dtype="uint8") ).reshape(-1,8)[:,2:]
value_weights = np.sum(values*weights, axis=1)[:,None]
selection_list = np.concatenate((values, value_weights), axis=1)
# remove entries that exeed 14:
selection_list = selection_list[selection_list[:,-1]<=14]
print(selection_list)
```

```
[[ 0  0  0  0  0  0  0]
 [ 0  0  0  0  0  1  3]
 [ 0  0  0  0  1  0  5]
 [ 0  0  0  0  1  1  8]
 [ 0  0  0  1  0  0  5]
 [ 0  0  0  1  0  1  8]
 [ 0  0  0  1  1  0 10]
 [ 0  0  0  1  1  1 13]
 [ 0  0  1  0  0  0  7]
 [ 0  0  1  0  0  1 10]
 [ 0  0  1  0  1  0 12]
 [ 0  0  1  1  0  0 12]
 [ 0  1  0  0  0  0  9]
 [ 0  1  0  0  0  1 12]
 [ 0  1  0  0  1  0 14]
```

```
 [ 0  1  0  1  0  0 14]
 [ 1  0  0  0  0  0 12]]
```

i) A minimum cover for our Knapsack is for example $C = \{2, 4, 6\}$ with the cover inequality:

$$x_2 + x_4 + x_6 \leq 2 \tag{25}$$

ii) To extend C we see that $\{j : a_j \geq a_i \forall i \in C\} = \{1\}$. Therefore, the extended cover is given by E(C)={1,2,4,6} and has the inequality:

$$x_1 + x_2 + x_4 + x_6 \leq 2 \tag{26}$$

iii) Consider again $C = \{2, 4, 6\}$. For the first lifting coefficient, we have the inequality:

$$\alpha_1 x_1 + x_2 + x_4 + x_6 \leq 2 \tag{27}$$

Which is clearly fulfilled if $x_1 = 0$ and for $x_1 = 1$ we get the new Knapsack problem:

$$\max x_2 + x_4 + x_6 \tag{28}$$
$$\text{s.t } 9x_2 + 5x_4 + 3x_6 \leq 14 - 12 = 2 \tag{29}$$

We can easily see that $\max x_2 + x_4 + x_6 = 0$ and therefore $\alpha_1 = 2$ resulting in the inequality:

$$2x_1 + x_2 + x_4 + x_6 \leq 2 \tag{30}$$

We continue to calculate the next lifting coefficient:

$$2x_1 + x_2 + \alpha_3 x_3 + x_4 + x_6 \leq 2 \tag{31}$$

Again this inequality is valid for $x_3 = 0$ and for $x_3 = 1$ we solve the Knapsack:

$$\max 2x_1 + x_2 + x_4 + x_6 \tag{32}$$
$$\text{s.t } 12x_1 + 9x_2 + 5x_4 + 3x_6 \leq 14 - 7 = 7 \tag{33}$$

The solution has $\max 2x_1 + x_2 + x_4 + x_6 = 1$ such that $\alpha_3 = 1$ and the lifted inequality is:

$$2x_1 + x_2 + x_3 + x_4 + x_6 \leq 2 \tag{34}$$

For the calculation of the last lifting coefficient, we have the inequality:

$$2x_1 + x_2 + x_3 + x_4 + \alpha_5 x_5 + x_6 \leq 2 \tag{35}$$

The inequality is valid for $x_5 = 0$. When setting $x_5 = 1$ the Knapsack for the sub-problem is:

$$\max 2x_1 + x_2 + x_3 + x_4 + x_6 \tag{36}$$
$$\text{s.t } 12x_1 + 9x_2 + 7x_3 + 5x_4 + 3x_6 \leq 14 - 5 = 9 \tag{37}$$

Now $\max 2x_1 + x_2 + x_3 + x_4 + x_6 = 2$ since we can set $x_4 = 1, x_6 = 1$. Therefore $\alpha_5 = 0$ and the final lifted inequality is given by:

$$2x_1 + x_2 + x_3 + x_4 + x_6 \leq 2 \tag{38}$$

1.3 Exercise 8 Totally Unimodularity

i) The node-edge incidence matrix of all simple, undirected graphs is totally unimodular This is not true. Consider the counterexample:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{39}$$

here we have $det(A_1) = 2$ and therefore A_1 is not totally unimodular. Since this matrix corresponds to a node-edge incidence matrix of all simple, undirected graph the statement cant be true.

ii) The node-arc incidence matrix of all bipartite, directed, simple, graphs is totally unimodular. This is true since each column has exactly a 1 and -1 and all other entries in a column are 0. Therefore each row has at most two non-zero coefficients and if we have the partition $M_1 = V$, $M_2 = \emptyset$. By that we will have for any j that $\sum_{i \in M_1} u_{i,j} - \sum_{i \in M_2} u_{i,j} = 1 - 1 + 0 = 0$. Proposition 6 therefore shows, that all matrices of this type must be totally unimodular.

[ ]: