Exercise 3

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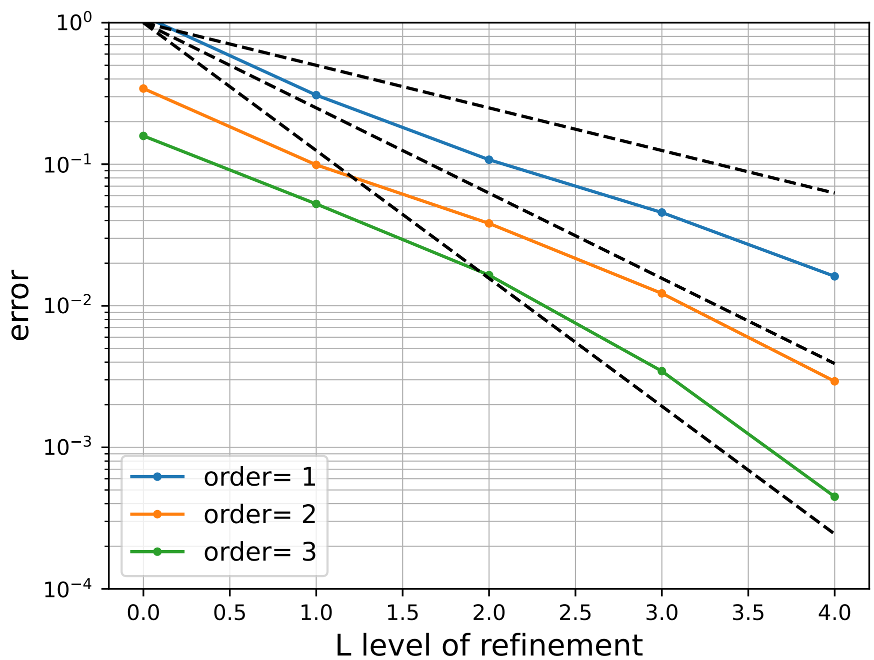
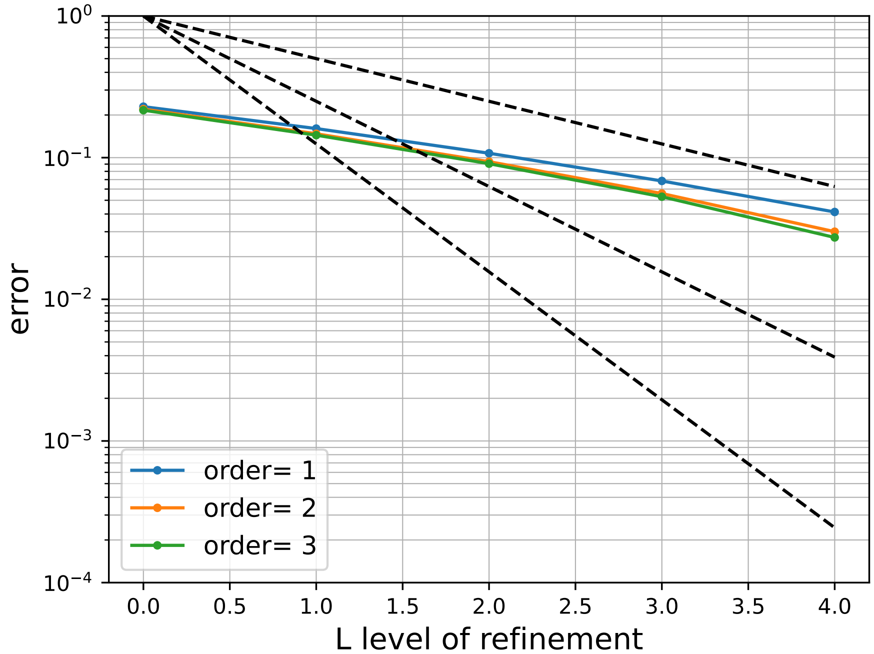
Part1



Part2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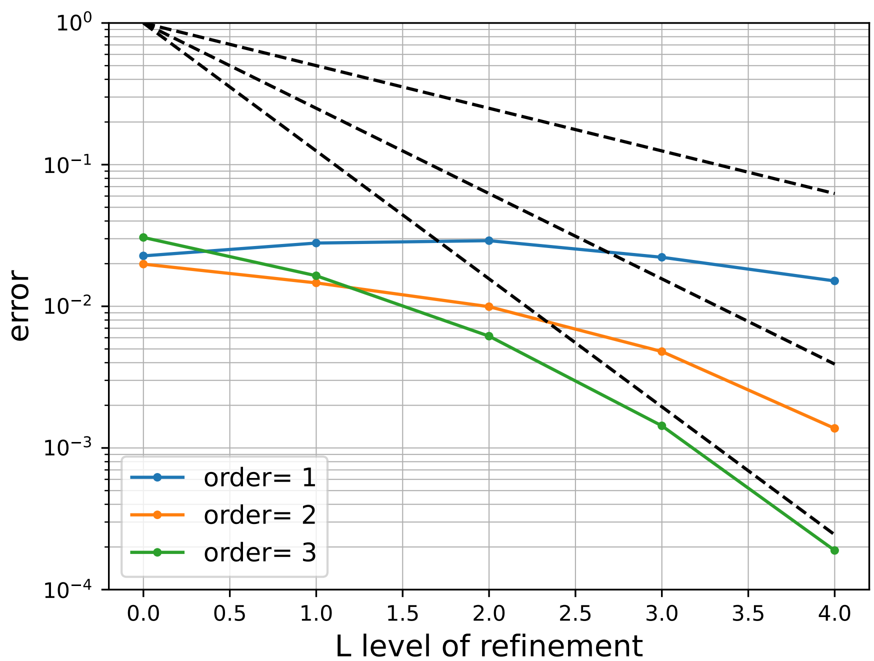
10^{-1}
10^{-2}
10^{-3}
10^{-4}
error
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
L level of refinement
order= 1
order= 2
order= 3

Part3


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
error
0.0
0.5
1.0
1.5
2.0
2.5
3.0
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4.0
L level of refinement
order= 1
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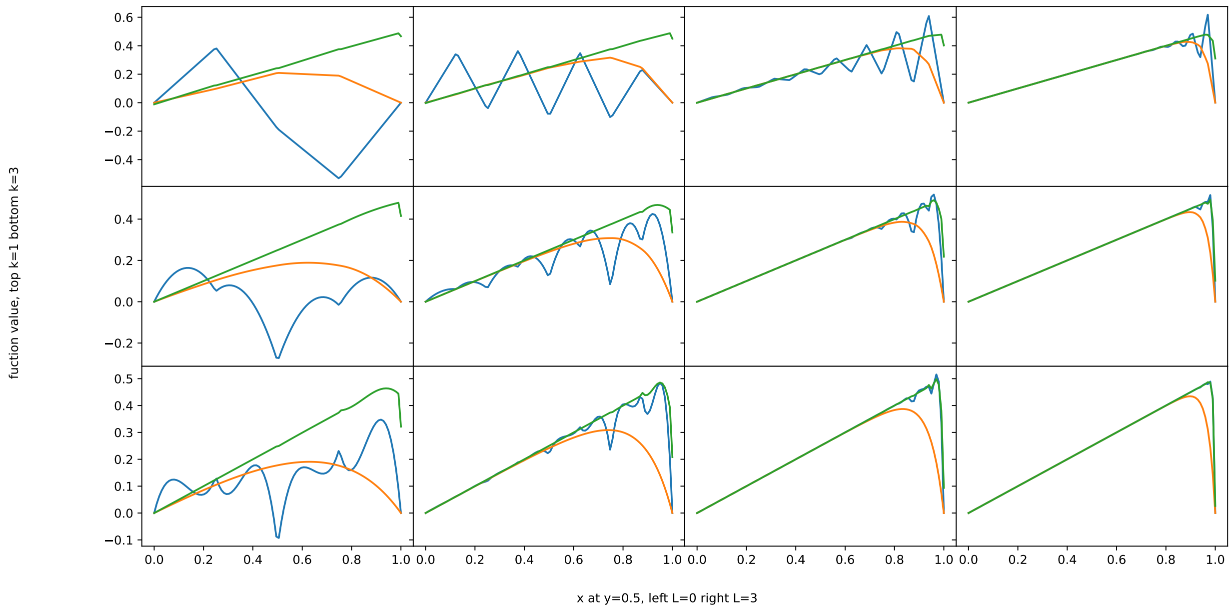
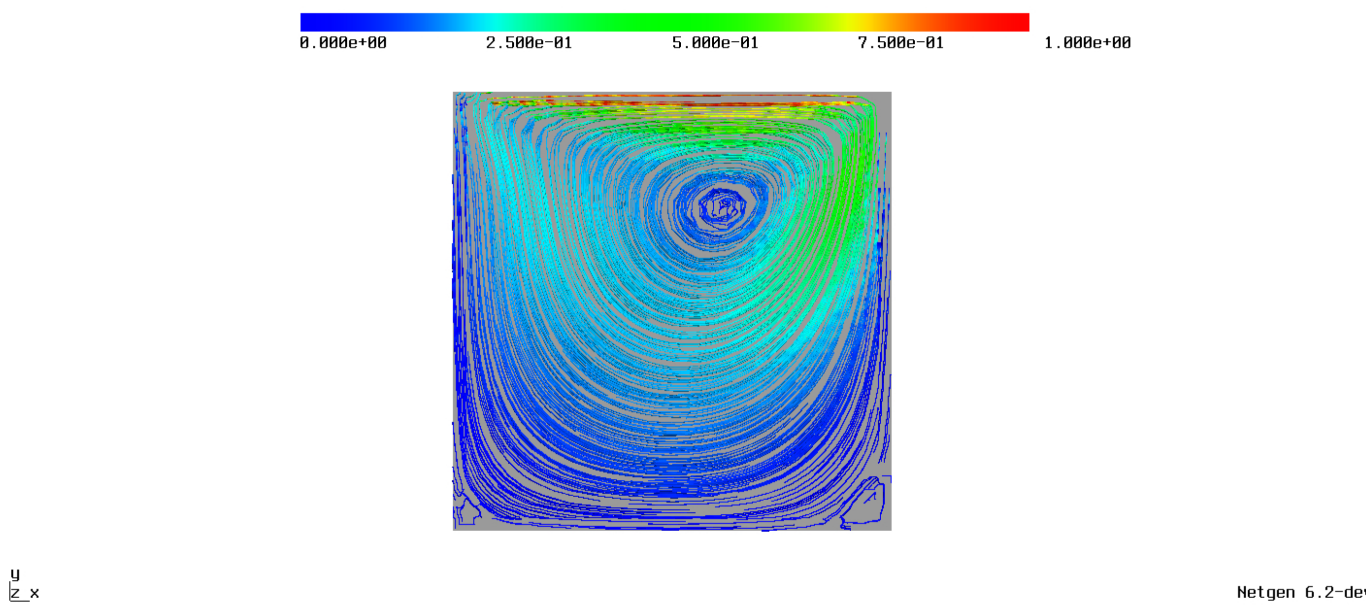
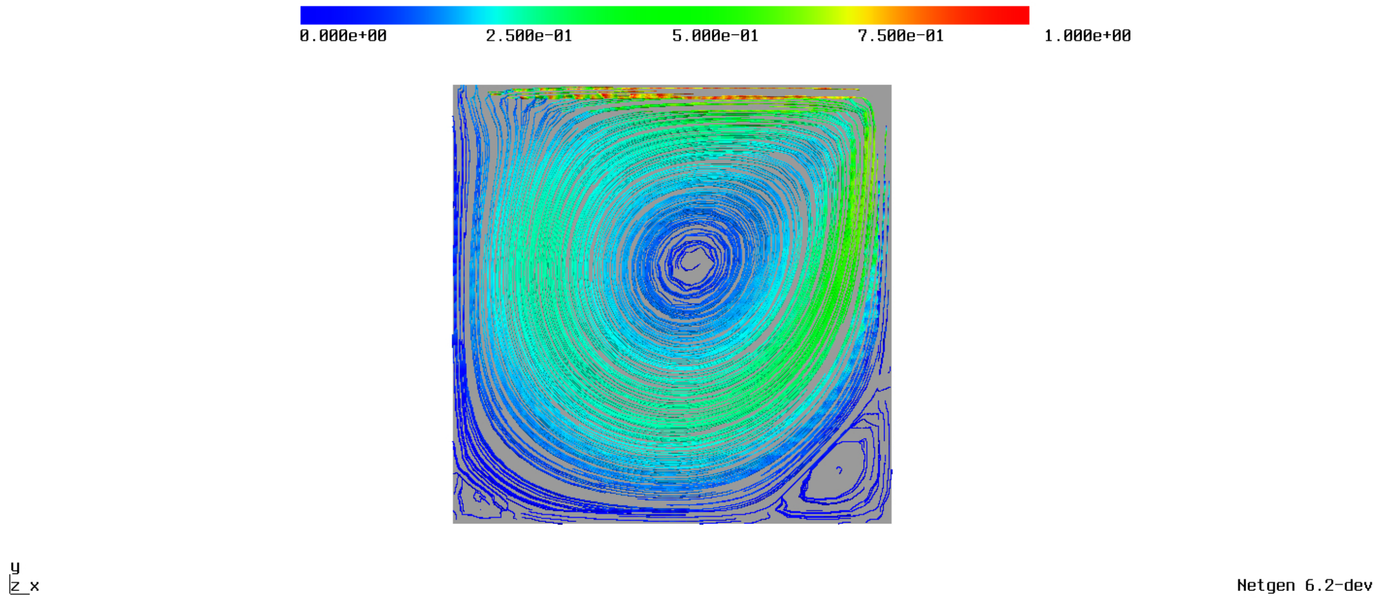


Part4

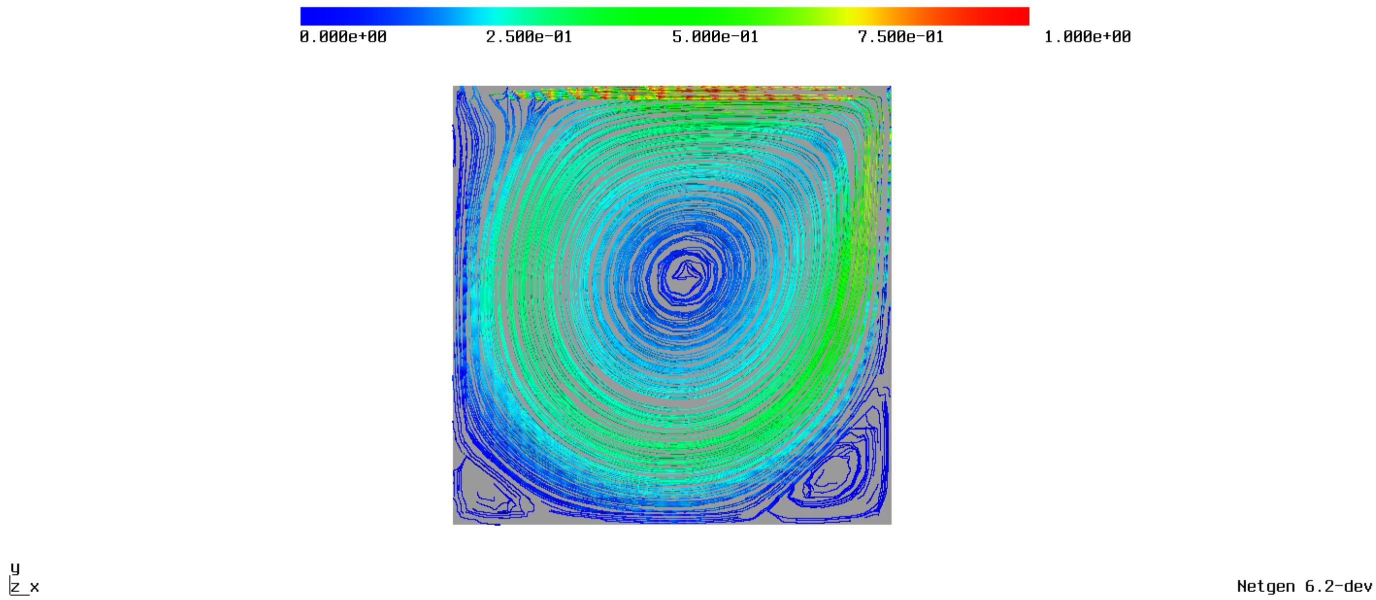
solution for $\nu = 0.01$ at $L = 3$



solution for $\nu = 0.002$ at $L = 3$



solution for $\nu = 0.001$ at $L = 3$



Iterations needed for convergence

Method, L	0	1	2	3
$\nu = 0.01$				
Newton	5	5	5	5
Picard	15	16	16	16
$\nu = 0.002$				
Newton	-	10	8	8
Picard	79	32	34	34
$\nu = 0.001$				
Newton	-	-	-	-
Picard	-	48	37	38

Part5

Weak form of NS-equation

Assume we given the NS-equations in weak form with $(u,p) \in V \times Q = H_0^1(\Omega)^2 \times Ł_0^2(\Omega)$:

$$(\frac{du}{dt}, v) + a(u,v) + c(u,u,v) + b(v,p) + b(u,q) + \epsilon(p,q) = f(v) \quad \forall (v,q) \in V \times Q \tag{1}$$

Time discretisation by θ-scheme

$$\begin{aligned}(u^{n+1}, v) + \theta\tau a(u^{n+1}, v) + &\theta\tau c(u^{n+1}, u^{n+1}, v) + \tau b(v, p^{n+1}) + \tau b(u^{n+1}, q)\tau(p^{n+1}, q) = \\ &\tau f(v) + (u^n, v) - (1-\theta)\tau a(u^n, v) - (1-\theta)\tau c(u^n, u^n, v)\end{aligned} \tag{2}$$

Next assume $u_{i+1}^{n+1} = u_i^{n+1} + \delta u_i^{n+1}$ and $p_{i+1}^{n+1} = p_i^{n+1} + \delta p_i^{n+1}$ where $u_0^{n+1} = u_K^{n+1}$ and $p_0^{n+1} = p_K^{n+1}$ for $K \to \infty$. Note further:

$$c(u_{i+1}^{n+1}, u_{i+1}^{n+1}, v) = c(u_i^{n+1}, \delta u_i^{n+1}, v) + c(\delta u_i^{n+1}, u_i^{n+1}, v) + c(u_i^{n+1}, u_i^{n+1}, v) + c(\delta u_i^{n+1}, \delta u_i^{n+1}, v) \tag{3}$$

Then for a Newton method we assume $c(\delta u_i^{n+1}, \delta u_i^{n+1}, v)$ and for the picard iteration additionally $c(\delta u_i^{n+1}, u_i^{n+1}, v)$.

Newton iteration

$$\begin{aligned}&(\delta u_i^{n+1}, v) + \theta\tau\left[a(\delta u_i^{n+1}, v) + c(\delta u_i^{n+1}, u_i^{n+1}, v) + c(u_i^{n+1}, \delta u_i^{n+1}, v))\right] + \tau\left[b(v, \delta p_i^{n+1}) + b(\delta u_i^{n+1}, q) + \epsilon(\delta p_i^{n+1}, q)\right] = \\ &(u_0^{n+1} - u_i^{n+1}, v) + \tau f(v) + (1-\theta)\tau\left[-a(u_0^{n+1}, v) - c(u_0^{n+1}, u_0^{n+1}, v)\right] + \theta\tau\left[-a(u_i^{n+1}, v) - c(u_i^{n+1}, u_i^{n+1}, v)\right] \\ &+ \tau\left[-b(v, p_i^{n+1}) - b(u_i^{n+1}, q) - \epsilon(p_i^{n+1}, q)\right]\end{aligned} \tag{4}$$

Picard iteration

$$\begin{aligned}&(\delta u_i^{n+1}, v) + \theta\tau\left[c(\delta u_i^{n+1}, u_i^{n+1}, v) + c(u_i^{n+1}, \delta u_i^{n+1}, v))\right] + \tau\left[b(v, \delta p_i^{n+1}) + b(\delta u_i^{n+1}, q) + \epsilon(\delta p_i^{n+1}, q)\right] = \\ &(u_0^{n+1} - u_i^{n+1}, v) + \tau f(v) + (1-\theta)\tau\left[-a(u_0^{n+1}, v) - c(u_0^{n+1}, u_0^{n+1}, v)\right] + \theta\tau\left[-a(u_i^{n+1}, v) - c(u_i^{n+1}, u_i^{n+1}, v)\right] \\ &+ \tau\left[-b(v, p_i^{n+1}) - b(u_i^{n+1}, q) - \epsilon(p_i^{n+1}, q)\right]\end{aligned} \tag{5}$$

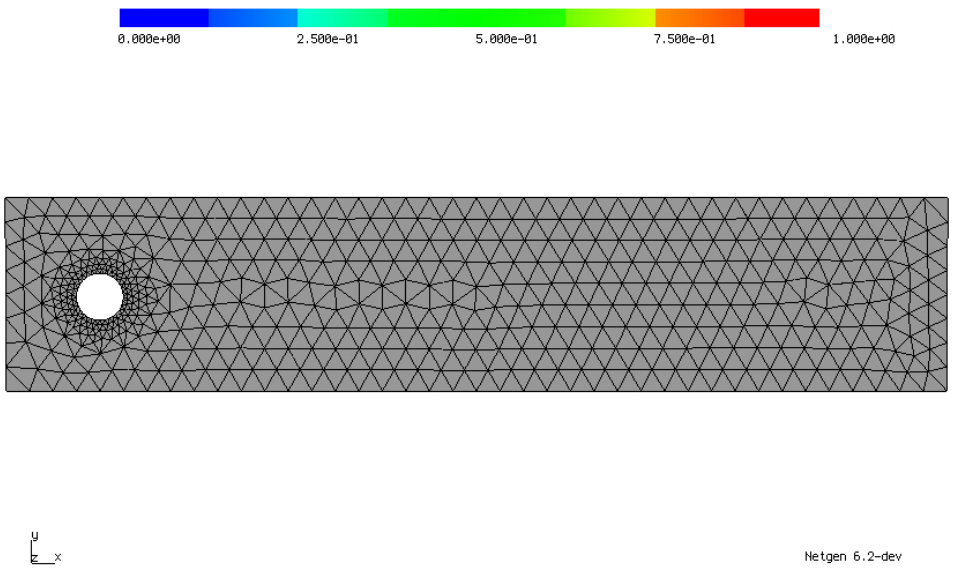
Space discretisation by Tailor-Hood element

The Taylor-Hood pair of order k is given by:

$$V_h := \{v \in H_0^1(\Omega)^2 : \; v|_T \in P^k(T)^2 \quad \forall T \in \mathcal{T}_h\}$$
$$Q_h := \{q \in L_0^2(\Omega) \cap C(\bar{\Omega}) : \; q|_T \in P^k(T) \quad \forall T \in \mathcal{T}_h\} \tag{6}$$

Mesh

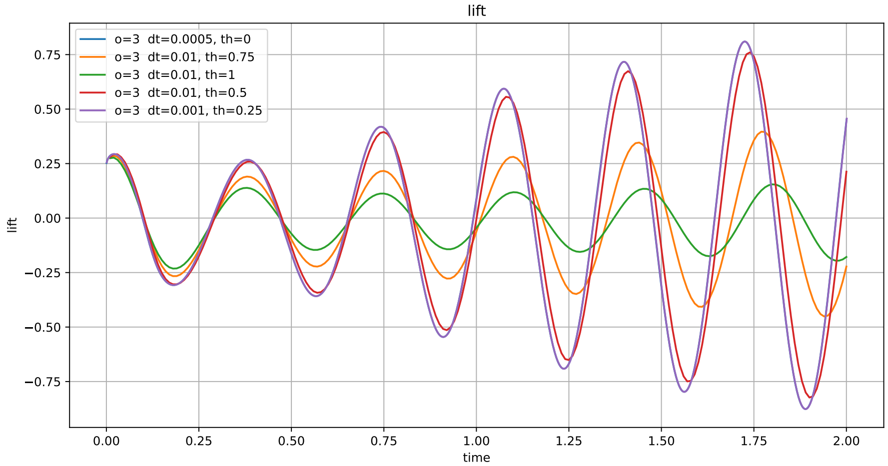
We choose a trtraheder mesh with 972 Elements and 550 vertices. The global mesh size is 0.05 where the local redinement at the cylinder wall is choosen to be 0.009. The mesh is shown below:

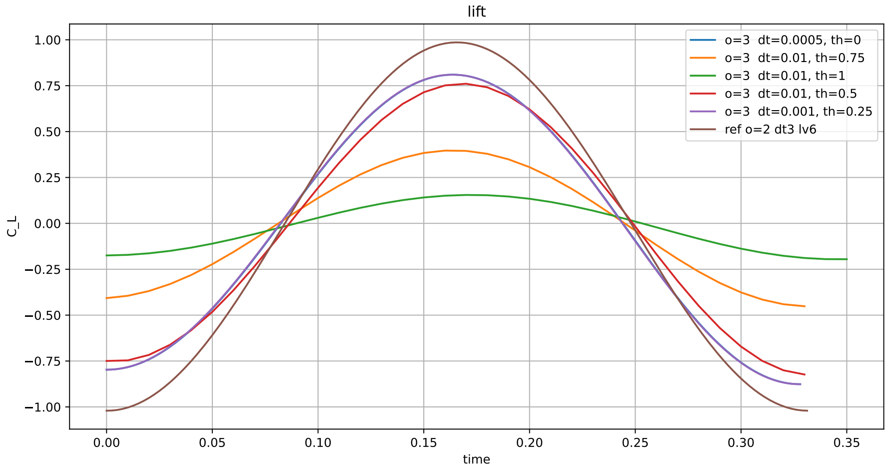


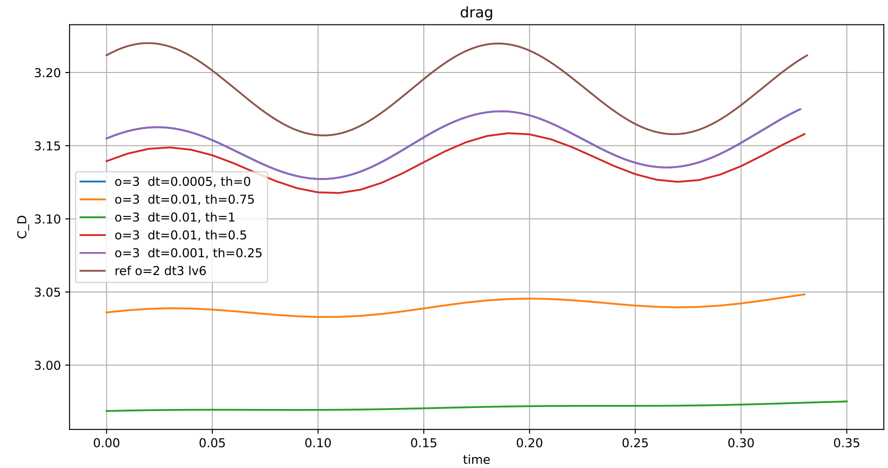
To improve the solution quality p-refinements were conducted for different orders k of the Taylor hood space where in contrast to that h-refinements were choosen for the reference solutions. In the Table below we compare cases where h-refinement of the reference and p-refinements of our solution yield a simillar number of free-dofs.

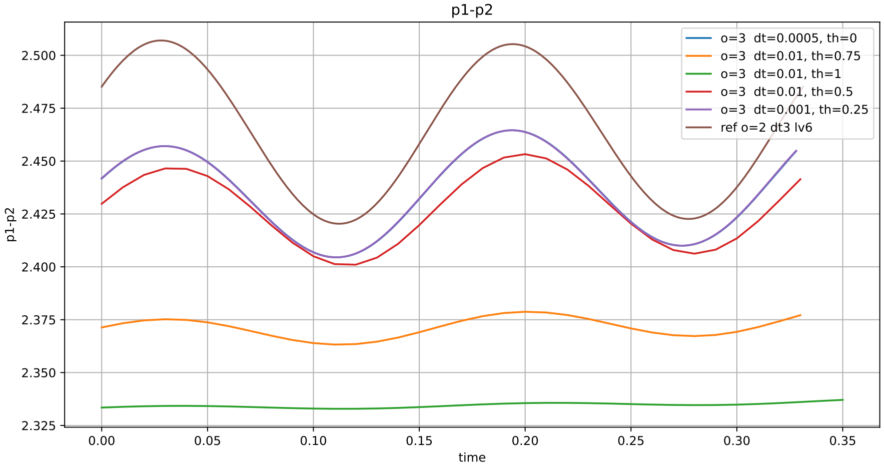
Lv ref.	order this	free-dofs ref.	free-dofs this
2	2	2704	4221
3	3	10608	10504
-	4	-	19709
4	5	42016	31836
6	-	667264	-

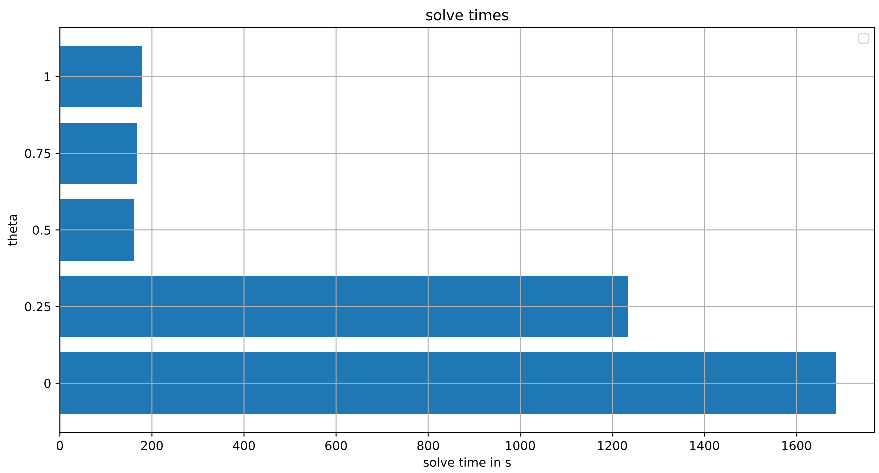
Investigating influence of θ values on solution for $t \in [0, 2]$



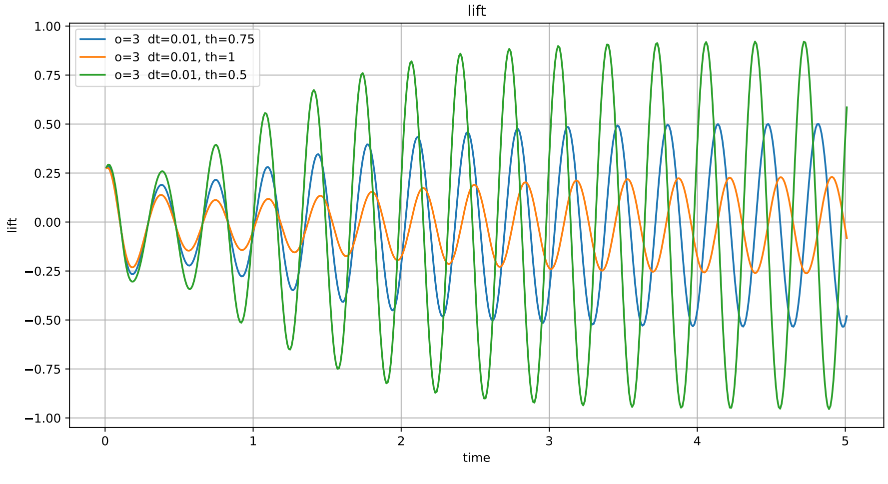


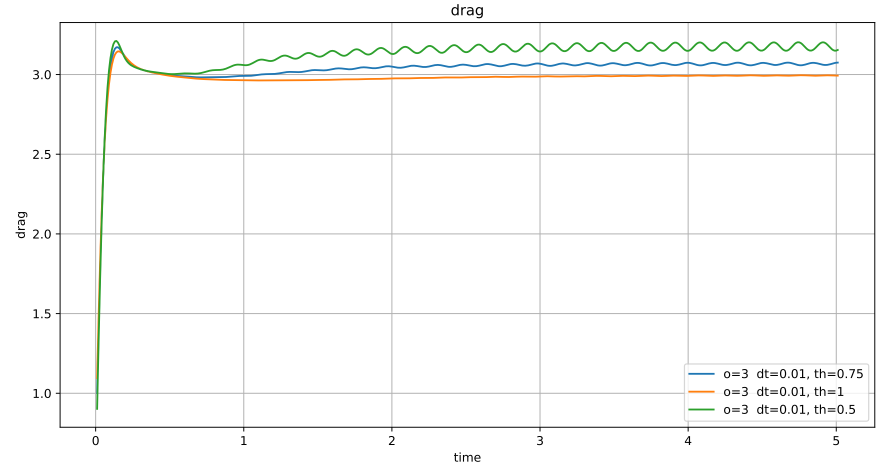


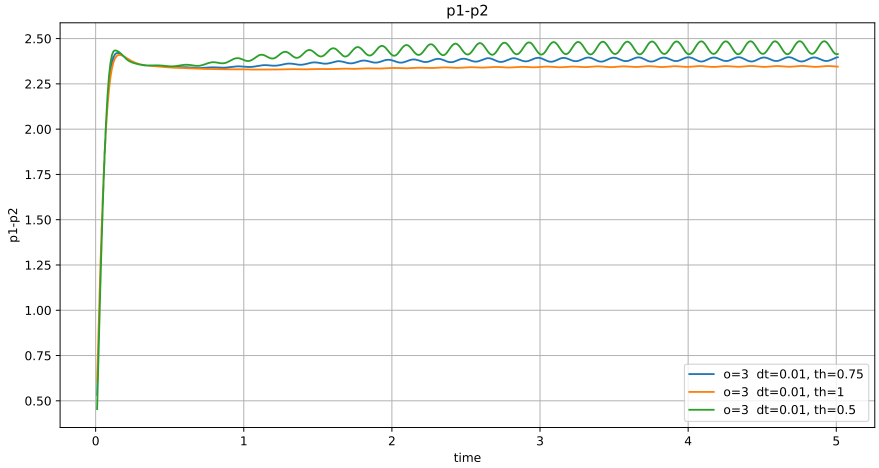


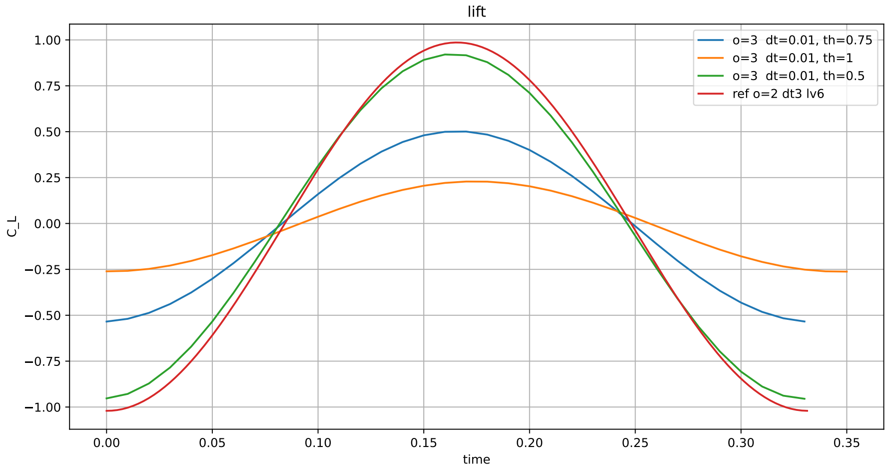


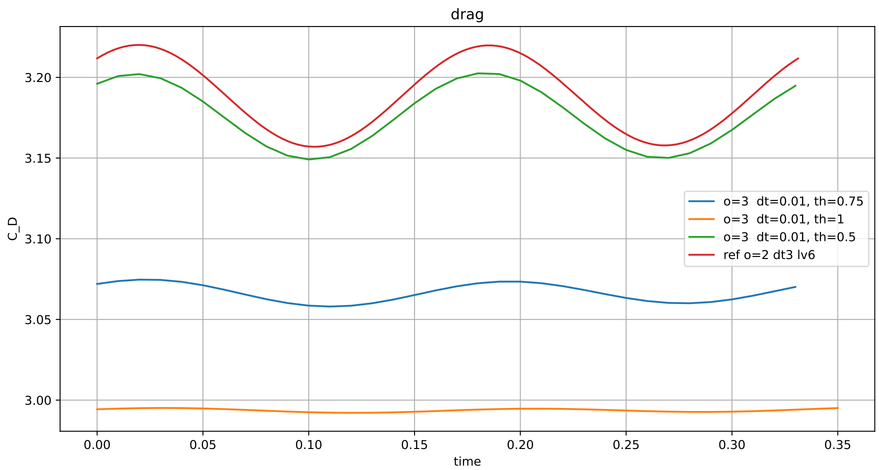
Comparing $\theta \geq 0.5$ methods for $t \in [0, 5]$

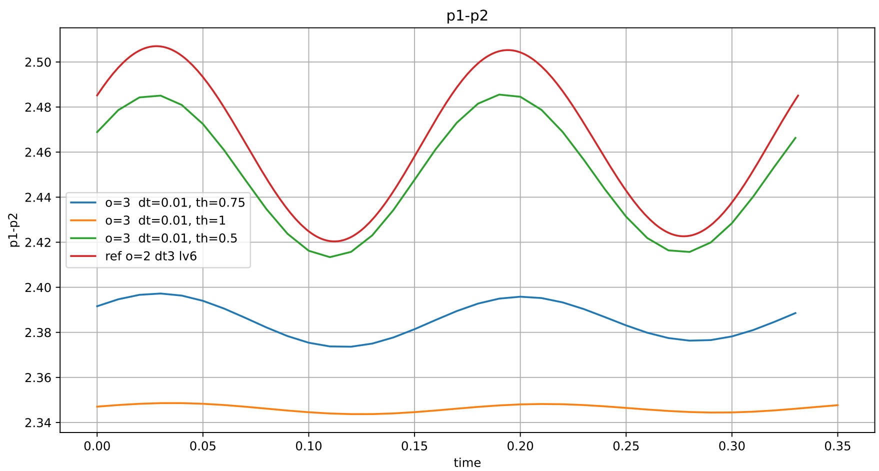



drag
drag
time
o=3 dt=0.01, th=0.75
o=3 dt=0.01, th=1
o=3 dt=0.01, th=0.5

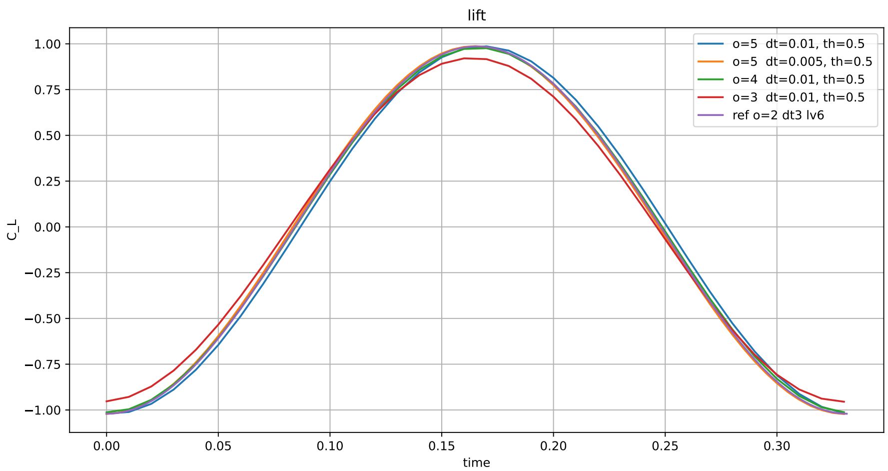

p1-p2
p1-p2
time
o=3 dt=0.01, th=0.75
o=3 dt=0.01, th=1
o=3 dt=0.01, th=0.5

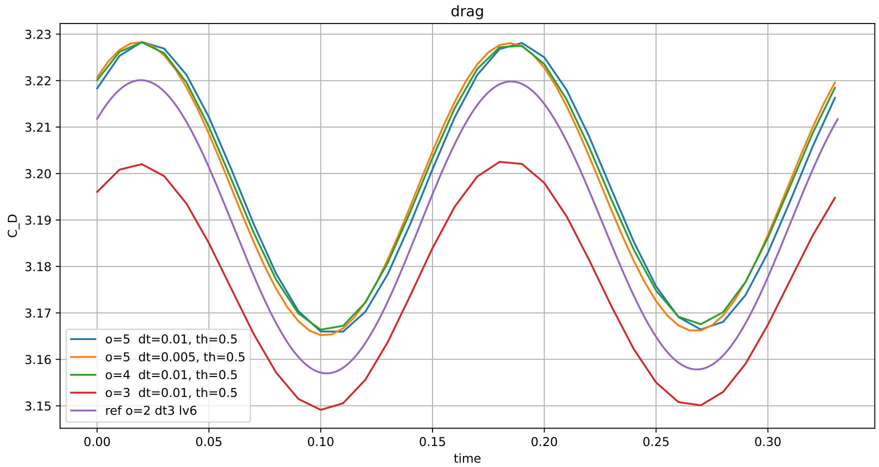

lift
C_L
time
o=3 dt=0.01, th=0.75
o=3 dt=0.01, th=1
o=3 dt=0.01, th=0.5
ref o=2 dt3 lv6

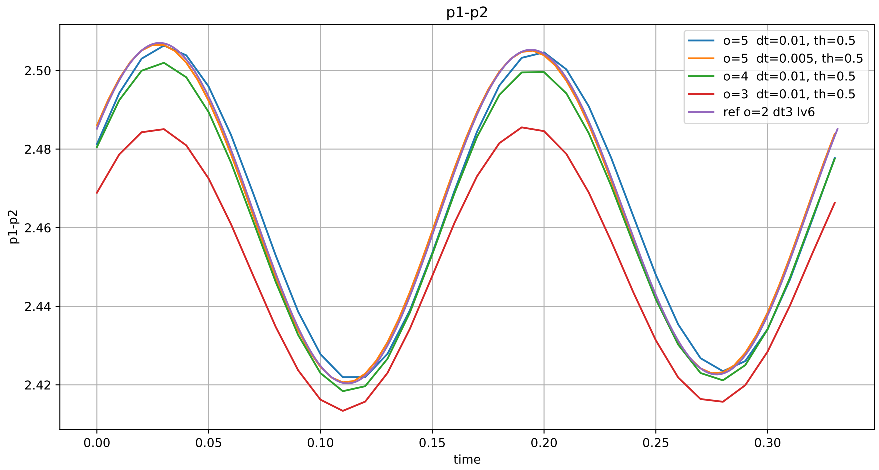




p-refinement at $\theta = 0.5$ method for $t \in [0, 5]$




drag
C_D
time
o=5 dt=0.01, th=0.5
o=5 dt=0.005, th=0.5
o=4 dt=0.01, th=0.5
o=3 dt=0.01, th=0.5
ref o=2 dt3 lv6


p1-p2
p1-p2
time
o=5 dt=0.01, th=0.5
o=5 dt=0.005, th=0.5
o=4 dt=0.01, th=0.5
o=3 dt=0.01, th=0.5
ref o=2 dt3 lv6