474

б) $\lim\limits_{x\to 0} \frac{tg\,x}{x} = \lim\limits_{x\to 0} \frac{\sin x}{x \cdot \cos x} = 1 \cdot \lim\limits_{x\to 0} \frac{1}{\cos x} =$

$= 1 \cdot 1 = 1$

в) $\lim\limits_{x\to 0} x \cdot ctg\,3x = \lim\limits_{x\to 0} x \cdot \frac{\cos 3x}{\sin 3x} =$

$= \lim\limits_{x\to 0} \frac{3x}{3\sin 3x} \cdot \cos 3x = \frac{1}{3} \cdot \lim\limits_{x\to 0} \cos 3x = \frac{1}{3}$

Семинар „Использование второго замечательного предела"

$$\lim_{x\to a} u(x)^{v(x)} = [\lim_{x\to a} u(x)]^{\lim\limits_{x\to a} v(x)} = b^c$$

если $\exists \lim\limits_{x\to a} u(x) = b \quad (b>0)$

$\exists \lim\limits_{x\to a} v(x) = c$

$\lim\limits_{x\to a} d^x = d^a$

$\lim\limits_{x\to a} \ln x = \ln a$

Как бороться с неопределённостями $(0^0, \infty^0, 1^\infty)$:

$\lim\limits_{x\to a} u(x)^{v(x)} = \lim\limits_{x\to a} e^{v(x)\ln u(x)}$

$\ln u(x)$

II зам. предел

$\lim\limits_{y\to 0} (1+y)^{\frac{1}{y}} = e$

$\lim\limits_{z\to\infty} \left(1+\frac{1}{z}\right)^z = e$

$$\lim_{x\to a} u(x)^{v(x)} = \left\{\begin{matrix} 1^{\infty} \\ u(x)\to 1 \\ v(x)\to\infty \end{matrix}\right\} =$$

$$= \lim_{x\to a}\left(1+\underbrace{(u(x)-1)}_{\downarrow\, 0}\right)^{\frac{1}{u(x)-1}\cdot(u(x)-1)v(x)} =$$

$$= e^{\lim\limits_{x\to a} v(x)(u(x)-1)}$$

506, 507, 508, 512, 514, 515, 517, 518, 521, 529 (+ следст.)

506

а) $$\lim_{x\to 0}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \frac{1}{2}$$

б) $$\lim_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \lim_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1}{1+\sqrt{x}}} = \sqrt{\frac{2}{3}}$$

в) $$\lim_{x\to +\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \lim_{x\to +\infty}\left(\frac{1+x}{2+x}\right)^{\frac{\frac{1}{x}+\frac{1}{\sqrt{x}}}{\frac{1}{x}-1}} = 1^{0} = 1$$

$$* \lim_{x\to\infty}\left(\frac{1+x}{2+x}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x}+1}{\frac{2}{x}+1}\right) = 1$$

$$** \lim_{x\to\infty}\frac{1-\sqrt{x}}{1-x} = \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{\sqrt{x}}}{\frac{1}{x}-1} = 0$$

507

$$\lim_{x\to\infty}\left(\frac{x+2}{2x-1}\right)^{x^2} = \lim_{x\to\infty}\left(\frac{1+\frac{2}{x}}{2-\frac{1}{x}}\right)^{x^2\to\infty} = 0$$

$\frac{2}{x} \to 0$, $\frac{1}{x} \to 0$

508

$$\lim_{x\to\infty}\left(\frac{3x^2-x+1}{2x^2+x+1}\right)^{\frac{x^3}{1-x}} = \lim_{x\to\infty}\left(\frac{3-\frac{1}{x}+\frac{1}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}}\right)^{\frac{x^3}{1-x}} =$$

$$= \lim_{x\to\infty}\left(\frac{3-\frac{1}{x}+\frac{1}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}}\right)^{\frac{x^2}{\frac{1}{x}-1}} = \left(\frac{3}{2}\right)^{-\infty} =$$

$$= \left(\frac{2}{3}\right)^{\infty} = 0$$

$\to \frac{2}{3} < 1$

512

$$\frac{x^2+1}{x^2-2} = \frac{x^2-2+3}{x^2-2} = 1+\frac{3}{x^2-2}$$

$$\lim_{x\to\infty}\left(\frac{x^2+1_{/x^2}}{x^2_{/x^2}-2_{/x^2}}\right)^{x^2} = \lim_{x\to\infty}\left(\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}}\right)^{\frac{3}{x^2-2}\cdot x^2} =$$

$$= e^{\lim_{x\to\infty}\frac{3x^2_{/x^2}}{x^2_{/x^2}-2_{/x^2}}} = e^3$$

$\to 3$, $\to 1$, $\to 0$, $\frac{3}{1} = 3$

514

$$\lim_{x\to 0}\sqrt[x]{1-2x} = \lim_{x\to 0}\left(1(-2x)\right)^{\frac{1}{x}} =$$

$$= \lim_{x\to 0}\left(1+(-2x)\right)^{\frac{1}{-2x}\cdot(-2)} = e^{-2}$$

$$\frac{x-a+2a}{x-a} = 1+\frac{2a}{x-a}$$

515

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^{x} = \lim_{x\to\infty}\left(\left(1+\frac{2a}{x-a}\right)^{\frac{x-a}{2a}}\right)^{x\cdot\frac{2a}{x-a}} =$$

$$= e^{\lim_{x\to\infty}\frac{2a\cdot x}{x-a}} = e^{\lim_{x\to\infty}\frac{2a}{1-\frac{a}{x}}} = e^{2a}$$

517

$$\lim_{x\to 0}(1+x^2)^{ctg^2 x} = \lim_{x\to 0}\left((1+x^2)^{\frac{1}{x^2}}\right)^{ctg^2 x\cdot x^2} =$$

$$= e^{\lim_{x\to 0} ctg^2 x\cdot x^2} = e^{\lim_{x\to 0} x^2\cdot\frac{\cos^2 x}{\sin^2 x}} = e^{1} = e$$

518

$$\lim_{x\to 1}(1+\sin\pi x)^{ctg\,\pi x} =$$

$$= \lim_{x\to 1}\left(\left(1+\underbrace{\sin\pi x}_{\to 0}\right)^{\frac{1}{\sin\pi x}}\right)^{\cos\pi x} =$$

$$= e^{\lim_{x\to 1}\cos\pi x} = e^{-1}$$

521 $\lim_{x\to 0}\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}} =$

$= \lim_{x\to 0}\left(1+\frac{\cos x-\cos 2x}{\cos 2x}\right)^{\frac{1}{x^2}} =$

$= \lim_{x\to 0}\left(1+\frac{2\sin\frac{3x}{2}\sin\frac{x}{2}}{\cos 2x}\right)^{\frac{1}{x^2}} =$

$= \lim_{x\to 0}\left(1+\frac{2\sin\frac{3x}{2}\sin\frac{x}{2}}{\cos 2x}\right)^{\frac{\cos 2x}{2\sin\frac{3x}{2}\sin\frac{x}{2}}\cdot\frac{1}{x^2}\cdot\frac{2\sin\frac{3x}{2}\sin\frac{x}{2}}{\cos 2x}} =$

$= e^{\lim_{x\to 0}\frac{2\sin\frac{3x}{2}\sin\frac{x}{2}}{x^2\cdot\cos 2x}} = e^{\lim_{x\to 0}\frac{2\sin\frac{3x}{2}\sin\frac{x}{2}\cdot\frac{3}{4}x^2}{\cos x\cdot x^2\cdot\frac{3}{4}}} =$

$= e^{\frac{3}{2}}$

$\frac{\sin\frac{x}{2}}{\frac{x}{2}}$

$\frac{\sin\frac{3x}{2}}{\frac{3x}{2}} \longrightarrow \frac{3}{4}x^2$

528

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} =$$

$$= \lim_{x\to 0}\frac{1}{x}\left(\ln(1+x)\right) = \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} =$$

$$= \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right) = \ln e = 1$$

$$* \lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{1}{x}(e^x-1) =$$

$e^x = t+1 = e^{\ln(t+1)}$

$x = \ln(t+1)$

$$= \left\{ e^x - 1 = t \right\} = \lim_{t\to 0}\frac{t}{\ln(t+1)} = 1$$

$$** \lim_{x\to 0}\frac{(1+x)^a-1}{x} = \lim_{x\to 0}\frac{e^{a\ln(x+1)}-1}{x} =$$

$$= \left\{ t = a\ln(1+x) \right\} =$$

Запомнить!

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{(1+x)^a-1}{x} = a$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

Синус и косинус суммы и разности

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\cos\beta + \sin\alpha\sin\beta$

Преобр. sin и cos

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$

476, 492, 495, 502, 511, 513, 519, 522, 576

Сумма/разность sin/cos

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta = 2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$$

476

$$\lim_{x\to 0} \frac{\sin 5x-\sin 3x}{\sin x} = \lim_{x\to 0}\frac{2\sin x\cdot\cos 4x}{\sin x} = \lim_{x\to 0} 2\cos 4x = 2$$

Дв. угол

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ | $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ | $\cos 2\alpha = 2\cos^2\alpha - 1$ | $\cos 2\alpha = 1-2\sin^2\alpha$

492

$$\lim_{x\to 0}\frac{\sin(\alpha+x)\cdot\sin(\alpha+2x)-\sin^2\alpha}{x} =$$

$$= \lim_{x\to 0}\frac{\frac{\cos(-x)}{2} - \frac{\cos(2\alpha+3x)}{2} - \sin^2\alpha}{x} = \frac{\cos x - \cos(2\alpha+3x) + \cos 2\alpha - 1}{2x} =$$

$$= \lim_{x\to 0}\frac{-(1-\cos x) - (\cos(2\alpha+3x)-\cos 2\alpha)}{2x} =$$

$$= \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2} + 2\sin(2\alpha+\frac{3x}{2})\sin\frac{3x}{2}}{2\cdot\left(\frac{x}{2}\right)^2\cdot\frac{2}{x}\cdot 2x\cdot\frac{3}{2}\cdot\frac{2}{3}} = \frac{3}{2}\sin 2\alpha$$

502

$$\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}^{\,*}}{1-\cos x^{**}} =$$

$*$ Пусть $\alpha = x^2$, тогда получаем $\sqrt{1-\cos\alpha}$;
учтём, что $\cos 2y = 1 - 2\sin^2 y$;
Получаем: $\sqrt{1-(1-2\sin^2\frac{\alpha}{2})} = \sqrt{2\sin^2\frac{\alpha}{2}} = \sqrt{2}\sin\frac{\alpha}{2}$;
Вернём x^2, получим: $\sqrt{2}\sin\frac{x^2}{2}$

$**$ $1-\cos x = 1-\cos^2\frac{x}{2}+\sin^2\frac{x}{2}$ (т.к. $\cos 2y =$
$= \cos^2 y - \sin^2 y$); $1-\cos^2\frac{x}{2}+\sin^2\frac{x}{2} = 2\sin^2\frac{x}{2}$

$$= \lim_{x\to 0}\frac{\sqrt{2}\sin\frac{x^2}{2}}{2\sin^2\frac{x}{2}} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}-\sqrt{2}\sin\frac{x^2}{2}}{2\sin^2\frac{x}{2}} =$$

$$= \lim_{x\to 0}\left(1+\left(-\frac{\sqrt{2}\sin\frac{x^2}{2}}{2\sin^2\frac{x}{2}}\right)\right)^1 =$$

$$= \lim_{x\to 0}\left(1+\left(-\frac{\sqrt{2}\sin\frac{x^2}{2}}{2\sin^2\frac{x}{2}}\right)\right)^{\left(\frac{-2\sin^2\frac{x}{2}}{\sqrt{2}\sin\frac{x^2}{2}}\right)\left(-\frac{\sqrt{2}\sin\frac{x^2}{2}}{2\sin^2\frac{x}{2}}\right)} =$$

$$= e^{\lim\limits_{x\to 0}\frac{-\sqrt{2}\sin\frac{x^2}{2}}{2\sin^2\frac{x}{2}}}$$

(511)

$$\lim_{x\to\infty}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}} =$$

$$\lim_{x\to c}\left(f(x)^{g(x)}\right) = \left(\lim_{x\to c}(f(x))\right)^{\lim_{x\to c}(g(x))}$$

$$\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}\to 0} = 1 \qquad \lim_{x\to\infty}\frac{x-1}{x+1} \qquad \frac{1-\frac{1}{x}\to 0}{1+\frac{1}{x}\to 0} = 1$$

$$= \left(\lim_{x\to\infty}\frac{x^2-1}{x^2+1}\right) = 1^1 = 1$$

(513)

$$\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{1}{x}} =$$

$$= \lim_{x\to\infty}\left(1+\frac{-x^2+5x+1}{2x^2-3x-2}\right)^{\frac{2x^2-3x-2}{-x^2+5x+1}\cdot\frac{-x^2+5x+1}{2x^2-3x-2}\cdot\frac{1}{x}} =$$

$$= e^{\lim_{x\to\infty}\frac{-x^2+5x+1}{2x^2-3x-2}\cdot\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{-1+\frac{5}{x}+\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}}\cdot\frac{1}{x}} = e^0 = 1$$

(519)

$$\lim_{x\to 0}\left(\frac{1+\operatorname{tg}x}{1+\sin x}\right)^{\frac{1}{\sin x}} = \lim_{x\to 0}\left(1+\frac{\operatorname{tg}x-\sin x}{1+\sin x}\right)^{\frac{1}{\sin x}} =$$

$$= \lim_{x\to 0}\left(1+\frac{\operatorname{tg}x-\sin x}{1+\sin x}\right)^{\frac{1+\sin x}{\operatorname{tg}x-\sin x}\cdot\frac{\operatorname{tg}x-\sin x}{1+\sin x}\cdot\frac{1}{\sin x}} =$$

$$= e^{\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{1+\sin x}\cdot\frac{1}{\sin x}} = e^{\lim_{x\to 0}\frac{\cancel{\sin x}(1-\cos x)}{\cos x(1+\sin x)\cancel{\sin x}}} = e^0 = 1$$

522

$$\lim_{x \to \frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} = \lim_{x \to \frac{\pi}{4}} (\operatorname{tg} x)^{\frac{2 \operatorname{tg} x}{1-\operatorname{tg}^2 x}} =$$

$$\operatorname{tg} 2x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}$$

$$= \lim_{x \to \frac{\pi}{4}} (1 + \operatorname{tg} x - 1)^{\frac{1}{\operatorname{tg} x - 1} \cdot \frac{-2 \operatorname{tg} x}{\operatorname{tg}^2 x + 1}} =$$

$$= e^{\lim_{x \to \frac{\pi}{4}} \frac{-2 \operatorname{tg} x}{\operatorname{tg}^2 x + 1}} = e^{-1}$$

$$\operatorname{sh} x = \frac{1}{2}(e^x - e^{-x});$$

$$\to \operatorname{ch} x = \frac{1}{2}(e^x + e^{-x}) \; ; \; \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x}$$

576

a) $$\lim_{x \to 0} \frac{\operatorname{sh} x}{x} = \lim_{x \to 0} \frac{e^x - e^{-x}}{2x}$$

495

$$\lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{1 - 2\cos x} = \lim_{y \to 0} \frac{\sin y}{1 - 2\cos(\frac{\pi}{3} + y)}$$

$y = x - \frac{\pi}{3}$ $\quad g(y) = 1 - 2\cos(\frac{\pi}{3} + y) = 1 - 2(\cos\frac{\pi}{3}\cos y -$

$y \to 0$

$$-\sin\frac{\pi}{3}\sin y) = 1 - \cos y + \sqrt{3}\sin y$$

$$\lim_{y \to 0} \frac{\sin y}{1 - \cos y + \sqrt{3}\sin y} = \lim_{y \to 0} \frac{\frac{\sin y}{y}}{\frac{1 - \cos y}{y} + \frac{\sqrt{3}\sin y}{y}} = \frac{1}{\sqrt{3}}$$

$$\lim_{y \to 0} \frac{1 - \cos y}{y}$$