Неопределённый интеграл.

$$\int \frac{P_n(x)}{Q_m(x)}\,dx$$

1) если $n \geq m$, выделить целую часть
2) $n < m$:

$$Q_m(x) = (x-a)^{\alpha} \ldots (x-b)^{\beta}(x^2 + px + q)^{\gamma} \ldots (x^2 + rx + s)^{\sigma}$$

не имеют вещественных корней

$$\frac{P_n(x)}{Q_m(x)} = \underbrace{\frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \ldots}_{\alpha \text{ штук}} +$$

$$+ \frac{A_1}{x-a} + \ldots + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \ldots +$$

$$+ \underbrace{\frac{B_1}{x-b}}_{\beta \text{ штук}} + \frac{M_\gamma x + N_\gamma}{(x^2+px+q)^\gamma} + \ldots +$$

$$+ \frac{M_1 x + N_1}{x^2+px+q} + \ldots$$

A_i, B_i, M_i, N_i – неопределённые коэффициенты

$$P_n(x) = p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0 = q_n x^n + q_{n-1} x^{n-1} + \ldots + q_1 x + q_0 \Rightarrow p_i x^i = q_i x^i \Rightarrow$$

$\Rightarrow$ t_i.

Пусть $\frac{P_n(x)}{Q_m(x)} = \frac{P_n(x)}{(x-a)^{\alpha} \cdot \varphi(x)} =$

где $\varphi(a) \neq 0$

$= \frac{A_\alpha}{(x-a)^\alpha} + \ldots + \frac{A_1}{x-a} + (\ldots)$

$A_\alpha = \frac{P_n(\alpha)}{\varphi(\alpha)}$

$\int \frac{dx}{x-a} = \ln|x-a| + c$

$\int \frac{dx}{(x-a)^\alpha} = \frac{1}{1-\alpha} \cdot \frac{1}{(x-a)^{\alpha-1}}$

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$\int \frac{dx}{x^2+px+q} = \frac{1}{\sqrt{q-\frac{p^2}{4}}} \int \frac{dt}{t^2+1}$

$\to x^2 + 2\frac{p}{2} \cdot x + \left(\frac{p}{2}\right)^2 + q - \left(\frac{p}{2}\right)^2 =$

$= \left(x + \frac{p}{2}\right)^2 + q - \left(\frac{p}{2}\right)^2 =$

$D: p^2 - 4q < 0$; $x + \frac{p}{2} = t\sqrt{q-\frac{p^2}{4}}$

$= (t^2+1)\left(q - \frac{p^2}{4}\right)$ $\quad dx = dt \cdot \sqrt{q - \frac{p^2}{4}}$

1869
1872
1880
~~1876~~
Д/з:
1870
1871
1877

N 1869

$$\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = \int \frac{x^3-5x^2+6x+1+5x^2-6x}{x^3-5x^2+6x}\,dx =$$

$$= \int \left(1 + \frac{5x^2-6x+1}{x^3-5x^2+6x}\right)dx = x + \int \frac{5x^2-6x+1}{x^3-5x^2+6x}\,dx =$$

$$= x + \int \frac{5(x-\frac{1}{5})(x-1)}{x(x-2)(x-3)}\,dx = x + \int\left(\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}\right)dx$$

К общ. знаменателю:

$$\frac{A(x-2)(x-3) + Bx(x-3) + Cx(x-2)}{x(x-2)(x-3)}$$

$$1 + \frac{5x^2-6x+1}{x(x-2)(x-3)} = 1 + \frac{A(x^2-5x+6) + B(x^2-3x) + C(x^2-2x)}{x(x-2)(x-3)}$$

$$5x^2-6x+1 = A(x^2-5x+6) + B(x^2-3x) + C(x^2-2x)$$

$$\begin{cases} 5 = A+B+C \\ -6 = -5A-3B-2C \\ 1 = 6A \end{cases} \Rightarrow \begin{cases} 5 = \frac{1}{6}+B+C \\ -6 = -\frac{5}{6} - 3B - 2C \\ A = \frac{1}{6} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{29}{6} = B+C \\ -\frac{31}{6} = -3B-2C \\ A = \frac{1}{6} \end{cases} \Rightarrow \begin{cases} A = \frac{1}{6} \\ 2B+2C = \frac{29}{3} \\ -3B-2C = -\frac{31}{6} \end{cases} \Rightarrow$$

$$-B = \frac{27}{6} \Rightarrow B = -\frac{27}{6}$$

$$\Rightarrow \begin{cases} A = \frac{1}{6} \\ B = -\frac{27}{6} = -\frac{9}{2} \\ C - \frac{9}{2} = \frac{29}{6} \end{cases} \Rightarrow \begin{cases} A = \frac{1}{6} \\ B = -\frac{9}{2} \\ C = \frac{56}{6} = \frac{28}{3} \end{cases}$$

Метод вычёркивания:

$$\frac{P(x)}{Q(x)} = \frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$A = \left.\frac{5x^2-6x+1}{(x-2)(x-3)}\right|_{x=0} = \frac{1}{6};$$

$$B = \left.\frac{5x^2-6x+1}{x(x-3)}\right|_{x=2} = -\frac{9}{2};$$

$$C = \left.\frac{5x^2-6x+1}{x(x-2)}\right|_{x=3} = \frac{28}{3}.$$

$$= x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C$$

№ 1872

$$\int \frac{x^2+1}{(x+1)^2(x-1)}dx = \int\frac{-1}{(x+1)^2}dx + \int\frac{1}{2}\cdot\frac{1}{x+1}dx + \int\frac{1}{2}\cdot\frac{1}{x-1}dx =$$

$$\frac{A_2}{(x+1)^2} \quad \frac{A_1}{x+1} \quad \frac{B_1}{x-1}$$

$$\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-1} =$$

$$A = \left.\frac{x^2+1}{x-1}\right|_{x=-1} = -1$$

$$C = \left.\frac{x^2+1}{(x+1)^2}\right|_{x=1} = \frac{1}{2}$$

$$= \frac{-1(x-1) + B(x^2-1) + \frac{1}{2}(x+1)^2}{(x+1)^2(x-1)}$$

$$x^2+1 = 1-x+B(x^2-1)+\frac{1}{2}x^2+x+\frac{1}{2}$$

$$1 = B + \frac{1}{2} \Rightarrow B = \frac{1}{2}$$

N 1880

$$\int \frac{dx}{x(1+x)(1+x+x^2)} =$$

$$\frac{1}{x(1+x)(1+x+x^2)} = \frac{A}{x} + \frac{B}{1+x} + \frac{Mx+N}{1+x+x^2}$$

$$A = \frac{1}{(x+1)(x^2+x+1)}\Big|_{x=0} = 1$$

$$B = \frac{1}{x(x^2+x+1)}\Big|_{x=-1} = -1$$

$$\frac{1}{x(x+1)(x^2+x+1)} = \frac{(x^2+x+1)(x+1) - (x^2+x+1)x + (Mx+N)(x^2+x)}{x(x+1)(x^2+x+1)}$$

$$1 = (x^2+x+1)(x+1) - (x^2+x+1)x + (Mx+N)(x^2+x)$$

$M = 0$

$N = -1$

посчитать отдельно

$$= \int \frac{dx}{x} - \int \frac{dx}{x+1} - \int \frac{dx}{x^2+x+1} =$$

$$= \ln|x| - \ln|x+1| - \frac{4}{3}\sqrt{\frac{3}{4}}\,\mathrm{arctg}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) + C$$

№1876 $\int \frac{x^2+5x+4}{x^4+5x^2+4}dx = \int \frac{\frac{5}{3}x+1}{x^2+1}dx + \int \frac{-\frac{5}{3}x}{x^2+4}dx =$

$$\frac{x^2+5x+4}{x^4+5x^2+4} = \frac{Mx+N}{x^2+1} + \frac{Px+Q}{x^2+4}$$

$(Mx+N)(x^2+4)+(Px+Q)(x^2+1) = x^2+5x+4$

$Mx^3+M4x+Nx^2+N4+Px^3+Px+Qx^2+Q = x^2+5x+4$

$x^3(M+P)+x^2(N+Q)+x(4M+P)+(4N+Q) =$

$= x^2+5x+4$

$$\begin{cases} M+P=0 \\ N+Q=1 \\ 4M+P=5 \\ 4N+Q=4 \end{cases} \Rightarrow \begin{cases} P=-M \\ N+Q=1 \\ M=\frac{5}{3} \\ 4M+Q=4 \end{cases} \Rightarrow \begin{cases} P=-\frac{5}{3} \\ Q=1-N \\ M=\frac{5}{3} \\ 4N+1-N=4 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} P=-\frac{5}{3} \\ Q=0 \\ M=\frac{5}{3} \\ N=1 \end{cases}$$

$= \frac{5}{6}\int \frac{d(x^2+1)}{x^2+1} + \int \frac{dx}{x^2+1} +$

$+ -\left(\frac{5}{6}\right)\int d \ldots =$

$= \frac{5}{6}\ln\left(\frac{x^2+1}{x^2+4}\right) + \mathrm{arctg}\, x + C$

№1870 $\int \frac{x^4}{x^4+5x^2+4}\,dx = \int \frac{x^4+5x^2+4-5x^2-4}{x^4+5x^2+4}\,dx$ 1870 1871 1877

$= \int \left(1 - \frac{5x^2+4}{x^4+5x^2+4}\right) dx = x - \int \frac{5x^2+4}{x^4+5x^2+4}\,dx =$

$= x - \int \frac{5x^2+4}{(x^2+1)(x^2+4)}\,dx$

$1 - \frac{5x^2+4}{(x^2+1)(x^2+4)} = 1 - \left(\frac{Mx+N}{x^2+1} + \frac{Px+Q}{x^2+4}\right)$

$(Mx+N)(x^2+4) + (Px+Q)(x^2+1) = 5x^2+4$

$Mx^3 + 4Mx + Nx^2 + 4N + Px^3 + Px + Qx^2 + Q = 5x^2+4$

$x^3(M+P) + x^2(N+Q) + x(4M+P) + (4N+Q) = 5x^2+4$

$\begin{cases} M+P=0 \\ N+Q=5 \\ 4M+P=0 \\ 4N+Q=4 \end{cases} \Rightarrow \begin{cases} P=-M \\ Q=5-N \\ 3M=0 \\ 3N+5=4 \end{cases} \Rightarrow \begin{cases} P=0 \\ Q=2 \\ M=0 \\ N=3 \end{cases}$

$= x - \int \left(\frac{3}{x^2+1} + \frac{2}{x^2+4}\right) dx = x - 3\,\mathrm{arctg}\,x -$

$- 2 \cdot \frac{1}{2}\,\mathrm{arctg}\,\frac{x}{2} + C = x - 3\,\mathrm{arctg}\,x - \mathrm{arctg}\,\frac{x}{2} + C$

№ 18+1

$x^3-4x+x+2=$
$=x(x-2)(x+2)$
$+(x+2)=$
$=(x+2)\cdot$
$\cdot(x(x-2+1)=$
$=(x+2)\cdot$
$\cdot(x-1)^2$

$$\int \frac{x}{x^3-3x+2}\,dx =$$

$$\frac{x}{x^3-3x+2} = \frac{A}{(x-1)^2} + \frac{B}{x+2}$$

$$A = \frac{x}{x+2}\bigg|_{x=1} = \frac{1}{3}$$

$$B = \frac{x}{(x-1)^2}\bigg|_{x=-2} = -\frac{2}{9}$$

$$= \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx -$$

$$-\frac{2}{9}\int \frac{1}{x+2}\,dx = \left\{\begin{array}{l} t_1 = x-1 \\ dx = \frac{1}{t'}\,dt = 1\cdot dt \\ t_2 = x+2 \\ dx = \frac{1}{t'}\,dt = 1\cdot dt \end{array}\right\} =$$

$$= \frac{1}{3}\int \frac{1}{t^2}\,dt - \frac{2}{9}\int \frac{1}{t}\,dt = \frac{1}{3}\cdot\frac{t^{-2+1}}{-2+1} -$$

$$-\frac{2}{9}\cdot \ln|t| + C = -\frac{1}{3(x-1)} - \frac{2}{9}\cdot \ln|x+2| + C =$$

$$= \frac{1}{3(1-x)} - \frac{2}{9}\ln|x+2| + C$$

N 1877 $\int \frac{1}{(x+1)(x^2+1)} dx =$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{M \cdot x + N}{x^2+1}$$

$$A = \frac{1}{x^2+1} \Big|_{x=-1} = \frac{1}{2}$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2x+2} + \frac{Mx+N}{x^2+1}$$

$$1 = x^2+1 + (2x+2)(Mx+N)$$
$$0 = x^2+1 + (2x+2)(Mx+N)$$

$$M = -\frac{1}{2}$$

$$N = 0$$

$$= \int \frac{1}{x+1} dx + \int \frac{-\frac{1}{2}x}{x^2+1} dx = \ln|x+1| + C +$$

$$+ \left(-\frac{1}{2}\right) \int \frac{x}{x^2+1} dx = \left\{ \begin{array}{l} t = x^2+1 \\ dx = \frac{1}{t'} dt = \frac{1}{2x} dt \end{array} \right\} =$$

$$= \ln|x+1| + C - \frac{1}{2} \int \frac{\cancel{x}}{x^2+1} \cdot \frac{1}{2\cancel{x}} dt =$$

$$= \ln|x+1| + C - \frac{1}{2} \int \underbrace{\frac{1}{x^2+1}}_{} \cdot \frac{1}{2} dt = \ldots + C -$$

$$- \frac{1}{4} \int \frac{1}{t} dt = \ln|x+1| - \frac{1}{4} \ln|x^2+1| + C$$