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WEIGHT SELECTION IN FIRST-ORDER LINEAR FILTERS

by

C. Frank Asquith

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WEIGHT SELECTION IN FIRST-ORDER LINEAR FILTERS

by

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ABSTRACT

This report describes methods for the proper design of first-order, recursive, fixed-weight, linear filters. Expressions are derived and listed for commonly used design parameters such as noise ratio, transient response, and truncation error. The performances of critically damped and steady-state optimum filters are compared. Design curves are given that can be used to select the weights of the steady-state optimum filter from total error requirements.

CONTENTS

ILLUSTRATIONS

1. Filter Equations

The first-order linear filters under consideration are those of the form

$$\hat{y}(n+1) = \bar{y} + \tau \bar{\dot{y}}(n)$$
$$\hat{\dot{y}}(n+1) = \bar{\dot{y}}(n) \tag{1}$$

$$\bar{y}(n) = \hat{y}(n) + w_1(n)[x(n) - \hat{y}(n)]$$
$$\bar{\dot{y}}(n) = \hat{\dot{y}}(n) + w_2(n)[x(n) - \hat{y}(n)] . \tag{2}$$

In state vector notation, these equations may be written

$$\hat{Y}(n+1) = \Phi(n) \bar{Y}(n) \tag{3}$$
$$\bar{Y}(n) = \hat{Y}(n) + W(n)[X(n) - H\hat{Y}(n)] . \tag{4}$$

In the preceding equations,

$$\hat{Y}(n+1) = \begin{bmatrix} \hat{y}(n+1) \\ \hat{\dot{y}}(n+1) \end{bmatrix}$$

is the predicted system state made at time nT for time $(n+1)T$ given n measurements. The equation

$$\bar{Y}(n+1) = \begin{bmatrix} \bar{y}(n) \\ \bar{\dot{y}}(n) \end{bmatrix}$$

is the smoothed system state made at time nT for time nT given n measurements. The equation

$$X(n) = [x(n)]$$

is the position measurement corrupted by white Gaussian noise. The term T is the sampling interval, and n is the time index. The equation

$$\Phi(n) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

is the state transition matrix for a constant velocity trajectory. The equation

$$H = [1 \quad 0]$$

is the observation matrix. The weighting matrix is

$$W(n) = \begin{bmatrix} w_1(n) \\ w_2(n) \end{bmatrix} \quad .$$

In the application considered for this filter, y and ẏ are estimated position and velocity, and x is measured position along one coordinate axis. Equation (1) or (3) is called the prediction equation, and Equation (2) or (4) is called the smoothing or correction equation. The objective of this paper is the selection of a constant weighting matrix W to get the best estimate of the predicted state $\hat{Y}$. In most of the literature dealing with a constant weighting matrix, w_1 is called g or α, and TW_2 is called h or β. The g-h notation will predominate in this paper.

The overall filter performance may be described by three factors. They are:

a)   Variance reduction ratio

b)   Transient response

c)   Truncation error.

A fourth and sometimes dominant factor is computional complexity. In tracking many targets by use of a time shared computer and radar, this factor limits the filter order to first or second, necessitates the use of constant weights, and makes smoothing in one coordinate at a time highly desirable.

2. Variance Reduction Ratio

The variance reduction ratio is the ratio of the rms noise output from the filter to the rms noise input. It is sometimes more appropriately called the noise amplification factor, since it may necessarily exceed unity if the filter requirement is for good dynamic response. In general, low noise output requires small weights, and good transient performance requires large weights. A mathematical expression can be derived for the first order filter in which $W_1 = g$ and $W_2 = h/T$ are constants. If equations (1) and (2) which occur at time nT are combined with the same equations for time (n-1)T, all quantities except predicted positions and measurements may be eliminated to yield

$$\hat{y}(n+1) = [g+h] \, x(n) - g \, x(n-1) + [2-g-h] \, \hat{y}(n)$$
$$+ [g-1] \, \hat{y}(n-1) \quad . \tag{5}$$

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Equation (5) gives predicted position in the standard feedback filter form

$$\hat{y}(n+1) = \sum_{i=0}^{k} a_i x(n-i) - \sum_{j=0}^{\ell} b_j \hat{y}(n-j) \ . \tag{6}$$

In Equation (5), the output is the weighted sum of the last two inputs and outputs.

Since the filter is linear, the Superposition Theorem may be invoked in calculating the noise ratio which is independent of any trajectory dynamical errors. If the input is sampled white noise with variance $\sigma_x^2 = E[x^2]$, in which $E[\ ]$ denotes "expected value of," then the output noise variance is $\sigma_{\hat{y}}^2 = E[\hat{y}^2(n+1)]$. Squaring Equation (5) and taking the expected value of the result gives

$$\sigma_{\hat{y}}^2 = (g+h)^2 \sigma_x^2 + g^2 \sigma_x^2 + (2-g-h)^2 \sigma_{\hat{y}}^2 + (g-1)^2 \sigma_{\hat{y}}^2$$

$$- 2g(2-g-h) E[x(n-1)\hat{y}(n)] + 2(2-g-h)(g-1) E[\hat{y}(n-1)\hat{y}(n)] \ . \tag{7}$$

In arriving at Equation (7), it was assumed that the input and output noise distributions are stationary so that

$$E[\hat{y}^2(n+1)] = E[\hat{y}^2(n)] = E[\hat{y}^2(n-1)] = \sigma_{\hat{y}}^2$$

and

$$E[x^2(n)] = E[x^2(n-1)] = \sigma_x^2 \ .$$

Also, the input noise samples are independent so that

$$E[x(n)\, x(n-1)] = 0$$

and predictions are independent of future measurements so that

$$E[x(n)\, \hat{y}(n)] = E[x(n)\, \hat{y}(n-1)] = E[x(n-1)\, \hat{y}(n-1)] = 0.$$

The remaining two terms in Equation (7) involve correlated quantities but can be evaluated from Equation (5). Writing Equation (5) for $\hat{y}(n)$, multiplying by $x(n-1)$, and taking the expected value of the result gives

$$E[x(n-1)\, \hat{y}(n)] = (g+h)\, \sigma_x^2 \ , \tag{8}$$

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in which use has been made of the same stationarity and independence assumptions as before. Again, writing Equation (5) for $\hat{y}(n)$, multiplying by $\hat{y}(n-1)$, and taking the expected value gives

$$E[\hat{y}(n)\,\hat{y}(n-1)] = \frac{1}{2-g}\left[(2-g-h)\,\sigma_{\hat{y}}^2 - g(g+h)\,\sigma_x^2\right] \,, \qquad (9)$$

after making the same assumptions as before and making use of Equation (8). Finally, substituting Equations (8) and (9) into Equation (7) gives

$$\frac{\sigma_{\hat{y}}^2}{\sigma_x^2} = \frac{2g^2 + gh + 2h}{g(4-2g-h)} \,. \qquad (10)$$

Equation (10) gives the ratio of the variance of output noise in predicted position to that of measurement noise for constant filter weights. This is one equation applicable to making a proper choice of g and h. Similar equations can be derived for smoothed position and predicted velocity. They are:

$$\frac{\sigma_{\bar{y}}^2}{\sigma_x^2} = \frac{2g^2 + 2h - 3gh}{g(4-2g-h)} \qquad (11)$$

$$\frac{\sigma_{\dot{\hat{y}}}^2}{\sigma_x^2} = \frac{1}{T^2}\,\frac{2h^2}{g(4-2g-h)} \,. \qquad (12)$$

Equations (10), (11), and (12) can also be derived from the system unit impulse response by using

$$\frac{\sigma_{\hat{y}}^2}{\sigma_x^2} = \sum_0^\infty h^2(n) \,, \qquad (13)$$

in which h(n) is the inverse transform of $H(Z) = Y(Z)/X(Z)$, and $Y(Z)$ is obtained by taking the transform of Equation (5) for predicted position. Equation (13) can be evaluated from the integral for the sum of a squared sample sequence [1].

$$\sum_0^\infty h^2(n) = \frac{1}{2\pi j}\oint H(z)\,H(z^{-1})\,z^{-1}\,dz \,, \qquad (14)$$

in which the path of integration is the unit circle in the Z-plane. The mean square noise ratio then becomes $2\pi j$ times the sum of the residues of

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Equation (14) at the interior poles of the integrand. However, it is felt that the first method of derivation more clearly points out the assumptions that have to be made in deriving Equations (10), (11), and (12). The same assumptions have to be made in deriving Equation (13).

The assumptions of stationarity should be kept in mind when applying equations such as Equation (10). The input stationarity assumption merely implies that the root mean square measurement noise does not change significantly during one sample period. This is entirely reasonable from the standpoint of the trajectory itself; that is, measured coordinates do not change fast enough between samples to affect the expected noise, which is a function of position. The change in measurement noise due to target scintillation is another matter. In practice, the target's effective radar cross section can vary with time such that significant deviations in the system signal-to-noise ratio occur at rates close to normally used tracking rates. However, there is evidence that, for periodic variations in signal to noise in the steady state, Equation (10) still gives the ratio of total mean square noise output to total mean square noise input [2]. Even though the input noise is stationary, the output noise need not be. There is a transient period following track initiation in the noise ratio itself about which not enough is known at this time. One reference indicates that Equation (10) gives an entirely erroneous result during this transition period [3]. Further work is planned in this area.

3. Transient Response

The specification of system transient response has always been somewhat of a problem. The classical method is the use of time constant, or time for the output to decay to $1/e$ of some initially stored value. Other criteria such as 10 to 90 percent rise time have been used. These ideas are still applicable as definitions of transient response in sampled data systems, but a more elegant approach is to calculate the sum-squared system error output caused by some input for which the output should eventually converge. As was indicated in Equation (14), if $f(nt)$ is a convergent series of data samples whose ultimate value is zero, then the sum of the squares of all the samples from zero to infinity is given by

$$S^2 = \frac{1}{2\pi j} \oint F(z) \ F(z^{-1}) \ z^{-1} \ dz \ . \tag{15}$$

In Equation (15), $F(z)$ is the z transform of $f(nt)$ defined by

$$F(z) = \sum_{n=0}^{\infty} f(nT) \ z^{-n} \ , \tag{16}$$

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and the path of integration is the unit circle.

The transform of Equation (5) is

$$\hat{Y}(z) = \frac{g + h - g\,z^{-1}}{1 - (2-g-h)z^{-1} - (g-1)z^{-2}} \quad . \tag{17}$$

Since the first-order filter will converge on a position ramp input (velocity step) without error, a reasonable input to be used as a basis for transient response calculation is

$$x(nT) = nT \quad , \tag{18}$$

whose transform is

$$X(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} \quad . \tag{19}$$

The desired prediction is

$$Y(z) = zX(z) = \frac{T}{(1 - z^{-1})^2} \quad . \tag{20}$$

The transform of the error is

$$\epsilon_{\hat{y}}(z) = \hat{Y}(z) - Y(z) = \frac{T}{1 - (2-g-h)z^{-1} - (g-1)z^{-2}} \tag{21}$$

after substituting Equations (15), (17), and (18). Evaluating the integral

$$\sum{}_{\hat{y}}^{2} = \frac{1}{2\pi j} \oint \epsilon_{\hat{y}}(z)\ \epsilon_{\hat{y}}(z^{-1})\ z^{-1}\,dz \quad , \tag{22}$$

by summing the residues at the two poles inside the unit circle, gives

$$\sum{}_{\hat{y}}^{2} = \frac{T^2(2-g)}{gh(4-2g-h)} \tag{23}$$

for the sum-squared error in predicted position of the first order filter with a ramp input. Similarly, it can be shown that for smoothed position and velocity,

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$$\sum_{\bar{y}}^{2} = \frac{T^2(2-g)(1-g)^2}{gh(4-2g-h)} \tag{24}$$

$$\sum_{\hat{\dot{y}}}^{2} = \frac{g^2(2-g) + 2h(1-g)}{gh(4-2g-h)} \quad . \tag{25}$$

4. Truncation Error

It was stated in the previous section that a first-order filter will converge on a constant velocity noise free input trajectory with zero error. It is also true that a constant acceleration input will be estimated with a constant error in the steady state, and trajectories with higher derivatives will cause errors that grow as long as the higher derivatives do not change sign. Fortunately, real target trajectories cannot possess monotonic higher derivatives along any given coordinate axis for very long and the position errors that result can be kept within reasonable limits. The steady-state error that is caused by higher derivatives than the first is called truncation error for the first-order filter.

The mathematical approach to estimating truncation error is to assume that that due to acceleration is predominant. This is equivalent to assuming that acceleration does not change significantly in one sample period. This is a good assumption for trajectories for which a first-order filter is useful. In practice, it is found that tracking filters higher than second order are rarely necessary and a first-order filter is frequently adequate. It is also true that the noise amplification ratio tends to increase with the order of the filter; that is, a low-order filter will follow a low-order curve with less noise output than a higher order filter. The order of the filter, then, like the choice of smoothing constants is a compromise between noise reduction and dynamic response.

The fact that truncation error for the first-order filter can be closely approximated for several representative trajectories by the response to the second derivative input has been demonstrated experimentally [4]. The analytical investigation can proceed as follows: Let the input be

$$x(nT) = A/2(nT)^2 \quad , \tag{26}$$

which corresponds to a parabolic trajectory in one coordinate with acceleration A. The Z transform is

$$X(z) = A/2 \ T^2 \ \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \qquad (27)$$

Then the transform of the desired output is

$$Y(z) = zX(z) = A/2 \ T^2 \ \frac{(1 + z^{-1})}{(1 - z^{-1})^3} \quad . \qquad (28)$$

By using Equations (17), (27), and (28) to form the error expression

$$\Delta \hat{Y}(z) = \hat{Y}(z) - Y(z) \ , \qquad (29)$$

and invoking the final value theorem, the steady-state error is obtained:

$$\Delta_{\hat{y}}(nT) = \lim_{\substack{nT \to \infty \\ z \to 1}} (1 - z^{-1}) \Delta_{\hat{y}}(z) \qquad (30)$$

The result is

$$\Delta_{\hat{y}}(\infty) = - AT^2/h \quad . \qquad (31)$$

A less sophisticated but simpler technique is to substitute

$$\hat{y}(n + 1) = A/2 \ (n + 1)^2 \ T^2 + \Delta_{\hat{y}}(n + 1)$$

$$\hat{y}(n) = A/2 \ n^2 \ T^2 + \Delta_{\hat{y}}(n)$$

$$\hat{y}(n - 1) = A/2 \ (n - 1)^2 \ T^2 + \Delta_{\hat{y}}(n - 1)$$

$$X(n) = A/2 \ n^2 \ T^2$$

$$X(n - 1) = A/2 \ (n - 1)^2 \qquad (32)$$

into Equation (5), set $\Delta_{\hat{y}}(n + 1) = \Delta_{\hat{y}}(n) = \Delta_{\hat{y}}(n - 1)$ in the steady state, and solve for $\Delta_{\hat{y}}(n + 1)$. The result is the same. As a matter of record, the truncation errors for smoothed position and velocity are

$$\Delta_{\hat{y}} = - AT^2 \ \frac{1-g}{h} \qquad (33)$$

$$\Delta_{\hat{y}} = - AT^2 \ \frac{2g - h}{2h} \quad . \qquad (34)$$

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5. Weight Selection

The weights g and h must be selected to give some suitable compromise between dynamic response and noise ratio which meets the overall system error specification. Sometimes the sampling period T can be varied within limits to aid in meeting the requirements. Two basic relationships between g and h have been used to ease the task of choosing the weights from all possible g-h pairs.

a. The Critically Damped Filter

As can be seen from Equation (17), the first-order filter has a second-degree polynomial in its transfer function. It has been recognized for many years that the critically damped condition in a system characterized by such a function has reasonably fast and well-behaved transient response. The poles of Equation (17) are

$$Z_{1,2} = \frac{1}{2} \left[2-g-h \pm \sqrt{(2-g-h)^2 + 4(g-1)} \right] , \qquad (35)$$

and critical damping occurs when

$$(2-g-h)^2 + 4(g-1) = 0 \qquad (36)$$

from which

$$h = 2-g \pm 2\sqrt{1-g} . \qquad (37)$$

Also, by applying Routh's method to the denominator polynomial of Equation (17) after using the bilinear transformation [5],

$$Z = \frac{S+1}{S-1} . \qquad (38)$$

The first-order filter is stable if

$$h > 0$$

$$g > 0$$

$$4-2g-h > 0 . \qquad (39)$$

The conditions in Equation (39) indicate that g and h lie within the triangle shown in Figure 1. The locus of Equation (37) is also shown in Figure 1 showing the combinations of g and h normally used for the critically damped filter. Actually, the useful part of the curve lies between h = o and h = 1, since for larger values both noise ratio and time response increase. For example, if the time response is defined to be the 1/e time constant in position error which results from a velocity step input, the inverse transform of Equation (21) with equal denominator roots gives for the time constant the value of n for which

$$\log_e [(n + 1)(1 - g) n/2] = -1 .$$ (40)

If Equation (40) is plotted against the noise ratio [Equation (10)], Figure 2 is obtained. The lower portion of the curve is the useful range and is obtained using the minus sign in Equation (37). The upper curve, obtained using the plus sign, gives more noise for the same time response.

b. The Optimum First-Order Filter

There are two weights to be determined in the first-order filter, and specifying one of them is equivalent to fixing one performance criterion, such as noise ratio. Then a relationship exists between g and h which minimizes the transient error for this specified noise ratio. This means that the function

$$\delta_{\hat{y}}^2 + \lambda \Sigma_{\hat{y}}^2$$ (41)

in which λ is a constant and $\delta_{\hat{y}}^2$ and $\Sigma_{\hat{y}}^2$ are given by Equations (10) and (23) must be differentiated with respect to g and h, set equal to zero, and h solved for in terms of g. The result is

$$h = g^2/(2-g) .$$ (42)

Further details are given by Benedict and Bordner [6]. It might also be pointed out that Benedict and Bordner showed that not only does Equation (42) minimize Equation (41) but that the filter given by Equation (5) is the optimum linear filter for tracking a noisy ramp among all the filters of the form in Equation (6). That is, no other combination of past measurement and predictions will do any better than the last two.

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c.   Kalman Steady-State Filter

If the Kalman filtering technique [7] is applied to the estimation of a constant velocity trajectory given noisy position measurements, Equations (3) and (4) give the predicted and corrected states. The weights are computed for each measurement from

$$W(n) = \widehat{P}(n) \; H^+ \; [H\widehat{P}\,H^T + R]^{-1}$$

$$\widehat{P}(n) = \Phi\overline{P}(n-1) \; \Phi^T + Q$$

$$\overline{P}(n-1) = [I - W(n-1)\,H] \; \widehat{P}(n-1) \; , \tag{43}$$

in which $\widehat{P}$ is the covariance error matrix in predicted state and P is the corrected covariance error matrix. Other matrixes were defined previously except for Q, which will be taken to be

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_M{}^2 \end{bmatrix} . \tag{44}$$

In defining Q in this manner, all deviations in the trajectory caused by derivatives higher than the first are treated as white noise with variance, and this maneuver noise is assumed to affect only the uncertainty in predicted velocity during the sample period over which it occurs.

Expanding the matrixes in Equation (43) leads to the following equations for the elements of $W(n)$ and $P(n)$:

$$W_1(n) = \widehat{P}_{11}(n)/[\widehat{P}_{11}(n) + R]$$

$$W_2(n) = \widehat{P}_{12}(n)/[\widehat{P}_{11}(n) + R] \tag{45}$$

$$\widehat{P}_{11}(n) = W_1(n-1)\,R + 2T\,W_2(n-1)\,R + T^2\,\widehat{P}_{22}(n-1) - T^2\,W_2(n-1)\,\widehat{P}_{12}(n-1)$$

$$\widehat{P}_{12}(n) = W_2(n-1)\,R + T\,\widehat{P}_{22}(n-1) - T\,W_2(n-1)\,\widehat{P}_{12}(n-1)$$

$$\widehat{P}_{22}(n) = \widehat{P}_{22}(n-1) - W_2(n-1)\,\widehat{P}_{12}(n-1) + M \; , \tag{46}$$

in which

$$R = \sigma_x{}^2 \text{ is the measurement variance and}$$

$$M = \sigma_M{}^2 \text{ is the maneuver variance} . \tag{47}$$

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If R and M are constant, $W(n)$ and $P(n)$ become constant matrixes as the time index n increases. In the steady state, the time indexes in Equation (46) may be dropped and the constant matrix elements solved for. If this is done, there is obtained

$$\widehat{P}_{11} - T \widehat{P}_{12} = W_1 R + T W_2 R$$

$$W_2 \widehat{P}_{12} = M ,$$
(48)

from which

$$TW_2 = W_1^2 / (2 - W_1)$$
(49)

$$T^2 \frac{M}{R} = \frac{W_1^4}{(2 - W_1)^2 (1 - W_1)} ,$$
(50)

or in g-h notation,

$$h = g^2 / (2 - g)$$
(51)

$$T^2 \frac{M}{R} = \frac{g^4}{(2 - g)^2 (1 - g)} .$$
(52)

Equation (51) is the same as Equation (42), which was derived by calculus of variations. Thus, this steady-state filter is the same as Benedict and Bordner's optimum fixed weight filter. However, the Kalman filter fixes the weights through Equation (52) which gives g in terms of the sampling period, measurement noise, and expected maneuver noise. In the optimum fixed weight filter, the final choice of weights is still a compromise between Equations (10) and (23). The steady-state Kalman makes the choice automatically if values can be selected for T, M, and R.

6. Filter Comparison and Design Curves

One objective of this paper is to compare the critically damped filter with the steady-state Kalman filter. Some basis for comparison must be chosen and that basis will be one of relative transient and truncation errors for equal noise ratios. Equations (37) and (42) have been plotted in Figure 3 showing h in terms of g for the two filters. In Figure 4, Equations (10), (23), and (31) for noise ratio, transient error, and truncation error are shown (normalized) for the critically damped filter as functions of g. The same quantities are shown for the steady-state filter in Figure 5. In Figure 6, the sum-squared transient errors and truncation errors for both filters are shown

as functions of mean-square noise ratio. Figure 6 shows that for a given output noise ratio the steady-state filter has a transient error about 85 percent as great as that of the critically damped filter and a truncation error about 72 percent as great.

In Figures 7(a) and (b), g and h are shown as functions of the parameter $T^2 M/R$ for the steady-state filter. These figures cover a wide range because M is normally a function of T^2 so that $T^2 M/R$ is a function of T^4. In general, Figure 7(a) is useful for tracking intervals on the order of a second and Figure 7(b) for intervals around 0.1 second.

Figures 7(a) and (b) are valid regardless of how M is chosen. In practice, T (or its allowable range) is known and a good estimate can be made of R, the measurement variance. A reasonable way to estimate M is to assume some maximum acceleration, A_{max}, for the trajectory so that the greatest velocity change in one sample period is $T A_{max}$. Let this quantity be related to σ_n by

$$B \sigma_M = T A_{max} \tag{53}$$

so that

$$M = \sigma_M^2 = \frac{1}{B^2} T^2 A_{max}^2 \tag{54}$$

and

$$T^2 \frac{M}{R} = T^4 A_{max}^2 \Big/ B^2 \sigma_x^2 \tag{55}$$

or

$$T \frac{\sigma_M}{\sigma_x} = \frac{T^2 A_{max}}{B \sigma_x} \quad . \tag{56}$$

If the maneuver were truly random from sample to sample, B would be set equal to 3 so that $T A_{max}$ would be the 3σ value of the maneuver noise. But a velocity maneuver is not really white noise. If the target accelerates at all, there will be a high degree of correlation between samples in the maneuver. There are techniques available for mathematically modeling the maneuver to account for this correlation, but this introduces more states to be estimated.

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Like the question of track initiation, this is a subject best left for future investigation. Some aid in selecting B can be obtained by plotting curves of truncation error and noise ratio as in Figure 8. The normalized 3σ noise error is 3δ from Equation (10) which is plotted as a function of g using Equation (43) for h. The normalized maximum truncation error can be obtained by equating Equation (56) to the square root of Equation (52):

$$\frac{T^2 A_{max}}{B\sigma_x} = \frac{g^2}{(2-g)\sqrt{1-g}} \quad .$$ (57)

Recognizing that the truncation error is

$$\Delta_{max} = T^2 \frac{A_{max}}{h}$$ (58)

and

$$h = g^2 / (2-g) \quad ,$$ (59)

then the normalized maximum truncation error is

$$\frac{\Delta_{max}}{\sigma_x} = \frac{B}{\sqrt{1-g}} \quad .$$ (60)

Equation (61) is plotted as a function of the parameter B in Figure 8 and shows that the filter weights the truncation and noise errors equally for B between 1 and 2. In Figure 9, B is plotted against the worst expected error, $3\delta + \Delta_{max}/\sigma_x$, with $T^2 A_{max}/\sigma_x$ as a parameter. From these curves, an estimate can be made of the value of B which minimizes the maximum error. In Figure 10, the total error is plotted as a function of $T^2 A_{max}/\sigma_x$ with B as a parameter. Figure 9 is useful when the absolute minimum total error must be obtained in order to maintain track. Figure 10 is more useful against maneuvering targets when the least obtainable error is not required.

Also in Figure 10 is shown the total error for the case $g = 1$, $h = 1$, or no smoothing, given by

$$\hat{y}(n+1) = 2x(n) - x(n-1) \quad ,$$ (61)

for which

$$\delta = \sqrt{5} \text{ and } \Delta = T^2 A \quad .$$ (62)

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It can be deduced from Figure 9 that when truncation error can be expected to exist throughout the trajectory, such as when tracking a ballistic missile, the choice of B = 1 or less is optimum. However, for small values of $T^2 A_{max}/\sigma_x$, B less than one does not drastically change the total error from what it is with B = 1. It is further seen from Figure 10 that for ratios of $T^2 A_{max}/\sigma_x$ greater than 4, the reduction in error over that of the no-feedback filter is completely negligible and a no-feedback filter should be used.

Against a maneuvering target, the largest value of B should be used such that the worst expected error, $\Delta_{max} + 3\sigma_x$, does not exceed the confines of the radar measurement volume. This will insure that track is not lost during a maximum maneuver while giving the least noise during no maneuver.

Incidentally, Figure 10 does not indicate that values of B to the right of the no smoothing curve are useless. As B decreases, the curves move across the page to the left then back toward the no smoothing line as B gets very small. No smoothing corresponds to B = 0.

As an example of the use of the curves, suppose that $T^2 A_{max}/\sigma_x$ is estimated to be 0.1 and the normalized coordinate extremity of the radar volume is 6.0. From Figure 9, the optimum value of B is about 0.6 and the worst error is 3.0, well within the radar volume. The noise ratio from Figure 8 is 2.15. If the radar volume dimension 6.0 is used for $\Delta_{max}/\sigma_x + 3\delta$, then, from Figure 10, B = 4 and the noise from Figure 8 is 1.22. Thus the no-maneuver error is about half what it is with the least maximum error condition, but track is maintained in either case.

7. Conclusion

The mathematical techniques useful in the analysis of first-order digital filters have been reviewed and demonstrated. The important performance measures (noise ratio, transient error, and truncation error) have been derived and listed for predicted position, smoothed position, and velocity. The assumptions made in these derivations have been noted and their effects discussed where possible. The steady-state optimum filter has been compared to the critically damped filter and shown to have less truncation and transient error for a given noise ratio. Design curves have been provided that can be used to select the smoothing weights if the measurement noise and trajectory dynamics can be estimated. This filter, however, in which the weights are constant, is not presently recommended for track initiation.

For ballistic missile tracking, the truncation and noise errors should be weighted about equally with the parameter B = 1. A larger value of B would be better against a maneuvering target, provided the total error does not exceed the radar measurement volume.
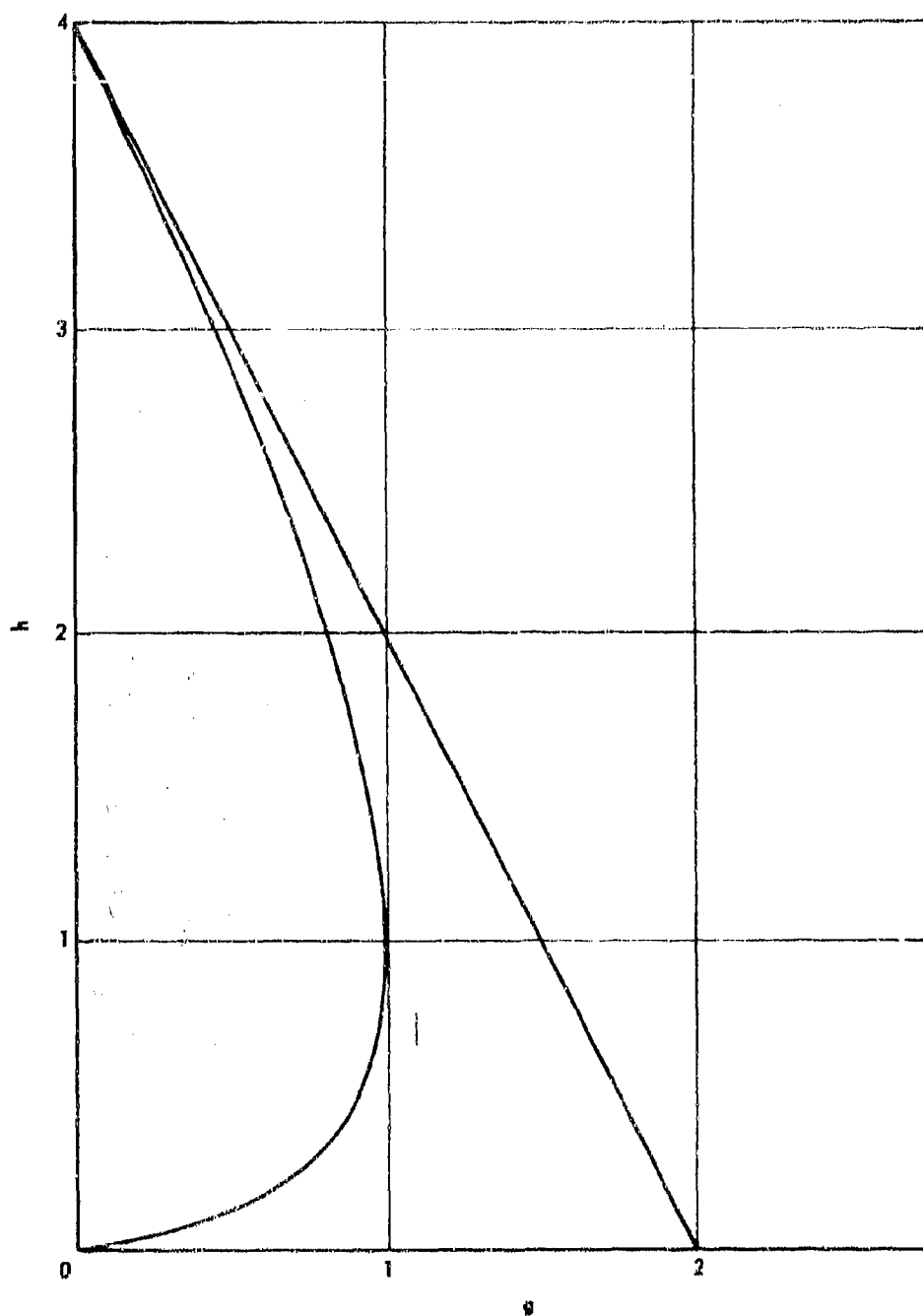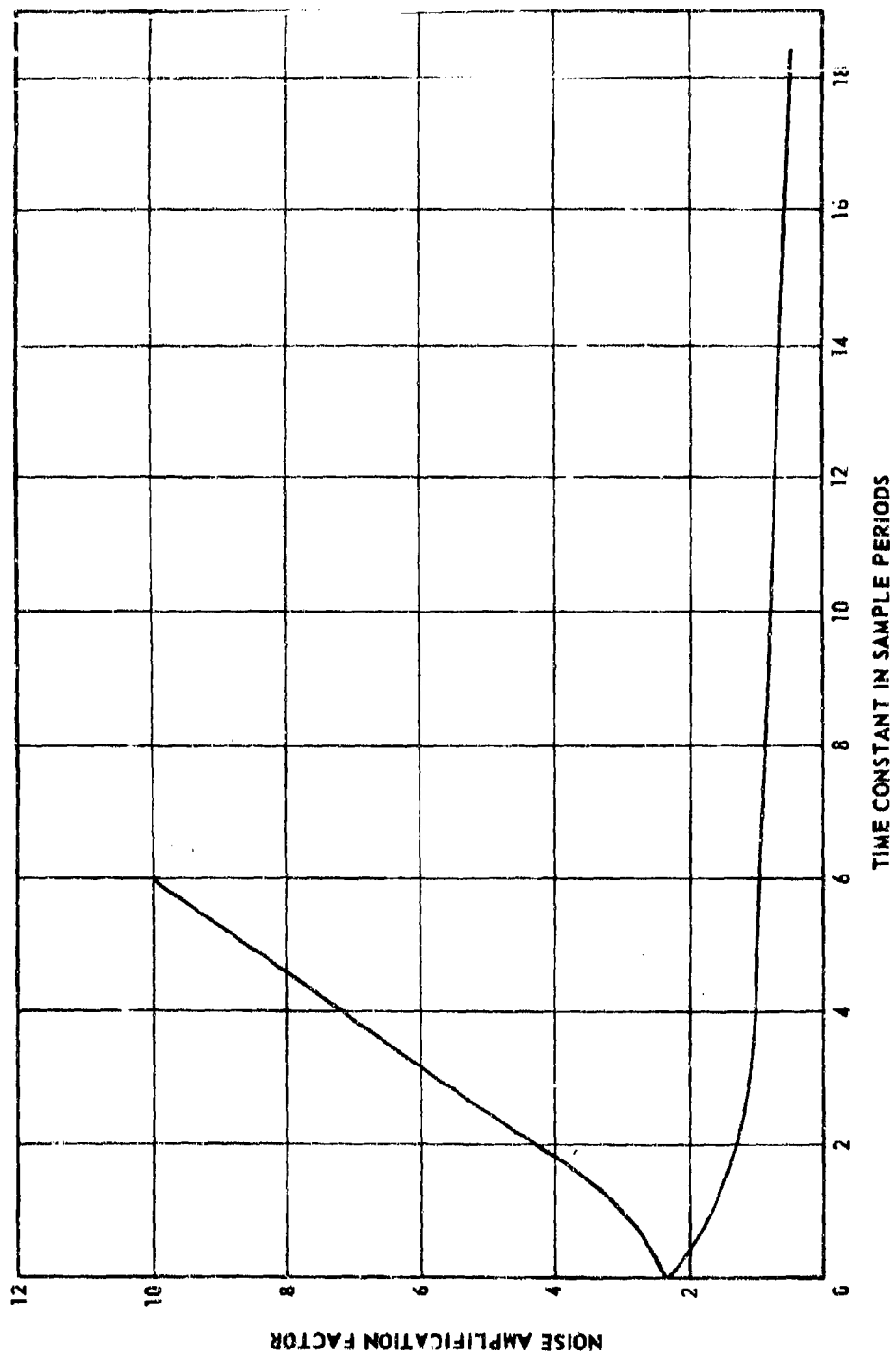
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FIGURE 1. g-h PLANE STABILITY TRIANGLE

FIGURE 2. SMOOTHING VERSUS TIME CONSTANT FOR CRITICALLY DAMPED g-h FILTER
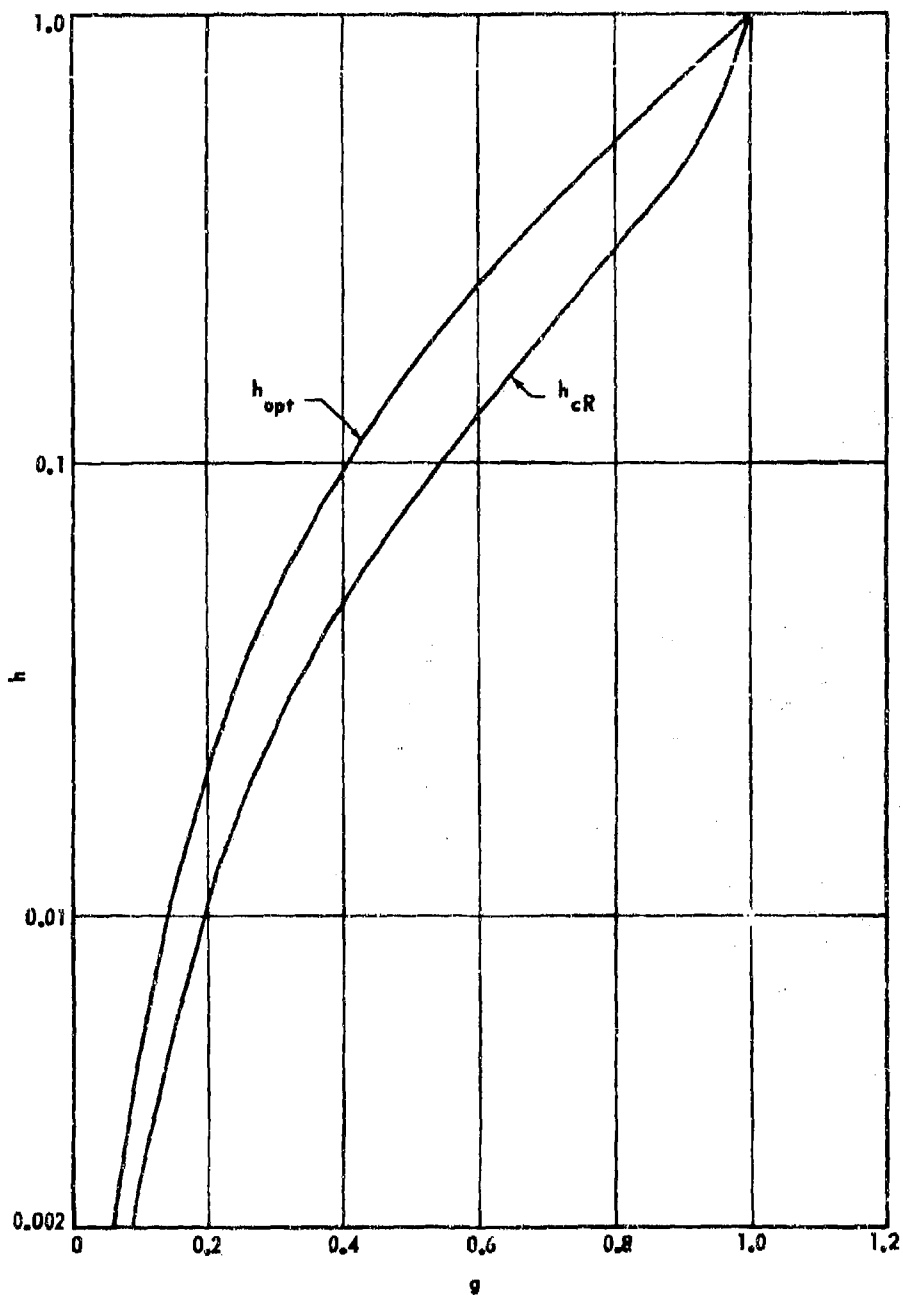
18

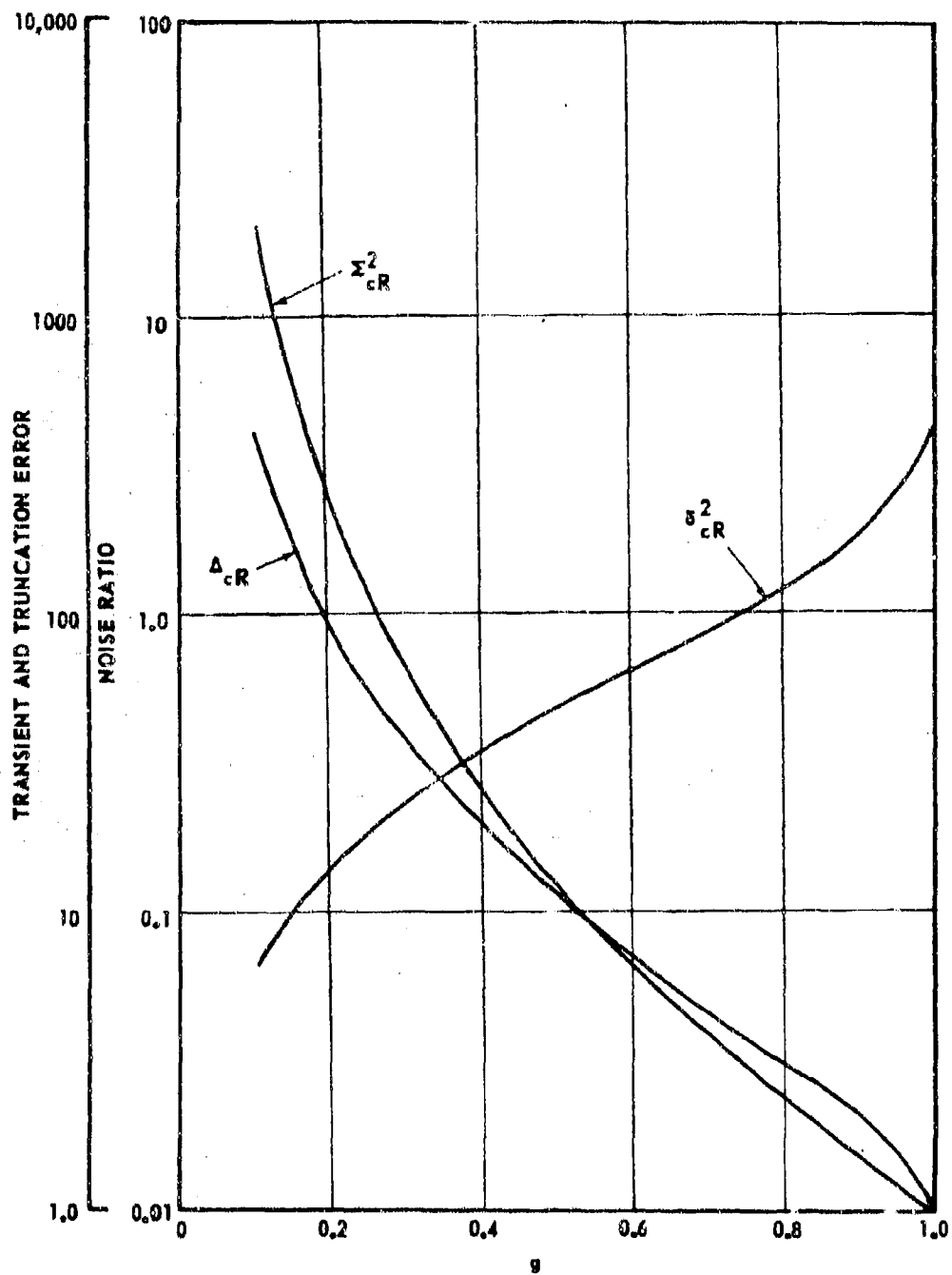FIGURE 3. h VERSUS g FOR THE OPTIMUM AND CRITICALLY
   DAMPED FILTERS

FIGURE 4. NOISE RATIO, TRANSIENT ERROR, AND TRUNCATION ERROR
FOR THE CRITICALLY DAMPED FILTER
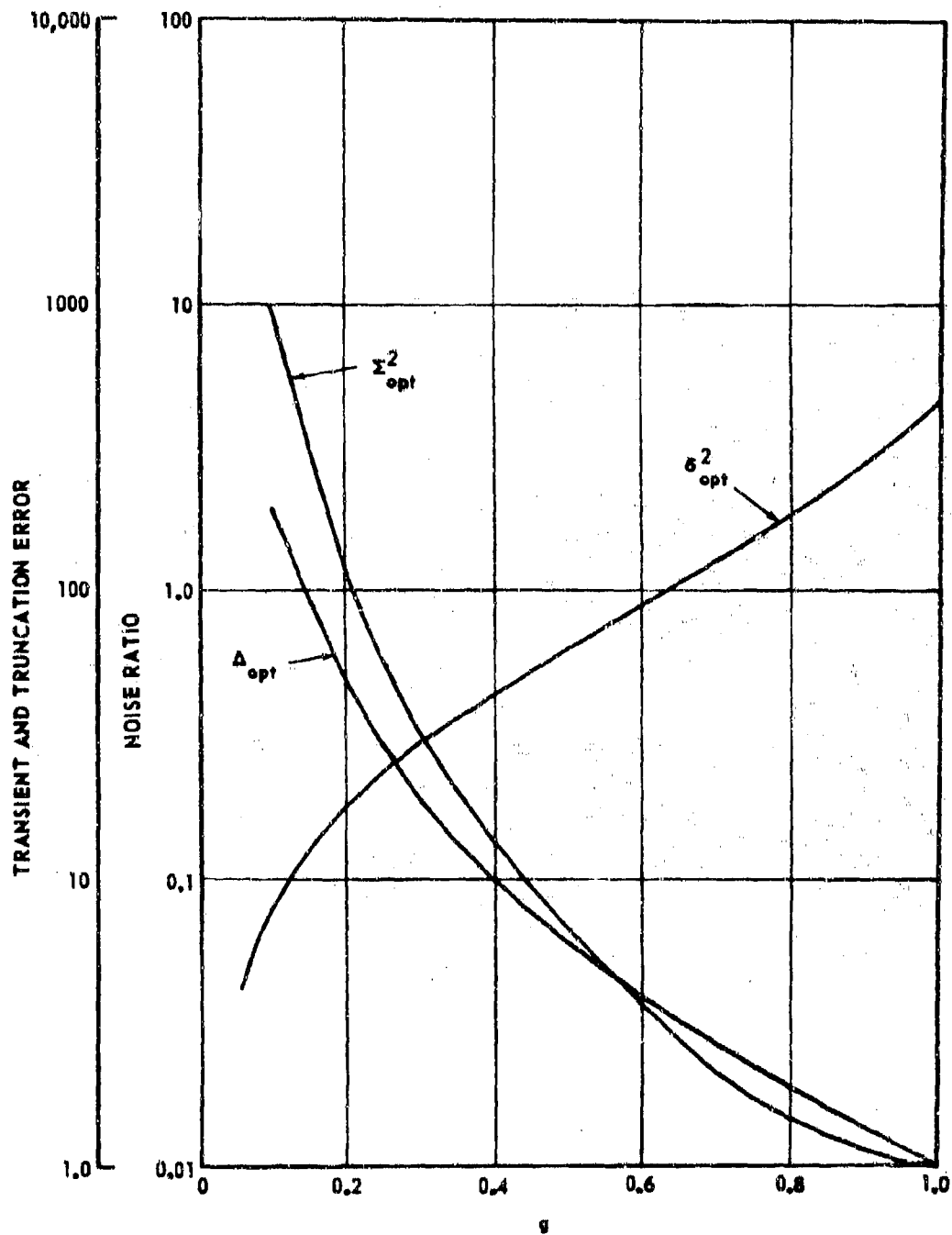
FIGURE 5. NOISE RATIO TRANSIENT ERROR AND TRUNCATION ERROR
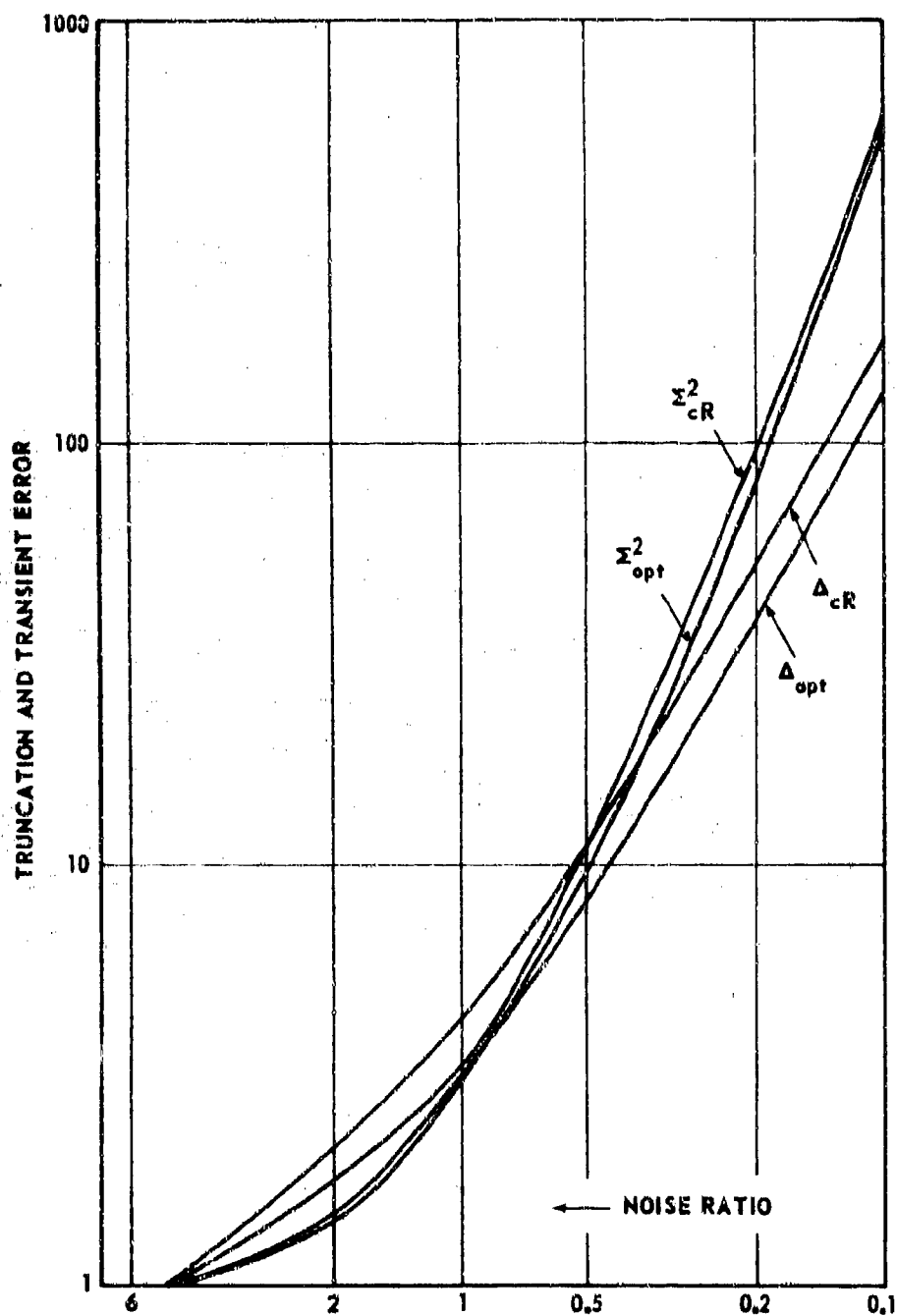FOR THE OPTIMUM FILTER

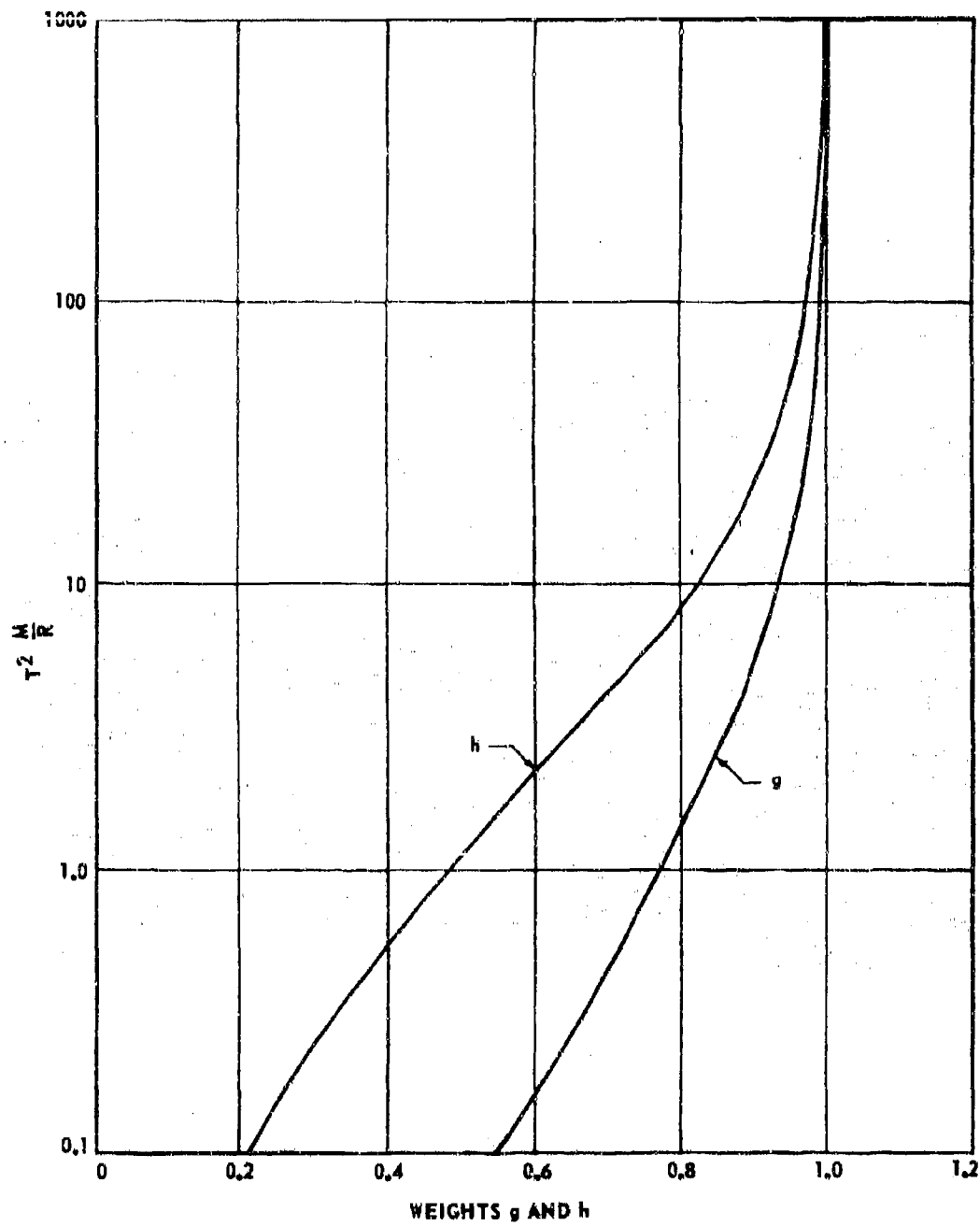FIGURE 6. TRANSIENT AND TRUNCATION ERRORS AS A
FUNCTION OF NOISE RATIO

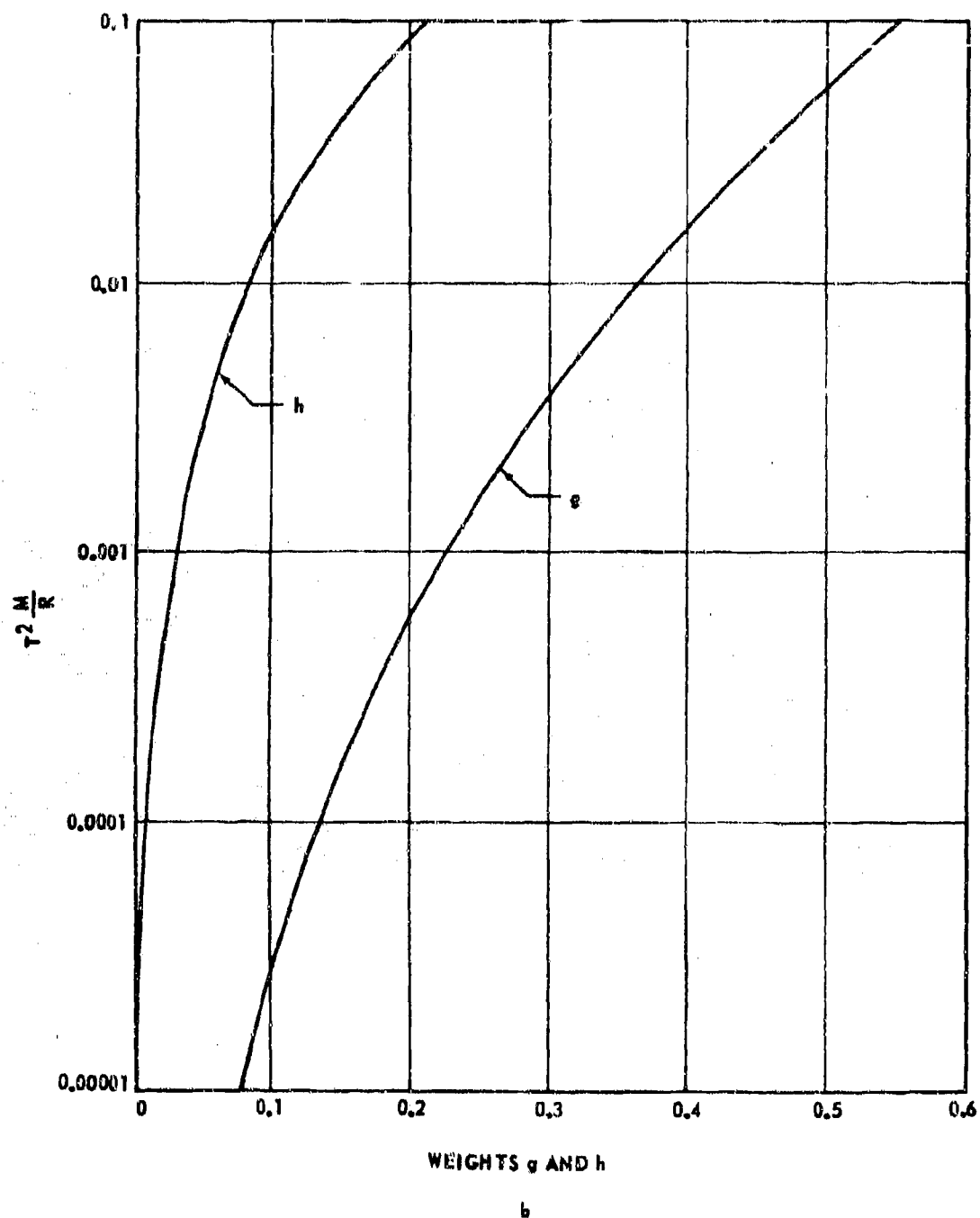FIGURE 7. g AND h AS A FUNCTION OF $T^2 \frac{M}{R}$

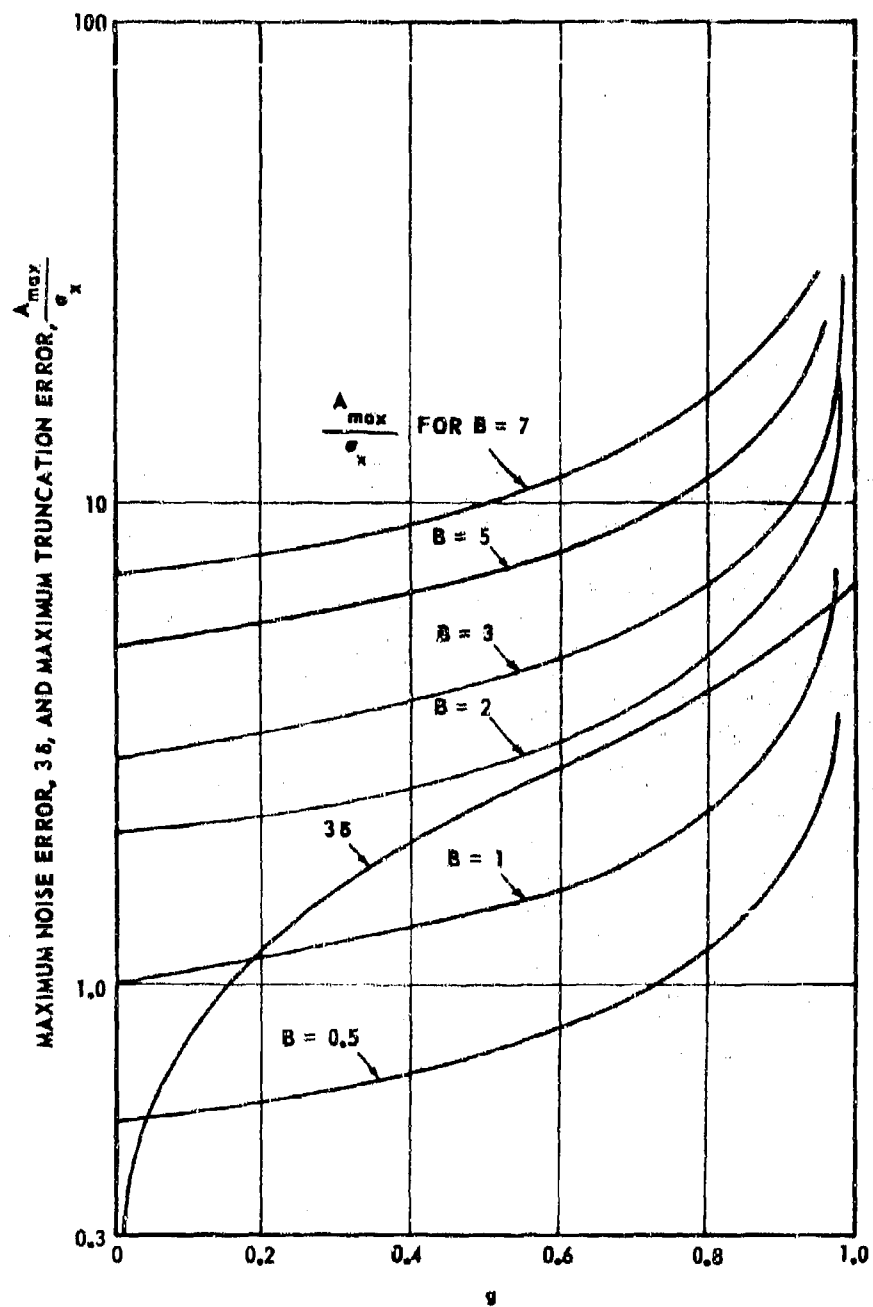FIGURE 7. g AND h AS A FUNCTION OF $T^2 \frac{M}{R}$ (Concluded)

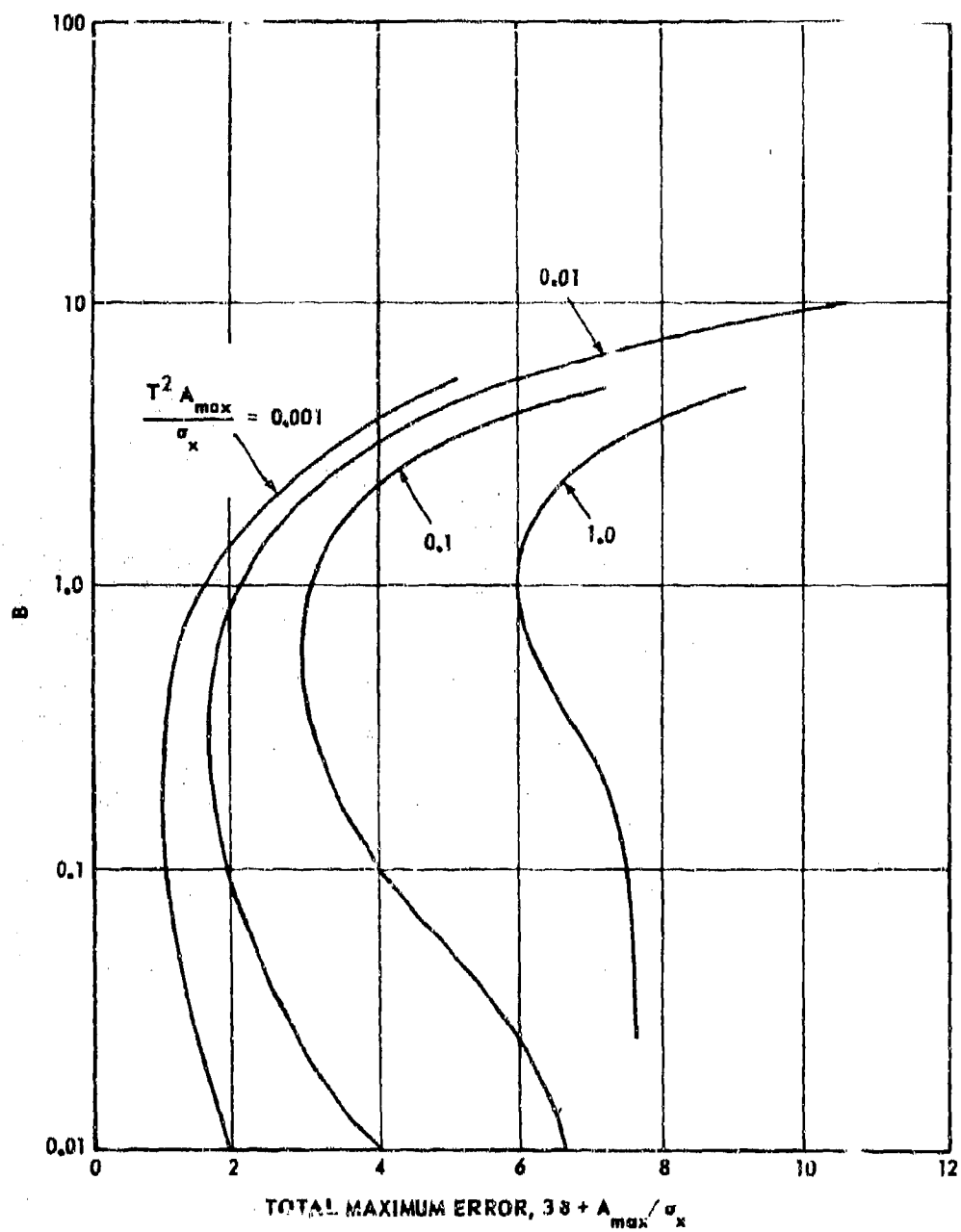FIGURE 8. MAXIMUM NOISE AND TRUNCATION ERRORS WITH B
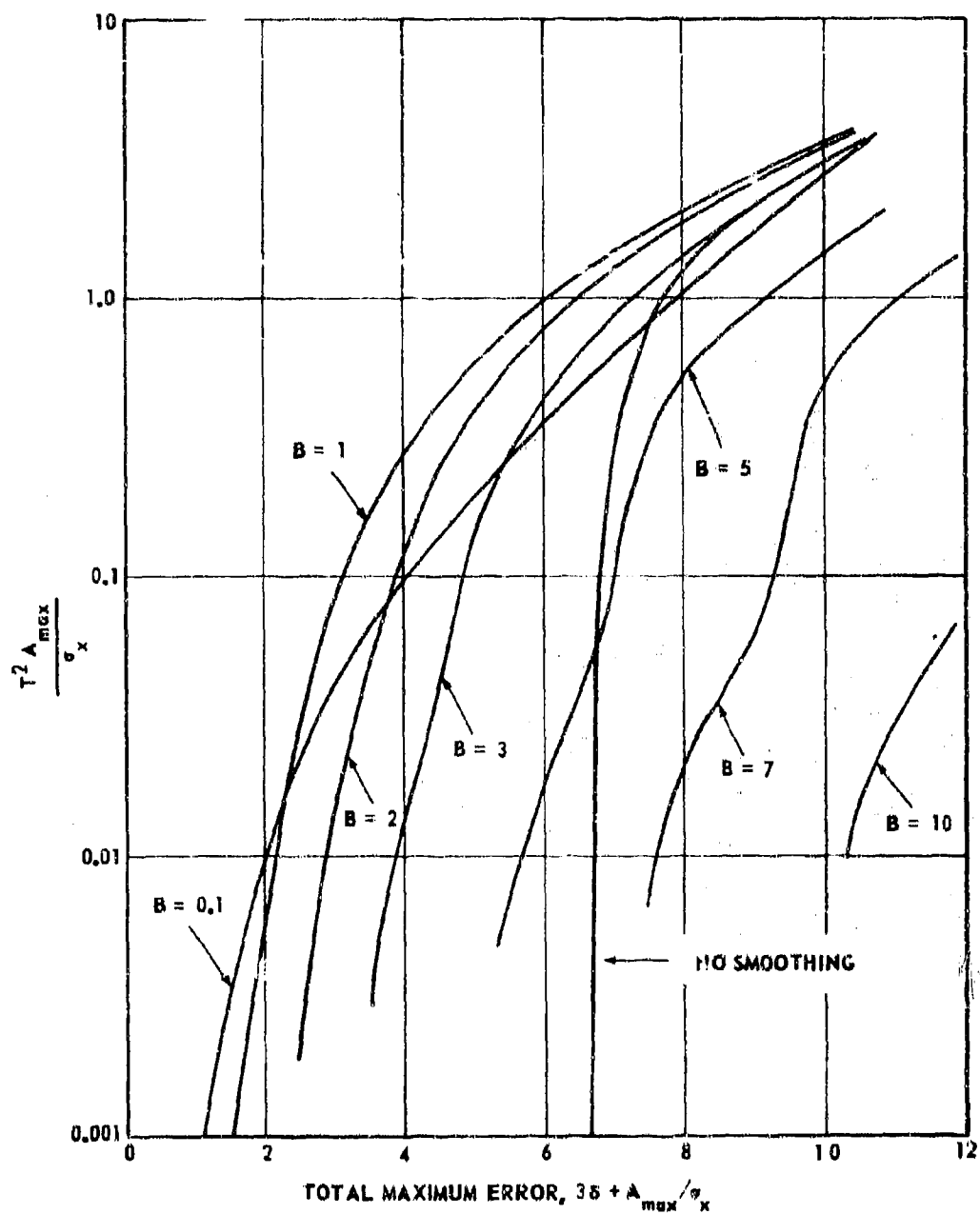AS A PARAMETER

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FIGURE 9. TOTAL MAXIMUM ERROR VERSUS B

FIGURE 10. TOTAL MAXIMUM ERROR VERSUS $T^2 \Lambda_{max}/\sigma_x$
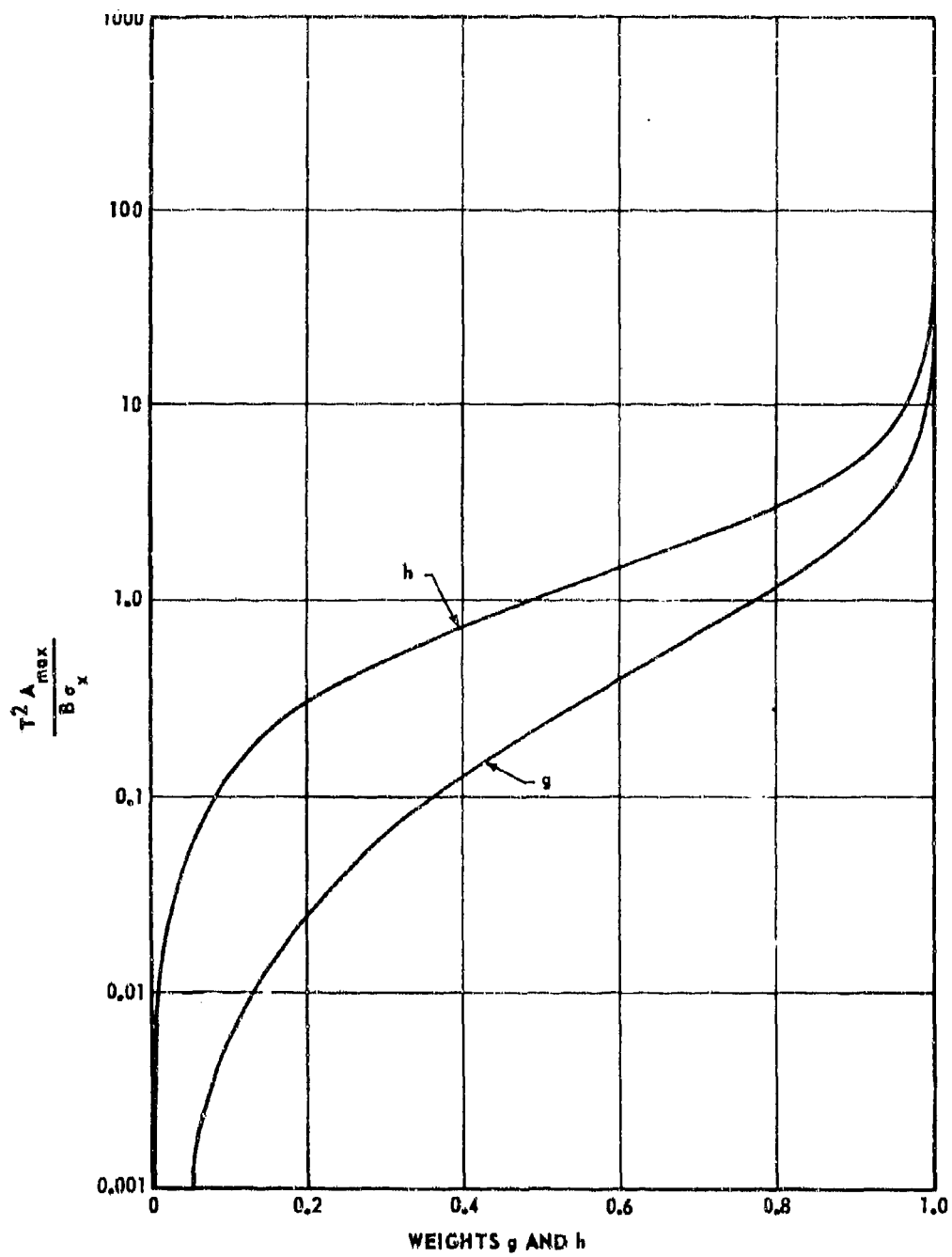
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FIGURE 11. g AND h VERSUS $\dfrac{\pi^2 A_{max}}{B\sigma_x}$

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13. ABSTRACT

This report describes methods for the proper design of first-order, recursive, fixed-weight, linear filters. Expressions are derived and listed for commonly used design parameters such as noise ratio, transient response, and truncation error. The performances of critically damped and steady-state optimum filters are compared. Design curves are given that can be used to select the weights of the steady-state optimum filter from total error requirements.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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