Recurrence Relations and Generating Functions

DEFINITION

A recurrence relation is a functional relation between the independent variable x, d
pendent variable $f(x)$ and the differences of various order of $f(x)$. A recurrence relation is al
called a difference equation and we will use these two terms interchangeably.

For example : The equation $f(x + 3h) + 3f(x + 2h) + 6f(x + h) + 9f(x) = 0$ is a recurrence relation.

It can also be written as

$$a_{r+3} + 3a_{r+2} + 6a_{r+1} + 9a_r = 0$$

or

$$y_{k+3} + 3y_{k+2} + 6y_{k+1} + 9y_k = 0.$$

For example : The Fibonacci sequence is defined by the recurrence relation $a_r = a_{r-2}$
a_{r-1}, $r \geq 2$, with the initial conditions $a_0 = 1$ and $a_1 = 1$.

ORDER OF THE RECURRENCE RELATION

The order of the recurrence relation or difference equation is defined to be the differ
ence between the highest and lowest subscripts of $f(x)$ or a_r or y_k.

For example : The equation $13a_r + 20a_{r-1} = 0$ is a first order recurrence relation.

For example : The equation $8f(x) + 4f(x + 1) + 8f(x + 2) = k(x)$ is a second order difference equation.

DEGREE OF THE DIFFERENCE EQUATION

The degree of a difference equation is defined to be the highest power of $f(x)$ or a_r or y_k.

For example : The equation $y_{k+3}^3 + 2y_{k+2}^2 + 2y_{k+1} = 0$ has the degree 3, as the highest power of y_k is 3.

For example : The equation $a_r^4 + 3a_{r-1}^3 + 6a_{r-2}^2 + 4a_{r-3} = 0$ has the degree 4, as the highest power of a_r is 4.

For example : The equation $y_{k+3} + 2y_{k+2} + 4y_{k+1} + 2y_k = k(x)$ has the degree 1, because the highest power of y_k is 1 and its order is 3.

For example : The equation $f(x + 2h) - 4f(x + h) + 2f(x) = 0$ has the degree 1 and its order is 2.

LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation is called linear if its degree is one.

The general form of linear recurrence relation with constant coefficient is

$$C_0 y_{n+r} + C_1 y_{n+r-1} + C_2 y_{n+r-2} + \ldots + C_r y_n = R(n)$$

where $C_0, C_1, C_2, \ldots, C_n$ are constants and $R(n)$ is same function of independent variable n.

A solution of a recurrence relation in any function which satisfies the given equation.

LINEAR HOMOGENEOUS RECURRENCE RELATION WITH CONSTANT COEFFICIENTS

The equation is said to be linear homogeneous difference equation if and only if $R(n) = 0$ and it will be of order n.

The equation is said to be linear non-homogeneous difference equation if $R(n) \neq 0$.

For example : The equation $a_{r+3} + 6a_{r+2} + 12a_{r+1} + 8a_r = 0$ is a linear homogeneous equation of order 3.

For example : The equation $a_{r+2} - 4a_{r+1} + 4a_r = 3r + 2^r$ is a linear non-homogeneous equation of order 2.

A linear homogeneous difference equation with constant coefficients is given by

$$C_0 y_n + C_1 y_{n-1} + C_2 y_{n-2} + \ldots + C_r y_{n-r} = 0 \qquad \ldots(i)$$

where $C_0, C_1, C_2, \ldots, C_r$ are constants.

The solution of the equation (i) is of the form $A\alpha_1^K$, where α_1 is the characteristic root and A is constant.

Substitute the values of $A\alpha^K$ for y_n in equation (i), we have

$$C_0 A\alpha^K + C_1 A\alpha^{K-1} + C_2 A\alpha^{K-2} + \ldots + C_r A\alpha^{K-r} = 0 \qquad \ldots(ii)$$

After simplifying equation (ii), we have

$$C_0 \alpha^r + C_1 \alpha^{r-1} + C_2 \alpha^{r-2} + \ldots + C_r = 0 \qquad \ldots(iii)$$

The equation (iii) is called the characteristic equation of the difference equation.

If α_1 is one of the roots of the characteristic equation, then $A\alpha_1^K$ is a homogeneous solution to the difference equation.

To find the solution of the linear homogeneous difference equations, we have the four cases, that are discussed as follows.

Case I. If the characteristic equation has n distinct real roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

Thus, $\alpha_1^K, \alpha_2^K, \ldots\ldots, \alpha_n^K$ are all solutions of equation (i).

Also, we have, $A_1 \alpha_1^K, A_2 \alpha_1^K, \ldots\ldots, A_n \alpha_n^K$ are all solutions of equation (i). The sums of solutions are also solutions.

Hence, the homogeneous solution of the difference equation is

$$y_K = A_1\alpha_1^K + A_2\alpha_2^K + \ldots + A_n \alpha_n^K.$$

Case II. If the characteristic equation has repeated real roots.

If $\alpha_1 = \alpha_2$, then $(A_1 + A_2 K)\alpha_1^K$ is also a solution.

If $\alpha_1 = \alpha_2 = \alpha_3$, then $(A_1 + A_2 K + A_3 K^2)\alpha_1^K$ is also a solution.

Similarly, if root α_1 is repeated n times, then

$$(A_1 + A_2 K + A_3 K^2 + \ldots + A_n K^{n-1})\alpha_1^K$$

is the solution of the homogeneous equation.

y_k A_1, $+A_2$ A_1

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Case III. If the characteristic equation has one imaginary root.

If $\alpha + i\beta$ is the root of the characteristic equation, then $\alpha - i\beta$ is also the root, where α and β are real.

Thus, $(\alpha + i\beta)^K$ and $(\alpha - i\beta)^K$ are solutions of the equations. This implies

$$(\alpha + i\beta)^K A_1 + (\alpha - i\beta)^K A_2$$

is also a solution of the characteristic equation, where A_1 and A_2 are constants which are to be determined.

Case IV. If the characteristic equation has repeated imaginary roots.

When the characteristic equation has repeated imaginary roots,

$$(C_1 + C_2 K)(\alpha + i\beta)^K + (C_3 + C_4 K)(\alpha - i\beta)^K$$

is the solution of the homogeneous equation.

Example 1. *Solve the difference equation* $a_r - 3a_{r-1} + 2a_{r-2} = 0$.

Sol. The characteristic equation is given by

$$s^2 - 3s + 2 = 0 \quad \text{or} \quad (s-1)(s-2) = 0$$

$r-2 = s^0 = 1$

$r-1 = s^1$

$\Rightarrow \quad s = 1, 2$

Therefore, the homogeneous solution of the equation is

$$a_r = C_1 + C_2 \cdot 2^r.$$

Example 2. *Solve the difference equation* $a_r - 6a_{r-1} + 8a_{r-2} = 0$.

Sol. The characteristic equation is

$$s^2 - 6s + 8 = 0 \quad \text{or} \quad (s-2)(s-4) = 0$$

$\Rightarrow \quad s = 2, 4$

Therefore, the homogeneous solution of the equation is

$$a_r = C_1 \cdot 2^r + C_2 \cdot 4^r.$$

Example 3. *Solve the difference equation* $9y_{K+2} - 6y_{K+1} + y_K = 0$.

Sol. The characteristic equation is

$$9s^2 - 6s + 1 = 0 \quad \text{or} \quad (3s-1)^2 = 0$$

$\Rightarrow \quad s = \frac{1}{3}$ and $\frac{1}{3}$

Therefore, the homogeneous solution is given by

$$y_K = (C_1 + C_2 K) \cdot \left(\frac{1}{3}\right)^K.$$

Example 4. *Solve the difference equation* $a_r - 4a_{r-1} + 4a_{r-2} = 0$.

Sol. The characteristic equation is given by

$$s^2 - 4s + 4 = 0 \quad \text{or} \quad (s-2)^2 = 0$$

$\Rightarrow \quad s = 2$ and 2

Therefore, the homogeneous solution of equation is $a_r = (C_1 + C_2 r) \cdot 2^r$.

Example 5. *Solve the difference equation* $a_r + a_{r-1} + a_{r-2} = 0$.

Sol. The characteristic equation is $s^2 + s + 1 = 0$

The roots of this characteristic equation are imaginary, *i.e.*, $s = \frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$.

Therefore, the homogeneous solution of the equation is

$$a_r = \left[\frac{-1+i\sqrt{3}}{2}\right]^r C_1 + \left[\frac{-1-i\sqrt{3}}{2}\right]^r C_2.$$

$1-4.1.1$

$1-4 = \sqrt{3}.$

$\sqrt{3}i$

Example 6. *Solve the difference equation* $y_K - y_{K-1} - y_{K-2} = 0$.

Sol. The characteristic equation is $s^2 - s - 1 = 0$

$$s = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Therefore, the homogeneous solution of the equation is

$$y_K = C_1 \left[\frac{1+\sqrt{5}}{2}\right]^K + C_2 \left[\frac{1-\sqrt{5}}{2}\right]^K.$$

Example 7. *Solve the difference equation* $a_{r+4} + 2a_{r+3} + 3a_{r+2} + 2a_{r+1} + a_r = 0$.

Sol. The characteristic equation is $s^4 + 2s^3 + 3s^2 + 2s + 1 = 0$

or $$(s^2 + s + 1)(s^2 + s + 1) = 0$$

or $$s = \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore, the homogeneous solution is given by

$$a_r = (C_1 + C_2 r)\left(\frac{-1+i\sqrt{3}}{2}\right)^r + (C_3 + C_4 r)\left(\frac{-1-i\sqrt{3}}{2}\right)^r.$$

Example 8. *Solve the difference equation* $y_{K+4} + 4y_{K+3} + 8y_{K+2} + 8y_{K+1} + 4y_K = 0$.

Sol. The characteristic equation is $s^4 + 4s^3 + 8s^2 + 8s + 4 = 0$

or $$(s^2 + 2s + 2)(s^2 + 2s + 2) = 0$$

or $$s = -1 \pm i, -1 \pm i$$

Therefore, the homogeneous solution is given by

$$y_K = (C_1 + C_2 K)(-1+i)^K + (C_3 + C_4 K)(-1-i)^K.$$

PARTICULAR SOLUTION

(*a*) Homogeneous Linear Difference Equations

We can find the particular solution of the difference equation, when the equation is of homogeneous linear type by putting the values of the initial conditions in the homogeneous solution.

Example 9. *Solve the difference equation* $2a_r - 5a_{r-1} + 2a_{r-2} = 0$ *and find particular solution such that* $a_0 = 0$ *and* $a_1 = 1$.

Sol. The characteristic equation is $2s^2 - 5s + 2 = 0$

or $$(2s - 1)(s - 2) = 0$$

or $$s = \frac{1}{2} \text{ and } 2.$$

Therefore, the homogeneous solution is given by

$$a_{r(h)} = C_1 \left(\frac{1}{2}\right)^r + C_2 \cdot 2^r \quad \ldots(i)$$

Putting $r = 0$ and $r = 1$ in equation (*i*), we get

$$a_0 = C_1 + C_2 = 0 \quad \ldots(a)$$

$$a_1 = \frac{1}{2} C_1 + 2C_2 = 1 \quad \ldots(b)$$

Solving eq. (*a*) and (*b*), we have

$$C_1 = -\frac{2}{3} \text{ and } C_2 = \frac{2}{3}$$

Hence, the particular solution is

$$a_{r(P)} = -\frac{2}{3} \cdot \left(\frac{1}{2}\right)^r + \frac{2}{3} \cdot (2)^r.$$

Example 10. *Solve the difference equation $a_r - 4a_{r-1} + 4a_{r-2} = 0$ and find the particular solution, given that $a_0 = 1$ and $a_1 = 6$.*

Sol. The characteristic equation is

$$s^2 - 4s + 4 = 0 \text{ or } (s-2)^2 = 0$$

or $$s = 2, 2$$

Therefore, the homogeneous solution is given by

$$a_{r(n)} = (C_1 + C_2 r) \cdot 2^r \quad \ldots(i)$$

Putting $r = 0$ and $r = 1$ in equation (*i*), we get

$$a_0 = (C_1 + 0) \cdot 2^0 = 1 \quad \therefore \quad C_1 = 1$$

$$a_1 = (C_1 + C_2) \cdot 2 = 6 \quad \therefore \quad C_1 + C_2 = 3 \Rightarrow C_2 = 2$$

Hence, the particular solution is $a_{r(P)} = (1 + 2r) \cdot 2^r$.

Example 11. *Solve the difference equation $9a_r - 6a_{r-1} + a_{r-2} = 0$ satisfying the conditions $a_0 = 0$ and $a_1 = 2$.*

Sol. The characteristic equation is

$$9s^2 - 6s + 1 = 0 \text{ or } (3s-1)^2 = 0$$

or $$s = \frac{1}{3}, \frac{1}{3}$$

Therefore, the homogeneous solution is given by

$$a_{r(h)} = (C_1 + C_2 r) \cdot \left(\frac{1}{3}\right)^r \quad \ldots(i)$$

Putting $r = 0$ and $r = 1$ in equation (*i*), we get

$$a_0 = C_1 = 0 \quad \ldots(a)$$

$$a_1 = (C_1 + C_2) \cdot \frac{1}{3} = 2. \quad \therefore \quad C_1 + C_2 = 6 \Rightarrow C_2 = 6$$

Hence, the particular solution is

$$a_{r(P)} = 6r \cdot \left(\frac{1}{3}\right)^r.$$

Example 12. *Solve the difference equation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ satisfying the conditions $a_0 = 0$ and $a_1 = 6$.*

Sol. The characteristic equation is

$$s^2 - 7s + 10 = 0 \quad \text{or} \quad (s-5)(s-2) = 0 \quad \text{or } s = 2, 5$$

Therefore, the homogeneous solution is given by

$$a_{r(h)} = C_1 \cdot 2^r + C_2 \cdot 5^r. \qquad ...(i)$$

Putting $r = 0$ and $r = 1$ in equation (*i*), we get

$$a_0 = C_1 + C_2 = 0 \qquad ...(a)$$

$$a_1 = 2C_1 + 5C_2 = 6 \qquad ...(b)$$

From eq. (*a*) $\quad C_1 = -C_2$

From eq. (*b*) $\quad 3C_2 = 6 \quad \therefore \quad C_2 = 2$ and $C_1 = -2$.

Hence, the particular solution is $a_{r(P)} = -2 \cdot 2^r + 2 \cdot 5^r$.

(*b*) **Non-Homogeneous Linear Difference Equations.** There are two methods to find the particular solution of non-homogeneous linear difference equations. These are as follows :

1. Undetermined coefficients method
2. E and Δ operator method.

1. Undetermined Coefficients Method. This method is used to find particular solution of non-homogeneous linear difference equations, whose R.H.S. term R(*n*) consists of terms of special forms.

In this method, firstly we assume the general form of the particular solution according to the form of R(*n*) containing a number of unknown constant coefficients, which have to be determined. Then according to the difference equation, we will determine the exact solution.

The general form of particular solution to be assumed for the special forms of R(*n*), to find the exact solution is shown in the table.

Form of R(*n*)	General form to be assumed
Z, here z is a constant	A
Z^r, here Z is a constant	$A \cdot Z^r$
P(*r*), a polynomial of degree *n*	$A_0 r^n + A_1 r^{n-1} + ... + A_n$
Z^r. P(*r*), here P(*r*) is a polynomial of *n*th degree in *r*. Z is a constant.	$[A_0 r^n + A_1 r^{n-1} + ... + A_n]. Z^r$

Example 13. *Find the particular solution of the difference equation*

$$a_{r+2} - 3a_{r+1} + 2a_r = Z^r, \qquad ...(i)$$

where Z is some constant.

Sol. The general form of solution is = $A \cdot Z^r$

Now putting this solution on L.H.S. of equation (*i*), we get

$$= AZ^{r+2} - 3AZ^{r+1} + 2AZ^r = (Z^2 - 3Z + 2)AZ^r \qquad ...(ii)$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$(Z^2 - 3Z + 2)A = 1$$

or
$$A = \frac{1}{Z^2 - 3Z + 2} = \frac{1}{(Z-1)(Z-2)} \quad (Z \neq 1, Z \neq 2)$$

Therefore, the particular solution is $\dfrac{Z^r}{(Z-1)(Z-2)}$.

Example 14. *Find the particular solution of the difference equation*

$$a_{r+2} - 5a_{r+1} + 6a_r = 5^r.$$

Sol. Let us assume the general form of the solution $= A \,.\, 5^r$. ...(*i*)

Now to find the value of A, put this solution on L.H.S. of the equation (*i*), then this becomes

$$= A.\, 5^{r+2} - 5 \,.\, A5^{r+1} + 6 \,.\, A5^r$$
$$= 25A \,.\, 5^r - 25A \,.\, 5^r + 6A \,.\, 5^r$$
$$= 6A \,.\, 5^r$$ ...(*ii*)

Equating equation (*ii*) to R.H.S. of equation (*i*), we get

$$A = \frac{1}{6}$$

Therefore, the particular solution of the difference equation is $= \dfrac{1}{6} \,.\, 5^r$.

Example 15. *Find the particular solution of the difference equation*

$$a_{r+2} - 4a_r = r^2 + r - 1.$$ ...(*i*)

Sol. The homogeneous solution of the difference equation is given by

$$a_{r(h)} = C_1(2)^r + C_2(-2)^r$$

To find the particular solution, let us assume the general form of the solution is

$$= A_1 r^2 + A_2 r + A_3.$$

Putting this solution in L.H.S. of equation (*i*), we get

$$= A_1 (r+2)^2 + A_2 (r+2) + A_3 - 4A_1 r^2 - 4A_2 r - 4A_3$$
$$= -3A_1 r^2 + (4A_1 - 3A_2)r + (4A_1 + 2A_2 - 3A_3)$$ ...(*ii*)

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$-3A_1 = 1$$
$$4A_1 - 3A_2 = 1$$
$$4A_1 + 2A_2 - 3A_3 = -1$$

After solving these three equations, we get

$$A_1 = -\frac{1}{3},\ A_2 = -\frac{7}{9},\ A_3 = -\frac{17}{27}$$

Therefore, the particular solution is $= -\dfrac{r^2}{3} - \dfrac{7}{9}r - \dfrac{17}{27}$.

$S^2 - 2S + 1 = 0$
$S^2 - S - S + 1 = 0$
$S(S-1) - 1(S-1)$
$(S-1)(S-1)$

Example 16. *Find the particular solution of the difference equation*

$$a_{r+2} - 2a_{r+1} + a_r = 3r + 5.$$ ...(*i*)

Sol. The homogeneous solution of the difference equation is given by

$$a_{r(h)} = C_1 + C_2 r$$ ...(*ii*)