Th.→ Any two right (left) cosets of a subgroup are either disjoint or idential.

Proof→. Suppose H is a subgroup of a group G and let Ha and Hb be two right cosets of H in G. Suppose Ha and Hb are not disjoint. Then there exists at least one element, say, c such that

$c \in Ha$ and $c \in Hb$. Let $c = h_1 a$ and $c = h_2 b$ where $h_1, h_2 \in H$.

Then $\quad h_1 a = h_2 b$

or $\quad h_1^{-1} h_1 a = h_1^{-1} h_2 b$

or $\quad e a = (h_1^{-1} h_2) b$

or $\quad a = (h_1^{-1} h_2) b$

Since H is a subgroup, therefore $h_1^{-1} h_2 \in H$.
Let $h_1^{-1} h_2 = h_3$. Then $a = h_3 b$.

Now, $\quad Ha = H h_3 b = (H h_3) b$

$\quad = Hb \qquad [\because h_3 \in H \Rightarrow H h_3 = H]$

Therefore the two right (left) cosets are identical if they are not disjoint. Thus either $Ha \cap Hb = \phi$ or $Ha = Hb$.

* Similarly, we can prove that either $aH \cap bH = \phi$ or $aH = bH$.

Th.→. If H is a subgroup of a group G, then G is equal to the union of all right cosets of H in G, i.e.,

$G = H \cup Ha \cup Hb \cup Hc \cdots$, where $a, b, c, \ldots$, are elements of G.

Proof→ G is a group. Therefore each element of any right coset of H in G is an element of G. Hence the union of all right cosets of H in G is a subset of G.

Also, if x is any element of G, then $x \in Hx$. Therefore x belongs to the union of all right cosets of H in G. Hence G is a subset of the union of all right cosets of H in G.

Therefore, G is equal to the union of all right cosets of H in G. Symbolically, we have $G = \bigcup_{x \in G} Hx$.

Similarly, we can prove that G is also equal to the union of all left cosets of H in G.

Th. → If H is a subgroup of G, there is a one-to-one correspondence between any two right cosets of H in G.

Proof. – Let $a, b \in G$. Then Ha and Hb are any two right cosets of H in G.

Let $f: Ha \to Hb$ be defined by

$$f(ha) = hb \ \forall h \in H.$$

The function f is one-one → If $h_1, h_2 \in H$, then $h_1 a, h_2 a \in Ha$. Also, by definition of f, we have $f(h_1 a) = h_1 b$ and $f(h_2 a) = h_2 b$.

Now, $f(h_1 a) = f(h_2 a)$

$\Rightarrow h_1 b = h_2 b$

$\Rightarrow h_1 = h_2$ [By right cancellation Law in G]

$\Rightarrow h_1 a = h_2 a$.

$\therefore$ f is one-one since only equal elements of Ha can have the same image in Hb.

The function f is onto → Let $h'b$ be any ~~element~~ arbitrary element of Hb. Then

$h'b \in Hb \Rightarrow h' \in H \Rightarrow h'a \in Ha$.

Now $f(h'a) = h'b$, by definition of f.

Thus $h'b \in Hb$

$\Rightarrow$ that there exists $h'a \in Ha$ such that $f(h'a) = h'b$. Therefore f is onto Hb.

Hence the result.

Similarly, there is a one-for one correspondence between any two left cosets of H in G.

Lagrange's Theorem →

The order of each subgroup of a finite group is a divisor of the order of the group.

Proof → Let G be a group of finite order n. Let H be a subgroup of G and let $o(H) = m$. Suppose $h_1, h_2, \ldots, h_m$ are the m members of H.

Let $a \in G$.

Then Ha is a right coset of H in G and we have —

$$Ha = \{h_1 a, h_2 a, \ldots, h_m a\}$$

Ha has m distinct members, since

$$h_i a = h_j a$$

$$\Rightarrow h_i = h_j$$

Therefore, each right coset of H in G has m distinct members.

Any two distinct right cosets of H in G are disjoint i.e., they have no element in common.

Since G is a finite group, the number of distinct right cosets of H in G will be finite, say, equal to k.

The union of these k distinct right cosets of H in G is equal to G.

Thus if $Ha_1, Ha_2, \ldots, Ha_k$ are the k-distinct right cosets of H in G, then

$$G = Ha_1 \cup Ha_2 \cup \ldots \cup Ha_k$$

$\Rightarrow$ the number of elements in G = no. of elements in Ha_1 + no. of elements in Ha_2 + $\ldots$ + no. of elements in Ha_k

[$\because$ two distinct right cosets are mutually disjoint]

$\Rightarrow o(G) = km$

$\Rightarrow n = km$

$\Rightarrow k = \frac{n}{m} \Rightarrow m$ is a divisor of n

$\Rightarrow o(H) =$ is a divisor of $o(G)$.

Hence the theorem.