CHAPTER 4
Power Series and Contour Integration

4.1. SERIES OF COMPLEX TERMS

A series of the form $(a_1 + ib_1) + (a_2 + ib_2) + ... + (a_n + ib_n) + ...$
where $a_1, b_1, a_2, b_2, ...$ are real numbers, is called a series of complex terms and can be expressed as

$$\sum_{n=1}^{\infty} a_n + i \sum_{n=1}^{\infty} b_n \qquad ...(1)$$

The series (1) is convergent if Σa_n and Σb_n both are convergent.

If the series Σa_n and Σb_n converge to a and b respectively, then the series (1) is said to converge to $a + ib$.

The necessary (not sufficient) condition for convergence of series (1) is $\lim_{n \to \infty} (a_n + ib_n) = 0$.

The series (1) is said to be **absolutely convergent** if the series
$| a_1 + ib_1 | + | a_2 + ib_2 | + ... + | a_n + ib_n | + ...$ is convergent.

Since $| a_n | \le | a_n + ib_n |$ and $| b_n | \le | a_n + ib_n |$, we conclude that an absolutely convergent series is convergent. The converse is not true.

If the series of functions $u_1(z) + u_2(z) + ... + u_n(z) + ... = \sum_{n=1}^{\infty} u_n(z)$ ...(2)

converges to $S(z)$ and $S_n(z)$ is its nth partial sum, then the series (2) is said to be **uniformly convergent** in a region R, if given any positive number ε, there exists a positive number N, depending on ε but not on z, such that for every z in R,

$$| S(z) - S_n(z) | < \varepsilon \text{ for all } n > N.$$

A series of the form $\quad a_0 + a_1(z - a) + a_2(z - a)^2 + ... + a_n(z - a)^n + ...$

$$= \sum_{n=0}^{\infty} a_n (z - a)^n \qquad ...(3)$$

is called a **power series** in $(z - a)$.

If the series (3) is convergent for all values of z satisfying $| z - a | < R$, *i.e.*, for all values of z lying inside the circle with centre at $z = a$ and radius R, then this circle is called the **circle of convergence** and R is called the **radius of convergence.**

The power series (3) is uniformly convergent in a region R if there exists a series of positive terms ΣM_n such that $| a_n(z - a)^n | \le M_n$ for all values of z in the region R.

This is called **Wierstrass M-test** for uniform convergence of a power series.

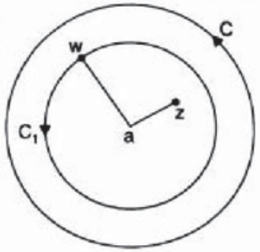
4.2. TAYLOR'S SERIES

If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^n}{n!} f^n(a) + \ldots$$

(P.T.U. 2006)

Proof. Let z be any point inside the circle C. Draw a circle C_1 with centre at a and radius smaller than that of C such that z is an interior point of C_1. Let w be any point on C_1, then



$$|z-a| < |w-a| \quad i.e., \left|\frac{z-a}{w-a}\right| < 1$$

Now, $\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$

Expanding R.H.S. by binomial theorem as $\left|\frac{z-a}{w-a}\right| < 1$, we get

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \ldots + \left(\frac{z-a}{w-a}\right)^n + \ldots\right] \quad \ldots(1)$$

This series converges uniformly since $\left|\frac{z-a}{w-a}\right| < 1$. Multiplying both sides of (1) by $\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, over C_1, we get

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a} dw + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2} dw + \frac{(z-a)^2}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^3} dw +$$

$$\ldots\ldots + \frac{(z-a)^n}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw + \ldots \quad \ldots(2)$$

Since $f(w)$ is analytic on and inside C_1, by Cauchy's formula,

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z} dw = f(z)$$

Also $\quad f^n(a) = \frac{n!}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \quad i.e., \quad \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^n(a)}{n!}$

$\therefore$ From (2), we have

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^n}{n!} f^n(a) + \ldots \quad \ldots(3)$$

which is the required *Taylor's series* for $f(z)$ about $z = a$.

Cor. 1. Putting $z = a + h$ in (3), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^n}{n!} f^n(a) + \ldots$$

Cor. 2. If $a = 0$, the series (3) becomes

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \ldots\ldots + \frac{z^n}{n!} f^n(0) + \ldots$$

This series is called **Maclaurin's series.**

4.3. LAURENT'S SERIES

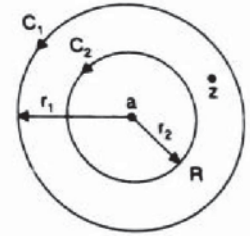
If $f(z)$ is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively having centre at a, then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

where $\quad a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw;\ n = 0, 1, 2, \ldots$

and $\quad a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw;\ n = 1, 2, 3, \ldots$ (U.P.T.U. 2008)

Proof. Let z be any point in the region R, then by Cauchy's integral formula for double connected region, we have



$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z} dw \quad \ldots(1)$$

For the first integral in (1), w lies on C_2

$\therefore \quad |z-a| < |w-a| \quad i.e., \left|\frac{z-a}{w-a}\right| < 1$

Now $\quad \frac{1}{w-a} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left(1 - \frac{z-a}{w-a}\right)^{-1}$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \ldots\ldots\right]$$

Multiplying both sides by $\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, along the circle C_1, we get

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a} dw + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2} dw + \frac{(z-a)^2}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^3} dw + \ldots\ldots$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \quad \ldots(2) \quad \left[\because\ a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n = 0, 1, 2, \ldots\ldots\right]$$

For the second integral in (1), w lies on C_2

$\therefore \quad |w-a| < |z-a| \quad i.e., \left|\frac{w-a}{z-a}\right| < 1$

Now $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = -\frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\ldots\ldots\right]$$

Multiplying both sides by $\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, along the circle C_2, we get

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\oint_{C_2} f(w)\,dw + \frac{1}{(z-a)^2}\cdot\frac{1}{2\pi i}\oint_{C_2}(w-a)\,f(w)\,dw$$

$$+ \frac{1}{(z-a)^3}\cdot\frac{1}{2\pi i}\oint_{C_2}(w-a)^2 f(w)\,dw + \ldots\ldots$$

$$= a_{-1}(z-a)^{-1} + a_{-2}(z-c)^{-2} + a_{-3}(z-a)^{-3} + \ldots\ldots$$

$$= \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \quad \ldots(3) \quad \left[\because\ a_{-n} = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw,\ n = 1, 2, 3, \ldots\ldots\right]$$

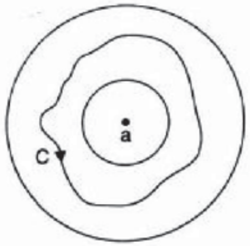
Substituting from (2) and (3) in (1), we get

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

Note 1. In the statement of Laurent's series, $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw \neq \frac{f^n(a)}{n!}$ because $f(z)$ is not given to be analytic inside C_1.

Note 2. In case $f(z)$ is analytic inside C_1, then $a_{-n} = 0$ and $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw = \frac{f^n(a)}{n!}$ and Laurent's series reduces to Taylor's series.

Note 3. If C is any simple closed curve which lies in the ring-shaped region R and encloses the circle C_1, then



$$\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw = \oint_{C}\frac{f(w)}{(w-a)^{n+1}}\,dw$$

and $$\oint_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw = \oint_{C}\frac{f(w)}{(w-a)^{-n+1}}\,dw$$

$\therefore$ Laurent's series can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n, \quad \text{where } a_n = \frac{1}{2\pi i}\oint_{C}\frac{f(w)}{(w-a)^{n+1}}\,dw.$$

Note 4. The process of finding the coefficient a_n by complex integration is complicated. In practice, we expand the function $f(z)$ by binomial theorem or by some other method to obtain Taylor's or Laurent's series.

ILLUSTRATIVE EXAMPLES

Example 1. *Expand* $\frac{1}{z^2-3z+2}$ *in the region*

(a) $|z| < 1$ (V.T.U. 2006; Kerala 2005)

(b) $1 < |z| < 2$

(c) $|z| > 2$

(d) $0 < |z-1| < 1$. (U.P.T.U. 2006, 2008)

Sol. Here $$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \quad \text{(Partial Fractions)}$$

(a) If $|z| < 1$, then $\left|\frac{z}{2}\right| < 1$

$$\therefore \quad f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{1-z}$$

[Arranged suitably to make the binomial expansions valid]

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\ldots\right) + (1+z+z^2+z^3+\ldots\ldots)$$

(By binomial theorem)

$$= \frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3+\ldots\ldots, \text{ which is a Taylor's series.}$$

(b) If $1 < |z| < 2$, then $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$\therefore \quad f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\ldots\right)$$

(By binomial theorem)

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\ldots\right) - (z^{-1}+z^{-2}+z^{-3}+\ldots\ldots)$$

which is a Laurent's series.

(c) If $|z| > 2$, then $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$

$$\therefore \quad f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\ldots\right)$$

(By binomial theorem)

$= z^{-2} + 3z^{-3} + 7z^{-4} + \ldots\ldots$ which is a Laurent's series.

(*d*) For $0 < |z-1| < 1$, we have

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1} = -\frac{1}{1-(z-1)} - \frac{1}{z-1} = -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots\ldots].$$

Example 2. *Find the series expansion of* $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ *about* $z = 0$ *in the region*

(*i*) $|z| < 2$ (*ii*) $2 < |z| < 3$.

(M.D.U. May 2007, Dec. 2007)

Sol. Here $f(z) = \dfrac{z^2-1}{z^2+5z+6} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$ (Partial Fractions)

(*i*) When $|z| < 2$, we have $\left|\dfrac{z}{2}\right| < 1$ and hence $\left|\dfrac{z}{3}\right| < 1$.

$$\therefore \quad f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\ldots\right] - \frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\right]$$

(By binomial theorem)

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

which is a Taylor's series.

(*ii*) When $2 < |z| < 3$, we have $\left|\dfrac{2}{z}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$.

$$\therefore \quad f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\ldots\right] - \frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

which is a Laurent's series.

Example 3. *Expand* $\dfrac{1}{(z^2+1)(z^2+2)}$ *as a Laurent's series valid for*

(*i*) $0 < |z| < 1$ (*ii*) $1 < |z| < \sqrt{2}$ (*iii*) $|z| > \sqrt{2}$.

(M.D.U. Dec. 2010; P.T.U. 2005)

Sol. Here $f(z) = \dfrac{1}{(z^2+1)(z^2+2)} = \dfrac{1}{z^2+1} - \dfrac{1}{z^2+2}$ (Partial Fractions)

(*i*) When $0 < |z| < 1$, we have $z^2 < 1$ and hence $z^2 < 2$ so that $\dfrac{z^2}{2} < 1$.

$$\therefore \quad f(z) = \frac{1}{1+z^2} - \frac{1}{2\left(1+\frac{z^2}{2}\right)} = (1+z^2)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)^{-1}$$

$$= (1 - z^2 + z^4 - z^6 + z^8 - \ldots) - \frac{1}{2}\left[1-\frac{z^2}{2}+\left(\frac{z^2}{2}\right)^2-\left(\frac{z^2}{2}\right)^3+\ldots\right]$$

$$= \left(1-\frac{1}{2}\right) - \left(1-\frac{1}{2^2}\right)z^2 + \left(1-\frac{1}{2^3}\right)z^4 - \left(1-\frac{1}{2^4}\right)z^6 + \ldots$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left(1-\frac{1}{2^n}\right)z^{2n-2}$$

(*ii*) When $1 < |z| < \sqrt{2}$, we have $z^2 > 1$ and $z^2 < 2$ so that $\dfrac{1}{z^2} < 1$ and $\dfrac{z^2}{2} < 1$.

$$\therefore \quad f(z) = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{2\left(1+\frac{z^2}{2}\right)} = \frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)^{-1}$$

$$= \frac{1}{z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\ldots\right] - \frac{1}{2}\left[1-\frac{z^2}{2}+\left(\frac{z^2}{2}\right)^2-\left(\frac{z^2}{2}\right)^3+\ldots\right]$$

$$= \left[\frac{1}{z^2}-\left(\frac{1}{z^2}\right)^2+\left(\frac{1}{z^2}\right)^3-\left(\frac{1}{z^2}\right)^4+\ldots\right] - \frac{1}{2} + \left[\frac{z^2}{2^2}-\frac{(z^2)^2}{2^3}+\frac{(z^2)^3}{2^4}-\ldots\right]$$

$$= -\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{z^2}\right)^n + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(z^2)^n}{2^{n+1}}$$

$$= -\frac{1}{2} - \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z^{2n}} + \frac{z^{2n}}{2^{n+1}}\right]$$

(*iii*) When $|z| > \sqrt{2}$, we have $z^2 > 2$ and hence $z^2 > 1$ so that $\frac{1}{z^2} < 1$ and $\frac{2}{z^2} < 1$.

$$\therefore \quad f(z) = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{z^2\left(1+\frac{2}{z^2}\right)} = \frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{z^2}\left(1+\frac{2}{z^2}\right)^{-1}$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \ldots\right] - \frac{1}{z^2}\left[1 - \frac{2}{z^2} + \left(\frac{2}{z^2}\right)^2 - \left(\frac{2}{z^2}\right)^3 + \ldots\right]$$

$$= \left(\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8} + \ldots\right) - \left(\frac{1}{z^2} - \frac{2}{z^4} + \frac{2^2}{z^6} - \frac{2^3}{z^8} + \ldots\right)$$

$$= \frac{2-1}{z^4} - \frac{2^2-1}{z^6} + \frac{2^3-1}{z^8} - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n - 1}{(z^2)^{n+1}}.$$

Example 4. *Show that when* $|z+1| < 1$, $z^{-2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$.

(M.D.U. Dec. 2011)

Sol. $$f(z) = z^{-2} = \frac{1}{z^2} = \frac{1}{[(z+1)-1]^2} = \frac{1}{[1-(z+1)]^2} = [1-(z+1)]^{-2}$$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \ldots\ldots$$

[By binomial theorem, since $|z+1| < 1$]

$$= 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n.$$

Example 5. *Expand* $\cos z$ *in a Taylor's series about* $z = \frac{\pi}{4}$.

Sol. Here $f(z) = \cos z$, $f'(z) = -\sin z$, $f''(z) = -\cos z$, $f'''(z) = \sin z$, ...

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad f'''\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ldots\ldots$$

Hence $\cos z = f(z) = f\left[\frac{\pi}{4} + \left(z - \frac{\pi}{4}\right)\right]$

$$= f\left(\frac{\pi}{4}\right) + \left(z-\frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \ldots\ldots$$

$$= \frac{1}{\sqrt{2}}\left[1 - \left(z-\frac{\pi}{4}\right) - \frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3 + \ldots\ldots\right]$$

Example 6. *Expand the function* $\frac{\sin z}{z-\pi}$ *about* $z = \pi$.

Sol. Putting $z - \pi = t$, we have

$$\frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{t} = \frac{-\sin t}{t}$$

$$= -\frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\ldots\ldots\right) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \ldots\ldots = -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \ldots\ldots$$

Example 7. *Expand* $f(z) = \frac{z}{(z+1)(z+2)}$ *about* $z = -2$.

Sol. To expand $f(z)$ about $z = -2$, *i.e.*, in powers of $z + 2$, we put $z + 2 = t$.

$$\therefore \quad f(z) = \frac{z}{(z+1)(z+2)} = \frac{t-2}{(t-1)t} = \frac{2-t}{t(1-t)} = \frac{2-t}{t}(1-t)^{-1}$$

$$= \frac{2-t}{t}(1 + t + t^2 + t^3 + \ldots\ldots) \qquad \text{for } 0 < |t| < 1$$

$$= \frac{1}{t}(2 + t + t^2 + t^3 + \ldots\ldots) = \frac{2}{t} + 1 + t + t^2 + \ldots\ldots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \ldots\ldots \qquad \text{for } 0 < |z+2| < 1$$

which is the required Laurent's series.

Example 8. *Expand* $\frac{e^{2z}}{(z-1)^3}$ *about the singularity* $z = 1$ *in Laurent's series.*

Sol. To expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$, *i.e.*, in powers of $z - 1$, we put $z - 1 = t$ or $z = t + 1$

$$\therefore \quad f(z) = \frac{e^{2(t+1)}}{t^3} = \frac{e^2}{t^3} \cdot e^{2t}$$

$$= \frac{e^2}{t^3}\left[1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \ldots\ldots\right]$$

$$= e^2\left[\frac{1}{t^3} + \frac{2}{t^2} + \frac{2}{t} + \frac{4}{3} + \frac{2}{3}t + \ldots\ldots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{z-1} + \frac{4}{3} + \frac{2}{3}(z-1) + \ldots\ldots\right]$$

$$= e^2\left[(z-1)^{-3} + 2(z-1)^{-2} + 2(z-1)^{-1} + \frac{4}{3} + \frac{2}{3}(z-1) + \ldots\ldots\right].$$

Example 9. *Find Taylor's expansion of*

(*i*) $\dfrac{1}{(z+1)^2}$ *about the point* $z=-i$ (*ii*) $\dfrac{2z^3+1}{z^2+z}$ *about the point* $z=i$

(M.D.U. May 2011)

Sol. (*i*) To expand $f(z)=\dfrac{1}{(z+1)^2}$ about $z=-i$, *i.e.*, in powers of $z+i$, we put $z+i=t$

or $z=t-i$.

$$\therefore \quad f(z)=\frac{1}{(t-i+1)^2}=\frac{1}{[t+(1-i)]^2}=\frac{1}{\left[(1-i)\left(1+\frac{t}{1-i}\right)\right]^2}$$

$$=\frac{1}{(1-i)^2}\left(1+\frac{t}{1-i}\right)^{-2}$$

$$=\frac{1}{-2i}\left[1-\frac{2t}{1-i}+\frac{3t^2}{(1-i)^2}-\frac{4t^3}{(1-i)^3}+\ldots\right]$$

$$=\frac{i}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{nt^{n-1}}{(1-i)^{n-1}}$$

$$=\frac{i}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n(z+i)^{n-1}}{(1-i)^{n-1}}.$$

(*ii*) To expand $f(z)=\dfrac{2z^3+1}{z^2+z}=2z-2+\dfrac{2z+1}{z(z+1)}$

$$=2z-2+\frac{1}{z}+\frac{1}{z+1} \qquad \text{[By partial fractions]}$$

about $z=i$, *i.e.*, in powers of $z-i$, we put $z-i=t$ or $z=t+i$

$$\therefore \quad f(z)=2(t+i)-2+\frac{1}{t+i}+\frac{1}{t+i+1} \qquad \ldots(1)$$

Now, $$\frac{1}{t+i}=\frac{1}{i\left(1+\frac{t}{i}\right)}=\frac{1}{i}\left(1+\frac{t}{i}\right)^{-1}$$

$$=\frac{1}{i}\left(1-\frac{t}{i}+\frac{t^2}{i^2}-\frac{t^3}{i^3}+\ldots\right)=\frac{1}{i}-\frac{t}{i^2}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+\ldots$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{t^{n-1}}{i^n}$$

Also, $$\frac{1}{t+i+1}=\frac{1}{(1+i)\left(1+\frac{t}{1+i}\right)}=\frac{1}{1+i}\left(1+\frac{t}{1+i}\right)^{-1}$$

$$=\frac{1}{1+i}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\ldots\right]$$

$$=\frac{1}{1+i}-\frac{t}{(1+i)^2}+\frac{t^2}{(1+i)^3}-\frac{t^3}{(1+i)^4}+\ldots$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{t^{n-1}}{(1+i)^n}$$

Substituting in (1), we have

$$f(z)=-2+2i+2t+\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{i^n}+\frac{1}{(1+i)^n}\right]t^{n-1}$$

$$=-2(1-i)+2(z-i)+\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{i^n}+\frac{1}{(1+i)^n}\right](z-i)^{n-1}.$$

EXERCISE 4.1

Expand the following functions as a Taylor's series:

1. $\log(1+z)$ about $z=0$. **2.** $\sin z$ about $z=\dfrac{\pi}{4}$. **3.** $\dfrac{z}{(z+1)(z+2)}$ about $z=2$.

Expand the following functions in Laurent's series or Taylor's series:

4. $\dfrac{1}{z-2}$, for $|z|>2$. **5.** $\dfrac{1}{z^2-4z+3}$, for $1<|z|<3$. **6.** $\dfrac{1}{z(z-1)(z-2)}$, for $|z|>2$.

(*M.D.U. Dec. 2006*)

7. $\dfrac{1-\cos z}{z^3}$, about $z=0$. **8.** $\dfrac{e^z}{(z-1)^2}$, about $z=1$. (*M.D.U. 2005, 2008*)

9. $\dfrac{1}{z(z-1)(z-2)}$, for $|z-1|<1$

10. $\dfrac{1}{z(e^z-1)}$, for $0<|z|<2\pi$.

11. $\dfrac{z^2-1}{(z+2)(z+3)}$, when

(*i*) $|z| < 2$ (*ii*) $2 < |z| < 3$ (*iii*) $|z| > 3$

12. $\dfrac{1}{(z+1)(z+3)}$, when

(*i*) $|z| < 1$ (*ii*) $1 < |z| < 3$

(*iii*) $|z| > 3$ (*iv*) $1 < |z+1| < 2$. (*M.D.U. Dec. 2009*)

13. $\dfrac{7z-2}{z(z+1)(z-2)}$, when

(*i*) $0 < |z+1| < 1$ (*ii*) $1 < |z+1| < 3$

(*iii*) $|z+1| > 3$ (*M.D.U. May 2009, Dec. 2010*)

14. $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in the region

(*i*) $|z| < 1$ (*ii*) $1 < |z| < 4$ (*iii*) $|z| > 4$

(*U.P.T.U. 2008*)

Answers

1. $z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \ldots\ldots$

2. $\dfrac{1}{\sqrt{2}}\left[1 + \left(z-\dfrac{\pi}{4}\right) - \dfrac{1}{2!}\left(z-\dfrac{\pi}{4}\right)^2 - \dfrac{1}{3!}\left(z-\dfrac{\pi}{4}\right)^3 + \ldots\ldots\right]$

3. $\left(\dfrac{1}{2}-\dfrac{1}{3}\right) - \left(\dfrac{1}{2^3}-\dfrac{1}{3^2}\right)(z-2) + \left(\dfrac{1}{2^5}-\dfrac{1}{3^3}\right)(z-2)^2 - \ldots\ldots = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{2^{2n-1}} - \frac{1}{3^n}\right)(z-2)^{n-1}$

4. $\dfrac{1}{z} + \dfrac{2}{z^2} + \dfrac{2^2}{z^3} + \dfrac{2^3}{z^4} + \ldots\ldots$

5. $-\dfrac{1}{6} - \dfrac{1}{6}\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{3^n} - \frac{1}{2}\sum_{n=1}^{\infty} z^{-n}$

6. $z^{-3} + 3z^{-4} + 7z^{-5} + \ldots\ldots + (2^n - 1)z^{-(n+2)} + \ldots\ldots$

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{z^{2n-3}}{(2n)!}$

8. $e\left[(z-1)^{-2} + (z-1)^{-1} + \dfrac{1}{2!} + \dfrac{1}{3!}(z-1) + \dfrac{1}{4!}(z-1)^2 + \ldots\ldots\right]$

9. $-\dfrac{1}{z-1} - \displaystyle\sum_{n=1}^{\infty} (z-1)^{2n-1}$.

10. $z^{-2} - \dfrac{1}{2}z^{-1} + \dfrac{1}{12} - \dfrac{z^2}{720} + \ldots\ldots$

11. (*i*) $1 + \dfrac{3}{2}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n\left(\frac{z}{3}\right)^n$ (*ii*) $1 + \dfrac{3}{z}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n\left(\frac{z}{3}\right)^n$

(*iii*) $1 + \dfrac{3}{z}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty} (-1)^n\left(\frac{3}{z}\right)^n$

12. (*i*) $\dfrac{1}{2}\left[\displaystyle\sum_{n=0}^{\infty} (-1)^n z^n - \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n\right]$ (*ii*) $\dfrac{1}{2}\left[\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} (-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n\right]$

(*iii*) $\dfrac{1}{2}\left[\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} (-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n\right]$ (*iv*) $\dfrac{1}{2(z+1)}\displaystyle\sum_{n=0}^{\infty} (-1)^n\left(\frac{z+1}{2}\right)^n$.

13. (*i*) $-\dfrac{3}{z+1} - \displaystyle\sum_{n=0}^{\infty} (z+1)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$ (*ii*) $\dfrac{1}{z+1}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$

(*iii*) $\dfrac{1}{z+1}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} + \frac{2}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$

14. (*i*) $-1 + \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}(1+4^{-n})\, z^n$ (*ii*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n} z^n + \sum_{n=1}^{\infty} (-1)^n z^{-n}$

(*iii*) $1 + \displaystyle\sum_{n=1}^{\infty} (-1)^n (1+4^n)\, z^{-n}$.

RESIDUES

4.4. SINGULAR POINTS

A point at which a function $f(z)$ ceases to be analytic is called a singular point or singularity of $f(z)$. For example $z = 2$ is a singular point of $f(z) = \dfrac{1}{z-2}$.

4.5. ISOLATED SINGULAR POINT AND NON-ISOLATED SINGULAR POINT

A singular point $z = a$ of a function $f(z)$ is called an isolated singular point if there exists a small circle with centre a which contains no other singular point of $f(z)$. Otherwise it is called non-isolated singular point.

For example; consider $f(z) = \dfrac{z}{z^2-1}$

The function $f(z)$ is analytic every where except at $z = -1, 1$ $\therefore$ $z = -1, 1$ are the isolated singularities of $f(z)$ as there are no other singularities of $f(z)$ in the neighbourhood of $z = -1$ and 1.

Again consider $f(z) = \cot\left(\dfrac{\pi}{z}\right) = \dfrac{1}{\tan\dfrac{\pi}{z}}$

It is not analytic at points where $\tan\dfrac{\pi}{z} = 0$ *i.e.*, $\dfrac{\pi}{z} = n\pi$ or $z = \dfrac{1}{n}$

Thus $z = 1, \frac{1}{2}, \frac{1}{3}, ..., 0$ are the singularities of $f(z)$, all of which are isolated except $z = 0$

$\because$ in the neighbourhood of $z = 0$, there are infinite number of other singularities $\left(z = \frac{1}{n} \text{ where } n \text{ is large}\right)$

$\therefore$ $z = 0$ is non-isolated singularity of the given function.

4.6. TYPES OF SINGULARITIES

Let $f(z)$ be analytic within a domain D except at $z = a$ (an isolated singularity). Then with $z = a$ as centre we can draw two circles C (with small radius) and C′ (with large radius) within D so that $f(z)$ is analytic within the annulus between C and C′. If z is any point in the annulus then we can expand $f(z)$ is a Laurent's series about $z = a$.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \left[a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ... + a_{-m}(z-a)^{-m} + ...\infty\right] \quad ...(1)$$

It gives rise to three types of singularities

(i) Removable Singularity

If in $f(z)$ the series has no negative power terms *i.e.*, $a_{-1}, -a_{-2}, a_{-3}, ... \infty$ are all zero then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$ In this case $z = a$ is called a removable singularity.

Note: This type of singularity can be made to disappear by re-defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$.

e.g., $f(z) = \frac{\sin(z-a)}{z-a}$ has removable singularity at $z = a$ because

$$\frac{\sin(z-a)}{z-a} = \frac{1}{z-a}\left\{(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} ......\infty\right\} = 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} ......\infty$$

has no term containing negative powers of $(z - a)$. However this singularity can be removed and the function can be made analytic by redefining the function as follows:

$$f(z) = \frac{\sin(z-a)}{z-a}, \text{ when } z \neq a$$

$$= 1, \text{ when } z = a$$

(ii) Essential Singularity. If in (1) *i.e.*, in the expansion of $f(z)$ the series with negative powers does not terminate *i.e.*, has an infinite number of terms then $z = a$ is called an essential singularity of $f(z)$

e.g.,
$$f(z) = \sin\frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} ......\infty$$

$$= (z-a)^{-1} - \frac{1}{3!}(z-a)^{-3} + \frac{1}{5!}(z-a)^{-5} ......\infty$$

has an infinite number of negative power of $z - a$

(iii) Pole

If in (1) *i.e.*, in $f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$, the series with negative powers has a finite number of terms, say m, so that $a_{-m} \neq 0$ and all further coefficients *i.e.*, $a_{-m-1}, a_{-m-2}, a_{-m-3}, ...$ are zero, then $z = a$ is called a pole of order m *e.g.*, $f(z) = \frac{\sin(z-a)}{(z-a)^4}$ has a pole of order 3 at $z = a$.

$$\because \quad f(z) = \frac{1}{(z-a)^4} \cdot \left\{(z-a) - \frac{1}{3!}(z-a)^3 + \frac{1}{5!}(z-a)^5 ...\infty\right\}$$

$$= \frac{1}{(z-a)^3} - \frac{1}{3!}\frac{1}{z-a} + \frac{1}{5!}(z-a)...\infty$$

$$= (z-a)^{-3} - \frac{1}{3!}(z-a)^{-1} + \frac{1}{5!}(z-1)...\infty$$

It has a finite number of (only two) negative power terms and all other are zero

$\therefore$ $z = a$ is a pole of $f(z)$ of order 3. ($a_{-3} = 1 \neq 0$ and $a_{-4} = a_{-5} = ... = 0$)

Note. Poles are always isolated singularities

4.7. HOW TO DETECT REMOVABLE SINGULAR POINTS

If $\underset{z \to a}{\text{Lt}} f(z)$ exists and is finite then $z = a$ is a **removable singular point**

4.8. RESIDUE AT A POLE

If $z = a$ is an isolated singularity of $f(z)$ then $f(z)$ can be expressed expanded in Laurents series about $z = a$,

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + a_{-3}(z-a)^{-3} + ...\infty$$

Then coefficient of $(z - a)^{-1}$ *i.e.*, a_{-1} **is called the residue of $f(z)$ at $z = a$** and is written as

$$a_{-1} = \text{Res}\{f(z), a\}$$

Since $$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

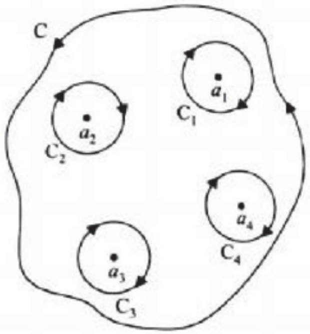
$$\therefore \quad \oint_C f(z)\,dz = 2\pi i \,\text{Res}\{f(z), a\}$$

4.9. CAUCHY'S RESIDUE THEOREM

(M.D.U. Dec. 2008)

If f(z) is analytic at all points inside and on a simple closed curve C, except at a finite number of isolated singular points within C, then

$\oint_C f(z)\,dz = 2\pi i$ *(sum of residues at singular points within C).*



Proof. Around each of the isolated singular points $a_1, a_2, \ldots\ldots, a_n$ draw small non-intersecting circles $C_1, C_2, \ldots\ldots, C_n$ lying wholly inside C, with centres at $z = a_1, a_2, \ldots\ldots, a_n$ respectively.

Since $f(z)$ is analytic in the multiply connected region bounded by C, $C_1, C_2, \ldots\ldots, C_n$ we have by Cauchy's theorem for multiply connected region.

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots\ldots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i\,[\text{Res}\,\{(z), a_1\} + \text{Res}\,\{f(z), a_2\} + \ldots\ldots + \text{Res}\,\{f(z), a_n\}]$$

$$= 2\pi i\,(\text{sum of residues at } a_1, a_2, \ldots\ldots, a_n).$$

4.10. CALCULATION OF RESIDUES

(1) *If f(z) has a simple pole (i.e., pole of order 1) at z = a, then* Res $\{f(z), a\} = \lim_{z \to a} (z - a) f(z)$.

Since $z = a$ is a pole of order 1, the Laurent's series becomes

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + a_{-1}(z - a)^{-1}.$$

Multiplying both sides by $(z - a)$, we get

$$(z - a) f(z) = a_0(z - a) + a_1(z - a)^2 + a_2(z - a)^3 + \ldots\ldots + a_{-1}$$

$$\therefore \quad \lim_{z \to a} (z - a) f(z) = a_{-1} = \text{Res}\,\{f(z), a\}$$

(2) *If f(z) has a pole of order m at z = a, then*

$$\textbf{Res}\,\{\mathbf{f(z), a}\} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)]$$

Since $z = a$ is a pole of order m, the Laurent's series becomes

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + a_{-1}(z - a)^{-1} + a_{-2}(z - a)^{-2} + \ldots\ldots + a_{-m}(z - a)^{-m}$$

Multiplying both sides by $(z - a)^m$, we get

$$(z - a)^m f(z) = a_0(z - a)^m + a_1(z - a)^{m+1} + a_2(z - a)^{m+2} + \ldots\ldots + a_{-1}(z - a)^{m-1} + a_{-2}(z - a)^{m-2} + \ldots\ldots + a_{-m}$$

Differentiating both sides $(m - 1)$ times w.r.t. z and taking the limit as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)] = a_{-1}(m - 1)\,!$$

or $$\frac{1}{(m-1)!} \underset{z \to a}{\text{Lt}} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)] = a_{-1} = \text{Res}\,\{f(z), a\}.$$

(3) *Another formula for Res* $\{f(z), a\}$

Let $$f(z) = \frac{\phi(z)}{\psi(z)}, \text{ where } \psi(z) = (z - a)\,F(z),\ F(a) \neq 0$$

Then Res $\{f(z), a\} = \lim_{z \to a} (z - a) f(z)$

$$= \lim_{z \to a} \frac{(z - a)\,\phi(z)}{\psi(z)} = \lim_{z \to a} \frac{(z - a)\,\phi\,\{a + (z - a)\}}{\psi\,\{a + (z - a)\}}$$

[Expanding $\phi(z)$ and $\psi(z)$ in ascending powers of $(z - a)$]

$$= \lim_{z \to a} \frac{(z - a)[\phi(a) + (z - a)\,\phi'(a) + \ldots]}{\psi(a) + (z - a)\,\psi'(a) + \dfrac{(z - a)^2}{2!}\psi''(a) + \ldots}$$

$$= \lim_{z \to a} \frac{(z - a)[\phi(a) + (z - a)\,\phi'(a) + \ldots]}{(z - a)\left[\psi'(a) + \dfrac{z - a}{2!}\psi''(a) + \ldots\right]} \quad \text{since } \psi(a) = 0$$

$$= \lim_{z \to a} \frac{\phi(a) + (z - a)\,\phi'(a) + \ldots}{\psi'(a) + \dfrac{z - a}{2!}\psi''(a) + \ldots} = \frac{\phi(a)}{\psi'(a)} = \lim_{z \to a} \frac{\phi(z)}{\psi'(z)}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the nature of singularities of the following functions:*

(i) $f(z) = \dfrac{z - \sin z}{z^3}$ at $z = 0$ (ii) $f(z) = e^{\frac{1}{z-a}}$ at $z = a$

(iii) $f(z) = \dfrac{1}{\sin z - \cos z}$ at $z = \dfrac{\pi}{4}$.

Sol. (i) $$f(z) = \frac{z - \sin z}{z^3} = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\ldots\infty\right)\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} \ldots\ldots\infty$$

Since there are no negative power terms of z $\therefore$ $z = 0$ is a removable singularity.

Aliter:
$$\lim_{z\to 0} \frac{z - \sin z}{z^3} \qquad \left[\frac{0}{0} \text{ form, apply L-Hospital's rule}\right]$$

$$= \lim_{z\to 0} \frac{1-\cos z}{3z^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z\to 0} \frac{\sin z}{6z} = \frac{1}{6} \qquad \left[\lim_{z\to 0} \frac{\sin z}{z} = 1\right]$$

Limit is finite $\therefore$ $z = 0$ is a removable singularity

(ii)
$$f(z) = e^{\frac{1}{z-a}} = 1 + \frac{1}{z-a} + \frac{1}{2!}\frac{1}{(z-a)^2} + \frac{1}{3!}\frac{1}{(z-a)^3} + \ldots\ldots \infty$$

$$= 1 + (z-a)^{-1} + \frac{1}{2!}(z-a)^{-2} + \frac{1}{3!}(z-a)^{-3} + \ldots\ldots \infty$$

It has infinite number of negative power terms of $z - a$ $\therefore$ $z = a$ is an essential singularity.

(iii) $f(z) = \dfrac{1}{\sin z - \cos z}$ at $z = \dfrac{\pi}{4}$

Put $z = \dfrac{\pi}{4} + t$

$$\therefore \quad \phi(t) = \frac{1}{\sin\left(\frac{\pi}{4}+t\right) - \cos\left(\frac{\pi}{4}+t\right)} = \frac{1}{\sqrt{2}\sin t} = \frac{1}{\sqrt{2}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \infty\right]}$$

$$= \frac{1}{\sqrt{2}t}\left[1 - \left(\frac{t^2}{3!} - \frac{t^4}{5!} + \ldots \infty\right)\right]^{-1}$$

$$= \frac{1}{\sqrt{2}t}\left[1 + \left(\frac{t^2}{3!} - \frac{t^4}{5!} + \ldots \infty\right) + \left(\frac{t^2}{3!} - \frac{t^4}{5!} + \ldots \infty\right)^2 + \ldots\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{t} + \frac{1}{3!}t + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)t^3 + \ldots\right]$$

Since $a_{-1} = \dfrac{1}{\sqrt{2}} \neq 0$ and $a_{-2} = a_{-3} = \ldots = 0$ $\therefore$ $t = 0$ is a simple pole of $f(t)$. Hence $z = \dfrac{\pi}{4}$ is a simple pole of $f(z)$.

Example 2. *What type of singularity have the following functions:*

(i) $\dfrac{e^{2z}}{(z-1)^4}$ (ii) $ze^{\frac{1}{z^2}}$ (iii) $\dfrac{1}{1-e^z}$

(M.D.U. May 2011)

Sol. (i)
$$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4}, \quad \text{where } t = z - 1$$

$$= \frac{e^2 . e^{2t}}{t^4}$$

$$= \frac{e^2}{t^4}\left[1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \ldots\ldots\right]$$

$$= e^2\left[t^{-4} + 2t^{-3} + 2t^{-2} + \frac{4}{3}t^{-1} + \frac{2}{3} + \frac{4}{15}t + \ldots\ldots\right]$$

$$= e^2\left[(z-1)^{-4} + 2(z-1)^{-3} + 2(z-1)^{-2} + \frac{4}{3}(z-1)^{-1} + \frac{2}{3} + \frac{4}{15}(z-1) + \ldots\ldots\right]$$

It has a finite number of negative power terms.

$$a_{-4} = e^2 \neq 0 \quad \text{and} \quad a_{-5} = a_{-6} = \ldots\ldots = 0$$

$\Rightarrow$ $z = 1$ is a pole of order 4.

(ii)
$$ze^{\frac{1}{z^2}} = z\left[1 + \frac{\frac{1}{z^2}}{1!} + \frac{\left(\frac{1}{z^2}\right)^2}{2!} + \frac{\left(\frac{1}{z^2}\right)^3}{3!} + \ldots\ldots\right] = z + z^{-1} + \frac{1}{2}z^{-3} + \frac{1}{6}z^{-5} + \ldots\ldots$$

It has an infinite number of negative powers of z

$\Rightarrow$ $z = 0$ is an essential singularity of $ze^{\frac{1}{z^2}}$.

(iii)
$$\frac{1}{1-e^z} = \frac{1}{1 - \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots\right)}$$

$$= \frac{1}{-z\left(1 + \frac{z}{2} + \frac{z^2}{6} + \ldots\ldots\right)} = -\frac{1}{z}\left[1 + \left(\frac{z}{2} + \frac{z^2}{6} + \ldots\ldots\right)\right]^{-1}$$

$$= -\frac{1}{z}\left[1 - \left(\frac{z}{2} + \frac{z^2}{6} + \ldots\ldots\right) + \left(\frac{z}{2} + \frac{z^2}{6} + \ldots\ldots\right)^2 \ldots\ldots\right]$$

$$= -\frac{1}{z}\left[1 - \frac{z}{2} + \left(\frac{1}{4} - \frac{1}{6}\right)z^2 \ldots\ldots\right] = -z^{-1} + \frac{1}{2} - \frac{1}{12}z + \ldots\ldots$$

It has a finite number of negative power terms.

$a_{-1} = -1 \neq 0$ and $a_{-2} = a_{-3} = \ldots\ldots = 0$

Also $1 - e^z = 0 \Rightarrow e^z = 1 = e^{2n\pi i}$, n being an integer.

$z = 2n\pi i \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$

$\therefore \dfrac{1}{1-e^z}$ has a simple poles at $z = 0, \pm 2\pi i, \pm 4\pi i, \ldots\ldots$

Example 3. *Find the residue at z = 0 of*

(i) $f(z) = z \cos \dfrac{1}{z}$ *(ii)* $f(z) = \operatorname{cosec}^2 z$ *(iii)* $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}$

Sol. (*i*)
$$f(z) = z\cos\frac{1}{z} = z.\left\{1 - \frac{1}{2!}.\frac{1}{z^2} + \frac{1}{4!}.\frac{1}{z^4}\ldots\ldots\infty\right\}$$
$$= z - \frac{1}{2!}.\frac{1}{z} + \frac{1}{4!}.\frac{1}{z^3}\ldots\ldots\infty$$
$$= z - \frac{1}{2!}.z^{-1} + \frac{1}{4!}z^{-3}\ldots\ldots\infty$$

which is Laurent's expansion about $z = 0$

$$\therefore \quad a_{-1} = \text{coeff. of } z^{-1} = -\frac{1}{2}$$

By definition of residue (art 4.8)

Residue of $f(z)$ at $z = 0$ is $-\dfrac{1}{2}$.

(*ii*) $f(z) = \operatorname{cosec}^2 z = \dfrac{1}{(\sin z)^2} = (\sin z)^{-2}$

$$= \left[z - \frac{z^3}{3!} + \frac{z^5}{5!}\ldots\infty\right]^{-2} = \frac{1}{z^2}\left[1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\infty\right)\right]^{-2}$$
$$= \frac{1}{z^2}\left[1 + 2\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\infty\right) + 3\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\infty\right)\ldots\right]^2$$
$$= \frac{1}{z^2}\left[1 + \frac{1}{3}z^2 + \frac{1}{15}z^4 + \ldots\infty\right]$$
$$= \frac{1}{z^2} + \frac{1}{3} + \frac{1}{15}z^2 + \ldots\infty = z^{-2} + \frac{1}{3} + \frac{1}{15}z^2 + \ldots\infty$$

Here $\quad a_{-1} = \text{Coeff. of } z^{-1} = 0$

$\therefore$ Res. of $f(z)$ at $z = 0$ is 0

(*iii*) $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}$

$z = 0$ is a simple pole of $f(z)$

$$\therefore \quad \text{Res}\,[f(z), 0)] = \underset{z\to 0}{\text{Lt}}\ z.\frac{1+e^z}{\sin z + z\cos z} = \underset{z\to 0}{\text{Lt}}\ \frac{1+e^z}{\dfrac{\sin z}{z} + \cos z} = \frac{1+1}{1+1} = \frac{2}{2} = 1.$$

Example 4. *Determine the poles of the function* $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ *and the residue at each pole.* (J.N.T.U. 2005)

Sol. The function $f(z)$ has a pole of order 2 at $z = 1$ and a simple pole at $z = -2$.

Residue of $f(z)$ at $z = 1$ is

$$\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$
$$= \lim_{z\to 1}\frac{(z+2).2z - z^2.1}{(z+2)^2} = \lim_{z\to 1}\frac{z^2+4z}{(z+2)^2} = \frac{5}{9}$$

Residue of $f(z)$ at $z = -2$, is

$$= \lim_{z\to -2}[(z+2)f(z)] = \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{4}{9}.$$

Example 5. *Find the sum of the residues of the function* $f(z) = \dfrac{\sin z}{z\cos z}$ *at its poles inside the circle* $|z| = 2$.

Sol. The function $f(z)$ has simple poles at $z = 0, \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots\ldots$ Of these, $z = 0, \pm\dfrac{\pi}{2}$ lie inside $|z| = 2$.

Residue of $f(z)$ at $z = 0$ is $\lim\limits_{z\to 0}[z.f(z)] = \lim\limits_{z\to 0}\dfrac{\sin z}{\cos z} = 0$

Residue of $f(z)$ at $z = \dfrac{\pi}{2}$ is

$$\lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)f(z) = \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\sin z}{z\cos z} \qquad \Big|\ \text{Form } \frac{0}{0}$$
$$= \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\cos z + \sin z}{\cos z - z\sin z} \qquad \text{[By L-Hospital's Rule]}$$
$$= \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$$

Similarly, residue of $f(z)$ at $z = -\dfrac{\pi}{2}$ is $\dfrac{2}{\pi}$.

$\therefore$ Sum of residues $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0$.

Note. Using Art. 4.10 (3). Another formula for Res $\{f(z), a\}$

$$\text{Res}\left\{f(z), \frac{\pi}{2}\right\} = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(z \cos z)} = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{\cos z - z \sin z} = \frac{1}{0 - \frac{\pi}{2} \times 1} = -\frac{2}{\pi}$$

Similarly, Res $\left\{f(z), -\frac{\pi}{2}\right\} = \frac{2}{\pi}$

Example 6. *Evaluate* $\oint_C \frac{e^z}{(z+1)^2}\, dz$, *where C is the circle* $|z - 1| = 3$.

Sol. Here $f(z) = \frac{e^z}{(z+1)^2}$ has only one singular point $z = -1$ which is a pole of order 2 and it lies inside the circle $|z - 1| = 3$.

Residue of $f(z)$ at $z = -1$ is $\lim_{z \to -1} \frac{d}{dz}[(z+1)^2 f(z)] = \lim_{z \to -1} \frac{d}{dz}(e^z) = \lim_{z \to -1} e^z = e^{-1}$

$\therefore$ By residue theorem, we have $\oint_C \frac{e^z}{(z+1)^2}\, dz = 2\pi i(e^{-1}) = \frac{2\pi i}{e}$.

Example 7. *Evaluate* $\oint_C \frac{2z - 1}{z(z+1)(z-3)}\, dz$, *where C is the circle* $|z| = 2$.

Sol. Here $f(z) = \frac{2z-1}{z(z+1)(z-3)}$ has three simple poles at $z = 0, -1, 3$ of which only $z = 0, -1$ lie inside the circle $|z| = 2$.

Residue of $f(z)$ at $z = 0$ is $\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{2z-1}{(z+1)(z-3)} = \frac{1}{3}$

Residue of $f(z)$ at $z = -1$ is $\lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{2z-1}{z(z-3)} = -\frac{3}{4}$

$\therefore$ By residue theorem,

$$\oint_C \frac{2z-1}{z(z+1)(z-3)}\, dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i\left(\frac{1}{3} - \frac{3}{4}\right) = -\frac{5\pi i}{6}$$

Example 8. *Evaluate* $\oint_C \frac{z-3}{z^2 + 2z + 5}\, dz$, *where C is the circle*

(*i*) $|z| = 1$ (*ii*) $|z + 1 - i| = 2$
(*iii*) $|z + 1 + i| = 2$. (M.D.U. May 2006, May 2007, Dec. 2007)

Sol. The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by

$$z^2 + 2z + 5 = 0 \quad i.e., \quad z = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

(*i*) Both the poles lie outside the circle $|z| = 1$.

$\therefore$ By Cauchy's theorem, we have $\oint_C \frac{z-3}{z^2+2z+5}\, dz = 0$

(*ii*) Only the pole $z = -1 + 2i$ lies inside the circle

$|z + 1 - i| = 2$ *i.e.*, $|z - (-1 + i)| = 2$ whose centre is at $-1 + i$ and radius is 2.

Residue of $f(z)$ at $z = -1 + 2i$ is

$$\lim_{z \to -1+2i} (z + 1 - 2i)\, f(x) = \lim_{z \to -1+2i} (z + 1 - 2i) \frac{z-3}{z^2 + 2z + 5}$$

$$= \lim_{z \to -1+2i} \frac{(z+1-2i)(z-3)}{(z+1+2i)(z+1-2i)}$$

$[\because \; z^2 + 2z + 5 = (z+1)^2 - 4i^2 = (z + 1 + 2i)(z + 1 - 2i)$
or roots of $z^2 + 2z + 5 = 0$ are $-1 \pm 2i$
$\therefore \; z^2 + 2z + 5 = (z - (-1 + 2i))(z - (-1 - 2i))$
$= (z + 1 - 2i)(z + 1 - 2i)]$

$$= \lim_{z \to -1+2i} \frac{z-3}{z+1+2i} = \frac{-1+2i-3}{-1+2i+1+2i} = \frac{2(i-2)}{4i} = \frac{i-2}{2i}$$

$\therefore$ By residue theorem, $\oint_C \frac{z-3}{z^2+2z+5}\, dz = 2\pi i\left(\frac{i-2}{2i}\right) = \pi(i-2)$

(*iii*) Only the pole $z = -1 - 2i$ lies inside the circle $|z + 1 + i| = 2$ *i.e.*, $|z - (-1 - i)| = 2$ whose centre is at $-1 - i$ and radius 2.

Residue of $f(z)$ at $z = -1 - 2i$ is

$$\lim_{z \to -1-2i} (z + 1 + 2i)\, f(z) = \lim_{z \to -1-2i} (z + 1 + 2i) \frac{z-3}{z^2+2z+5}$$

$$= \lim_{z \to -1-2i} \frac{(z+1+2i)(z-3)}{(z+1+2i)(z+1-2i)}$$

$$= \lim_{z \to -1-2i} \frac{z-3}{z+1-2i} = \frac{-1-2i-3}{-1-2i+1-2i} = \frac{-2(i+2)}{-4i} = \frac{i+2}{2i}$$

Example 9. *Evaluate* $\oint_C \frac{e^z}{\cos \pi z}$, *where C is the unit circle* $|z| = 1$.

(M.D.U. 2005, 2007, 2008)

Sol. Here $f(z) = \frac{e^z}{\cos \pi z}$ has simple poles at $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\ldots$ of which only $z = \pm\frac{1}{2}$ lie inside the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{1}{2}$ is

$$\lim_{z \to \frac{1}{2}} (z - \tfrac{1}{2}) f(z) = \lim_{z \to \frac{1}{2}} \frac{(z - \frac{1}{2}) e^z}{\cos \pi z} \qquad \left| \text{ Form } \frac{0}{0} \right.$$

$$= \lim_{z \to \frac{1}{2}} \frac{(z - \frac{1}{2}) e^z + e^z}{-\pi \sin \pi z} \qquad | \text{ By L' Hospital's Rule}$$

$$= \frac{e^{1/2}}{-\pi}.$$

Similarly, residue of $f(z)$ at $z = -\dfrac{1}{2}$ is $\dfrac{e^{-1/2}}{\pi}$.

$\therefore$ By residue theorem, $\displaystyle\oint_C \frac{e^z}{\cos \pi z}\, dz = 2\pi i$ (sum of residues)

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \tfrac{1}{2}.$$

EXERCISE 4.2

Determine the poles of the following functions and the residue at each pole:

1. $\dfrac{2z+1}{z^2 - z - 2}$ 2. $\dfrac{z+1}{z^2(z-2)}$ 3. $\dfrac{z^2}{(z-1)(z-2)^2}$ 4. $\dfrac{1 - e^{2z}}{z^4}$

5. $\dfrac{e^z}{z^2 + \pi^2}$ 6. $\dfrac{z}{\cos z}$ 7. $\dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$. *(J.N.T.U. 2005)*

Evaluate the following integrals:

8. $\displaystyle\oint_C \frac{z^2 + 2z - 2}{z - 4}\, dz$, where C is a closed curve containing the point $z = 4$ in its interior.

9. $\displaystyle\oint_C \frac{1 - 2z}{z(z-1)(z-2)}\, dz$, where C is the circle $|z| = 1.5$. *(M.D.U. Dec. 2006)*

10. $\displaystyle\oint_C \frac{z}{(z-1)(z-2)^2}\, dz$, where C is the circle $|z - 2| = \frac{1}{2}$. *(Madras 2006)*

11. $\displaystyle\oint_C \frac{12z - 7}{(z-1)^2 (2z+3)}\, dz$, where C is the circle $|z| = 2$.

12. $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}\, dz$, where C is the circle $|z| = 3$.

(U.P.T.U. 2005; V.T.U. 2006; M.D.U. Dec. 2008)

13. $\displaystyle\oint_C \frac{1 - \cos 2(z-3)}{(z-3)^3}\, dz$, where C: $|z - 3| = 1$.

14. $\displaystyle\oint_C \frac{z \sec z}{(1-z)^2}\, dz$, where C is the circle $|z| = 1.5$.

15. $\displaystyle\oint_C \frac{dz}{z^2 + 9}$, where C is

(*i*) $|z + 3i| = 2$ (*ii*) $|z| = 5$ *(M.D.U. May 2009)*

16. $\displaystyle\oint_C \frac{z^2}{(z-1)^2 (z+2)}\, dz$, where C: $|z| = \dfrac{5}{2}$.

17. $\displaystyle\oint_C \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}\, dz$, where C: $|z| = 10$. *(U.P.T.U. 2009)*

18. $\displaystyle\oint_C \frac{z^3}{(z-1)^4 (z-2)(z-3)}\, dz$, where C: $|z| = \dfrac{5}{2}$.

19. $\displaystyle\oint_C \frac{dz}{\sinh z}$, where C is the circle $|z| = 4$.

20. $\displaystyle\oint_C \tan z\, dz$, where C is the circle $|z| = 2$.

Answers

1. $z = -1, 2; \frac{1}{3}, \frac{5}{3}$ **2.** $z = 0, 2; -\frac{3}{4}, \frac{3}{4}$ **3.** $z = 1, 2; 1, 0$ **4.** $z = 0; -\frac{4}{3}$

5. $z = \pm \pi i; \pm \frac{i}{2\pi}$ **6.** $z = (2n+1)\frac{\pi}{2}; (-1)^{n+1} (2n+1) \frac{\pi}{2}$, where n is an integer.

7. $z = -1, \pm 2i; -\frac{14}{25}, \frac{7 \pm i}{25}$ **8.** $44\pi i$ **9.** $3\pi i$

10. $-2\pi i$ **11.** 0 **12.** $4\pi i(\pi + 1)$ **13.** $4\pi i$

14. $2\pi i \sec 1(1 + \tan 1)$ **15.** $-\frac{\pi}{3}, 0$ **16.** $2\pi i$

17. 0 **18.** $-\frac{27\pi i}{8}$ **19.** $-2\pi i$ **20.** $-4\pi i$

4.11. APPLICATION OF RESIDUES TO EVALUATE REAL INTEGRALS

The residue theorem provides a simple and elegant method for evaluating many important definite integrals of real variables. Some of these are illustrated below:

(1) **Integrals of the type** $\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$.

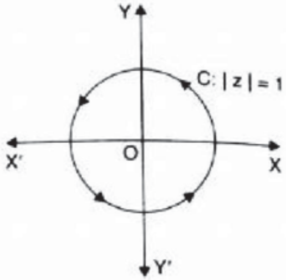
where $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Such integrals can be reduced to complex line integrals by the substitution $z = e^{i\theta}$, so that

$$dz = ie^{i\theta}\, d\theta, \text{ i.e., } d\theta = \frac{dz}{iz}.$$

Also $$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$



As θ varies from 0 to 2π, z moves once round the unit circle in the anti-clockwise direction.

$$\therefore \quad \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

where C is the unit circle $|z| = 1$.

The integral on the right can be evaluated by using the residue theorem.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. (P.T.U. 2006)

Sol. Put $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$

Then $$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \oint_C \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz} = \frac{2}{i}\oint_C \frac{1}{z^2+4z+1}\, dz$$

where C is the circle $|z| = 1$.

The poles of the integrand are the roots of $z^2 + 4z + 1 = 0$, which are $z = -2 \pm \sqrt{3}$.

Of the two poles, only $z = -2+\sqrt{3}$ lies inside the circle C.

Residue at $z = -2 + \sqrt{3}$ is

$$= \lim_{z\to\alpha}(z-\alpha)\frac{1}{z^2+4z+1}, \quad \text{where } \alpha = -2+\sqrt{3} \qquad \left|\ \text{Form } \frac{0}{0}\right.$$

$$= \lim_{z\to\alpha}\frac{1}{2z+4} = \frac{1}{2\alpha+4} = \frac{1}{2\sqrt{3}}$$

$\therefore$ By residue theorem, $$\oint_C \frac{dz}{z^2+4z+1} = 2\pi i\left(\frac{1}{2\sqrt{3}}\right) = \frac{\pi i}{\sqrt{3}}$$

Hence $$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \frac{2}{i}\oint_C \frac{dz}{z^2+4z+1} = \frac{2}{i}\left(\frac{\pi i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

Example 2. *Evaluate* $\int_0^{2\pi}\frac{d\theta}{1-2a\sin\theta+a^2}$, $0 < a < 1$.

(Kerala 2005; M.D.U. Dec. 2008, Dec. 2011)

Sol. Put $z = e^{i\theta}$ so that $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$.

Then, $$\int_0^{2\pi}\frac{d\theta}{1-2a\sin\theta+a^2} = \oint_C \frac{1}{1-2a\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)+a^2}\frac{dz}{iz}$$

$$= -\oint_C \frac{dz}{az^2-i(a^2+1)z-a} = -\oint_C\frac{dz}{(az-i)(z-ia)}$$

where C is the circle $|z| = 1$.

The integrand has simple poles at $z = \frac{i}{a}$ and $z = ia$ of which only $z = ia$ lies inside the circle C. $[\because \ 0 < a < 1]$

Residue at $z = ia$ is $\lim_{z\to ia}(z-ia)\frac{1}{(az-i)(z-ia)} = \lim_{z\to ia}\frac{1}{az-i} = \frac{1}{i(a^2-1)}$

$\therefore$ By residue theorem,

$$\oint_C \frac{dz}{(az-i)(z-ia)} = 2\pi i\cdot\frac{1}{i(a^2-1)} = \frac{2\pi}{a^2-1}$$

Hence $$\int_0^{2\pi}\frac{d\theta}{1-2a\sin\theta+a^2} = -\oint_C\frac{dz}{(az-i)(z-ia)} = \frac{2\pi}{1-a^2}.$$

Example 3. *Prove that* $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$, *where* $0 < b < a$.

Sol. Let $$I = \int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \int_0^{2\pi}\frac{1-\cos 2\theta}{2(a+b\cos\theta)}\,d\theta$$

$$= \text{Real part of } \int_0^{2\pi}\frac{1-e^{2i\theta}}{2a+2b\cos\theta}\,d\theta$$

Put $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$

Then $$\int_0^{2\pi}\frac{1-e^{2i\theta}}{2a+2b\cos\theta}\,d\theta = \oint_C \frac{1-z^2}{2a+b\left(z+\frac{1}{z}\right)}\frac{dz}{iz} = \oint_C\frac{1-z^2}{i(bz^2+2az+b)}\,dz$$

where C is the circle $|z| = 1$.

The poles of the integrand are the roots of $bz^2 + 2az + b = 0$, *viz.*

$$z = \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$$

Let $$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \quad \text{and} \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Clearly, $|\beta| > 1$ so that $z = \alpha$ is the only simple pole inside C.

Also $\quad bz^2 + 2az + b = b(z - \alpha)(z - \beta)$

Residue at $z = \alpha$ is

$$\lim_{z \to \alpha} (z - \alpha) \cdot \frac{1 - z^2}{ib(z - \alpha)(z - \beta)} = \lim_{z \to \alpha} \frac{1 - z^2}{ib(z - \beta)} = \frac{1 - \alpha^2}{ib(\alpha - \beta)}$$

$$= \frac{\alpha\left(\frac{1}{\alpha} - \alpha\right)}{ib(\alpha - \beta)} = \frac{\alpha(\beta - \alpha)}{ib(\alpha - \beta)} \qquad [\because \alpha\beta = 1]$$

$$= -\frac{\alpha}{ib} = \frac{a - \sqrt{a^2 - b^2}}{ib^2}$$

$\therefore$ By residue theorem,

$$\oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)}\, dz = 2\pi i \cdot \frac{a - \sqrt{a^2 - b^2}}{ib^2} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$

Hence I = Real part of $\displaystyle\oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)}\, dz = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$.

Example 4. *Evaluate* $\displaystyle\int_0^{\pi} \frac{d\theta}{a + b\cos\theta}$, *where* $a > |b|$.

Sol. Since $\quad \displaystyle\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$, if $f(2a - x) = f(x)$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} \qquad [\because \cos(2\pi - \theta) = \cos\theta]$$

or
$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \qquad \ldots(1)$$

Putting $z = e^{i\theta}$, so that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$, we have

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \oint_C \frac{1}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \oint_C \frac{2dz}{i(bz^2 + 2az + b)}$$

where C is the circle $|z| = 1$.

Proceeding as in example 3, residue at $z = \alpha$ is

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{2}{ib(z - \alpha)(z - \beta)} = \lim_{z \to \alpha} \frac{2}{ib(z - \beta)} = \frac{2}{ib(\alpha - \beta)}$$

$$= \frac{2}{ib} \cdot \frac{b}{2\sqrt{a^2 - b^2}} = \frac{1}{i\sqrt{a^2 - b^2}}$$

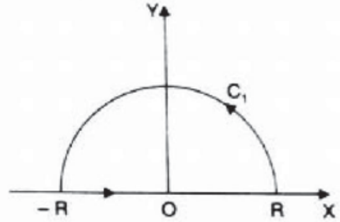
$\therefore$ By residue theorem,

$$\oint_C \frac{2dz}{i(bz^2 + 2az + b)} = 2\pi i \cdot \frac{1}{i\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Hence from (1), $\displaystyle\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}$.

(2) Integrals of the type $\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx$, where f(x) and F(x) are polynomials in x such that $\dfrac{x f(x)}{F(x)} \to 0$ as $x \to \infty$ and F(x) has no zeros on the real axis.

Consider the integral $\displaystyle\oint_C \frac{f(z)}{F(z)}\, dz$ over the closed contour C consisting of the real axis from $-R$ to R and the semi-circle C_1 of radius R in the upper half plane.



We take R large enough so that all the poles of $\dfrac{f(z)}{F(z)}$ in the upper half plane lie within C.

By residue theorem, we have

$$\oint_C \frac{f(z)}{F(z)}\, dz = 2\pi i \left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

or
$$\oint_{C_1} \frac{f(z)}{F(z)}\, dz + \int_{-R}^{R} \frac{f(x)}{F(x)}\, dx = 2\pi i \left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right] \qquad \ldots(1)$$

($\because$ on the real axis, $z = x$)

If we put $z = Re^{i\theta}$ in the first integral on the left side, then R is constant on C_1 and as z moves along C_1, θ varies from 0 to π.

$$\therefore \quad \int_{C_1} \frac{f(z)}{F(z)}\, dz = \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta}\, i\, d\theta$$

For large R, $\displaystyle\int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta}\, i d\theta$ is of the order $\dfrac{Rf(R)}{F(R)}$

$$\therefore \quad \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta}\, i d\theta \to 0 \text{ when } R \to \infty$$

Hence from (1), we have

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx = 2\pi i \times \left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx \ (a > 0, b > 0)$.

(P.T.U. 2007; Anna 2005, 2009)

Sol. Consider the integral $\oint_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz$ over the closed contour C consisting of the real axis from – R to R and the semi- circle C_1 of radius R in the upper half plane.

$$\therefore \quad \oint_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz = \int_{C_1}\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz + \int_{-R}^{R}\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz \qquad ...(1)$$

(*i*) To evaluate $\oint_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz$

Poles of $\phi(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$ are given by $z^2 + a^2 = 0$, $z^2 + b^2 = 0$

i.e., $z = \pm ia, \pm ib$

of these, only $z = ia$ and $z = ib$ lie in the upper half plane.

Residue of $\phi(z)$ at $z = ia$

$$= \lim_{z\to ia}(z-ia)\frac{z^2}{(z+ia)(z-ia)(z^2+b^2)}$$

$$= \lim_{z\to ia}\frac{z^2}{(z+ia)(z^2+b^2)} = \frac{-a^2}{2ia(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Residue of $\phi(z)$ at $z = ib$

$$= \lim_{z\to ib}(z-ib)\frac{z^2}{(z^2+a^2)(z+ib)(z-ib)}$$

$$= \lim_{z\to ib}\frac{z^2}{(z^2+a^2)(z+ib)} = \frac{-b^2}{(-b^2+a^2)(2ib)} = \frac{-b}{2i(a^2-b^2)}$$

$\therefore$ By Cauchy Residue Theorem

$$\oint_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz = 2\pi i \text{ [Sum of residues of } \phi(z) \text{ within C]}$$

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right] = \pi\left(\frac{a-b}{a^2-b^2}\right) = \frac{\pi}{a+b} \qquad ...(2)$$

(*ii*) Now $\int_{-R}^{R}\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz = \int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx \qquad ...(3)$

[$\because$ Along x-axis, $z = x$ and x varies from –R to R where R $\to \infty$]

(*iii*) Also $\int_{C_1}\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz$ Put $z = Re^{i\theta}$ (any point on C_1)

$$= \int_0^{\pi}\frac{R^2e^{2i\theta}}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}\cdot Re^{i\theta}\cdot id\theta$$

$$= \int_0^{\pi}\frac{iR^3e^{3i\theta}}{R^2\left(e^{2i\theta}+\frac{a^2}{R^2}\right)R^2\left(e^{2i\theta}+\frac{b^2}{R^2}\right)}d\theta$$

$$= \int_0^{\pi}\frac{ie^{3i\theta}}{R\left(e^{2i\theta}+\frac{a^2}{R^2}\right)\left(e^{2i\theta}+\frac{b^2}{R^2}\right)}d\theta \to 0 \text{ as } R\to\infty$$

$$\therefore \quad \int_{C_1}\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz = 0 \qquad ...(4)$$

From (1), (2), (3) and (4)

$$\frac{\pi}{a+b} = 0 + \int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$$

Hence $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a+b}$

Example 2. *Evaluate* $\int_0^{\infty}\frac{dx}{x^4+1}$ (*U.P.T.U. 2007*)

Sol. $\int_0^{\infty}\frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+1} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)dx \text{ if } f(x) \text{ is even}\right] \qquad ...(1)$

Consider the integral $\oint_C \frac{dz}{z^4+1}$ over the closed contour C consisting of the real axis from $-R$ to R and the semi-circle C_1 of radius R in the upper half plane.

$$\therefore \quad \oint_C \frac{dz}{z^4+1} = \int_{C_1} \frac{dz}{z^4+1} + \int_{-R}^{R} \frac{dz}{z^4+1} \quad \ldots(2)$$

(i) To evaluate $\oint_C \frac{dz}{z^4+1}$.

Poles of $\phi(z) = \frac{1}{z^4+1}$ are obtained by solving $z^4 + 1 = 0$

Now $z^4 + 1 = 0$

$$\Rightarrow \quad z = (-1)^{\frac{1}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}} = [\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]^{\frac{1}{4}}$$

$$= \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4} \quad \text{(By De Moivre's theorem)}$$

where $n = 0, 1, 2, 3$

When $n = 0$, $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

When $n = 1$, $z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

When $n = 2$, $z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

When $n = 3$, $z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Of these, only $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = e^{\frac{i\pi}{4}}$ and $z = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = e^{\frac{3i\pi}{4}}$ lie in the upper half of z-plane.

Residue of $\phi(z)$ at $z = e^{\frac{i\pi}{4}}$ is

$$\lim_{z \to e^{\frac{i\pi}{4}}} \frac{z - e^{\frac{i\pi}{4}}}{z^4+1} \quad \left| \text{Form } \frac{0}{0} \right.$$

$$= \lim_{z \to e^{i\frac{\pi}{4}}} \frac{1}{4z^3} \quad | \text{ By L-Hospital's Rule}$$

$$= \frac{1}{4e^{\frac{3i\pi}{4}}} = \frac{1}{4}e^{-\frac{3i\pi}{4}} = \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]$$

$$= \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -\frac{1+i}{4\sqrt{2}}$$

Residue of $\phi(z)$ at $z = e^{\frac{3i\pi}{4}}$ is

$$\lim_{z \to e^{\frac{3i\pi}{4}}} \frac{z - e^{\frac{3i\pi}{4}}}{z^4+1} \quad \left| \text{Form } \frac{0}{0} \right.$$

$$= \lim_{z \to e^{\frac{3i\pi}{4}}} \frac{1}{4z^3} \quad | \text{ By L-Hospital's Rule}$$

$$= \frac{1}{4z^{9i\frac{\pi}{4}}} = \frac{1}{4e^{2\pi i + i\frac{\pi}{4}}} = \frac{1}{4e^{i\frac{\pi}{4}}} \quad (\because e^{2\pi i} = 1)$$

$$= \frac{1}{4}e^{-i\frac{\pi}{4}} = \frac{1}{4}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{1-i}{4\sqrt{2}}$$

$\therefore$ By Cauchy Residue Theorem

$$\oint_C \frac{dz}{z^4+1} = 2\pi i \text{ [Sum of residues of } \phi(z) \text{ within C]}$$

$$= 2\pi i\left(-\frac{1+i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right) = 2\pi i\left(\frac{-2i}{4\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}} \quad \ldots(3)$$

(ii) Now $\int_{-R}^{R} \frac{dz}{z^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$ ...(4)

[$\because$ Along x-axis, $z = x$ and x varies from $-R$ to R where $R \to \infty$]

(iii) Also $\int_{C_1} \frac{dz}{z^4+1}$ Put $z = Re^{i\theta}$ (any point on C_1)

$$= \int_0^{\pi} \frac{R e^{i\theta} . id\theta}{R^4 e^{4i\theta}+1} = \int_0^{\pi} \frac{ie^{i\theta}d\theta}{R^3\left(e^{4i\theta}+\frac{1}{R^4}\right)} \to 0 \text{ as } R \to \infty$$

$$\therefore \quad \int_{C_1} \frac{dz}{z^4+1} = 0 \quad \ldots(5)$$

From (2), (3), (4) and (5)

$$\frac{\pi}{\sqrt{2}} = 0 + \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$$

$$\Rightarrow \quad \frac{\pi}{\sqrt{2}} = 2\int_0^{\infty} \frac{dx}{x^4+1} \quad \text{[Using (1)]}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

The above method can also be applied to some cases, where f (x) contains trigonometric functions (sin ax or cos ax) *i.e.*, integrals of the type $\int_{-\infty}^{\infty} \frac{\sin ax}{F(x)} dx$ or $\int_{-\infty}^{\infty} \frac{\cos ax}{F(x)} dx$, where $a \geq 0$.

Example 3. *Evaluate* $\int_0^{\infty} \frac{\cos ax}{x^2+1} dx \ (a \geq 0)$.

Sol. $$\int_0^{\infty} \frac{\cos ax}{x^2+1} dx = \text{Real part of } \int_0^{\infty} \frac{e^{iax}}{x^2+1} dx$$

Consider $\oint_C \frac{e^{iaz}}{z^2+1} dz$, where C consists of

(*i*) the semicircle C_1 of radius R in the upper half plane

(*ii*) the real axis from – R to R

$$\therefore \quad \oint_C \frac{e^{iaz}}{z^2+1} dz = \oint_{C_1} \frac{e^{iaz}}{z^2+1} dz + \int_{-R}^{R} \frac{e^{iaz}}{z^2+1} dz \quad \ldots(1)$$

(*i*) To evaluate $\oint_C \frac{e^{iaz}}{z^2+1} dz$

Poles of the integral are given by $z^2 + 1 = 0 \Rightarrow z = \pm i$

Only $z = i$ lies within C

$\therefore$ Residue at $z = i$ (pole of order one) $= \underset{z \to i}{\text{Lt}} (z-i) \frac{e^{iaz}}{(z^2+1)} = \underset{z \to i}{\text{Lt}} \frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$

$$\therefore \quad \oint_C \frac{e^{iaz}}{z^2+1} dz = 2\pi i . \frac{e^{-a}}{2i} = \pi e^{-a} \quad \ldots(2)$$

(*ii*) To evaluate $\oint_{C_1} \frac{e^{iaz}}{z^2+1} dz$ Put $z = Re^{i\theta}$

Now $$z . \frac{e^{iaz}}{z^2+1} = Re^{i\theta} \frac{e^{iaR(\cos\theta + i\sin\theta)}}{R^2 e^{i2\theta}+1} = \frac{e^{i\theta} . e^{iaR\cos\theta} . e^{-aR\sin\theta}}{R . \left[e^{i2\theta} + \frac{1}{R^2}\right]}$$

Since $\left| e^{iaR\cos\theta} e^{i\theta} \right| = |\cos(aR\cos\theta) + i\sin(aR\cos\theta)| . 1 \leq 1$

$$\left| e^{-iaR\sin\theta} \right| \leq 1, \left| e^{2i\theta} + \frac{1}{R^2} \right| \neq 0$$

$$\therefore \quad z\frac{e^{iaz}}{z^2+1} \to 0 \text{ as } R \to \infty$$

$$\therefore \quad \int_{C_1} \frac{e^{iaz}}{z^2+1} dz \to 0 \quad \ldots(3)$$

(*iii*) $$\int_{-R}^{R} \frac{e^{iaz}}{z^2+1} dz = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx \quad \ldots(4)$$

[$\because$ Along x-axis, $z = x$ and x varies from – R to R where $R \to \infty$]

$\therefore$ From (1), (2), (3), (4) $\pi e^{-a} = 0 + \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx$

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx = \text{Real part of } \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-a}$$

$$\therefore \quad \int_0^{\infty} \frac{\cos ax}{x^2+1} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx = \frac{\pi e^{-a}}{2}.$$

EXERCISE 4.3

Evaluate the following integrals by contour integration:

1. $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

2. $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$, where $a > |b|$ (U.P.T.U. 2009)

3. $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$ (M.D.U. Dec. 2010)

4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$ (U.P.T.U. 2007)

5. $\int_0^{\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}$, $(0 < r < 1)$ (Madras 2006; J.N.T.U. 2006)

6. $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}$, $(0 < a < 1)$ (M.D.U. May 2007)

7. $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$, $(a > b > 0)$

8. $\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$, $(a^2 < 1)$

9. $\int_0^{\pi} \frac{d\theta}{17 - 8\cos\theta}$

10. $\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta}$, $(a > 0)$ (U.P.T.U. 2005)

11. $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2}$

12. $\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}$, $(a > b)$

13. $\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$

14. $\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta}\, d\theta$

15. (a) $\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta$ (b) $\int_0^{2\pi} \frac{1 + \sin\theta}{3 + \cos\theta}\, d\theta$

16. $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$

17. $\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2 + 1)(x^2 + 4)}$ (M.D.U. Dec. 2006)

18. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$

19. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$ (J.N.T.U. 2006)

20. $\int_0^{\infty} \frac{dx}{x^6 + 1}$ (J.N.T.U. 2005; M.D.U. May 2006)

21. $\int_0^{\infty} \frac{x^2}{x^6 + 1}\, dx$ (M.D.U. 2005, 2008)

22. $\int_0^{\infty} \frac{dx}{(a^2 + x^2)^2}$ (M.D.U. Dec. 2009)

23. $\int_0^{\infty} \frac{x^2\, dx}{(x^2 + 9)(x^2 + 4)^2}$

24. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3}$ (M.D.U. Dec. 2010)

25. $\int_{-\infty}^{\infty} \frac{x^2}{(1 + x^2)^3}\, dx$

Answers

1. $\frac{\pi}{2}$	2. $\frac{2\pi}{\sqrt{a^2 - b^2}}$	3. $\frac{\pi}{12}$	4. $\frac{\pi}{6}$
5. $\frac{\pi}{1 - r^2}$	6. $\frac{2\pi a^2}{1 - a^2}$	7. $\frac{2\pi}{\sqrt{a^2 - b^2}}$	8. $\frac{2\pi}{\sqrt{1 - a^2}}$
9. $\frac{\pi}{15}$	10. $\frac{\pi}{\sqrt{1 + a^2}}$	11. $\frac{5\pi}{32}$	12. $\frac{2\pi a}{(a^2 - b^2)^{3/2}}$
13. $\frac{2\pi}{3}$	14. 0	15. (a) 0	(b) $\frac{\pi}{\sqrt{2}}$
16. $\frac{\pi}{4}$	17. $\frac{\pi}{3}$	18. $\frac{5\pi}{12}$	19. $\frac{\pi}{\sqrt{2}}$
20. $\frac{\pi}{3}$	21. $\frac{\pi}{6}$	22. $\frac{\pi}{4a^3}$	23. $\frac{\pi}{200}$
24. $\frac{3\pi}{8}$	25. $\frac{\pi}{8}$.		