CHAPTER 1

Fourier Series

1.1. PERIODIC FUNCTIONS

A function $f(x)$ which satisfies the relation $f(x + T) = f(x)$ for all x is called a periodic function. The *smallest positive number* T, for which this relation holds, is called the **period** of $f(x)$.

If T is the period of $f(x)$, then $f(x) = f(x + T) = f(x + 2T) = \ldots\ldots = f(x + nT) = \ldots\ldots$

Also $f(x) = f(x - T) = f(x - 2T) = \ldots\ldots = f(x - nT) = \ldots\ldots$

$\therefore\ f(x) = f(x \pm nT)$, where n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

For example, $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π while $\tan x$ and $\cot x$ are periodic functions with period π. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic. If T_1 and T_2 are the periods of $f(x)$ and $g(x)$, then the period of $a\,f(x) + b\,g(x)$ is the least common multiple of T_1 and T_2.

For example, $\cos x$, $\cos 2x$, $\cos 3x$ are periodic functions with periods 2π, π and $\frac{2\pi}{3}$ respectively.

$\therefore\ f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic with period 2π, the L.C.M. of 2π, π and $\frac{2\pi}{3}$.

1.2. FOURIER SERIES

Periodic functions are of common occurrence in many physical and engineering problems; for example, in conduction of heat and mechanical vibrations. It is useful to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form $\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots$ $+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots$

within a desired range of values of the variable. Such a series is known as *Fourier Series*. Thus, any function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed in the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n ($n = 1, 2, 3, \ldots\ldots$) are constants, called the Fourier co-efficients of $f(x)$.

Note. To determine a_0, a_n and b_n, we shall need the following results: (m and n are integers)

(*i*) $\displaystyle\int_c^{c+2\pi} \sin nx\, dx = -\left[\frac{\cos nx}{n}\right]_c^{c+2\pi} = 0, \quad \int_c^{c+2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0, n \neq 0$

(*ii*) $\displaystyle\int_c^{c+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\sin(m+n)x + \sin(m-n)x]\, dx$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

(*iii*) $\displaystyle\int_c^{c+2\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

(*iv*) $\displaystyle\int_c^{c+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\cos(m-n)x - \cos(m+n)x]\, dx$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_c^{c+2\pi} = 0, m \neq n$$

(*v*) $\displaystyle\int_c^{c+2\pi} \cos^2 nx\, dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, \quad \int_c^{c+2\pi} \sin^2 nx\, dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, n \neq 0$

(*vi*) $\displaystyle\int_c^{c+2\pi} \sin nx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\, dx = -\frac{1}{2}\left[\frac{\cos 2nx}{2n}\right]_c^{c+2\pi} = 0, n \neq 0$

(*vii*) To integrate the product of two functions, one of which is a positive integral power of x, we apply the *generalised rule of integration by parts*. If dashes denote differentiation and suffixes denote integration w.r.t. x, the rule can be stated as follows:

$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$ where u and v are functions of x. *i.e.*, Integral of the product of two functions

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd, signs alternately + ve and – ve.

[Simplification should be done only when the integration is over.]

For example, $\displaystyle\int x^3 e^{-2x}\, dx = x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left[\frac{e^{-2x}}{(-2)^2}\right] + 6x\left[\frac{e^{-2x}}{(-2)^3}\right] - 6\left[\frac{e^{-2x}}{(-2)^4}\right]$

$$= e^{-2x}\left[-\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{3}{8}\right] = -\frac{1}{8}e^{-2x}(4x^3 + 6x^2 + 6x + 3)$$

$$\int x^2 \cos nx\, dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx.$$

(*viii*) $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$, where n is an integer.

(*ix*) **Even and Odd Functions**

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ *e.g.*, x^2, $\cos x$, $\sin^2 x$ are even functions.

The graph of an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ *e.g.*, x^3, $\sin x$, $\tan^3 x$ are odd functions.

The graph of an odd function is symmetrical about the origin.

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

Also, $\int_{-c}^{c} f(x)\,dx = 0$, when $f(x)$ is an odd function

and $\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$, when $f(x)$ is an even function.

1.3. EULER'S FORMULAE

The Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{...(1)}$$

In finding the co-efficients a_0, a_n and b_n, we assume that the series on the right hand side of (1) is uniformly convergent for $c < x < c + 2\pi$ and it can be integrated term by term in the given interval.

To find $\mathbf{a_0}$. Integrate both sides of (1) w.r.t. x, between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\pi - c) + 0 + 0 \qquad \text{[By formulae (i) above]}$$

$$= a_0\pi$$

$$\therefore \qquad \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

To find $\mathbf{a_n}$, multiply both sides of (1) by $\cos nx$ and integrate w.r.t. x, between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + a_n\pi + 0 \qquad \text{[By formulae (i), (v) and (vi)]}$$

$$= a_n\pi$$

$$\therefore \qquad \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

To find b_n, multiply both sides of (1) by sin nx and integrate w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$
$$+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi \qquad \text{[By formulae (}i\text{), (}vi\text{) and (}v\text{)]}$$
$$= b_n\pi$$

$$\therefore \quad \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

Hence $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$; and $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$ ...(I)

These values of a_0, a_n and b_n are called Euler's formulae.

Cor. 1. *If c = 0, the interval becomes $0 < x < 2\pi$ and the formulae I reduce to*

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx;\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx;\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2. *If $c = -\pi$, the interval becomes $-\pi < x < \pi$, and the formulae I reduce to*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx;\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx;\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cor. 3. *When f(x) is an odd function* $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

Since cos nx is an even function, therefore, $f(x)$ cos nx is an odd function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Since sin nx is an odd function, therefore, $f(x)$ sin nx is an even function.

$$\therefore \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms,

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

When f(x) is an even function $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

Since cos nx is an even function, therefore, $f(x)$ cos nx is an even function.

$$\therefore \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Since $\sin nx$ is an odd function, therefore, $f(x) \sin nx$ is an odd function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms,

i.e., $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx.$$

An Important Note. If a periodic function f is defined in the open interval (a, b), then how to find $f(a)$ and $f(b)$, the values of f at the end points?

Consider, $f(x) = x + x^2, \; -\pi < x < \pi$

where f is periodic with period 2π.

$$f(-\pi) = \frac{1}{2}\ (\text{LHL} + \text{RHL}) = \frac{1}{2}\ [f(-\pi - 0) + f(-\pi + 0)]$$

$$= \frac{1}{2}\ [f(2\pi + (-\pi - 0)) + f(-\pi + 0)] \qquad \mid \because \text{ Period of } f \text{ is } 2\pi$$

$$= \frac{1}{2}\ [f(\pi - 0) + f(-\pi + 0)]$$

$$= \frac{1}{2}\ [(\pi + \pi^2) + (-\pi) + (-\pi)^2] = \pi^2$$

Clearly, $f(-\pi) \neq -\pi + \pi^2$

Similarly, $$f(\pi) = \frac{1}{2}\ [\text{LHL} + \text{RHL}]$$

$$= \frac{1}{2}[f(\pi - 0) + f(\pi + 0)] = \frac{1}{2}[f(\pi - 0) + f(-2\pi + (\pi + 0)]$$

$$\mid \because \text{ Period of } f \text{ is } 2\pi$$

$$= \frac{1}{2}[f(\pi - 0) + f(-\pi + 0)] = \frac{1}{2}[(\pi + \pi^2) + (-\pi + (-\pi)^2)] = \pi^2$$

Clearly, $f(\pi) \neq \pi + \pi^2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the Fourier series to represent* $f(x) = \left(\frac{\pi - x}{2}\right)^2, 0 \le x \le 2\pi$.

(M.D.U. 2010, 2012, Dec. 2014; K.U.K. Jan. 2013)

Hence obtain the following relations:

(*i*) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (M.D.U. Dec. 2014)

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$ (M.D.U., 2010, Dec. 2014)

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Sol. Let $f(x)=\frac{1}{4}(\pi-x)^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

By Euler's formulae, we have

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\,dx=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}=-\frac{1}{12\pi}[-\pi^3-\pi^3]=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cos nx\,dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-\{-2(\pi-x)\}\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(0+\frac{2\pi\cos 2n\pi}{n^2}+0\right)-\left(0-\frac{2\pi\cos 0}{n^2}+0\right)\right]=\frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right]=\frac{1}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\sin nx\,dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)-\{-2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n}-0+\frac{2\cos 2n\pi}{n^3}\right)-\left(-\frac{\pi^2}{n}-0+\frac{2\cos 0}{n^3}\right)\right]$$

$$=\frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0$$

$$\therefore\quad f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\ldots \quad\ldots(1)$$

Deductions

(i) Putting $x=0$ in equation (1), we get

$$f(0)=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\Rightarrow\quad \frac{\pi^2}{4}=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\Rightarrow\quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{6} \quad\ldots(2)$$

(*ii*) Putting $x = \pi$ in equation (1), we get

$$f(\pi) = \frac{\pi^2}{12} + \left[\left(\frac{-1}{1^2}\right) + \frac{1}{2^2} + \left(\frac{-1}{3^2}\right) + \frac{1}{4^2} + \ldots\ldots\right]$$

$$\Rightarrow \quad 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12} \qquad \ldots(3)$$

(*iii*) Adding (2) and (3), we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{1}{2}\left(\frac{\pi^2}{4}\right) = \frac{\pi^2}{8}.$$

Note. In the above example, if $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$, then

$$f(0) = \frac{1}{2}[f(0-0) + f(0+0)]$$

$$= \frac{1}{2}[f(2\pi - 0) + f(0+0)] \qquad [\because \; f \text{ is periodic with period } 2\pi]$$

$$= \frac{1}{2}\left[\left(\frac{\pi - 2\pi}{2}\right)^2 + \left(\frac{\pi - 0}{2}\right)^2\right] = \frac{1}{2}\left(\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = \frac{\pi^2}{4}.$$

Example 2. *Expand $f(x) = x \sin x$, $0 < x < 2\pi$ as a Fourier series.*

(K.U.K. 2010; M.D.U. 2010, 2013; G.B.T.U. 2010)

Sol. Let $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{2\pi} = \frac{1}{\pi}[-2\pi] = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(2 \cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, n \neq 1$$

When $n = 1$, we have $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2 \sin nx \sin x) \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos (n-1)\,x - \cos (n+1)\,x] \, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin (n-1)\,x}{n-1} - \frac{\sin (n+1)\,x}{n+1}\right\} - 1.\left\{-\frac{\cos (n-1)\,x}{(n-1)^2} + \frac{\cos (n+1)\,x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0, n \neq 1$$

When $n = 1$, we have $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x) \, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx + 0$$

$$= -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots\ldots$$

Example 3. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that*

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

(G.B.T.U. 2012; K.U.K. May 2013; DCRUST, Murthal May 2014)

Sol. Let $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}(x - x^2)\,dx = \dfrac{1}{\pi}\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 - 2\pi)\frac{\cos n\pi}{n^2} - (1 + 2\pi)\frac{\cos n\pi}{n^2}\right] = \frac{1}{\pi}\left(-4\pi\,.\frac{\cos n\pi}{n^2}\right)$$

$$= -4\frac{(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - \pi)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3} + (-\pi - \pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\,.\frac{\cos n\pi}{n}\right] = -2\frac{(-1)^n}{n}$$

$$\therefore \quad x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\ldots\right] - 2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \ldots\ldots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right]$$

Putting $x = 0$, we get $0 = -\dfrac{\pi^2}{3} + 4\left(\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots\right)$

$$\Rightarrow \qquad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

Example 4. *Obtain the Fourier series for the function $f(x) = x^2$, $-\pi \le x \le \pi$. Hence show that*

(i) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots\ldots = \displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$ (K.U.K. Jan. 2014)

(ii) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots = \displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^2} = \dfrac{\pi^2}{12}$ (M.D.U. 2011)

(iii) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$.

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $\quad f(x) = x^2 = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

Then $\quad a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi} x^2\,dx = \dfrac{2}{\pi}\left[\dfrac{x^3}{3}\right]_0^{\pi} = \dfrac{2}{3}\pi^2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[2\pi\,.\,\frac{\cos n\pi}{n^2}\right] = 4\frac{(-1)^n}{n^2}$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right) \qquad \ldots(1)$$

Putting $x = \pi$ in (1), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right) \quad \Rightarrow \quad \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\right)$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6} \qquad \text{[Result } (i)]$$

Putting $x = 0$ in (1), we get $0 = \dfrac{\pi^2}{3} - 4\left(\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots\right)$

$$\therefore \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12} \qquad \text{[Result } (ii)]$$

Adding (*i*) and (*ii*), we get $2\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots\right) = \dfrac{\pi^2}{4}$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}\,. \qquad \text{[Result } (iii)]$$

Example 5. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.*

Sol. Let $\quad f(x) = e^{-x} = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$

Then $\quad a_0 = \dfrac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \dfrac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \dfrac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \dfrac{1 - e^{-2\pi}}{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{n}{1+n^2}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$\therefore \quad e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\ldots\right)\right]$$

Example 6. *Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.*

Sol. Let $f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \frac{1}{a\pi}(e^{a\pi} - e^{-a\pi}) = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi} = \frac{1}{\pi(a^2+n^2)}[ae^{a\pi}\cos n\pi - ae^{-a\pi}\cos n\pi]$$

$$= \frac{a\cos n\pi\,(e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

Similarly, $b_n = \frac{-2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$

$$\therefore \quad e^{ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\cos nx - \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\sin nx$$

$$= \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} - a\left(\frac{\cos x}{a^2+1^2} - \frac{\cos 2x}{a^2+2^2} + \frac{\cos 3x}{a^2+3^2} - \ldots\ldots\right)\right.$$
$$\left. + \left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - \ldots\ldots\right)\right]$$

Deduction. Putting $x = 0$ and $a = 1$, we get

$$1 = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{1+1^2} - \frac{1}{1+2^2} + \frac{1}{1+3^2} - \frac{1}{1+4^2} + \ldots\ldots\right)\right]$$

$$\Rightarrow \qquad \frac{\pi}{\sinh \pi} = 2\left(\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots\ldots\right)$$

Example 7. *Express $f(x) = |x|, -\pi < x < \pi$, as Fourier series.* (M.D.U. 2013, Dec. 2015)

Sol. Since $f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \qquad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Note. Putting $x = 0$ in the above result, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Example 8. *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$.*

Deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots = \frac{\pi - 2}{4}$. (M.D.U. 2012)

Sol. Since $x \sin x$ is an even function of x, $b_n = 0$

Let $$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{\pi} = \frac{2}{\pi}(-\pi\cos\pi) = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} x(2 \cos nx \sin x) \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin (n+1) x - \sin (n-1)x] \, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos (n+1)\pi}{n+1} + \frac{\cos (n-1)\pi}{n-1}\right\}\right], \; n \neq 1$$

$$= \frac{\cos (n-1)\pi}{n-1} - \frac{\cos (n+1)\pi}{n+1}$$

When n is odd, $n \neq 1$, $n-1$ and $n+1$ are even

$$\therefore \qquad a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $n-1$ and $n+1$ are odd

$$\therefore \qquad a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $\qquad n = 1$, we have $a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \qquad x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \ldots\ldots\right)$$

Putting $\qquad x = \frac{\pi}{2}$, we get $\frac{\pi}{2} = 1 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \ldots\ldots\right)$

$$\Rightarrow \qquad \frac{\pi}{2} - 1 = 2\left(\frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \ldots\ldots\right) \quad \Rightarrow \quad \frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots$$

Example 9. *Show that for* $-\pi < x < \pi$,

$$\sin ax = \frac{2 \sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \ldots\ldots\right).$$

Sol. Since $\sin ax$ is an odd function of x, $a_0 = 0$ and $a_n = 0$.

Let $\quad \sin ax = \sum_{n=1}^{\infty} b_n \sin nx$

Then $\qquad b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} [\cos (n-a) x - \cos (n+a) x] \, dx$

$$= \frac{1}{\pi}\left[\frac{\sin (n-a)x}{n-a} - \frac{\sin (n+a)x}{n+a}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin (n-a)\pi}{n-a} - \frac{\sin (n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = -\frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$$= (-1)^{n+1} \cdot \frac{2n \sin a\pi}{\pi(n^2 - a^2)}$$

$$\therefore \quad \sin ax = \frac{2 \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - a^2} \sin nx$$

$$= \frac{2 \sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \ldots\ldots\right).$$

Example 10. *Obtain Fourier series for the function f(x) given by*

$$f(x) = 1 + \frac{2x}{\pi}, -\pi \le x \le 0$$

$$= 1 - \frac{2x}{\pi}, 0 \le x \le \pi.$$

(U.K.T.U. 2011)

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (K.U.K. Jan. 2013)

Sol. When $-\pi \le x \le 0, \quad 0 \le -x \le \pi$

$$\therefore \quad f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

When $0 \le x \le \pi, \quad -\pi \le -x \le 0$

$$\therefore \quad f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)$$

$\Rightarrow$ $f(x)$ is an even function of x in $[-\pi, \pi]$. This is also clear from its graph which is symmetrical above the y-axis.

$$\therefore \quad b_n = 0$$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

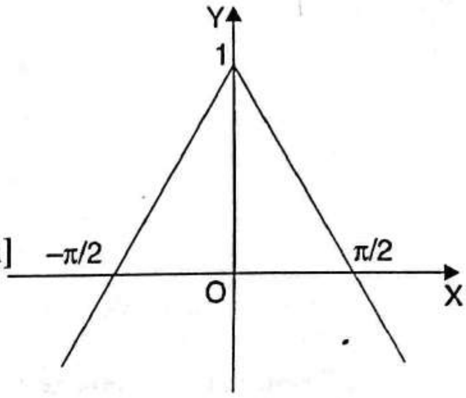
then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{\pi^2 n^2}[1 - (-1)^n]$$

$$\therefore \quad f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} [1-(-1)^n] \frac{\cos nx}{n^2}$$



$$= \frac{4}{\pi^2}\left(\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \dots\dots\right)$$

$[\because \;\; 1-(-1)^n = 0$ when n is even$]$

$$= \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right)$$

Putting $x = 0$, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots = \frac{\pi^2}{8}$, since $f(0) = 1$.

Example 11. *Show that for* $-\pi \le x \le \pi$,

$$\cos cx = \frac{\sin c\pi}{\pi}\left[\frac{1}{c} - \frac{2c\cos x}{c^2-1^2} + \frac{2c\cos 2x}{c^2-2^2} - \dots\dots\right]$$

where c is non-integral. Hence deduce that

$$\pi \operatorname{cosec}(c\pi) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n+c} + \frac{1}{n+1-c}\right].$$ (M.D.U. 2011)

Sol. Since $\cos cx$ is an even function of x, $b_n = 0$

Let $$\cos cx = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \dots(1)$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos cx\, dx = \frac{2}{\pi}\left[\frac{\sin cx}{c}\right]_0^{\pi}$$

$$= \frac{2\sin c\pi}{c\pi},$$ since c is non-integral, $\sin c\pi \neq 0$

Also, $$a_n = \frac{2}{\pi}\int_0^{\pi} \cos cx \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} [\cos(n+c)x + \cos(n-c)x]\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+c)x}{n+c} + \frac{\sin(n-c)x}{n-c}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+c)\pi}{n+c} + \frac{\sin(n-c)\pi}{n-c}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n\pi+c\pi)}{n+c} + \frac{\sin(n\pi-c\pi)}{n-c}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n\sin c\pi}{n+c} + \frac{(-1)^n\sin(-c\pi)}{n-c}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n \sin c\pi}{n+c} - \frac{(-1)^n \sin c\pi}{n-c}\right] = \frac{(-1)^n \sin c\pi}{\pi}\left(\frac{1}{n+c} - \frac{1}{n-c}\right)$$

$$= \frac{(-1)^n \sin c\pi}{\pi}\left[\frac{-2c}{n^2-c^2}\right] = \frac{(-1)^n \sin c\pi}{\pi} \cdot \frac{2c}{c^2-n^2}$$

$\therefore$ From (1), $$\cos cx = \frac{\sin c\pi}{c\pi} + \frac{2c \sin c\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{c^2-n^2} \cos nx$$

$$= \frac{\sin c\pi}{\pi}\left[\frac{1}{c} + 2c\left\{-\frac{\cos x}{c^2-1^2} + \frac{\cos 2x}{c^2-2^2} - \ldots\ldots\right\}\right]$$

$$= \frac{\sin c\pi}{\pi}\left[\frac{1}{c} - \frac{2c \cos x}{c^2-1^2} + \frac{2c \cos 2x}{c^2-2^2} - \ldots\ldots\right]$$

Deduction Put $x = 0$

$$1 = \frac{\sin c\pi}{\pi}\left[\frac{1}{c} - \frac{2c}{c^2-1^2} + \frac{2c}{c^2-2^2} - \ldots\ldots\right]$$

$$\Rightarrow \qquad \pi \operatorname{cosec}(c\pi) = \frac{1}{c} - \frac{(c+1)+(c-1)}{(c+1)(c-1)} + \frac{(c+2)+(c-2)}{(c+2)(c-2)} - \ldots\ldots$$

$$= \frac{1}{c} - \frac{1}{c-1} - \frac{1}{c+1} + \frac{1}{c-2} + \frac{1}{c+2} - \frac{1}{c-3} + \ldots\ldots$$

$$= \left(\frac{1}{c} + \frac{1}{1-c}\right) - \left(\frac{1}{c+1} + \frac{1}{2-c}\right) + \left(\frac{1}{c+2} + \frac{1}{3-c}\right) - \ldots\ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n+c} + \frac{1}{n+1-c}\right]$$

Hence, $$\pi \operatorname{cosec}(c\pi) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n+c} + \frac{1}{n+1-c}\right]$$

EXERCISE 1.1

1. Expand in a Fourier series the function $f(x) = x$ in the interval $0 < x < 2\pi$.
2. Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$.
3. Find the Fourier series for the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$.

(G.B.T.U. 2010)

Hence show that:

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$. (U.K.T.U. 2011)

(ii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$

4. Prove that for all values of x between $-\pi$ and π, $x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots\right]$

(M.D.U. 2011)

5. Obtain the Fourier series to represent e^x in the interval $0 < x < 2\pi$.

6. Find the Fourier series to represent e^x in the interval $-\pi < x < \pi$. *(M.D.U. 2011)*

7. Find the Fourier series to represent the function $f(x) = |\sin x|, -\pi < x < \pi$.

8. Expand $f(x) = |\cos x|$ as a Fourier series in the interval $-\pi < x < \pi$.

(M.D.U. 2010, Dec. 2013; M.T.U. 2011)

9. Prove that in the interval $-\pi < x < \pi$, $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx$.

10. Prove that for $-\pi < x < \pi$, $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \ldots\ldots$

11. (*a*) Obtain a Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$.

[**Hint.** For all integral values of n, $\cos\left(n+\frac{1}{2}\right)\pi = \cos(2n+1)\frac{\pi}{2} = 0 = \cos\left(n-\frac{1}{2}\right)\pi$.

$$\sqrt{1-\cos x} = \sqrt{2\sin^2\frac{x}{2}} = \sqrt{2}\left|\sin\frac{x}{2}\right| = \begin{cases} -\sqrt{2}\sin\frac{x}{2}, & -\pi < x \le 0 \\ \sqrt{2}\sin\frac{x}{2}, & 0 \le x < \pi \end{cases}\Bigg]$$

(*b*) Obtain a Fourier series for $\sqrt{1-\cos x}$ in the interval $(0, 2\pi)$ and hence find the value of

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots$$

12. Express $f(x) = \cos wx$, $-\pi < x < \pi$, where w is a fraction, as a Fourier series. Hence, prove that

$$\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \ldots\ldots$$

13. Find the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when

$$\begin{aligned} f(x) &= \pi + x, && -\pi < x < 0 \\ &= \pi - x, && 0 < x < \pi. \end{aligned}$$

14. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh\pi}$.

15. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.

16. Given $f(x) = \begin{cases} -x+1 & \text{for } -\pi \le x \le 0 \\ x+1 & \text{for } 0 \le x \le \pi \end{cases}$

Is the function even or odd ? Find the Fourier series for $f(x)$ and deduce the value of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

17. Find the Fourier series expansion for $f(x) = x + \frac{x^2}{4}, -\pi \le x \le \pi$.

18. Find the Fourier series of $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$ in the interval $(0, 2\pi)$. Hence deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$. *(M.T.U. 2012)*

19. Find the Fourier series to represent the function $f(x)$ given by $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$

and hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. *(DCRUST, Murthal May 2014)*

20. Obtain the Fourier series expansion for $f(x) = \pi^2 - x^2$ in the interval $-\pi < x < \pi$. *(M.D.U. May 2014)*

21. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} \cos x - \sin x, & -\pi < x \le 0 \\ \cos x + \sin x, & 0 \le x < \pi \end{cases}$$

22. Find the Fourier series expansion of $f(x)$ in $(-\pi, \pi)$, where $f(x) = \sin^3 x + \cos^3 x$.

23. Find the Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$.

Answers

1. $f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}$ 2. $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

3. $f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right)$

5. $e^x = \frac{e^{2\pi} - 1}{2\pi} + \frac{e^{2\pi} - 1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos nx}{1+n^2} - \frac{n}{1+n^2} \sin nx\right)$

6. $e^x = \frac{2 \sinh \pi}{\pi}$

$$\left[\frac{1}{2} - \left(\frac{1}{2}\cos x - \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x - \ldots\ldots\right) - \left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \ldots\ldots\right)\right]$$

7. $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\ldots + \frac{\cos 2nx}{4n^2 - 1} + \ldots\ldots\right)$

8. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\ldots\right)$

11. (*a*) $\sqrt{1 - \cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{4n^2 - 1}$ (*b*) Same as in part (*a*); $\frac{1}{2}$

12. $\cos wx = \frac{2w \sin w\pi}{\pi} \left(\frac{1}{2w^2} + \frac{\cos x}{1^2 - w^2} - \frac{\cos 2x}{2^2 - w^2} + \frac{\cos 3x}{3^2 - w^2} - \ldots\ldots\right)$

13. $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$

14. $e^{-ax} = \frac{2\sinh a\pi}{\pi}\left[\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \ldots\ldots\right) - \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} - \ldots\ldots\right)\right]$

$\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \ldots\ldots\right]$

16. Even. $f(x) = \frac{\pi}{2} + 1 - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right); \frac{\pi^2}{8}$.

17. $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$.

18. $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

19. $f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right)$.

20. $f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$.

21. $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=2,4,6,\ldots} \frac{n}{n^2-1}\cos nx$

22. $f(x) = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x + \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

23. $f(x) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x$.

1.4. DIRICHLET'S CONDITIONS

(M.D.U. May 2014, Dec. 2014, May 2015)

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function $f(x)$ can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where a_0, a_n, b_n are constants, provided

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuities in any one period.

(iii) $f(x)$ has a finite number of maxima and minima.

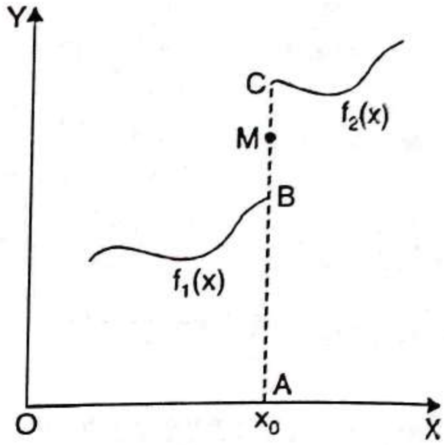
When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left.

i.e., $\frac{1}{2}[f(x+0)+f(x-0)]$

where $f(x+0)$ and $f(x-0)$ denote the limit on the right and the limit on the left respectively.

1.5. FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In Art. 1.3, we derived Euler's formulae for a_0, a_n, b_n on the assumption that $f(x)$ is continous in $(c, c+2\pi)$. However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for a_0, a_n, b_n are to be evaluated by breaking up the range of integration.



Let $f(x)$ be defined by $f(x) = f_1(x),\ c < x < x_0$

$= f_2(x),\ x_0 < x < c + 2\pi$

where x_0 is the point of finite discontinuity in the interval $(c, c+2\pi)$.

The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0-0)$ and $f(x_0+0)$ exist but are unequal. The sum of the Fourier series $= \frac{1}{2}[f(x_0-0)+f(x_0+0)]$ $= \frac{1}{2}(AB + AC) = AM$, where M is the mid-point of BC.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier series to represent the function $f(x)$ given by*

$$f(x) = x \quad \text{for} \quad 0 \le x \le \pi$$

and $$= 2\pi - x \text{ for} \quad \pi \le x \le 2\pi.$$ (G.B.T.U. 2010)

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots = \frac{\pi^2}{8}$.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$

$$= \frac{1}{\pi}\left[\left| \frac{x^2}{2} \right|_0^\pi + \left| 2\pi x - \frac{x^2}{2} \right|_\pi^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_0^\pi x\cos nx\, dx + \int_\pi^{2\pi}(2\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| x\,.\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right|_0^\pi + \left|(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) + \left(-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{2}{\pi n^2}\,[(-1)^n - 1] = \begin{cases} -\dfrac{4}{\pi n^2}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_0^\pi x\sin nx\, dx + \int_\pi^{2\pi}(2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right|_0^\pi \right.$$

$$\left. + \left|(2\pi - x)\times\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right|_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\pi\cos n\pi}{n}\right] = 0$$

$$\therefore \qquad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Putting $x = 0$, we get $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

Example 2. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi, \end{cases}$

prove that $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$. (M.D.U. Dec. 2015; G.B.T.U. 2012)

Hence show that

(i) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots = \frac{1}{2}$ (ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots = \frac{\pi - 2}{4}$.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\,dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}, \; n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \dfrac{1}{2\pi}\left(-\dfrac{1}{n+1} + \dfrac{1}{n-1} + \dfrac{1}{n+1} - \dfrac{1}{n-1}\right), \text{ when } n \text{ is odd} \\ \dfrac{1}{2\pi}\left(\dfrac{1}{n+1} - \dfrac{1}{n-1} + \dfrac{1}{n+1} - \dfrac{1}{n-1}\right), \text{ when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, \text{ when } n \text{ is odd, } i.e., n = 3, 5, 7, \ldots.. \\ -\dfrac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even} \end{cases}$$

When $n = 1$, we have $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\cos x\,dx = \dfrac{1}{2\pi}\int_0^{\pi}\sin 2x\,dx = \dfrac{1}{2\pi}\left[-\dfrac{\cos 2x}{2}\right]_0^{\pi} = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0, \; n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

∴ $$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots..\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1} \quad \ldots(1)$$

Putting $x = 0$ in (1), we have $0 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

$$\Rightarrow \qquad \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots$$

Putting $x = \frac{\pi}{2}$ in (1), we have $1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2 - 1}$

$$\Rightarrow \qquad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$\Rightarrow \qquad \frac{\pi - 2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\ldots = \frac{\pi - 2}{4}.$$

EXERCISE 1.2

1. Find the Fourier series to represent the function

$$f(x) = -k \quad \text{when} \quad -\pi < x < 0$$
$$= k \quad \text{when} \quad 0 < x < \pi$$

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ *(G.B.T.U. 2010, 2013)*

2. (*a*) Develop $f(x)$ in a Fourier series in the interval $(-\pi, \pi)$ if $f(x) = 0$ when $-\pi < x < 0$
$= 1$ when $0 < x < \pi$.

Deduce that sum of the Gregory series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ is $\frac{\pi}{4}$.

(*b*) Find the Fourier series of the function defined by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \pi, & 0 \le x < \pi \end{cases}$$ *(M.D.U. 2011)*

3. Find the Fourier series expansion for $f(x)$, if $f(x) = -\pi, \quad -\pi < x < 0$
$= x, \quad 0 < x < \pi$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. *(M.D.U. Dec. 2013; U.K.T.U. 2012; G.B.T.U. 2011)*

[**Hint.** For the deduction, put $x = 0$ in the expansion of $f(x)$.

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\frac{\pi}{2}$$]

4. Find the Fourier expansion of the function defined in one period by the relations

$$f(x) = 1 \text{ for } 0 < x < \pi$$
$$= 2 \text{ for } \pi < x < 2\pi$$

and deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

5. Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

which is assumed to be periodic with period 2π. *(M.T.U. 2011)*

6. Find the Fourier series of the following function:

$$f(x) = x^2, \quad 0 \le x \le \pi$$
$$= -x^2, \quad -\pi \le x \le 0.$$

7. An alternating current after passing through a rectifier has the form

$$i = I_0 \sin x \quad \text{for} \quad 0 \le x \le \pi$$
$$= 0 \quad \text{for} \quad \pi \le x \le 2\pi$$

where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.

8. Obtain Fourier series for the function

$$f(x) = \begin{cases} x & \text{for } -\pi < x < 0 \\ -x & \text{for } 0 < x < \pi \end{cases}$$ (*M.T.U. 2013*)

and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$ (*G.B.T.U. 2010*)

9. Find the Fourier series for the function

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}. \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$. (*G.B.T.U. 2012*)

Answers

1. $f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

2. (*a*) $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

   (*b*) $f(x) = \frac{\pi}{2} + 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

3. $f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \sin 3x - \frac{\sin 4x}{4} + \ldots\ldots\right)$

4. $f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

5. $f(x) = \frac{\pi^2}{6} - 2\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right)$

$$-\frac{1}{\pi}\left[\left(\frac{4}{1^3} - \frac{\pi^2}{1}\right)\sin x + \frac{\pi^2}{2}\sin 2x + \left(\frac{4}{3^3} - \frac{\pi^2}{3}\right)\sin 3x + \frac{\pi^2}{4}\sin 4x + \ldots\ldots\right]$$

6. $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots\ldots$

7. $i = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \ldots\ldots\right)$

8. $f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$

9. $f(x) = \frac{2}{\pi}\left(\sin x - \sin 2x + \frac{\sin 3x}{3} - \ldots\ldots\right)$

1.6. CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier series over an interval of length $2l$ and not 2π. In order to apply foregoing theory, this interval must be transformed into an interval of length 2π. This can be achieved by a transformation of the variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$. To change the interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \quad \text{so that when } x = c,\ z = \frac{\pi c}{l} = d \text{ (say)}$$

and when $\quad x = c + 2l, \quad z = \frac{\pi(c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi.$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the function $f\left(\frac{lz}{\pi}\right)$ $= \mathrm{F}(z)$, say, of period 2π in $(d, d + 2\pi)$ and the latter function can be expressed as the Fourier series

$$\mathrm{F}(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \ldots(1)$$

where $a_0 = \frac{1}{\pi}\int_d^{d+2\pi} \mathrm{F}(z)\,dz$; $a_n = \frac{1}{\pi}\int_d^{d+2\pi} \mathrm{F}(z)\cos nz\,dz$; and $b_n = \frac{1}{\pi}\int_d^{d+2\pi} \mathrm{F}(z)\sin nz\,dz \quad \ldots(2)$

Now making the inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}\,dx$

When $z = d$, $x = c$ and when $z = d + 2\pi$, $x = c + 2l$.

The expression (1) becomes $\mathrm{F}(z) = \mathrm{F}\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

and the co-efficients a_0, a_n, b_n from (2) reduce to

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx; \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx; \quad \text{and} \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Hence the Fourier series for f(x) in the interval c < x < c + 2l is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$ and $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx.$

Cor. 1. If we put $c = 0$, the interval becomes $0 < x < 2l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx; \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx; \quad \text{and} \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Cor. 2. If we put $c = -l$, the interval becomes $-l < x < l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{and} \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

If f(x) is an even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{and} \quad b_n = 0$$

If f(x) is an odd function, we have $a_0 = 0, \quad a_n = 0$

and
$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find Fourier expansion for the function $f(x) = x - x^2$, $-1 < x < 1$.*

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos n\pi x + \sum\limits_{n=1}^{\infty} b_n \sin n\pi x$

Then
$$a_0 = \int_{-1}^{1}(x - x^2)\,dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} x^2\,dx = 0 - 2\int_0^1 x^2\,dx = -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$$

$$a_n = \int_{-1}^{1}(x - x^2)\cos n\pi x\,dx = \int_{-1}^{1} x\cos n\pi x\,dx - \int_{-1}^{1} x^2\cos n\pi x\,dx$$

$$= 0 - 2\int_0^1 x^2\cos n\pi x\,dx = -2\left[x^2\cdot\frac{\sin n\pi x}{n\pi} - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1}(x - x^2)\sin n\pi x\,dx = \int_{-1}^{1} x\sin n\pi x\,dx - \int_{-1}^{1} x^2\sin n\pi x\,dx$$

$$= 2\int_0^1 x\sin n\pi x\,dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1.\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

$$\therefore \quad x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \ldots\ldots\right)$$
$$+ \frac{2}{\pi}\left(\frac{\sin\pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \ldots\ldots\right)$$

Example 2. *Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.*

(M.D.U. 2011)

Sol. Since $f(x)$ is an even function, $b_n = 0$.

Let $f(x) = x^2 - 2 = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \dfrac{n\pi x}{2}$

Then $a_0 = \dfrac{2}{2}\int_0^2 (x^2 - 2)\, dx = \left[\dfrac{x^3}{3} - 2x\right]_0^2 = \dfrac{8}{3} - 4 = -\dfrac{4}{3}$

$$a_n = \frac{2}{2}\int_0^2 (x^2 - 2)\cos\frac{n\pi x}{2}\, dx$$

$$= \left[(x^2 - 2)\cdot\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \frac{16\cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$\therefore$ $x^2 - 2 = -\dfrac{2}{3} - \dfrac{16}{\pi^2}\left(\cos\dfrac{\pi x}{2} - \dfrac{1}{4}\cos \pi x + \dfrac{1}{9}\cos\dfrac{3\pi x}{2} - \ldots\ldots\right).$

Example 3. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

Sol. Let $f(x) = e^{-x} = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$

Then $a_0 = \dfrac{1}{l}\int_{-l}^{l} e^{-x}\, dx = \dfrac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \dfrac{1}{l}(e^l - e^{-l}) = \dfrac{2\sinh l}{l}$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\, dx = \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$\left[\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{l}{l^2+(n\pi)^2}\left[-e^{-l}\cos n\pi + e^{l}\cos n\pi\right] = -\frac{2l\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^l - e^{-l}}{2}\right) = \frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= -\frac{1}{l^2+(n\pi)^2}\left[\frac{n\pi}{l}(e^{-l}-e^{l})\cos n\pi\right] = \frac{2n\pi\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^l-e^{-l}}{2}\right) = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$\therefore \quad e^{-x} = \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \ldots\ldots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\ldots\right)\right].$$

Example 4. *Obtain Fourier series for the function* $f(x) = \pi x, \quad 0 \le x \le 1$
$= \pi(2-x), \quad 1 \le x \le 2.$

(M.D.U. May 2013)

Sol. Let $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \displaystyle\int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2-\frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x\,.\,\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\,.\,\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ or $-\dfrac{4}{n^2\pi}$ according as n is even or odd.

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right).$$

Note. Putting $x = 0$, we have $f(0) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots\right)$

or $$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

EXERCISE 1.3

1. Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$.
2. Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when $f(x) = 0, \quad -2 < x < 0$
$= 1, \quad 0 < x < 2.$
3. Develop $f(x)$ in a Fourier series in the interval $(0, 2)$ if $f(x) = x, \quad 0 < x < 1$
$= 0, \quad 1 < x < 2.$
4. Find the Fourier expansion for $f(x) = \pi x$ from $x = -c$ to $x = c$.
5. Find the Fourier expansion for the function $f(x) = x - x^3$ in the interval $-1 < x < 1$.
6. (*a*) Find the Fourier series for the function given by $f(t) = t, \quad 0 < t < 1$
$= 1 - t, \quad 1 < t < 2.$

Hence deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ *(G.B.T.U. 2011)*

(*b*) Find the Fourier series for the function

$$f(x) = \begin{cases} x, & -1 < x \le 0 \\ x+2, & 0 < x < 1 \end{cases}$$

where $f(x) = f(x+2)$

(*c*) Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$ *(K.U.K. Dec. 2015)*

7. Find a Fourier series to represent x^2 in the interval $(-l, l)$.
8. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-2, 2)$.
9. Expand: $$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \\ 1, & 1 < x < \frac{3}{2} \\ x - 1, & \frac{3}{2} < x < 2 \end{cases}$$

as a Fourier series.

10. Find the Fourier series for the function $f(x) = \begin{cases} 0 & \text{when } -2 < x < -1 \\ k & \text{when } -1 < x < 1 \\ 0 & \text{when } 1 < x < 2. \end{cases}$
11. A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negative portion of the wave.

Expand the resulting periodic function $u(t) = \begin{cases} 0 & \text{when } -\frac{T}{2} < t < 0 \\ E \sin \omega t & \text{when } 0 < t < \frac{T}{2} \end{cases}$

and $T = \dfrac{2\pi}{\omega}$, in a Fourier series.

Hence deduce that $\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$. (*G.B.T.U. 2011*)

12. Obtain the Fourier series expansion of

$f(x) = \left(\dfrac{\pi - x}{2}\right)$ for $0 < x < 2$. (*M.T.U. 2011*)

13. Find the Fourier series expansion of $f(x) = 1 + |x|$ defined in $-3 < x < 3$. (*M.T.U. 2011*)

14. Obtain the Fourier series for $f(x) = \begin{cases} -1 & \text{in } -1 < x < 0 \\ 2x & \text{in } 0 < x < 1 \end{cases}$

Hence show that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

and $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots = \dfrac{\pi}{4}$.

[**Hint:** For deductions, put $x = 0$ and $x = \dfrac{1}{2}$]

15. Obtain the Fourier series expansion of $f(x) = \begin{cases} 1 & \text{in } 0 < x < 1 \\ 2 & \text{in } 1 < x < 3 \end{cases}$.

16. Obtain the Fourier series expansion of $f(x) = x^2$ defined in the interval $0 < x < 2l$. Hence deduce that

(*i*) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$ (*ii*) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots = \dfrac{\pi^2}{12}$

(*iii*) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

[**Hint:** (*i*) Put $x = 0$ (*ii*) Put $x = l$ (*iii*) Add]

Answers

1. $1 - t^2 = \dfrac{2}{3} + \dfrac{4}{\pi^2}\left(\cos \pi t - \dfrac{\cos 2\pi t}{2^2} + \dfrac{\cos 3\pi t}{3^2} - \ldots\ldots\right)$

2. $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi}\left(\sin \dfrac{\pi x}{2} + \dfrac{1}{3} \sin \dfrac{3\pi x}{2} + \dfrac{1}{5} \sin \dfrac{5\pi x}{2} + \ldots\ldots\right)$

3. $f(x) = \dfrac{1}{4} - \dfrac{2}{\pi^2}\left(\cos \pi x + \dfrac{\cos 3\pi x}{3^2} + \dfrac{\cos 5\pi x}{5^2} + \ldots\ldots\right) + \dfrac{1}{\pi}\left(\sin \pi x - \dfrac{\sin 2\pi x}{2} + \dfrac{\sin 3\pi x}{3} - \ldots\ldots\right)$

4. $f(x) = 2c\left[\sin\left(\dfrac{\pi x}{c}\right) - \dfrac{1}{2}\sin\left(\dfrac{2\pi x}{c}\right) + \dfrac{1}{3}\sin\left(\dfrac{3\pi x}{c}\right) - \ldots\ldots\right]$

5. $f(x) = \dfrac{12}{\pi^3}\left(\sin \pi x - \dfrac{\sin 2\pi x}{2^3} + \dfrac{\sin 3\pi x}{3^3} - \ldots\ldots\right)$

6. (a) $f(t) = -\dfrac{4}{\pi^2}\left(\cos \pi t + \dfrac{\cos 3\pi t}{3^2} + \dfrac{\cos 5\pi t}{5^3} + \ldots\ldots\right) + \dfrac{2}{\pi}\left(\sin \pi t + \dfrac{\sin 3\pi t}{3} + \ldots\ldots\right)$

(b) $f(x) = 1 + \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1-2(-1)^n}{n} \sin n\pi x = 1 + \dfrac{2}{\pi}\left(3 \sin \pi x - \dfrac{\sin 2\pi x}{2} + \sin 3\pi x - \dfrac{\sin 4\pi x}{4} + \ldots\ldots\right)$

(c) $2x - x^2 = -\sum_{n=1}^{\infty} \dfrac{9}{n^2\pi^2} \cos \dfrac{2n\pi x}{3} + \sum_{n=1}^{\infty} \dfrac{3}{n\pi} \sin \dfrac{2n\pi x}{3}$

(For deduction, put $x = \dfrac{3}{2}$)

7. $x^2 = \dfrac{l^2}{3} - \dfrac{4l^2}{\pi^2}\left(\dfrac{\cos \pi x/l}{1^2} - \dfrac{\cos 2\pi x/l}{2^2} + \dfrac{\cos 3\pi x/l}{3^2} - \dfrac{\cos 4\pi x/l}{4^2} + \ldots\ldots\right)$

8. $e^{-x} = \sinh 2\left[\dfrac{1}{2} - 4\left(\dfrac{1}{2^2+\pi^2}\cos\dfrac{\pi x}{2} - \dfrac{1}{2^2+2^2\pi^2}\cos \pi x + \dfrac{1}{2^2+3^2\pi^2}\cos\dfrac{3\pi x}{2} - \ldots\ldots\right)\right.$

$\left. - 2\pi\left(\dfrac{1}{2^2+\pi^2}\sin\dfrac{\pi x}{2} - \dfrac{2}{2^2+2^2\pi^2}\sin \pi x + \dfrac{3}{2^2+3^2\pi^2}\sin\dfrac{3\pi x}{2} - \ldots\ldots\right)\right]$

9. $f(x) = \dfrac{7}{16} + \dfrac{1}{\pi}\left(\dfrac{1}{\pi} - \dfrac{1}{2}\right)\cos \pi x - \dfrac{1}{\pi}\left(\dfrac{1}{\pi} + \dfrac{3}{2}\right)\sin \pi x + \dfrac{3}{2\pi^2}\cos 2\pi x + \dfrac{1}{4\pi}\sin 2\pi x + \ldots\ldots$

10. $f(x) = \dfrac{k}{2} + \dfrac{2k}{\pi}\left(\cos\dfrac{\pi x}{2} - \dfrac{1}{3}\cos\dfrac{3\pi x}{2} + \dfrac{1}{5}\cos\dfrac{5\pi x}{2} - \ldots\ldots\right)$

11. $u(t) = \dfrac{E}{\pi} + \dfrac{E}{2}\sin \omega t - \dfrac{2E}{\pi}\left(\dfrac{\cos 2\omega t}{1.3} + \dfrac{\cos 4\omega t}{3.5} + \dfrac{\cos 6\omega t}{5.7} + \ldots\ldots\right)$

12. $f(x) = \dfrac{\pi - 1}{2} + \dfrac{1}{\pi}\sum_{n=1}^{\infty} \dfrac{\sin n\pi x}{n}$.

13. $f(x) = \dfrac{5}{2} - \dfrac{12}{\pi^2}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\right)$

14. $f(x) = -\dfrac{4}{\pi^2}\left(\cos \pi x + \dfrac{\cos 3\pi x}{3^2} + \dfrac{\cos 5\pi x}{5^2} + \ldots\right)$

$+ \dfrac{2}{\pi}\left(2\sin \pi x - \dfrac{\sin 2\pi x}{2} + \dfrac{2\sin 3\pi x}{3} - \dfrac{\sin 4\pi x}{4} + \dfrac{2\sin 5\pi x}{5} - \ldots\right)$

15. $f(x) = \dfrac{5}{3} - \dfrac{1}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\sin\dfrac{2n\pi}{3}\cos\dfrac{2n\pi x}{3} + \dfrac{1}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\left(\cos\dfrac{2n\pi}{3} - 1\right)\sin\dfrac{2n\pi x}{3}$

16. $f(x) = \dfrac{4l^2}{3} + \dfrac{4l^2}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2}\cos\dfrac{n\pi x}{l} - \dfrac{4l^2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\sin\dfrac{n\pi x}{l}$

1.7. HALF RANGE SERIES

Sometimes it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or more generally in the range $(0, l)$ in a Fourier series of period $2l$.

If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

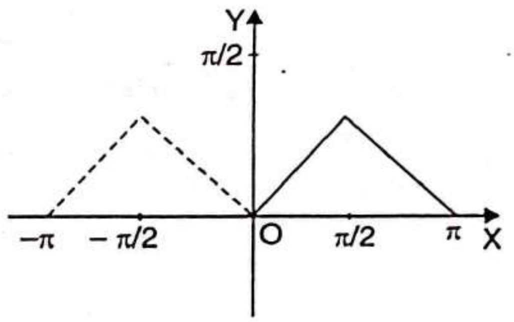
If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

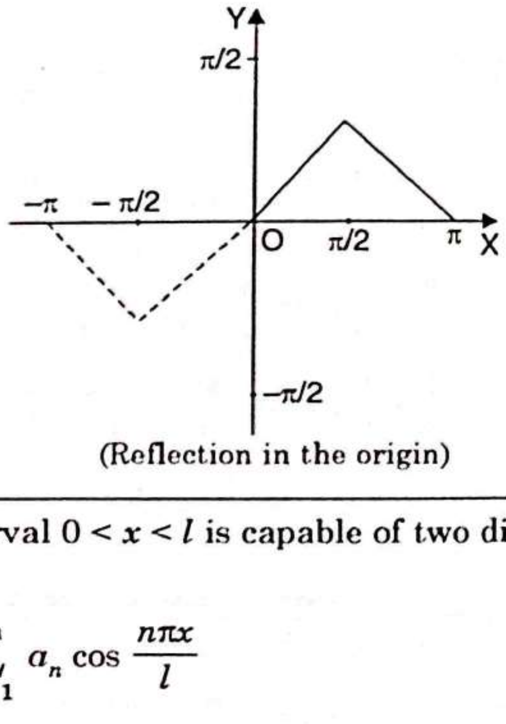
For example, consider the function

$$f(x) = x, \qquad 0 < x < \frac{\pi}{2}$$

$$= \pi - x \qquad \frac{\pi}{2} < x < \pi$$



(Reflection in the y-axis)



(Reflection in the origin)

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half-range series.

The half-range cosine series is $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

where $\quad a_0 = \dfrac{2}{l} \displaystyle\int_0^l f(x)\, dx; \quad a_n = \dfrac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx.$

The half-range sine series is $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, where $b_n = \dfrac{2}{l} \displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$

Cor. If the range is $0 < x < \pi$, then

(*i*) The half-range cosine series is $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

where $\quad a_0 = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x)\, dx; \quad a_n = \dfrac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx.$

(*ii*) The half-range sine series is $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$, where $\quad b_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x) \sin nx\, dx.$

ILLUSTRATIVE EXAMPLES

Example 1. *Expand $\pi x - x^2$ in a half-range sine series in the interval $(0, \pi)$ upto the first three terms.*

Sol. Let $\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$, then

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\,[1 - (-1)^n]$$

$$= 0 \quad \text{or} \quad \frac{8}{\pi n^3} \text{ according as } n \text{ is even or odd.}$$

$$\therefore \quad \pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\ldots\right).$$

Example 2. *If* $f(x) = x, \quad 0 < x < \frac{\pi}{2}$

$$= \pi - x, \quad \frac{\pi}{2} < x < \pi$$

show that (i) $f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right].$ (M.D.U. 2012)

Sol. (*i*) **For the half-range sine series**

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$$

(ii) **For the half-range cosine series**

Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\, dx = \dfrac{2}{\pi}\left[\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x \cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[x \cdot \frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x) \cdot \frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n} \sin\frac{n\pi}{2} + \frac{1}{n^2} \cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n} \sin\frac{n\pi}{2} + \frac{1}{n^2} \cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2} \cos\frac{n\pi}{2} - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[2 \cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$\therefore \quad a_1 = 0,\ a_2 = \dfrac{2}{\pi . 2^2}(2 \cos \pi - \cos 2\pi - 1) = \dfrac{-2}{\pi . 1^2}$,

$a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \dfrac{2}{\pi . 6^2}(2 \cos 3\pi - \cos 6\pi - 1) = \dfrac{-2}{\pi . 3^2}$,

$a_7 = 0,\ a_8 = 0,\ a_9 = 0,\ a_{10} = \dfrac{2}{\pi . 10^2}(2 \cos 5\pi - \cos 10\pi - 1) = \dfrac{-2}{\pi . 5^2}$, ...

Hence $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{1^2} + \dfrac{\cos 6x}{3^2} + \dfrac{\cos 10x}{5^2} + \ldots\ldots\right]$.

Example 3. *Find a series of cosines of multiples of x which will represent x sin x in the interval (0, π) and show that* $\dfrac{1}{1 \cdot 3} - \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} - \ldots\ldots = \dfrac{\pi - 2}{4}$. (M.T.U. 2013)

Sol. Let $x \sin x = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \dfrac{2}{\pi}\int_0^{\pi} x \sin x\, dx = \dfrac{2}{\pi}\Big[x(-\cos x) - 1 . (-\sin x)\Big]_0^{\pi} = \dfrac{2}{\pi}[-\pi \cos \pi] = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\, (2 \cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin (n+1)\,x - \sin (n-1)\,x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos (n+1)x}{n+1}+\frac{\cos (n-1)x}{n-1}\right\}-1.\left\{-\frac{\sin (n+1)x}{(n+1)^2}-\frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos (n+1)\pi}{n+1}+\frac{\pi\cos (n-1)\pi}{n-1}\right], \quad \text{when } n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{1}{n-1}-\frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-1.\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \quad x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1.3}-\frac{\cos 3x}{2.4}+\frac{\cos 4x}{3.5}-\ldots\ldots\right).$$

Putting $x = \frac{\pi}{2}$, we get $\frac{\pi}{2} = 1 - 2\left(\frac{-1}{1.3}+\frac{1}{3.5}+\frac{-1}{5.7}-\ldots\ldots\right)$

$$\Rightarrow \qquad 1+\frac{2}{1\cdot3}-\frac{2}{3\cdot5}+\frac{2}{5\cdot7}-\ldots\ldots = \frac{\pi}{2}$$

$$\Rightarrow \qquad \frac{2}{1\cdot3}-\frac{2}{3\cdot5}+\frac{2}{5\cdot7}-\ldots\ldots = \frac{\pi}{2}-1$$

Hence $$\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\ldots\ldots = \frac{\pi-2}{4}.$$

Example 4. *Obtain the half-range sine series for e^x in $0 < x < 1$.*

Sol. Let $e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x$, (since $l = 1$)

Then $$b_n = 2\int_0^1 e^x \sin n\pi x\,dx = 2\left[\frac{e^x}{1+(n\pi)^2}(\sin n\pi x - n\pi\cos n\pi x)\right]_0^1$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= 2\left[\frac{e}{1+(n\pi)^2}(-n\pi\cos n\pi)-\frac{1}{1+(n\pi)^2}(-n\pi)\right]$$

$$= \frac{2}{1+n^2\pi^2}\,[-en\pi\,(-1)^n + n\pi] = \frac{2n\pi}{1+n^2\pi^2}\,[1 - e\,(-1)^n]$$

Hence $e^x = 2\pi \displaystyle\sum_{n=1}^{\infty} \frac{n[1-e(-1)^n]}{1+n^2\pi^2}$

$$= 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + \ldots\ldots\right].$$

Example 5. *Develop* $\sin\left(\frac{\pi x}{l}\right)$ *in half-range cosine series in the range* $0 < x < l$.

Sol. Let $\sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then
$$a_0 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right]_0^l = -\frac{2}{\pi}\,[\cos\pi - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^l, \qquad \text{when } n \neq 1$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When $n = 1$, we have

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\,dx = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{-\cos\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right]_0^l = -\frac{1}{2\pi}\,(\cos 2\pi - \cos 0)$$

$$= -\frac{1}{2\pi}\,(1 - 1) = 0.$$

When n is odd, $n \neq 1$, $a_n = \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$, also $a_1 = 0$

When n is even, $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{4}{\pi(n+1)(n-1)}$$

$$\therefore \qquad \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7} + \ldots\ldots\right].$$

Example 6. *Obtain a half-range cosine series for*

$$f(x) = kx \qquad \text{for} \quad 0 \le x \le \frac{l}{2}$$

$$= k(l - x) \qquad \text{for} \quad \frac{l}{2} \le x \le l.$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$ (M.D.U. Dec. 2014)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2}{l}\left[\left|\frac{kx^2}{2}\right|_0^{l/2} + \left|k\left(lx - \frac{x^2}{2}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\cdot\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cdot\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left|kx\cdot\frac{1}{n\pi}\sin\frac{n\pi x}{l} + k\cdot\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_0^{l/2}\right.$$

$$\left. + \left|k(l-x)\,.\,\frac{1}{n\pi}\sin\frac{n\pi x}{l} - k\,.\,\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left|\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right|\right.$$

$$\left. + \left|\frac{-kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right|\right]$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right] = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1 \quad \therefore \quad a_n = 0 \Rightarrow a_1 = a_3 = a_5 = \ldots\ldots = 0$

When n is even, $a_2 = \frac{2kl}{2^2\pi^2}[2\cos\pi - 1 - \cos 2\pi] = -\frac{8kl}{2^2\pi^2}$;

$$a_4 = \frac{2kl}{4^2\pi^2}[2\cos 2\pi - 1 - \cos 4\pi] = 0$$

$$a_6 = \frac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - \cos 6\pi]$$

$$= \frac{2kl}{6^2\pi^2}(-2-1-1) = -\frac{8kl}{6^2\pi^2} \text{ and so on.}$$

$\therefore$
$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\ldots\right) \quad \ldots(1)$$

Putting $x = l, f(l) = 0$

$\therefore$ From (1), we have $0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots\right)$

$$\Rightarrow \quad \frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots = \frac{\pi^2}{32} \Rightarrow \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right] = \frac{\pi^2}{32}$$

Hence
$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

EXERCISE 1.4

1. (a) Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

   (b) Prove that for $0 < x < l$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\right)$$

2. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$.

   (M.D.U. Dec. 2015; K.U.K. Jan. 2014)

3. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence show that

   (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$

   (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}$.

4. (a) Express $\sin x$ as a cosine series in $0 < x < \pi$.

(b) Show that a constant function c can be expanded in an infinite series

$$\frac{4c}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right) \text{ in the range } 0 < x < \pi.$$

(c) Find the Fourier half-range sine series of $f(x) = 1,\ 0 \le x \le 2$.

5. If
$$f(x) = \begin{cases} \dfrac{\pi}{3}, & 0 \le x \le \dfrac{\pi}{3} \\ 0, & \dfrac{\pi}{3} < x < \dfrac{2\pi}{3} \\ -\dfrac{\pi}{3}, & \dfrac{2\pi}{3} \le x \le \pi \end{cases}$$

then show that $f(x) = \dfrac{2}{\sqrt{3}}\left[\cos x - \dfrac{\cos 5x}{5} + \dfrac{\cos 7x}{7} - \ldots\ldots\right]$.

6. If
$$f(x) = \begin{cases} mx, & 0 \le x \le \dfrac{\pi}{2} \\ m(\pi - x), & \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

then show that $f(x) = \dfrac{4m}{\pi}\left[\dfrac{\sin x}{1^2} - \dfrac{\sin 3x}{3^2} + \dfrac{\sin 5x}{5^2} - \ldots\ldots\right]$.

7. Express $f(x) = x$ as a half-range.

(i) sine series in $0 < x < 2$. *(M.D.U. Dec. 2013, Dec. 2014, May 2015; U.K.T.U. 2011)*

(ii) cosine series in $0 < x < 2$. *(K.U.K. 2010, May 2013; M.D.U. 2010)*

8. Find the Fourier sine and cosine series of

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$$

9. Show that the series $\dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{l}{n} \sin \dfrac{2n\pi x}{l}$ represents $\dfrac{1}{2}\, l - x$ when $0 < x < l$.

10. Find the half-range sine series for $f(x) = \begin{cases} \dfrac{1}{4} - x, & 0 < x < \dfrac{1}{2} \\ x - \dfrac{3}{4}, & \dfrac{1}{2} < x < 1. \end{cases}$ *(U.K.T.U. 2012)*

11. Represent the following function by Fourier sine series

$$f(x) = \begin{cases} 1 \text{ when } 0 < x < \dfrac{l}{2} \\ 0 \text{ when } \dfrac{l}{2} < x < l. \end{cases}$$

12. Find the half-range sine series for the function $f(t) = t - t^2,\ 0 < t < 1$. *(M.T.U. 2011)*

13. Prove that for $0 < x < \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right)$$

14. Let $f(x) = \begin{cases} \omega x, & \text{when } 0 \le x \le \dfrac{l}{2} \\ \omega(l - x), & \text{when } \dfrac{l}{2} \le x \le l \end{cases}$

Show that $f(x) = \dfrac{4\omega l}{\pi^2} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{l}$

Hence obtain the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$ *(G.B.T.U. 2011)*

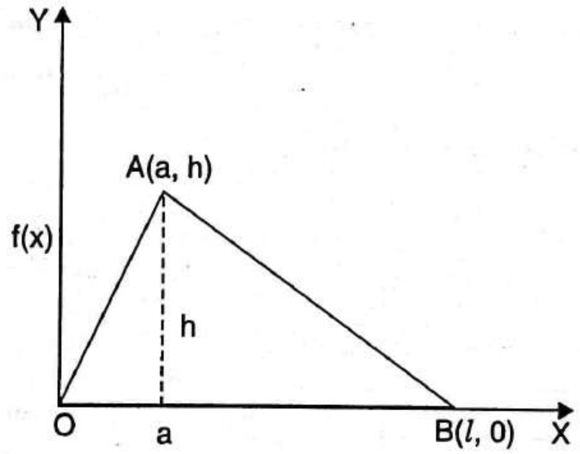
15. If $f(x) = \begin{cases} \sin x, & \text{for } 0 \le x < \dfrac{\pi}{4} \\ \cos x, & \text{for } \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \end{cases}$ expand $f(x)$ in a series of sines. *(M.T.U. 2011)*

16. Find the half range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$. Hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$ *(M.T.U. 2012)*

17. Obtain the half-range sine series of the function $f(x) = x \sin x$ in $0 < x < \pi$. *(K.U.K. Dec. 2015)*

18. For the function defined by the graph OAB, find the half-range Fourier sine series.



Answers

1. (a) $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\ldots\right)$; $2\left(\sin x - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \ldots\ldots\right)$

2. $\dfrac{\pi^2}{3} - 4\left[\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \dfrac{\cos 4x}{4^2} + \ldots\ldots\right]$

3. $\dfrac{1}{3} + \dfrac{4}{\pi^2}\left(\dfrac{\cos \pi x}{1^2} + \dfrac{\cos 2\pi x}{2^2} + \dfrac{\cos 3\pi x}{3^2} + \ldots\ldots\right)$ **4.** (a) $\dfrac{2}{\pi} - \dfrac{4}{\pi}\left[\dfrac{\cos 2x}{1 \cdot 3} + \dfrac{\cos 4x}{3 \cdot 5} + \dfrac{\cos 6x}{5 \cdot 7} + \ldots\ldots\right]$

(c) $1 = \dfrac{4}{\pi}\left(\sin \dfrac{\pi x}{2} + \dfrac{1}{3} \sin \dfrac{3\pi x}{2} + \dfrac{1}{5} \sin \dfrac{5\pi x}{2} + \ldots\ldots\right)$

7. (i) $\frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\frac{1}{4}\sin\frac{4\pi x}{2}+\ldots\ldots\right)$

(ii) $1-\frac{8}{\pi^2}\left[\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\ldots\ldots\right]$

8. (i) $f(x)=\frac{2}{\pi}\left(\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\ldots\right)+\left(\frac{\sin 2x}{2}-\frac{\sin 4x}{4}+\frac{\sin 6x}{6}-\ldots\ldots\right)$

(ii) $f(x)=\frac{\pi}{8}+\frac{2}{\pi}\left[\left(\frac{\pi}{2}-1\right)\cos x-\frac{1}{2}\cos 2x-\left(\frac{\pi}{6}+\frac{1}{3^2}\right)\cos 3x+\left(\frac{\pi}{10}-\frac{1}{5^2}\right)\cos 5x\ldots\ldots\right]$

10. $f(x)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin \pi x+\left(\frac{1}{3\pi}+\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+\cdots$

11. $f(x)=\frac{2}{\pi}\left[\sin\frac{\pi x}{l}+\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\ldots\ldots\right]$

12. $\frac{8}{\pi^3}\left(\frac{\sin \pi t}{1^3}+\frac{\sin 3\pi t}{3^3}+\frac{\sin 5\pi t}{5^3}+\ldots\ldots\right)$

14. $\frac{\pi^2}{8}$

15. $\frac{4\sqrt{2}}{\pi}\left(\frac{\sin 2x}{1\cdot 3}-\frac{\sin 6x}{5\cdot 7}+\frac{\sin 10x}{9\cdot 11}\ldots\ldots\right)$

16. $f(x)=\frac{8l^2}{\pi^3}\left(\frac{1}{1^3}\sin\frac{\pi x}{l}+\frac{1}{3^3}\sin\frac{3\pi x}{l}+\frac{1}{5^3}\sin\frac{5\pi x}{l}+\cdots\right)$.

17. $f(x)=\left(\frac{\pi}{2}+\frac{2}{\pi}\right)\sin x-\frac{8}{\pi}\left[\frac{2\sin 2x}{(3\times 1)^2}+\frac{4\sin 4x}{(5\times 3)^2}+\frac{6\sin 6x}{(7\times 5)^2}+\ldots..\right]$.

18. $\frac{2l^2h}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$.

1.8. PARSEVAL'S THEOREM ON FOURIER CONSTANTS

If the Fourier series of f(x) over an interval $c<x<c+2l$ is given as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\}$$

then
$$\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Proof. The Fourier series of $f(x)$ in $c<x<c+2l$ is given as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\} \qquad \ldots(1)$$

where $a_0=\frac{1}{l}\int_c^{c+2l}f(x)\,dx$; $a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\frac{n\pi x}{l}\,dx$; $b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\frac{n\pi x}{l}\,dx \qquad \ldots(2)$

Multiplying both sides of (1) by $f(x)$, we have

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n\pi x}{l}$$

Integrating both sides w.r.t. x, between the limits c to $c + 2l$, we have

$$\int_c^{c+2l} [f(x)]^2 \, dx = \frac{a_0}{2} \int_c^{c+2l} f(x) \, dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{a_0}{2} . la_0 + \sum_{n=1}^{\infty} a_n (la_n) + \sum_{n=1}^{\infty} b_n (lb_n) \qquad \text{[Using (2)]}$$

$$\Rightarrow \quad \int_c^{c+2l} [f(x)]^2 \, dx = \frac{la_0^2}{2} + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \quad \frac{1}{2l} \int_c^{c+2l} [f(x)]^2 \, dx = \frac{1}{2l} \left\{ \frac{la_0^2}{2} + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$$

or
$$\frac{1}{2l} \int_c^{c+2l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \qquad \text{(Parseval's identity)}$$

Hence the proof.

Note. *Parseval's identities in different cases:*

(*i*) If $c = 0$, the interval becomes $0 < x < 2l$ and Parseval's identity reduces to

$$\frac{1}{2l} \int_0^{2l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

If $c = -l$, the interval becomes $-l < x < l$ and Parseval's identity reduces to

$$\frac{1}{2l} \int_{-l}^{l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

(*ii*) If $f(x)$ is an even function in $(-l, l)$ then $\frac{2}{l} \int_0^{l} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(*iii*) If $f(x)$ is an odd function in $(-l, l)$ then $\frac{2}{l} \int_0^{l} [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2$

(*iv*) If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ in $(0, l)$ then $\frac{2}{l} \int_0^{l} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(*v*) If $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ in $(0, l)$ then $\frac{2}{l} \int_0^{l} [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Sol. We require half-range Fourier sine series for 1 in $(0, \pi)$

Let $$1 = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Then $$b_n = \frac{2}{\pi}\int_0^{\pi} (1) \sin nx \, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

Now $b_n = 0$ when n is even; and $b_n = \dfrac{4}{n\pi}$ when n is odd.

Substituting in (1), we get

$$\therefore \quad 1 = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} \sin (2m-1)x \quad \text{or} \quad 1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right) \qquad \ldots(2)$$

Now from Parseval's theorem on Fourier constants

$$\int_c^{c+2l} [f(x)]^2 \, dx = 2l\left[\frac{{a_0}^2}{4} + \frac{l}{2}\sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2)\right] \qquad \ldots(3)$$

Applying (3) to half-range sine series for 1 in $(0, \pi)$

$$c = 0,\ 2l = \pi,\ f(x) = 1,\ a_0 = 0,\ a_n = 0, \text{ and } b_n = \frac{4}{(2m-1)\pi},\ m = 1, 2 \ldots\ldots$$

We get, $$\int_0^{\pi} (1)^2 \, dx = \pi \cdot \frac{1}{2} \sum_{m=1}^{\infty} \frac{16}{(2m-1)^2} \cdot \pi^2$$

$$\Rightarrow \qquad \Big[x\Big]_0^{\pi} = \frac{8}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right\} \quad \text{or} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Hence the result.

Example 2. *Find Fourier series of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that*

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots$$

Sol. The Fourier series of x^2 in $(-\pi, \pi)$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad ...(1)$$

Here $$a_0 = \frac{2\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0,\ f(x) = x^2$$

Now by Parseval's identity from (1), we get

$$\int_{-\pi}^{\pi} (x^2)^2\, dx = 2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4}\right]$$

$$\Rightarrow \quad \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \pi \sum_{n=1}^{\infty} \frac{16}{n^4} \quad \text{or} \quad \frac{2\pi^5}{5} - \frac{2\pi^5}{9} = \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

or $$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{or} \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots = \frac{\pi^4}{90}.$$

EXERCISE 1.5

1. If $f(x)$ has the Fourier series expansion

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ in } a \le x \le a + 2l$$

show that $\displaystyle\int_a^{a+2l} [f(x)]^2\, dx = 2l\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$.

2. If $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ in $0 < x < l$, then show that

$$\int_0^l [f(x)]^2\, dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right].$$

3. If $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0, l)$, then show that $\displaystyle\int_0^l [f(x)]^2\, dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2$.

4. Prove that in the range $(0, l)$, $x = \dfrac{l}{2} - \dfrac{4l}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2m-1)^2} \cos\frac{(2m-1)\pi x}{l}$ and deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}.$$

5. Show that for $0 < x < \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right)$$

and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

1.9. TYPICAL WAVEFORMS

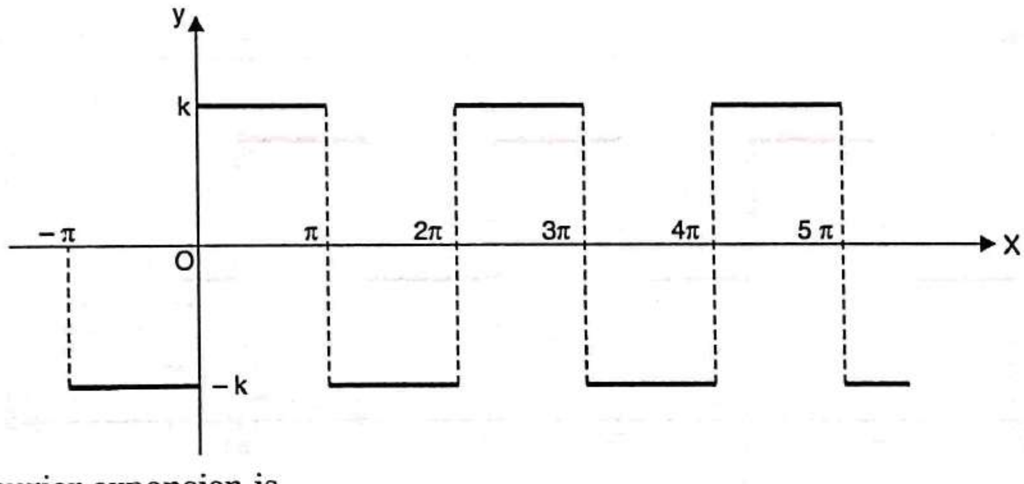
A periodic waveform is a waveform that repeats a basic pattern. It is a single-valued periodic function. Therefore it can be developed as a Fourier series.

We give below some typical waveforms usually met with in communication engineering and also the corresponding Fourier series. The student is urged to construct the Fourier series in each case.

I. Square (or Rectangular) Waveform *(M.D.U. May 2015)*

It is a periodic function of the form given below.

(*i*) $$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } \quad 0 < x < \pi \end{cases}, \quad f(x + 2\pi) = f(x)$$

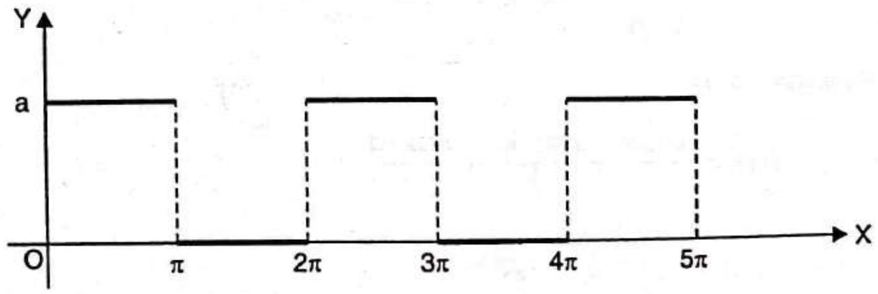


Its Fourier expansion is

$$f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right]$$

[See Question 1 in Exercise 1.2]

(*ii*) $$f(x) = \begin{cases} a & \text{when } 0 < x < \pi \\ 0 & \text{when } \pi < x < 2\pi \end{cases}, \quad f(x + 2\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

Exercise. Draw and write the name of the wave form for the following function:

$$f(x) = \begin{cases} a & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi \end{cases}$$ *(M.D.U. May 2014)*

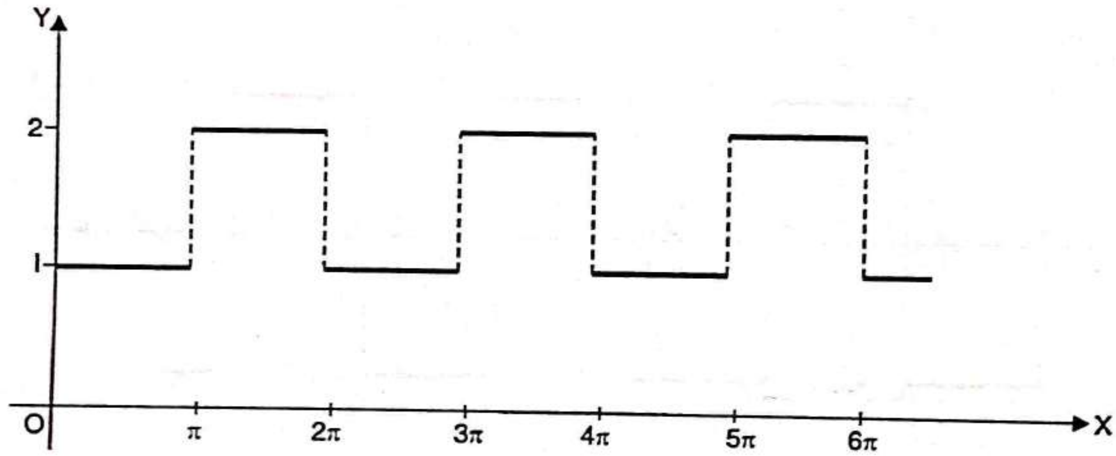
(iii) $$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } \quad 0 < x < \pi \end{cases}, \ f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

[See Question 2 in Exercise 1.2]

(iv) $$f(x) = \begin{cases} 1 & \text{for } \ 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases}, \ f(x + 2\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

(v) $$f(x) = \begin{cases} -\dfrac{\pi}{4} & \text{for } -\pi < x < 0 \\ \dfrac{\pi}{4} & \text{for } 0 < x < \pi \end{cases}, \ f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots$$

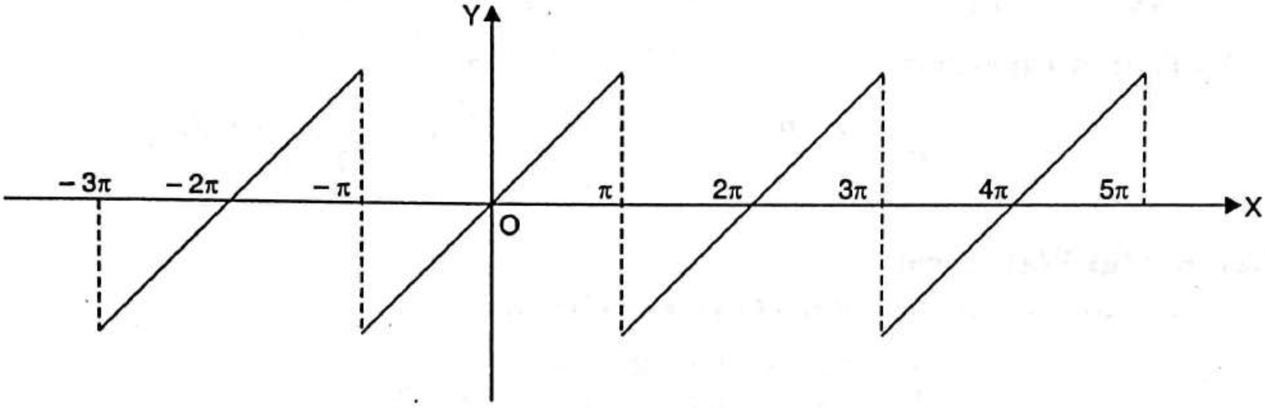
II. Saw-toothed Waveform

It is a periodic function of the form given below.

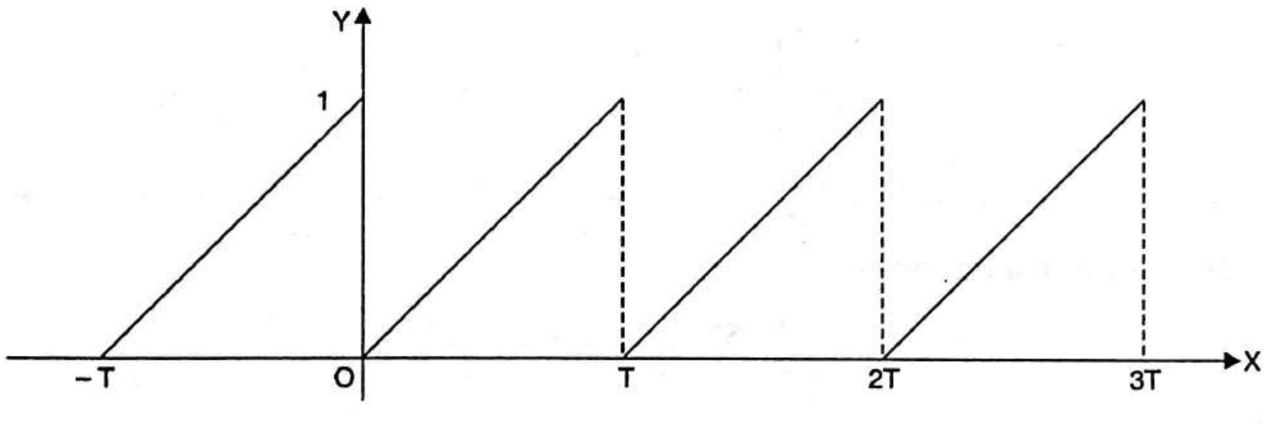
(*i*) $f(x) = x, -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

Its Fourier expansion is

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right)$$



(*ii*) $f(x) = \frac{1}{T}x$ when $0 < x < T$ and $f(x + T) = f(x)$



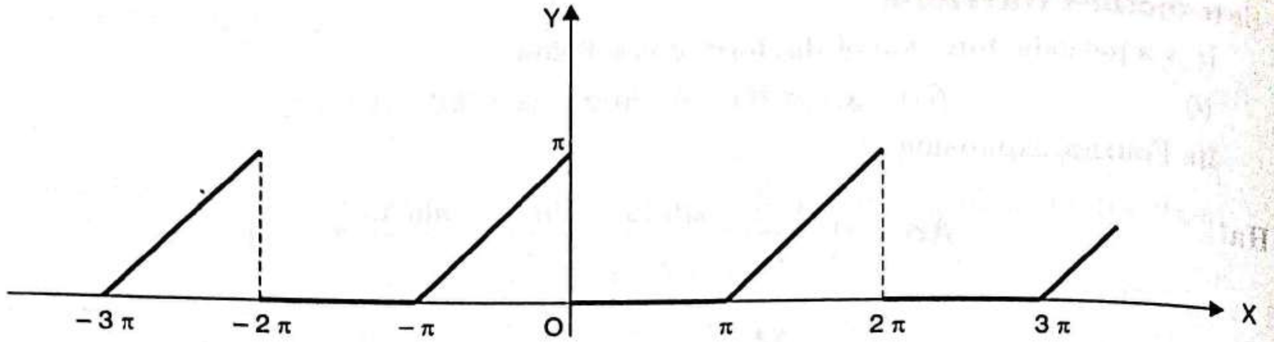
Its Fourier expansion is

$$f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin n\omega x}{n}, \qquad \text{where } \omega = \frac{2\pi}{T}.$$

III. Modified Saw-toothed Waveform

It is a periodic function of the form given below.

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < 0 \\ 0 & \text{for } \ 0 \le x < \pi \end{cases}, \ f(x + 2\pi) = f(x)$$



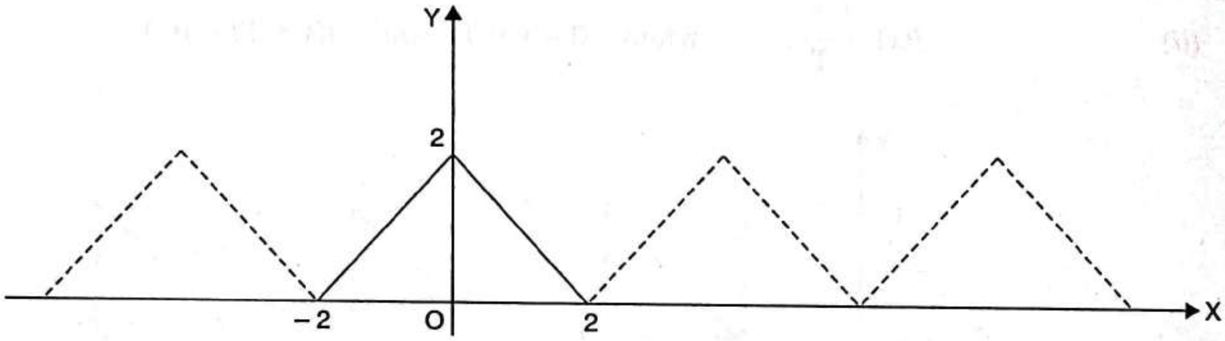
Its Fourier expansion is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right) - \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \ldots\ldots\right)$$

IV. Triangular Waveform

It is a periodic function of the form given below.

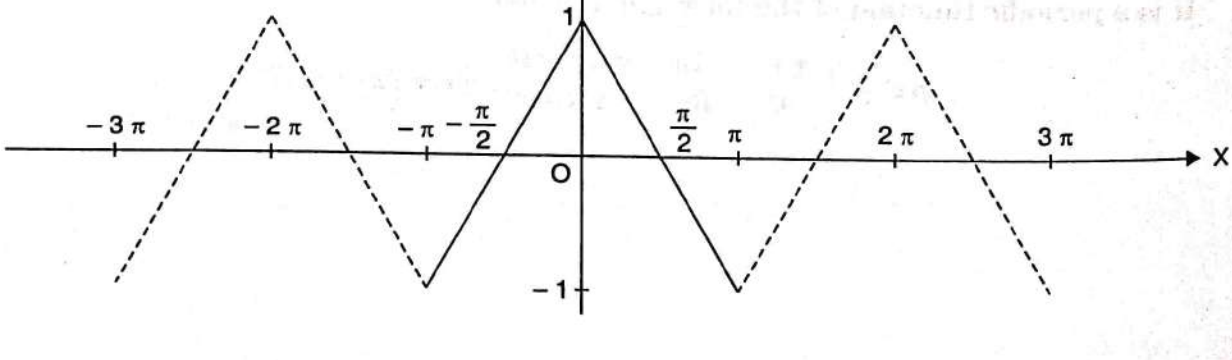
(i) $$f(x) = \begin{cases} 2 + x & \text{for } -2 \le x \le 0 \\ 2 - x & \text{for } \;\; 0 < x \le 2 \end{cases}, \; f(x+4) = f(x)$$



Its Fourier expansion is

$$f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left[(2n-1)\frac{\pi x}{2}\right]$$

(ii) $$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & \text{for } -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & \text{for } \;\; 0 < x \le \pi \end{cases}, \; f(x + 2\pi) = f(x)$$



Its Fourier expansion is

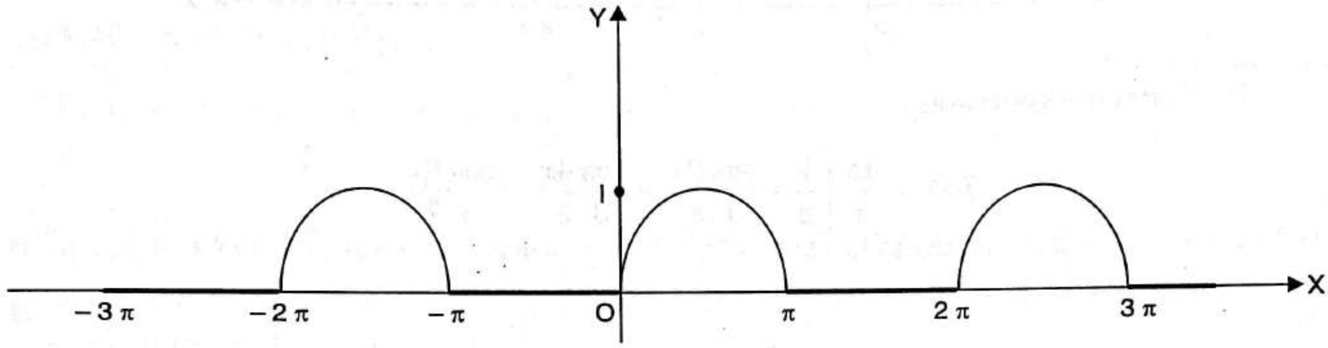
$$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

[See Example 10 before Exercise 1.1]

V. Half Rectified Waveform

It is a periodic function of the form given below.

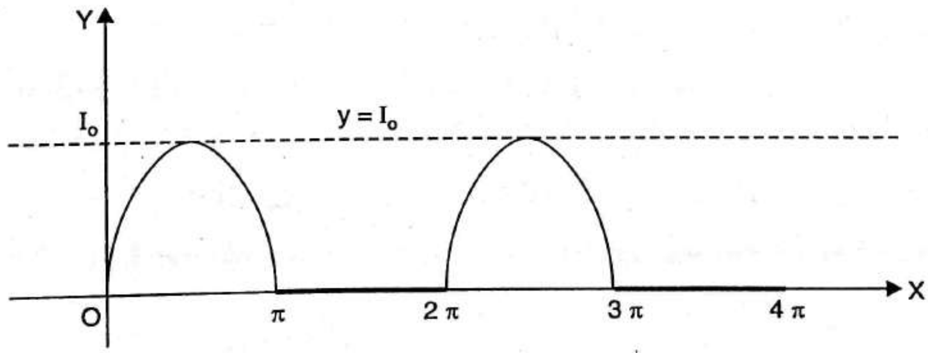
(*i*) $$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ \sin x & \text{for } \ 0 \le x \le \pi \end{cases}$$



Its Fourier expansion is

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$$

(*ii*) $$f(x) = \begin{cases} I_0 \sin x & \text{for } \ 0 \le x \le \pi \\ 0 & \text{for } \pi \le x \le 2\pi \end{cases}, \ f(x + 2\pi) = f(x)$$



Its Fourier expansion is

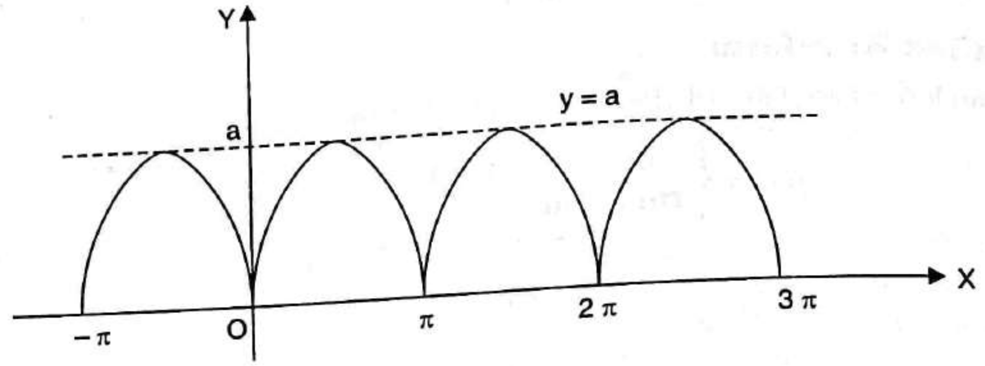
$$f(x) = \frac{I_0}{\pi} + \frac{1}{2}I_0 \sin x - \frac{2I_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\,x}{4n^2 - 1}$$

[See Question 6 in Exercise 1.2]

VI. Full Rectified Waveform

It is a periodic function of the form given below.

$$f(x) = a \sin x \text{ for } 0 \le x \le \pi, \ f(x + \pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} - \dots\dots\right]$$

□□□

Chapter 2

Fourier Transforms

2.1. INTEGRAL TRANSFORMS

The integral transform of a function $f(x)$ is defined by the equation

$$I\{f(x)\} = \bar{f}(s) = \int_a^b f(x)\, K(s, x)\, dx,$$

where $K(s, x)$ is a known function of s and x, called the **kernel** of the transform; s is called the **parameter** of the transform and $f(x)$ is called the **inverse transform** of $\bar{f}(s)$.

Some of the well-known transforms are given below:

(*i*) **Laplace Transform.** When $K(s, x) = e^{-sx}$, we have the Laplace transform of $f(x)$.

Thus $$L\{f(x)\} = \bar{f}(s) = \int_0^\infty f(x)\, e^{-sx}\, dx$$

(*ii*) **Fourier Transform.** When $K(s, x) = e^{isx}$, we have the Fourier transform of $f(x)$. Thus

$$F(s) = \int_{-\infty}^\infty f(x)\, e^{isx}\, dx$$

(*iii*) **Hankel Transform.** When $K(s, x) = x\, J_n(sx)$, we have the Hankel transform of $f(x)$.

Thus $$H_n(s) = \int_0^\infty f(x)\, x\, J_n(sx)\, dx$$

where $J_n(sx)$ is the Bessel function of the first kind and order n.

(*iv*) **Mellin Transform.** When $K(s, x) = x^{s-1}$, we have the Mellin transform of $f(x)$. Thus

$$M(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$$

(*v*) **Fourier Sine Transform.** When $K(s, x) = \sin sx$, we have the Fourier sine transform of $f(x)$. Thus

$$F_s(s) = \int_0^\infty f(x) \sin sx\, dx$$

(*vi*) **Fourier Cosine Transform.** When $K(s, x) = \cos sx$, we have the Fourier cosine transform of $f(x)$. Thus

$$F_c(s) = \int_0^\infty f(x) \cos sx\, dx$$

We have already discussed Laplace transform and its applications to the solution of ordinary differential equations. In the present chapter, we shall discuss the Fourier integrals and Fourier transforms which are useful in solving boundary value problems arising in engineering *e.g.* conduction of heat, theory of communication, wave propagation etc. Fourier series are helpful in problems involving periodic functions. However, in many practical problems, the function is non-periodic. A suitable representation for non-periodic functions can be obtained by considering the limiting form of Fourier series when the fundamental period is made infinite. In such case, the Fourier series becomes a Fourier integral which can be expressed in terms of Fourier transform which transforms a non-periodic function.

The effect of applying an integral transform to a partial differential equation is to reduce the number of independent variables by one. The choice of a particular transform is decided by the nature of the boundary conditions and the facility with which the transform can be inverted to give $f(x)$.

2.2. FOURIER INTEGRAL THEOREM

Statement. *If*

(*i*) *$f(x)$ satisfies Dirichlet's conditions in every interval $(-c, c)$ however large.*

(*ii*) $\int_{-\infty}^{\infty} |f(x)|\, dx$ *converges;*

then
$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

The integral on the right hand side is called **Fourier Integral** *of $f(x)$.*

Proof. Consider a function $f(x)$ satisfying Dirichlet's conditions in every interval $(-c, c)$, however large. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \quad ...(1)$$

where $a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\,dt$, $a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt$ and $b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt$

Substituting the values of a_0, a_n and b_n in (1), we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c}\left[\cos\frac{n\pi x}{c}\cos\frac{n\pi t}{c} + \sin\frac{n\pi x}{c}\sin\frac{n\pi t}{c}\right] f(t)\,dt$$
$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c}\cos\frac{n\pi(t-x)}{c}\,.\,f(t)\,dt \quad ...(2)$$

If we assume that the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges,

then
$$\lim_{c\to\infty}\left[\frac{1}{2c}\int_{-c}^{c} f(t)\,dt\right] = 0, \text{ since } \left|\frac{1}{2c}\int_{-c}^{c} f(t)\,dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty}|f(t)|\,dt$$

Putting $\frac{\pi}{c} = \Delta\lambda$, the second term in (2) becomes

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\lambda\int_{-c}^{c}\cos\{n\Delta\lambda(t-x)\}\,f(t)\,dt$$

This is of the form $\sum_{n=1}^{\infty} F(n\Delta\lambda)\,\Delta\lambda$ whose limit as $\Delta\lambda \to 0$, is $\int_0^{\infty} F(\lambda)\,d\lambda$.

Hence as $c \to \infty$, (2) reduces to
$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda} \quad ...(3)$$
which is known as **Fourier Integral** of $f(x)$. *(M.D.U. May 2015)*

Equation (3) is true at a point of continuity. At a point of discontinuity the value of the integral on the right is

$$\tfrac{1}{2}\,[f(x+0)+f(x-0)].$$

2.3. FOURIER SINE AND COSINE INTEGRALS

We know that $\cos\lambda(t-x) = \cos\lambda t\cos\lambda x + \sin\lambda t\sin\lambda x$

$\therefore$ Fourier integral of $f(x)$ can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\,\{\cos\lambda t\cos\lambda x + \sin\lambda t\sin\lambda x\}\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x\int_{-\infty}^\infty f(t)\cos\lambda t\,dt\,d\lambda + \frac{1}{\pi}\int_0^\infty \sin\lambda x\int_{-\infty}^\infty f(t)\sin\lambda t\,dt\,d\lambda \quad ...(4)$$

When f(x) is an odd function, $f(t)\cos\lambda t$ is odd while $f(t)\sin\lambda t$ is even. Thus the first integral in (4) vanishes and, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda} \quad ...(5)$$

This is called **Fourier sine integral.**

When f(x) is an even function, $f(t)\cos\lambda t$ is even while $f(t)\sin\lambda t$ is odd. Thus the second integral in (4) vanishes and, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\int_0^\infty f(t)\cos\lambda t\,dt\,d\lambda} \quad ...(6)$$

This is called **Fourier cosine integral.**

2.4. COMPLEX FORM OF FOURIER INTEGRAL

Since $\cos\lambda(t-x)$ is an even function of λ, we have from (3)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda \quad ...(7)$$

$$\left[\because\ 2\int_0^a f(x)\,dx = \int_{-a}^a f(x)\,dx, \text{if } f(x) \text{ is even}\right]$$

Also $\sin\lambda(t-x)$ is an odd function of λ, so that

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\sin\lambda\,(t-x)\,dt\,d\lambda \quad ...(8)$$

$$\left[\because\ \int_{-a}^a f(x)\,dx = 0, \text{if } f(x) \text{ is odd}\right]$$

Multiplying (8) by i and adding to (7), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,\{\cos \lambda\,(t-x) + i\,\sin \lambda(t-x)\}\,dt\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\lambda(t-x)}\,dt\,d\lambda = \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda}\int_{-\infty}^{\infty} f(t)\,e^{i\lambda t}\,dt\,d\lambda}$$

which is known as the **complex form of Fourier integral.**

ILLUSTRATIVE EXAMPLES

Example 1. *Express the function* $f(x) = \begin{cases} 1 & for\ \ |x| \le 1, \\ 0 & for\ \ |x| > 1, \end{cases}$

as a Fourier integral. Hence evaluate $\displaystyle\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\,d\lambda$.

(M.D.U. 2012, May 2013, Dec. 2014)

Sol. The Fourier integral for $f(x)$ is

$$\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos \lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1} \cos \lambda\,(t-x)\,dt\,d\lambda \qquad \left[\because\ f(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}\right]$$

$$= \frac{1}{\pi}\int_0^{\infty} \left[\frac{\sin \lambda(t-x)}{\lambda}\right]_{-1}^{1} d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\sin \lambda(1-x) - \sin \lambda(-1-x)}{\lambda}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\sin \lambda(1+x) + \sin \lambda(1-x)}{\lambda}\,d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\,d\lambda$$

$$\therefore \quad \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

At $|x| = 1$, *i.e.*, $x = \pm 1$, $f(x)$ is discontinuous and the integral has the value

$$\frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}.$$

Note. Putting $x = 0$, we get $\displaystyle\int_0^{\infty} \frac{\sin \lambda}{\lambda}\,d\lambda = \frac{\pi}{2}$ or $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Example 2. *Express* $f(x) = \begin{cases} 1 & for\ \ 0 \le x \le \pi, \\ 0 & for\ \ x > \pi, \end{cases}$ *as a Fourier sine integral and hence evaluate*

$$\int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)\,d\lambda.$$

Sol. The Fourier sine integral for $f(x)$ is

$$\frac{2}{\pi}\int_0^{\infty} \sin(\lambda x) \int_0^{\infty} f(t) \sin(\lambda t)\, dt\, d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x) \left[\int_0^{\pi} f(t) \sin(\lambda t)\, dt + \int_{\pi}^{\infty} f(t) \sin(\lambda t)\, dt\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x) \int_0^{\pi} \sin(\lambda t)\, dt\, d\lambda \qquad \text{[on substituting for } f(t)]$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x) \left[-\frac{\cos(\lambda t)}{\lambda}\right]_0^{\pi} d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(\lambda\pi)}{\lambda} \sin(\lambda x)\, d\lambda$$

$$\therefore \qquad f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda)\, d\lambda$$

$$\Rightarrow \qquad \int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda)\, d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \dfrac{\pi}{2} & \text{for } \; 0 \le x < \pi \\ 0 & \text{for } \; x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, we have

$$f(x) = \frac{1}{2}[f(\pi - 0) + f(\pi + 0)] = \frac{1}{2}(1 + 0) = \frac{1}{2}$$

$$\therefore \qquad \int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda)\, d\lambda = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}.$$

Example 3. *Using Fourier Integral representation, show that:*

$$\int_0^{\infty} \frac{\cos x\alpha + \alpha \sin x\alpha}{1+\alpha^2}\, d\alpha = \begin{cases} 0 & , \text{ if } x < 0 \\ \dfrac{\pi}{2} & , \text{ if } x = 0 \\ \pi e^{-x}, & \text{ if } x > 0 \end{cases}.$$

Sol. Fourier Integral for

$$f(x) = \begin{cases} 0\;, & \text{if } \; x < 0 \\ e^{-x}, & \text{if } \; x > 0 \end{cases}$$

is $\quad \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{0} f(t)\cos\lambda(t-x)\, dt + \int_0^{\infty} f(t) \cos\lambda(t-x)\, dt\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{0} 0 \cos\lambda(t-x)\, dt + \int_0^{\infty} e^{-t} \cos\lambda(t-x)\, dt\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} e^{-t} \cos\lambda(t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\frac{e^{-t}}{1+\lambda^2}\{-\cos\lambda(t-x) + \lambda\sin\lambda(t-x)\}\right]_0^{\infty} d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos \alpha x + \alpha \sin \alpha x}{1+\alpha^2}\, d\alpha \qquad \text{(Replacing } \lambda \text{ by } \alpha)$$

$$\Rightarrow \qquad \int_0^\infty \frac{\cos x\alpha + \alpha \sin x\alpha}{1+\alpha^2}\, d\alpha = \pi f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases}$$

When $x = 0$, $\displaystyle\int_0^\infty \frac{\cos x\alpha + \alpha \sin x\alpha}{1+\alpha^2}\, d\alpha = \int_0^\infty \frac{1}{1+\alpha^2}\, d\alpha = \left[\tan^{-1}\alpha\right]_0^\infty = \frac{\pi}{2}$

$$\therefore \qquad \int_0^\infty \frac{\cos x\alpha + \alpha \sin x\alpha}{1+\alpha^2}\, d\alpha = \begin{cases} 0, & \text{if } x < 0 \\ \dfrac{\pi}{2}, & \text{if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases}.$$

EXERCISE 2.1

Using Fourier integral representation, show that (1 – 8):

1. $\displaystyle\int_0^\infty \frac{\lambda \sin x\lambda}{k^2+\lambda^2}\, d\lambda = \frac{\pi}{2} e^{-kx}$, $x > 0, k > 0$ (M.D.U. May 2015)

2. $\displaystyle\int_0^\infty \frac{\cos x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x}, x \ge 0$ (K.U.K. Jan. 2014)

3. $\displaystyle\int_0^\infty \frac{\sin \pi\lambda \sin x\lambda}{1-\lambda^2}\, d\lambda = \begin{cases} \dfrac{\pi}{2}\sin x, & \text{when } 0 \le x \le \pi \\ 0, & \text{when } x > \pi \end{cases}$

4. $\displaystyle\int_0^\infty \frac{\cos \lambda x}{\lambda^2 + a^2}\, d\lambda = \frac{\pi}{2a} e^{-ax}, a > 0, \quad x \ge 0.$

5. $\displaystyle\int_0^\infty \left(\frac{\lambda^2+2}{\lambda^4+4}\right)\cos \lambda x\, d\lambda = \frac{\pi}{2} e^{-x}\cos x$, if $x > 0$.

6. $\displaystyle\int_0^\infty \left(\frac{\lambda^3}{\lambda^4+4}\right)\sin \lambda x\, d\lambda = \frac{\pi}{2} e^{-x}\cos x$, if $x > 0$.

7. $\displaystyle\int_0^\infty \frac{\cos \frac{\pi\lambda}{2} \cos \lambda x}{1-\lambda^2}\, d\lambda = \begin{cases} \dfrac{\pi}{2}\cos x, & \text{if } |x| < \dfrac{\pi}{2} \\ 0, & \text{if } |x| > \dfrac{\pi}{2} \end{cases}.$

8. $\displaystyle\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+\alpha^2)(\lambda^2+\beta^2)}\, d\lambda = \frac{\pi}{2}\left(\frac{e^{-\alpha x} - e^{-\beta x}}{\beta^2-\alpha^2}\right).$

9. Find Fourier sine integral representation of

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ k, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

where k is a constant.

10. Find the Fourier integral representation for the following functions:

(i) $f(x) = \begin{cases} \dfrac{\pi}{2}\cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ (ii) $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$

(iii) $f(x) = e^{-|x|}, -\infty < x < \infty.$

Answers

9. $f(x) = \dfrac{2k}{\pi} \displaystyle\int_0^\infty \left(\dfrac{\cos \lambda - \cos 2\lambda}{\lambda} \right) \sin \lambda x \, d\lambda$

10. (*i*) $f(x) = \displaystyle\int_0^\infty \dfrac{\lambda \sin \lambda\pi}{1-\lambda^2} \cos \lambda x \, d\lambda$ (*ii*) $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \dfrac{\sin \lambda \, a \cos \lambda x}{\lambda} \, d\lambda$

(*iii*) $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \dfrac{1}{1+\lambda^2} \cos \lambda x \, d\lambda.$

2.5. FOURIER TRANSFORMS AND INVERSION FORMULAE

(1) *Fourier sine transform and its inversion formula*

Fourier sine integral is $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \, dt \, d\lambda$

Replacing λ by s, we get $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \sin sx \int_0^\infty f(t) \sin st \, dt \, ds$

Denoting the value of the inner integral by $F_s(s)$, we have

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds \qquad \dots(1)$$

where
$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx \qquad \dots(2)$$

The function $F_s(s)$ as defined by equation (2) is known as the **Fourier sine transform** of $f(x)$ in $0 < x < \infty$.

The function $f(x)$ as defined by equation (1) is called the **inverse Fourier sine transform** of $F_s(s)$. Thus equation (1) gives the inversion formula for the Fourier sine transform.

Note. Some authors write the above formulae as:

$$F_s(s) \text{ or } F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

and
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

(2) *Fourier cosine transform and its inversion formula*

Fourier cosine integral is $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda$

Replacing λ by s, we get $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \cos sx \int_0^\infty f(t) \cos st \, dt \, ds$

Denoting the value of the inner integral by $F_c(s)$, we have

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(s) \cos sx \, ds \qquad \dots(3)$$

where
$$F_c(s) = \int_0^\infty f(x) \cos sx \, dx \qquad \dots(4)$$

The function $F_c(s)$ as defined by equation (4) is known as the **Fourier cosine transform** of $f(x)$ in $0 < x < \infty$.

The function $f(x)$ as defined by equation (3) is called the **inverse Fourier cosine transform** of $F_c(s)$. Thus equation (3) gives the inversion formula for the Fourier cosine transform.

Note. Some authors write the above formulae as:

$$F_c(s) \text{ or } F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

(3) *Complex Fourier transform and its inversion formula*

Complex form of Fourier integral is $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda x} \int_{-\infty}^\infty f(t) e^{i\lambda t} \, dt \, d\lambda$

Replacing λ by s, we get $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-isx} \int_{-\infty}^\infty f(t) e^{ist} \, dt \, ds$

Denoting the value of the inner integral by $F(s)$, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(s) e^{-isx} \, ds \quad \ldots(5)$$

where

$$F(s) = \int_{-\infty}^\infty f(x) e^{isx} \, dx \quad \ldots(6)$$

The function $F(s)$ as defined by equation (6) is known as the **Fourier transform** of $f(x)$.

The function $f(x)$ as defined by equation (5) is called the **inverse Fourier transform** of $F(s)$. Thus equation (5) gives the inversion formula for the Fourier transform.

Note. Some authors write the above formulae as:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} \, dx$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(s) e^{-isx} \, ds$$

(4) *Finite Fourier sine transform and its inversion formula*

The **finite Fourier sine transform** of $f(x)$ in $0 < x < c$ is defined as

$$F_s(n) = \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier sine transform** of $F_s(n)$ and is given by

$$f(x) = \frac{2}{c} \sum_{n=1}^\infty F_s(n) \sin \frac{n\pi x}{c}.$$

(5) *Finite Fourier cosine transform and its inversion formula*

The **finite Fourier cosine transform** of $f(x)$ in $0 < x < c$ is defined as

$$F_c(n) = \int_0^c f(x) \cos \frac{n\pi x}{c} dx$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier cosine transform** of $F_c(n)$ and is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine transform of* $e^{-|x|}$. *Hence evaluate* $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$.

Sol. In the interval $(0, \infty)$, x is positive so that $e^{-|x|} = e^{-x}$. Fourier sine transform of $f(x) = e^{-x}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx = \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \left[\frac{e^{-x}}{1+s^2} (- \sin sx - s \cos sx) \right]_0^\infty = \frac{s}{1+s^2}$$

Using inversion formula for Fourier sine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} \sin sx\, ds \quad \text{or} \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx\, ds$$

Replacing x by m, we have $e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin ms\, ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+x^2} dx$

Hence, $$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}.$$

Example 2. *Find the Fourier sine transform of*

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2. \\ 0, & \text{for } x > 2 \end{cases}$$

Sol. Fourier transform of $f(x)$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx$$

$$= \int_0^1 f(x) \sin sx \, dx + \int_1^2 f(x) \sin sx \, dx + \int_2^\infty f(x) \sin sx \, dx$$

$$= \int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx + \int_2^\infty 0 \, dx$$

$$= \left\{x.\frac{-\cos sx}{s} - 1.\frac{-\sin sx}{s^2}\right\}_0^1 + \left\{(2-x).\frac{-\cos sx}{s} - (-1).\frac{-\sin sx}{s^2}\right\}_1^2$$

$$= \left(-\frac{\cos s}{s} + \frac{\sin s}{s^2}\right) + \left(-\frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right)$$

$$= \frac{2\sin s - \sin 2s}{s^2} = \frac{2\sin s - 2\sin s \cos s}{s^2} = \frac{2\sin s\,(1-\cos s)}{s^2}.$$

Example 3. *Find the Fourier sine transform of* $\frac{e^{-ax}}{x}$.

Sol. Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = I \quad \text{(say)} \qquad \ldots(1)$$

Differentiating w.r.t. s, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x} \sin sx\right) dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} . x \cos sx \, dx = \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s \sin sx)\right]_0^\infty$$

$$= 0 - \frac{1}{a^2+s^2}(-a) = \frac{a}{s^2+a^2}$$

Integrating w.r.t. s, we get

$$I = \tan^{-1}\frac{s}{a} + c \qquad \ldots(2)$$

Now, when $s = 0$, from (1), $I = 0$

$\therefore$ From (2), $0 = 0 + c \Rightarrow c = 0$

$$\therefore \quad I = \tan^{-1}\frac{s}{a} \quad \text{or} \quad F_s\left\{\frac{e^{-ax}}{x}\right\} = \tan^{-1}\frac{s}{a}.$$

Example 4. *Find Fourier sine transform of* $\frac{1}{x(x^2+a^2)}$.

(M.D.U. May 2013, Dec. 2014)

Sol. Fourier sine transform of $f(x) = \frac{1}{x(x^2+a^2)}$ is

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx$$

$$= \int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx = I \quad \text{(say)} \quad \ldots(1)$$

Differentiating w.r.t. s, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{x\cos sx}{x(x^2+a^2)}\,dx = \int_0^\infty \frac{\cos sx}{x^2+a^2}\,dx \quad \ldots(2)$$

Differentiating again w.r.t. s, we have

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{-x\sin sx}{x^2+a^2}\,dx = \int_0^\infty \frac{-x^2\sin sx}{x(x^2+a^2)}\,dx$$

$$= \int_0^\infty \frac{[a^2-(x^2+a^2)]\sin sx}{x(x^2+a^2)}\,dx = a^2\int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx - \int_0^\infty \frac{\sin sx}{x}\,dx$$

$$= a^2 I - \frac{\pi}{2}$$

$\Rightarrow$ $\quad (D^2-a^2)\,I = -\frac{\pi}{2}$, where $D = \frac{d}{ds}$

Its A.E. is $\quad D^2-a^2=0$ whence $D = \pm a$

$$\text{C.F.} = c_1e^{as} + c_2e^{-as}$$

$$\text{P.I.} = \frac{1}{D^2-a^2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}\cdot\frac{1}{D^2-a^2}e^{0s} = -\frac{\pi}{2}\cdot\frac{1}{-a^2} = \frac{\pi}{2a^2}$$

$\therefore$ $\quad I = \text{C.F.} + \text{P.I.}$

$\Rightarrow$ $\quad I = c_1e^{as} + c_2e^{-as} + \frac{\pi}{2a^2} \quad \ldots(3)$

$\therefore$ $\quad \frac{dI}{ds} = ac_1e^{as} - ac_2e^{-as} \quad \ldots(4)$

When $\quad s = 0$, from (1), $I = 0$

From (3), $\quad I = c_1 + c_2 + \frac{\pi}{2a^2}$

$\therefore$ $\quad c_1 + c_2 + \frac{\pi}{2a^2} = 0 \quad \ldots(5)$

When $s = 0$, from (2), $\dfrac{dI}{ds} = 1\left[\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}\right]_0^\infty = \dfrac{\pi}{2a}$

From (4), $$\frac{dI}{ds} = ac_1 - ac_2$$

$$\therefore \quad ac_1 - ac_2 = \frac{\pi}{2a}$$

or $$c_1 - c_2 - \frac{\pi}{2a^2} = 0 \quad ...(6)$$

Solving (5) and (6), $c_1 = 0$, $c_2 = -\dfrac{\pi}{2a^2}$

$$\therefore \quad I = -\frac{\pi}{2a^2}e^{-as} + \frac{\pi}{2a^2}$$

or $$F_s\{f(x)\} = \frac{\pi}{2a^2}(1 - e^{-as}).$$

Example 5. *Find the Fourier sine transform of* $\dfrac{1}{x}$.

Sol. Fourier sine transform of $\dfrac{1}{x}$ is given by

$$F_s\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x}\sin sx\,dx$$

Putting $sx = \theta$ *i.e.,* $x = \dfrac{\theta}{s}$, we have

$$F_s\left(\frac{1}{x}\right) = \int_0^\infty \frac{s}{\theta}\sin\theta\,\frac{d\theta}{s} = \int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}.$$

Example 6. *Find the Fourier sine and cosine transforms of* x^{n-1}, $n > 0$.

(M.D.U. 2011)

Sol. We know that

$$F_s(x^{n-1}) = \int_0^\infty x^{n-1}\sin sx\,dx \quad ...(1)$$

and $$F_c(x^{n-1}) = \int_0^\infty x^{n-1}\cos sx\,dx \quad ...(2)$$

Now $$\Gamma(n) = \int_0^\infty e^{-t}\,t^{n-1}\,dt,\ n > 0$$

Putting $t = ax$, $a > 0$, we have

$$\Gamma(n) = \int_0^\infty e^{-ax}(ax)^{n-1}\,a\,dx = a^n\int_0^\infty e^{-ax}x^{n-1}\,dx$$

$$\Rightarrow \qquad \int_0^\infty e^{-ax}\, x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$$

Putting $a = is$, we have

$$\int_0^\infty e^{-isx}\, x^{n-1}\, dx = \frac{\Gamma(n)}{(is)^n}$$

$$\Rightarrow \quad \int_0^\infty (\cos sx - i \sin sx)\, x^{n-1}\, dx = \left(\frac{i}{i^2}\right)^n \frac{\Gamma(n)}{s^n} = (-i)^n \frac{\Gamma(n)}{s^n}$$

$$= \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}\right)^n \frac{\Gamma(n)}{s^n}$$

$$= \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}\right) \frac{\Gamma(n)}{s^n}$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty x^{n-1} \cos sx\, dx = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2}$$

and $$\int_0^\infty x^{n-1} \sin sx\, dx = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2}$$

$\therefore$ From (1), $F_s(x^{n-1}) = \dfrac{\Gamma(n)}{s^n} \sin \dfrac{n\pi}{2}$

and from (2), $F_c(x^{n-1}) = \dfrac{\Gamma(n)}{s^n} \cos \dfrac{n\pi}{2}$

Note. Taking $n = \frac{1}{2}$, we get

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \sin \frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{s}} \cdot \frac{1}{\sqrt{2}} \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \sqrt{\frac{\pi}{2s}}$$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos \frac{\pi}{4} = \sqrt{\frac{\pi}{2s}}.$$

Example 7. *Find the Fourier cosine transform of* e^{-x^2}.

Sol. Fourier cosine transform of e^{-x^2} is given by

$$F_c\{e^{-x^2}\} = \int_0^\infty e^{-x^2} \cos sx\, dx = I \text{ (say)} \qquad \ldots(1)$$

Differentiating w.r.t. s, we have

$$\frac{dI}{ds} = -\int_0^\infty x e^{-x^2} \sin sx\, dx = \frac{1}{2}\int_0^\infty (\sin sx)\,(-2x\, e^{-x^2})\, dx$$

$$= \frac{1}{2}\left[\left\{\sin sx\, e^{-x^2}\right\}_0^\infty - s\int_0^\infty \cos sx\, e^{-x^2}\, dx\right]$$

(Integrating by parts)

$$= -\frac{s}{2}\int_0^\infty e^{-x^2} \cos sx\, dx = -\frac{s}{2} I$$

or $$\frac{dI}{I} = -\frac{s}{2}\, ds$$

Integrating, we have $\log I = -\dfrac{s^2}{4} + \log A$ or $I = A\, e^{-\frac{s^2}{4}}$ ...(2)

Now when $s = 0$, from (1), $I = \displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

$\therefore$ From (2), $$\frac{\sqrt{\pi}}{2} = A$$

Hence $$I = F_c\{e^{-x^2}\} = \frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{4}}.$$

Note. $I = \displaystyle\int_0^\infty e^{-x^2} dx$ Put $x^2 = t$ *i.e.,* $x = \sqrt{t}$

$$= \int_0^\infty \frac{e^{-t}}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^\infty t^{-1/2}\, e^{-t} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Example 8. *Find the Fourier cosine transform of*

$$f(x) = \begin{cases} \cos x\,, & 0 < x < a \\ 0\,, & x \geq a \end{cases}.$$

Sol. Fourier cosine transform of $f(x)$ is

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos sx\, dx = \int_0^a f(x)\cos sx\, dx + \int_a^\infty f(x)\cos sx\, dx$$

$$= \int_0^a \cos x \cos sx\, dx + \int_a^\infty 0\, dx = \frac{1}{2}\int_0^a 2\cos x \cos sx\, dx + 0$$

$$= \frac{1}{2}\int_0^a [\cos(1+s)x + \cos(1-s)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(1+s)x}{1+s} + \frac{\sin(1-s)x}{1-s}\right]_0^a = \frac{1}{2}\left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}\right].$$

Example 9. *Find the Fourier cosine transform of* $f(x) = \dfrac{1}{1+x^2}$. (M.D.U. 2011)

Hence derive Fourier sine transform of $\phi(x) = \dfrac{1}{x(1+x^2)}$.

Sol. Fourier cosine transform of $f(x) = \dfrac{1}{1+x^2}$ is

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I \quad \text{(say)} \qquad \ldots(1)$$

Differentiating w.r.t. s, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}\,dx = \int_0^\infty \frac{-x^2 \sin sx}{x(1+x^2)}\,dx$$

$$= \int_0^\infty \frac{[1-(1+x^2)]\sin sx}{x(1+x^2)}\,dx = \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx - \int_0^\infty \frac{\sin sx}{x}\,dx$$

$$= \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx - \frac{\pi}{2} \qquad \ldots(2)$$

Differentiating again w.r.t. s, we have

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}\,dx = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I$$

$\Rightarrow$ $(D^2 - 1)\,I = 0$, where $D = \dfrac{d}{ds}$

Its A.E. is $D^2 - 1 = 0$ whence $D = \pm 1$

$\therefore$

$$I = c_1 e^{s} + c_2 e^{-s} \qquad \ldots(3)$$

$$\frac{dI}{ds} = c_1 e^{s} - c_2 e^{-s} \qquad \ldots(4)$$

When $s = 0$, from (1), $I = \displaystyle\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}$

From (3), $I = c_1 + c_2$

$$\therefore \quad c_1 + c_2 = \frac{\pi}{2} \qquad \ldots(5)$$

When $s = 0$, from (2), $\dfrac{dI}{ds} = -\dfrac{\pi}{2}$

From (4), $\dfrac{dI}{ds} = c_1 - c_2$

$$\therefore \quad c_1 - c_2 = -\frac{\pi}{2} \qquad \ldots(6)$$

Solving (5) and (6), $c_1 = 0, \quad c_2 = \frac{\pi}{2}$

$$\therefore \qquad I = \frac{\pi}{2} e^{-s} \Rightarrow F_c\{f(x)\} = \frac{\pi}{2} e^{-s}$$

Putting the values of c_1 and c_2 in (4)

$$\frac{dI}{ds} = -\frac{\pi}{2} e^{-s}$$

$$\therefore \text{ From (2),} \qquad -\frac{\pi}{2} e^{-s} = \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx - \frac{\pi}{2}$$

$$\Rightarrow \qquad \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx = \frac{\pi}{2} (1 - e^{-s})$$

$$\Rightarrow \qquad F_s\{\phi(x)\} = \frac{\pi}{2} (1 - e^{-s}).$$

Note. In example 9 above, we have proved that

$$\int_0^\infty \frac{\cos sx}{1+x^2} dx = \frac{\pi}{2} e^{-s}$$

Differentiating w.r.t. s, we get

$$\int_0^\infty \frac{\partial}{\partial s}\left(\frac{\cos sx}{1+x^2}\right) dx = -\frac{\pi}{2} e^{-s}$$

$$\Rightarrow \qquad \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = -\frac{\pi}{2} e^{-s}$$

$$\Rightarrow \qquad \int_0^\infty \frac{x}{1+x^2} \sin sx \, dx = \frac{\pi}{2} e^{-s}$$

$$\Rightarrow \qquad F_s\{\psi(x)\} = \frac{\pi}{2} e^{-s} \qquad \text{where } \psi(x) = \frac{x}{1+x^2}$$

$\Rightarrow$ Fourier sine transform of $\frac{x}{1+x^2}$ is $\frac{\pi}{2} e^{-s}$.

Example 10. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Hence evaluate (M.D.U. 2011)

(*i*) $\int_{-\infty}^\infty \frac{\sin as \cos sx}{s} ds$ (*ii*) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. (M.D.U. Dec. 2013)

Sol. Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{a} f(x)\, e^{isx\, dx} + \int_{a}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-a} 0\,.\, e^{isx}\, dx + \int_{-a}^{a} 1\,.\, e^{isx}\, dx + \int_{a}^{\infty} 0\,.\, e^{isx}\, dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{e^{ias} - e^{-ias}}{is}$$

$$= \frac{2}{s}\left(\frac{e^{ias} - e^{-ias}}{2i}\right) = \frac{2 \sin as}{s}, \qquad | \; s \neq 0$$

For $s = 0$, we find $F\{f(x)\} = 2a$

By inversion formula for Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{f(x)\}.\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin as}{s}.\, e^{-isx}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as\,(\cos sx - i \sin sx)}{s}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \sin sx}{s}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$$

[Second integral vanishes since the integrand is an odd function of s]

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \pi f(x) = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases} \qquad \text{...(1)}$$

Since the integrand in (1) is an even function of s, we have

$$2\int_{0}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \begin{cases} \pi/2, & |x| < a \\ 0, & |x| > a \end{cases}$$

Putting $x = 0$, $\int_0^\infty \frac{\sin as}{s}\, ds = \begin{cases} \frac{\pi}{2}, & a > 0 \\ 0, & a < 0 \end{cases}$

Putting $a = 1$, $\int_0^\infty \frac{\sin s}{s}\, ds = \frac{\pi}{2}$

$\Rightarrow$ $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

Example 11. *Find the Fourier transform of* $f(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

and use it to evaluate $\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2}\, dx.$ (M.D.U. 2010, May 2013)

Sol. Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{isx} dx + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \int_1^\infty f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx + \int_1^\infty 0\, dx$$

$$= \left[(1 - x^2) \cdot \frac{e^{isx}}{is} - (-2x) \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= \left[-\frac{2}{s^2}(e^{is} + e^{-is}) + \frac{2}{is^3}(e^{is} - e^{-is})\right]$$

$$= \left[-\frac{4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is} - e^{-is}}{2i}\right)\right]$$

$$= 4\left[-\frac{\cos s}{s^2} + \frac{\sin s}{s^3}\right] = -\frac{4}{s^3}\,(s \cos s - \sin s)$$

By inversion formula for Fourier transform, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F\{f(x)\} \cdot e^{-isx}\, ds$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3}\right) (\cos sx - i \sin sx)\, ds$$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,ds+\frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\sin sx\,ds$$

$$= -\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,ds$$

(Since the integrand in the first integral is even and that in the second integral is odd)

$$\Rightarrow \quad \int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,ds = -\frac{\pi}{4}f(x) = \begin{cases}-\frac{\pi}{4}(1-x^2), & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$$

Putting $x=\frac{1}{2}$, we have $\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}\,ds = -\frac{\pi}{4}\left(1-\frac{1}{4}\right) = -\frac{3\pi}{16}$

Hence $\int_{0}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$.

Example 12. *Find the inverse Fourier transform of $F(s) = e^{-|s|y}$.*

Sol. The inverse Fourier transform of $F(s) = e^{-|s|y}$ is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|s|y}\cdot e^{-isx}\,ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{sy}\cdot e^{-isx}\,ds + \int_{0}^{\infty} e^{-sy}\cdot e^{-isx}\,ds\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{s(y-ix)}\,ds + \int_{0}^{\infty} e^{-s(y+ix)}\,ds\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{s(y-ix)}}{y-ix}\right\}_{-\infty}^{0} + \left\{\frac{e^{-s(y+ix)}}{-(y+ix)}\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y-ix}+\frac{1}{y+ix}\right] = \frac{1}{2\pi}\left(\frac{2y}{y^2+x^2}\right)$$

$$= \frac{y}{\pi(y^2+x^2)}.$$

Example 13. *Solve the integral equation* $\int_0^{\infty} f(x)\cos px\,dx = \begin{cases}1-p, & 0\le p\le 1\\ 0, & p>1\end{cases}$

Hence deduce that $\int_0^{\infty}\frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$. (K.U.K. May 2013)

Sol. Here $\int_0^{\infty} f(x)\cos px\,dx = F_c(p)$

$$\therefore \quad F_c(p) = \begin{cases}1-p, & 0\le p\le 1\\ 0, & p>1\end{cases}$$

By inversion formula for Fourier cosine transform, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(p)\cos px\,dp$$

$$= \frac{2}{\pi}\left[\int_0^1 F_c(p)\cos px\,dp + \int_1^\infty F_c(p)\cos px\,dp\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-p)\cos px\,dp + \int_1^\infty 0\,dp\right] \quad \text{(Integrating by parts)}$$

$$= \frac{2}{\pi}\left[(1-p).\frac{\sin px}{x} - (-1).\frac{-\cos px}{x^2}\right]_0^1 = \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right] = \frac{2(1-\cos x)}{\pi x^2}.$$

Deduction. Since $\int_0^\infty f(x)\cos px\,dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$, where $f(x) = \dfrac{2(1-\cos x)}{\pi x^2}$

$$\therefore \quad \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos x}{x^2}\right)\cos px\,dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

When $p = 0$, we have $\dfrac{2}{\pi}\int_0^\infty \dfrac{1-\cos x}{x^2}dx = 1$ or $\int_0^\infty \dfrac{2\sin^2\frac{x}{2}}{x^2}dx = \dfrac{\pi}{2}$

Putting $\dfrac{x}{2} = t$ so that $dx = 2dt$, we have $\int_0^\infty \dfrac{\sin^2 t}{t^2} = \dfrac{\pi}{2}$.

Example 14. *Find the finite Fourier sine transform of*

$$f(x) = \begin{cases} \dfrac{2k}{l}x, & 0 \le x \le \dfrac{l}{2} \\ \dfrac{2k}{l}(l-x), & \dfrac{l}{2} \le x \le l \end{cases}.$$

Sol. Finite Fourier sine transform of $f(x)$ in $0 \le x \le l$ is

$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \int_0^{l/2}\frac{2k}{l}x\sin\frac{n\pi x}{l}dx + \int_{l/2}^l \frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2k}{l}\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \frac{2k}{l}\int_{l/2}^l (l-x)\sin\frac{n\pi x}{l}dx \quad \text{(Integrating by parts)}$$

$$= \frac{2k}{l}\left[x.\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 1.\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{l/2} + \frac{2k}{l}\left[(l-x).\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1).\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/2}^l$$

$$= \frac{2k}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]+\frac{2k}{l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2k}{l}\cdot\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Example 15. *Find the finite Fourier cosine transform of*

$$f(x) = \left(1-\frac{x}{\pi}\right)^2,\ 0 \le x < \pi.$$

Sol. Finite Fourier cosine transform of $f(x)$ in $0 \le x \le \pi$ is

$$F_c(n) = \int_0^\pi f(x)\cos\frac{n\pi x}{\pi}\,dx = \int_0^\pi \left(1-\frac{x}{\pi}\right)^2 \cos nx\,dx \qquad \text{(Integrating by parts)}$$

$$= \left[\left(1-\frac{x}{\pi}\right)^2 \cdot \frac{\sin nx}{n} - 2\left(1-\frac{x}{\pi}\right)\left(-\frac{1}{\pi}\right)\cdot\frac{-\cos nx}{n^2} + 2\left(-\frac{1}{\pi}\right)\left(-\frac{1}{\pi}\right)\cdot\frac{-\sin nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi n^2} \text{ when } n \ne 0.$$

If $n = 0$, then $F_c(0) = \int_0^\pi \left(1-\frac{x}{\pi}\right)^2 dx \qquad [\because \cos 0x = 1]$

$$= \left[\frac{\left(1-\frac{x}{\pi}\right)^3}{3x\left(-\frac{1}{\pi}\right)}\right]_0^\pi = -\frac{\pi}{3}(0-1) = \frac{\pi}{3}$$

$$\therefore \qquad F_c(n) = \begin{cases} \dfrac{2}{\pi n^2}, & \text{if } n = 1, 2, 3, \ldots \\ \dfrac{\pi}{3}, & \text{if } n = 0. \end{cases}$$

Example 16. *Find $f(x)$, if $F_s(n) = \dfrac{1-\cos n\pi}{n^2\pi^2}$, where $0 \le x \le \pi$.*

Sol. Here $F_s(n) = \dfrac{1-\cos n\pi}{n^2\pi^2}$ is the finite Fourier sine transform of $f(x)$ in $0 \le x \le \pi$.

$\therefore \qquad f(x) =$ inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2\pi^2}\right)\sin nx \qquad (\because \text{ here } c = \pi)$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2}\right)\sin nx.$$

EXERCISE 2.2

1. Find the Fourier sine and cosine transform of the function $f(x) = e^{-x}$ and hence show that

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m} \quad \text{and} \quad \int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}.$$

2. Find the Fourier sine transform of

(i) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ (ii) $f(x) = \begin{cases} 0, & 0 \le x < a \\ x, & a \le x \le b \\ 0, & x > b. \end{cases}$

3. Find the Fourier cosine transform of

(i) $f(x) = e^{-x} + e^{-2x}, x > 0$ (ii) $f(x) = e^{-\frac{x^2}{2}}$

(iii) $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$ (iv) $f(x) = 2e^{-5x}, + 5e^{-2x}$

(v) $f(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases}$ (vi) $\left(1-\frac{x}{\pi}\right)^2$

4. (a) Find the Fourier sine and cosine transforms of

(i) $f(x) = \begin{cases} 1, & \text{for } 0 \le x < a \\ 0, & \text{for } x \ge a \end{cases}$ (ii) $f(x) = e^{-ax}, a > 0$ (M.D.U. 2012)

(M.D.U. Dec. 2013, May 2014)

[**Note.** Remember the results of Q. 4.]

(b) Find the Fourier cosine transform of $f(x) = e^{-ax}, a > 0$ and hence find the Fourier sine transform of $x\,e^{-ax}$.

(K.U.K. Dec. 2015)

[**Hint.** $F_c(e^{-ax}) = \dfrac{a}{s^2+a^2} \Rightarrow \displaystyle\int_0^\infty e^{-ax}\cos sx\,dx = \frac{a}{s^2+a^2}$

Differentiating w.r.t. s, we get $\displaystyle\int_0^\infty \frac{\partial}{\partial s}(e^{-ax}\cos sx)\,dx = \frac{-2as}{(s^2+a^2)^2}$

$$\Rightarrow \int_0^\infty e^{-ax}(-x\sin sx)\,dx = \frac{-2as}{(s^2+a^2)^2} \Rightarrow \int_0^\infty (xe^{-ax})\sin sx\,dx = \frac{2as}{(s^2+a^2)^2}$$

$$\therefore \quad F_s(x\,e^{-ax}) = \frac{2as}{(s^2+a^2)^2}\Big]$$

5. Find the Fourier transforms of the following functions:

(i) $f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a \end{cases}$ (ii) $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$ (K.U.K. Jan. 2013)

(iii) $f(x) = e^{-|x|}$ (iv) $f(x) = e^{-\frac{x^2}{2}}, -\infty < x < \infty.$

(K.U.K. 2010; M.D.U. Dec. 2013)

(v) $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$ (vi) $f(x) = \begin{cases} 0, & x < \alpha \\ 1, & \alpha < x < \beta \\ 0, & x > \beta \end{cases}$

6. (*i*) Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$.

(*ii*) Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 + \frac{x}{a}, & \text{for } -a < x < 0 \\ 1 - \frac{x}{a}, & \text{for } \quad 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

(*iii*) Find the Fourier transform of the function

$$f(t) = \begin{cases} 1\ , & \text{for } -2 < t < -1 \\ 2\ , & \text{for } -1 < t < 1 \\ 1\ , & \text{for } \quad 1 < t < 2 \\ 0\ , & \text{otherwise} \end{cases}$$

7. Using inverse Fourier sine transform, find $f(x)$ if

(*i*) $F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$ (*ii*) $F_s(\lambda) = \frac{\lambda}{1+\lambda^2}$.

8. Find the finite Fourier sine and cosine transforms of the following functions:

(*i*) $f(x) = 2x,\ 0 \le x \le 4$ (*ii*) $f(x) = x(l - x),\ 0 \le x \le l$

(*M.D.U. May 2013, Dec. 2014*)

(*iii*) $f(x) = x^2,\ 0 \le x \le 2$ (*iv*) $f(x) = a\left(1 - \frac{x}{l}\right),\ 0 \le x \le l$

(*v*) $f(x) = \begin{cases} kx\ , & 0 \le x \le \frac{\pi}{2} \\ k(\pi - x)\ , & \frac{\pi}{2} \le x \le \pi \end{cases}$.

9. Find the finite Fourier sine transform of the following functions:

(*i*) $f(x) = \cos x,\ 0 \le x \le \pi$ (*ii*) $f(x) = \begin{cases} \frac{2x}{3}, & 0 \le x \le \frac{\pi}{3} \\ \frac{\pi - x}{3}, & \frac{\pi}{3} \le x \le \pi \end{cases}$.

10. If $f(x) = \sin kx$, where $0 \le x \le \pi$ and k is a positive integer, show that

$$F_s(n) = \begin{cases} 0\ , & \text{if } n \ne k \\ \frac{\pi}{2}, & \text{if } n = k \end{cases}.$$

11. Find the finite Fourier cosine transform of the following functions:

(*i*) $f(x) = \sin x,\ 0 \le x \le \pi$ (*ii*) $f(x) = \begin{cases} kx\ , & 0 \le x \le \frac{l}{2} \\ k(l - x)\ , & \frac{l}{2} \le x \le l \end{cases}$.

12. Find $f(x)$, if

(i) $F_s(n) = \dfrac{2l}{\pi n} \sin^2 \dfrac{n\pi}{4}$, $0 \le x \le l$ (ii) $F_s(n) = \dfrac{1}{n^2} \sin \dfrac{n\pi}{3}$, $0 \le x \le \pi$

(iii) $F_s(n) = \dfrac{2l^3}{n^3\pi^3}(1 - \cos n\pi)$, $0 \le x \le l$

(iv) $F_c(n) = \dfrac{cl}{n\pi} \sin \dfrac{n\pi a}{l}$ and $F_c(0) = ac$, where $0 \le x \le l$

(v) $F_c(n) = -\dfrac{l^3}{n^2\pi^2}(1 + \cos n\pi)$ and $F_c(0) = \dfrac{l^3}{6}$, where $0 \le x \le l$

(vi) $F_c(n) = \dfrac{\cos\left(\dfrac{2n\pi}{3}\right)}{(2n+1)^2}$, where $0 \le x \le 1$.

13. Solve the following integral equations:

(i) $\displaystyle\int_0^\infty f(x) \sin px \, dx = \begin{cases} 1, & 0 \le p < 1 \\ 2, & 1 \le p < 2 \\ 0, & p \ge 2 \end{cases}$ *(M.D.U. 2012)*

(ii) $\displaystyle\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}, \lambda > 0$ *(DCRUST Murthal May 2014; M.D.U. 2011)*

(iii) $\displaystyle\int_0^\infty f(x) \sin \lambda x \, dx = \begin{cases} 1 - \lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ (iv) $\displaystyle\int_0^\infty f(x) \sin \lambda x \, dx = \frac{\lambda}{\lambda^2 + k^2}$.

Answers

2. (i) $\dfrac{1}{2}\left[\dfrac{\sin(1-s)a}{1-s} - \dfrac{\sin(1+s)a}{1+s}\right]$ (ii) $\dfrac{a \cos sa - b \cos sb}{s} + \dfrac{\sin sb - \sin sa}{s^2}$

3. (i) $\dfrac{6 + 3s^2}{4 + 5s^2 + s^4}$ (ii) $\sqrt{\dfrac{\pi}{2}}\, e^{-\frac{s^2}{2}}$

(iii) $\dfrac{2 \cos s(1 - \cos s)}{s^2}$ (iv) $10\left(\dfrac{1}{s^2 + 25} + \dfrac{1}{s^2 + 4}\right)$

(v) $\dfrac{1}{s^2}\left(2 \cos \dfrac{s}{2} - 1 - \cos s\right)$ (vi) $\dfrac{2}{\pi s^2}$

4. (i) $\dfrac{1 - \cos as}{s}, \dfrac{\sin as}{s}$ (ii) $\dfrac{s}{s^2 + a^2}, \dfrac{a}{s^2 + a^2}$

5. (i) $2i\left(\dfrac{\sin as - as \cos as}{s^2}\right)$ (ii) $\dfrac{2}{s^3}[(a^2s^2 - 2) \sin as + 2as \cos as]$

(iii) $\dfrac{2}{1 + s^2}$ (iv) $\sqrt{2\pi}\, e^{-\frac{s^2}{2}}$

(v) $\dfrac{i}{(k+s)}[e^{i(k+s)a} - e^{i(k+s)b}]$ (vi) $\dfrac{i}{s}(e^{is\alpha} - e^{is\beta})$

6. (i) $\dfrac{2 \sin s}{s}$; $\dfrac{\pi}{2}$ (ii) $\dfrac{2}{as^2}(1 - \cos as)$

(iii) $\dfrac{2}{s} \sin s(1 + 2 \cos s)$

7. (i) $f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a}$ (ii) $f(x) = e^{-x}$

8. (i) $-\frac{32}{n\pi} \cos n\pi = -\frac{32(-1)^n}{n\pi}$; $\frac{32[(-1)^n - 1]}{n^2\pi^2}$, $F_c(0) = 16$

(ii) $\frac{2l^3}{n^3\pi^3} [1 - (-1)^n]$; $-\frac{l^3}{n^2\pi^2} [(-1)^n + 1]$, $F_c(0) = \frac{l^3}{6}$

(iii) $\frac{16}{n^3\pi^3} [(-1)^n - 1] - \frac{8}{n\pi} (-1)^n$; $\frac{16}{n^2\pi^2} (-1)^n$, $F_c(0) = \frac{8}{3}$

(iv) $\frac{al}{n\pi}$; $\frac{al}{n^2\pi^2} [1 - (-1)^n]$, $F_c(0) = \frac{al}{2}$

(v) $\frac{2k}{n^2} \sin \frac{n\pi}{2}$; $\frac{k}{n^2}\left[2 \cos \frac{n\pi}{2} - (-1)^n - 1\right]$, $F_c(0) = \frac{k\pi^2}{4}$

9. (i) $\frac{n}{n^2 - 1} [1 + (-1)^n]$ (ii) $\frac{1}{n^2} \sin \frac{n\pi}{3}$

11. (i) $-\frac{1}{n^2 - 1}$ $[1 + (-1)^n]$, $F_c(0) = 2$ (ii) $\frac{kl^2}{n^2\pi^2}\left[2 \cos \frac{n\pi}{2} - 1 - (-1)^n\right]$, $F_c(0) = \frac{kl^2}{4}$

12. (i) $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{n\pi}{4} \sin \frac{n\pi x}{l}$ (ii) $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \sin nx$

(iii) $f(x) = \frac{4l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} (1 - \cos n\pi) \sin \frac{n\pi x}{l}$ (iv) $f(x) = \frac{ac}{l} + \frac{2c}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi a}{l} \cos \frac{n\pi x}{l}$

(v) $f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (1 + \cos n\pi) \cos \frac{n\pi x}{l}$

(vi) $f(x) = 1 + 2 \sum_{n=1}^{\infty} \frac{\cos \frac{2n\pi}{3}}{(2n+1)^2} \cos n\pi x$

13. (i) $f(x) = \frac{2}{\pi x}$ $(1 + \cos x - 2 \cos 2x)$ (ii) $f(x) = \frac{2}{\pi(1 + x^2)}$

(iii) $f(x) = \frac{2}{\pi}\left(\frac{x - \sin x}{x^2}\right)$ (iv) $f(x) = e^{-kx}$, $x > 0$.

2.6. PROPERTIES OF FOURIER TRANSFORMS

1. Linearity Property. *If F(s) and G(s) are Fourier transforms of f(x) and g(x) respectively, then*

$$\mathbf{F[af(x) + bg(x)] = aF(s) + bG(s)}$$

where a and b are constants.

Proof. By definition of Fourier transform, we have

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \, . \, e^{isx} \, dx$$

and

$$G(s) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x) \, . \, e^{isx} \, dx$$

$$\therefore \qquad F[af(x) + bg(x)] = \int_{-\infty}^{\infty} [af(x) + bg(x)]\, e^{isx} dx$$

$$= a\int_{-\infty}^{\infty} f(x) \,.\, e^{isx} dx + b\int_{-\infty}^{\infty} g(x) \,.\, e^{isx}\, dx = aF(s) + bG(s).$$

Cor. (*i*) If $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

$$F_s[af(x) + bg(x)] = aF_s(s) + bG_s(s)$$

where a and b are constants.

(*ii*) If $F_c(s)$ and $G_c(s)$ are the Fourier cosine transforms of $f(x)$ and $g(x)$ respectively, then

$$F_c[af(x) + bg(x)] = aF_c(s) + bG_c(s)$$

where a and b are constants.

2. Change of scale property (Similarity Theorem). *If F(s) is the complex Fourier transform of f(x), then*

$$\mathbf{F[f(ax)] = \frac{1}{a}F\left(\frac{s}{a}\right), a \neq 0.}$$

Proof. By definition of complex Fourier transform, we have

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx \qquad \ldots(1)$$

$$\therefore \qquad F[f(ax)] = \int_{-\infty}^{\infty} f(ax) \,.\, e^{isx}\, dx \qquad \ldots(2)$$

Putting $\quad ax = t \quad i.e., \quad x = \dfrac{t}{a}$, we have $dx = \dfrac{dt}{a}$.

When $x \to -\infty$, $t \to -\infty$ and when $x \to \infty$, $t \to \infty$

$\therefore$ From (2) we have

$$F[f(ax)] = \int_{-\infty}^{\infty} f(t)\, e^{ist/a} \,.\, \frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t}\, dt = \frac{1}{a} F\left(\frac{s}{a}\right) \quad \text{[by (2)]}$$

Cor. (*i*) If $F_s(s)$ is the Fourier sine transform of $f(x)$, then

$$F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right).$$

(*ii*) If $F_c(s)$ is the Fourier cosine transform of $f(x)$, then

$$F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right).$$

3. Shifting Property. *If F(s) is the complex Fourier transform of f(x), then*

$$\mathbf{F[f(x - a)] = e^{isa}F(s).}$$

Proof. By definition of complex Fourier transform, we have

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \,.\, e^{isx}\, dx$$

$$\therefore \qquad F[f(x - a)] = \int_{-\infty}^{\infty} f(x - a)\, e^{isx}\, dx \qquad \ldots(1)$$

Putting $x - a = t \quad i.e., \quad x = a + t$, we have $dx = dt$

When $x \to -\infty$, $t \to -\infty$ and when $x \to \infty$, $t \to \infty$

$\therefore$ From (1), we have

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(t)\, e^{is(a+t)}\, dt = \int_{-\infty}^{\infty} f(t) \,.\, e^{isa} \,.\, e^{ist}\, dt$$

$$= e^{isa} \int_{-\infty}^{\infty} f(t) \,.\, e^{ist}\, dt = e^{isa}\, F(s).$$

3. (a) Shifting on time axis. *If $F(s)$ is the complex Fourier transform of $f(t)$ and t_0 is any real number, then*

$$\mathbf{F[f(t-t_0)] = e^{ist_0} F(s).}$$

Proof. By definition of complex Fourier transform, we have

$$F(s) = F[f(t)] = \int_{-\infty}^{\infty} f(t) \,.\, e^{ist}\, dt$$

$$\therefore \quad F[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)\, e^{ist}\, dt \qquad \ldots(1)$$

Putting $t - t_0 = T$ *i.e.*, $t = t_0 + T$, we have $dt = dT$

When $t \to -\infty$, $T \to -\infty$ and when $t \to \infty$, $T \to \infty$

$\therefore$ From (1), we have

$$F[f(t-t_0)] = \int_{-\infty}^{\infty} f(T)\, e^{is(t_0+T)}\, dT$$

$$= \int_{-\infty}^{\infty} f(T)\, e^{ist_0} \,.\, e^{isT}\, dT = e^{ist_0} \int_{-\infty}^{\infty} f(T)\, e^{isT}\, dT$$

$$= e^{ist_0}\, F(s).$$

Remark. Inverse Fourier transform of $e^{ist_0} F(s)$ is $f(t-t_0)$.

3. (b) Shifting on frequency axis. *If $F(s)$ is the complex Fourier transform of $f(t)$, and s_0 is any real number, then*

$$\mathbf{F[e^{is_0t} f(t)] = F(s+s_0).}$$

Proof. By definition of complex Fourier transform, we have

$$F(s) = F[f(t)] = \int_{-\infty}^{\infty} f(t) \,.\, e^{ist}\, dt$$

$$\therefore \quad F[e^{is_0t} f(t)] = \int_{-\infty}^{\infty} e^{is_0t} f(t) \,.\, e^{ist}\, dt = \int_{-\infty}^{\infty} f(t) \,.\, e^{i(s+s_0)t}\, dt = F(s+s_0).$$

Remark. Inverse Fourier transform of $F(s+s_0)$ is $e^{is_0t} f(t)$.

4. Modulation Theorem. *If $F(s)$ is the complex Fourier transform of $f(x)$, then*

$$\mathbf{F[f(x) \cos ax] = \frac{1}{2} [F(s+a) + F(s-a)].}$$

Proof. By definition of complex Fourier transform, we have

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$\therefore$
$$F[f(x) \cos ax] = \int_{-\infty}^{\infty} f(x) \cos ax \, . e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) e^{isx}\, dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \int_{-\infty}^{\infty} f(x) . e^{i(s-a)x}\, dx \right]$$

$$= \frac{1}{2}\, [F(s + a) + F(s - a)].$$

Cor. If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

(*i*) $F_s[f(x) \cos ax] = \frac{1}{2} [F_s(s + a) + F_s(s - a)]$

(*ii*) $F_s[f(x) \sin ax] = \frac{1}{2} [F_c(s - a) - F_c(s + a)]$

(*iii*) $F_c[f(x) \cos ax] = \frac{1}{2} [F_c(s + a) + F_c(s - a)]$

(*iv*) $F_c[f(x) \sin ax] = \frac{1}{2} [F_s(s + a) - F_s(s - a)]$.

5. **If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of f(x) respectively, then**

(*i*) $$F_s\,\{x\, f(x)\} = -\frac{d}{ds}\,\{F_c\,(s)\}$$

(*ii*) $$F_c\{x\, f(x)\} = \frac{d}{ds}\,\{F_s\,(s)\}$$

Proof. (*i*) $$\frac{d}{ds}\,\{F_c\,(s)\} = \frac{d}{ds}\left\{ \int_0^{\infty} f(x) \cos sx\, dx \right\}$$

$$= \int_0^{\infty} f(x)\, (-\,x \sin sx)\, dx$$

$$= -\int_0^{\infty} \{x\, f(x)\} \sin sx\, dx$$

$$= -\,F_s\,\{x\, f(x)\}$$

$\Rightarrow$
$$F_s\,\{x\, f(x)\} = -\frac{d}{ds}\,\{F_c\,(s)\}$$

(*ii*) $$\frac{d}{ds}\,\{F_s\,(s)\} = \frac{d}{ds}\left\{ \int_0^{\infty} f(x) \sin sx\, dx \right\}$$

$$= \int_0^{\infty} f(x)\, (x \cos sx)\, dx$$

$$= \int_0^\infty \{x\, f(x)\} \cos sx\, dx$$
$$= F_c \{x\, f(x)\}$$
$$\Rightarrow \qquad F_c \{x\, f(x)\} = \frac{d}{ds} \{F_s (s)\}.$$

Example 1. *Find the Fourier transform of e^{-x^2}. Hence find the Fourier transform of*

(*i*) e^{-ax^2}, $(a > 0)$ (*ii*) $e^{-\frac{x^2}{2}}$

(*iii*) $e^{-4(x-3)^2}$ (*iv*) $e^{-x^2} \cos 2x$.

Sol. Fourier transform of $f(x) = e^{-x^2}$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) . e^{isx}\, dx = \int_{-\infty}^{\infty} e^{-x^2} . e^{isx} dx$$
$$= \int_{-\infty}^{\infty} e^{-(x^2 - isx)}\, dx = \int_{-\infty}^{\infty} e^{-\left\{\left(x - \frac{is}{2}\right)^2 + \frac{s^2}{4}\right\}} dx$$
$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} dx$$
$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-z^2}\, dz \quad \text{where } z = x - \frac{is}{2}$$
$$= 2\, e^{-\frac{s^2}{4}} \int_0^{\infty} e^{-z^2}\, dz = 2\, e^{-\frac{s^2}{4}} . \frac{\sqrt{\pi}}{2}$$
$$= \sqrt{\pi}\, e^{-\frac{s^2}{4}} = F(s)$$

(*i*) $$e^{-ax^2} = e^{-(\sqrt{a}x)^2} = f(\sqrt{a}x)$$

By change of scale property, we have

$$F\{f(\sqrt{a}x)\} = \frac{1}{\sqrt{a}} F\left(\frac{s}{\sqrt{a}}\right)$$
$$\Rightarrow \qquad F(e^{-ax^2}) = \frac{1}{\sqrt{a}} \sqrt{\pi}\, e^{-\frac{1}{4}\left(\frac{s}{\sqrt{a}}\right)^2} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{s^2}{4a}}$$

(*ii*) Putting $a = \frac{1}{2}$ in deduction (*i*), we have

$$F\left(e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}\, e^{-s^2/2}$$

(*iii*) $e^{-4x^2} = e^{-(2x)^2} = f(2x)$

By change of scale property, we have

$$F\{f(2x)\} = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$\Rightarrow \qquad F\{e^{-4x^2}\} = \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}\left(\frac{s}{2}\right)^2} = \frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{16}}$$

By shifting property $F\{f(x-a)\} = e^{isa} F(s)$

$$\therefore \qquad F\{e^{-4(x-3)^2}\} = e^{3is} \cdot \frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{16}} = \frac{\sqrt{\pi}}{2} e^{\left(3is - \frac{s^2}{16}\right)}$$

(*iv*) By modulation theorem,

$$F\{f(x) \cos ax\} = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$\therefore \qquad F\{e^{-x^2} \cos 2x\} = \frac{1}{2}\left[\sqrt{\pi}\, e^{-\frac{1}{4}(s+2)^2} + \sqrt{\pi}\, e^{-\frac{1}{4}(s-2)^2}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[e^{-\frac{1}{4}(s+2)^2} + e^{-\frac{1}{4}(s-2)^2}\right].$$

Example 2. *Find the Fourier sine and cosine transform of $x\, e^{-ax}$.*

Sol. Let us first find the Fourier sine and cosine transforms of e^{-ax}.

$$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx \, dx = \left[\frac{e^{-ax}}{a^2+s^2}(-a \sin sx - s \cos sx)\right]_0^\infty = \frac{s}{a^2+s^2}$$

and

$$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx \, dx = \left[\frac{e^{-ax}}{a^2+s^2}(-a \cos sx + s \sin sx)\right]_0^\infty = \frac{a}{a^2+s^2}$$

$$\therefore \qquad F_s(xe^{-ax}) = -\frac{d}{ds}\{F_c(e^{-ax})\} = -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(a^2+s^2)^2}$$

and

$$F_c(xe^{-ax}) = \frac{d}{ds}\left\{F_s(e^{-ax})\right\} = \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right)$$

$$= \frac{(a^2+s^2) \cdot 1 - s(2s)}{(a^2+s^2)^2} = \frac{a^2-s^2}{(a^2+s^2)^2}.$$

2.7. CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$$

2.8. CONVOLUTION THEOREM FOR FOURIER TRANSFORMS (or Falting Theorem)

(M.D.U. 2010, 2011, 2012, May 2014; K.U.K. 2010, 2013; DCRUST Murthal May 2014)

The Fourier transform of the convolution of f(x) and g(x) is equal to the product of the Fourier transforms of f(x) and g(x), i.e.,

$$\mathbf{F[f(x) * g(x)] = F[f(x)].\ F[g(x)] = F(s)G(s).}$$

Proof. By definition of Fourier transform

$$F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x).e^{isx}\,dx,$$

$$G(s) = F[g(x)] = \int_{-\infty}^{\infty} g(x).e^{isx}\,dx$$

and by definition of convolution,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$$

$$\therefore \qquad F[f(x) * g(x)] = \int_{-\infty}^{\infty} [f(x) * g(x)]\,e^{isx}dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right] e^{isx}dx$$

$$= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u)\,e^{isx}\,dx\right]du$$

(By changing the order of integration)

Putting $x - u = t$ i.e., $x = u + t$, we have $dx = dt$.
When $x \to -\infty$, $t \to -\infty$ and when $x \to \infty$, $t \to \infty$

$$\therefore \qquad F[f(x) * g(x)] = \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t)\,e^{is(u+t)}\,dt\right]du$$

$$= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t).e^{isu}.e^{ist}\,dt\right]du$$

$$= \left[\int_{-\infty}^{\infty} f(u)\,e^{isu}\,du\right]\left[\int_{-\infty}^{\infty} g(t)\,e^{ist}\,dt\right]$$

$$= \left[\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right]\left[\int_{-\infty}^{\infty} g(x).e^{isx}\,dx\right]$$

$$= F[f(x)]\ F[g(x)] = F(s)\ G(s).$$

Remark. The following properties of convolution can be easily proved.

(*i*) $f(x) * g(x) = g(x) * f(x)$

(*ii*) $f(x) * [g(x) * h(x)] = [f(x) * g(x)] * h(x)$

(*iii*) $f(x) * [g(x) + h(x)] = f(x) * g(x) + f(x) * h(x)$.

2.9. RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

If $$f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases}, \quad \textit{then } F[f(t)] = L[g(t)].$$

Proof. By definition of Fourier transform, we have

$$F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{ist}\, dt$$

$$= \int_{-\infty}^{0} 0.e^{ist}\, dt + \int_{0}^{\infty} e^{-xt} g(t) . e^{ist}\, dt \qquad \text{[By def. of } f(t)]$$

$$= \int_{0}^{\infty} e^{-(x-is)t} g(t)\, dt$$

$$= \int_{0}^{\infty} e^{-pt} g(t)\, dt, \quad \text{where } p = x - is$$

$$= L[g(t)].$$

2.10. PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

(*i*) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, \overline{G}(s)\, ds = \int_{-\infty}^{\infty} f(x)\, \overline{g}(x)\, dx$$

(*ii*) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

where bar stands for the complex conjugate.

Proof. (*i*) By definition of Fourier transform

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) . e^{isx}\, dx$$

and $$G(s) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x) . e^{isx}\, dx$$

By inversion formula for Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) . e^{-isx}\, ds$$

and $$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) . e^{-isx}\, ds$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\, \overline{g}(x)\, dx = \int_{-\infty}^{\infty} f(x) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G}(s) . e^{isx}\, ds \right] dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G}(s) \left[\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \right] ds$$

(Changing the order of integration)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G}(s)\, [F(s)]\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{G}(s)\,ds.$$

(*ii*) By definition of Fourier transform

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

By inversion formula for Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{e}^{isx}\,ds$$

$$\therefore \qquad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} f(x)\,\bar{f}(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F}(s)\,e^{isx}\,ds\right]dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F}(s)\left[\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right]ds$$

(Changing the order of integration)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F}(s)\,[F(s)]\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

Remark. The following Parseval's identities for Fourier sine and cosine transforms can be easily proved:

(*i*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} F_s(s)\,G_s(s)\,ds = \int_0^{\infty} f(x)\,g(x)\,dx$

(*ii*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} [F_s(s)]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx$

(*iii*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} F_c(s)\,G_c(s)\,ds = \int_0^{\infty} f(x)\,g(x)\,dx$

(*iv*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} [F_c(s)]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx.$

ILLUSTRATIVE EXAMPLES

Example 1. *Using Parseval's identities, prove that*

(*i*) $\displaystyle\int_0^{\infty} \frac{dt}{(4+t^2)(9+t^2)} = \frac{\pi}{60}$ (*ii*) $\displaystyle\int_0^{\infty} \frac{t^2}{(4+t^2)(9+t^2)}\,dt = \frac{\pi}{10}$

(*iii*) $\displaystyle\int_0^{\infty} \frac{dt}{(4+t^2)^2} = \frac{\pi}{32}$ (*iv*) $\displaystyle\int_0^{\infty} \frac{t^2}{(9+t^2)^2}\,dt = \frac{\pi}{12}.$

Sol. Let $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$

Then $\quad F_c(s) = \dfrac{2}{4+s^2}, \quad G_c(s) = \dfrac{3}{9+s^2}$

$F_s(s) = \dfrac{s}{4+s^2}, \quad G_s(s) = \dfrac{s}{9+s^2}$

$$\left[\because f(x) = e^{-ax} \Rightarrow F_c(s) = \frac{a}{a^2+s^2}, F_s(s) = \frac{s}{a^2+s^2}\right]$$

(*i*) Using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\frac{2}{4+s^2}\right)\left(\frac{3}{9+s^2}\right) ds = \int_0^\infty e^{-2x} . e^{-3x}\, dx$

$\Rightarrow \quad \dfrac{12}{\pi}\displaystyle\int_0^\infty \frac{ds}{(4+s^2)(9+s^2)} = \int_0^\infty e^{-5x}\, dx$

$$= \left[\frac{e^{-5x}}{-5}\right]_0^\infty = -\frac{1}{5}(0-1) = \frac{1}{5}$$

$\Rightarrow \quad \displaystyle\int_0^\infty \frac{ds}{(4+s^2)(9+s^2)} = \frac{\pi}{60}$

$\therefore \quad \displaystyle\int_0^\infty \frac{dt}{(4+t^2)(9+t^2)} = \frac{\pi}{60}.$

(*ii*) Using Parseval's identity for fourier sine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\frac{s}{4+s^2}\right)\left(\frac{s}{9+s^2}\right) ds = \int_0^\infty e^{-2x} . e^{-3x}\, dx$

$\Rightarrow \quad \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{s^2}{(4+s^2)(9+s^2)}\, ds = \int_0^\infty e^{-5x}\, dx$

$$= \left[\frac{e^{-5x}}{-5}\right]_0^\infty = -\frac{1}{5}(0-1) = \frac{1}{5}$$

$\Rightarrow \quad \displaystyle\int_0^\infty \frac{s^2}{(4+s^2)(9+s^2)}\, ds = \frac{\pi}{10}$

$\therefore \quad \displaystyle\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)}\, dt = \frac{\pi}{10}.$

(*iii*) Using Parseval's identity for Fourier cosine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^\infty [F_c(s)]^2\, ds = \int_0^\infty [f(x)]^2\, dx$$

We have $$\frac{2}{\pi}\int_0^\infty \left(\frac{2}{4+s^2}\right)^2 ds = \int_0^\infty \left(e^{-2x}\right)^2 dx$$

$$\Rightarrow \quad \frac{8}{\pi}\int_0^\infty \frac{ds}{(4+s^2)^2} = \int_0^\infty e^{-4x}\, dx$$

$$= \left[\frac{e^{-4x}}{-4}\right]_0^\infty = -\frac{1}{4}(0-1) = \frac{1}{4}$$

$$\Rightarrow \quad \int_0^\infty \frac{ds}{(4+s^2)^2} = \frac{\pi}{32}$$

$$\therefore \quad \int_0^\infty \frac{dt}{(4+t^2)^2} = \frac{\pi}{32}.$$

(*iv*) Using Parseval's identity for Fourier sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^\infty [G_s(s)]^2\, ds = \int_0^\infty [g(x)]^2\, dx$$

We have $$\frac{2}{\pi}\int_0^\infty \left(\frac{s}{9+s^2}\right)^2 ds = \int_0^\infty (e^{-3x})^2\, dx$$

$$\Rightarrow \quad \frac{2}{\pi}\int_0^\infty \frac{s^2}{(9+s^2)^2}\, ds = \int_0^\infty e^{-6x}\, dx$$

$$= \left[\frac{e^{-6x}}{-6}\right]_0^\infty = -\frac{1}{6}(0-1) = \frac{1}{6}$$

$$\Rightarrow \quad \int_0^\infty \frac{s^2}{(9+s^2)^2}\, ds = \frac{\pi}{12}$$

$$\therefore \quad \int_0^\infty \frac{t^2}{(9+t^2)^2}\, dt = \frac{\pi}{12}.$$

Example 2. *Using Parseval's identity, prove that*

$$\int_0^\infty \frac{\sin at}{t(a^2+t^2)}\, dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

Sol. Let $$f(x) = e^{-ax},\ a > 0 \quad \text{and} \quad g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$$

then $$F_c(s) = \frac{a}{a^2+s^2} \text{ and } G_c(s) = \frac{\sin as}{s}.$$

Using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

We have $$\frac{2}{\pi}\int_0^\infty \left(\frac{a}{a^2+s^2}\right)\left(\frac{\sin as}{s}\right) ds = \int_0^\infty e^{-ax} \cdot g(x)\, dx$$

$$\Rightarrow \quad \frac{2a}{\pi}\int_0^\infty \frac{\sin as}{s(a^2+s^2)}\,ds = \int_0^a e^{-ax}\,.\,g(x)\,dx + \int_a^\infty e^{-ax}\,.\,g(x)\,dx$$

$$= \int_0^a e^{-ax}\,.\,1\,dx + \int_a^\infty e^{-ax}\,.\,0\,dx$$

$$= \left[\frac{e^{-ax}}{-a}\right]_0^a + 0 = -\frac{1}{a}(e^{-a^2}-1) = \frac{1}{a}(1-e^{-a^2})$$

$$\Rightarrow \quad \int_0^\infty \frac{\sin as}{s(a^2+s^2)}\,ds = \frac{\pi}{2a^2}\,(1-e^{-a^2})$$

$$\therefore \quad \int_0^\infty \frac{\sin at}{t(a^2+t^2)}\,dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

Example 3. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

and hence find the value of $\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$. (M.D.U. 2010, May 2014)

Sol. Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^\infty f(x)e^{isx}\,dx$$

$$= \int_{-\infty}^{-1} f(x)e^{isx}\,dx + \int_{-1}^{1} f(x)\,e^{isx}\,dx + \int_1^\infty f(x)\,e^{isx}\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1}(1-|x|)\,e^{isx}\,dx + \int_1^\infty 0\,dx$$

$$= \int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)\,dx$$

$$= \int_{-1}^{1}(1-|x|)\cos sx\,dx + \int_{-1}^{1}(1-|x|)(i\sin sx)\,dx$$

$$= 2\int_0^1 (1-|x|)\cos sx\,dx + 0$$

[$\because$ $(1-|x|)\cos sx$ is an even function of x where as $(1-|x|)\sin sx$ is an odd function of x]

$$= 2\int_0^1 (1-x)\cos sx\,dx \qquad [\because \ |x| = x \ \text{ where } x>0]$$

$$= 2\left[(1-x)\,.\,\frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right] = 2\left(\frac{1-\cos s}{s^2}\right) = F(s)$$

Using Parseval's identity for Fourier transform,

i.e., $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

We have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(1-\cos s)^2}{s^4}\, ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\Rightarrow \quad \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\Rightarrow \quad \frac{4}{\pi}\int_{0}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = 2\int_{0}^{1} (1-|x|)^2\, dx$$

($\because$ Both integrands are even functions)

$$\Rightarrow \quad \frac{4}{\pi}\int_{0}^{\infty} \frac{(2\sin^2 s/2)^2}{s^4}\, ds = 2\int_{0}^{1} (1-x)^2\, dx$$

$$\Rightarrow \quad \frac{16}{\pi}\int_{0}^{\infty} \frac{\sin^4 s/2}{s^4}\, ds = 2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{2}{3}$$

Putting $\frac{s}{2} = t$ *i.e.,* $s = 2t$, we have

$$\frac{16}{\pi}\int_{0}^{\infty} \frac{\sin^4 t}{16t^4}\,(2dt) = \frac{2}{3}$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{\sin^4 t}{t^4}\, dt = \frac{\pi}{3}.$$

EXERCISE 2.3

1. Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$. *(M.D.U. 2011)*
2. Using Parseval's identities, prove that

   (*i*) $\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ (*ii*) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\, dx = \frac{\pi}{4}$.

   [**Hint.** Use Parseval's identity for Fourier sine and cosine transforms of $f(x) = e^{-x}$]
3. Using Parseval's identity, show that $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$. *(K.U.K. May 2013)*

   [**Hint.** Use Parseval's identity for Fourier cosine transforms of $f(x) = e^{-ax}$, $g(x) = e^{-bx}$]

4. If $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and $F(s) = \dfrac{2 \sin as}{s}$, $(s \neq 0)$, then prove that $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi a}{2}$.

5. Using Parseval's identity, prove that

(i) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$ (ii) $\displaystyle\int_0^\infty \frac{1-\cos 2x}{x^2}\, dx = \pi.$

6. Using Parseval's identity, prove that

(i) $\displaystyle\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx = \frac{\pi}{2}$ (ii) $\displaystyle\int_0^\infty \frac{\sin^4 t}{t^2}\, dt = \frac{\pi}{4}.$

2.11. FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x, t)] = \int_{-\infty}^{\infty} u e^{isx}\, dx$$

(i) Fourier transform of $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ *(M.D.U. May 2015)*

Suppose $u \to 0$ as $x \to \pm\infty$, then the Fourier transform of $\dfrac{\partial u}{\partial x}$ is given by

$$F\left[\frac{\partial u}{\partial x}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} . e^{isx}\, dx = \int_{-\infty}^{\infty} e^{isx} . \frac{\partial u}{\partial x} dx \qquad \text{(Integrating by parts)}$$

$$= \left[e^{isx} . u\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ise^{isx} . u\, dx$$

$$= 0 - is \int_{-\infty}^{\infty} u . e^{isx}\, dx \qquad [\because \quad u \to 0 \text{ as } x \to \pm\infty]$$

$$= -is\, F(u)$$

Hence $\mathbf{F}\left[\dfrac{\partial \mathbf{u}}{\partial \mathbf{x}}\right] = -\, \mathbf{is\, F(u)}.$

(ii) Fourier transform of $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$ *(M.D.U. May 2015)*

Suppose u and $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$, then the Fourier transform of $\dfrac{\partial^2 u}{\partial x^2}$ is given by

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} . e^{isx}\, dx = \int_{-\infty}^{\infty} e^{isx} . \frac{\partial^2 u}{\partial x^2} dx$$

(Applying general rule of integration by parts)

$$= \left[e^{isx} . \frac{\partial u}{\partial x} - ise^{isx} . u\right]_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} e^{isx} . u\, dx$$

$$= 0 - s^2 \int_{-\infty}^{\infty} u.\, e^{isx}\, dx \qquad \left[\because\ u \text{ and } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \pm\infty\right]$$

$$= -\, s^2 F(u).$$

Hence $$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\, \mathbf{s}^2 \mathbf{F}(\mathbf{u}) = (-\, \mathbf{is})^2\ \mathbf{F}(\mathbf{u}).$$

(iii) **Fourier transform of** $\frac{\partial^n \mathbf{u}}{\partial \mathbf{x}^n}$

Suppose $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}} \to 0$ as $x \to \pm\infty$, then the Fourier transform of $\frac{\partial^n u}{\partial x^n}$ is given by

$$F\left[\frac{\partial^n u}{\partial x^n}\right] = \int_{-\infty}^{\infty} \frac{\partial^n u}{\partial x^n} .\, e^{isx}\, dx = \int_{-\infty}^{\infty} e^{isx} .\, \frac{\partial^n u}{\partial x^n} dx$$

(Applying general rule of integration by parts)

$$= \left[e^{isx} .\, \frac{\partial^{n-1} u}{\partial x^{n-1}} - ise^{isx} .\, \frac{\partial^{n-2} u}{\partial x^{n-2}} + (is)^2 e^{isx} .\, \frac{\partial^{n-3} u}{\partial x^{n-3}} - \ldots\ldots + (-\, is)^{n-1} u \right]_{-\infty}^{\infty}$$

$$+ (-\, is)^n \int_{-\infty}^{\infty} e^{isx} .\, u dx$$

$$= 0 + (-\, is)^n \int_{-\infty}^{\infty} u .\, e^{isx}\, dx = (-\, is)^n\, F(u)$$

Hence $$\mathbf{F}\left[\frac{\partial^n \mathbf{u}}{\partial \mathbf{x}^n}\right] = (-\, \mathbf{is})^n\, \mathbf{F}(\mathbf{u}).$$

2.12. FOURIER SINE AND COSINE TRANSFORMS OF $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

The Fourier sine and cosine transforms of the function $u(x, t)$ are given by

$$F_s[u(x, t)] = \int_0^{\infty} u \sin sx\, dx$$

and $$F_c[u(x, t)] = \int_0^{\infty} u \cos sx\, dx .$$

(i) **Fourier sine transform of** $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

Suppose u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, then the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$ is given by

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} .\, \sin sx\, dx = \int_0^{\infty} \sin sx .\, \frac{\partial^2 u}{\partial x^2}\, dx \qquad \text{(Integrating by parts)}$$

$$= \left[\sin sx \cdot \frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty (s \cos sx) \cdot \frac{\partial u}{\partial x} dx$$

$$= 0 - s \int_0^\infty (\cos sx) \frac{\partial u}{\partial x} dx \qquad \left[\because \quad \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

(Integrating again by parts)

$$= -s \left[\{\cos sx \cdot u\}_0^\infty - \int_0^\infty (-s \sin sx) \cdot u \, dx\right]$$

$$= -s \left[0 - (u)_{x=0} + s \int_0^\infty u \sin sx \, dx\right] \qquad [\because \quad u \to 0 \text{ as } x \to \infty]$$

$$= s(u)_{x=0} - s^2 F_s(u)$$

Hence $\mathbf{F_s}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s(u)_{x=0} - s^2 F_s(u)}$.

(*ii*) **Fourier cosine transform of** $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

Suppose u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, then the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$ is given by

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \cos sx \, dx = \int_0^\infty \cos sx \cdot \frac{\partial^2 u}{\partial x^2} dx \qquad \text{(Integrating by parts)}$$

$$= \left[\cos sx \cdot \frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty -s \sin sx \cdot \frac{\partial u}{\partial x} dx$$

$$= \left[0 - \left(\frac{\partial u}{\partial x}\right)_{x=0}\right] + s \int_0^\infty \sin sx \frac{\partial u}{\partial x} dx \qquad \left[\because \quad \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s \int_0^\infty \sin sx \cdot \frac{\partial u}{\partial x} dx \qquad \text{(Integrating again by parts)}$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s \left[\{\sin sx \cdot u\}_0^\infty - \int_0^\infty s \cos sx \cdot u \, dx\right]$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 \int_0^\infty u \cos sx \, dx \qquad [\because \quad u \to 0 \text{ as } x \to \infty]$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 F_c(u)$$

Hence $\mathbf{F_c}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x=0}} - \mathbf{s^2 F_c(u)}$.

2.13. APPLICATION OF FOURIER TRANSFORMS TO BOUNDARY VALUE PROBLEMS

Fourier transforms are very useful in solving boundary value problems. We take Fourier transform of the given partial differential equation using given boundary and initial conditions. The required solution is then obtained by taking corresponding inverse transform. The choice of particular transform to be employed depends on the boundary conditions of the problem.

(*i*) *If the interval is* $-\infty < x < \infty$ and if boundary conditions are

$$u \text{ and } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \pm\infty$$

use infinite Fourier transform.

(*ii*) *If the interval is* $0 < x < \infty$ and

(*a*) boundary conditions are u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and $u(x, t) = 0$ or $f(t)$ at $x = 0$ and for all t, use Fourier sine transform.

(*b*) boundary conditions are u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and $\frac{\partial u}{\partial x} = 0$ or $f(t)$ at $x = 0$ and for all t, use Fourier cosine transform.

For the interval $0 < x < \infty$, we always assume u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, even if it is not given in the problem.

(*iii*) *If the interval is* $0 < x < L$ and

(*a*) boundary conditions are $u(0, t) = u(L, t) = 0$ for all t, use finite Fourier sine transform.

(*b*) boundary conditions are $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$ for all t, use finite Fourier cosine transform.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$

subject to the conditions

(*i*) $u = 0$, *when* $x = 0$, $t > 0$ (*ii*) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$, *when* $t = 0$

and (*iii*) $u(x, t)$ *is bounded.* (M.D.U. Dec. 2015; K.U.K. Jan. 2013)

Sol. Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ ...(1)

Boundary condition is $u(0, t) = 0$

Initial conditions are $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$ ...(2)

and $u(x, t)$ is bounded.

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (1). Thus,

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx.$$ Integrating by parts

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u \sin sx \, dx = \left[\sin sx \, . \, \frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty s \cos sx \, . \, \frac{\partial u}{\partial x} \, . \, dx$$

$$= 0 - s\int_0^\infty \cos sx \, . \, \frac{\partial u}{\partial x} \, . \, dx$$

$\left[\because \quad u(x, t) \text{ is bounded } \quad \therefore \quad \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \infty\right]$

$$= - s\left[\left\{\cos sx \, . \, u\right\}_0^\infty - \int_0^\infty - s \sin sx \, . u \, dx\right]$$

$$= - s\left[0 + s\int_0^\infty u \sin sx \, dx\right]$$

$[\because \quad \text{when } x \to \infty, \, u \to 0 \text{ and when } x = 0, \, u = u(0, t) = 0]$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u \sin sx \, dx = - s^2 \int_0^\infty u \sin sx \, dx$$

$$\Rightarrow \quad \frac{d\overline{u}_s}{dt} = - s^2 \overline{u}_s \qquad \text{where } \overline{u}_s = \overline{u}_s(s, t) = F_s\,[u(x, t)]$$

Separating the variables,

$$\frac{d\overline{u}_s}{\overline{u}_s} = - s^2 dt$$

Integrating, $\quad \log \overline{u}_s = - s^2 t + \log c$

$$\Rightarrow \quad \log_e\left(\frac{\overline{u}_s}{c}\right) = - s^2 t \quad \text{or} \quad \overline{u}_s = ce^{-s^2 t} \qquad ...(3)$$

Putting $t = 0$ in (3),

$$c = \overline{u}_s\,(s, 0) = F_s\,[u\,(x, 0)] = \int_0^\infty u(x, 0) \sin sx \, dx$$

$$= \int_0^1 u(x, 0) \sin sx \, dx + \int_1^\infty u(x, 0) \sin sx \, dx$$

$$= \int_0^1 1 \,.\, \sin sx \, dx + \int_1^\infty 0 \,.\, \sin sx \, dx \qquad \text{[from (2)]}$$

$$= \left[-\frac{\cos sx}{s} \right]_0^1 = \frac{1 - \cos s}{s}$$

$\therefore$ From (3), $\quad \overline{u}_s (s, t) = \left(\dfrac{1 - \cos s}{s} \right) e^{-s^2 t}$

Taking its inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \left(\frac{1 - \cos s}{s} \right) e^{-s^2 t} \sin sx \, ds$$

which is the required solution.

Example 2. *Solve* $\dfrac{\partial V}{\partial t} = K \dfrac{\partial^2 V}{\partial x^2}$ *for* $x > 0$, $t > 0$ *under the boundary conditions* $V = V_0$ *when* $x = 0$, $t > 0$ *and the initial condition* $V = 0$ *when* $t = 0$, $x > 0$.

Sol. Given $\quad \dfrac{\partial V}{\partial t} = K \dfrac{\partial^2 V}{\partial x^2}, \quad x > 0, t > 0 \qquad ...(1)$

Boundary condition is $\quad V(0, t) = V_0, t > 0$

Initial condition is $\quad V(x, 0) = 0, x > 0 \qquad ...(2)$

Since $V(0, t)$ is given, we take Fourier sine transform of both sides of (1). Thus

$$F_s \left(\frac{\partial V}{\partial t} \right) = F_s \left(K \frac{\partial^2 V}{\partial x^2} \right)$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial V}{\partial t} \sin sx \, dx = K \int_0^\infty \frac{\partial^2 V}{\partial x^2} \sin sx \, dx. \qquad \text{Integrating by parts}$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^\infty V \sin sx \, dx = K \left[\left\{ \sin sx . \frac{\partial V}{\partial x} \right\}_0^\infty - \int_0^\infty s \cos sx \,.\, \frac{\partial V}{\partial x} dx \right]$$

$$= K \left[0 - s \int_0^\infty \cos sx \,.\, \frac{\partial V}{\partial x} dx \right] \qquad \left[\because \frac{\partial V}{\partial x} \to 0 \text{ when } x \to \infty \right]$$

$$= - Ks \left[\left\{ \cos sx \,.\, V \right\}_0^\infty - \int_0^\infty - s \sin sx \,.\, V dx \right]$$

$$= - Ks \left[- V_0 + s \int_0^\infty V \sin sx \, dx \right]$$

$[\because$ when $x \to \infty$, $V \to 0$ and when $x = 0$, $V = V(0, t) = V_0]$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty \mathrm{V}\sin sx\,dx = \mathrm{K}s\mathrm{V}_0 - \mathrm{K}s^2\int_0^\infty \mathrm{V}\sin sx\,dx$$

$$\Rightarrow \quad \frac{d\overline{\mathrm{V}}_s}{dt} = \mathrm{K}s\mathrm{V}_0 - \mathrm{K}s^2\overline{\mathrm{V}}_s \quad \text{where } \overline{\mathrm{V}}_s = \overline{\mathrm{V}}_s(s, t) = \mathrm{F}_s[\mathrm{V}(x, t)]$$

$$\Rightarrow \quad \frac{d\overline{\mathrm{V}}_s}{dt} + \mathrm{K}s^2\,\overline{\mathrm{V}}_s = \mathrm{K}s\mathrm{V}_0$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int \mathrm{K}s^2 dt} = e^{\mathrm{K}s^2 t}$$

$\therefore$ Its solution is

$$\overline{\mathrm{V}}_s\, e^{\mathrm{K}s^2 t} = \int \mathrm{K}s\mathrm{V}_0 . e^{\mathrm{K}s^2 t}\,dt + c$$

$$= \mathrm{K}s\mathrm{V}_0 \cdot \frac{e^{\mathrm{K}s^2 t}}{\mathrm{K}s^2} + c$$

or

$$\overline{\mathrm{V}}_s = \frac{\mathrm{V}_0}{s} + ce^{-\mathrm{K}s^2 t} \qquad \text{...(3)}$$

Putting $t = 0$ in (3), we have $\overline{\mathrm{V}}_s\,(s, 0) = \dfrac{\mathrm{V}_0}{s} + c$

$$\Rightarrow \quad c = -\frac{\mathrm{V}_0}{s} + \overline{\mathrm{V}}_s\,(s, 0) = -\frac{\mathrm{V}_0}{s} + \mathrm{F}_s\,[\mathrm{V}(x, 0)]$$

$$= -\frac{\mathrm{V}_0}{s} + \int_0^\infty 0 \sin sx\,dx \qquad \text{[from (2)]}$$

$$= -\frac{\mathrm{V}_0}{s}$$

$\therefore$ From (3), $\overline{\mathrm{V}}_s\,(s, t) = \dfrac{\mathrm{V}_0}{\mathrm{S}}\left(1 - e^{-\mathrm{K}s^2 t}\right)$

Taking its inverse Fourier sine transform, we get

$$\mathrm{V}(x, t) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{V}_0}{s}\left(1 - e^{-\mathrm{K}s^2 t}\right)\sin sx\,ds$$

$$= \frac{2\mathrm{V}_0}{\pi}\left[\int_0^\infty \frac{\sin sx}{s}\,ds - \int_0^\infty \frac{e^{-\mathrm{K}s^2 t}}{s}\sin sx\,ds\right]$$

$$= \frac{2\mathrm{V}_0}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{e^{-\mathrm{K}s^2 t}}{s}\sin sx\,ds\right] \qquad \left[\because \int_0^\infty \frac{\sin sx}{s}\,ds = \frac{\pi}{2}\right]$$

or

$$\mathrm{V}(x, t) = \mathrm{V}_0\left[1 - \frac{2}{\pi}\int_0^\infty \frac{e^{-\mathrm{K}s^2 t}}{s}\sin sx\,ds\right]$$

which is the required solution.

Example 3. *The temperature u in a semi-infinite rod $0 \le x < \infty$ is determined by the differential equation* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *subject to the conditions:*

(i) $u = 0$ *when* $t = 0, x \ge 0$.

(ii) $\frac{\partial u}{\partial x} = -\mu$ *(a constant) when* $x = 0$ *and* $t > 0$. (K.U.K. Dec. 2015)

Determine the temperature formula.

Sol. Given $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Boundary condition is $\frac{\partial u}{\partial x} = -\mu$ when $x = 0,\ t > 0$

Initial condition is $u(x, 0) = 0$ ...(2)

Since $\frac{\partial u}{\partial x}$ at $x = 0$ is given, we take Fourier cosine transform of both sides of (1). Thus

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$\Rightarrow$ $\int_0^\infty \frac{\partial u}{\partial t}\cos sx\,dx = k\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\,dx$. Integrating by parts

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u\cos sx\,dx = k\left[\left\{\cos sx\,.\frac{\partial u}{\partial x}\right\}_0^\infty - \int_0^\infty -s\sin sx\,.\frac{\partial u}{\partial x}\,dx\right]$$

$$= k\left[0-(-\mu) + s\int_0^\infty \sin sx\frac{\partial u}{\partial x}\,dx\right]$$

$$\left[\because \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \infty \text{ and } \frac{\partial u}{\partial x} = -\mu \text{ when } x = 0\right]$$

$$= k\left[\mu + s\left\{\left|\sin sx\,.u\right|_0^\infty - \int_0^\infty s\cos sx\,.u\,dx\right\}\right]$$

$$= k\left[\mu + s\{0\} - s\int_0^\infty s\cos sx\,.u\,dx\right] \qquad [\because \ u \to 0 \text{ when } x \to \infty]$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u\cos sx\,dx = k\mu - ks^2\int_0^\infty u\cos sx\,dx$$

$$\Rightarrow \quad \frac{d\overline{u}_c}{dt} = k\mu - ks^2\overline{u}_c \qquad \text{where } \overline{u}_c = \overline{u}_c(s,t) = F_c[u(x,t)]$$

$$\Rightarrow \qquad \frac{d\bar{u}_c}{dt} + ks^2\bar{u}_c = k\mu$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int ks^2 dt} = e^{ks^2 t}$$

$\therefore$ Its solution is

$$\bar{u}_c \cdot e^{ks^2t} = \int k\mu \cdot e^{ks^2t}\, dt + c = k\mu \cdot \frac{e^{ks^2t}}{ks^2} + c$$

or
$$\bar{u}_c = \frac{\mu}{s^2} + ce^{-ks^2t} \qquad \text{...(3)}$$

Putting $t = 0$ in (3), we have

$$\bar{u}_c(s, 0) = \frac{\mu}{s^2} + c$$

$$\Rightarrow \qquad c = -\frac{\mu}{s^2} + \bar{u}_c(s, 0) = -\frac{\mu}{s^2} + F_c[u(x, 0)]$$

$$= -\frac{\mu}{s^2} + \int_0^\infty 0 \cos sx\, dx \qquad \text{[from (2)]}$$

$$= -\frac{\mu}{s^2}$$

$\therefore$ From (3), $\bar{u}_c(s, t) = \frac{\mu}{s^2}\left(1 - e^{-ks^2t}\right)$

Taking its inverse Fourier cosine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{\mu}{s^2}\left(1 - e^{-ks^2t}\right)\cos sx\, ds$$

or
$$u(x, t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos sx}{s^2}\left(1 - e^{-ks^2t}\right) ds$$

which is the required solution.

Example 4. *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases},$$

determine the temperature at any instant x and at any instant t.

Sol. To determine the temperature $\theta(x, t)$, we have to solve the one-dimensional heat-flow equation

$$\frac{\partial\theta}{\partial t} = c^2\frac{\partial^2\theta}{\partial x^2}, \quad t > 0 \qquad \text{...(1)}$$

subject to the initial condition $\theta(x, 0) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$ ...(2)

Taking Fourier transform of (1), we get

$$\int_{-\infty}^{\infty} \frac{\partial\theta}{\partial t} e^{isx}\, dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2\theta}{\partial x^2} e^{isx}\, dx$$

or
$$\frac{d}{dt}\int_{-\infty}^{\infty} \theta e^{isx}\, dx = c^2(-s^2\overline{\theta})$$

or
$$\frac{d\overline{\theta}}{dt} = -c^2s^2\overline{\theta} \text{ where } \overline{\theta} = \overline{\theta}(s,t) = \text{F}[\theta(x,t) \qquad \ldots(3)$$

Now taking the Fourier transform of (2), we get

$$\overline{\theta}(s,0) = \int_{-\infty}^{\infty} \theta(x,0)\, e^{isx}\, dx = \int_{-a}^{a} \theta_0 e^{isx}\, dx = \theta_0 \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \theta_0 \left[\frac{e^{isa} - e^{-isa}}{is}\right] = \frac{2\theta_0}{s}\left[\frac{e^{isa} - e^{-isa}}{2i}\right]$$

$$= \frac{2\theta_0 \sin sa}{s} \qquad \ldots(4)$$

From (3), $\dfrac{d\overline{\theta}}{\overline{\theta}} = -c^2s^2dt$

Integrating $\log \overline{\theta} = -c^2s^2t + \log \text{A}$ or $\overline{\theta} = \text{A}e^{-c^2s^2t}$

Since $\overline{\theta} = \dfrac{2\theta_0 \sin sa}{s}$ when $t = 0$, from (4), we get $\text{A} = \dfrac{2\theta_0 \sin sa}{s}$

$\therefore \quad \overline{\theta} = \dfrac{2\theta_0 \sin sa}{s} e^{-c^2s^2t}$

Taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\theta_0 \sin as}{s} . e^{-c^2s^2t} . e^{-isx}\, ds$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} (\cos xs - i \sin xs)\, ds$$

$$= \frac{\theta_0}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \cos xs\, ds - i\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \sin xs\, ds\right]$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \cos xs\, ds$$

(The second integral vanishes since its integrand is an odd function)

$$= \frac{2\theta_0}{\pi}\int_{0}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \cos xs\, ds = \frac{\theta_0}{\pi}\int_{0}^{\infty} \frac{e^{-c^2s^2t}}{s} . 2 \sin as \cos xs\, ds$$

$$= \frac{\theta_0}{\pi}\int_{0}^{\infty} e^{-c^2s^2t}\left(\frac{\sin (a+x)s + \sin (a-x)s}{s}\right) ds$$

which is the required solution.

Example 5. *Use the method of Fourier transform to determine the displacement* $y(x, t)$ *of an infinite string, given that the string is initially at rest and that the initial displacement is* $f(x)$, $(-\infty < x < \infty)$.

Sol. The equation for the vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

subject to the initial conditions

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{and} \quad y(x, 0) = f(x) \quad ...(2)$$

Taking Fourier transform of (1), we get

$$\frac{d^2 \bar{y}}{dt^2} = c^2 (-s^2 \bar{y}) \qquad \text{where} \quad \bar{y} = \mathrm{F}[y(x, t)]$$

or
$$\frac{d^2 \bar{y}}{dt^2} + c^2 s^2 \bar{y} = 0$$

Its solution is $\bar{y} = \mathrm{A} \cos cst + \mathrm{B} \sin cst$ ...(3)

where A, B are constants.

Now taking Fourier transform of (2), we get

$$\frac{\partial \bar{y}}{\partial t} = 0 \quad \text{and} \quad \bar{y} = \mathrm{F}(s) \quad \text{when } t = 0$$

Putting $t = 0$, $\bar{y} = \mathrm{F}(s)$ in (3), we get $\mathrm{A} = \mathrm{F}(s)$

Also
$$\frac{\partial \bar{y}}{\partial t} = -cs\mathrm{A} \sin cst + cs\mathrm{B} \cos cst$$

Putting $t = 0$, $\frac{\partial \bar{y}}{\partial t} = 0$, we get $\mathrm{B} = 0$

$$\therefore \quad \bar{y} = \mathrm{F}(s) \cos cst$$

Taking inverse Fourier transform, we get

$$y(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{F}(s) \cos cst \,.\, e^{-isx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{F}(s) \left[\frac{e^{icst} + e^{-icst}}{2}\right] . \, e^{-isx} \, dx$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} [\mathrm{F}(s) \, e^{-is(x - ct)} + \mathrm{F}(s) \, e^{-is(x + ct)}] \, dx$$

$$= \frac{1}{2} [f(x - ct) + f(x + ct)] \qquad \left[\because f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{F}(s) \, e^{-isx} \, dx\right]$$

Example 6. *Use the complex form of Fourier transform to show that*

$$u = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(u) \, e^{-\frac{(x-u)^2}{4t}} \, du$$

is the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; -\infty < x < \infty, \; t > 0 \,; \qquad u = f(x) \text{ when } t = 0.$$

Sol. Given $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \quad \text{...(1)}$$

Initial condition is $u(x, 0) = f(x)$ ...(2)

Taking Fourier transform of both sides of (1)

$$\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{isx}\,dx=\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}e^{isx}\,dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty}ue^{isx}dx=\left[e^{isx}\cdot\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}ise^{isx}\cdot\frac{\partial u}{\partial x}dx$$

$$=0-is\int_{-\infty}^{\infty}e^{isx}\cdot\frac{\partial u}{\partial x}dx \qquad \left[\because \frac{\partial u}{\partial x}\to 0 \text{ as } x\to\pm\infty\right]$$

$$=-is\left[\left\{e^{isx}.u\right\}_{-\infty}^{\infty}-\int_{-\infty}^{\infty}ise^{isx}.u\,dx\right]$$

$$=-is\left[0-is\int_{-\infty}^{\infty}u\,e^{isx}\,dx\right] \qquad [\because\ u\to 0 \text{ as } x\to\pm\infty]$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty}ue^{isx}\,dx=i^2s^2\int_{-\infty}^{\infty}ue^{isx}\,dx$$

$$\Rightarrow \quad \frac{d\overline{u}}{dt}=-s^2\overline{u}$$

$$\Rightarrow \quad \frac{d\overline{u}}{\overline{u}}=-s^2dt$$

Integrating $\quad \log\overline{u}=-s^2t+\log c$

$$\Rightarrow \quad \overline{u}=ce^{-s^2t} \quad \text{...(3)}$$

Putting $\quad t=0,\ \overline{u}(s,0)=c$

$$\therefore \quad c=\overline{u}(s,0)=\int_{-\infty}^{\infty}u(x,0)\,e^{isx}dx$$

$$=\int_{-\infty}^{\infty}f(x)\,e^{isx}dx \qquad \text{[from (2)]}$$

$$\therefore \quad \text{From (3),} \quad \overline{u}=e^{-s^2t}\int_{-\infty}^{\infty}f(x)\,e^{isx}dx$$

$$=e^{-s^2t}\int_{-\infty}^{\infty}f(u)e^{isu}du \qquad \text{[by changing the variable } x \text{ to } u]$$

Taking its inverse Fourier transform, we get

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-s^2 t}\int_{-\infty}^{\infty} f(u)\, e^{isu}\, du\right] e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s^2 t} f(u) e^{isu} . e^{-isx}\, du\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s^2 t} f(u) e^{isu - isx}\, ds\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-s^2 t + is(u-x)}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s^2 + is\left(\frac{x-u}{t}\right)\right]}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s^2 + 2s\,.\,\frac{i(x-u)}{2t} + \frac{i^2 (x-u)^2}{4t^2} - \frac{i^2 (x-u)^2}{4t^2}\right]}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s + i\frac{x-u}{2t}\right]^2 - \frac{(x-u)^2}{4t}}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s + i\frac{x-u}{2t}\right]^2} . e^{-\frac{(x-u)^2}{4t}}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4t}}\left[\int_{-\infty}^{\infty} e^{-t\left[s + i\frac{x-u}{2t}\right]^2}\, ds\right] du$$

Put $\sqrt{t}\left(s + i\,\frac{x-u}{2t}\right) = y$, then $\sqrt{t}\; ds = dy$ or $ds = \frac{dy}{\sqrt{t}}$

$\therefore$
$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) . e^{-\frac{(x-u)^2}{4t}}\left[\int_{-\infty}^{\infty} e^{-y^2}\frac{dy}{\sqrt{t}}\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(u)\, e^{-\frac{(x-u)^2}{4t}}}{\sqrt{t}}\sqrt{\pi}\, du \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4t}}\, du.$$

Example 7. *Using suitable transforms, solve the differential equation* $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$, $0 \le x \le \pi$, $t \ge 0$ *where* $V(0, t) = 0 = V(\pi, t)$ *and* $V(x, 0) = V_0$ *constant.*

Sol. Given $$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \ 0 \le x \le \pi, \ t \ge 0 \qquad ...(1)$$

Boundary conditions are $V(0, t) = 0 = V(\pi, t)$

Initial condition is $V(x, 0) = V_0$ ...(2)

Since the interval $0 \le x \le \pi$ is finite and $V(0, t)$ and $V(\pi, t)$ are given, we use finite Fourier sine transform.

Let $\overline{V}_s(n, t)$ denote finite Fourier sine transform of $V(x, t)$, then

$$\overline{V}_s(n, t) = \int_0^\pi V(x, t) \sin\left(\frac{n\pi x}{\pi}\right) dx = \int_0^\pi V(x, t) \sin nx \, dx$$

Taking finite Fourier sine transform of both sides of (1), we have

$$\int_0^\pi \frac{\partial V}{\partial t} \sin nx \, dx = \int_0^\pi \frac{\partial^2 V}{\partial x^2} \sin nx \, dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\pi V \sin nx \, dx = \left[\sin nx \, . \frac{\partial V}{\partial x}\right]_0^\pi - \int_0^\pi n \cos nx \, . \frac{\partial V}{\partial x} dx$$

$$= 0 - n \int_0^\pi \cos nx \, . \frac{\partial V}{\partial x} dx$$

$$= - n \left[\left\{\cos nx \, . V\right\}_0^\pi - \int_0^\pi - n \sin nx \, . V dx\right]$$

$$= - n \left[0 + n \int_0^\pi V \sin nx \, dx\right] \qquad [\because \ V = 0 \text{ at } x = 0 \text{ and } \pi]$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\pi V \sin nx \, dx = - n^2 \int_0^\pi V \sin nx \, dx$$

$$\Rightarrow \quad \frac{d\overline{V}_s}{dt} = - n^2 \overline{V}_s$$

$$\Rightarrow \quad \frac{d\overline{V}_s}{\overline{V}_s} = - n^2 dt$$

Integrating, $\log \overline{V}_s = - n^2 t + \log c$

$$\therefore \quad \overline{V}_s = ce^{-n^2 t} \qquad ...(3)$$

Putting $t = 0$ in (3),

$$c = \overline{V}_s(n, 0) = \int_0^{\pi} V(x, 0) \sin nx \, dx$$

$$= \int_0^{\pi} V_0 \sin nx \, dx \qquad \text{[from (2)]}$$

$$= V_0 \left[-\frac{\cos nx}{n} \right]_0^{\pi} = \frac{V_0}{n} [1 - \cos n\pi]$$

$$= \frac{V_0}{n} [1 - (-1)^n]$$

$\therefore$ From (3), $\overline{V}_s(n, t) = \dfrac{V_0}{n} [1 - (-1)^n] \, e^{-n^2 t}$

Taking its inverse finite Fourier sine transform, we get

$$V(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \overline{V}_s(n, t) \sin nx$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{V_0}{n} [1 - (-1)^n] \, e^{-n^2 t} \sin nx$$

$$= \frac{2V_0}{\pi} \sum_{n = 1, 3, 5, \dots}^{\infty} \frac{1}{n} (2) \, e^{-n^2 t} \sin nx$$

$$[\because \; 1 - (-1)^n = 0 \text{ for } n = 2, 4, 6, \dots]$$

or

$$V(x, t) = \frac{4 V_0}{\pi} \sum_{n = 1, 3, 5, \dots}^{\infty} \frac{1}{n} e^{-n^2 t} \sin nx$$

which is the required solution.

Example 8. *Using finite Fourier transform, find the solution of the wave equation* $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ *subject to the conditions :*

$$u(0, t) = u(\pi, t) = 0, \; t > 0$$

$$u(x, 0) = 3 \sin x + 4 \sin 4x \quad \text{and} \quad u_t(x, 0) = 0 \text{ for } 0 < x < \pi.$$

Sol. Given $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le \pi, \; t > 0$ ...(1)

and

$$u(0, t) = u(\pi, t) = 0, \; t > 0$$

$$\left.\begin{aligned} u(x, 0) &= 3 \sin x + 4 \sin 4x \\ u_t(x, 0) &= 0 \end{aligned}\right\} \text{for } 0 < x < \pi \qquad \text{...(2)}$$

Taking finite Fourier sine transform of both sides of (1), we have

$$\int_0^\pi \frac{\partial^2 u}{\partial t^2} \sin nx\, dx = a^2 \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx\, dx$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\int_0^\pi u \sin nx\, dx = a^2\left[\left\{\sin nx.\frac{\partial u}{\partial x}\right\}_0^\pi - \int_0^\pi n\cos nx.\frac{\partial u}{\partial x}\, dx\right]$$

$$= a^2\left[0 - n\int_0^\pi \cos nx\, \frac{\partial u}{\partial x}\, dx\right]$$

$$= -a^2 n\left[\left\{\cos nx.u\right\}_0^\pi - \int_0^\pi - n\sin nx.u\, dx\right]$$

$$= -a^2 n\left[0 + n\int_0^\pi u \sin nx\, dx\right] \qquad [\because \quad u(0, t) = u(\pi, t) = 0]$$

$$= -a^2 n^2 \int_0^\pi u \sin nx\, dx$$

$$\Rightarrow \quad \frac{d^2\overline{u}_s}{dt^2} + a^2 n^2\, \overline{u}_s = 0$$

where $\overline{u}_s = \overline{u}_s\ (n, t) = \int_0^\pi u(x, t) \sin\left(\frac{n\pi x}{\pi}\right)dx = \int_0^\pi u\ (x, t) \sin nx\, dx$

A.E. is $D^2 + a^2n^2 = 0 \Rightarrow D = \pm\, ian$

$\therefore \qquad \overline{u}_s = c_1 \cos ant + c_2 \sin ant \qquad$ ...(3)

Put $t = 0$ in (3), $\overline{u}_s(n, 0) = c_1$

$$\Rightarrow \quad c_1 = \overline{u}_s(n, 0) = \int_0^\pi u(x, 0) \sin nx\, dx$$

$$= \int_0^\pi (3 \sin x + 4 \sin 4x) \sin nx\, dx \qquad \text{[from (2)]}$$

$$= 3\int_0^\pi \sin nx \sin x\, dx + 4\int_0^\pi \sin nx \sin 4x\, dx$$

$$= \frac{3}{2}\int_0^\pi 2 \sin nx \sin x\, dx + 2\int_0^\pi 2 \sin nx \sin 4x\, dx$$

$$= \frac{3}{2}\int_0^\pi [\cos (n-1)\, x - \cos (n+1)\, x]\, dx + 2\int_0^\pi [\cos(n-4)\, x - \cos (n+4)\, x]\, dx$$

$$= \frac{3}{2}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi + 2\left[\frac{\sin(n-4)x}{n-4} - \frac{\sin(n+4)x}{n+4}\right]_0^\pi$$

$= 0$ except for $n = 1$ and $n = 4$

For $n = 1$, $\quad c_1 = \bar{u}_s(n, 0) = \int_0^\pi u(x, 0) \sin x \, dx$

$$= \int_0^\pi (3 \sin x + 4 \sin 4x) \sin x \, dx$$

$$= 3 \int_0^\pi \sin^2 x \, dx + 2 \int_0^\pi 2 \sin 4x \sin x dx$$

$$= 3 \int_0^\pi \frac{1 - \cos 2x}{2} dx + 2 \int_0^\pi (\cos 3x - \cos 5x) \, dx$$

$$= \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi + 2\left[\frac{\sin 3x}{3} - \frac{\sin 5x}{5}\right]_0^\pi = \frac{3}{2}\pi$$

For $n = 4$, $\quad c_1 = \bar{u}_s(n, 0) = \int_0^\pi u(x, 0) \sin 4x \, dx$

$$= \int_0^\pi (3 \sin x + 4 \sin 4x) \sin 4x \, dx$$

$$= \frac{3}{2} \int_0^\pi 2 \sin 4x \sin x \, dx + 4 \int_0^\pi \sin^2 4x \, dx$$

$$= \frac{3}{2} \int_0^\pi (\cos 3x - \cos 5x) dx + 4 \int_0^\pi \frac{1 - \cos 8x}{2} dx$$

$$= \frac{3}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 5x}{5}\right]_0^\pi + 2\left[x - \frac{\sin 8x}{8}\right]_0^\pi = \pi$$

$$\therefore \quad c_1 = \begin{cases} \dfrac{3\pi}{2} & \text{for } n = 1 \\ 2\pi & \text{for } n = 4 \end{cases}$$

Again from (3), $\quad \dfrac{\partial \bar{u}_s}{\partial t} = -c_1 an \sin ant + c_2 an \cos ant$

Putting $t = 0$, $\quad \dfrac{\partial \bar{u}_s}{\partial t} = 0 \qquad \left[\because \text{ when } t = 0, \dfrac{\partial u}{\partial t} = 0 \;\; \therefore \;\; \dfrac{\partial \bar{u}_s}{\partial t} = 0 \text{ for } t = 0\right]$

$0 = c_2 an \Rightarrow c_2 = 0$

$\therefore$ (3) reduces to $\bar{u}_s = c_1 \cos ant$ ...(4)

where $c_1 = \dfrac{3\pi}{2}$ for $n = 1$ and $c_1 = 2\pi$ for $n = 4$

Taking inverse finite Fourier sine transform of (4), we have

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \overline{u}_s\ (n, t) \sin nx = \frac{2}{\pi} \sum_{n=1}^{\infty} c_1 \cos ant \sin nx$$

$$= \frac{2}{\pi} \left[\underset{(\text{for } n = 1)}{\frac{3\pi}{2} \cos at \sin x} + \underset{(\text{for } n = 4)}{2\pi \cos 4\, at \sin 4x} \right]$$

or $$u(x, t) = 3 \cos at \sin x + 4 \cos 4\, at \sin 4x$$

which is the required solution.

Example 9. *Using finite Fourier transform, solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *subject to the conditions:*

(a) $u_x(0, t) = u_x(6, t) = 0, \qquad 0 < x < 6,\ t > 0$

(b) $u(x, 0) = x(6 - x), \qquad 0 < x < 6.$

Sol. Since the boundary conditions are u_x (0, t) = u_x (6, 0) = 0, we take finite Fourier cosine transform.

Let $\overline{u}_c$ (n, t) denote finite Fourier cosine transform of u(x, t), then

$$\overline{u}_c\ (n, t) = \int_0^6 u(x, t) \cos\left(\frac{n\pi x}{6}\right) dx \qquad [\because\ L = 6]$$

Taking finite Fourier cosine transform of both sides of the given equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

we get $$\int_0^6 \frac{\partial u}{\partial t} \cos \frac{n\pi x}{6} dx = \int_0^6 \frac{\partial^2 u}{\partial x^2} \cos \frac{n\pi x}{6} dx$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^6 u \cos \frac{n\pi x}{6} dx = \left[\cos \frac{n\pi x}{6} \cdot \frac{\partial u}{\partial x} \right]_0^6 - \int_0^6 -\frac{n\pi}{6} \sin \frac{n\pi x}{6} \cdot \frac{\partial u}{\partial x} dx$$

$$= 0 + \frac{n\pi}{6} \int_0^6 \sin \frac{n\pi x}{6} \cdot \frac{\partial u}{\partial x} dx \left[\because\ \frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = 6 \right]$$

$$= \frac{n\pi}{6} \left[\left\{ \sin \frac{n\pi x}{6} \cdot u \right\}_0^6 - \int_0^6 \frac{n\pi}{6} \cos \frac{n\pi x}{6} \cdot u\, dx \right]$$

$$= \frac{n\pi}{6} \left[0 - \frac{n\pi}{6} \int_0^6 u \cos \frac{n\pi x}{6} dx \right]$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^6 u \cos \frac{n\pi x}{6} dx = -\frac{n^2 \pi^2}{36} \int_0^6 u \cos \frac{n\pi x}{6} dx$$

$$\Rightarrow \quad \frac{d\overline{u}_c}{dt} = -\frac{n^2\pi^2}{36}\overline{u}_c$$

$$\Rightarrow \quad \frac{d\overline{u}_c}{\overline{u}_c} = -\frac{n^2\pi^2}{36}dt$$

Integrating, $\log \overline{u}_c = -\frac{n^2\pi^2}{36} t + \log A$ where $A = A(n)$

$$\Rightarrow \quad \overline{u}_c = Ae^{-\frac{n^2\pi^2}{36}t} \Rightarrow \overline{u}_c(n, t) = A(n)e^{-\frac{n^2\pi^2}{36}t} \quad \text{...(1)}$$

Putting $t = 0$, $\overline{u}_c(n, 0) = A(n)$

$\therefore$ $A(n) = \overline{u}_c(n, 0) =$ finite Fourier cosine transform of $u(x, 0)$

$$= \int_0^6 u(x, 0) \cos\frac{n\pi x}{6}dx = \int_0^6 x(6-x)\cos\frac{n\pi x}{6}dx \quad \text{...(2)}$$

$$= \left[(6x - x^2).\frac{\sin\frac{n\pi x}{6}}{\frac{n\pi}{6}} - (6-2x).\frac{-\cos\frac{n\pi x}{6}}{\left(\frac{n\pi}{6}\right)^2} + (-2).\frac{-\sin\frac{n\pi x}{6}}{\left(\frac{n\pi}{6}\right)^3}\right]_0^6$$

$$= -6.\left(\frac{6}{n\pi}\right)^2 \cos n\pi - 6.\left(\frac{6}{n\pi}\right)^2$$

$$= -\frac{216}{n^2\pi^2}(1 + \cos n\pi)$$

$\therefore$ From (1), $$\overline{u}_c = -\frac{216}{n^2\pi^2}(1+\cos n\pi)\, e^{-\frac{n^2\pi^2}{36}t} \quad \text{...(3)}$$

To find inverse finite Fourier cosine transform, we need $\overline{u}_c(0, t)$.

From (1), $\overline{u}_c(0, t) = A(0)$

From (2), $A(0) = \overline{u}_c(0, 0)$

$$= \int_0^6 x(6-x)\,dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6$$

$$= 108 - 72 = 36$$

$\therefore$ $\overline{u}_c(0, t) = 36$

Taking inverse finite Fourier cosine transform of (3), we have

$$u(x, t) = \frac{1}{L} \bar{u}_c(0, t) + \frac{2}{L} \sum_{n=1}^{\infty} \bar{u}_c(n, t) \cos \frac{n\pi x}{L}$$

$$= \frac{1}{6}(36) + \frac{2}{6} \sum_{n=1}^{\infty} -\frac{216}{n^2\pi^2}(1 + \cos n\pi)\, e^{-\frac{n^2\pi^2}{36}t} \cos \frac{n\pi x}{6}$$

$[\because \quad L = 6]$

or
$$u(x, t) = 6 - \frac{72}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + \cos n\pi}{n^2} e^{-\frac{n^2\pi^2 t}{36}} \cos \frac{n\pi x}{6}$$

which is the required solution.

EXERCISE 2.4

1. Solve the partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0$$

subject to the following conditions:

(a) $u(0, t) = 0, \; t > 0$ (b) $u(x, 0) = e^{-x}, \; x > 0$

(c) u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$.

2. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0$ where $u(x, t)$ satisfies the conditions:

(a) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0, \; t > 0$ (b) $u(x, 0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

(c) $|\, u(x, t) \,| < M$.

3. The initial temperature along the length of an infinite bar is given by

$$u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

If the temperature $u(x, t)$ satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, \; t > 0$, find the temperature at any point of the bar at any time t.

4. Determine the distribution of temperature in the semi-infinite medium $x \ge 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

5. (a) If the flow of heat is linear so that the variation of θ (temperature) with z and y may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation

$$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$$

by using Fourier transform, where $-\infty < x < \infty$ and $\theta = f(x)$ when $t = 0$, $f(x)$ being a function of x.

(*b*) If the initial temperature of an infinite bar is given by

$$\mu(x, 0) = \begin{cases} 1, & \text{for } -c < x < c \\ 0, & \text{otherwise} \end{cases}$$

determine the temperature of the infinite bar at any point x and at any time $t > 0$.

6. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$ subject to the conditions

(*a*) $u(x, 0) = 1$, $0 < x < \pi$ (*b*) $u(0, t) = u(\pi, t) = 0$, $t > 0$

using appropriate Fourier transform.

7. Use finite Fourier transform to solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ given that

$$u(0, t) = 0,\ u(\pi, t) = 0,\ u(x, 0) = 2x,\ 0 < x < \pi,\ t > 0.$$

8. Use finite Fourier transform to solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ 0 < x < 4,\ t > 0$$

subject to the conditions

(*a*) $u(x, 0) = 2x$, $0 < x < 4$ (*b*) $u(0, t) = u(4, t) = 0$, $t > 0$.

9. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 6$, $t > 0$ subject to the conditions

(*a*) $u(0, t) = u(6, t) = 0$, $t > 0$ (*b*) $u(x, 0) = \begin{cases} 1, & 0 < x < 3 \\ 0, & 3 < x < 6 \end{cases}$.

10. Solve $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, given

(*a*) $u(0, t) = u(\pi, t) = 0$ for $t > 0$ (*b*) $u(x, 0) = \frac{1}{10}\sin x + \frac{1}{100}\sin 4x$

(c) $u_t(x, 0) = 0$ for $0 < x < \pi$.

11. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 6$, $t > 0$ subject to conditions $u_x(0, t) = u_x(6, t) = 0$, $u(x, 0) = 2x$.

12. Solve by using finite Fourier transform

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 2$$

subject to the conditions:

$$u(0, t) = u(2, t) = 0,\ u(x, 0) = x(2 - x) \text{ and } u_t(x, 0) = 0.$$

Answers

1. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{s}{1+s^2} e^{-2s^2t} \sin sx\, ds$

2. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\dfrac{s \sin s + \cos s - 1}{s^2}\right) e^{-s^2t} \cos sx ds$

3. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty e^{-s^2t} \left(\dfrac{\sin(1+x)s + \sin(1-x)s}{s}\right) ds$

4. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \bar{f}_s(s)\, e^{-c^2s^2t} \sin sx\, ds$ where $\bar{f}_s(s) = F_s[f(x)]$

5. (a) $\theta(x, t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty \bar{f}(s)\, e^{-ks^2t - isx} ds$, where $\bar{f}(s) = F[f(x)]$

(b) $\mu(x, t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty \dfrac{2}{s} \sin cse^{-k^2s^2t} \cdot e^{-isx}\, ds = \dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{e^{-k^2s^2t}}{s} [\sin(c+x)s + \sin(c-x)s]\, ds$

6. $u(x, t) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^\infty \dfrac{1-\cos n\pi}{n} e^{-n^2t} \sin nx$ or $\dfrac{4}{\pi}\displaystyle\sum_{n=1,3,5\ldots\ldots}^\infty \dfrac{1}{n} e^{-n^2t} \sin nx$

7. $u(x, t) = 4\displaystyle\sum_{n=1}^\infty \dfrac{1}{n}(-1)^{n+1} e^{-n^2t} \sin nx$ 8. $u(x, t) = \dfrac{16}{\pi}\displaystyle\sum_{n=1}^\infty \dfrac{(-1)^{n+1}}{n} e^{-\frac{kn^2\pi^2t}{16}} \sin \dfrac{n\pi x}{4}$

9. $u(x, t) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^\infty \dfrac{1}{n}\left(1 - \cos\dfrac{n\pi}{2}\right) e^{-\frac{n^2\pi^2t}{36}} \sin \dfrac{n\pi x}{6}$

10. $u(x, t) = \dfrac{1}{10} \cos 2t \sin x + \dfrac{1}{100} \cos 8t \sin 4x$ **11.** $u(x, t) = 6 + \dfrac{24}{\pi^2}\displaystyle\sum_{n=1}^\infty \dfrac{(-1)^n - 1}{n^2} e^{-\frac{n^2\pi^2t}{36}} \cos \dfrac{n\pi x}{6}$

12. $u(x, t) = \dfrac{16}{\pi^3}\displaystyle\sum_{n=1}^\infty \dfrac{1-\cos n\pi}{n^3} \cos \dfrac{3n\pi t}{2} \sin \dfrac{n\pi x}{2}$.

2.14. FOURIER TRANSFORM OF AN INTEGRAL

Theorem. Let $f(t)$ be piecewise continuous on every interval $[-l, l]$ and $\int_{-\infty}^\infty |f(t)|dt$ converge. Let $F[f(t)] = F(s)$ and $F(s)$ satisfies $F(0) = 0$. Then

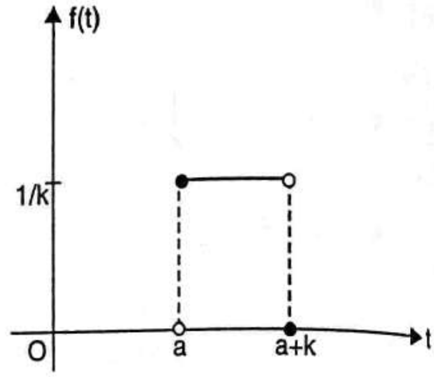
$$F\left[\int_{-\infty}^t f(T)\, dT\right] = \frac{1}{is} F(s).$$

2.15. FOURIER TRANSFORM OF DIRAC-DELTA FUNCTION

Dirac-delta function (or unit impulse function) $\delta(t-a)$ is defined as

$$\delta(t-a) = \lim_{k \to 0} \delta_k(t-a) \text{ where}$$

$$\delta_k(t-a) = \begin{cases} 0, & \text{for } t < a \\ \dfrac{1}{k}, & \text{for } a \le t < a+k \\ 0, & \text{for } t \ge a+k \end{cases}$$



$$F\{\delta(t-a)\} = \int_{-\infty}^{\infty} \delta(t-a) e^{ist}\, dt$$

$$= \lim_{k \to 0} \int_a^{a+k} \frac{1}{k} e^{ist}\, dt$$

$$= \lim_{k \to 0} \frac{1}{k} \left[\frac{e^{ist}}{is} \right]_a^{a+k}$$

$$= \lim_{k \to 0} \frac{e^{is(a+k)} - e^{isa}}{isk} = \lim_{k \to 0} e^{isa} \left(\frac{e^{isk} - 1}{isk} \right)$$

$$= e^{isa} \times 1 \qquad \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \right]$$

$$= e^{isa}.$$

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