CHAPTER 3

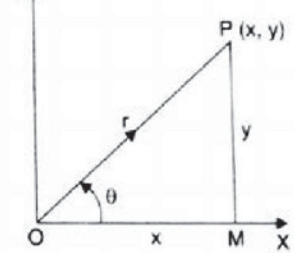
Functions of a Complex Variable

3.1. INTRODUCTION

A complex number z is an ordered pair (x, y) of real numbers and is written as

$$z = x + iy, \quad \text{where } i = \sqrt{-1}.$$

The real numbers x and y are called the real and imaginary parts of z. In the Argand's diagram, the complex number z is represented by the point $P(x, y)$. If (r, θ) are the polar coordinates of P, then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $|z|$. Also $\theta = \tan^{-1} \dfrac{y}{x}$ is called the argument of z and is denoted by arg. z. Every non-zero complex number z can be expressed as



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

If $z = x + iy$, then the complex number $x - iy$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

Clearly, $\qquad |\bar{z}| = |z|, |z|^2 = z\bar{z},$

$$R(z) = \frac{z + \bar{z}}{2}, \quad I(z) = \frac{z - \bar{z}}{2i}.$$

3.2. FUNCTION OF A COMPLEX VARIABLE

If x and y are real variables, then $z = x + iy$ is called a **complex variable.** If corresponding to each value of a complex variable $z(= x + iy)$ in a given region R, there correspond one or more values of another complex variable $w\ (= u + iv)$, then w is called a function of the complex variable z and is denoted by

$$w = f(z) = u + iv$$

For example, if $\qquad w = z^2, \quad$ where $z = x + iy$ and $w = f(z) = u + iv$

then $\qquad u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

$\Rightarrow \qquad u = x^2 - y^2 \quad \text{and} \quad v = 2xy$

Thus u and v, the real and imaginary parts of w, are functions of the real variables x and y.

$\therefore \qquad w = f(z) = u(x, y) + iv(x, y)$

If to each value of z there corresponds one and only one value of w, then w is called a *single-valued function* of z. If to each value of z there correspond more than one values of w, then w is called a *multi-valued function of* z.

To represent $w = f(z)$ graphically, we take two Argand diagrams: one to represent the point z and the other to represent w. The former diagram is called the xOy-plane or the z-plane and the latter uOv-plane or the w-plane.

3.3. EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

Def. The exponential function of the complex variable $z = x + iy$, where x and y are real, is defined as

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \infty$$

where, $e = 2.718$ is the base of natural logarithms

Replacing z by $x + iy$, we have

$$e^{x+iy} = 1 + \frac{x+iy}{1!} + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \ldots$$

Putting, $x = 0$, we get

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \ldots$$

$$= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} + \ldots \qquad \left[\because \begin{array}{ll} i^2 = -1, & i^3 = -i \\ i^4 = 1, & i^5 = i \end{array}\right]$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right) = \cos y + i \sin y$$

$$\therefore \quad e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

[**Note.** We have proved $e^{iy} = \cos y + i \sin y$

Changing i to $-i$ (*i.e.*, taking conjugates of both sides), $e^{-iy} = \cos y - i \sin y$.]

3.4. PERIODICITY

e^z IS A PERIODIC FUNCTION, WHERE z IS A COMPLEX VARIABLE

Proof. Let $\quad z = x + iy$

then, by definition $\quad e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x [\cos (2n\pi + y) + i \sin (2n\pi + y)]$

$$= e^x \cdot e^{i(2n\pi + y)}$$

$$= e^{x + i(2n\pi + y)} = e^{(x+iy) + 2n\pi i} = e^{z + 2n\pi i}$$

i.e., e^z remains unchanged when z is increased by any multiple of $2\pi i$.

$\Rightarrow$ e^z is a periodic function with period $2\pi i$.

ILLUSTRATIVE EXAMPLES

Example 1. *Split up into real and imaginary parts:*

(*i*) $e^{5 + \frac{1}{2} i\pi}$ (*ii*) $e^{(5+3i)^2}$ (*iii*) e^{e^z}.

Sol. (*i*) $\quad e^{5 + \frac{1}{2} i\pi} = e^5 \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right] = e^5 [0 + i \,.\, 1] = ie^5$

$\therefore \quad \text{Re}\,[e^{5 + \frac{1}{2} i\pi}] = 0, \quad \text{Im}\,[e^{5 + \frac{1}{2} i\pi}] = e^5.$

(*ii*) $\quad (5 + 3i)^2 = 25 + 9i^2 + 30i = 16 + 30i \qquad [\because \; i^2 = -1]$

$\therefore \quad e^{(5+3i)^2} = e^{16 + 30i} = e^{16} (\cos 30 + i \sin 30)$

$\therefore \quad \text{Re}\,[e^{(5+3i)^2}] = e^{16} \cos 30, \; \text{Im}\,[e^{(5+3i)^2}] = e^{16} \sin 30.$

(*iii*) $\quad e^{e^z} = e^{e^{x+iy}} = e^{e^x(\cos y + i \sin y)} = e^{(e^x \cos y) + i(e^x \sin y)}$

$$= e^{e^x \cos y} [\cos (e^x \sin y) + i \sin (e^x \sin y)]$$

$\therefore \quad \text{Re}(e^{e^z}) = e^{e^x \cos y} \,.\, \cos (e^x \sin y)$

$\quad \text{Im}\,(e^{e^z}) = e^{e^x \cos y} \,.\, \sin (e^x \sin y).$

Example 2. *Find all values of z which satisfy* $e^z = 1 + i$.

Sol. Since $\quad e^z = e^{x+iy} = e^x (\cos y + i \sin y)$

$\therefore \quad e^z = 1 + i \;\Rightarrow\; e^x(\cos y + i \sin y) = 1 + i$

Equating real parts $\quad e^x \cos y = 1 \qquad \ldots(1)$

Equating imaginary parts $\quad e^x \sin y = 1 \qquad \ldots(2)$

Squaring and adding (1) and (2), we get

$$e^{2x} (\cos^2 y + \sin^2 y) = 1 + 1 \quad \text{or} \quad e^{2x} = 2$$

or $\quad 2x = \log 2 \quad \therefore \quad x = \frac{1}{2} \log 2$

Dividing (2) by (1), $\quad \tan y = 1 = \tan \frac{\pi}{4}$

$\Rightarrow \quad y = n\pi + \frac{\pi}{4}$, where n is an integer.

$\therefore \quad z = x + iy = \frac{1}{2} \log 2 + i\left(n\pi + \frac{\pi}{4}\right)$, where $n = 0, \pm 1, \pm 2, \ldots\ldots$

Example 3. *Prove that* $[\sin (\alpha - \theta) + e^{-i\alpha} \sin \theta]^n = \sin^{n-1} \alpha \,[\sin (\alpha - n\theta) + e^{-i\alpha} \sin n\theta]$.

Sol. L.H.S. $= [(\sin \alpha \cos \theta - \cos \alpha \sin \theta) + (\cos \alpha - i \sin \alpha) \sin \theta]^n$

$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n = [\sin \alpha (\cos \theta - i \sin \theta)]^n$

$= [\sin \alpha \,.\, e^{-i\theta}]^n = \sin^n \alpha \,.\, e^{-in\theta}$

R.H.S. $= \sin^{n-1} \alpha \,[(\sin \alpha \cos n\theta - \cos \alpha \sin n\theta) + (\cos \alpha - i \sin \alpha) \sin n\theta]$

$= \sin^{n-1} \alpha \,[\sin \alpha \cos n\theta - i \sin \alpha \sin n\theta] = \sin^n \alpha \,[\cos n\theta - i \sin n\theta]$

$= \sin^n \alpha \,.\, e^{-in\theta}$

$\therefore \quad$ L.H.S. = R.H.S.

Example 4. *Given* $\frac{1}{\rho} = \frac{1}{LPi} + CPi + \frac{1}{R}$ *where L, P, R are real, express* ρ *in the form* $Ae^{i\theta}$ *giving the values of A and* θ.

Sol.

$$\frac{1}{\rho} = \frac{1}{LPi} + CPi + \frac{1}{R}$$

$$\Rightarrow \quad \frac{1}{\rho} = \frac{R + LP^2CR(-1) + LPi}{LPRi}$$

$$\Rightarrow \quad \rho = \frac{LPRi}{(R - LP^2CR) + LPi}$$

$$= \frac{LPRi}{(R - LP^2CR) + LPi} \times \frac{(R - LP^2CR) - LPi}{(R - LP^2CR) - LPi}$$

$$= \frac{L^2P^2R + iLPR(R - LP^2CR)}{(R - LP^2CR)^2 + L^2P^2}$$

$$= A(\cos\theta + i\sin\theta), \text{ say}$$

Equating real and imaginary parts, we get

$$A\cos\theta = \frac{L^2P^2R}{(R - LP^2CR)^2 + L^2P^2} \qquad ...(1)$$

$$A\sin\theta = \frac{LPR(R - LP^2CR)}{(R - LP^2CR)^2 + L^2P^2} \qquad ...(2)$$

Squaring and adding (1) and (2),

$$\Rightarrow \quad A^2 = \frac{L^4P^4R^2 + L^2P^2R^2(R - LP^2CR)^2}{[(R - LP^2CR)^2 + L^2P^2]^2}$$

$$= \frac{L^2P^2R^2[L^2P^2 + (R - LP^2CR)^2]}{[(R - LP^2CR)^2 + L^2P^2]^2}$$

$$= \frac{L^2P^2R^2}{(R - LP^2CR)^2 + L^2P^2}$$

$$\Rightarrow \quad A = \frac{LPR}{\sqrt{(R - LP^2CR)^2 + L^2P^2}}$$

Dividing (2) by (1),

$$\tan\theta = \frac{R - LP^2CR}{LP}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left[\frac{R(1 - LP^2C)}{LP}\right]$$

Hence $\rho = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$

where $A = \dfrac{LPR}{\sqrt{(R - LP^2CR)^2 + L^2P^2}}$ and $\theta = \tan^{-1}\left[\dfrac{R(1 - LP^2C)}{LP}\right]$.

3.5. TRIGONOMETRIC FUNCTIONS OF A COMPLEX VARIABLE

1. Definitions. For all real values of x, we know that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

Adding and subtracting, we get $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$; $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

These are called Euler's Exponential values of $\sin x$ and $\cos x$, where $x \in R$.

If $z = x + iy$ then trigonometric functions of z are defined as follows:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}, \quad \sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \quad \text{cosec } z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

2. Euler's Theorem.

For all values of θ, real or complex, $e^{i\theta} = \cos\theta + i\sin\theta$.

For all values of θ, real or complex $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\therefore \quad \cos\theta + i\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{2e^{i\theta}}{2} = e^{i\theta}.$$

Hence $e^{i\theta} = \cos\theta + i\sin\theta$ for all values of θ.

3. Periodicity of Circular Functions.

(*a*) **To prove that sin z and cos z are periodic functions with period 2π.**

We know that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

If n is any integer, then

$$\sin(z + 2n\pi) = \frac{e^{i(z + 2n\pi)} - e^{-i(z + 2n\pi)}}{2i}$$

$$= \frac{e^{iz}\,.\,e^{2n\pi i} - e^{-iz}\,.\,e^{-2n\pi i}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} \qquad [\because \; e^{2n\pi i} = 1 = e^{-2n\pi i}]$$

$$= \sin z$$

$\Rightarrow$ $\sin z$ remains unchanged when z is increased by any multiple of 2π.

$\therefore$ **sin z is a periodic function with period 2π.**

Similarly, $\cos z$ can be shown to be a periodic function with period 2π.

(*b*) **To prove that tan z is a periodic function with period π.**

We know that $\tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

If n is any integer, $\tan(z + n\pi) = \dfrac{e^{i(z + n\pi)} - e^{-i(z + n\pi)}}{i[e^{i(z + n\pi)} + e^{-i(z + n\pi)}]} = \dfrac{e^{iz}\,.\,e^{in\pi} - e^{-iz}\,.\,e^{-in\pi}}{i[e^{iz}\,.\,e^{in\pi} + e^{-iz}\,.\,e^{n\pi}]}$

Multiplying the numerator and denominator by $e^{in\pi}$

$$= \frac{e^{iz}\,.\,e^{2n\pi i} - e^{-iz}}{i\,[e^{iz}\,.\,e^{2n\pi i} + e^{-iz}]} = \frac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})} = \tan z \qquad |\because \; e^{2n\pi i} = 1$$

$\Rightarrow$ tan z remains unchanged when z is increased by any multiple of π.

$\therefore$ **tan z is a periodic function with period π.**

3.6. TRIGONOMETRIC IDENTITIES

If z is a complex variable, prove that

(i) $\sin^2 z + \cos^2 z = 1$ *(ii)* $\sin 2z = 2 \sin z \cos z$

(iii) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

(iv) $\tan 2z = \dfrac{2\tan z}{1 - \tan^2 z}$ *(v)* $\sin(-z) = -\sin z$

(vi) $\sin 3z = 3 \sin z - 4 \sin^3 z$ *(vii)* $\tan 3z = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$.

Proof. *(i)* L.H.S. $= \sin^2 z + \cos^2 z = \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2$

$$= -\frac{1}{4}(e^{2iz} + e^{-2iz} - 2) + \frac{1}{4}(e^{2iz} + e^{-2iz} + 2) = \frac{1}{2} + \frac{1}{2} = 1 = \text{R.H.S.}$$

(ii) R.H.S. $= 2 \sin z \cos z = 2\,.\,\dfrac{e^{iz} - e^{-iz}}{2i}\,.\,\dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{e^{2iz} - e^{-2iz}}{2} = \sin 2z = $ L.H.S.

(iii) $\cos^2 z - \sin^2 z = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2$

$$= \frac{1}{4}\,[(e^{2iz} + e^{-2iz} + 2) + (e^{2iz} + e^{-2iz} - 2)] = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

$$2\cos^2 z - 1 = 2\left(\frac{e^{iz} + e^{-iz}}{2i}\right)^2 - 1 = \frac{1}{2}(e^{2iz} + e^{-2iz} + 2) - 1 = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

$$1 - 2\sin^2 z = 1 - 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = 1 + \frac{1}{2}(e^{2iz} + e^{-2iz} - 2) = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

Hence the result.

(iv) R.H.S. $= \dfrac{2\tan z}{1 - \tan^2 z} = \dfrac{2\,.\,\dfrac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}}{1 - \left[\dfrac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}\right]^2} = \dfrac{2\,(e^{iz} - e^{-iz})\,(e^{iz} + e^{-iz})}{i\,[(e^{iz} + e^{-iz})^2 + (e^{iz} - e^{-iz})^2]}$

$$= \frac{2\,(e^{2iz} - e^{-2iz})}{i.2\,(e^{2iz} + e^{-2iz})} = \frac{e^{2iz} - e^{-2iz}}{i\,(e^{2iz} + e^{-2iz})} = \tan 2z = \text{L.H.S.}$$

(v) $\sin(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i} = \dfrac{e^{-iz} - e^{iz}}{2i} = -\dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z.$

(vi) $\sin 3z = \dfrac{e^{3iz} - e^{-3iz}}{2i} = \dfrac{x^3 - y^3}{2i}$, where $x = e^{iz}, y = e^{-iz}$

$$= \frac{(x - y)^3 + 3xy\,(x - y)}{2i} = \frac{1}{2i}\,[(e^{iz} - e^{-iz})^3 + 3\,.\,e^{iz}\,.\,e^{-iz}\,(e^{iz} - e^{-iz})]$$

$$= \frac{1}{2i}\,[(2i \sin z)^3 + 3(2i \sin z)] = \frac{1}{2i}\,[-8i \sin^3 z + 6i \sin z] = 3 \sin z - 4 \sin^3 z.$$

(vii) R.H.S. $= \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$

$$= \frac{3\,.\left[\dfrac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}\right] - \left[\dfrac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}\right]^3}{1 - 3\,.\left[\dfrac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}\right]^2} = \frac{3\left[\dfrac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}\right] + \dfrac{1}{i}\,.\left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^3}{1 + 3\,.\left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^2}$$

$$= \frac{3\,\dfrac{x}{iy} + \dfrac{1}{i}\,.\left(\dfrac{x}{y}\right)^3}{1 + 3\left(\dfrac{x}{y}\right)^2}, \text{ where } x = e^{iz} - e^{-iz},\ y = e^{iz} + e^{-iz}$$

$$= \frac{3xy^2 + x^3}{iy^3}\,.\,\frac{y^2}{y^2 + 3x^2} = \frac{x\,(3y^2 + x^2)}{iy\,(y^2 + 3x^2)} = \frac{x\,(3e^{2iz} + 3e^{-2iz} + 6 + e^{2iz} + e^{-2iz} - 2)}{iy\,(e^{2iz} + e^{-2iz} + 2 + 3e^{2iz} + 3e^{-2iz} - 6)}$$

$$= \frac{x\,(4e^{2iz} + 4e^{-2iz} + 4)}{iy\,(4e^{2iz} + 4e^{-2iz} - 4)} = \frac{(e^{iz} - e^{-iz})(e^{2iz} + e^{-2iz} + 1)}{i\,(e^{iz} + e^{-iz})(e^{2iz} + e^{-2iz} - 1)}$$

$$= \frac{e^{3iz} - e^{-3iz}}{i\,(e^{3iz} + e^{-3iz})} \qquad \left[\begin{array}{l} \because\ (a - b)(a^2 + b^2 + ab) = a^3 - b^3 \\ \ (a + b)(a^2 + b^2 - ab) = a^3 + b^3 \end{array}\right.$$

$= \tan 3z.$

EXERCISE 3.1

1. If $z = x + iy$, find the real and imaginary parts of *(i)* $e^{\bar{z}}$ *(ii)* $\exp(z^2)$ *(iii)* $\dfrac{e^{i\theta}}{1 - \lambda e^{i\phi}}$.
2. Prove that

   (i) $\sin(\alpha + n\theta) - e^{i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$ *(ii)* $[\sin(\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha\,.\,e^{-in\theta}$

   (iii) $\dfrac{z^2 - 1}{z^2 + 1} = i \tan \theta$, where $z = e^{i\theta}$.
3. Prove that by a proper choice of p and q, $pe^{2i\theta} + qe^{-2i\theta}$ can be made equal to $5 \cos 2\theta - 7 \sin 2\theta$.
4. If z is a complex number, prove that

   (i) $\cos(-z) = \cos z$ *(ii)* $\tan(-z) = -\tan z$

   (iii) $\cos 3z = 4\cos^3 z - 3\cos z$ *(iv)* $\tan z = \dfrac{\sin 2z}{1 + \cos 2z}$.

5. If z_1, z_2 are complex numbers, show that

(i) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (ii) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

(iii) $\tan(z_1 + z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$ (iv) $\sin z_1 + \sin z_2 = 2 \sin \dfrac{z_1 + z_2}{2} \cos \dfrac{z_1 - z_2}{2}$

(v) $\cos z_1 - \cos z_2 = 2 \sin \dfrac{z_1 + z_2}{2} \sin \dfrac{z_2 - z_1}{2}$.

6. Show that

(i) $\cos(\alpha + i\beta) = \dfrac{1}{2}(e^{-\beta} + e^{\beta}) \cos \alpha + \dfrac{i}{2}(e^{-\beta} - e^{\beta}) \sin \alpha$

(ii) $\sin(\alpha - i\beta) = \dfrac{1}{2}(e^{-\beta} + e^{\beta}) \sin \alpha + \dfrac{i}{2}(e^{-\beta} - e^{\beta}) \cos \alpha$.

7. If α and β are the imaginary cube roots of unity, prove that

$$\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{-\frac{x}{2}}\left(\cos \frac{\sqrt{3}}{2} x + \sqrt{3} \sin \frac{\sqrt{3}}{2} x\right).$$

Answers

1. (i) $e^x \cos y, -e^x \sin y$ (ii) $e^{x^2 - y^2} \cos 2xy, e^{x^2 - y^2} \sin 2xy$.

(iii) $\dfrac{\cos\theta - \lambda \cos(\theta - \phi)}{1 - 2\lambda \cos\phi + \lambda^2}, \dfrac{\sin\theta - \lambda \sin(\theta - \phi)}{1 - 2\lambda \cos\phi + \lambda^2}$

3. $p = \dfrac{1}{2}(5 + 7i), q = \dfrac{1}{2}(5 - 7i)$.

3.7. LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

Definition. If $\omega = e^z$, where z and ω are complex numbers, then z is called a logarithm of ω to the base e. Thus $\log_e \omega = z$.

1. Prove that $\log_e \omega$ is a many-valued function.

Proof. We know that $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$

Let $e^z = \omega$, then $e^{z + 2n\pi i} = e^z \cdot e^{2n\pi i} = e^z \cdot 1 = \omega$

$\therefore$ By definition $\log_e \omega = z + 2n\pi i$, where n is zero, or any + ve or – ve integer.

Thus if z be a logarithm of ω, so is $z + 2n\pi i$.

Hence the logarithm of a complex number has infinite values and is thus a many-value function.

Note. The value $z + 2n\pi i$ is called the *general value* of $\log_e \omega$ and is denoted by $\mathbf{Log}_e\, \omega$.

Thus $\quad \mathrm{Log}_e\, \omega = z + 2n\pi i = 2n\pi i + \log_e \omega$

If $\omega = x + iy$, then $\mathbf{Log}\,(\mathbf{x + iy}) = \mathbf{2n\pi i + \log (x + iy)}$.

If we put n = 0, in the general value, we get the principal value of z, i.e., $\log_e \omega$.

2. Prove that $\log(-N) = \pi i + \log N$, where N is positive.

Proof. $\quad -N = N(-1) = N(\cos \pi + i \sin \pi) = N \cdot e^{i\pi}$

$\therefore \quad \log(-N) = \log(N \cdot e^{i\pi}) = \log N + \log e^{i\pi} = \log N + \pi i$.

3. Separate Log $(\alpha + i\beta)$ into real and imaginary parts.

Proof. Let $\alpha + i\beta = r(\cos\theta + i \sin\theta)$ so that $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1} \dfrac{\beta}{\alpha}$

$\therefore \quad \mathrm{Log}(\alpha + i\beta) = 2n\pi i + \log(\alpha + i\beta) = 2n\pi i + \log[r(\cos\theta + i \sin\theta)]$

$= 2n\pi i + \log(r e^{i\theta}) = 2n\pi i + \log r + \log e^{i\theta} = 2n\pi i + \log r + i\theta$

$= 2n\pi i + \log \sqrt{\alpha^2 + \beta^2} + i \tan^{-1} \dfrac{\beta}{\alpha} = 2n\pi i + \dfrac{1}{2} \log(\alpha^2 + \beta^2) + i \tan^{-1} \dfrac{\beta}{\alpha}$

$\therefore \quad \mathrm{Re}[\mathrm{Log}(\alpha + i\beta)] = \dfrac{1}{2} \log(\alpha^2 + \beta^2)$

$\mathrm{Im}[\mathrm{Log}(\alpha + i\beta)] = 2n\pi + \tan^{-1} \dfrac{\beta}{\alpha}$.

Note. Putting $n = 0$, the principal value of $\mathbf{log}\,(\alpha + i\beta) = \dfrac{1}{2} \mathbf{log}\,(\alpha^2 + \beta^2) + \mathbf{i}\, \mathbf{tan}^{-1} \dfrac{\beta}{\alpha}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $\log(1 + re^{i\theta}) = \dfrac{1}{2} \log(1 + 2r\cos\theta + r^2) + i \tan^{-1} \dfrac{r \sin\theta}{1 + r\cos\theta}$.

Deduce that $\log(1 + \cos\theta + i \sin\theta) = \log\left(2 \cos \dfrac{\theta}{2}\right) + i \dfrac{\theta}{2}$.

Sol. $\log(1 + re^{i\theta}) = \log[1 + r(\cos\theta + i \sin\theta)] = \log[(1 + r\cos\theta) + i(r \sin\theta)]$

$= \dfrac{1}{2} \log[(1 + r\cos\theta)^2 + (r \sin\theta)^2] + i \tan^{-1} \dfrac{r \sin\theta}{1 + r\cos\theta}$

$= \dfrac{1}{2} \log[1 + 2r\cos\theta + r^2 \cos^2\theta + r^2 \sin^2\theta] + i \tan^{-1} \dfrac{r \sin\theta}{1 + r\cos\theta}$

$= \dfrac{1}{2} \log[1 + 2r\cos\theta + r^2] + i \tan^{-1} \dfrac{r \sin\theta}{1 + r\cos\theta}$ ...(i)

Now $\quad \log(1 + \cos\theta + i \sin\theta) = \log(1 + e^{i\theta})$

Putting $r = 1$ in (i),

$\log(1 + \cos\theta + i \sin\theta) = \dfrac{1}{2} \log(1 + 2\cos\theta + 1) + i \tan^{-1} \dfrac{\sin\theta}{1 + \cos\theta}$

$= \dfrac{1}{2} \log[2(1 + \cos\theta)] + i \tan^{-1} \dfrac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$

$= \dfrac{1}{2} \log\left[2 \cdot 2 \cos^2 \dfrac{\theta}{2}\right] + i \tan^{-1}\left(\tan \dfrac{\theta}{2}\right) = \dfrac{1}{2} \log\left[\left(2 \cos \dfrac{\theta}{2}\right)^2\right] + i \dfrac{\theta}{2}$

$= \dfrac{1}{2} \cdot 2 \log\left(2 \cos \dfrac{\theta}{2}\right) + i \cdot \dfrac{\theta}{2} = \log\left(2 \cos \dfrac{\theta}{2}\right) + i \cdot \dfrac{\theta}{2}$.

Example 2. *Find the general value of* $\log(-3)$.

Sol. $\because \quad -3 = 3(-1) = 3 \text{ cis } \pi = 3 e^{i\pi}$

$\therefore \quad \mathrm{Log}(-3) = \mathrm{Log}(3e^{i\pi}) = 2n\pi i + \log(3 e^{i\pi})$

$= 2n\pi i + \log 3 + \log e^{i\pi} = 2n\pi i + \log 3 + i\pi = \log 3 + i(2n + 1)\pi$.

Example 3. *Separate into real and imaginary parts Log (4 + 3i).* (M.D.U. Dec. 2006)

Sol. Let $4 + 3i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts $r\cos\theta = 4;\ r\sin\theta = 3$

Squaring and adding, $r^2 = 16 + 9 = 25 \quad \therefore \quad r = 5$

Dividing, $\tan\theta = \dfrac{3}{4} \qquad \therefore \quad \theta = \tan^{-1}\dfrac{3}{4}$

$$\therefore \quad \text{Log}(4+3i) = \text{Log}[r(\cos\theta + i\sin\theta)] = \text{Log}(re^{i\theta}) = 2n\pi i + \log(re^{i\theta})$$

$$= 2n\pi i + \log r + \log e^{i\theta} = 2n\pi i + \log 5 + i\theta = \log 5 + 2n\pi i + i\tan^{-1}\frac{3}{4}$$

$\therefore \quad \text{Re}[\text{Log}(4+3i)] = \log 5$

$\text{Im}[\text{Log}(4+3i)] = \left(2n\pi + \tan^{-1}\dfrac{3}{4}\right).$

Example 4. *Prove that* $\tan\left(i\log\dfrac{a-ib}{a+ib}\right) = \dfrac{2ab}{a^2-b^2}.$

Sol. Let $a + ib = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts $r\cos\theta = a,\ r\sin\theta = b$

Dividing, $\tan\theta = \dfrac{b}{a}$ ...(i)

Also $a - ib = r(\cos\theta - i\sin\theta)$

$$\text{L.H.S.} = \tan\left[i\log\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)}\right] = \tan\left[i\log\frac{e^{-i\theta}}{e^{i\theta}}\right]$$

$$= \tan[i\log e^{-2i\theta}] = \tan[i(-2i\theta)\log e]$$

$$= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\dfrac{b}{a}}{1-\dfrac{b^2}{a^2}} \qquad [\because \text{ of } (i)]$$

$$= \frac{2ab}{a^2-b^2}.$$

Example 5. *Express Log (Log i) in the form A + iB.*

Sol. $\quad i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = e^{i\pi/2}$

$$\therefore \quad \text{Log}\, i = 2n\pi i + \log e^{i\pi/2} = 2n\pi i + i\frac{\pi}{2} = i(4n+1)\frac{\pi}{2}$$

$$\therefore \quad \text{Log}(\text{Log}\, i) = \text{Log}\left[i(4n+1)\frac{\pi}{2}\right] = 2m\pi i + \log\left[i(4n+1)\frac{\pi}{2}\right]$$

$$= 2m\pi i + \log i + \log(4n+1)\frac{\pi}{2}$$

$$= 2m\pi i + \log e^{i\pi/2} + \log(4n+1)\frac{\pi}{2} = 2m\pi i + i\frac{\pi}{2} + \log(4n+1)\frac{\pi}{2}$$

$$= \log(4n+1)\frac{\pi}{2} + i(4m+1)\frac{\pi}{2}.$$

3.8. THE GENERAL EXPONENTIAL FUNCTION

Definition. The general exponential function a^z is defined by the equation $a^z = e^{z\,\text{Log}\,a}$, where a and z are any numbers, real or complex.

1. Prove that a^z is a many valued function.

Proof. Since $\text{Log}\, a = 2n\pi i + \log a$

$$\therefore \quad a^z = e^{z(2n\pi i + \log a)}$$

Hence a^z is a many valued function and its principal value is obtained by putting $n = 0$.

2. Separate $(a + ib)^{x+iy}$ into real and imaginary parts.

Proof. $(a+ib)^{x+iy} = e^{(x+iy)\,\text{Log}(a+ib)}$

$$= e^{(x+iy)[2n\pi i + \log(a+ib)]}$$

$$= e^{(x+iy)\left[2n\pi i + \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}\right]}$$

$$= e^{\left[\frac{x}{2}\log(a^2+b^2) - 2n\pi y - y\tan^{-1}\frac{b}{a}\right] + i\left[2n\pi x + x\tan^{-1}\frac{b}{a} + \frac{y}{2}\log(a^2+b^2)\right]}$$

$$= e^{\alpha + i\beta}$$

where, $\alpha = \dfrac{x}{2}\log(a^2+b^2) - 2n\pi y - y\tan^{-1}\dfrac{b}{a},\ \beta = \dfrac{y}{2}\log(a^2+b^2) + 2n\pi x + x\tan^{-1}\dfrac{b}{a}$

$\Rightarrow \quad (a+ib)^{x+iy} = e^{\alpha}(\cos\beta + i\sin\beta)$

$\therefore \quad \text{Re}[(a+ib)^{x+iy}] = e^{\alpha}\cos\beta$

and $\text{Im}[(a+ib)^{x+iy}] = e^{\alpha}\sin\beta$

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that i^i is wholly real and find its principal value. Also show that the values of i^i form a G.P.*

Sol. $\quad i^i = e^{i\,\text{Log}\, i}$ [By definition]

$$= e^{i(2n\pi i + \log i)} = e^{i[2n\pi i + \log(\cos\pi/2 + i\sin\pi/2)]}$$

$$= e^{i[2n\pi i + \log e^{i\pi/2}]} = e^{i[2n\pi i + i\pi/2]} = e^{i^2(4n+1)\pi/2} = e^{-(4n+1)\pi/2}$$

which is wholly real.

The principal value of $i^i = e^{-\pi/2}$ (Putting $n = 0$)

Putting $n = 0, 1, 2, \ldots\ldots$ the values of i^i are $e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots\ldots$

which form a G.P. whose common ratio is $e^{-2\pi}$.

Example 2. *If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.* (K.U.K. 2005)

Sol. $\alpha + i\beta = i^{\alpha+i\beta} = e^{(\alpha+i\beta)\,\text{Log}\, i}$

$$= e^{(\alpha+i\beta)[2n\pi i + \log i]} = e^{(\alpha+i\beta)[2n\pi i + \log(\cos\pi/2 + i\sin\pi/2)]}$$

$$= e^{(\alpha+i\beta)[2n\pi i + \log e^{i\pi/2}]} = e^{(\alpha+i\beta)[2n\pi i + i\pi/2]} = e^{-\beta(4n+1)\pi/2 + i\alpha(4n+1)\pi/2}$$

$$= e^{-\beta(4n+1)\pi/2} \cdot e^{i\alpha(4n+1)\pi/2} = e^{-\beta(4n+1)\pi/2}\left[\cos(4n+1)\frac{\alpha\pi}{2} + i\sin(4n+1)\frac{\alpha\pi}{2}\right]$$

$[\because \quad e^{i\theta} = \cos\theta + i\sin\theta]$

Equating real and imaginary parts

$$\alpha = e^{-(4n+1)\beta\pi/2} \cdot \cos\,(4n+1)\frac{\alpha\pi}{2}; \quad \beta = e^{-(4n+1)\beta\pi/2} \cdot \sin\,(4n+1)\frac{\alpha\pi}{2}$$

Squaring and adding, $\alpha^2 + \beta^2 = e^{-(4n+1)\beta\pi}\left[\cos^2(4n+1)\dfrac{\alpha\pi}{2} + \sin^2(4n+1)\dfrac{\alpha\pi}{2}\right]$

$$= e^{-(4n+1)\beta\pi}.$$

Example 3. *Considering only the principal value, prove that the real part of*

$$(1+i\sqrt{3})^{1+i\sqrt{3}} \text{ is } 2e^{-\pi/\sqrt{3}}\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right).$$

Sol. $(1+i\sqrt{3})^{1+i\sqrt{3}} = e^{(1+i\sqrt{3})\log(1+i\sqrt{3})} = e^{(1+i\sqrt{3})\left[\frac{1}{2}\log(1+3)+i\tan^{-1}\sqrt{3}\right]}$

$$= e^{(1+i\sqrt{3})[\frac{1}{2}\log 4 + i\pi/3]} = e^{(1+i\sqrt{3})[\frac{1}{2}.2\log 2 + i\pi/3]}$$

$$= e^{(\log 2 - \pi/\sqrt{3}) + i(\pi/3+\sqrt{3}\log 2)} = e^{\log 2 - \pi/\sqrt{3}} \cdot e^{i(\pi/3+\sqrt{3}\log 2)}$$

$$= e^{\log 2}\, e^{-\pi/\sqrt{3}}\left[\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right]$$

$$= 2e^{-\pi/\sqrt{3}}\left[\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right] \quad [\because\ e^{\log f(x)} = f(x)]$$

$\Rightarrow$ Real part of $(1+i\sqrt{3})^{1+i\sqrt{3}}$ is $2e^{-\pi/\sqrt{3}}\cos\left(\dfrac{\pi}{3}+\sqrt{3}\log 2\right)$.

Example 4. *If* $i^{i^{\cdots \text{ad inf}}} = A + iB$ *and only principal values are considered, prove that*

(a) $\tan\dfrac{\pi A}{2} = \dfrac{B}{A}$ (b) $A^2 + B^2 = e^{-\pi B}$. (M.D.U. Dec. 2010)

Sol. $i^{i^{\cdots \text{ad inf}}} = A + iB \quad \Rightarrow \quad i^{A+iB} = A + iB$

Now $A + iB = i^{A+iB} = e^{(A+iB)\log i}$ (Taking principal values only)

$$= e^{(A+iB)\log(\cos\pi/2 + i\sin\pi/2)} = e^{(A+iB)\log(e^{i\pi/2})}$$

$$= e^{(A+iB)(i\pi/2)} = e^{-(\beta\pi/2) + i\,.\,(A\pi/2)}$$

$$= e^{-(B\pi/2)} \cdot e^{i\,A\pi/2} = e^{-B\pi/2}\left(\cos\frac{A\pi}{2} + i\sin\frac{A\pi}{2}\right)$$

Equating real and imaginary parts

$$A = e^{-B(\pi/2)}\cos\frac{A\pi}{2} \qquad \ldots(i)$$

$$B = e^{-B(\pi/2)}\sin\frac{A\pi}{2} \qquad \ldots(ii)$$

Dividing (*ii*) by (*i*), $\tan\dfrac{A\pi}{2} = \dfrac{B}{A}$ ...(I)

Squaring and adding (*i*) and (*ii*), $A^2 + B^2 = e^{-B\pi}\left(\cos^2\dfrac{A\pi}{2} + \sin\dfrac{A\pi}{2}\right) = e^{-B\pi}.$ ...(II)

Example 5. *If* $(a+ib)^p = m^{x+iy}$, *then prove that* $\dfrac{y}{x} = \dfrac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2+b^2)}$ *when only principal values are considered.*

Sol. $(a+ib)^p = m^{x+iy}$

Taking log of both sides, $\log(a+ib)^p = \log m^{x+iy}$

or $p\log(a+ib) = (x+iy)\log m$

or $p\left[\dfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}\dfrac{b}{a}\right] = x\log m + iy\log m$

(Considering only the principal values)

Equating real and imaginary parts $x\log m = \frac{1}{2}p\log(a^2+b^2)$ ...(*i*)

$$y\log m = p\tan^{-1}\frac{b}{a} \qquad \ldots(ii)$$

Dividing (*ii*) by (*i*), $\dfrac{y}{x} = \dfrac{p\tan^{-1}\frac{b}{a}}{\frac{1}{2}p\log(a^2+b^2)} = \dfrac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$.

Example 6. *If* $\tan\log(x+iy) = a+ib$ *and* $a^2+b^2 \neq 1$, *then prove that*

$$\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}.$$

Sol. $\tan\log(x+iy) = a+ib$ ...(*i*)

$\Rightarrow$ $\tan\log(x-iy) = a-ib$ ...(*ii*)

Now $\tan\log(x^2+y^2) = \tan\log(x+iy)(x-iy)$

$$= \tan[\log(x+iy)+\log(x-iy)] = \frac{\tan\log(x+iy)+\tan\log(x-iy)}{1-\tan\log(x+iy).\tan\log(x-iy)}$$

$$= \frac{a+ib+a-ib}{1-(a+ib)(a-ib)} = \frac{2a}{1-a^2-b^2}, \text{ where } a^2+b^2 \neq 1.$$

EXERCISE 3.2

1. Find the general value of

   (*i*) $\log(-i)$ (*ii*) $\log(1+i)$.

2. Prove that

   (*i*) $i\log\left(\dfrac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$ (*ii*) $\cos\left[i\log\left(\dfrac{a+ib}{a-ib}\right)\right] = \dfrac{a^2-b^2}{a^2+b^2}$

   (*iii*) $i^i = e^{-(4n+1)\frac{\pi}{2}}$ (*iv*) $\operatorname{Log} i^i = -\left(2n+\dfrac{1}{2}\right)\pi$.

3. Show that

   (*i*) $\log(1+i\tan\alpha) = \log\sec\alpha + i\alpha$ (*ii*) $\operatorname{Log}_e\left(\dfrac{3-i}{3+i}\right) = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$.

(*iii*) $\log (6 + 8i) = \log 10 + i \tan^{-1} \dfrac{4}{3}$ (*M.D.U. Dec.* 2005)

(*iv*) $(\sqrt{i})^{\sqrt{i}} = e^{-\frac{\pi}{4\sqrt{2}}} \left(\cos \dfrac{\pi}{4\sqrt{2}} + i \sin \dfrac{\pi}{4\sqrt{2}} \right)$

(*v*) $\log (e^{i\alpha} + e^{i\beta}) = \log \left(2 \cos \dfrac{\alpha - \beta}{2} \right) + i \left(\dfrac{\alpha + \beta}{2} \right)$.

4. Prove that $\sin \left[i \log \left(\dfrac{1 + ie^{-i\theta}}{1 - ie^{i\theta}} \right) \right]$ is wholly real.

5. Prove that $\sin (\log i^i) = -1$.

6. Prove that $\tan^{-1} x = \dfrac{1}{2i} \log \left(\dfrac{1 + ix}{1 - ix} \right)$.

7. If $\log \log (x + iy) = p + iq$, show that $y = x \tan [\tan q \log \sqrt{x^2 + y^2}\,]$.

8. Prove that the principal value of

$\dfrac{(a + ib)^{p+iq}}{(a - ib)^{p-iq}}$ is $\cos 2 (p\alpha + q \log r) + i \sin 2 (p\alpha + q \log r)$, where $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} \dfrac{b}{a}$.

9. If $\dfrac{(1 + i)^{x+iy}}{(1 - i)^{x-iy}} = \alpha + i\beta$, prove that one of the values of $\tan^{-1} \dfrac{\beta}{\alpha}$ is $\dfrac{1}{2} \pi x + y \log 2$.

10. Prove that $\dfrac{(1 + i)^{1-i}}{(1 - i)^{1+i}} = \sin (\log 2) + i \cos (\log 2)$.

11. Prove that the real part of the principal value of $i^{\log (1 + i)}$ is $e^{-\frac{\pi^2}{8}} \cos \left(\dfrac{\pi}{4} \log 2 \right)$.

12. Prove that $\text{Log}_i\, i = \dfrac{4m + 1}{4n + 1}$, where m and n are integers.

13. If (x, y) is a point on the circle having its centre at the origin and radius a, prove that

$$\log \frac{x + iy - a}{x + iy + a} = \frac{1}{2} \log \frac{a - x}{a + x} + \frac{\pi}{2} i.$$

14. Prove that $\log \dfrac{1}{1 - e^{i\theta}} = \log \left(\dfrac{1}{2} \text{cosec} \dfrac{\theta}{2} \right) + i \left(\dfrac{\pi}{2} - \dfrac{\theta}{2} \right)$.

15. If $(a + ib)^{c + id}$ is wholly real and principal values are considered, prove that

$$(a + ib)^{c + id} = (a^2 + b^2)^{\frac{c^2 + d^2}{2c}}.$$

16. If $(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n) = A + iB$, prove that

(*i*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots\ldots (a_n^2 + b_n^2) = A^2 + B^2$

(*ii*) $\tan^{-1} \dfrac{b_1}{a_1} + \tan^{-1} \dfrac{b_2}{a_2} + \ldots\ldots + \tan^{-1} \dfrac{b_n}{a_n} = \tan^{-1} \dfrac{B}{A}$.

Answers

1. (*i*) $(4n - 1) \dfrac{\pi i}{2}$, (*ii*) $\dfrac{1}{2} \log 2 + i(8n + 1) \dfrac{\pi}{4}$.

3.9. HYPERBOLIC FUNCTIONS

1. Definitions. For all values of x, real or complex

(*i*) the quantity $\dfrac{e^x - e^{-x}}{2}$ is called *hyperbolic sine of* x and is written as $\sinh x$

(*ii*) the quantity $\dfrac{e^x + e^{-x}}{2}$ is called *hyperbolic cosine of* x and is written as $\cosh x$.

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$; $\cosh x = \dfrac{e^x + e^{-x}}{2}$

The other hyperbolic functions are defined in terms of hyperbolic sine and cosine as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \qquad \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Note. $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1 - 1}{2} = 0$; $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1 + 1}{2} = 1$

$\cosh x + \sinh x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2} = e^x$; $\cosh x - \sinh x = \dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2} = e^{-x}$.

2. (*a*) Relations between hyperbolic and trigonometric functions

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}; \ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Putting $\theta = ix$ in these equations, we get

$$\cos (ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x$$

$$\sin (ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^x}{2i} = \frac{-(e^x - e^{-x})}{2i}$$

$$= \frac{i^2(e^x - e^{-x})}{2i} = i \,.\, \frac{e^x - e^{-x}}{2} = i \sinh x$$

$$\tan (ix) = \frac{\sin (ix)}{\cos (ix)} = \frac{i \sinh x}{\cosh x} = i \tanh x$$

$$\cot (ix) = \frac{\cos (ix)}{\sin (ix)} = \frac{\cosh x}{i \sinh x} = \frac{i \cosh x}{i^2 \sinh x} = -i \coth x$$

$$\sec (ix) = \frac{1}{\cos (ix)} = \frac{1}{\cosh x} = \text{sech}\, x$$

$$\text{cosec}\, (ix) = \frac{1}{\sin (ix)} = \frac{1}{i \sinh x} = \frac{1}{i^2 \sinh x} = -i\, \text{cosech}\, x.$$

(*b*) **By Definition,** $\sinh \theta = \dfrac{e^\theta - e^{-\theta}}{2}$; $\cosh \theta = \dfrac{e^\theta + e^{-\theta}}{2}$; $\tanh \theta = \dfrac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}$

Putting $\theta = ix$, we get

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i \cdot \frac{e^{ix} - e^{-ix}}{2i} = i \sin x;$$

$$\cosh(ix) = \frac{e^{ix} - e^{-ix}}{2} = \cos x$$

$$\tanh(ix) = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = i \cdot \frac{\frac{e^{ix} - e^{-ix}}{2i}}{\frac{e^{ix} + e^{-ix}}{2}} = i \cdot \frac{\sin x}{\cos x} = i \tan x$$

3. Prove that hyperbolic functions are periodic and find their periods.

(*a*) We know that $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\therefore \sinh(x + 2n\pi i) = \frac{e^{x + 2n\pi i} - e^{-(x + 2n\pi i)}}{2}, \text{ where } n \text{ is any integer}$$

$$= \frac{1}{2}[e^x \cdot e^{2n\pi i} - e^{-x} \cdot e^{-2n\pi i}] = \frac{1}{2}[e^x \cdot 1 - e^{-x} \cdot 1] = \frac{e^x - e^{-x}}{2} = \sinh x$$

Thus $\sinh x$ remains unchanged when x is increased by any multiple of $2\pi i$. **Hence sinh x is a periodic function and its period is $2\pi i$.**

(*b*) $$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh(x + 2n\pi i) = \frac{e^{x + 2n\pi i} + e^{-(x + 2n\pi i)}}{2}, \quad \text{where } n \text{ is any integer}$$

$$= \tfrac{1}{2}[e^x \cdot e^{2n\pi i} + e^{-x} \cdot e^{-2n\pi i}] = \tfrac{1}{2}[e^x \cdot 1 + e^{-x} \cdot 1] = \frac{e^x + e^{-x}}{2} = \cosh x$$

Thus $\cosh x$ remains unchanged when x is increased by any multiple of $2\pi i$. Hence **cosh x is a periodic function and its period is $2\pi i$.**

(*c*) $$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\tanh(x + n\pi i) = \frac{e^{x + n\pi i} - e^{-(x + n\pi i)}}{e^{x + n\pi i} + e^{-(x + n\pi i)}}, \text{ where } n \text{ is any integer}$$

$$= \frac{e^x \cdot e^{n\pi i} - e^{-x} \cdot e^{-n\pi i}}{e^x \cdot e^{n\pi i} + e^{-x} \cdot e^{-n\pi i}}$$

Multiplying the numerator and denominator by $e^{n\pi i}$

$$= \frac{e^x \cdot e^{2n\pi i} - e^{-x}}{e^x \cdot e^{2n\pi i} + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$$

$$[\because \quad e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1]$$

Thus $\tanh x$ remains unchanged when x is increased by any multiple of πi. Hence **tanh x is a periodic function and its period is πi.**

Note. cosech x, sech x and coth x being reciprocals of $\sinh x$, $\cosh x$ and $\tanh x$ respectively, are also periodic functions with periods $2\pi i$, $2\pi i$ and πi respectively.

3.10. FORMULAE OF HYPERBOLIC FUNCTIONS

1. Prove that (*a*) $\cosh^2 x - \sinh^2 x = 1$, (*b*) $\text{sech}^2 x + \tanh^2 x = 1$, (*c*) $\coth^2 x - \text{cosech}^2 x = 1$

Proof. (*a*) For all values of θ, $\cos^2 \theta + \sin^2 \theta = 1$

Putting $\theta = ix$, we get $\cos^2(ix) + \sin^2(ix) = 1$

or $\quad (\cosh x)^2 + (i \sinh x)^2 = 1 \quad [\because \cos ix = \cosh x; \sin(ix) = i \sinh x]$

or $\quad \cosh^2 x - \sinh^2 x = 1 \quad [\because i^2 = -1]$

(*b*) We know that $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\cosh^2 x$, we have

$$1 - \tanh^2 x = \text{sech}^2 x \Rightarrow \text{sech}^2 x + \tanh^2 x = 1$$

(*c*) We know that $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\sinh^2 x$, we have

$$\coth^2 x - 1 = \text{cosech}^2 x \Rightarrow \coth^2 x - \text{cosech}^2 x = 1$$

2. Prove that (*a*) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(*b*) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(*c*) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

Proof. (*a*) $\sinh(x \pm y) = \dfrac{1}{i} \sin i(x \pm y) \qquad \left[\because \sinh x = \dfrac{1}{i} \sin ix\right]$

$$= \frac{1}{i}(\sin ix \cos iy \pm \cos ix \sin iy)$$

$$= \frac{1}{i}(i \sinh x \cosh y \pm \cosh x \cdot i \sinh y)$$

$$[\because \sin i\theta = i \sinh \theta; \cos i\theta = \cosh \theta]$$

$$= \sinh x \cosh y \pm \cosh x \sinh y$$

(*b*) $\cosh(x \pm y) = \cos i(x \pm y) \qquad [\because \cosh x = \cos ix]$

$$= \cos ix \cos iy \mp \sin ix \sin iy = \cosh x \cosh y \mp i \sinh x \cdot i \sinh y$$

$$= \cosh x \cosh y \mp (-\sinh x \cdot \sinh y) \qquad [\because i^2 = -1]$$

$$= \cosh x \cosh y \pm \sinh x \sinh y$$

(*c*) $$\tanh(x \pm y) = \frac{\sinh(x \pm y)}{\cosh(x \pm y)} = \frac{\sinh x \cosh y \pm \cosh x \sinh y}{\cosh x \cosh y \pm \sinh x \sinh y}$$

Dividing the numerator and denominator by $\cosh x \cosh y = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

3. Prove that (*a*) $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2 \tanh x}{1 - \tanh^2 x}$

(b) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

(c) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

Proof. (a) We know that $\sin 2\theta = 2\sin\theta\cos\theta$

Putting $\theta = ix$, we get $\sin(2ix) = 2\sin(ix)\cos(ix)$ or $i\sinh 2x = 2 \,.\, i\sinh x \,.\, \cosh x$

or $$\mathbf{\sinh 2x = 2\sinh x\cosh x}$$

Also $$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

Putting $\theta = ix$, we get $\sin(2ix) = \dfrac{2\tan ix}{1+\tan^2 ix} = \dfrac{2\,.\,i\tanh x}{1+(i\tanh x)^2}$

or $$i\sinh 2x = \frac{2i\tanh x}{1-\tanh^2 x} \quad\text{or}\quad \sinh 2x = \frac{2\tanh x}{1-\tanh^2 x}$$

(b) We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Putting $\theta = ix$, we get $\cos(2ix) = \cos^2(ix) - \sin^2(ix)$ or $\cosh 2x = (\cosh x)^2 - (i\sinh x)^2$

or $$\mathbf{\cosh 2x = \cosh^2 x + \sinh^2 x}$$

We know that $\cos 2\theta = 2\cos^2\theta - 1$

Putting $\theta = ix$, we get $\cos(2ix) = 2\cos^2(ix) - 1$ or $\mathbf{\cosh 2x = 2\cosh^2 x - 1}$

Cor. $$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

We know that $\cos 2\theta = 1 - 2\sin^2\theta$

Putting $\theta = ix$, we get $\cos(2ix) = 1 - 2\sin^2(ix)$

or $$\mathbf{\cosh 2x} = 1 - 2(i\sinh x)^2 = \mathbf{1 + 2\sinh^2 x}$$

Cor. $$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

We know that $$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

Putting $\theta = ix$, we get $\cos(2ix) = \dfrac{1-\tan^2(ix)}{1+\tan^2(ix)} = \dfrac{1-(i\tanh x)^2}{1+(i\tanh x)^2}$

or $$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$$

(c) We know that $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Putting $\theta = ix$, we get

$$\tan(2ix) = \frac{2\tan(ix)}{1-\tan^2(ix)}$$

$$i\tanh 2x = \frac{2i\tanh x}{1-(i\tanh x)^2} = \frac{2i\tanh x}{1+\tanh^2 x}$$

$\therefore$ $$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}.$$

4. Prove that (a) $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

(b) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (c) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

Proof. (a) We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Putting $\theta = ix$, we get $\sin(3ix) = 3\sin(ix) - 4\sin^3(ix)$

or $$i\sinh 3x = 3i\sinh x - 4(i\sinh x)^3$$

or $$i\sinh 3x = 3i\sinh x + 4\,i\sinh^3 x \qquad [\because\ i^3 = -i]$$

or $$\sinh 3x = 3\sinh x + 4\sinh^3 x$$

(b) We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Putting $\theta = ix$, we get $\cos(3ix) = 4\cos^3(ix) - 3\cos(ix)$

or $$\cosh 3x = 4\cosh^3 x - 3\cosh x$$

(c) We know that $$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Putting $\theta = ix$, we get $\tan(3ix) = \dfrac{3\tan(ix) - \tan^3(ix)}{1-3\tan^2(ix)}$

or $$i\tanh 3x = \frac{3\,.\,i\tanh x - (i\tanh x)^3}{1-3(i\tanh x)^2}$$

or $$i\tanh 3x = \frac{3\,.\,i\tanh x + i\tanh^3 x}{1+3\tanh^2 x} \quad\text{or}\quad \tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}.$$

5. Prove that

(i) $2\sinh A\cosh B = \sinh(A+B) + \sinh(A-B)$

(ii) $2\cosh A\sinh B = \sinh(A+B) - \sinh(A-B)$

(iii) $2\cosh A\cosh B = \cosh(A+B) + \cosh(A-B)$

(iv) $2\sinh A\sinh B = \cosh(A+B) - \cosh(A-B)$

Proof. We shall prove only the last result.

The first three are left as an exercise for the student.

We know that $2\sin x\sin y = \cos(x-y) - \cos(x+y)$

Putting $x = iA$; $y = iB$, we get

$$2\sin(iA)\,.\,\sin(iB) = \cos i(A-B) - \cos i(A+B)$$

or $$2\,.\,i\sinh A\,.\,i\sinh B = \cosh(A-B) - \cosh(A+B)$$

or $$-2\sinh A\sinh B = \cosh(A-B) - \cosh(A+B) \qquad [\because\ i^2 = -1]$$

or $$2\sinh A\sinh B = \cosh(A+B) - \cosh(A-B).$$

6. Prove that

(i) $\sinh C + \sinh D = 2\sinh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$

(ii) $\sinh C - \sinh D = 2\cosh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$

(iii) $\cosh C + \cosh D = 2\cosh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$

(iv) $\cosh C - \cosh D = 2\sinh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$

Proof. We shall prove only the last result. The first three are left as an exercise for the student.

We know that $\cos x - \cos y = 2\sin\dfrac{x+y}{2}\sin\dfrac{y-x}{2}$

Putting $x = i\text{A}$ and $y = i\text{B}$, we get

$$\cos(i\text{A}) - \cos(i\text{B}) = 2\sin\left(i\,\frac{\text{A}+\text{B}}{2}\right)\sin\left(i\,\frac{\text{B}-\text{A}}{2}\right)$$

$$\Rightarrow \quad \cosh \text{A} - \cosh \text{B} = 2i\sinh\frac{\text{A}+\text{B}}{2}\,.\,i\sinh\frac{\text{B}-\text{A}}{2}$$

$$= -2\sinh\frac{\text{A}+\text{B}}{2}\sinh\frac{\text{B}-\text{A}}{2} = 2\sinh\frac{\text{A}+\text{B}}{2}\sinh\frac{\text{A}-\text{B}}{2}$$

$$[\because \quad \sinh(-x) = \sinh x]$$

7. Prove that

$$\tanh(x+y+z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x\tanh y\tanh z}{1 + \tanh x\tanh y + \tanh y\tanh z + \tanh z\tanh x}$$

Proof. We know that,

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$$

Putting $\alpha = ix;\ \beta = iy;\ \gamma = iz$, we get

$$\tan i(x+y+z) = \frac{\tan(ix) + \tan(iy) + \tan(iz) - \tan(ix)\tan(iy)\tan(iz)}{1 - \tan(ix)\tan(iy) - \tan(iy)\tan(iz) - \tan(iz)\tan(ix)}$$

$$i\tanh(x+y+z) = \frac{i\tanh x + i\tanh y + i\tanh z - i\tanh x\,.\,i\tanh y\,.\,i\tanh z}{1 - i\tanh x\,.\,i\tanh y - i\tanh y\,.\,i\tanh z - i\tanh z\,.\,i\tanh x}$$

or

$$\tanh(x+y+z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x\tanh y\tanh z}{1 + \tanh x\tanh y + \tanh y\tanh z + \tanh z\tanh x}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Separate into real and imaginary parts*

(a) $\sin(x+iy)$ (b) $\cos(x+iy)$

(c) $\tan(x+iy)$ (d) $\cot(x+iy)$

(e) $\sec(x+iy)$ (f) $\operatorname{cosec}(x+iy)$.

Sol. (a) $\sin(x+iy) = \sin x\cos iy + \cos x\sin iy$

$= \sin x\cosh y + \cos x\,.\,i\sinh y = \sin x\cosh y + i\,.\cos x\sinh y$

(b) $\cos(x+iy) = \cos x\cos iy - \sin x\sin iy$

$= \cos x\cosh y - \sin x\,.\,i\sinh y = \cos x\cosh y - i\,.\sin x\sinh y$

(c) $$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} \quad \left[\begin{array}{l}\because\ 2\sin\text{A}\cos\text{B} = \sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})\\ \ \ \ 2\cos\text{A}\cos\text{B} = \cos(\text{A}+\text{B}) + \cos(\text{A}-\text{B})\end{array}\right]$$

$$= \frac{\sin 2x + i\,.\sinh 2y}{\cos 2x + \cosh 2y} = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\,.\,\frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

(d) $$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)} = \frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\sin 2x - \sin 2iy}{\cos 2iy - \cos 2x} \quad \left[\begin{array}{l}\because\ 2\cos\text{A}\sin\text{B} = \sin(\text{A}+\text{B}) - \sin(\text{A}-\text{B})\\ \ \ \ 2\sin\text{A}\sin\text{B} = \cos(\text{A}-\text{B}) - \cos(\text{A}+\text{B})\end{array}\right]$$

$$= \frac{\sin 2x - i\,.\sinh 2y}{\cosh 2y - \cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x} - i\,.\,\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

(e) $$\sec(x+iy) = \frac{1}{\cos(x+iy)} = \frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2(\cos x\cos iy + \sin x\sin iy)}{\cos 2x + \cos 2iy} = \frac{2(\cos x\cosh y + \sin x\,.\,i\sinh y)}{\cos 2x + \cosh 2y}$$

$$= \frac{2\cos x\cosh y}{\cos 2x + \cosh 2y} + i\,.\,\frac{2\sin x\sinh y}{\cos 2x + \cosh 2y}$$

(f) $$\operatorname{cosec}(x+iy) = \frac{1}{\sin(x+iy)} = \frac{2\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$

$$= \frac{2(\sin x\cos iy - \cos x\sin iy)}{\cos 2iy - \cos 2x} = \frac{2(\sin x\cosh y + \cos x\,.\,i\sinh y)}{\cosh 2y - \cos 2x}$$

$$= \frac{2\sin x\cosh y}{\cosh 2y - \cos 2x} - i\,.\,\frac{2\cos x\sinh y}{\cosh 2y - \cosh 2x}.$$

Example 2. *Separate the following into real and imaginary parts*:

(a) $\sinh(x+iy)$ (b) $\cosh(x+iy)$

(c) $\tanh(x+iy)$ (d) $\coth(x+iy)$

(e) $\operatorname{sech}(x+iy)$ (f) $\operatorname{cosech}(x+iy)$.

Sol. (a) $\sinh(x+iy) = \dfrac{1}{i}\sin i(x+iy)$ $\quad [\because\ i\sinh\theta = \sin i\theta]$

$= \dfrac{i}{i^2}\sin(ix-y) = -i(\sin ix\cos y - \cos ix\sin y)$

$= -i(i\sinh x\cos y - \cosh x\sin y) = \sinh x\cos y + i\cosh x\sin y$

(b) $\cosh(x+iy) = \cos i(x+iy)$ $\quad [\because\ \cosh\theta = \cos i\theta]$

$= \cos(ix-y) = \cos ix\cos y + \sin ix\sin y$

$= \cosh x\cos y + i\sinh x\sin y$

(c) $\tanh(x+iy) = \dfrac{1}{i}\tan i(x+iy)$ $\quad [\because\ i\tanh\theta = \tan i\theta]$

$$= \frac{i}{i^2}\tan(ix-y) = -i\,\frac{\sin(ix-y)}{\cos(ix-y)} = -i\,.\,\frac{2\sin(ix-y)\cos(ix+y)}{2\cos(ix-y)\cos(ix+y)}$$

$$= -i\,.\,\frac{\sin 2ix - \sin 2y}{\cos 2ix + \cos 2y} = -i\,.\,\frac{i\sinh 2x - \sin 2y}{\cosh 2x + \cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i \cdot \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

(*d*) $\coth (x + iy) = \dfrac{\cosh (x+iy)}{\sinh (x+iy)} = \dfrac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)} = i \cdot \dfrac{\cos (ix-y)}{\sin (ix-y)}$

$$= i \cdot \frac{2 \sin (ix+y) \cos (ix-y)}{2 \sin (ix+y) \sin (ix-y)} = i \cdot \frac{\sin 2ix + \sin 2y}{\cos 2y - \cos 2ix}$$

$$= i \cdot \frac{i \sinh 2x + \sin 2y}{\cos 2y - \cosh 2x} = \frac{-\sinh 2x}{\cos 2y - \cosh 2x} + i \cdot \frac{\sin 2y}{\cos 2y - \cosh 2x}$$

$$= \frac{\sinh 2y}{\cosh 2x - \cos 2y} - i \cdot \frac{\sin 2y}{\cosh 2x - \cos 2y}.$$

(*e*) $\text{sech} (x + iy) = \dfrac{1}{\cosh (x+iy)} = \dfrac{1}{\cos i(x+iy)}$

$$= \frac{1}{\cos (ix-y)} = \frac{2 \cos (ix+y)}{2 \cos (ix+y) \cos (ix-y)} = \frac{2(\cos ix \cos y - \sin ix \sin y)}{\cos 2ix + \cos 2y}$$

$$= \frac{2(\cosh x \cos y - i \sinh x \sin y)}{\cosh 2x + \cos 2y}$$

$$= \frac{2 \cosh x \cos y}{\cosh 2x + \cos 2y} - i \cdot \frac{2 \sinh x \sin y}{\cosh 2x + \cos 2y}.$$

(*f*) $\text{cosech} (x + iy) = \dfrac{1}{\sinh (x+iy)} = \dfrac{1}{\frac{1}{i}\sin i(x+iy)} = \dfrac{i}{\sin (ix-y)}$

$$= i \cdot \frac{2 \sin (ix+y)}{2 \sin (ix+y) \sin (ix-y)}$$

$$= i \cdot \frac{2(\sin ix \cos y + \cos ix \sin y)}{\cos 2y - \cos 2ix} = i \cdot \frac{2(i \sinh x \cos y + \cosh x \sin y)}{\cos 2y - \cosh 2x}$$

$$= -\frac{2 \sinh x \cos y}{\cos 2y - \cosh 2x} + i \cdot \frac{2 \cosh x \sin y}{\cos 2y - \cosh 2x}$$

$$= \frac{2 \sinh x \cos y}{\cosh 2x - \cos 2y} - i \cdot \frac{2 \cosh x \sin y}{\cosh 2x - \cos 2y}.$$

Example 3. *If* $u = \log \tan \left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, *then prove that*

(*i*) $\tanh \frac{u}{2} = \tan \frac{\theta}{2}$ (P.T.U. 2006) (*ii*) $\cosh u = \sec \theta$.

Sol. $u = \log \tan \left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

(*i*) $e^u = \tan \left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$\Rightarrow \quad e^{u/2} \cdot e^{u/2} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \quad \Rightarrow \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}}$$

By componendo and dividendo

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{\left(1 + \tan \frac{\theta}{2}\right) - \left(1 - \tan \frac{\theta}{2}\right)}{\left(1 + \tan \frac{\theta}{2}\right) + \left(1 - \tan \frac{\theta}{2}\right)} \quad \Rightarrow \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2}$$

(*ii*) $\cosh u = \dfrac{1 + \tanh^2 \frac{u}{2}}{1 - \tanh^2 \frac{u}{2}} = \dfrac{1 + \tan^2 \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$ [Using part (*i*)]

$$= \frac{1}{\cos \theta} = \sec \theta.$$

Example 4. *If* $\sin (A + iB) = x + iy$, *prove that*

(*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (*ii*) $x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1$.

Sol. $x + iy = \sin (A + iB)$

$= \sin A \cos iB + \cos A \sin iB = \sin A \cosh B + i \cos A \sinh B$

Equating real and imaginary parts on both sides

$x = \sin A \cosh B;\ y = \cos A \sinh B$ ...(*i*)

From (*i*), $\dfrac{x}{\cosh B} = \sin A;\ \dfrac{y}{\sinh B} = \cos A$

Squaring and adding, $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A = 1$

Also from (*i*), $\dfrac{x}{\sin A} = \cosh B;\ \dfrac{y}{\cos A} = \sinh B$

Squaring and subtracting, $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$

or $x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1$.

Example 5. *If* $x + iy = \cosh (u + iv)$ *show that*

(*i*) $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$ (*ii*) $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = 1$.

(M.D.U. Dec. 2010)

Sol. $x + iy = \cosh (u + iv) = \cos i(u + iv)$ [$\because \cosh \theta = \cos i\theta$]

$= \cos (iu - v) = \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v$

Equating the real and imaginary parts

$x = \cosh u \cos v;\ y = \sinh u \sin v$ ...(*i*)

From (*i*), $\dfrac{x}{\cosh u} = \cos v,\ \dfrac{y}{\sinh u} = \sin v$

Squaring and adding, $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1$

From (*i*), $\dfrac{x}{\cos v} = \cosh u;\ \dfrac{y}{\sin v} = \sinh u$

Squaring and subtracting, $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = \cosh^2 u - \sinh^2 u = 1.$

Example 6. *If $x + iy = \tan (A + iB)$; prove that*

(*i*) $x^2 + y^2 + 2x \cot 2A = 1$ (*ii*) $x^2 + y^2 - 2y \coth 2B + 1 = 0.$

Sol. $x + iy = \tan (A + iB)$

Changing i into $-i$, we get $x - iy = \tan (A - iB)$

Now $\tan 2A = \tan [(A + iB) + (A - iB)]$

$$= \frac{\tan (A + iB) + \tan (A - iB)}{1 - \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)} = \frac{2x}{1 - (x^2 + y^2)}$$

or $\dfrac{1}{\cot 2A} = \dfrac{2x}{1 - (x^2 + y^2)}$ or $1 - (x^2 + y^2) = 2x \cot 2A$

or $x^2 + y^2 + 2x \cot 2A = 1$ ...(I)

Again $\tan (2iB) = \tan [(A + iB) - (A - iB)]$

$$= \frac{\tan (A + iB) - \tan (A - iB)}{1 + \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)} = \frac{2iy}{1 + x^2 + y^2}$$

or $i \tanh 2B = \dfrac{2iy}{1 + x^2 + y^2}$ or $\dfrac{1}{\coth 2B} = \dfrac{2y}{1 + x^2 + y^2}$

or $1 + x^2 + y^2 = 2y \coth 2B$

Hence $x^2 + y^2 - 2y \coth 2B + 1 = 0$ ...(II)

Example 7. *If $a + ib = \tanh \left(v + \dfrac{i\pi}{4}\right)$, prove that $a^2 + b^2 = 1$.*

Sol. Given $a + ib = \tanh \left(v + \dfrac{i\pi}{4}\right)$ ...(1)

Changing i to $-i$, we get

$$a - ib = \tanh \left(v - \frac{i\pi}{4}\right) \quad ...(2)$$

Multiplying (1) and (2), we have

$$a^2 - i^2 b^2 = \tanh \left(v + \frac{i\pi}{4}\right) \tanh \left(v - \frac{i\pi}{4}\right)$$

$$\Rightarrow \quad a^2 + b^2 = \frac{1}{i} \tan i \left(v + \frac{i\pi}{4}\right) \cdot \frac{1}{i} \tan i \left(v - \frac{i\pi}{4}\right)$$

$$= \frac{1}{i^2} \tan \left(iv - \frac{\pi}{4}\right) \tan \left(iv + \frac{\pi}{4}\right)$$

$$= -\frac{\sin \left(iv - \frac{\pi}{4}\right) \sin \left(iv + \frac{\pi}{4}\right)}{\cos \left(iv - \frac{\pi}{4}\right) \cos \left(iv + \frac{\pi}{4}\right)} = -\frac{\sin^2 (iv) - \sin^2 \frac{\pi}{4}}{\cos^2 (iv) - \sin^2 \frac{\pi}{4}}$$

$[\because \ \sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta\ ;\ \cos (\alpha + \beta) \cos (\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta]$

$$= -\frac{(i \sinh v)^2 - \frac{1}{2}}{\cosh^2 v - \frac{1}{2}} = -\frac{-\sinh^2 v - \frac{1}{2}}{(1 + \sinh^2 v) - \frac{1}{2}}$$

$$= \frac{\sinh^2 v + \frac{1}{2}}{\sinh^2 v + \frac{1}{2}} = 1$$

Hence, $a^2 + b^2 = 1$

Example 8. *If $\tan (\theta + i\phi) = \cos \alpha + i \sin \alpha = e^{i\alpha}$, prove that*

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4} \text{ and } \phi = \frac{1}{2} \log \tan \left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$ (M.D.U. May 2011)

Sol. $\tan (\theta + i\phi) = \cos \alpha + i \sin \alpha$ ...(*i*)

Changing i into $-i$, we get

$\tan (\theta - i\phi) = \cos \alpha - i \sin \alpha$ ...(*ii*)

Now $\tan 2\theta = \tan [(\theta + i\phi) + (\theta - i\phi)] = \dfrac{\tan (\theta + i\phi) + \tan (\theta - i\phi)}{1 - \tan (\theta + i\phi) \tan (\theta - i\phi)}$

$$= \frac{(\cos \alpha + i \sin \alpha) + (\cos \alpha - i \sin \alpha)}{1 - (\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha)}$$

$$= \frac{2 \cos \alpha}{1 - (\cos^2 \alpha - i^2 \sin^2 \alpha)} = \frac{2 \cos \alpha}{1 - (\cos^2 \alpha + \sin^2 \alpha)}$$

$$= \frac{2 \cos \alpha}{1 - 1} = \frac{2 \cos \alpha}{0} = \infty = \tan \frac{\pi}{2}$$

$\therefore$ $2\theta = n\pi + \dfrac{\pi}{2}$ $[\because \ \tan \theta = \tan \alpha \ \Rightarrow \ \theta = n\pi + \alpha]$

or $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

Also $\tan 2i\phi = \tan [(\theta + i\phi) - (\theta - i\phi)]$

$$= \frac{\tan (\theta + i\phi) - \tan (\theta - i\phi)}{1 + \tan (\theta + i\phi) \tan (\theta - i\phi)} = \frac{(\cos \alpha + i \sin \alpha) - (\cos \alpha - i \sin \alpha)}{1 + (\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha)}$$

$$= \frac{2i \sin \alpha}{1 + (\cos^2 \alpha + \sin^2 \alpha)} = \frac{2i \sin \alpha}{1 + 1} = i \sin \alpha$$

$i \tanh 2\phi = i \sin \alpha$ or $\tanh 2\phi = \sin \alpha$

or $$\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}}=\frac{\sin\alpha}{1} \quad \text{or} \quad \frac{e^{2\phi}+e^{-2\phi}}{e^{2\phi}-e^{-2\phi}}=\frac{1}{\sin\alpha}$$

By componendo and dividendo

$$\frac{2e^{2\phi}}{2e^{-2\phi}}=\frac{1+\sin\alpha}{1-\sin\alpha} \quad \text{or} \quad e^{4\phi}=\frac{1+\sin\alpha}{1-\sin\alpha}$$

or $$e^{4\phi}=\frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}=\left[\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right]^2$$

or $$e^{2\phi}=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}=\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}=\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Taking logarithms of both sides $\log e^{2\phi}=\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$ or $2\phi=\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$

$$\therefore \quad \phi=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

Example 9. *Separate into real and imaginary parts log sin (x + iy).*

Sol. $$\begin{aligned}\log\sin(x+iy)&=\log(\sin x\cos iy+\cos x\sin iy)\\&=\log(\sin x\cosh y+i\cos x\sinh y)\\&=\log(\alpha+i\beta),\text{ where } \alpha=\sin x\cosh y,\ \beta=\cos x\sinh y\\&=\frac{1}{2}\log(\alpha^2+\beta^2)+i\tan^{-1}\frac{\beta}{\alpha}\\&=\frac{1}{2}\log(\sin^2x\cosh^2y+\cos^2x\sinh^2y)+i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)\\&=\frac{1}{2}\log\left[\frac{1-\cos 2x}{2}\cdot\frac{\cosh 2y+1}{2}+\frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y-1}{2}\right]\\&\qquad+i\tan^{-1}(\cot x\tanh y)\\&=\tfrac{1}{2}\log\left[\tfrac{1}{4}(2\cosh 2y-2\cos 2x)\right]+i\tan^{-1}(\cot x\tanh y)\\&=\tfrac{1}{2}\log\left[\tfrac{1}{2}(\cosh 2y-\cos 2x)\right]+i\tan^{-1}(\cot x\tanh y).\end{aligned}$$

Example 10. *If $z = x + iy$ is a complex variable, then prove that sin z and cos z are not bounded.*

Sol. We know that

$$\sin z=\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$$

$$\begin{aligned}\therefore\quad |\sin z|^2&=|\sin x\cosh y+i\cos x\sinh y|^2\\&=\sin^2x\cosh^2y+\cos^2x\sinh^2y\\&=\sin^2x(1+\sinh^2y)+(1-\sin^2x)\sinh^2y\\&=\sin^2x+\sinh^2y\end{aligned}$$

Since x is real, $0\le\sin^2x\le1$. But $\sinh y=\frac{e^y-e^{-y}}{2}$ is not bounded.

$\Rightarrow$ $|\sin z|$ and hence $\sin z$ is not bounded.

Similarly, $|\cos z|^2=\cos^2x+\sinh^2y$

$\Rightarrow$ $|\cos z|$ and hence $\cos z$ is not bounded.

Hence for a complex variable z, $\sin z$ and $\cos z$ can have any value.

Remark. $|\sin z|\le 1$ and $|\cos z|\le 1$ only when z is real.

Example 11. *Find all values of z such that*

(i) $\sinh z = 0$ (ii) $\cosh z = 0.$ (M.D.U. Dec. 2010)

Sol. We know that

$$\sinh z=\sinh(x+iy)=\sinh x\cos y+i\cosh x\sin y$$

and $$\cosh z=\cosh(x+iy)=\cosh x\cos y+i\sinh x\sin y \qquad \text{[See Example 2 (a) and (b)]}$$

$$\begin{aligned}\therefore\quad |\sinh z|^2&=|\sinh x\cos y+i\cosh x\sin y|^2\\&=\sinh^2x\cos^2y+\cosh^2x\sin^2y\\&=\sinh^2x(1-\sin^2y)+(1+\sinh^2x)\sin^2y\\&=\sinh^2x+\sin^2y\end{aligned}$$

Now $\sinh z=0 \Rightarrow |\sinh z|=0$

$\Rightarrow$ $\sinh^2x+\sin^2y=0$

$\Rightarrow$ $\sinh x=0$ and $\sin y=0$

$\Rightarrow$ $x=0$ and $y=n\pi$, where n is any integer.

$\Rightarrow$ $z=x+iy=n\pi i$

$\therefore$ $\sinh z=0$ only when z is purely imaginary and $z=n\pi i$.

Similarly, $|\cosh z|^2=\sinh^2x+\cos^2y$

Now $\cosh z=0 \Rightarrow |\cosh z|=0$

$\Rightarrow$ $\sinh^2x+\cos^2y=0$

$\Rightarrow$ $\sinh x=0$ and $\cos y=0$

$\Rightarrow$ $x=0$ and $y=(2n+1)\frac{\pi}{2}$, where n is any integer.

$\Rightarrow$ $z=x+iy=(2n+1)\frac{\pi}{2}i$

$\therefore$ $\cosh z=0$ only when z is purely imaginary and $z=(2n+1)\frac{\pi}{2}i$.

Example 12. *Find all values of z such that sin z = 4.*

Sol. Let $z=x+iy$, then $\sin z=4$

$\Rightarrow$ $\sin(x+iy)=4$

$\Rightarrow$ $\sin x\cosh y+i\cos x\sinh y=4$

$\Rightarrow$ $\sin x\cosh y=4$ ...(1)

and $\cos x\sinh y=0$ ...(2)

by comparing real and imaginary parts.

From (2), we have $\cos x=0$ or $\sinh y=0$

$\Rightarrow$ $x=(2n+1)\frac{\pi}{2}$, where n is any integer or $y=0$

When $y=0$, from (1), we get

$\sin x=4$ $(\because \cosh 0=1)$

which is not possible since x is real.

When $x = (2n+1)\frac{\pi}{2}$, from (1), we get

$$\sin\left(n\pi + \frac{\pi}{2}\right)\cosh y = 4 \qquad \text{or} \qquad (-1)^n \sin\frac{\pi}{2}\cosh y = 4$$

or $$(-1)^n \cosh y = 4 \qquad ...(3)$$

When n is odd, $\cosh y = -4$ which is not possible since $\cosh y > 0$ for every y.

$\therefore$ n must be an even integer.

From (3) $\quad \cosh y = 4 \quad$ or $\quad y = \cosh^{-1} 4$

Hence $$z = x + iy$$

$$= (2n+1)\frac{\pi}{2} + i\cosh^{-1} 4, \qquad \text{where } n \text{ is an even integer}$$

or $$z = (4k+1)\frac{\pi}{2} + i\cosh^{-1} 4, \qquad \text{where } k \text{ is any integer.}$$

3.11. INVERSE HYPERBOLIC FUNCTIONS

1. Prove that $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

Proof. Let $\quad \sinh^{-1} x = y$, then $x = \sinh y$

$$\Rightarrow \qquad x = \frac{e^y - e^{-y}}{2} = \frac{e^{2y}-1}{2e^y} \quad \Rightarrow \quad e^{2y} - 2x\,e^y - 1 = 0$$

It is a quadratic in e^y

$$\therefore \qquad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Rejecting the negative sign, $e^y = x + \sqrt{x^2+1}$

Taking logarithms, $\quad y = \log(x + \sqrt{x^2+1}) \quad$ or $\quad \sinh^{-1} x = \log(x + \sqrt{x^2+1})$

2. Prove that $\cosh^{-1} x = \log(x + \sqrt{x^2-1})$

Proof. Let $\quad \cosh^{-1} x = y$ then $x = \cosh y$

or $$x = \frac{e^y + e^{-y}}{2} = \frac{e^{2y}+1}{2e^y} \quad \text{or} \quad e^{2y} - 2x\,e^y + 1 = 0$$

It is a quadratic in e^y

$$\therefore \qquad e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

Rejecting the negative sign $e^y = x + \sqrt{x^2-1}$

$$\therefore \qquad y = \log(x + \sqrt{x^2-1}) \quad \text{or} \quad \cosh^{-1} x = \log(x + \sqrt{x^2-1})$$

3. Prove that $\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}$

Proof. Let $\quad y = \tanh^{-1} x$, then $x = \tanh y \quad$ or $\quad \frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}} \quad$ or $\quad \frac{1}{x} = \frac{e^y + e^{-y}}{e^y - e^{-y}}$

By componendo and dividendo $\frac{1+x}{1-x} = \frac{e^y}{e^{-y}} = e^{2y}$

$$\therefore \qquad 2y = \log\frac{1+x}{1-x} \quad \text{or} \quad y = \frac{1}{2}\log\frac{1+x}{1-x}$$

$$\therefore \qquad \tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

4. Prove that:

(*i*) $\mathbf{\sinh^{-1} x + \sinh^{-1} y = \sinh^{-1}(x\sqrt{1+y^2} + y\sqrt{1+x^2})}$

(*ii*) $\mathbf{\cosh^{-1} x + \cosh^{-1} y = \cosh^{-1}(xy + \sqrt{(x^2-1)(y^2-1)})}$

Proof. (*i*) Let $\sinh^{-1} x = u \quad$ and $\quad \sinh^{-1} y = v$

so that $\quad x = \sinh u \quad$ and $\quad y = \sinh v$

Now $\quad \sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v$

$$= \sinh u\sqrt{1+\sinh^2 v} + \sinh v\sqrt{1+\sinh^2 u}$$

$$= x\sqrt{1+y^2} + y\sqrt{1+x^2}$$

$$\Rightarrow \qquad u + v = \sinh^{-1}(x\sqrt{1+y^2} + y\sqrt{1+x^2})$$

$$\Rightarrow \quad \sinh^{-1} x + \sinh^{-1} y = \sinh^{-1}(x\sqrt{1+y^2} + y\sqrt{1+x^2})$$

(*ii*) Let, $\quad \cosh^{-1} x = u \quad$ and $\quad \cosh^{-1} y = v$

so that $\quad x = \cosh u \quad$ and $\quad y = \cosh v$

Now, $\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v$

$$= \cosh u \cosh v + \sqrt{\cosh^2 u - 1}\,.\sqrt{\cosh^2 v - 1}$$

$$= xy + \sqrt{(x^2-1)(y^2-1)}$$

$$\Rightarrow \qquad u + v = \cosh^{-1}\left(xy + \sqrt{(x^2-1)(y^2-1)}\right)$$

$$\Rightarrow \quad \cosh^{-1} x + \cosh^{-1} y = \cosh^{-1}\left(xy + \sqrt{(x^2-1)(y^2-1)}\right)$$

Example 1. *Separate into real and imaginary parts*

(*i*) *$\sin^{-1}(\cos\theta + i\sin\theta)$, $0 < \theta < \frac{\pi}{2}$*

(*ii*) *$\tan^{-1}(x + iy)$.*

Sol. (*i*) Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

$$\therefore \qquad \cos\theta + i\sin\theta = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i\cos x \sinh y$$

Equating real and imaginary parts, we have

$$\cos\theta = \sin x \cosh y \quad ...(i)$$

and $$\sin\theta = \cos x \sinh y \quad ...(ii)$$

Squaring (i) and (ii) and adding, we have

$$1 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$= \sin^2 x (1 + \sinh^2 y) + (1 - \sin^2 x) \sinh^2 y = \sin^2 x + \sinh^2 y$$

or $$\sinh^2 y = 1 - \sin^2 x = \cos^2 x$$

$\therefore$ From (ii), $$\sin^2\theta = \cos^2 x \sinh^2 y = \cos^4 x \Rightarrow \cos^2 x = \sin\theta$$

$\Rightarrow$ $$\cos x = \sqrt{\sin\theta}$$

$\therefore$ Real part $$x = \cos^{-1}\sqrt{\sin\theta}$$

From (ii), $$\sinh y = \frac{\sin\theta}{\cos x} = \frac{\sin\theta}{\sqrt{\sin\theta}} = \sqrt{\sin\theta}$$

$\therefore$ $$y = \sinh^{-1}(\sqrt{\sin\theta})$$
$$= \log[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}] \quad \Big| \because \sinh^{-1} x = \log(x + \sqrt{x^2+1})$$

$\Rightarrow$ Imaginary part $y = \log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(ii) Let $$\tan^{-1}(x+iy) = u + iv \quad ...(i)$$

then $$\tan^{-1}(x-iy) = u - iv \quad ...(ii)$$

Adding (i) and (ii), we have

$$2u = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$$
$$= \tan^{-1}\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \tan^{-1}\frac{2x}{1-x^2-y^2}$$

$\therefore$ $$u = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$$

Subtracting (ii) from (i), we have

$$2iv = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$$
$$= \tan^{-1}\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \tan^{-1}\frac{2iy}{1+x^2+y^2}$$

$$\Rightarrow \tan 2iv = \frac{2iy}{1+x^2+y^2} \Rightarrow i\tanh 2v = \frac{2iy}{1+x^2+y^2} \Rightarrow v = \frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}.$$

Example 2. *Find all values of z such that* $\sinh z = e^{i\frac{\pi}{3}}$.

Sol. Let $z = x + iy$, then

$$\sinh z = e^{i\frac{\pi}{3}} \Rightarrow \sinh(x+iy) = e^{i\frac{\pi}{3}}$$

or $$\sinh x \cos y + i\cosh x \sin y = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Equating real and imaginary parts,

$$\sinh x \cos y = \frac{1}{2} \Rightarrow \sinh x = \frac{1}{2\cos y} \quad ...(1)$$

$$\cosh x \sin y = \frac{\sqrt{3}}{2} \Rightarrow \cosh x = \frac{\sqrt{3}}{2\sin y} \quad ...(2)$$

Using $\cosh^2 x - \sinh^2 x = 1$, we have

$$\frac{3}{4\sin^2 y} - \frac{1}{4\cos^2 y} = 1 \quad \text{or} \quad 3\cos^2 y - \sin^2 y = 4\sin^2 y\cos^2 y$$

or $$3(1-\sin^2 y) - \sin^2 y = 4\sin^2 y(1-\sin^2 y)$$

or $$4\sin^4 y - 8\sin^2 y + 3 = 0$$

or $$(2\sin^2 y - 3)(2\sin^2 y - 1) = 0$$

$\therefore$ $$\sin^2 y = \frac{3}{2} \quad \text{or} \quad \frac{1}{2}$$

Rejecting $\sin^2 y = \frac{3}{2}$, since $\sin^2 y \le 1$ for real y, we have

$$\sin^2 y = \frac{1}{2} \Rightarrow \sin y = \pm\frac{1}{\sqrt{2}}$$

When $\sin y$ is negative, from (2), $\cosh x$ is also negative which is impossible.
($\because$ $\cosh x > 0$ for all x)

$\therefore$ $$\sin y = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$\Rightarrow$ $$y = n\pi + (-1)^n\frac{\pi}{4}, \text{ where } n \text{ is an integer, is the general value of } y.$$

Case I. If n is even, $y = n\pi + \frac{\pi}{4}$

Taking $n = 2k$, $$\cos y = \cos\left(2k\pi + \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

From (1), $$\sinh x = \frac{1}{\sqrt{2}}$$

$$x = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) = \log\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2}+1}\right) = \log\frac{\sqrt{3}+1}{\sqrt{2}}$$

$\therefore$ $$z = x + iy = \log\frac{\sqrt{3}+1}{\sqrt{2}} + i\left(n\pi + \frac{\pi}{4}\right)$$

Case II. If n is odd, $y = n\pi - \frac{\pi}{4}$

Taking $n = 2k + 1$, $\cos y = \cos\left[(2k+1)\pi - \frac{\pi}{4}\right]$

$$= \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

From (1), $\sinh x = -\dfrac{1}{\sqrt{2}}$

$$x = \sinh^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \log\left(-\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2}+1}\right) = \log\frac{\sqrt{3}-1}{\sqrt{2}}$$

$$\therefore \quad z = x + iy = \log\frac{\sqrt{3}-1}{\sqrt{2}} + i\left(n\pi - \frac{\pi}{4}\right).$$

Example 3. *Show that* $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\text{cosec}\,\theta)$. (K.U.K. 2005)

Sol. Let $\tanh^{-1}(\cos\theta) = \phi$ then $\cos\theta = \tanh\phi$

$\Rightarrow \quad \cos^2\theta = \tanh^2\phi \quad \Rightarrow \quad 1 - \sin^2\theta = \tanh^2\phi$

$\Rightarrow \quad 1 - \tanh^2\phi = \sin^2\theta \quad \Rightarrow \quad \text{sech}^2\phi = \sin^2\theta$

$\Rightarrow \quad \text{sech}\,\phi = \sin\theta$

Taking reciprocals $\cosh\phi = \text{cosec}\,\theta$

$\Rightarrow \quad \phi = \cosh^{-1}(\text{cosec}\,\theta)$

$\Rightarrow \quad \tanh^{-1}(\cos\theta) = \cosh^{-1}(\text{cosec}\,\theta).$

EXERCISE 3.3

1. Prove that
   (i) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$; n being a positive integer.
   (ii) $\left(\dfrac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x.$
2. If $y = \log\tan x$, show that $\sinh ny = \dfrac{1}{2}(\tan^n x - \cot^n x)$.
3. (a) If $\tan y = \tan\alpha\tanh\beta$ and $\tan z = \cot\alpha\tanh\beta$, prove that $\tan(y+z) = \sinh 2\beta\,\text{cosec}\,2\alpha$.
   (b) Prove that:
   (i) $\overline{\sin z} = \sin\bar{z}$ (ii) $\overline{\cos z} = \cos\bar{z}$
   (iii) $\overline{\tan z} = \tan\bar{z}$ (M.D.U. Dec. 2009)
4. If $\tan\theta = \tanh x\cot y$ and $\tan\phi = \tanh x\tan y$, prove that $\dfrac{\sin 2\theta}{\sin 2\phi} = \dfrac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}$.
5. If $c\cosh(\theta + i\phi) = x + iy$, prove that
   (i) $x^2\,\text{sech}^2\theta + y^2\,\text{cosech}^2\theta = c^2$ (ii) $x^2\sec^2\phi - y^2\,\text{cosec}^2\phi = c^2$.
6. If $\tan(x+iy) = A + iB$, show that $\dfrac{A}{B} = \dfrac{\sin 2x}{\sinh 2y}$.
7. If $\sin(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$, prove that
   (i) $\rho^2 = \dfrac{1}{2}(\cosh 2\phi - \cos 2\theta)$ (ii) $\tan\alpha = \tanh\phi\cot\theta$.

8. If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that $\cos^2\theta = \pm\sin\alpha$.
9. If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that
   (i) $\sin^2\theta = \pm\sin\alpha$ (ii) $\cos 2\theta + \cosh 2\phi = 2$.
10. (a) If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that $\cos 2\theta\cosh 2\phi = 3$.
    (b) If $\tanh\alpha = \cos\theta$, prove that $\alpha = \log(\text{cosec}\,\theta + \cot\theta)$
    (c) Prove that $\log\left\{\dfrac{\cos(x-iy)}{\cos(x+iy)}\right\} = 2i\tan^{-1}(\tan x\tanh y)$
    (d) If $\sin x\cosh y = \cos\theta$ and $\cos x\sinh y = \sin\theta$, prove that
    (i) $\sinh^2 y = \cos^2 x$ (ii) $\cos^4 x = \sin^2\theta$
11. (a) If $\cos(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$, prove that $e^{2\phi} = \dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$. (K.U.K. 2005)
    (b) If $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, prove that $\theta = -i\log\tan\left(\dfrac{\pi}{4} + \dfrac{iu}{2}\right)$. (K.U.K. 2006)
12. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that $e^{2\phi} = \pm\cot\dfrac{\alpha}{2}$ and $2\theta = \left(n + \dfrac{1}{2}\right)\pi + \alpha$.
    (K.U.K. Dec. 2009, Dec. 2010)
13. If $\tan(x+iy) = \sin(u+iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tanh v}$. (S.V.T.U. 2006)
14. If $\tan(x+iy) = \cosh(\alpha + i\beta)$, prove that $\tanh\alpha\tan\beta = \text{cosec}\,2x\sinh 2y$.
15. If $C\tan(x+iy) = A + iB$, prove that $\tan 2x = \dfrac{2CA}{C^2 - A^2 - B^2}$. (M.D.U. Dec. 2011)
16. (a) Prove that $(1 + \cosh x + \sinh x)^n = 2^n\cosh^n\dfrac{x}{2}\left(\cosh\dfrac{nx}{2} + \sinh\dfrac{nx}{2}\right)$.
    (b) Prove that $\sinh^{-1}(\tan\theta) = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$
17. If $\cosh x = \sec\theta$, prove that
    (i) $\tanh^2\dfrac{x}{2} = \tan^2\dfrac{\theta}{2}$ (M.D.U. Dec., 2008) (ii) $x = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$
18. If $\tan\dfrac{x}{2} = \tanh\dfrac{u}{2}$, prove that
    (i) $\cos x\cosh u = 1$ (ii) $\tan x = \sinh u$
    (iii) $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$.
19. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \dfrac{4x}{x^2 + y^2}$.
20. If $\sin[\log(A + iB)] = x + iy$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.
    (M.D.U. May 2011)
21. Separate into real and imaginary parts:
    (i) $e^{\cosh(x+iy)}$ (ii) $\sin^2(x+iy)$ (iii) $\log\cos(x+iy)$.

22. If $\tan(x+iy) = \theta + i\phi$, prove that $\theta^2 + \phi^2 = \dfrac{\cosh^2 y - \cos^2 x}{\cosh^2 y - \sin^2 x}$.

23. Prove that: $\tan \dfrac{u+iv}{2} = \dfrac{\sin u + i \sinh v}{\cos u + \cosh v}$.

24. If $x + iy = \cos(u + iv)$, show that

(i) $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$ (ii) $(1-x)^2 + y^2 = (\cosh v - \cos u)^2$.

25. (a) If $\cos^{-1}(u+iv) = \alpha + i\beta$, prove that $\cos^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - (1 + u^2 + v^2)\,x + u^2 = 0.$$

(b) If $\sin^{-1}(u+iv) = \alpha + i\beta$, prove that $\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - x\,(1 + u^2 + v^2) + u^2 = 0.$$

26. Prove that $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2} \log \tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$. (M.D.U. Dec. 2008)

27. Find $\tanh x$ if $5 \sinh x - \cosh x = 5$.

[**Hint.** Divide both sides by $\cosh x$, square, replace $\operatorname{sech}^2 x$ by $(1 - \tanh^2 x)$ and solve for $\tanh x$]

28. If $\cos^{-1}(x + iy) = \alpha + i\beta$, show that

(i) $x^2 \sec^2 \alpha - y^2 \operatorname{cosec}^2 \alpha = 1$ (ii) $x^2 \operatorname{sech}^2 \beta + y^2 \operatorname{cosech}^2 \beta = 1$.

29. Find all the roots of the equation:

(i) $\cos z = 2$ (ii) $\tanh z + 2 = 0$

(iii) $\sin z = \cosh 4$ (iv) $\sinh z = i$.

30. If $|\cos(u + iv)| = 1$, show that $\sin^2 u = \sinh^2 v$.

31. If $\log \sin(\theta + i\phi) = \alpha + i\beta$, prove that

(i) $2 \cos 2\theta = e^{2\phi} + e^{-2\phi} - 4e^{2\alpha}$ (ii) $\cos(\theta - \beta) = e^{2\phi} \cos(\theta + \beta)$.

32. Solve $\tan z = e^{i\alpha}$, where α is real. (M.D.U. May 2009)

33. If $\cosh^{-1}(x + iy) + \cosh^{-1}(x - iy) = \cosh^{-1} a$, prove that $2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1$.

Answers

21. (i) $e^{\cosh x \cos y}\,[\cos(\sinh x \sin y) + i \sin(\sinh x \sin y)]$

(ii) $\dfrac{1}{2}\,[(1 - \cos 2x \cosh 2y) + i \sin 2x \sinh 2y]$

(iii) $\dfrac{1}{2} \log\left[\dfrac{1}{2}(\cos 2x + \cosh 2y)\right] - i \tan^{-1}(\tan x \tanh y)$

27. $\dfrac{4}{5}, -\dfrac{3}{5}$

29. (i) $2n\pi + i \log(2 + \sqrt{3})$

(ii) $-\dfrac{1}{2} \log 3 + i\left(n + \dfrac{1}{2}\right)\pi$

(iii) $n\pi + (-1)^n \left(\dfrac{\pi}{2} - 4i\right)$

(iv) $i\,(4n + 1)\,\dfrac{\pi}{2}$

32. $z = \dfrac{n\pi}{2} + \dfrac{\pi}{4} + \dfrac{i}{2} \log \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$ [**Hint.** See Example 7.]

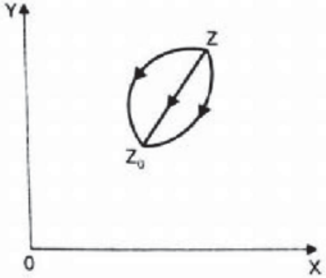
3.12. LIMIT OF A FUNCTION

(P.T.U. May 2006)

Let $f(z)$ be a single-valued function of z defined in a neighbourhood of $z = z_0$, then $f(z)$ is said to have the limit l as z approaches z_0 (along any path, straight or curved) if given an arbitrary real number $\varepsilon > 0$, however small, there exists a real number $\delta > 0$ such that

$$|f(z) - l| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

and we write $\lim\limits_{z \to z_0} f(z) = l$



Remark 1. δ usually depends on ε.

Remark 2. In real variables, $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path, straight or curved. The limit must be independent of the manner in which z approaches z_0. If we get two different limits as $z \to z_0$ along two different paths then limit does not exist.

3.13. THEOREMS ON LIMITS

1. If $\lim\limits_{z \to z_0} f(z)$ exists, then it is unique.

2. If $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and $z_0 = x_0 + iy_0$, then $\lim\limits_{z \to z_0} f(z) = l = u_0 + iv_0$

if and only if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0.$$

3. If $\lim\limits_{z \to z_0} f(z) = l$ and c is a constant, real or complex, then $\lim\limits_{z \to z_0} cf(z) = c \lim\limits_{z \to z_0} f(z) = cl$.

4. If $\lim\limits_{z \to z_0} f(z) = l_1$ and $\lim\limits_{z \to z_0} g(z) = l_2$, then

(i) $\lim\limits_{z \to z_0} [f(z) + g(z)] = \lim\limits_{z \to z_0} f(z) + \lim\limits_{z \to 0} g(z) = l_1 + l_2$

(ii) $\lim\limits_{z \to z_0} [f(z) - g(z)] = \lim\limits_{z \to z_0} f(z) - \lim\limits_{z \to z_0} g(z) = l_1 - l_2$

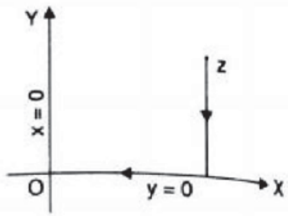
(iii) $\lim\limits_{z \to z_0} [f(z)g(z)] = \lim\limits_{z \to z_0} f(z) \,.\, \lim\limits_{z \to z_0} g(z) = l_1 l_2$

(iv) $\lim\limits_{z \to z_0} \dfrac{1}{g(z)} = \dfrac{1}{\lim\limits_{z \to z_0} g(z)} = \dfrac{1}{l_2}$ provided $l_2 \neq 0$

(v) $\lim\limits_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim\limits_{z \to z_0} f(z)}{\lim\limits_{z \to z_0} g(z)} = \dfrac{l_1}{l_2}$ provided $l_2 \neq 0$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $\lim_{z \to 0} \frac{\bar{z}}{z}$ *does not exist.*

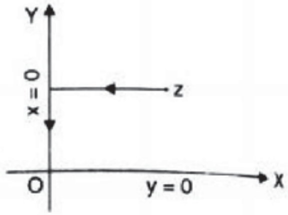


Sol. If the limit exists, then it must be independent of the manner in which z approaches 0.

Consider the path $y \to 0$ followed by $x \to 0$, we get

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0}\left[\lim_{y \to 0} \frac{x - iy}{x + iy}\right] = \lim_{x \to 0} \frac{x}{x} = 1$$

Now consider the path $x \to 0$ followed by $y \to 0$, we get



$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0}\left[\lim_{x \to 0} \frac{x - iy}{x + iy}\right] = \lim_{y \to 0} \frac{-iy}{iy}$$

$$= -1$$

As $z \to 0$ along two different paths, we get different limits. Hence the limit does not exist.

Example 2. *Show that* $\lim_{z \to 0} \frac{(Re\, z - Im\, z)^2}{|z|^2}$ *does not exist.*

Sol. Let $z \to 0$ along the path $y = mx$, then

$$\lim_{z \to 0} \frac{(\text{Re } z - \text{Im } z)^2}{|z|^2} = \lim_{z \to 0} \frac{(x - y)^2}{x^2 + y^2} = \lim_{x \to 0} \frac{(x - mx)^2}{x^2 + m^2x^2}$$

$$= \lim_{x \to 0} \frac{(1 - m)^2 x^2}{(1 + m^2)x^2} = \frac{(1 - m)^2}{1 + m^2}$$

which depends on m. For different values of m, we have different paths and different limits. Hence the limit does not exist.

Example 3. *Prove that* $\lim_{z \to 1+i} \left(\frac{z - 1 - i}{z^2 - 2z + 2}\right)^2 = -\frac{1}{4}$.

Sol. $$\lim_{z \to 1+i} \left(\frac{z - 1 - i}{z^2 - 2z + 2}\right)^2 = \lim_{z \to 1+i} \left[\frac{z - 1 - i}{(z - 1)^2 - i^2}\right]^2 = \lim_{z \to 1+i} \left[\frac{z - 1 - i}{(z - 1 + i)(z - 1 - i)}\right]^2$$

$$= \lim_{z \to 1+i} \frac{1}{(z - 1 + i)^2} = \frac{1}{\lim_{z \to 1+i} (z - 1 + i)^2}$$

$$= \frac{1}{(1 + i - 1 + i)^2} = \frac{1}{4i^2} = -\frac{1}{4}.$$

EXERCISE 3.4

Show that the following limits do not exist:

1. $\lim_{z \to 0} \frac{z}{|z|}$

2. $\lim_{z \to 0} \frac{z^2}{|z|^2}$

3. $\lim_{z \to 0} \frac{\text{Re } z}{|z|}$

4. $\lim_{z \to 0} \frac{\text{Im}(z)^2}{|z|^2}$

Find the following limits:

5. $\lim_{z \to 1-i} (z^2 + 4z - 7)$

6. $\lim_{z \to -2i} \frac{(2z + 3)(z - 1)}{z^2 - 2z + 4}$

7. $\lim_{z \to -i} (2x^3 - 5y^3 i)$

8. $\lim_{z \to 1+i} \frac{z^3 - 1}{z^2 - 1}$

9. $\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

10. $\lim_{z \to 1-i} (z^2 - \bar{z}^2)$

11. $\lim_{z \to \infty} \frac{iz^3 + iz - 1}{(2z + 3i)(z - i)^2}$. (*P.T.U. May 2007*)

Answers

5. $-3 - 6i$ 6. $-\frac{1}{2} + \frac{11}{4}i$ 7. $5i$
8. $\frac{7}{5} + \frac{4}{5}i$ 9. $1 - \frac{1}{2}i$ 10. $-4i$ 11. $\frac{i}{2}$.

3.14. CONTINUITY OF A FUNCTION

Let $f(z)$ be a single-valued function of z defined in a neighbourhood of $z = z_0$, then $f(z)$ is said to continuous at $z = z_0$ if given an arbitrary real number $\varepsilon > 0$, however small, there exists a real number $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

In other words, $f(z)$ is continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

Thus three conditions must be satisfied so that $f(z)$ is continuous at $z = z_0$

1. $f(z)$ is defined at z_0 *i.e.*, $f(z_0)$ exists
2. $\lim_{z \to z_0} f(z)$ exists
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

If any of the above conditions is not satisfied then $f(z)$ is said to be *discontinuous* at $z = z_0$.

3.15. REMOVABLE DISCONTINUITY

If $f(z_0)$ exists and $\lim_{z \to z_0} f(z) = l$ exists but $f(z_0) \neq l$, then $z = z_0$ is called a point of *removable discontinuity*.

By redefining $f(z)$ at $z = z_0$ such that $f(z_0) = l$, the function can be made continuous. If $\lim_{z \to z_0} f(z)$ does not exist, discontinuity cannot be removed. Existence of limit is the necessary condition for removable discontinuity.

3.16. CONTINUITY IN A REGION

A function $f(z)$ is said to be *continuous in a region* if it is continuous at all points of the region.

Remark 1. To examine the continuity of $f(z)$ at $z = \infty$, we put $z = \frac{1}{w}$ and examine the continuity of $f\left(\frac{1}{w}\right)$ at $w = 0$.

Remark 2. If $f(z)$ and $g(z)$ are continuous at $z = z_0$, then so also are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)\,g(z)$ and $\frac{f(z)}{g(z)}$, where $g(z_0) \neq 0$.

Remark 3. If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in the region.

If $f(z) = u(x, y) + iv(x, y)$, then $f(z)$ is continuous if and only if $u(x, y)$ and $v(x, y)$ are separately continuous functions of x and y.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the function $f(z)$ defined by*

$$f(z) = \begin{cases} \dfrac{Re(z^2)}{|z|^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not continuous at $z = 0$.

Sol. Given $f(0) = 0$

Let $z \to 0$ along the path $y = mx$, then

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\text{Re}(z^2)}{|z|^2} = \lim_{z \to 0} \frac{x^2 - y^2}{x^2 + y^2} \qquad [\because\ z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy]$$

$$= \lim_{x \to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{(1 - m^2)x^2}{(1 + m^2)x^2} = \frac{1 - m^2}{1 + m^2}$$

which depends on m. For different values of m, we have different paths and different limits. Hence the limit does not exist and the function is not continuous at $z = 0$.

Example 2. *Find the value of $f(z_0)$ so that the function*

$$f(z) = \frac{z^2 - 2z + 2}{z^2 + 2i} \text{ is continuous at } z_0 = 1 - i.$$

Sol.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 1-i} \frac{z^2 - 2z + 2}{z^2 + 2i} = \lim_{z \to 1-i} \frac{(z - 1)^2 + 1}{z^2 - (1 - i)^2}$$

$$= \lim_{z \to 1-i} \frac{(z - 1)^2 - i^2}{z^2 - (1 - i)^2} = \lim_{z \to 1-i} \frac{(z - 1 + i)(z - 1 - i)}{(z + 1 - i)(z - 1 + i)}$$

$$= \lim_{z \to 1-i} \frac{z - 1 - i}{z + 1 - i} = \frac{\lim_{z \to 1-i} (z - 1 - i)}{\lim_{z \to 1-i} (z + 1 - i)}$$

$$= \frac{1 - i - 1 - i}{1 - i + 1 - i} = \frac{-2i}{2(1 - i)} = \frac{-i}{1 - i} \times \frac{1 + i}{1 + i} = \frac{1 - i}{2}$$

Now $f(z)$ will be continuous at z_0 if

$$f(z_0) = \lim_{z \to z_0} f(z) = \frac{1 - i}{2}.$$

EXERCISE 3.5

1. Examine the continuity of the following functions:

(*i*) $f(z) = \begin{cases} \dfrac{z^2 + 1}{z + i}, & z \neq -i \\ 0, & z = -i \end{cases}$ at $z = -i$ (*ii*) $f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z - i}, & z \neq i \\ 0, & z = i \end{cases}$ at $z = i$

(*iii*) $f(z) = \begin{cases} \dfrac{z^2 + 4}{z - 2i}, & z \neq 2i \\ 2 + 3i, & z = 2i \end{cases}$ at $z = 2i$.

2. Show that the following functions are continuous for all z.

(*i*) $\sin z$ (*ii*) e^z.

[**Hint:** Express $f(z)$ as $u(x, y) + iv(x, y)$ and show that $u(x, y)$ and $v(x, y)$ are continuous for all real values of x and y.]

3. Show that the function

$$f(z) = \begin{cases} \dfrac{\text{Im}(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not continuous at $z = 0$. (*M.D.U. Dec. 2005*)

4. Find the value of $f(i)$ so that the function

$$f(z) = \frac{iz^3 - 1}{z - i} \text{ is continuous at } z = i.$$

Answers

1. (*i*) Not continuous (*ii*) Continuous (*iii*) Not continuous 4. $-3i$.

3.17. DIFFERENTIABILITY

Let $w = f(z)$ be a single-valued function of the complex variable $z (= x + iy)$.

Let $w + \Delta w = f(z + \Delta z)$, then

$$\Delta w = f(z + \Delta z) - f(z)$$

and

$$\frac{\Delta w}{\Delta z} = \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z},$$

if it exists, is called the derivative of $f(z)$ and is denoted by $f'(z)$.

Thus $$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

The function $f(z)$ is said to be differentiable at $z = z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

This limit is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$.

Thus $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Equivalently, by putting $z - z_0 = \Delta z$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

3.18. ANALYTIC FUNCTION

(K.U.K. Dec., 2010; M.D.U. 2006, 2007, May 2008; P.T.U. May 2007; U.P.T.U. 2006)

If a single-valued function $f(z)$ possesses a unique derivative at every point of a region R, then $f(z)$ is called an **analytic function** or a **regular function** or a **holomorphic function** of z in R.

A point where the function ceases to be analytic is called a singular point.

3.19. NECESSARY AND SUFFICIENT CONDITIONS FOR f(z) TO BE ANALYTIC

The necessary and sufficient conditions for the functions

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are continuous functions of x and y in the region R.*

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The conditions in (ii) are known as **Cauchy-Riemann equations** *or briefly C-R equations.* (K.U.K. Dec. 2010 ; M.D.U. 2006, 2007, 2008; U.P.T.U. 2008)

Proof. (a) **Necessary Condition.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R, then $\frac{dw}{dz} = f'(z)$ exists uniquely at every point of that region.

Let δx and δy be the increments in x and y respectively. Let δu, δv and δz be the corresponding increments in u, v and z respectively. Then,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad ...(1)$$

Since the function $w = f(z)$ is analytic in the region R, the limit (1) must exist independent of the manner in which $\delta z \to 0$, *i.e.*, along whicheve- path δx and $\delta y \to 0$.

First, let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$.

[since $z = x + iy$, $z + \delta z = (x + \delta x) + i(y + \delta y)$ and $\delta z = \delta x + i\delta y$]

$\therefore$ From (1), $$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad ...(2)$$

Now, let $\delta z \to 0$ along a line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\,\delta y$.

$\therefore$ From (1), $$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\,\delta y} + i\frac{\delta v}{i\,\delta y}\right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad ...(3) \quad \left[\because \frac{1}{i} = -i\right]$$

From (2) and (3), we have $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Equating the real and imaginary parts, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Hence the necessary condition for f(z) to be analytic is that the C-R equations must be satisfied.

(b) **Sufficient Condition.** Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic, *i.e.*, $f'(z)$ exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= \left[u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\right] + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\right]$$

$$= [u(x, y) + iv(x, y)] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

or $$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y \quad \text{[Using C-R equations]}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\delta y \quad [\because -1 = i^2]$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \qquad [\because \ \delta x + i\delta y = \delta z]$$

$$\Rightarrow \qquad \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \qquad f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Note 1. The real and imaginary parts of an analytic function are called **conjugate functions.** Thus, if $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then $u(x, y)$ and $v(x, y)$ are conjugate functions. The relation between two conjugate functions is given by C-R equations.

Note 2. When a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary way as if z is a real variable.

Thus, $\quad f(z) = z^2 \Rightarrow f'(z) = 2z$

$\quad f(z) = \sin z \Rightarrow f'(z) = \cos z$ etc.

3.20. CAUCHY-RIEMANN EQUATIONS IN POLAR COORDINATES

(V.T.U. 2006; M.D.U. 2005; K.U.K. Dec. 2009; U.P.T.U. 2008)

Let (r, θ) be the polar coordinates of the point whose cartesian coordinates are (x, y), then

$$x = r\cos\theta, \ y = r\sin\theta,$$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore \qquad u + iv = f(z) = f(re^{i\theta}) \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. r, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. θ, we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad \text{[Using (2)]}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

or $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$ which is the polar form of C-R equations.

3.21. HARMONIC FUNCTIONS

(P.T.U. May 2007; U.P.T.U. 2007; M.D.U. May 2005)

Any solution of the Laplace's equation, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called a harmonic function.

Let $f(z) = u + iv$ be analytic in some region of the z-plane, then u and v satisfy C-R equations.

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{...(1)}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. x and (2) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{...(3)}$$

and $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \text{...(4)}$$

Assuming $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(5)}$$

Now, differentiating (1) partially w.r.t. y and (2) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \text{...(6)}$$

and $$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \qquad \text{...(7)}$$

Assuming $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and subtracting (7) from (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{...(8)}$$

Equations (5) and (8) show that the real and imaginary parts u and v of an analytic function satisfy the Laplace's equation.

Hence u and v are known as harmonic functions.

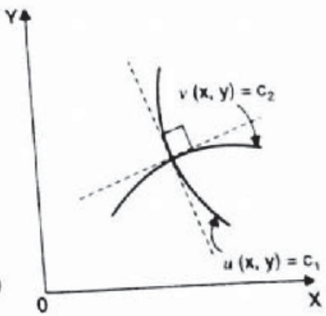
3.22. ORTHOGONAL SYSTEM

Every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system. (U.P.T.U. 2009)

Consider the two families of curves

$$u(x, y) = c_1 \qquad \text{...(1)}$$

and $$v(x, y) = c_2 \qquad \text{...(2)}$$



Differentiating (1) w.r.t. x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (say)}$$

Similarly, from (2), we get $$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2 \text{ (say)}$$

$$\therefore \qquad m_1 m_2 = \frac{\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial v}{\partial x}}{\dfrac{\partial u}{\partial y} \cdot \dfrac{\partial v}{\partial y}} \qquad ...(3)$$

Since $f(z)$ is analytic, u and v satisfy C-R equations

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \text{ From (3), } \qquad m_1 m_2 = \frac{\dfrac{\partial v}{\partial y} \cdot \dfrac{\partial v}{\partial x}}{-\dfrac{\partial v}{\partial x} \cdot \dfrac{\partial v}{\partial y}} = -1$$

Thus the product of the slopes of the curves (1) and (2) is -1. Hence the curves intersect at right angles, *i.e.*, they form an orthogonal system.

3.23. APPLICATION OF ANALYTIC FUNCTIONS TO FLOW PROBLEMS

Since the real and imaginary parts of an analytic function satisfy the Laplace's equation in two variables, these conjugate functions provide solutions to a number of field and flow problems.

For example, consider the two dimensional irrotational motion of an incompressible fluid, in planes parallel to xy-plane.

Let V be the velocity of a fluid particle, then it can be expressed as

$$\mathrm{V} = v_x \hat{i} + v_y \hat{j} \qquad ...(1)$$

Since the motion is irrotational, there exists a scalar function $\phi(x, y)$, such that

$$\mathrm{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} \qquad ...(2)$$

From (1) and (2), we have $$v_x = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial \phi}{\partial y} \qquad ...(3)$$

The scalar function $\phi(x, y)$, which gives the velocity components, is called the **velocity potential function** or simply the **velocity potential.**

Also the fluid being incompressible, div V = 0

$$\Rightarrow \qquad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot (v_x \hat{i} + v_y \hat{j}) = 0$$

$$\Rightarrow \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad ...(4)$$

Substituting the values of v_x and v_y from (3) in (4), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0 \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thus the function ϕ is harmonic and can be treated as real part of an analytic function

$$w = f(z) = \phi(x, y) + i\,\psi(x, y)$$

For interpretation of conjugate function $\psi(x, y)$, the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial \psi}{\partial x}}{\dfrac{\partial \psi}{\partial y}} = \frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial x}} \qquad \text{[By C-R equations]}$$

$$= \frac{v_y}{v_x} \qquad \text{[By (3)]}$$

This shows that the resultant velocity $\sqrt{v_x^2 + v_y^2}$ of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$ *i.e.*, the fluid particles move along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function*. The curves represented by $\phi(x, y) = c$ are called *equipotential lines*.

Since $\phi(x, y)$ and $\psi(x, y)$ are conjugate functions of analytic function $w = f(z)$, the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$, intersect each other orthogonally.

Now, $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \qquad \text{[By C-R equations]}$$

$$= v_x - iv_y \qquad \text{[By (3)]}$$

$$\therefore \quad \text{The magnitude of resultant velocity} = \left|\frac{dw}{dz}\right| = \sqrt{v_x^{\,2} + v_y^{\,2}}$$

The function $w = f(z)$ which fully represents the flow pattern is called the *complex potential*.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are called *equipotential lines* and *lines of force* respectively. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

ILLUSTRATIVE EXAMPLES

Example 1. *Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.*

Sol. Let $f(z) = u + iv$, then $u = r^2 \cos 2\theta$, $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \quad \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \qquad 2r\cos 2\theta = pr\cos p\theta \quad \text{and} \quad 2r \sin p\theta = 2r\sin 2\theta$$

Both these equations are satisfied if $p = 2$.

Note. For a function $f(z)$ to be analytic, the first order partial derivatives of u and v must be continuous in addition to C-R equations.

Example 2. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, even though Cauchy-Riemann equations are satisfied there at.* (U.P.T.U. 2005)

Sol. Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Clearly, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied at the origin.

Now $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence the function $f(z)$ is not analytic at the origin.

Example 3. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \; z \neq 0 \text{ and } f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. (M.D.U. Dec. 2009, May 2011)

Sol. Here $$f(z) = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, \; z \neq 0$$

Let $f(z) = u + iv = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}$, then $u = \frac{x^3 - y^3}{x^2 + y^2}$, $v = \frac{x^3 + y^3}{x^2 + y^2}$

Since $z \neq 0 \Rightarrow x^2 + y^2 \neq 0$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus, u, v and hence *$f(z)$ are continuous functions when $z \neq 0$.* To test them for continuity at $z = 0$, on changing u, v to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$u = r(\cos^3\theta - \sin^3\theta) \text{ and } v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$$\therefore \quad \lim_{z \to 0} u = \lim_{r \to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

Similarly, $\lim_{z \to 0} v = 0$

$$\therefore \quad \lim_{z \to 0} f(z) = 0 = f(0)$$

$\Rightarrow$ **f(z) is continuous at z = 0.**

Hence $f(z)$ is continuous for all values of z.

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{y - 0}{y} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now, $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{0 + 2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2} \quad \ldots(1)$$

Also, let $z \to 0$ along the x-axis (*i.e.*, $y = 0$), then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^3} = 1 + i \quad \ldots(2)$$

Since the limits (1) and (2) are different, $f'(0)$ does not exist.

Example 4. *Prove that the function $\sinh z$ is analytic and find its derivative.* (P.T.U. 2007)

Sol. Here $f(z) = u + iv = \sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$

$\therefore$ $u = \sinh x \cos y$ and $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \; \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y, \; \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic every where.

Now,
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \cosh x \cos y + i \sinh x \sin y = \cosh (x + iy) = \cosh z.$$

Example 5. *Determine the analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$.*

(Bombay, 2005; M.D.U. 2005, 2007, Dec. 2011)

Sol. Let $f(z) = u + iv$ be the analytic function, where $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$\therefore \quad \frac{\partial u}{\partial x} = 2e^{2x}(x \cos 2y - y \sin 2y) + e^{2x}\cos 2y$$
$$= e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y) \quad \ldots(1)$$
$$\frac{\partial u}{\partial y} = e^{2x}(-2x \sin 2y - \sin 2y - 2y \cos 2y)$$
$$= -e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y) \quad \ldots(2)$$

Since $f(z)$ is analytic, u and v must satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y)$$

Integrating w.r.t. y, treating x as constant, we get

$$v = e^{2x}\left[2x \cdot \frac{\sin 2y}{2} - \left\{2y\left(-\frac{\cos 2y}{2}\right) - (2)\left(-\frac{\sin 2y}{4}\right)\right\} + \frac{\sin 2y}{2}\right] + \phi(x)$$
$$= e^{2x}(x \sin 2y + y \cos 2y) + \phi(x) \quad \ldots(3)$$

where $\phi(x)$ is an arbitrary function of x.

$$\therefore \quad \frac{\partial v}{\partial x} = 2e^{2x}(x \sin 2y + y \cos 2y) + e^{2x}(\sin 2y) + \phi'(x)$$
$$= e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y) + \phi'(x)$$

But
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y) \quad \text{[From (2)]}$$

$\therefore \quad \phi'(x) = 0$

$\Rightarrow \quad \phi(x) = c$, an arbitrary constant.

$\therefore$ From (3), $\quad v = e^{2x}(x \sin 2y + y \cos 2y) + c$

$$f(z) = u + iv = e^{2x}(x \cos 2y - y \sin 2y) + ie^{2x}(x \sin 2y + y \cos 2y) + ic$$
$$= e^{2x}[(x + iy)\cos 2y + i(x + iy)\sin 2y] + ic$$
$$= (x + iy)e^{2x}(\cos 2y + i \sin 2y) + ic$$
$$= ze^{2x} \cdot e^{2iy} + ic = ze^{2(x+iy)} + ic = ze^{2z} + ic.$$

(Milne-Thomson's Method)

This method determines the analytic function $f(z)$ when u or v is given.

Since $\quad z = x + iy,\ \bar{z} = x - iy \quad$ so that $x = \frac{z + \bar{z}}{2},\ y = \frac{z - \bar{z}}{2i}$

Let
$$f(z) = u(x, y) + iv(x, y) \quad \ldots(1)$$
$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Considering this as an identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0)$$

which is the same as (1) if we replace x by z and y by 0.

Thus to express any function in terms of z, replace x by z and y by 0.

Now $\quad f(z) = u + iv$

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{[C-R equations]}$$
$$= e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y)$$
$$+ ie^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y)$$

On replacing x by z and y by 0 on R.H.S., we get $f'(z) = e^{2z}(2z + 1)$

Integrating w.r.t. z, we have $f(z) = (2z + 1)\frac{e^{2z}}{2} - 2 \cdot \frac{e^{2z}}{4} + ic = ze^{2z} + ic$,

taking the constant of integration as imaginary since u does not contain any constant.

Example 6. *Determine the analytic function $w = u + iv$, if $v = \log (x^2 + y^2) + x - 2y$.*

Sol. Here $\quad v = \log (x^2 + y^2) + x - 2y$

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1$$
$$\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$$

Since $\quad w = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{[C-R equations]}$$
$$= \left(\frac{2y}{x^2 + y^2} - 2\right) + i\left(\frac{2x}{x^2 + y^2} + 1\right)$$

Replacing x by z and y by 0, we get $\frac{dw}{dz} = -2 + i\left(\frac{2}{z} + 1\right) = (i - 2) + \frac{2i}{z}$

Integrating w.r.t. z, we have $\quad w = (i - 2)z + 2i \log z + c$.

Example 7. *Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express u + iv as an analytic function of z.*

Sol. Here $u = e^{-2xy}\sin(x^2 - y^2)$

$$\therefore \quad \frac{\partial u}{\partial x} = -2y\,e^{-2xy}\sin(x^2 - y^2) + 2x\,e^{-2xy}\cos(x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2 e^{-2xy}\sin(x^2 - y^2) - 4xy\,e^{-2xy}\cos(x^2 - y^2) + 2e^{-2xy}\cos(x^2 - y^2) - 4xy\,e^{-2xy}\cos(x^2 - y^2) - 4x^2 e^{-2xy}\sin(x^2 - y^2) \quad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -2x\,e^{-2xy}\sin(x^2 - y^2) - 2y\,e^{-2xy}\cos(x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2 e^{-2xy}\sin(x^2 - y^2) + 4xy\,e^{-2xy}\cos(x^2 - y^2) - 2e^{-2xy}\cos(x^2 - y^2) + 4xy\,e^{-2xy}\cos(x^2 - y^2) - 4y^2 e^{-2xy}\sin(x^2 - y^2) \quad \ldots(2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ which proves that } u \text{ is harmonic.}$$

Now, let $f(z) = u + iv$

then $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ [C-R equations]

$$= [-2y\,e^{-2xy}\sin(x^2 - y^2) + 2x\,e^{-2xy}\cos(x^2 - y^2)] + i\,[2x\,e^{-2xy}\sin(x^2 - y^2) + 2y\,e^{-2xy}\cos(x^2 - y^2)]$$

Replacing x by z and y by 0, we get

$$f'(z) = 2z\cos z^2 + 2iz\sin z^2 = 2z(\cos z^2 + i\sin z^2) = 2z\,e^{iz^2}$$

Integrating w.r.t. z, we have

$$f(z) = -i\,e^{iz^2} + ic$$

which expresses $u + iv$ as an analytic function of z.

Since $u + iv = -i\,e^{iz^2} + ic = -i\,e^{i(x+iy)^2} + ic$

$$= -i\,e^{i(x^2 - y^2 + 2ixy)} + ic = -ie^{-2xy}\,.\,e^{i(x^2 - y^2)} + ic$$

$$= -ie^{-2xy}[\cos(x^2 - y^2) + i\sin(x^2 - y^2)] + ic$$

$$= e^{-2xy}\sin(x^2 - y^2) + i\,[-e^{-2xy}\cos(x^2 - y^2)] + ic$$

$$\therefore \quad v = -e^{-2xy}\cos(x^2 - y^2) + c.$$

Example 8. *An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.*

Sol. Let $\psi(x, y)$ be a stream function.

By C-R equations $\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y} = -3x^2 + 3y^2$ ...(1)

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 6xy \quad \ldots(2)$$

Integrating (1) w.r.t. x, treating y as constant, we get

$$\psi = -x^3 + 3xy^2 + F(y)$$

so that $\dfrac{\partial \psi}{\partial y} = 6xy + F'(y)$ ...(3)

From (2) and (3), $6xy + F'(y) = 6xy$ or $F'(y) = 0 \quad \therefore \quad F(y) = c$

Hence $\psi = -x^3 + 3xy^2 + c.$

Example 9. *If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Sol. We have $u - v = (x - y)(x^2 + 4xy + y^2)$

$$= x^3 + 3x^2y - 3xy^2 - y^3$$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \quad \ldots(1)$$

and $\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2$

or $-\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2$ ...(2)

Subtracting (2) from (1)

$$2\frac{\partial u}{\partial x} = 12xy \quad \text{or} \quad \frac{\partial u}{\partial x} = 6xy$$

Adding (1) and (2)

$$-2\frac{\partial v}{\partial x} = 6x^2 - 6y^2 \quad \text{or} \quad \frac{\partial v}{\partial x} = 3y^2 - 3x^2$$

Thus $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 6xy + i(3y^2 - 3x^2)$

Replacing x by z and y by 0, we get

$$f'(z) = -3iz^2$$

Integrating $f(z) = -iz^3 + c.$

Example 10. *If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Sol. Given $f(z) = u + iv$ ...(1)

$\Rightarrow \quad i\,f(z) = iu - v$ ...(2)

Adding (1) and (2), we get

$$(1 + i)\,f(z) = (u - v) + i(u + v)$$

or $F(z) = U + iV$

where $F(z) = (1 + i)\,f(z)$, $U = u - v$ and $V = u + v$

$$\therefore \quad V = u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} \quad \text{(Given)}$$

$$= \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Now, we proceed to find F(z) whose imaginary part is given.

$$\frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial V}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Since $f(z)$ is analytic, so is F(z).

$$\therefore \quad F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} \quad \text{by C-R equations}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, replacing x by z and y by 0 on R.H.S., we get

$$F'(z) = i\,\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} = \frac{-2i}{1 - \cos 2z} = \frac{-2i}{2\sin^2 z}$$

$$= -i\,\text{cosec}^2 z$$

Integrating w.r.t.z, we have

$$F(z) = i\cot z + c$$

$$\Rightarrow \quad (1+i)f(z) = i\cot z + c$$

$$\Rightarrow \quad f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{1-i^2}\cot z + \frac{c}{1+i}$$

$$\therefore \quad f(z) = \frac{1}{2}(1+i)\cot z + C, \text{ where } C = \frac{c}{1+i}.$$

Example 11. *Find the analytic function $f(z) = u + iv$, given $v = \left(r - \frac{1}{r}\right)\sin\theta$, $r \neq 0$.*

Sol. Using C-R equations in polar coordinates, we have

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta \quad \ldots(1)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta \quad \ldots(2)$$

Now (1) gives $\quad \frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta$

Integrating w.r.t. r

$$u = \left(r + \frac{1}{r}\right)\cos\theta + \phi(\theta), \quad \text{where } \phi(\theta) \text{ is an arbitrary function of } \theta.$$

$$\Rightarrow \quad \frac{\partial u}{\partial \theta} = -\left(r + \frac{1}{r}\right)\sin\theta + \phi'(\theta)$$

Also, from (2) $\quad \frac{\partial u}{\partial \theta} = -\left(r + \frac{1}{r}\right)\sin\theta$

$$\therefore \quad -\left(r + \frac{1}{r}\right)\sin\theta + \phi'(\theta) = -\left(r + \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow \quad \phi'(\theta) = 0 \text{ or } \phi(\theta) = c$$

$$\therefore \quad u = \left(r + \frac{1}{r}\right)\cos\theta + c$$

Hence $\quad f(z) = u + iv$

$$= \left(r + \frac{1}{r}\right)\cos\theta + c + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Example 12. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.* (M.D.U. May 2009; Bombay 2005 S; P.T.U. 2005; U.P.T.U. 2008)

Sol. Let $f(z) = u + iv$ be an analytic function.

Since $|f(z)|$ = constant = c (say), $(c \neq 0)$

$$\therefore \quad |f(z)|^2 = u^2 + v^2 = c^2 \quad \ldots(1)$$

Differentiating (1) partially w.r.t. x and y respectively, we have

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \ldots(2)$$

and

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \quad \text{or} \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \ldots(3)$$

Using Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

equation (3) becomes $\quad -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad \ldots(4)$

Squaring and adding (2) and (4), we have

$$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

or $$c^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]=0$$

or $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=0 \qquad [\because c \neq 0]$$

or $$|f'(z)|^2=0 \qquad \left[\text{Since } f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]$$

$\therefore\ f'(z)=0 \Rightarrow f(z)=\text{constant}.$

Example 13. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,|f'(z)|^2.$$

(J.N.T.U. 2006; M.D.U. Dec. 2011; U.P.T.U. 2005, 2007)

Sol. Let $f(z)=u+iv$ so that $|f(z)|=\sqrt{u^2+v^2}$

or $$|f(z)|^2=u^2+v^2=\phi(x,y) \text{ (say)}$$

$$\therefore \quad \frac{\partial \phi}{\partial x}=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2}=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $$\frac{\partial^2\phi}{\partial y^2}=2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \quad ...(1)$$

Since $f(z)=u+iv$ is a regular function of z, u and v satisfy C-R equations and Laplace's equation.

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

$\therefore$ From (1), we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=2\left[0+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+0+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \quad ...(2)$$

Now $$f(z)=u+iv$$

$$\therefore \quad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \quad \text{and} \quad |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

From (2), we get $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi=4\,|f'(z)|^2$ or $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,|f'(z)|^2.$

Example 14. *If $f(z)$ is a holomorphic function of z, show that*

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2=|f'(z)|^2.$$

(M.D.U. May 2007, Dec. 2010 K.U.K. Dec. 2009; P.T.U. 2005; U.P.T.U. 2009)

Sol. Let $$f(z)=u+iv$$

then $$|f(z)|=(u^2+v^2)^{1/2}=\phi(x,y) \text{ (say)}$$

$$\frac{\partial\phi}{\partial x}=\frac{1}{2}(u^2+v^2)^{-1/2}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)$$

$$=\frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)$$

Similarly, $$\frac{\partial\phi}{\partial y}=\frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}\right)$$

$$=\frac{1}{\sqrt{u^2+v^2}}\left(-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right) \qquad \text{[Using C-R equations]}$$

$$\therefore \quad \left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\frac{1}{u^2+v^2}\left[\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)^2\right]$$

$$=\frac{1}{u^2+v^2}\left[(u^2+v^2)\left(\frac{\partial u}{\partial x}\right)^2+(u^2+v^2)\left(\frac{\partial v}{\partial x}\right)^2\right]=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

$$\Rightarrow \quad \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \quad ...(1)$$

Also $$f(z)=u+iv$$

$$\Rightarrow \quad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\therefore \qquad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \qquad \ldots(2)$$

From (1) and (2), we have

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Example 15. *If $z = x + iy$, show that* $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

Sol. We have $z = x + iy \quad \therefore \quad \bar{z} = x - iy$

$$\Rightarrow \qquad x = \frac{1}{2}(z + \bar{z}),\ y = \frac{1}{2i}(z - \bar{z}) = -\frac{i}{2}(z - \bar{z})$$

$$\therefore \qquad \frac{\partial x}{\partial z} = \frac{1}{2},\ \frac{\partial x}{\partial \bar{z}} = \frac{1}{2},\ \frac{\partial y}{\partial z} = -\frac{i}{2},\ \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Now
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and
$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\therefore \qquad \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Hence
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Note. Remember the result of Example 15.

Example 16. *If $f(z)$ is an analytic function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0.$$

Sol. We know that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Also
$$\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2$$

$$= \frac{1}{2}\log[f'(z)\,.\,f'(\bar{z})] = \frac{1}{2}[\log f'(z) + \log f'(\bar{z})]$$

$$\therefore \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 2\frac{\partial^2}{\partial z \partial \bar{z}}[\log f'(z) + \log f'(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right] = 0.$$

EXERCISE 3.6

1. (*a*) Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

   (*b*) Determine p such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ be an analytic function.

2. (*a*) Show that $f(z) = xy + iy$ is everywhere continuous but not analytic.

   (*b*) Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. (*M.D.U. Dec. 2006*)

   (*c*) If $w = \log z$, find $\frac{dw}{dz}$ and determine where w is non-analytic. (*J.N.T.U. 2005*)

3. Show that the polar form of Cauchy-Riemann equation are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

   Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$ (*M.D.U. Dec. 2010; K.U.K.2009*)

4. If $f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2},\ z \neq 0,\ f(0) = 0$, prove that $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

5. (*a*) Show that the function $f(z)$ defined by $f(z) = \frac{x^2 y^3 (x + iy)}{x^6 + y^{10}},\ z \neq 0,\ f(0) = 0$, is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin.

   (*b*) Show that $f(z) = \begin{cases} xy^2(x + iy) \div (x^2 + y^4), & z \neq 0 \\ 0, & z = 0 \end{cases}$

   is not analytic at $z = 0$, although C – R equations are satisfied at the origin.

   (*c*) Examine the nature of the function

   $$f(z) = \begin{cases} \dfrac{x^2 y^5 (x + iy)}{x^4 + y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

   in a region including the origin.

6. Determine which of the following functions are analytic:

   (*i*) e^z (*ii*) $\sin z$ (*iii*) $\cosh z$

   (*iv*) $\frac{1}{z}$ (*v*) $\frac{x + iy}{x^2 + y^2}$ or $\frac{z}{|z|^2}$ (*vi*) $\frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x}$.

7. (*a*) Show that $f(z) = |z|^2$ is continuous everywhere but not analytic at any point other than the origin. (*P.T.U. 2005*)

   (*b*) Show that the function $f(z) = z\,|z|$ is not analytic anywhere.

8. Show that $u + iv = \frac{x - iy}{x - iy + a}$, where $a \neq 0$, is not an analytic function of $z = x + iy$, whereas $u - iv$ is such a function.

9. Determine the analytic function whose real part is

(i) $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ (M.D.U., Dec. 2008)

(ii) $\log \sqrt{x^2 + y^2}$ (K.U.K., Dec. 2009)

(iii) $e^x (x \cos y - y \sin y)$ (iv) $\cos x \cosh y$

(v) $e^{-x} (x \sin y - y \cos y)$ (vi) $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ (M.D.U., Dec. 2010)

(U.P.T.U. 2006, 2008)

(vii) $x \sin x \cosh y - y \cos x \sinh y$ (viii) $e^x [(x^2 - y^2) \cos y - 2xy \sin y]$

(V.T.U. 2006) (M.D.U. May 2005, Dec. 2010)

10. Find the regular function whose imaginary part is

(i) $\dfrac{x - y}{x^2 + y^2}$ (ii) $\cos x \cosh y$ (iii) $\sinh x \cos y$

(P.T.U. 2005; M.D.U. Dec. 2011) (M.D.U. Dec. 2006)

(iv) $e^{-x} (x \sin y - y \cos y)$ (v) $e^{-x} (x \cos y + y \sin y)$. (U.P.T.U., 2009)

11. Find the real part of the regular function whose imaginary part is $\dfrac{2 \sin x \sin y}{\cos 2x + \cosh 2y}$. (Bombay 2006)

12. (a) Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.

(b) Show that the function $v(x, y) = \ln (x^2 + y^2) + x - 2y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$. (M.D.U., May 2009)

13. An electrostatic field in the xy-plane is given by the potential function $\phi = x^2 - y^2$, find the stream function.

14. If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ determine the function ϕ. (K.U.K., Dec. 2010)

15. If the potential function is $\log (x^2 + y^2)$, find the flux function and the complex potential function. (M.D.U., May 2011)

16. In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$, find the velocity potential ϕ.

17. If $f(z) = u + iv$ is an analytic function, find $f(z)$ if

(i) $u - v = e^x (\cos y - \sin y)$ (K.U.K., Dec. 2010; M.D.U., 2006, 2007, May 2008)

(ii) $u + v = \dfrac{x}{x^2 + y^2}$, when $f(1) = 1$

(iii) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$, when $f\left(\dfrac{\pi}{2}\right) = 0$.

18. If $f(z)$ is an analytic function of z, prove that

(i) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) | \text{R} f(z) |^2 = 2 \, | f'(z) |^2$

(ii) $\nabla^2 | f(z) |^2 = 4 | f'(z) |^2$ (Madras 2006)

19. Prove that $\psi = \log [(x - 1)^2 + (y - 2)^2]$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z.

20. (a) Find the analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$.
(b) Find the analytic function $f(z) = u + iv$, given $u = a (1 + \cos \theta)$.

21. Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic cojugates.

Answers

1. (a) $a = 2, b = -1, c = -1, d = 2$ (b) $p = -1$

2. (c) $\dfrac{1}{z}$, at $z = 0$

5. (c) Not differentiable at the origin and hence not analytic in the region.

6. (i), (ii) and (iii); (iv), (v) and (vi) except when $z = 0$

9. (i) $z^3 + 3z^2 + 1 + ic$ (ii) $\log z + ic$ (iii) $ze^z + ic$
(iv) $\cos z + ic$ (v) $i(ze^{-z} + c)$ (vi) $\cot z + ic$
(vii) $z \sin z + ic$ (viii) $z^2e^z + ic$.

10. (i) $\dfrac{1+i}{z} + c$ (ii) $i \cos z + c$ (iii) $i \sinh z + c$
(iv) $-ze^{-z} + c$ (v) $ize^{-z} + c$

11. $\dfrac{2 \cos x \cosh y}{\cos 2x + \cosh 2y} + c$.

12. (a) $v = x^2 - y^2 + 2xy - 2y - 3x + c$, $f(z) = (1 + i)z^2 - (2 + 3i)z + ic$
(b) $u(x, y) = -2x - y - 2\tan^{-1} \dfrac{y}{x} + c$; $f(z) (i - 2) z + 2i \log z + c$

13. $\psi = 2xy + c$ 14. $-2xy + \dfrac{y}{x^2 + y^2} + c$ 15. $2 \tan^{-1}\left(\dfrac{y}{x}\right) + c, 2 \log z + ic$

16. $\dfrac{1}{2} \log (x^2 + y^2) + c$ 17. (i) $e^z + c$ (ii) $\dfrac{1}{2}\left[\dfrac{1+i}{z} + (1 - i)\right]$ (iii) $\dfrac{1}{2}\left(1 - \cot \dfrac{z}{2}\right)$

19. $-2 \tan^{-1}\left(\dfrac{y - 2}{x - 1}\right)$, $2i \log (z - 1 - 2i)$

20. (a) $i(r^2e^{2i\theta} - re^{i\theta} + 2) + c$ (b) $a(1 + \cos \theta + i \sin \theta \log r + \phi (\theta)$.

3.24. COMPLEX INTEGRATION

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide the curve C into n parts at the points

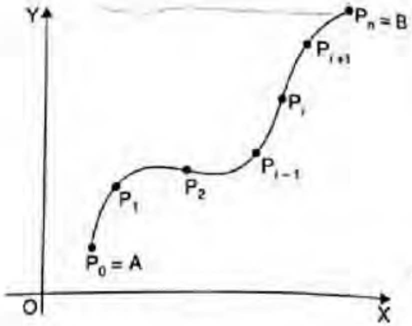
$$\text{A} = \text{P}_0(z_0), \text{P}_1(z_1), \ldots\ldots, \text{P}_i(z_i), \ldots\ldots, \text{P}_n(z_n) = \text{B}.$$

Let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $P_{i-1} P_i$. Then the limit of the sum

$$\sum_{i=1}^{n} f(\xi_i)\, \delta z_i$$

as $n \to \infty$ and each $\delta z_i \to 0$, if it exists, is called the **line integral of f(z)** along the curve C. It is denoted by

$$\int_C f(z)\, dz \qquad \text{(M.D.U. May 2005)}$$



In case the points P_0 and P_n coincide so that C is a closed curve, then this integral is called **contour integral** and is denoted by $\oint_C f(z)\, dz$.

If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + idy$, we have

$$\int_C f(z)\, dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i \int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Moreover, the value of the integral depends on the path of integration unless the integrand is analytic.

When the same path of integration is used in each integral, then

$$\int_a^b f(z)\, dz = -\int_b^a f(z)\, dz$$

If c is a point on the arc joining a and b, then

$$\int_a^b f(z)\, dz = \int_a^c f(z)\, dz + \int_c^b f(z)\, dz.$$

ILLUSTRATIVE EXAMPLES

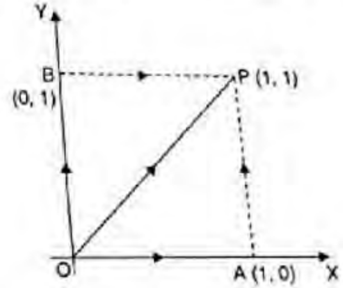
Example 1. *Evaluate* $\int_0^{1+i} (x - y + ix^2)\, dz$.

(a) along the straight line from $z = 0$ to $z = 1 + i$

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

(c) along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to real axis from $z = i$ to $z = 1 + i$.

Sol. (a) Along the straight line OP joining O($z = 0$) and P($z = 1 + i$), $y = x$, $dy = dx$ and x varies from 0 to 1.



$$\therefore \int_0^{1+i} (x - y + ix^2)\, dz = \int_0^{1+i} (x - y + ix^2)(dx + idy)$$

$$= \int_0^1 (x - x + ix^2)(dx + idx) = \int_0^1 ix^2(1 + i)\, dx$$

$$= (-1 + i) \int_0^1 x^2\, dx = (-1 + i)\left[\frac{x^3}{3}\right]_0^1$$

$$= (-1 + i)\left(\frac{1}{3}\right) = -\frac{1}{3} + \frac{1}{3} i.$$

(b) Along the path OAP, where A is $z = 1$

$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_{OA} (x - y + ix^2)\, dz + \int_{AP} (x - y + ix^2)\, dz \qquad ...(1)$$

Now along OA, $y = 0$, $dz = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_{OA} (x - y + ix^2)\, dz = \int_0^1 (x + ix^2)\, dx = \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{1}{3} i$$

Also, along AP, $x = 1$, $dz = idy$ and y varies from 0 to 1

$$\therefore \quad \int_{AP} (x - y + ix^2)\, dz = \int_0^1 (1 - y + i)\, idy = \left[(-1 + i)y - i\frac{y^2}{2}\right]_0^1 = -1 + i - i = -1 + \frac{1}{2} i$$

Hence from (1), $\displaystyle\int_0^{1+i} (x - y + ix^2)\, dz = \left(\frac{1}{2} + \frac{1}{3} i\right) + \left(-1 + \frac{1}{2} i\right) = -\frac{1}{2} + \frac{5}{6} i.$

(c) Along the path OBP, where B is $z = i$

$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_{OB} (x - y + ix^2)\, dz + \int_{BP} (x - y + ix^2)\, dz \qquad ...(2)$$

Now along OB, $x = 0$, $dz = idy$ and y varies from 0 to 1

$$\therefore \quad \int_{OB} (x - y + ix^2)\, dz = \int_0^1 (-y)\, idy = -i\left[\frac{y^2}{2}\right]_0^1 = -\frac{1}{2} i$$

Also, along BP, $y = 1$, $dz = dx$ and x varies from 0 to 1

$$\therefore \quad \int_{BP} (x - y + ix^2)\, dz = \int_0^1 (x - 1 + ix^2)\, dx = \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{1}{3} i$$

Hence from (2), $\displaystyle\int_0^{1+i} (x - y + ix^2)\, dz = -\frac{1}{2} i + \left(-\frac{1}{2} + \frac{1}{3} i\right) = -\frac{1}{2} - \frac{1}{6} i.$

Note. The values of the integral are different along the three different paths.

Example 2. *Evaluate* $\int_0^{1+i} (x^2 - iy)\, dz$ *along the paths*

(a) $y = x$ (b) $y = x^2$.

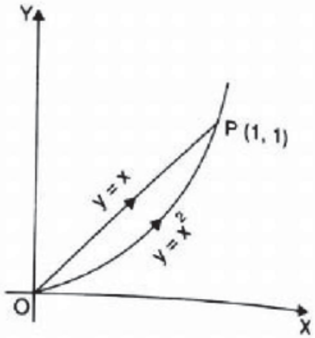
Sol. (*a*) Along the line $y = x$,

$$dy = dx \text{ so that } dz = dx + i\,dx = (1+i)\,dx$$

$$\therefore \quad \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix)(1+i)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\,\frac{x^2}{2}\right]_0^1 = (1+i)\left(\frac{1}{3} - \frac{1}{2}i\right)$$

$$= \frac{(1+i)(2-3i)}{6} = \frac{5}{6} - \frac{1}{6}i.$$



(*b*) Along the parabola $y = x^2$, $dy = 2x\,dx$ so that

$$dz = dx + 2ix\,dx = (1 + 2ix)\,dx$$

and x varies from 0 to 1.

$$\therefore \quad \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1 + 2ix)\,dx$$

$$= (1-i)\int_0^1 x^2(1+2ix)\,dx = (1-i)\left[\frac{x^3}{3} + i\,\frac{x^4}{2}\right]_0^1$$

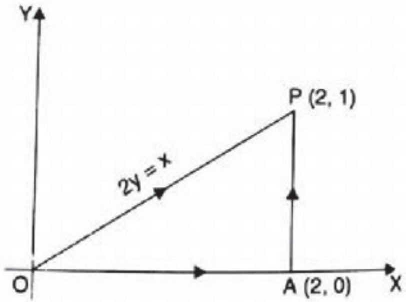
$$= (1-i)\left(\frac{1}{3} + \frac{1}{2}i\right) = \frac{(1-i)(2+3i)}{6} = \frac{5}{6} + \frac{1}{6}i.$$

Example 3. *Evaluate* $\int_0^{2+i} (\bar{z})^2\,dz$, *along*

(*a*) *the real axis to 2 and then vertically to* $2 + i$.

(*b*) *along the line* $2y = x$.

Sol. $(\bar{z})^2 = (x - iy)^2 = (x^2 - y^2) - 2ixy$



(*a*) Along the path OAP, where A is (2, 0) and P is (2, 1).

$$\int_0^{2+i} (\bar{z})^2\,dz$$

$$= \int_{OA} (x^2 - y^2 - 2ixy)\,dz + \int_{AP} (x^2 - y^2 - 2ixy)\,dz \qquad \ldots(1)$$

Now, along OA, $y = 0$, $dz = dx$ and x varies from 0 to 2

$$\therefore \quad \int_{OA} (x^2 - y^2 - 2ixy)\,dz = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Also, along AP, $x = 2$, $dz = i\,dy$ and y varies from 0 to 1

$$\therefore \quad \int_{AP} (x^2 - y^2 - 2ixy)\,dz = \int_0^1 (4 - y^2 - 4iy)\,i\,dy$$

$$= \left[4iy - i\,\frac{y^3}{3} + 2y^2\right]_0^1 = 4i - \frac{1}{3}i + 2 = 2 + \frac{11}{3}i$$

Hence from (1), we have $\displaystyle\int_0^{2+i} (\bar{z})^2\,dz = \frac{8}{3} + 2 + \frac{11}{3}i = \frac{14}{3} + \frac{11}{3}i.$

(*b*) Along the line OP, $2y = x$, $dx = 2dy$ so that $dz = 2dy + i\,dy = (2+i)\,dy$ and y varies from 0 to 1.

$$\therefore \quad \int_0^{2+i} (\bar{z})^2\,dz = \int_0^{2+i} (x^2 - y^2 - 2ixy)\,dz = \int_0^1 (4y^2 - y^2 - 4iy^2)(2+i)\,dy$$

$$= (2+i)(3-4i)\int_0^1 y^2\,dy = (10 - 5i)\left[\frac{y^3}{3}\right]_0^1 = \frac{10}{3} - \frac{5}{3}i.$$

Example 4. *Integrate* $f(z) = x^2 + ixy$ *from* A(1, 1) *to* B(2, 4) *along the curve* $x = t$, $y = t^2$.

Sol. Equations of the path of integration are $x = t$, $y = t^2$

$$\therefore \quad dx = dt, \quad dy = 2t\,dt$$

At A(1, 1), $t = 1$ and at B(2, 4), $t = 2$

$$\therefore \quad \int_A^B f(z)\,dz = \int_A^B (x^2 + ixy)(dx + i\,dy) = \int_1^2 (t^2 + it^3)(dt + 2it\,dt)$$

$$= \int_1^2 (t^2 - 2t^4)\,dt + i\int_1^2 3t^3\,dt = \left[\frac{t^3}{3} - \frac{2t^5}{5}\right]_1^2 + i\left[\frac{3t^4}{4}\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{64}{5}\right) - \left(\frac{1}{3} - \frac{2}{5}\right) + i\left(12 - \frac{3}{4}\right) = -\frac{151}{15} + \frac{45}{4}i.$$

Example 5. *Prove that*

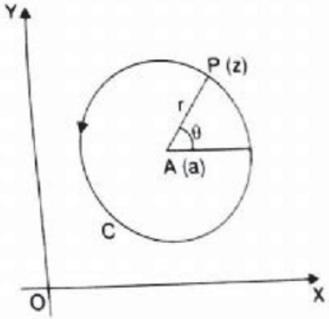
(*i*) $\displaystyle\oint_C \frac{dz}{z-a} = 2\pi i$

(*ii*) $\displaystyle\oint_C (z-a)^n\,dz = 0$ [*n is an integer* $\neq -1$], *where C is the circle* $|z - a| = r$.

Sol. The equation of the circle C is

$$|z - a| = r \quad \text{or} \quad z - a = re^{i\theta}$$

where θ varies from 0 to 2π as z describes C once in the anti-clockwise direction.



Also $\quad dz = ire^{i\theta}\,d\theta.$

$$(i) \quad \oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

$$(ii) \quad \oint_C (z-a)^n\,dz = \int_0^{2\pi} r^n e^{ni\theta} \cdot ire^{i\theta}\,d\theta$$

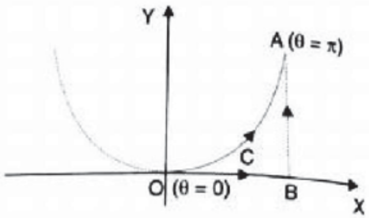
$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$$

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} \qquad [\because \; n \neq -1]$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i2(n+1)\pi} - 1\right]$$

$$= 0. \qquad [\because \; e^{i2(n+1)\pi} = \cos 2(n+1)\pi + i \sin 2(n+1)\pi = 1 + i0 = 1]$$

Example 6. *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where C is the arc of the cycloid* $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ *between the points* $(0, 0)$ *and* $(\pi a, 2a)$.



Sol. The function $f(z) = z^2 + 3z + 2$ is a polynomial and therefore analytic in z-plane. Hence the line integral of $f(z)$ between the points $O(0, 0)$ and $A(\pi a, 2a)$ is independent of the path joining these points. Let us choose the path of integration as:

(*i*) From $O(0, 0)$ to $B(\pi a, 0)$ along the real axis; followed by

(*ii*) From $B(\pi a, 0)$ to $A(\pi a, 2a)$ vertically

$$\therefore \quad \int_C f(z)\, dz = \int_{OB} f(z)\, dz + \int_{BA} f(z)\, dz \qquad ...(i)$$

Now along OB, $y = 0$

$$\therefore \quad z = x + iy = x, \quad dz = dx$$

and x varies from 0 to πa.

$$f(z) = z^2 + 3z + 2 = x^2 + 3x + 2$$

$$\therefore \quad \int_{OB} f(z)\, dz = \int_0^{\pi a} (x^2 + 3x + 2)\, dx = \left[\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right]_0^{\pi a}$$

$$= \frac{\pi^3 a^3}{3} + \frac{3\pi^2 a^2}{2} + 2\pi a = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12)$$

Also, along BA, $x = \pi a$

$$\therefore \quad z = x + iy = \pi a + iy,$$

$$dz = i dy \text{ and } y \text{ varies from 0 to } 2a.$$

$$f(z) = z^2 + 3z + 2 = (\pi a + iy)^2 + 3(\pi a + iy) + 2$$

$$\therefore \quad \int_{BA} f(z)\, dz = \int_0^{2a} [(\pi a\ .\ iy)^2 + 3(\pi a + iy) + 2]\, i dy$$

$$= \left[\left\{\frac{(\pi a + iy)^3}{3i} + \frac{3(\pi a + iy)^2}{2i} + 2y\right\} i\right]_0^{2a}$$

$$= \frac{1}{3}(\pi a + i2a)^3 + \frac{3}{2}(\pi a + i2a)^2 + 2i\ .\ 2a - \frac{1}{3}\pi^3 a^3 - \frac{3}{2}\pi^2 a^2$$

$$= \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia - \frac{1}{3}\pi^3 a^3 - \frac{3}{2}\pi^2 a^2$$

$\therefore$ From (1), we have

$$\int_C (z^2 + 3z + 2)\, dz = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia - \frac{1}{3}\pi^3 a^3 - \frac{3}{2}\pi^2 a^2$$

$$= \frac{1}{3}\pi^3 a^3 + \frac{3}{2}\pi^2 a^2 + 2\pi a + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2a)^2 + 4ia - \frac{1}{3}\pi^3 a^3 - \frac{3}{2}\pi^2 a^2$$

$$= \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 2a(\pi + 2i)$$

EXERCISE 3.7

1. Evaluate $\int_0^{3+i} z^2\, dz$, along

   (*a*) the line $y = \frac{x}{3}$ (*b*) the real axis to 3 and then vertically to $3 + i$

   (*c*) the parabola $x = 3y^2$.

2. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$, along

   (*a*) the straight line joining $(1 - i)$ to $(2 + i)$. (*b*) the curve $x = t + 1$, $y = 2t^2 - 1$.

3. Evaluate $\int_0^{4+2i} \bar{z}\, dz$, along the curve given by $z = t^2 + it$. (*M.D.U. Dec. 2006*)

4. Evaluate $\oint_C |z|^2\, dz$, around the square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

5. Show that $\oint_C (z + 1)\, dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$. (*M.D.U. Dec. 2005, May 2008*)

6. Evaluate $\int_C (y - x - 3x^2 i)\, dz$, where C is the straight line from $z = 0$ to $z = 1 + i$.

7. (*a*) Evaluate $\int_C (z - z^2)\, dz$, where C is the upper half of the circle $|z| = 1$. What is the value of this integral if C is the lower half of above circle ?

   (*b*) Evaluate $\int_C (z - z^3)\, dz$, where C is the upper half of the circle $|z| = 1$. (*M.D.U. Dec. 2011*)

   (*c*) Evaluate $\int_C (z - z^2)\, dz$ where C is the upper half of the circle $|z - 2| = 3$. What is the value of the integral if C is the lower half of the above given circle ? (*M.D.U. Dec. 2009*)

8. Prove that $\int_C \frac{1}{z}\, dz = -\pi i$ or πi according as C is the semi-circular arc $|z| = 1$ from -1 to 1 above or below the real axis. (*M.D.U. May 2005*)

9. Show that for every path between the limits,

$$\int_{-2}^{-2+i} (2 + z)^2\, dz = -\frac{i}{3}$$

10. Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\, dz$ along the line joining the points $(1, -1)$ and $(2, 3)$.

11. Evaluate $\int_C |z|\,dz$, where C is the contour

(*a*) straight line from $z = -i$ to $z = i$

(*b*) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.

(*c*) circle given by $|z + 1| = 1$ described in the clockwise sense.

12. Evaluate $\oint_C \log z\,dz$, where C is the unit circle $|z| = 1$.

Answers

1. (*a*) $6 + \frac{26}{3}i$	(*b*) $6 + \frac{26}{3}i$	(*c*) $6 + \frac{26}{3}i$	
2. (*a*) $4 + 8i$	(*b*) $4 + \frac{25}{3}i$	3. $10 - \frac{8}{3}i$	4. $-1 + i$
6. $1 - i$	7. (*a*) $\frac{2}{3}, -\frac{2}{3}$	(*b*) 0	(*c*) 30, − 30
10. $\frac{1}{6}(-103 + 64i)$	11. (*a*) i	(*b*) $2i$	(*c*) $\frac{8}{3}i$
12. $2\pi i$			

3.25. SIMPLY AND MULTIPLY CONNECTED REGIONS (*M.D.U. May 2006*)

A curve is called *simple closed curve* if it does not cross itself (Fig. 1). A curve which crosses itself is called a *multiple curve* (Fig. 2).

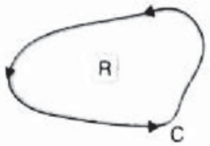


Fig. 1

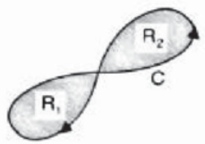


Fig. 2

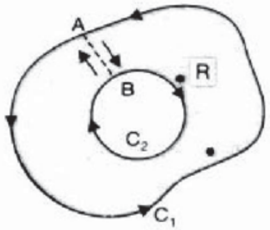


Fig. 3

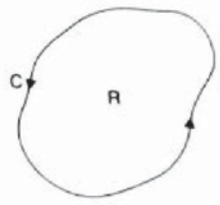
A region is called *simply connected* if every closed curve in the region encloses points of the region only, *i.e.*, every closed curve lying in it can be contracted indefinitely without passing out of it. A region which is not simply connected is called a *multiply connected* region. In plain terms, a simply connected region is one which has no holes. Fig. 3 shows a multiply connected region R enclosed between two separate curves C_1 and C_2. (There can be more than two separate curves). We can convert a multiply connected region into a simply connected one, by giving it one or more cuts (*e.g.*, along the dotted line AB).

3.26. CAUCHY'S INTEGRAL THEOREM (*M.D.U. Dec. 2010; U.P.T.U. 2006*)

Statement. *If f(z) is an analytic function and f ′(z) is continuous at each point within and on a simple closed curve C, then*

$$\oint_C f(z)\,dz = 0$$

Proof. Let R be the region bounded by the curve C.



Let $\quad f(z) = u(x, y) + iv(x, y)$, then

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \qquad \ldots(1)$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R. Hence by Green's Theorem, we have

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \ldots(2)$$

Now $f(z)$ being analytic at each point of the region R, by Cauchy-Riemann equations, we have

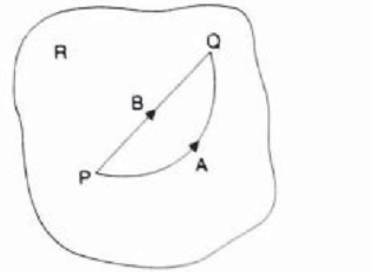
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in (2) vanish.

Hence $\quad \oint_C f(z)\,dz = 0.$

Cor. 1. *If f(z) is analytic in a region R and P and Q are two points in R, then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q and lying entirely in R.*

Let PAQ and PBQ be any two paths joining P and Q.



By Cauchy's theorem,

$$\int_{PAQBP} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{PAQ} f(z)\,dz + \int_{QBP} f(z)\,dz = 0$$

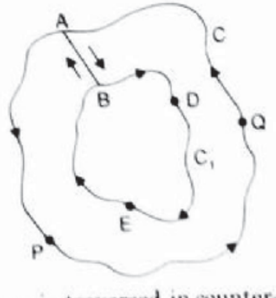
$$\Rightarrow \quad \int_{PAQ} f(z)\,dz - \int_{PBQ} f(z)\,dz = 0$$

Hence $\quad \int_{PAQ} f(z)\,dz = \int_{PBQ} f(z)\,dz.$

Cor. 2. *If f(z) is analytic in the region bounded by two simple closed curves C and C_1, then*

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

Let AB be a cross-cut joining the curves C and C_1, then the doubly connected region becomes simply connected.



By Cauchy's theorem, $\int_{APQABDEBA} f(z)\,dz = 0$

$$\Rightarrow \int_{APQA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BDEB} f(z)\,dz + \int_{BA} f(z)\,dz = 0$$

$$\Rightarrow \quad \oint_C f(z)\,dz + \int_{AB} f(z)\,dz - \int_{BEDB} f(z)\,dz - \int_{AB} f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$$

(integrals around a closed curve are taken positive when the curve is traversed in counter-clockwise direction)

Hence $\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$

The theorem can be extended.

If a closed curve C contains non-intersecting closed curves $C_1, C_2, ..., C_n$, then by introducing cross-cuts, it can be shown that

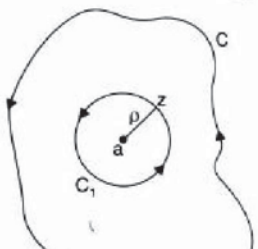
$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + ... + \oint_{C_n} f(z)\,dz. \qquad \text{(M.D.U. May 2006)}$$

3.27. CAUCHY'S INTEGRAL FORMULA

(V.T.U. 2007; U.P.T.U. 2006, 2007, 2008)

Statement. *If f(z) is analytic within and on a closed curve C and a is any point within C, then*

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz.$$



Proof. Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at $z = a$.

Draw a circle C_1 with a as centre and radius ρ such that C_1 lies entirely inside C.

Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

∴ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \qquad ...(1)$$

Now, the equation of circle C_1 is

$$|z - a| = \rho \quad \text{or} \quad z - a = \rho e^{i\theta}$$

so that $\qquad dz = i\rho e^{i\theta}\,d\theta$

$$\oint_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\,.\,i\rho e^{i\theta}\,d\theta = i\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta$$

Hence by (1), we have $\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta \qquad ...(2)$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, $\rho \to 0$, then from (2),

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)$$

Hence $\quad f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \quad$ which is the required Cauchy's integral formula.

Cor. By Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a} \qquad ...(1)$$

where a is any point within C and may be treated as a parameter.

Differentiating both sides of (1) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

Similarly, $\quad f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz$

$$f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}\,dz$$

and in general, $\quad f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$

Thus, *if a function of a complex variable has a first derivative in a simply connected region, all its higher derivatives exist in that region. This property is not exhibited by the functions of real variables.*

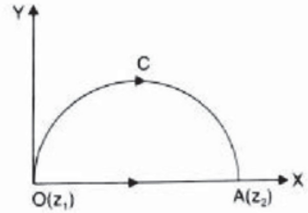
ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\oint_C (x^2 - y^2 + 2ixy)\,dz$, *where C is the contour* $|z| = 1$.

Sol. $f(z) = x^2 - y^2 + 2ixy = (x+iy)^2 = z^2$ is analytic everywhere within and on $|z| = 1$.

∴ By Cauchy's integral theorem, $\oint_C f(z)\,dz = 0$.

Example 2. *Evaluate* $\int_C (3z^2 + 4z + 1)\,dz$, *where C is the arc of the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ *between (0, 0) and (2πa, 0).*



Sol. Here, $f(z) = 3z^2 + 4z + 1$ is analytic everywhere so that the integral is independent of the path of integration and depends only on the end points $z_1 = 0 + i0$ and $z_2 = 2\pi a + i0$.

$$\therefore \quad \int_C (3z^2 + 4z + 1)\,dz = \int_0^{2\pi a} (3z^2 + 4z + 1)\,dz$$

$$= \Big[z^3 + 2z^2 + z\Big]_0^{2\pi a} = 8\pi^3a^3 + 8\pi^2a^2 + 2\pi a = 2\pi a\,(4\pi^2a^2 + 4\pi a + 1).$$

Example 3. *Evaluate* $\oint_C \frac{e^{-z}}{z+1}\,dz$, *where C is the circle*

(*a*) $|z| = 2$ $\qquad$ (*b*) $|z| = \frac{1}{2}$.

Sol. $f(z) = e^{-z}$ is an analytic function.

(*a*) The point $a = -1$ lies inside the circle $|z| = 2$.

∴ By Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(-1)}\,dz = f(-1) \;\Rightarrow\; \oint_C \frac{e^{-z}}{z+1}\,dz = 2\pi i e.$$

(*b*) The point $a = -1$ lies outside the circle $|z| = \frac{1}{2}$.

$\therefore$ The function $\dfrac{e^{-z}}{z+1}$ is analytic within and on C.

By Cauchy's integral theorem, we have $\oint_C \dfrac{e^{-z}}{z+1} dz = 0$.

Example 4. *Evaluate* $\oint_C \dfrac{\cos \pi z^2}{(z-1)(z-2)} dz$, *where C is the circle* $|z| = 3$.

Sol. The integrand has singularities where

$$(z-1)(z-2) = 0 \quad i.e., \quad \text{at } z = 1 \quad \text{and} \quad z = 2.$$

Both these points lie within the circle $|z| = 3$

$f(z) = \cos \pi z^2$ is an analytic function.

Also
$$\frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$\therefore \quad \oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -\oint_C \frac{\cos \pi z^2}{z-1} dz + \oint_C \frac{\cos \pi z^2}{z-2} dz \qquad ...(1)$$

$$= -2\pi i f(1) + 2\pi i f(2), \text{ where } f(z) = \cos \pi z^2$$

$$= -2\pi i \cos \pi + 2\pi i \cos 4\pi = 2\pi i + 2\pi i = 4\pi i.$$

Example 5. *Evaluate* $\oint_C \dfrac{3z^2 + z}{z^2 - 1} dz$, *where C is the circle* $|z - 1| = 1$.

(M.D.U. Dec. 2006)

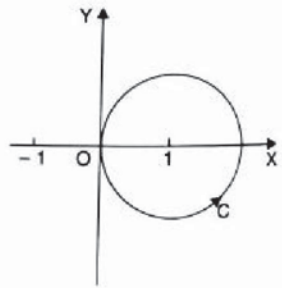
Sol. The integrand has singularities, where $z^2 - 1 = 0$ *i.e.,* at $z = 1$ and $z = -1$.

The circle $|z - 1| = 1$ has centre at $z = 1$ and radius 1 and includes the point $z = 1$.

$f(z) = 3z^2 + z$ in an analytic function.

Also
$$\frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\therefore \quad \oint_C \frac{3z^2 + z}{z^2 - 1} dz = \frac{1}{2}\oint_C \frac{3z^2 + z}{z-1} dz - \frac{1}{2}\oint_C \frac{3z^2 + z}{z+1} dz \qquad ...(1)$$



By Cauchy's integral formula,

$$\oint_C \frac{3z^2 + z}{z-1} dz = 2\pi i\, f(1), \quad \text{where } f(z) = 3z^2 + z$$

$$= 8\pi i$$

By Cauchy's integral theorem,

$$\oint_C \frac{3z^2 + z}{z+1} dz = 0$$

$\therefore$ From (1), we have

$$\oint_C \frac{3z^2 + z}{z^2 - 1} dz = 4\pi i.$$

Example 6. *Use Cauchy's integral formula to evaluate*

$\oint_C \dfrac{e^{2z}}{(z+1)^4} dz$, *where C is the circle* $|z| = 2$.

Sol. The integrand has a singularity at $z = -1$ which lies within the circle $|z| = 2$.

Now
$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}} dz \qquad ...(1)$$

Here $a = -1$, $n + 1 = 4$ *i.e.,* $n = 3$, $f(z) = e^{2z}$

$$\therefore \quad f^n(z) = 2^n e^{2z},\ f'''(a) = f'''(-1) = 2^3 e^{-2}$$

From (1), we have $f'''(-1) = \dfrac{3!}{2\pi i}\oint_C \dfrac{e^{2z}}{(z+1)^4} dz \Rightarrow 8e^{-2} = \dfrac{3}{\pi i}\oint_C \dfrac{e^{2z}}{(z+1)^4} dz$

Hence
$$\oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3e^2}.$$

Example 7. *If* $f(\xi) = \oint_C \dfrac{3z^2 + 7z + 1}{z - \xi} dz$, *where C is the circle* $x^2 + y^2 = 4$, *find the values of* $f(3)$, $f'(1-i)$ *and* $f''(1+i)$. (M.D.U. Dec. 2011)

Sol. The given circle C is $x^2 + y^2 = 4$ or $|z| = 2$.

The point $z = 3$ lies outside the circle $|z| = 2$ while $z = 1 - i$ and $z = 1 + i$ lie inside the circle.

Now, $f(3) = \oint_C \dfrac{3z^2 + 7z + 1}{z - 3} dz$ and $\dfrac{3z^2 + 7z + 1}{z - 3}$ is analytic within and on C.

$\therefore$ By Cauchy's integral theorem, we have

$$\oint_C \frac{3z^2 + 7z + 1}{z - 3} dz = 0 \Rightarrow f(3) = 0$$

Now let $\phi(z) = 3z^2 + 7z + 1$ which is analytic everywhere.

$\therefore$ By Cauchy's integral formula, we have

$$\phi(\xi) = \frac{1}{2\pi i}\oint_C \frac{\phi(z)}{z - \xi} dz, \quad \text{where } \xi \text{ is a point within C.}$$

$$\Rightarrow \quad 2\pi i \phi(\xi) = \oint_C \frac{3z^2 + 7z + 1}{z - \xi} dz = f(\xi)$$

$$\Rightarrow \quad f(\xi) = 2\pi i(3\xi^2 + 7\xi + 1) \quad \Rightarrow \quad f'(\xi) = 2\pi i(6\xi + 7)$$

$$\therefore \quad f'(1-i) = 2\pi i\,[6(1-i) + 7] = 2\pi i(13 - 6i) = 2\pi(6 + 13i)$$

Also
$$f''(\xi) = 12\pi i$$

$$\therefore \quad f''(1+i) = 12\pi i.$$

EXERCISE 3.8

1. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the
   (*i*) rectangle with vertices $-1, 1, 1 + i, -1 + i$,
   (*ii*) triangle with vertices (1, 2), (1, 4), (3, 2).

2. (a) Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots\ldots$ where C is a closed curve containing the point $z = a$.

(b) Evaluate $\oint_C \frac{z^2 - z + 1}{z - 1} dz$, where C is the circle $|z| = \frac{1}{2}$. *(M.D.U. Dec. 2011)*

3. Evaluate $\oint_C \frac{z^2 + 5}{z - 3} dz$, where C is the circle

(a) $|z| = 4$ (b) $|z| = 1$.

4. Evaluate $\oint_C \frac{e^z}{z - 2} dz$, where C is the circle.

(a) $|z| = 3$ (b) $|z| = 1$.

5. Evaluate $\oint_C \frac{3z^2 + 7z + 1}{z + 1} dz$, where C is the circle.

(a) $|z| = 1.5$ (b) $|z + i| = 1$ (c) $|z| = \frac{1}{2}$.

6. Evaluate $\oint_C \frac{\sin 3z}{z + \frac{\pi}{2}} dz$, where C is the circle $|z| = 5$.

7. Evaluate $\oint_C \frac{z^3 + z + 1}{z^2 - 3z + 2} dz$, where C is the ellipse $4x^2 + 9y^2 = 1$. *(M.D.U. Dec. 2005, May 2008)*

8. Evaluate $\oint_C \frac{\cos \pi z}{z^2 - 1} dz$ around a rectangle with vertices

(a) $2 \pm i, -2 \pm i$ (b) $-i, 2 - i, 2 + i, i$.

9. (i) Evaluate $\oint_C \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3} dz$, where C is $|z| = 1$. (M.D.U. May 2006, Dec. 2006)

(ii) Evaluate $\oint_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3} dz$, where C is the circle $|z| = 1$. *(M.D.U. May 2005)*

(iii) Evaluate $\oint_C \frac{z^3 - z}{(z - 2)^3} dz$, where C: $|z| = 3$

10. Evaluate $\oint_C \frac{z\, dz}{(z-1)(z-3)}$, where C is the circle

(a) $|z| = 3.5$ (b) $|z| = 3/2$.

11. Evaluate $\oint_C \frac{z + 4}{z^2 + 2z + 5} dz$, where C is the circle $|z| = 1$.

12. Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$. *(U.P.T.U. 2008)*

13. Evaluate $\oint_C \frac{e^z}{(z-1)(z-4)} dz$, where C is the circle $|z| = 2$.

14. Evaluate $\oint_C \frac{e^z}{z(z+1)} dz$, where C is the circle $|z| = \frac{1}{4}$.

15. Evaluate $\oint_C \frac{\cos z}{z - \pi} dz$, where C is the circle $|z - 1| = 3$.

16. Let $P(z) = a + bz + cz^2$ and $\oint_C \frac{P(z)}{z} dz = \oint_C \frac{P(z)}{z^2} dz = \oint_C \frac{P(z)}{z^3} dz = 2\pi i$ where C is the circle $|z| = 1$. Evaluate $P(z)$.

17. Evaluate $\oint_C \frac{e^{-z}}{z^2} dz$, where C is the circle $|z| = 1$.

18. (a) Evaluate $\oint_C \frac{e^{-2z}\, dz}{(z+1)^3}$, where C is the circle $|z| = 2$.

(b) Evaluate $\oint_C \frac{e^z}{(z-1)(z-2)^2} dz$, where C is the circle $|z| = 3$

19. If $f(\xi) = \oint_C \frac{4z^2 + z + 5}{z - \xi} dz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$. *(J.N.T.U. 2005; M.D.U., Dec. 2007)*

20. Evaluate, using Cauchy's integral formula:

(i) $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$.

(ii) $\int_C \frac{z}{z^2 - 3z + 2} dz$, where C is $|z - 2| = \frac{1}{2}$. *(U.P.T.U. 2009; M.D.U. 2007)*

(iii) $\int_C \frac{e^z\, dz}{(z+1)^2}$, where C is $|z - 1| = 3$. *(M.D.U. May 2006)*

(iv) $\int_C \frac{e^z}{(z^2 + \pi^2)^2} dz$, where C is $|z| = 4$. *(M.D.U. May 2011; U.P.T.U. 2008)*

(v) $\int_C \frac{z^3 - 2z + 1}{(z - i)^2} dz$, where C is $|z| = 2$.

(vi) $\int_C \frac{\log z}{(z-1)^3} dz$, where C is $|z - 1| = \frac{1}{2}$. *(M.D.U. Dec. 2010)*

(vii) $\int_C \frac{\tan z}{z} dz$, where C is the circle $|z| = 1.5$.

(viii) $\oint_C \frac{e^{3z}}{(z - \ln 2)^4} dz$, where C is the square with vertices at $\pm 1, \pm i$. *(M.D.U. Dec. 2009)*

Answers

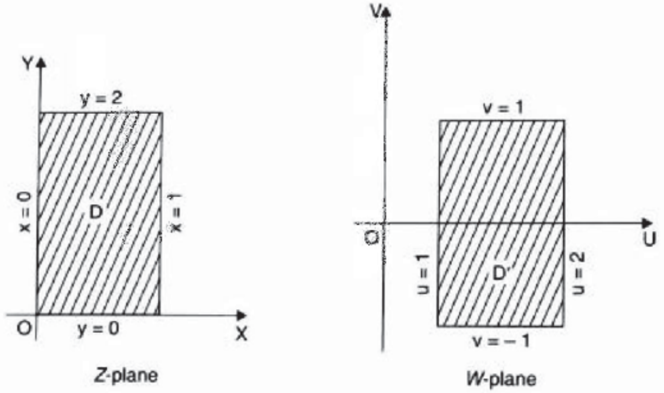
2. (a) 0 (b) 0 3. (a) $28\pi i$ (b) 0 4. (a) $2\pi i\, e^2$ (b) 0
5. (a) $-6\pi i$ (b) 0 (c) 0 6. $2\pi i$ 7. 0
8. (a) 0 (b) $-\pi i$ 9. (i) πi (ii) $\frac{21}{16}\pi i$ (iii) $12\pi i$
10. (a) $2\pi i$ (b) $-\pi i$ 11. 0 12. $4\pi i$
13. $-\frac{2}{3}\pi i e$ 14. $2\pi i$ 15. $-2\pi i$
16. $1 + z + z^2$ 17. $-2\pi i$
18. (a) $4\pi i e^2$ (b) $2\pi i e$ 19. $20\pi i$, $2\pi(i - 1)$, $-14\pi i$, $16\pi i$
20. (i) $2\pi i(e^4 - e^2)$ (ii) $4\pi i$ (iii) $\frac{2\pi i}{e}$
(iv) $\frac{i}{\pi}$ (v) $-10\pi i$ (vi) $-\pi i$
(vii) 0 (viii) $72\pi i$ [ln 2 = 0.6931].

CONFORMAL MAPPING*

(For K.U.K. Only)

3.28. TRANSFORMATION OR MAPPING

We know that the real function $y = f(x)$ can be represented graphically by a curve in the xy-plane. Also, the real function $z = f(x, y)$ can be represented by a surface in three dimensional space. However, this method or graphical representation fails in the case of complex functions because a complex function $w = f(z)$ *i.e.*, $u + iv = f(x + iy)$ involves four real variables, two independent variables x, y and two dependent variables u, v. Thus a four dimensional region is required to represent it graphically in the cartesian fashion. As it is not possible, we choose, two complex planes and call them z-plane and w-plane. In the z-plane, we plot the point $z = x + iy$ and in the w-plane, we plot the corresponding point $w = u + iv$. Thus the function $w = f(z)$ defines a correspondence between points of these two planes. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C in the w-plane, since to each (x, y) there corresponds a point (u, v). *The function $w = f(z)$ thus defines a mapping or transformation of the z-plane into the w-plane.*



For example, consider the transformation $w = z + (1 - i)$. Let us determine the region D' of the w-plane corresponding to the rectangular region D in the z-plane bounded by $x = 0$, $y = 0$, $x = 1$ and $y = 2$.

Since $w = z + (1 - i)$, we have

$$u + iv = (x + iy) + (1 - i) = (x + 1) + i(y - 1)$$

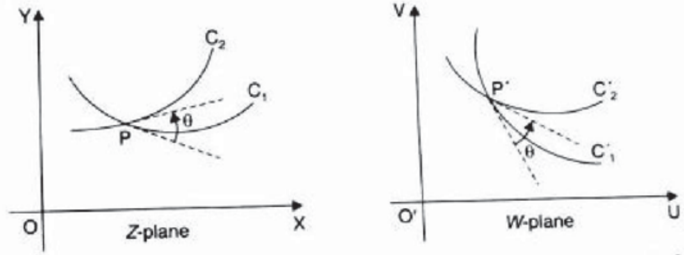
Thus $\qquad u = x + 1 \quad \text{and} \quad v = y - 1$

Hence the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$ in the z-plane are mapped onto the lines $u = 1$, $v = -1$, $u = 2$ and $v = 1$ in the w-plane. The regions D and D' are shown shaded in the figure.

*This part is **not included in** the syllabus of **M.D.U.,** Rohtak.

3.29. CONFORMAL TRANSFORMATION OR GEOMETRICAL REPRESENTATION OF w = f(z)

Suppose two curves C_1, C_2 in the z-plane intersect at the point P and the corresponding curves C_1', C_2' in the w-plane intersect at P' under the transformation $w = f(z)$. *If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P', both in magnitude and sense, then the transformation is said to be conformal at P.*



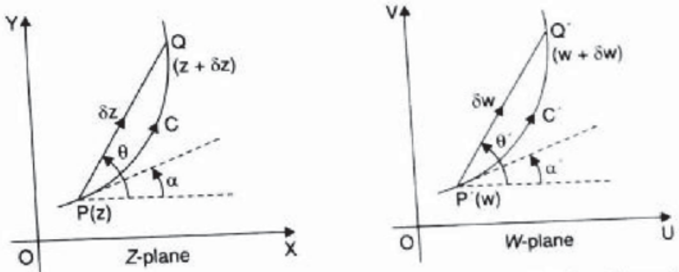
Definition. A transformation which preserves angles both in magnitude and sense between every pair of curves through a point is said to be conformal at the point.

The conditions under which the transformation $w = f(z)$ is conformal are given by the following theorem.

3.30. THEOREM

If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane, then the mapping $w = f(z)$ is conformal at all points of R.

Proof. Let P(z) be a point in the region R of the z-plane and P'(w) the corresponding point in the region R' of the w-plane. Suppose P moves on a curve C and P' moves on the corresponding curve C'. Let Q($z + \delta z$) be a neighbouring point on C and Q' ($w + \delta w$) the corresponding point on C' so that $\overrightarrow{PQ} = \delta z$ and $\overrightarrow{P'Q'} = \delta w$.



Then δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x-axis.

$$\therefore \qquad \delta z = re^{i\theta}$$

Similarly, $\delta w = r'e^{i\theta'}$ where r' is the modulus and θ' is the amplitude of δw.

$$\therefore \qquad \frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Let the tangent to C at P make an angle α with x-axis and the tangent to C′ at P′ make an angle α' with u-axis, then as $\delta z \to 0$, $\theta \to \alpha$ and $\theta' \to \alpha'$.

$$\therefore \qquad f'(z) = \frac{dw}{dz} = \underset{\delta z \to 0}{\text{Lt}} \frac{\delta w}{\delta z} = \underset{\delta z \to 0}{\text{Lt}} \left(\frac{r'}{r}\right) e^{i(\theta' - \theta)} \qquad \ldots(1)$$

Since $f'(z) \neq 0$, let $f'(z) = \rho e^{i\phi}$, then $\rho = | f'(z) |$ and ϕ = amplitude of $f'(z)$.

$$\therefore \text{ From (1),} \qquad \rho e^{i\phi} = \underset{\delta z \to 0}{\text{Lt}} \left(\frac{r'}{r}\right) e^{i(\theta' - \theta)}$$

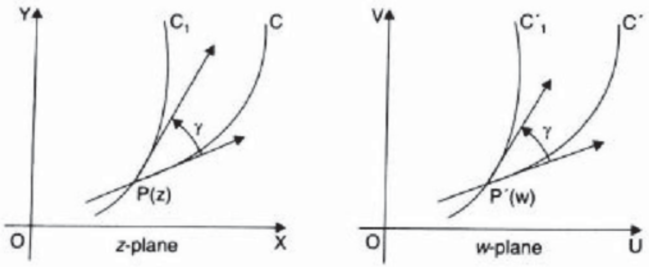
Thus
$$\rho = \underset{\delta z \to 0}{\text{Lt}} \frac{r'}{r} \qquad \ldots(2)$$

and
$$\phi = \underset{\delta z \to 0}{\text{Lt}} (\theta' - \theta) = \alpha' - \alpha \qquad \ldots(3)$$

Now let C_1 be another curve through P in the z-plane and C_1' the corresponding curve through P′ in the w-plane. If the tangent to C_1 at P makes an angle β with x-axis and the tangent to C_1' at P′ makes an angle β' with u-axis, then as in (3),

$$\phi = \beta' - \beta \qquad \ldots(4)$$

From (3) and (4), $\alpha' - \alpha = \beta' - \beta$ or $\beta - \alpha = \beta' - \alpha' = \gamma$



Thus angle between the curves before and after the mapping is preserved in magnitude and sense. Hence the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Note 1. A point at which $f'(z) = 0$ is called a **critical point** of the transformation.

Note 2. From (2) $\qquad \rho = \underset{\delta z \to 0}{\text{Lt}} \frac{r'}{r}$

It follows that under the conformal transformation $w = f(z)$, the lengths of arcs through P are magnified in the ratio ρ: 1, where $\rho = | f'(z) |$. Thus an infinitesimal length in the z-plane is magnified by the factor $| f'(z) |$ in the w-plane and consequently infinitesimal areas in the z-plane are magnified by the factor $| f'(z) |$ in the ω-plane.

Note 3. From (3), $\alpha' = \alpha + \phi$ shows that the tangent to the curve C at P is rotated through an angle ϕ under the given transformation.

Note 4. A harmonic function remains harmonic under a conformal transformation.

3.31. COEFFICIENT OF MAGNIFICATION

Coefficient of magnification for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by $| f'(\alpha + i\beta) |$ where dash represents derivative.

3.32. ANGLE OF ROTATION

Angle of rotation for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by amp. $[f'(\alpha + i\beta)]$.

Example 1. *For the conformal transformation $w = z^2$, show that*

(i) the coefficient of magnification at $z = 1 + i$ is $2\sqrt{2}$.

(ii) The angle of rotation at $z = 1 + i$ is $\frac{\pi}{4}$.

Sol. Here $\qquad f(z) = w = z^2$

$\therefore \qquad f'(z) = 2z$ and $f'(1 + i) = 2 + 2i$.

The coefficient of magnification at $z = 1 + i$ is $| f'(1 + i) | = \sqrt{4 + 4} = 2\sqrt{2}$.

The angle of rotation at $z = 1 + i$ is amp. $[f'(1 + i)] = \tan^{-1} \frac{2}{2} = \frac{\pi}{4}$.

3.33. SOME STANDARD TRANSFORMATIONS

1. Translation: w = z + c, where c is a complex constant

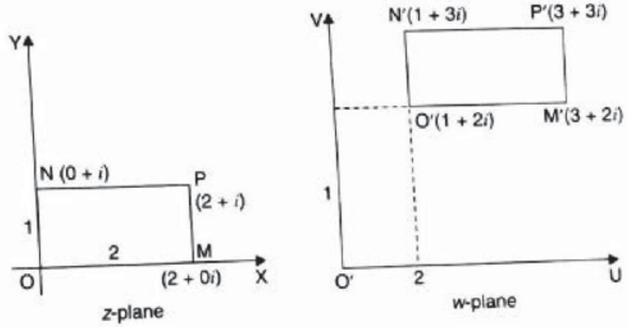
Let $\qquad z = x + iy, \quad c = a + ib$ and $w = u + iv$

then the transformation becomes $u + iv = (x + iy) + (a + ib) = (x + a) + i(y + b)$

so that $\qquad u = x + a$ and $v = y + b$

Thus the transformation is a mere translation of the axes and preserves the shape and size.

For example, the rectangle OMPN in z-plane is transformed to rectangle O′M′P′N′ in the w-plane under the transformation $w = z + (1 + 2i)$.



2. Rotation and Magnification: w = cz, where c is complex constant.

Let $c = \rho e^{i\alpha}$ $\quad z = re^{i\theta}$, $\quad$ and $w = \text{R}e^{i\phi}$

then the transformation becomes $\text{R}e^{i\phi} = \rho\, re^{i(\theta + \alpha)}$

$\therefore$ $\quad \text{R} = \rho r$ and $\phi = \theta + \alpha$

Thus the transformation maps a point (r, θ) in the z-plane into a point P′ $(\rho r, \theta + \alpha)$ in the w-plane. Hence the transformation consists of **magnification of the radius vector of P** by $\rho = |\,c\,|$ and its **rotation through an** angle $\alpha = \text{amp}\,(c)$.

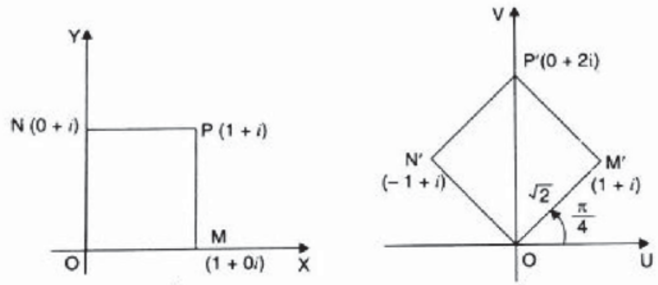
Thus under this transformation figure in w-plane is similar to the figure in z-plane (magnified by $|\,c\,|$) but rotated through an angle α.

Note 1. If $\alpha > 0$ then rotation is anticlock wise and if $\alpha < 0$ then rotation is clockwise.

Note 2. In $w = cz$; if C is real then $\alpha = 0$, then this transformation is only that of magnification (no rotation)

$\because$ in this case the two figures in z-plane and w-plane are similarly situated about their respective origins but figure in w-plane is |c| times figure in z plane. Such mapping is called Magnification.

For example the transformation $w = (1 + i)\, z$ maps the square OMPN bounded by $x = 0$, $y = 0, x = 1, y = 1$ in z plane to the square OM′P′N′ w-plane.



z-plane $\qquad$ w-plane

Here $\quad u + iv = (1 + i)\,(x + iy) = (x - y) + i\,(x + y)$

$\therefore$ $\quad u = x - y,\ v = x + y$

$\therefore$ $\quad x = 0$ maps into $u = -y,\quad v = y\quad$ *i.e.,* $\quad v = -u$

$\quad y = 0$ maps into $u = x,\quad v = x\quad$ *i.e.,* $\quad v = u$

$\quad x = 1$ maps into $u = 1 - y,\quad v = 1 + y\quad$ *i.e.,* $\quad u + v = 2$

$\quad y = 1$ maps into $u = x - 1,\quad v = x + 1\quad$ *i.e.,* $\quad v - u = 2$

$\therefore$ Square in z plane is mapped into square in the w plane bounded by $u = v$, $u = -v$, $u + v = 2$, $v - u = 2$

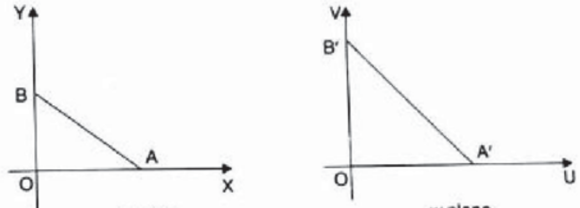
Verification: Here $i = 1 + i\quad \therefore\quad |\,c\,| = \sqrt{2}$

Each side of the square in w-plane is $|\,c\,|$ times (*i.e.,* $\sqrt{2}$) the side of the square is z-plane also amp of $c = \tan^{-1}\dfrac{1}{1} = \dfrac{\pi}{4}$ = and the sides of the square are rotated through an angle $\dfrac{\pi}{4}$.

This transformation is rotation as well as magnification.

Consider another example in which c is real:

Example: *The transformation $w = 2z$ maps the triangular region OAB bound by the lines $x = 0$, $y = 0$, $x + y = 1$ into a similar triangle OA′B′ in w- plane.*



z-plane $\qquad$ w-plane

Here $\quad w = 2z \qquad \therefore \qquad u + iv = 2(x + iy)$

$\therefore$ $\quad u = 2x \qquad\qquad v = 2y$

$\quad x = 0$ maps into $\quad u = 0$

$\quad y = 0$ maps into $\quad v = 0$

$\quad x + y = 1$ maps into $\dfrac{u}{2} + \dfrac{x}{2} = 1 \quad$ *i.e.,* $\quad u + v = 2$

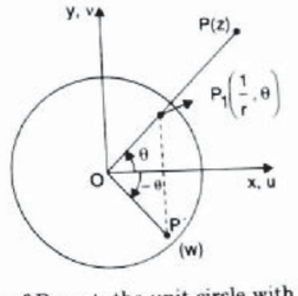
Varifiction: The two figures are similar but figure in w-plane is 2 times the figure is z-plane.

This transformation is only Magnification

3. Inversion: $\mathbf{w = \dfrac{1}{z}}$

Let $\qquad z = re^{i\theta}$ and $w = \text{R}e^{i\phi}$

then the transformation becomes $\text{R}e^{i\phi} = \dfrac{1}{r}e^{-i\theta}$ so that $\text{R} = \dfrac{1}{r}$ and $\phi = -\theta$.



Thus under the transformation $w = \dfrac{1}{z}$, a point P(r, θ) in z-plane is mapped into the point P $\left(\dfrac{1}{r}, -\theta\right)$.

Consider the w-plane superposed on the z-plane. If P is (r, θ) and P_1 is $\left(\dfrac{1}{r}, \theta\right)$, then

$\text{OP}_1 = \dfrac{1}{r} = \dfrac{1}{\text{OP}}\quad$ *i.e.,* OP . $\text{OP}_1 = 1$ so that P_1 is reverse of P w.r.t. the unit circle with centre O.

[The inverse of a point P w.r.t. a circle having centre O and radius k is defined as the point Q on OP such that OP, OQ = k^2]

The reflection P′ of P_1 in the real axis represents $w = \frac{1}{z}$. Thus the transformation $w = \frac{1}{z}$ is an **inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.**

Obviously, the transformation $w = \frac{1}{z}$ **maps the interior of the unit circle $|z| = 1$ into the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ into the interior of $|w| = 1$.**

However, the origin $z = 0$ is mapped to the point $w = \infty$, called the *point at infinity*.

Note. *This transformation* $w = \frac{1}{z}$ *maps a circle in z-plane to a circle in w-plane or to a straight line if the circle in z-plane passes through the origin.*

The general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

Let $w = u + iv = \frac{1}{z}$, then $z = \frac{1}{w}$ or $x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

so that $$x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

Substituting the values of x and y in (1), we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

or $$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

or $$\frac{1}{u^2+v^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

or $$c(u^2+v^2) + 2gu - 2fv + 1 = 0 \qquad ...(2)$$

If $c \neq 0$, the circle (1) does not pass through the origin and equation (2) represents a circle in the w-plane.

If $c = 0$, the circle (1) passes through the origin and equation (2) reduces to $2gu - 2fv + 1 = 0$ which is a straight line in the w-plane.

Regarding a straight line as a circle of infinite radius, we can say that the transformation $w = \frac{1}{z}$ maps circles into circles.

4. Bilinear Transformation: *(P.T.U., May 2005)*

A transformation of the form $$w = \frac{az+b}{cz+d} \qquad ...(1)$$

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called a **bilinear** or **Mobius transformation.**

The transformation given by (1) is conformal, since

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

The inverse mapping of (1) is $z = \frac{-dw+b}{cw-a}$ ...(2)

which is also a bilinear transformation.

The transformation (1) can be written as

$$cwz + wd - az - b = 0$$

which is linear both in w and z and hence the name bilinear transformation.

From (1), we observe that each point in the z-plane except the point $z = -\frac{d}{c}$ maps into a unique point in the w-plane. Similarly, from (2), we observe that each point in the w-plane except the point $w = \frac{a}{c}$ maps into a unique point in the z-plane. Considering the two exceptional points as points at infinity in respective planes, we can say that there is one to one correspondence between all points in the two plane.

Every bilinear transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ is the combination of basic transformations

(*i*) translation: $w = z + c$ (*ii*) rotation and magnification: $w = cz$

(*iii*) inversion: $w = \frac{1}{z}$

By actual division, we have $w = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z + \frac{d}{c}}$

Taking $w_1 = z + \frac{d}{c}$, $w_2 = \frac{1}{w_1}$, $w_3 = \frac{bc-ad}{c^2} w_2$, we get $w = \frac{a}{c} + w_3$

Thus, by these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane.

Since each of these auxiliary transformations maps circles into circles, hence a bilinear transformation also maps circles into circles.

Note 1. Cross Section: If four complex numbers z_1, z_2, z_3, z_4, are taken in order then $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross-section of z_1, z_2, z_3, z_4.

Note 2. *The cross-ratio is invariant under a bilinear transformation.*

Thus if w_1, w_2, w_3 and w_4 are the respective images of four distinct points z_1, z_2, z_3 and z_4, then

$$\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}.$$

Note 3. In the bilinear transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ dividing the numerator and denominator of the right hand side by one of the four constants, we observe that there are only three independent constants. Hence *three independent conditions are required to determine a bilinear transformation.*

ILLUSTRATIVE EXAMPLES

Example 1. *What is the region of w-plane into which the rectangular region is the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$ is mapped under the transformation $w = z + (2 - i)$?*

Sol. The given transformation is

$$w = z + (2 - i) \qquad i.e., \; u + iv = x + iy + 2 - i$$

$$\therefore \qquad u = x + 2 \qquad v = y - 1$$

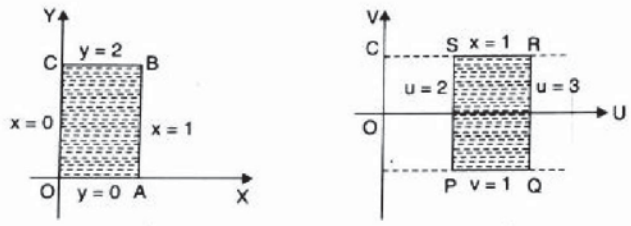
$x = 0$ maps into $u = 2$

$y = 0$ maps into $v = -1$

$x = 1$ maps into $u = 3$

$y = 2$ maps into $v = 1$

So the mapped region PQRS is also a rectangle bounded by $u = 2$, $v = -1$, $u = 3$, $v = 1$ (shown in the figure below)



z-plane *w*-plane

Example 2. *Consider the transformation $w = e^{\frac{i\pi}{4}}$ and determine the region in w-plane corresponding to the triangular region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the z-plane.*

Sol. The given transformation is

$$w = e^{\frac{i\pi}{4}} z = \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)(x + iy)$$

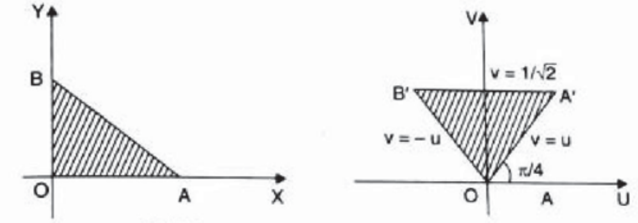
i.e.,
$$u + iv = \frac{1}{\sqrt{2}}(1 + i)(x + iy) = \frac{1}{\sqrt{2}}[(x - y) + i(x + y)]$$

$$\therefore \qquad u = \frac{1}{\sqrt{2}}(x - y); \quad v = \frac{1}{\sqrt{2}}(x + y)$$

$$x = 0 \text{ maps into } u = \frac{1}{\sqrt{2}}(-y), v = \frac{1}{\sqrt{2}}y; \quad \therefore \; v = -u$$

$$y = 0 \text{ maps into } u = \frac{1}{\sqrt{2}}x, v = \frac{1}{\sqrt{2}}x \quad \therefore \; v = u$$

$$x + y = 1 \text{ maps into } v = \frac{1}{\sqrt{2}}.1 \quad \therefore \; v = \frac{1}{\sqrt{2}}$$



z-plane *w*-plane

$\therefore$ The transformation $w = e^{\frac{i\pi}{4}} z$ transforms the triangle OAB in z-plane into a triangle OA'B' in w-plane rotated through an angle $\frac{\pi}{4}$.

Example 3. *Under the transformation $w = \frac{1}{z}$, find the image of the following curves*

(*i*) $| z - 2i | = 2$. (Kerala 2005)

(*ii*) $y - x + 1 = 0$. (P.T.U., May 2007)

Sol. The given transformation is

$$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w} \quad \text{or} \quad x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

so that
$$x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = -\frac{v}{u^2 + v^2} \qquad ...(1)$$

(*i*) The given curve is $| z - 2i | = 2$ or $| x + i(y - 2) | = 2$

or
$$x^2 + (y - 2)^2 = 2 \quad \text{or} \quad x^2 + y^2 - 4y = 0 \qquad ...(2)$$

which is a circle in the z-plane with centre (0, 2) and radius 2.

Substituting the values of x and y from (1) in (2), we get

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{4v}{u^2 + v^2} = 0 \quad \text{or} \quad \frac{u^2 + v^2}{(u^2 + v^2)^2} + \frac{4v}{u^2 + v^2} = 0$$

or
$$\frac{1}{u^2 + v^2} + \frac{4v}{u^2 + v^2} = 0$$

or $\quad 1 + 4v = 0$, a straight line which is the required image of the given curve.

(*ii*) $y - x + 1 = 0$ maps into

$$-\frac{v}{u^2 + v^2} - \frac{u}{u^2 + v^2} + 1 = 0 \; i.e., \; -u - v + u^2 + v^2 = 0$$

or $\quad u^2 + v^2 - u - v = 0$ which is a circle with centre at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$.

Example 4. *Find the image of infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also show the regions graphically.*

Sol. The given transformation is

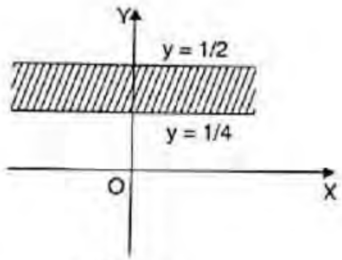
$$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w} \quad \text{or} \quad x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

so that
$$x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

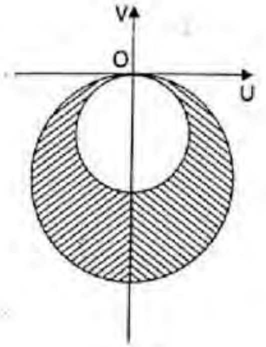
If
$$y = \frac{1}{4}, \text{ then } \quad \frac{-v}{u^2 + v^2} = \frac{1}{4}$$

or
$$u^2 + v^2 + 4v = 0 \quad \text{or} \quad u^2 + (v + 2)^2 = 4$$

If $y = \dfrac{1}{2}$, then $\dfrac{-v}{u^2 + v^2} = \dfrac{1}{2}$ or $u^2 + v^2 + 2v = 0$ or $u^2 + (v + 1)^2 = 1$



Z-plane



W-plane

Hence the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ is transformed into the region between the two circles

$$u^2 + (v + 2)^2 = 4, \text{ centre } (0, -2), \text{ radius } 2$$

and
$$u^2 + (v + 1)^2 = 1, \text{ centre } (0, -1), \text{ radius } 1.$$

Example 5. *Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the Lemniscate $\rho^2 = \cos 2\phi$.* (Bombay 2005; J.N.T.U. 2005)

Sol. Given transformation is $w = \dfrac{1}{z}$

$$w = \rho e^{i\phi} \qquad \text{Let } z = re^{i\theta}$$

$$\rho e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} \qquad \therefore \quad \rho = \frac{1}{r}, \phi = -\theta$$

Equation of the hyperbola is

$$x^2 - y^2 = 1 \quad i.e., \quad r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$$

or
$$r^2 \cos 2\theta = 1$$

or
$$\frac{1}{\rho^2} \cos 2(-\phi) = 1 \quad \text{or} \quad \rho^2 = \cos 2\phi.$$

Example 6. *Show that the transformation $w = \dfrac{2z + 3}{z - 4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$.*

Sol. The given transformation is $\quad w = \dfrac{2z + 3}{z - 4}$

The inverse transformation is $\quad z = \dfrac{4w + 3}{w - 2}$ ...(1)

Now the equation $x^2 + y^2 - 4x = 0$ can be written as $z\bar{z} - 2(z + \bar{z}) = 0$

Substituting for z and $\bar{z}$ from (1), we get

$$\frac{4w + 3}{w - 2} \cdot \frac{4\bar{w} + 3}{\bar{w} - 2} - 2\left(\frac{4w + 3}{w - 2} + \frac{4\bar{w} + 3}{\bar{w} - 2}\right) = 0$$

or
$$16w\bar{w} + 12w + 12\bar{w} + 9 - 2(4w\bar{w} + 3\bar{w} - 8w - 6 + 4w\bar{w} + 3w - 8\bar{w} - 6) = 0$$

or
$$22(w + \bar{w}) + 33 = 0 \quad \text{or} \quad 22(2u) + 33 = 0 \quad \text{or} \quad 4u + 3 = 0.$$

Example 7. *Show that $w = \dfrac{i - z}{i + z}$ maps the real axis of the z-plane into the circle $|w| = 1$ and the half plane $y > 0$ into the interior of the unit circle $|w| = 1$ in the w-plane.* (P.T.U. May 2007)

Sol. Given transformation is $w = \dfrac{i - z}{i + z}$

$$|w| = 1 \Rightarrow \left|\frac{i - z}{i + z}\right| = 1 \quad \text{or} \quad |i - z| = |i + z|$$

or
$$|i - x - iy| = |i + x + iy| \quad \text{or} \quad |-x + i(1 - y)| = |x + i(1 + y)|$$

or
$$\sqrt{x^2 + (1 - y)^2} = \sqrt{x^2 + (1 + y)^2} \quad \text{or} \quad (1 - y)^2 = (1 + y)^2$$

or
$$4y = 0 \quad \text{or} \quad y = 0$$

which is the equation of real axis in z-plane

Hence the real axis of z-plane is mapped into $|w| = 1$

Now the interior of $|w| = 1$ means

$$|w| < 1 \quad i.e., \quad \left|\frac{i - z}{i + z}\right| < 1 \quad i.e., \quad |i - z| < |i + z|$$

or
$$x^2 + (1 - y)^2 < x^2 + (1 + y)^2 \quad \text{or} \quad -4y < 0$$

or
$$y > 0$$

Hence the half plane $y > 0$ is mapped into the interior of the circle $|w| = 1$.

Example 8. *Show that the transformation $w = z^2$ maps the circle $|z - 1| = 1$ into the cardioid $\rho = 2(1 + \cos \phi)$, where $w = \rho e^{i\phi}$ in the w-plane.*

Sol. Let $z = re^{i\theta}$, then $w = z^2 \quad \Rightarrow \quad \rho e^{i\phi} = r^2 e^{2i\theta}$

so that
$$\rho = r^2 \quad \text{and} \quad \phi = 2\theta \qquad \text{...(1)}$$

Now the equation of circle in the z-plane is

$$|z - 1| = 1 \quad \text{or} \quad |x + iy - 1| = 1$$

or
$$|(x - 1) + iy| = 1 \quad \text{or} \quad (x - 1)^2 + y^2 = 1$$

or
$$x^2 + y^2 - 2x = 0 \quad \text{or} \quad r^2 - 2r \cos \theta = 0 \quad [\because \; z = x + iy = re^{i\theta} = r(\cos\theta + i \sin\theta)]$$

or
$$r = 2 \cos \theta \quad \text{or} \quad r^2 = 4 \cos^2 \theta$$

or
$$r^2 = 2(1 + \cos 2\theta) \quad \text{or} \quad \rho = 2(1 + \cos \phi) \qquad \text{[Using (1)]}$$

$\therefore$ The circle $|z - 1| = 1$ in z-plane transforms into the cardioid $\rho = 2(1 + \cos \phi)$ in w-plane.

Example 9. *Determine the region of the w-plane into which the first quadrant of z-plane is mapped by the transformation $w = z^2$.*

Sol. Proceeding as in example 5, $\phi = 2\theta$

For the first quadrant in z-plane, $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \phi < \pi$

Hence the first quadrant of z-plane is mapped into the upper half of w-plane.

Example 10. *Determine the region of the w-plane into which the region $\frac{1}{2} \le x \le 1$ and $\frac{1}{2} \le y \le 1$ is mapped by the transformation $w = z^2$.*

Sol. The given transformation is $w = z^2$ or $u + iv = (x + iy)^2 = (x^2 - y^2) + 2ixy$

so that $u = x^2 - y^2$ and $v = 2xy$

When $x = \frac{1}{4}$, we have $u = \frac{1}{4} - y^2$ and $v = y$ so that $v^2 = -\left(u - \frac{1}{4}\right)$

[a left-handed parabola with vertex $(\frac{1}{4}, 0)$ and latus rectum 1]

When $x = 1$, we have $u = 1 - y^2$ and $v = 2y$ so that $v^2 = -4(u - 1)$

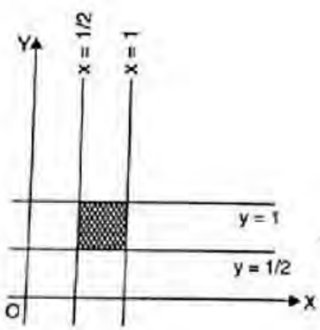
[a left-handed parabola with vertex (1, 0) and latus rectum 4]

When $y = \frac{1}{2}$, we have $u = x^2 - \frac{1}{4}$ and $v = x$ so that $v^2 = u + \frac{1}{4}$

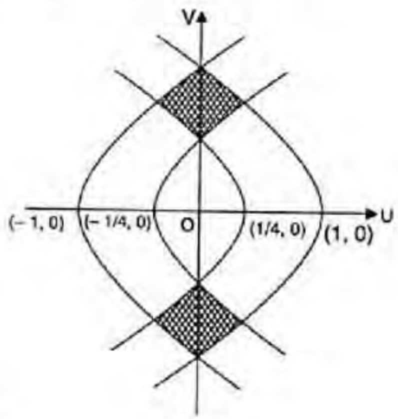
[a right-handed parabola with vertex $(-\frac{1}{4}, 0)$ and latus rectum 1]

When $y = 1$, we have $u = x^2 - 1$ and $v = 2x$ so that $v^2 = 4(u + 1)$

[a right-handed parabola with vertex (– 1, 0) and latus rectum 4]



Z-plane



W-plane

Thus the rectangular region bounded by the lines $x = \frac{1}{2}$, $x = 1$ and $y = \frac{1}{2}$, $y = 1$ maps into the region bounded by the parabolas

$$v^2 = (u - \frac{1}{4}),\ v^2 = -4(u - 1) \quad \text{and} \quad v^2 = u + \frac{1}{4},\ v^2 = 4(u + 1).$$

Example 11. *Show that the transformation $w = z + \frac{1}{z}$ maps the circle $|z| = c$ into the ellipse $u = \left(c + \frac{1}{c}\right)\cos\theta$, $v = \left(c - \frac{1}{c}\right)\sin\theta$. Discuss the case when $c = 1$.*

Sol. A point on the circle $|z| = c$ can be written as $z = ce^{i\theta}$

$\therefore$ $w = z + \frac{1}{z}$ becomes $u + iv = ce^{i\theta} + \frac{1}{c}e^{-i\theta}$

or $u + iv = c(\cos\theta + i\sin\theta) + \frac{1}{c}(\cos\theta - i\sin\theta) = \left(c + \frac{1}{c}\right)\cos\theta + i\left(c - \frac{1}{c}\right)\sin\theta$

$\therefore$ $u = \left(c + \frac{1}{c}\right)\cos\theta$ and $v = \left(c - \frac{1}{c}\right)\sin\theta$

which are the parametric equations of an ellipse.

When $c = 1$, we have $u = 2\cos\theta$ and $v = 0$

Since $|\cos\theta| \le 1$, we get $-2 \le u \le 2$ and $v = 0$

$\therefore$ The transformation gives a segment of the u-axis of length 4.

Example 12. *Show that the transformation $w = z + \frac{1}{z}$ converts the straight line $\arg z = \alpha$ $\left(|a| < \frac{\pi}{2}\right)$ into a branch of the hyperbola of eccentricity $\sec\alpha$.* (Bombay 2005 S)

Sol. Let $z = re^{i\theta}$ and $w = u + iv$

$$\therefore \quad u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$u = \left(r + \frac{1}{r}\right)\cos\theta,\ v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$\arg z = \alpha \Rightarrow \theta = \alpha$$

$$\therefore \quad u = \left(r + \frac{1}{r}\right)\cos\alpha,\ v = \left(r - \frac{1}{r}\right)\sin\alpha$$

Eliminate r; $r + \frac{1}{r} = \frac{u}{\cos\alpha}$, $r - \frac{1}{r} = \frac{v}{\sin\alpha}$

$$\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = \frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha}, \quad 4 = \frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha}$$

or $\frac{u^2}{4\cos^2\alpha} - \frac{v^2}{4\sin^2\alpha} = 1$ where $|\alpha| < \frac{\pi}{2}$ $\therefore$ this equation represents a branch of hyperbola in w-plane

Here $a^2 = 4\cos^2\alpha$, $b^2 = 4\sin^2\alpha$

Eccentricity e of the hyperbola is given by

$$b^2 = a^2 (e^2 - 1)$$

$$4 \sin^2 \alpha = 4 \cos^2 \alpha (e^2 - 1) \quad \therefore \quad e^2 - 1 = \tan^2 \alpha$$

$$\therefore \quad e^2 = 1 + \tan^2 \alpha = \sec^2 \alpha$$

$$\therefore \quad e = \sec \alpha.$$

$\therefore$ Eccentricity of the hyperbola = $\sec \alpha$

Example 13. *Discuss the transformation $w = e^z$ and show that it transforms the region between the real axis and a line parallel to real axis at $y = \pi$ into the upper half of the w-plane.*

Sol. Let $w = Re^{i\phi}$, then the given transformation becomes

$$Re^{i\phi} = e^{x+iy} = e^x \cdot e^{iy} \text{ so that } R = e^x \text{ and } \phi = y$$

The real axis *i.e.*, $y = 0$ maps into the positive u-axis $\phi = 0$ in the w-plane.

The line $y = \pi$ maps into the negative u-axis $\phi = \pi$ in the w-plane.

Thus the region between the lines $y = 0$ and $y = \pi$ maps into the upper half of w-plane.

Note. The region between the lines $y = 0$ and $y = -\pi$ maps into the lower half of w-plane.

The region between the lines $y = c$ and $y = c + 2\pi$ maps into the whole of the w-plane, since e^z is periodic with period $2\pi i$.

The imaginary axis $x = 0$ maps into a unit circle $R = e^0 = 1$ in the w-plane.

Example 14. *Show that the transformation $w = e^z$ is always conformal. Under the mapping, find the images of the regions.*

(*i*) *the line segment $0 < y < A$, $A < 2\pi$, $x < 0$*

(*ii*) *the rectangle bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = \pi$*

(*iii*) *The rectangular region bounded by the lines $a \le x \le b$; $c \le y \le d$.* (P.T.U., Dec. 2005)

Sol.

$$w = f(z) = e^z$$

$$f'(w) = e^z \neq 0 \text{ for any } z$$

$\therefore$ The transformation is conformable in the region of w *i.e.*, for all values of z

Let $\quad z = x + iy$ and w-$Re^{i\phi}$

$$\therefore \quad Re^{i\phi} = e^{x+iy} = e^x (e^{iy})$$

$$\therefore \quad R = e^x, \ \phi = y$$

(*i*) For $0 < y < A$, $A < 2\pi$, $x < 0$ we have $0 < \phi < A$ *i.e.*, $0 < \arg w < A < 2\pi$

and $\quad x < 0 \Rightarrow R < 1 \Rightarrow 0 < |w| < 1$

The image curves are given in Fig. (*i*).

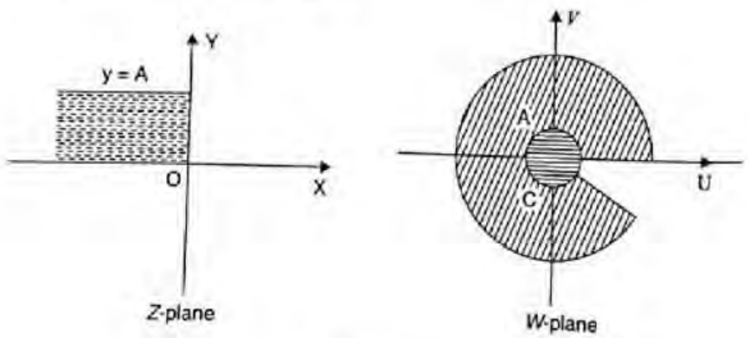


Fig. (*i*)

Image is the interior of the portion of the circle $|R| = 1$ whose angle of rotation is $A < 2\pi$.

(*ii*) The rectangle formed by $x = 0$, $y = 0$, $x = 1$ and $y = \pi$

$$x = 0 \Rightarrow R = 1$$

$$y = 0 \Rightarrow \phi = 0$$

$$x = 1 \Rightarrow R = e$$

$$y = \pi \Rightarrow \phi = \pi$$

If we take $w = u + iv$, then $u + iv = e^{x+iy} = e^x (\cos y + i \sin y)$

$$\therefore \quad u = e^x \cos y, \ v = e^x \sin y$$

$$x = 0, y = 0 \Rightarrow u = 1, v = 0$$

$$x = 1, y = 0 \Rightarrow u = e, v = 0$$

$$x = 1, y = \pi \Rightarrow u = -e, v = 0$$

$$x = 0, y = \pi \Rightarrow u = -1, v = 0.$$

The image curves are given in Fig. (*ii*),

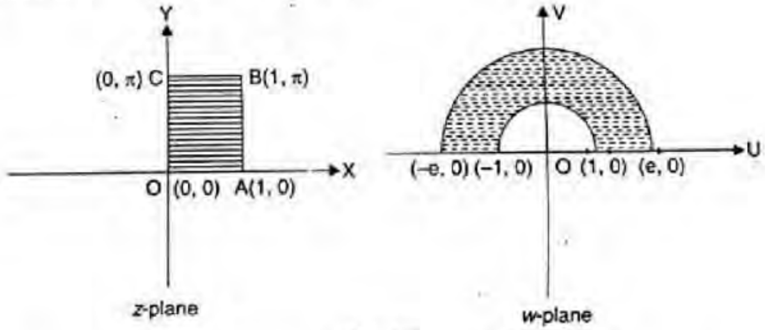


Fig. (*ii*)

Image is the region included between two semicircles $|R| = 1$ and $|R| = e$

(*iii*) The rectangle bounded by the lines $a \le x \le b$, $c \le y \le d$

$$a \le x \le b \Rightarrow e^a \le e^x \le e^b \Rightarrow e^a < R < e^b$$

$$c \le y \le d \Rightarrow c \le \phi \le d$$

$\therefore$ The image is shown in Fig. (*iii*)

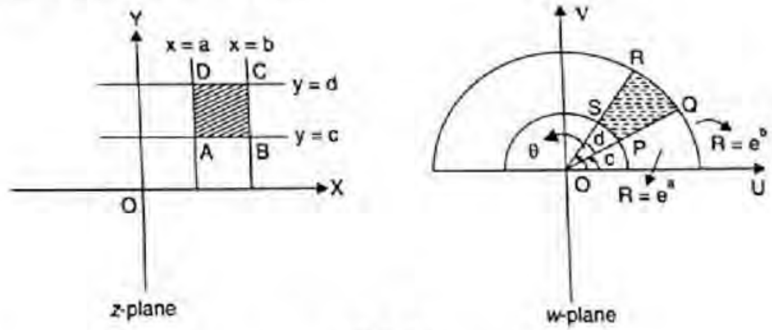


Fig. (*iii*)

The image is the region PQRS.

Example 15. *Discuss the transformation $w = \sin z$.*

Sol. The given transformation is

$$w = u + iv = \sin (x + iy)$$

or $$u + iv = \sin x \cosh y + i \cos x \sinh y$$

so that $$u = \sin x \cosh y$$

and $$v = \cos x \sinh y \qquad ...(1)$$

Eliminating y from equation (1), we get

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \qquad [\because \cosh^2 y - \sinh^2 y = 1]$$

Thus the straight lines $x = c$ in the z-plane are mapped into confocal hyperbolas in the w-plane.

Eliminating x from equation (1), we get $\dfrac{u^2}{\cosh^2 y} + \dfrac{v^2}{\sinh^2 y} = 1$

Thus the straight lines $y = c$ in the z-plane are mapped into confocal ellipses.

The lines $x = 0$ and $y = 0$ map into the lines $u = 0$ and $v = 0$ respectively in the w-plane.

Example 16. *Show that transformation $w = \cosh z$ maps the lines parallel to x-axis and lines parallel to y-axis into confocal central conics.*

Sol. Given transformation is $w = \cosh z$

i.e., $$u + iv = \cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y$$

$$\therefore \quad \left.\begin{aligned} u &= \cosh x \cos y \\ v &= \sinh x \sin y \end{aligned}\right] \qquad ...(1)$$

Lines parallel to x-axis are given by $y = k$

$$u = \cosh x \cos k$$
$$v = \sinh x \sin k$$

$$\therefore \quad \cosh x = \frac{u}{\cos k}$$

$$\sinh x = \frac{v}{\sin k}$$

Squaring and subtracting, we get

$$\frac{u^2}{\cos^2 k} - \frac{v^2}{\sin^2 k} = 1,$$ which is a hyperbola in w-plane.

Here $$a^2 = \cos^2 k,\ b^2 = \sin^2 k$$

$\therefore$ Eccentricity of the curve is given by $b^2 = a^2 (e^2 - 1)$ ...(2)

$$\sin^2 k = \cos^2 k\, (e^2 - 1) \text{ or } e^2 = 1 + \tan^2 k = \sec^2 k$$

$$\therefore \quad e = \sec k$$

Foci of (2) are $(\pm ae, 0) = (\pm \cos k \,.\, \sec k, 0) = (\pm 1, 0)$

Centre of (2) is (0, 0)

Now lines paralled to y- axis are $x = k'$

$\therefore$ From (1), $$u = \cosh k' \cos y$$
$$v = \sinh k' \sin y$$

$$\frac{u}{\cosh k'} = \cos x$$

$$\frac{v}{\sinh k'} = \sin x$$

Squaring and adding $\dfrac{u^2}{\cosh^2 k'} + \dfrac{v^2}{\sinh^2 k'} = 1$ which is an ellipse ...(3)

Here $a^2 = \cosh^2 k'$, $b^2 = \sinh^2 k'$

Eccentricity of ellipse is given by $b^2 = a^2 (1 - e^2)$ *i.e.,* $\sinh^2 k' = \cosh^2 k' (1 - e^2)$ or

$$e^2 = 1 - \tanh^2 k' = \operatorname{sech}^2 k'$$

$\therefore$ e sech k'

$\therefore$ Foci of (3) are $(\pm ae, 0) = (\pm \cosh k' \operatorname{sech} k', 0) = (\pm 1, 0)$

Centre of (3) is (0, 0)

$\therefore$ The centres and foci of both the conics are same.

Hence lines || to x-axis and y-axis maps into confocal central conics.

Example 17. *The bilinear transformation $w = \dfrac{az + b}{cz + d}$ transforms the circle arg $\dfrac{z - z_1}{z - z_2} = \lambda$ into similar circle of arg $\dfrac{w - w_1}{w - w_2}$ = constant where w_1, w_2 corresponds to z_1, z_2 respectively.*

Sol. Here $$w = \frac{az + b}{cz + d}$$

$$\therefore \quad w_1 = \frac{az_1 + b}{cz_1 + d}; \; w_2 = \frac{az_2 + b}{cz_2 + d}$$

So that $$\frac{w - w_1}{w - w_2} = \frac{\dfrac{az + b}{cz + d} - \dfrac{az_1 + b}{cz_1 + d}}{\dfrac{az + b}{cz + d} - \dfrac{az_2 + b}{cz_2 + d}} = \frac{cz_2 + d}{cz_1 + d} \cdot \frac{z - z_1}{z - z_2}$$

$$\therefore \quad \frac{w - w_1}{w - w_2} = \mu \frac{z - z_1}{z - z_2}$$ where μ is complex constant and $= \dfrac{cz_2 + d}{cz_1 + d}$

$$\arg \frac{w - w_1}{w - w_2} = \arg \mu \left(\frac{z - z_1}{z - z_2}\right)$$

$$= \arg \mu + \arg \frac{z - z_1}{z - z_2}$$

$$= \arg \mu + \lambda$$

$= k$ (say) where k is real because argument of a complex number is always real.

Now, $\arg \dfrac{w-w_1}{w-w_2} = k$; represents a circle in w-plane passing through the points w_1 and w_2 corresponding to z_1 and z_2 in the z-plane.

Example 18. *Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$.*

Hence find the image of $|z| < 1$. (Bombay 2006)

Sol. Let the required bilinear transformation be $w = \dfrac{az+b}{cz+d}$ ...(1)

Substituting the corresponding values of w and z in (1), we get

$$i = \frac{a+b}{c+d}, \quad 0 = \frac{ai+b}{ci+d}, \quad -i = \frac{-a+b}{-c+d}$$

These equations can be written as

$$(a+b) - i(c+d) = 0 \qquad \text{...(2)}$$

$$b + ia = 0 \qquad \text{...(3)}$$

and $$(-a+b) + i(-c+d) = 0 \qquad \text{...(4)}$$

From (3), $b = -ia$

Adding (2) and (4), $2b - 2ic = 0$ or $c = \dfrac{b}{i} = -a$ [Using (3)]

Subtracting (4) from (2), $2a - 2id = 0$ or $d = \dfrac{a}{i} = -ia$

Substituting for b, c, d in (1), we get $w = \dfrac{az-ia}{-az-ia}$ or $w = \dfrac{i-z}{i+z}$ ...(5)

which is the required bilinear transformation.

Now, from (5), $z = i\,\dfrac{1-w}{1+w}$

$\therefore$ $|z| < 1$ is mapped into the region

$$\left| i\,\frac{1-w}{1+w} \right| < 1 \quad \text{or} \quad \frac{|i||1-w|}{|1+w|} < 1$$

or $$|1-w| < |1+w| \qquad |\because\ |i| = 1$$

or $$|1-u-iv| < |1+u+iv| \quad \text{or} \quad (1-u)^2 + v^2 < (1+u)^2 + v^2$$

or $$u > 0.$$

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped into the entire half of the w-plane to the right of the imaginary axis.

EXERCISE 3.9

1. Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$.
2. Find the image of the triangle with vertices at i, $1+i$, $1-i$ in the z-plane under the transformation
$$w = 3z + 4 - 2i.$$
3. Determine the region in the w-plane in which the rectangle bounded by the lines $x = 0, y = 0, x = 2$ and $y = 3$ is mapped under the transformation $w = \sqrt{2}\,e^{i\frac{\pi}{4}}\,.\,z$
4. Consider the transformation $w = 2z$ and determine the region D′ of the w-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ in the z- plane is mapped under this transformation .
5. Find the image of the semi-infinite strip $x > 0$, $0 < y < 2$, under the transformation $w = iz + 1$. Show the region graphically.
6. (*a*) Find the image of the circle $|z - 3| = 5$ under the mapping $w = \dfrac{1}{z}$.

   (*b*) Show that under the transformation $w = \dfrac{1}{z}$, the circle $x^2 + y^2 - 6x = 0$, is transformed into a straight line in the w-plane.
7. Show that the map of the real axis of the z-plane on the w-plane by the transformation $w = \dfrac{1}{z+i}$ is a circle and find its centre and radius.
8. Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane.
9. Under the transformation $w = \dfrac{z-i}{1-iz}$, find the map of the circle $|z| = 1$ in the w- plane.
10. Show that the transformation $w = i\,\dfrac{1-z}{1+z}$, transforms the circle $|z| = 1$ into the real axis of w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.
11. Show that under the transformation $w = \dfrac{z-i}{z+i}$, real axis in the z-plane is mapped into the circle $|w| = 1$, What portion of the z-plane corresponds to the interior of the circle ?
12. Under the transformation $w = z^2$, obtain the map in the w-plane of the square with vertices (0, 0), (2, 0), (2, 2), (0, 2) in the z-plane.
13. Find the images of the straight lines $x = 0, y = 0, x = 1$ and $y = 1$ under the transformation $w = z^2$.
14. Determine the region of the w-plane in to which the triangle formed by $x = 1, y = 1$ and $x + y = 1$ is mapped under the transformation $w = z^2$.
15. Show that the transformation $w = \sqrt{z}$ maps the domain in the z-plane to the right of the line $x = a$ into the interior of a hyperbola in the w-plane.
16. Show that the map of the circle $|z| = 2$ under the transformation $w + 2i = z + \dfrac{1}{z}$ is an ellipse, find its axes and centre.

17. Show that the transformation $w = z + \frac{a^2 - b^2}{4z}$ transforms the circle $|z| = \frac{1}{2}(a + b)$ in the z-plane into an ellipse of semi axes a, b in the w-plane.

18. Find the bilinear transformation which maps:
(a) the points $z = 1, -i, -1$ onto the points $w = i, 0, -i$.
(b) the points $z = 1, i, -1$ onto the points $w = 0, 1, \infty$. *(V.T.U. 2006)*
(c) the points $z = 0, -i, -1$ into the points $w = i, 1, 0$. Find the image of the line $y = mx$ under this transformation.

19. Show that the condition for transformation $w = \frac{az + b}{cz + d}$ to make the circle $|w| = 1$, correspond to a straight line in the z-plane is $|a| = |c|$.

20. Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = 2, i, -2$ respectively. Also find the fixed points of the transformation.

Answers

1. $(u - 3)^2 + (v - 2)^2 = 4$
2. A triangle with vertices $(4, -1)$, $(7, 1)$ and $(7, -3)$.
3. Rectangular region bounded by the lines $v = -u$, $v = u$, $u + v = 4$, $v - u = 6$.
4. A triangular region bounded by the lines $u = 0$, $v = 0$, $u + v = 2$.
5. $-1 < u < 1, v > 0$
6. $\left| w + \frac{3}{16} \right| = \frac{5}{16}$
7. $\left(0, -\frac{1}{2}\right); \frac{1}{2}$
9. $v = 0$
11. $y > 0$
12. $(0, 0)$, $(4, 0)$, $(0,8)$, $(-4, 0)$; and isosceles triangle.
13. Negative real axis, positive real axis, $v^2 = 4(1 - u)$, $v^2 = 4(1 + u)$.
14. Region bounded by the parabolas $v^2 = 4(1 \pm u)$, $u^2 = 1 - 2v$
16. $\frac{5}{2}, \frac{3}{2}$; $(0, -2)$
18. (a) $w = \frac{z + i}{z - i}$ (b) $w = \frac{i(1 - z)}{1 + z}$
(c) $w = \frac{i(1 + z)}{1 - z}$; $m(u^2 + v^2) + 2u - m = 0$.
20. $w = \frac{2i - 6z}{iz - 3}$; For fixed points $w = z$ gives
$$iz^2 + 3z - 2i = 0 \quad \text{or} \quad z^2 - 3iz - 2 = 0$$
or $(z - i)(z - 2i) = 0 \quad \therefore \quad z = i, 2i$