Fourier Series

Fourier Series is an infinite series representation of periodic function in terms of trignometric function (sin x & cos x)

— The fourier series for the function $f(x)$ in the interval $c < x < c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n, are called fourier cofficients.

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \quad , \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Even function $f(x) = f(-x)$

eg: $\cos x$, x^2

$$\boxed{\int_{-l}^{l} f(x)\,dx = 2\int_0^l f(x)\,dx}$$

Odd function $f(x) = -f(-x)$

eg: $\sin x$, x, x^3, $\underset{(-)(+)}{x\cos x}$ = $\boxed{\int_{-l}^{l} f(x)\,dx = 0}$

When $f(x)$ is odd $(-\pi < x < \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx =$$

When $f(x)$ is even $(-\pi < x < \pi)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Question find the fourier series if $f(x) = x$ in the interval $-\pi < x < \pi$

Sol $f(x)$ is a odd function

so $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \int \frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n} - 0\right] = \frac{-2(-1)^n}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin nx\,dx$$

$$= \left(+2\sin x - \frac{2}{2}\sin nx + \frac{2}{3}\sin nx - \cdots\cdots\right)$$

Question Find Fourier Series of $f(x) = x^2$, $-\pi < x < \pi$ & deduce from it the relation

i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2}$

ii) $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - - - -$

iii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} - - - -$

Sol x^2 even function

so $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \left.\frac{2x^3}{\pi \times 3}\right]_0^{\pi} = \boxed{\frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \int \frac{2x \sin nx}{n}\,dx\right.$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \int \frac{2\cos nx}{n^2}\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\left(-\frac{\cos x}{1} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} - - - - -\right.$$

$\rightarrow$ Let $\cancel{x^2} = \cancel{\frac{\pi^2}{6}}$ $\quad \cancel{x} = \cancel{\frac{\pi}{\sqrt{6}}}$ Let $x = \pi$

$x^2 = \pi^2$

$$\pi^2 = x^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} - - - -\right)$$

$$\frac{2\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} - - -\right)$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} - - - - - \quad (1)$$

$\rightarrow$ Put $x = 0$

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} \cdots\right)$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} \cdots \quad \text{--- (2)}$$

Add (1) & (2)

$\rightarrow$

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right) + \left(1 - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right)$$

$$\frac{\pi^2}{4} = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots \quad \text{--- (3)}$$

Hence proved.

Question $f(x) = \frac{1}{4}(\pi - x)^2 \quad (0 < x < 2\pi)$

Prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Sol

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx =$$

$$\frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2\,dx = \frac{1}{4\pi}\left[\pi^2 x + \frac{x^3}{3} - \frac{2x^2\pi}{2}\right]_0^{2\pi}$$

$$\frac{1}{4\pi}\left[2\pi^3 + \frac{\pi^3}{3} - 4\pi^3\right] = \boxed{\frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^2\sin nx}{n} + \int\frac{2(\pi-x)\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^2\sin nx}{n} + \frac{2(\pi-x)\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[0 + \frac{2(\pi-2\pi)}{n^2} + \cancel{\frac{2}{n^2}} + \frac{2\pi}{n^2} - \cancel{\frac{2}{n^2}}\right]$$

$$= \frac{1}{4\pi}\left[\frac{-2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \boxed{\frac{+8}{n^2}}$$

$$a_n = \boxed{\frac{1}{n^2}}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{\cos nx}{n}\right) + 2\int\frac{(\pi-x)\cos nx\,dx}{n}\right]$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) + \frac{2(\pi-x)\sin nx}{n^2} + 2\int\frac{\sin nx\,dx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) + \frac{2(\pi-x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^2}{n} - \frac{2\pi(0)}{n^2} - \frac{2}{n^3} + \frac{\pi^2}{n} + \frac{2}{n^3}\right]$$

$$= 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + \sum_{n=1}^{\infty} 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

$\longrightarrow$ Put $x = 0$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right)$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots = \frac{\pi^2}{6} \longleftarrow (1)$$

$\longrightarrow$ $x = \pi$

$$0 = \frac{\pi^2}{12} - \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots \longrightarrow (2)$$

Adding (1) & (2)

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots \longrightarrow (3)$$

Question Expand $f(x) = x\sin x$, $0 < x < 2\pi$ as a fourier series.

Sol $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$

$$= \frac{1}{\pi}\Big[x(-\cos x) + \sin x\Big]_0^{2\pi} = \boxed{-2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx \sin x)\,dx$$

Q- $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

Sol $a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1}{\pi}(1-e^{-2\pi})$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[-\cos nx\, e^{-x} - \int \sin nx\, e^{-x}\,dx\right]_0^{2\pi}$$

$$\frac{1}{\pi}\left[-\cos nx(e^{-x}) + \sin nx\, e^{-x} - \int \cos nx\, e^{-x}\,dx\right.$$

using $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \boxed{\frac{1-e^{-2\pi}}{\pi(1+n^2)}}$$

$$\boxed{b_n = \frac{(1-e^{-2\pi})n}{\pi(1+n^2)}}$$

$$f(x) = \frac{(1-e^{-2\pi})}{2\pi} + \left(\frac{1-e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \left(\frac{1-e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}$$

$$f(x) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x - - -\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x - - -\right)\right]$$

Question $f(x) = e^{ax}$ $(-\pi < x < \pi)$ Hence derive series for $\frac{\pi}{\sinh \pi}$

Solution $f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \left(\frac{e^{ax} - e^{-ax}}{a\pi}\right) = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\, dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}a(-1)^n - ae^{-\pi x}\cancel{\cos}(-1)^n\right]$$

$$= \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+\pi^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\, dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin \cancel{b}x - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{-2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$f(x) = \frac{\cancel{2}\sinh a\pi}{\cancel{2}a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\cos nx\, dx$$

$$+ \sum_{n=1}^{\infty}\frac{-2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\sin nx\, dx$$

Question Discontinous functions

$$f(x) = x \qquad 0 \le x \le \pi$$
$$= 2\pi - x \qquad \pi \le x \le 2\pi$$

Sol

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] + \frac{1}{\pi}\left[\cancel{4\pi^2} - \cancel{\frac{4\pi^2}{2}} - \cancel{2\pi^2} + \frac{\pi^2}{2}\right]$$

$$= \boxed{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{(2\pi - x)\sin nx}{n} + \int \frac{\sin nx}{n}\,dx\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] + \frac{1}{\pi}\left[\frac{(2\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^2}{n^2} - \frac{1}{n^2}\right] + \frac{1}{\pi}\left[\frac{-1}{n^2} + \frac{(-1)^2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left((-1)^2 - (1)\right) \qquad \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Question $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$ prove $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1} \frac{\cos nx}{4n^2}$

Sol

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \int_0^{\pi} \sin x\, dx = \boxed{\frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(x+nx) + \sin(x-nx)\right]dx$$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$= \frac{1}{2\pi}\times\left[-\frac{\cos x(1+n)}{(1+n)} \div \frac{\cos x(1-x)}{(1-x)}\right]_0^{\pi}$$

$$\frac{1}{2\pi}\left[-\frac{(-1)^n}{1+n} - \frac{(-1)^n}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$\begin{cases} 0, & \text{odd} \\ \frac{-2}{\pi(n^2-1)} & \text{even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\, dx = \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = \boxed{0}$$

$$b_1 = \frac{1}{2}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} a_n \cos nx\, dx$$

$$\frac{1}{\pi} + \left(-\frac{2}{\pi}\right)\left[\frac{\cos 2n}{(2^2-1)} + \frac{\cos 4x}{(4^2-1)} - \cdots\cdots\right] + \frac{1}{2}\sin x$$

Change Of Interval

Consider a periodic function $f(x)$ defined in the interval $c < x < c+2l$

To change the interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l}$$

So when $x = c$

$$z = \frac{\pi c}{l} = \boxed{d} \text{ (say)}$$

& when $x = c + 2l$

$$z = \frac{\pi(c+2l)}{l} = \frac{\pi c}{l} + 2\pi = \boxed{d + 2\pi}$$

Thus the function $f(x)$ of period $2l$ in $(c, c+2l)$ is transformed to the function $f\left(\frac{lz}{\pi}\right) = F(z)$, say of period 2π in $(d, d+2\pi)$ & the latter function can be expressed as the fourier series

$$\boxed{F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz}$$

$$a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\,dz \qquad a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz\,dz$$

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin nz\,dz$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \quad , \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Question $f(x) = x - x^2 \quad -1 < x < 1$

Solution $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

$(l = 1)$

$$a_0 = \int_{-1}^{1} (x - x^2)\,dx = \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = \boxed{-\frac{2}{3}}$$

$$a_n = \int_{-1}^{1} (x - x^2)\cos n\pi x\,dx = \frac{x \sin n\pi x}{n\pi} - \int \frac{\sin \pi n x}{\pi n}dx - \int x^2 \cos n\pi x\,dx$$

$$= \frac{x \sin n\pi x}{n\pi} + \frac{\cos \pi n x}{\pi^2 n^2} - \frac{x^2 \sin n\pi x}{n\pi} + \int \frac{2x \sin n\pi x}{n\pi}\,dx$$

$$= \frac{x \sin n\pi x}{n\pi} + \frac{\cos \pi n x}{\pi^2 n^2} - \frac{x^2}{n\pi}\sin n\pi x + \frac{2x \cos n\pi x}{\pi^2 n^2} - \frac{2}{n\pi}\int \cos n\pi x\,dx$$

$$= -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2)\sin n\pi x\,dx = \int_{-1}^{1} x \sin n\pi x\,dx - \int_{-1}^{1} x^2 \sin n\pi x\,dx$$

$$= 2\int_0^1 x \sin n\pi x\,dx - 0 = 2\left[\frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\left(-\frac{(-1)^n}{n\pi} - \frac{1(0)}{n\pi}\right) = \frac{-2(-1)^n}{n\pi}$$

$$f(x) = -\frac{1}{3} + \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} - - - \right) + \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} - - -\right)$$

Question: $f(x) = x^2 - 2$, when $-2 \le x \le 2$

Sol $f(x) = x^2 - 2 = \frac{a^2}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$ $\quad l = 2$

$$a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)\,dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = -\frac{4}{3}$$

$$a_n = \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[(x^2-2)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2x\left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \frac{16\cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$$f(x) = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2}, \dots\right)$$

<u>Question</u> $f(x) = e^{-x}$ $(-l, l)$

$$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \frac{1}{l}\left(e^{l} - e^{-l}\right)\ \frac{2\sinh l}{l}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{\left(1+\frac{\pi^2 n^2}{l^2}\right)}\left(-1\cos\frac{n\pi x}{l} + \frac{\pi n}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

using $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \frac{l}{l^2+(n^2\pi^2)}\left[-e^{-l}\cos n\pi + e^{l}\cos n\pi\right] = \frac{2l\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^{l}-e^{-l}}{2}\right)$$

$$= \frac{2l\cos n\pi}{l^2+n^2\pi^2}(\sinh l)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

using $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$\frac{1}{l^2+(n\pi)^2}\left[\frac{n\pi}{l}\left(e^{-l}-e^{l}\right)\cos n\pi\right] = \frac{2n\pi\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^{l}-e^{-l}}{2}\right)$$

$$= \frac{2n\pi\cos n\pi}{l^2+n^2\pi^2}(\sinh l)$$

$$e^{-x} = \sinh l\left(\frac{2}{l}\right) + 2l\sinh l\sum_{n=1}^{\infty}\frac{\cos n\pi}{l^2+n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$+\ 2\pi\sinh l\sum_{n=1}^{\infty}\frac{\cos n\pi}{l^2+n^2\pi^2}\sin\frac{n\pi}{l}$$