# Named Distribution

| Named Distribution | Notation | Range | Use Case | PMF/PDF | CDF | Expected Value $E(X)$ | Variance $V(X)$ | Moment Generating Function $M(t)$ | Method of Momenet | Maximum Likelihood | Sufficient Statistic | Note |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bernoulli | $Ber(p)$ | $x = 0, 1$ | Single trials. | $p^x(1-p)^{1-x}$ | | $p$ | $p(1-p)$ | $(1-p)+pe^t$ | $\hat{p} = \bar{X}$ | $\hat{p} = \bar{X}$ | | |
| Binomial | $Bin(n, p)$ | $x = 0, 1, \ldots$<br>$n \in \mathbb{N}$ | $x$ is number of successes out of $n$ trials. | $\binom{n}{x}p^x(1-p)^{n-x}$ | | $np$ | $np(1-p)$ | $(1-p+pe^t)^n$ | | $\hat{p} = \frac{\bar{X}}{n}$ | | |
| Geometric | $Geo(p)$ | $x = 1, 2, \ldots$ | $x$ is number of trials until first success. | $(1-p)^{x-1}p$ | $1-(1-p)^x$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ | $\frac{e^t p}{1-(1-p)e^t}$ | $\hat{p} = \frac{1}{\bar{X}}$ | $\hat{p} = \frac{1}{\bar{X}}$ | | Memoryless[1] |
| Negative Binomial | $NegBin(r, p)$ | $x = r, r+1, \ldots$<br>$r \in \mathbb{N}$ | $x$ is number of trials until $r$ successes. | $\binom{x-1}{r-1}p^r(1-p)^{x-r}$ | | $\frac{r}{p}$ | $\frac{r(1-p)}{p^2}$ | $\left[\frac{e^t p}{1-(1-p)e^t}\right]^2$ | | | | |
| Hyper Geometric | $HypGeo(m, r, n)$ | $x = 1, \ldots, n$<br>$m, r, n \in \mathbb{N}$ | Pick $m$ out of $n$ balls with $r$ blacks. $x$ is the number of black ball picked. | $\frac{\binom{r}{x}\binom{n-r}{m-x}}{\binom{n}{m}}$ | | $m\left(\frac{r}{n}\right)$ | $m\left(\frac{r}{n}\right)\left(1-\frac{r}{n}\right)\left(\frac{n-m}{n-1}\right)$ | | | | | $n$ is population, $m$ is number of trials, $r$ is number of black ball. |
| Poisson | $Pois(\lambda)$ | $x = 0, 1, \ldots$<br>$\lambda > 0$ | Something happens on average $\lambda$ times for a period. $x$ is the number of observe happens. | $\frac{\lambda^x e^{-\lambda}}{x!}$ | | $\lambda$ | $\lambda$ | $e^{\lambda(e^t-1)}$ | | $\hat{\lambda} = \bar{X}$ | | Poisson can approximate Binomial when $n \to \infty$ and $p \to 0$, that is: $X \sim Bin(n, p)$ then $X \sim Pois(np)$ |
| Uniform | $Uni(a, b)$ | $x \in [a, b]$<br>$a, b \in \mathbb{R}$<br>$a \leq b$ | | $\frac{1}{\beta - \alpha}$ | $\frac{x-a}{b-a}$ | $\frac{b+a}{2}$ | $\frac{(b-a)^2}{12}$ | $\frac{e^{tb} - eta}{t(b-a)}$ | | $\hat{\beta} = \max\{X_1, \ldots, X_n\}$ | $\beta : \max\{X_1, \ldots, X_n\}$ | |
| Exponential | $Exp(\lambda)$ | $x \geq 0$<br>$\lambda > 0$ | Something happens on average $\lambda$ times for a period. $x$ is the time to wait for one occurance. | $\lambda e^{-\lambda x}$ | $1 - e^{-\lambda x}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\frac{\lambda}{\lambda - t}$ | $\hat{\lambda} = \frac{1}{\bar{X}}$ | $\hat{\lambda} = \frac{1}{\bar{X}}$ | | Memoryless[1], Often model time. |
| Gamma | $Ga(\alpha, \lambda)$ | $x > 0$<br>$\alpha > 0$<br>$\lambda > 0$ | Something happens on average $\lambda$ times for a period. $x$ is the time to wait for $\alpha$ occurances. | $\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$ | | $\frac{\alpha}{\lambda}$ | $\frac{\alpha}{\lambda^2}$ | $\left[\frac{\lambda}{\lambda - t}\right]^\alpha$ | | | $\alpha : \sum^n \ln X_i$ | $Ga(1, \lambda) = Exp(\lambda)$ |
| Normal (Gaussian) | $N(\mu, \sigma^2)$ | $x \in \mathbb{R}$<br>$\mu \in \mathbb{R}$<br>$\sigma \in \mathbb{R}$ | | $\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$ | | $\mu$ | $\sigma^2$ | $e^{\mu t}e^{(\sigma^2 t^2)/2}$ | | $\hat{\mu} = \bar{X}$<br>$\hat{\sigma^2} = \frac{1}{n}\sum_i(X_i - \bar{X})^2$ | | Symmetric. |
| Standard Normal | $Z, N(0, 1)$ | $x \in \mathbb{R}$ | | $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ | $\Phi(x)$ | $0$ | $1$ | $e^{t^2/2}$ | – | – | | Special Case of Normal. |
| Beta | $Beta(\alpha, \beta)$ | $x \in [0, 1]$<br>$\alpha > 0$<br>$\beta > 0$ | Modelling proportion. | $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | | | | | $Beta(1, 1) = Uni(0, 1)$ |
| Chi-Square | $\chi^2_n$ | $x \geq 0$<br>$n \in \mathbb{N}^+$ | A Transformation of Standard Normal. Only takes positive value, model sample variance. Hypothesis tests. | $\frac{x^{n/2-1}}{2^{n/2}\Gamma(n/2)}e^{-x/2}$ | | $n$ | $2n$ | $(1-2t)^{-n/2}$ | | | | $\chi^2_n \sim Ga\,(n/2, 0.5)$,<br>$U + V \sim \chi^2_{n+m}$ |
| T | $T_n$ | $t \geq 0$<br>$n \in \mathbb{N}^+$ | Model Standardized quantities. Bell-shaped, looks like Normal but with heavier tails. | $\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$ | | $0$ | | | | | | $T = \frac{Z}{\sqrt{U/n}}, Z \sim N(0, 1), U \sim \chi^2_n$ |
| F | $F_{m,n}$ | $w \geq 0$<br>$m \in \mathbb{N}^+$<br>$n \in \mathbb{N}^+$ | Model ratio of variance. | $\frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2}\left(1 + \frac{m}{n}w\right)^{-(m+n)/2}w^{(m/2)-1}$ | | | | | | | | $W = \frac{U/m}{V/n}, U \sim \chi^2_m, V \sim \chi^2_n$,<br>$X \sim T_n \implies X^2 \implies F_{1,n}$ |
| Sample Mean | $\bar{X}$ | | $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ | $\frac{1}{n}\sum_{i=1}^n X_i$ | | $\mu$ | $\frac{\sigma^2}{n}$ | | | | | $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T_{n-1}$ |
| Sample Variance | $S^2$ | | | $\frac{1}{n-1}\sum_{i=1}^n(X_i - \bar{X})^2$ | | $\sigma^2$ | $\frac{2\sigma^4}{n-1}$ | | | | | $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ |

[1]Memoryless : $P(X > n + k | X > n) = P(k)$

# 1 Formulas

### Formulas

- Permutation:
$$\frac{n!}{(n-r)!}$$
- Combinations:
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
- Geometric Series:
$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$
- Infinity Series:
$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$
- Exponential Result:
$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

# 2 Basic Properties

- PMF (Probability Mass Function) for Discrete RV:
  - $p(k) \ge 0$ for all $k$
  - $\sum_i p(k_i) = 1$
- PDF (Probability Density Function) for Continuous RV:
  - $f(x) \ge 0$ for all $x$
  - $\int_{\infty}^{\infty} f(x)\,dx = 1$
- CDF (Cumulative Distribution Function):
  - Non-decreasing function.
  - $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$

### Joint Distribution

Joint CDF:
$$F(x,y) = P(X \le x, Y \le y)$$

Joint PMF:
$$p(x,y) = P(X = x, Y = y)$$
Marginal PMF:
$$p_X(x) = \sum_y p(x,y)$$
$$p_Y(y) = \sum_x p(x,y)$$
- $p(x,y) \ge 0$ for all $x, y$
- $\sum_x \sum_y p(x,y) = 1$

Joint PDF:
$$f(x,y) = P(X = x, Y = y)$$
Marginal PDF:
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
- $f(x,y) \ge 0$ for all $x, y$
- $\int_x \int_y p(x,y)\partial y \partial x = 1$

# 3 Conditional

### Conditional Probability for Event

- Definition:
$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$
- Mutiplicative Law:
$$P(A\cap B) = P(B\cap A) = P(A|B)P(B)$$
- Law of Total Probability, Union of all $B_i$ is the $\Omega$
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

### Conditional Probability for Multivariate

- Definition:
$$p(X=x|Y=y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad f(X=x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$
- If $X$ and $Y$ are independent, then their margin PMF/PDF can factor into the product of their Marginal, and canceled by the denominator, thus we got:
$$p(X=x|Y=y) = p(X=x) \qquad f(X=x|Y=y) = f(X=x)$$
- Mutiplication Law:
$$p_{XY}(x,y) = p_{X|Y}(x|y)p_Y(y) \qquad f_{XY}(x,y) = f_{X|Y}(x|y)f_Y(y)$$
- Law of Total Probability:
$$p_X(x) = \sum_y p(x,y) = \sum_y p_{X|Y}(x|y)p_Y(y)$$
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)$$

### Bayes's Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

# 4 Independence

### Independence

- RV $X$ and $Y$ are independent iff:
$$\begin{array}{c} f(x,y) = f_X(x)f_Y(y) \\ p(x,y) = p_X(x)p_Y(y) \end{array} \quad or \quad M_{X,Y}(x,y) = M_X(x)M_Y(y)$$
- Event $A$ and $B$ are independent iff:
$$P(A\cup B) = P(A)P(B)$$

# 5 Transformation

### Normal Transformation (Normalize)

$$X \sim N(\mu,\sigma^2) \implies \frac{X-\mu}{\sigma} \sim N(0,1)$$

### Direct Transformation Method

Given $f_X(x)$, find $F_X(x)$, then construct $F_Y(y)$ in terms of $F_X(y)$, next we find the derivative of $F_Y(y)$ to find $f_Y(y)$.

### Monotone Transformation Method

If $Y = g(X)$, where $g$ is differentiable and strictly monotonic on some interval $I$, then the PMF/PDF of $Y$ is given as:
$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$$

### Probability Integral Transformation

If $Z = F_X(X)$ then $Z \sim Uni(0,1)$.

### Inverse Integral Transformation

If $X = F^{-1}(U)$ where $U \sim Uni(0,1)$ then $X$ has PDF $F(x)$.

### Convolution Method (Sum of two RV)

Given $Z = X + Y$, then we have:
$$p_Z(z) = \sum_x p(x, z-x) \qquad f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx$$
If $X$ and $Y$ are independent, we got:
$$p_Z(z) = \sum_x p_X(x)p_Y(z-x) \qquad f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx$$

### Bivariate Transformation Method

Suppose $X$ and $Y$ continuous RV and independent, we have two RV, defined as transformation of $X$ and $Y$, we first use $U, V$ represent $X, Y$:
$$U = g_1(X,Y) \quad V = g_2(X,Y) \quad \text{and} \quad X = h_1(U,V) \quad Y = h_2(U,V)$$
Then the joint PDF of $U$ and $V$ is given as:
$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|\det(J(u,v))|$$
Where $J(u,v)$ is the Jacobian Matrix, defined as:
$$J(u,v) = \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix}$$
And the determinate is calculated as:
$$\frac{\partial h_1}{\partial u} \times \frac{\partial h_2}{\partial v} - \frac{\partial h_1}{\partial v} \times \frac{\partial h_2}{\partial u}$$

# 6 Expected Value

$$E(X) = \sum_x xp(x) \qquad E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

### Properties of Expected Value

- Expectation of a constant:
$$E(E(X)) = E(X)$$
- Expectation of Linear Combinations of RV:
$$E\left(a + \sum_{i=1}^{n} b_i X_i\right) = a + \sum_{i=1}^{n} b_i E(X_i)$$
Especially:
$$E(aX+b) = aE(X) + b$$
When $a = 0$, $b = 1$ we got:
$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$
- Notice:
$$E(g(X)) \ne g(E(X))$$
- Expectation of product of RV, If $X$ and $Y$ are independent:
$$E(XY) = E(X)E(Y)$$
- Expected Value of a function of RV, suppose $Y = g(X)$:
$$E(Y) = \sum_x g(x)p(x) \qquad E(Y) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$
Especially:
$$E(X^2) = \sum_x x^2 p(x) \qquad E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

### Conditional Expectation

- Expectation of $Y$ given $X = x$ (fixed), and $h(Y)$ is a function of $Y$:
$$E(h(Y)|X=x) = \sum_{y_i} h(y_i)p(Y=y_i|X=x)$$
$$E(h(Y)|X=x) = \int_{-\infty}^{\infty} h(y)p(Y=y_i|X=x)$$
- Law of total Expectation:
$$E(Y) = E_X(E(Y|X))$$
The key here is $E(Y|X)$ is a function of $X$.

# 7 Variance

$$Var(X) = E([X-\mu]^2)$$

### Standard Deviation

$$Std(X) = \sqrt{Var(X)}$$

### Properties of Variance

- Alternative Variance Form:
$$Var(X) = E(X^2) - [E(X)]^2$$
- Variance of sum of RV:
$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$
$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$$
If $X$ and $Y$ are independent, the covariance term is 0, thus:
$$Var(X+Y) = Var(X-Y) = Var(X) + Var(Y)$$
In General:
$$Var\left(a + \sum_{i=1}^{n} b_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j Cov(X_i, X_j)$$
Especially:
$$Var(aX+b) = a^2 Var(X)$$
- If all $X_i$ are mutually independent, then:
$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$
- Variance of product of RV:
$$Var(XY) = E(X^2Y^2) - [E(XY)]^2$$

### Law of total Variance

$$Var(Y) = Var(E(Y|X)) + E(Var(Y|X))$$

### Covariance

- The Covariance of $X$ and $Y$ is defined as:
$$Cov(X,Y) = E((X - E(X)) * (Y - E(Y)))$$
Notice that covariance can be positive or negative, contrast to variance which can only take positive value.
- Alternative Covariance Form:
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
- Property 1:
$$Cov(a+X, Y) = Cov(X,Y)$$
- Property 2:
$$Cov(aX, bY) = abCov(X,Y)$$
- Property 3:
$$Cov(X, Y+Z) = Cov(X,Y) + Cov(X,Z)$$
- Property 4:
$$Cov(aX+bW, cY+dZ) = \begin{array}{l} ac * Cov(X,Y) + ad * Cov(X,Z) + \\ bc * Cov(W,Y) + bd * Cov(W,Z) \end{array}$$
In General, if $U = a + \sum_{i=1}^{n} b_i X_i$, $V = c + \sum_{i=1}^{m} d_i X_i$, we have:
$$Cov(U,V) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i d_j Cov(X_i, Y_j)$$
- Property 5:
$$Cov(X,X) = Var(X)$$
- Property 6: If $X$ and $Y$ are independent then:
$$Cov(X,Y) = 0$$
But $Cov(X,Y) = 0$ can't gives us $X$ and $Y$ independent.

### Correlation

- The Correlation of $X$ and $Y$ is defined as:
$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \qquad -1 \le \rho \le 1$$
- When $\rho$ is close to 1, then $X$ and $Y$ are positively associated.
- When $\rho$ is close to $-1$, then $X$ and $Y$ are negatively associated.
- When $\rho$ is equals to 0, then $X$ and $Y$ are not associated.

# 8 Markov and Chebyshev

### Markov's Inequality

If $X$ only defined on non negative values, then:
$$P(X \ge t) \le \frac{E(X)}{t}$$

### Chebyshev's Inequality

Let $\mu$ and $\sigma^2$ be the mean and variance, then for $t > 0$, we set $t = k\sigma$:
$$P(|X-\mu| > t) \le \frac{\sigma^2}{t^2} \qquad P(|X-\mu| > k\sigma) \le \frac{1}{k^2}$$

# 9 Moment Generating Function

$$M(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$$
$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

### Properties of Moment Generating Function

- MGF is unique for a distribution, so can prove the distribution that an RV follows.
- MGF can be used to calculate some form of Expectation. That is, the rth moment is:
$$E(X^r) = M^{(r)}(0)$$
So Variance can also be calculated as the second moment of $X$ subtract the square of the first moment of $X$, that is:
$$Var(X) = M^{(2)}(0) - [M^{(1)}(0)]^2$$
- rth central moment is defined as:
$$E([X - E(X)])^r$$
- MGF of a transformed function is:
$$M_{aX+b}(t) = e^{bt}M_X(at)$$
- If $X$ and $Y$ are independent RV, then:
$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

# 10 Gamma Function

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1}e^{-u}\,du$$

- when $\alpha$ is fraction : $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
- when $\alpha$ is integer : $\Gamma(\alpha+1) = \alpha!$

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | $1/2$ | $3/2$ | 1 | 2 | 3 | 4 |
| $\Gamma(a)$ | $\sqrt{\pi}$ | $\sqrt{\pi}/2$ | 1 | 1 | 2 | 6 |

# 11 Law of Large Number (LLN)

Let $X_1, X_2, \dots$ be independent RV, and $E(X_i) = \mu$, $Var(X_i) = \sigma^2$ (we only require the variance is finite), Let: $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ then for any $\varepsilon > 0$:
$$P\left(|\bar{X}_n - \mu| \ge \varepsilon\right) \to 0 \qquad \text{as} \quad n \to \infty$$
That is when $n \to \infty$, then the sample mean convergence in probability to the true mean $\bar{X}_n \to \mu$.

# 12 Monte Carlo Integration

We want to find:
$$I(g) = \int_0^1 g(x)\,dx$$
We first generate $n$ RV $X_1, \dots X_n$ from $Uni(0,1)$, then we have:
$$\bar{X}_n = \hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n} f(X_i) \qquad \text{as} \quad n \to \infty$$
The key here is that the Uniform distribution on $[0,1]$ has PDF 1, thus the expectation of sample mean is just the function we want.

# 13 Convergence in Distribution

Let $X_1, \dots, X_n$ be independent RV. Let $X$ be RV. Then $X_n$ convergence in distribution to $X$ if:
$$\lim_{n\to\infty} F_n(x) = F(x)$$
At all points which $F$ is continuous. We often use MGF to prove convergence in distribution, that is:
$$\lim_{n\to\infty} M_n(t) = M(t) \implies \lim_{n\to\infty} F_n(x) = F(x)$$
For $t$ in an open interval containing zero.

We can use Standard Normal to approximate Poisson, when $\lambda$ gets large enough, but we first need to Standardiz the Poisson.

# 14 Central Limit Theorem (CLT)

Let $X_1, X_2, \dots$ be independent RV, and $E(X_i) = 0$, $Var(X_i) = \sigma^2$ and common CDF/PDF and MGF defined in a neighbourhood of zero.

- If we want Sum of $X_i$, we define:
$$S_n = \sum_{i=1}^{n} X_i$$
Then we have:
$$\lim_{n\to\infty} P\left(\frac{S_n}{\sigma\sqrt{n}} \le x\right) = \Phi(x)$$
So, we have convergence in distribution of:
$$S_n \to N(0, n\sigma^2)$$
- If we want Average of $X_i$, we define:
$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$
Then we have:
$$\lim_{n\to\infty} P\left(\frac{\bar{X}_n}{\sigma/\sqrt{n}} \le x\right) = \Phi(x)$$
So, we have convergence in distribution of:
$$\bar{X}_n \to N(0, \sigma^2/n)$$

If we don't have Expected value 0, we can subtract off the mean and shift the distribution to make it have an expected value of 0.

## Normal Approximation

In practise, we can normalize the sum to make it a standard normal:

$$\frac{S - E(S)}{Std(S)} \sim N(0,1)$$

- Binomial: Let $X_1, \ldots, X_n$ be RV that follows Bernoulli distribution with parameter $p$. So their sum:

  $$S_n = \sum_{i=1}^{n} X_i$$

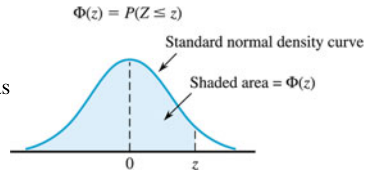
  follows a Binomial distribution, that is: $S_n \sim Bin(n,p)$, we have that:

  $$Z_n = \frac{S_n - np}{\sqrt{np(1-p)}}$$

  Where $Z_n \sim N(0,1)$.

## A.3 Standard Normal cdf

**Table A.3** Standard normal curve areas



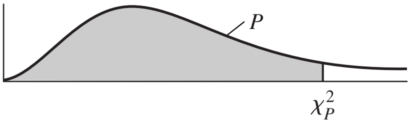

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **–3.4** | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |
| **–3.3** | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| **–3.2** | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| **–3.1** | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| **–3.0** | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| **–2.9** | .0019 | .0018 | .0017 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| **–2.8** | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| **–2.7** | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| **–2.6** | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| **–2.5** | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| **–2.4** | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| **–2.3** | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| **–2.2** | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| **–2.1** | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| **–2.0** | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| **–1.9** | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| **–1.8** | .0359 | .0352 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| **–1.7** | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| **–1.6** | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| **–1.5** | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| **–1.4** | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0722 | .0708 | .0694 | .0681 |
| **–1.3** | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| **–1.2** | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| **–1.1** | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| **–1.0** | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| **–0.9** | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| **–0.8** | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| **–0.7** | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| **–0.6** | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| **–0.5** | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| **–0.4** | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| **–0.3** | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3482 |
| **–0.2** | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| **–0.1** | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| **–0.0** | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |

(continued)

**Table A.3** (continued)

| *z* | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| **0.1** | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| **0.2** | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| **0.3** | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| **0.4** | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| **0.5** | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| **0.6** | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| **0.7** | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| **0.8** | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| **0.9** | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| **1.0** | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| **1.1** | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| **1.2** | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| **1.3** | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| **1.4** | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9278 | .9292 | .9306 | .9319 |
| **1.5** | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| **1.6** | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| **1.7** | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| **1.8** | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| **1.9** | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| **2.0** | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| **2.1** | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| **2.2** | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| **2.3** | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| **2.4** | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| **2.5** | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| **2.6** | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| **2.7** | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| **2.8** | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| **2.9** | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| **3.0** | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| **3.1** | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| **3.2** | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| **3.3** | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| **3.4** | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

## TABLE 3 Percentiles of the $\chi^2$ Distribution—Values of $\chi^2_P$ Corresponding to $P$




| $df$ | $\chi^2_{.005}$ | $\chi^2_{.01}$ | $\chi^2_{.025}$ | $\chi^2_{.05}$ | $\chi^2_{.10}$ | $\chi^2_{.90}$ | $\chi^2_{.95}$ | $\chi^2_{.975}$ | $\chi^2_{.99}$ | $\chi^2_{.995}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .000039 | .00016 | .00098 | .0039 | .0158 | 2.71 | 3.84 | 5.02 | 6.63 | 7.88 |
| 2 | .0100 | .0201 | .0506 | .1026 | .2107 | 4.61 | 5.99 | 7.38 | 9.21 | 10.60 |
| 3 | .0717 | .115 | .216 | .352 | .584 | 6.25 | 7.81 | 9.35 | 11.34 | 12.84 |
| 4 | .207 | .297 | .484 | .711 | 1.064 | 7.78 | 9.49 | 11.14 | 13.28 | 14.86 |
| 5 | .412 | .554 | .831 | 1.15 | 1.61 | 9.24 | 11.07 | 12.83 | 15.09 | 16.75 |
| 6 | .676 | .872 | 1.24 | 1.64 | 2.20 | 10.64 | 12.59 | 14.45 | 16.81 | 18.55 |
| 7 | .989 | 1.24 | 1.69 | 2.17 | 2.83 | 12.02 | 14.07 | 16.01 | 18.48 | 20.28 |
| 8 | 1.34 | 1.65 | 2.18 | 2.73 | 3.49 | 13.36 | 15.51 | 17.53 | 20.09 | 21.96 |
| 9 | 1.73 | 2.09 | 2.70 | 3.33 | 4.17 | 14.68 | 16.92 | 19.02 | 21.67 | 23.59 |
| 10 | 2.16 | 2.56 | 3.25 | 3.94 | 4.87 | 15.99 | 18.31 | 20.48 | 23.21 | 25.19 |
| 11 | 2.60 | 3.05 | 3.82 | 4.57 | 5.58 | 17.28 | 19.68 | 21.92 | 24.73 | 26.76 |
| 12 | 3.07 | 3.57 | 4.40 | 5.23 | 6.30 | 18.55 | 21.03 | 23.34 | 26.22 | 28.30 |
| 13 | 3.57 | 4.11 | 5.01 | 5.89 | 7.04 | 19.81 | 22.36 | 24.74 | 27.69 | 29.82 |
| 14 | 4.07 | 4.66 | 5.63 | 6.57 | 7.79 | 21.06 | 23.68 | 26.12 | 29.14 | 31.32 |
| 15 | 4.60 | 5.23 | 6.26 | 7.26 | 8.55 | 22.31 | 25.00 | 27.49 | 30.58 | 32.80 |
| 16 | 5.14 | 5.81 | 6.91 | 7.96 | 9.31 | 23.54 | 26.30 | 28.85 | 32.00 | 34.27 |
| 18 | 6.26 | 7.01 | 8.23 | 9.39 | 10.86 | 25.99 | 28.87 | 31.53 | 34.81 | 37.16 |
| 20 | 7.43 | 8.26 | 9.59 | 10.85 | 12.44 | 28.41 | 31.41 | 34.17 | 37.57 | 40.00 |
| 24 | 9.89 | 10.86 | 12.40 | 13.85 | 15.66 | 33.20 | 36.42 | 39.36 | 42.98 | 45.56 |
| 30 | 13.79 | 14.95 | 16.79 | 18.49 | 20.60 | 40.26 | 43.77 | 46.98 | 50.89 | 53.67 |
| 40 | 20.71 | 22.16 | 24.43 | 26.51 | 29.05 | 51.81 | 55.76 | 59.34 | 63.69 | 66.77 |
| 60 | 35.53 | 37.48 | 40.48 | 43.19 | 46.46 | 74.40 | 79.08 | 83.30 | 88.38 | 91.95 |
| 120 | 83.85 | 86.92 | 91.58 | 95.70 | 100.62 | 140.23 | 146.57 | 152.21 | 158.95 | 163.64 |

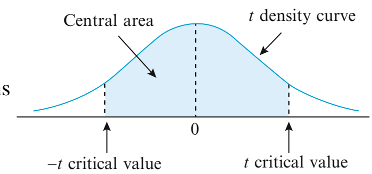
For large degrees of freedom,

$$\chi^2_P = \tfrac{1}{2}(z_P + \sqrt{2v - 1})^2 \text{ approximately,}$$

where $v$ = degrees of freedom and $z_P$ is given in Table 2.

## A.5 Critical Values for *t* Distributions

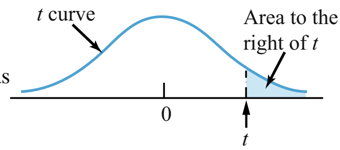
**Table A.5** Critical values for *t* distributions




| $\nu$ | Central area 80% | 90% | 95% | 98% | 99% | 99.8% | 99.9% |
|---|---|---|---|---|---|---|---|
| **1** | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.31 | 636.62 |
| **2** | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.326 | 31.598 |
| **3** | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 10.213 | 12.924 |
| **4** | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 | 8.610 |
| **5** | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 | 6.869 |
| **6** | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| **7** | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| **8** | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| **9** | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| **10** | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| **11** | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
| **12** | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 | 4.318 |
| **13** | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 | 4.221 |
| **14** | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 | 4.140 |
| **15** | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 | 4.073 |
| **16** | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.686 | 4.015 |
| **17** | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.646 | 3.965 |
| **18** | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.610 | 3.922 |
| **19** | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.579 | 3.883 |
| **20** | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.552 | 3.850 |
| **21** | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.527 | 3.819 |
| **22** | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.505 | 3.792 |
| **23** | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.485 | 3.767 |
| **24** | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.467 | 3.745 |
| **25** | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.450 | 3.725 |
| **26** | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.435 | 3.707 |
| **27** | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.421 | 3.690 |
| **28** | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.408 | 3.674 |
| **29** | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.396 | 3.659 |
| **30** | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.385 | 3.646 |
| **32** | 1.309 | 1.694 | 2.037 | 2.449 | 2.738 | 3.365 | 3.622 |
| **34** | 1.307 | 1.691 | 2.032 | 2.441 | 2.728 | 3.348 | 3.601 |
| **36** | 1.306 | 1.688 | 2.028 | 2.434 | 2.719 | 3.333 | 3.582 |
| **38** | 1.304 | 1.686 | 2.024 | 2.429 | 2.712 | 3.319 | 3.566 |
| **40** | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 3.307 | 3.551 |
| **50** | 1.299 | 1.676 | 2.009 | 2.403 | 2.678 | 3.262 | 3.496 |
| **60** | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 3.232 | 3.460 |
| **120** | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 3.160 | 3.373 |
| $\infty$ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 3.090 | 3.291 |

## A.6 Tail Areas of *t* Distributions

**Table A.6** *t* curve tail areas



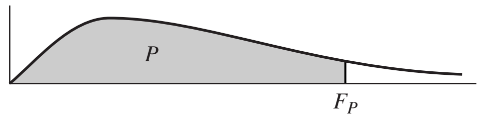

Degrees of Freedom (ν)

| *t* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 |
| **0.1** | .468 | .465 | .463 | .463 | .462. | .462 | .462 | .461 | .461 | .461 | .461 | .461 |
| **0.2** | .437 | .430 | .427 | .426 | .425 | .424 | .424 | .423 | .423 | .423 | .423 | .422 |
| **0.3** | .407 | .396 | .392 | .390 | .388 | .387 | .386 | .386 | .386 | .385 | .385 | .385 |
| **0.4** | .379 | .364 | .358 | .355 | .353 | .352 | .351 | .350 | .349 | .349 | .348 | .348 |
| **0.5** | .352 | .333 | .326 | .322 | .319 | .317 | .316 | .315 | .315 | .314 | .313 | .313 |
| **0.6** | .328 | .305 | .295 | .290 | .287 | .285 | .284 | .283 | .282 | .281 | .280 | .280 |
| **0.7** | .306 | .278 | .267 | .261 | .258 | .255 | .253 | .252 | .251 | .250 | .249 | .249 |
| **0.8** | .285 | .254 | .241 | .234 | .230 | .227 | .225 | .223 | .222 | .221 | .220 | .220 |
| **0.9** | .267 | .232 | .217 | .210 | .205 | .201 | .199 | .197 | .196 | .195 | .194 | .193 |
| **1.0** | .250 | .211 | .196 | .187 | .182 | .178 | .175 | .173 | .172 | .170 | .169 | .169 |
| **1.1** | .235 | .193 | .176 | .167 | .162 | .157 | .154 | .152 | .150 | .149 | .147 | .146 |
| **1.2** | .221 | .177 | .158 | .148 | .142 | .138 | .135 | .132 | .130 | .129 | .128 | .127 |
| **1.3** | .209 | .162 | .142 | .132 | .125 | .121 | .117 | .115 | .113 | .111 | .110 | .109 |
| **1.4** | .197 | .148 | .128 | .117 | .110 | .106 | .102 | .100 | .098 | .096 | .095 | .093 |
| **1.5** | .187 | .136 | .115 | .104 | .097 | .092 | .089 | .086 | .084 | .082 | .081 | .080 |
| **1.6** | .178 | .125 | .104 | .092 | .085 | .080 | .077 | .074 | .072 | .070 | .069 | .068 |
| **1.7** | .169 | .116 | .094 | .082 | .075 | .070 | .065 | .064 | .062 | .060 | .059 | .057 |
| **1.8** | .161 | .107 | .085 | .073 | .066 | .061 | .057 | .055 | .053 | .051 | .050 | .049 |
| **1.9** | .154 | .099 | .077 | .065 | .058 | .053 | .050 | .047 | .045 | .043 | .042 | .041 |
| **2.0** | .148 | .092 | .070 | .058 | .051 | .046 | .043 | .040 | .038 | .037 | .035 | .034 |
| **2.1** | .141 | .085 | .063 | .052 | .045 | .040 | .037 | .034 | .033 | .031 | .030 | .029 |
| **2.2** | .136 | .079 | .058 | .046 | .040 | .035 | .032 | .029 | .028 | .026 | .025 | .024 |
| **2.3** | .131 | .074 | .052 | .041 | .035 | .031 | .027 | .025 | .023 | .022 | .021 | .020 |
| **2.4** | .126 | .069 | .048 | .037 | .031 | .027 | .024 | .022 | .020 | .019 | .018 | .017 |
| **2.5** | .121 | .065 | .044 | .033 | .027 | .023 | .020 | .018 | .017 | .016 | .015 | .014 |
| **2.6** | .117 | .061 | .040 | .030 | .024 | .020 | .018 | .016 | .014 | .013 | .012 | .012 |
| **2.7** | .113 | .057 | .037 | .027 | .021 | .018 | .015 | .014 | .012 | .011 | .010 | .010 |
| **2.8** | .109 | .054 | .034 | .024 | .019 | .016 | .013 | .012 | .010 | .009 | .009 | .008 |
| **2.9** | .106 | .051 | .031 | .022 | .017 | .014 | .011 | .010 | .009 | .008 | .007 | .007 |
| **3.0** | .102 | .048 | .029 | .020 | .015 | .012 | .010 | .009 | .007 | .007 | .006 | .006 |
| **3.1** | .099 | .045 | .027 | .018 | .013 | .011 | .009 | .007 | .006 | .006 | .005 | .005 |
| **3.2** | .096 | .043 | .025 | .016 | .012 | .009 | .008 | .006 | .005 | .005 | .004 | .004 |
| **3.3** | .094 | .040 | .023 | .015 | .011 | .008 | .007 | .005 | .005 | .004 | .004 | .003 |
| **3.4** | .091 | .038 | .021 | .014 | .010 | .007 | .006 | .005 | .004 | .003 | .003 | .003 |
| **3.5** | .089 | .036 | .020 | .012 | .009 | .006 | .005 | .004 | .003 | .003 | .002 | .002 |
| **3.6** | .086 | .035 | .018 | .011 | .008 | .006 | .004 | .004 | .003 | .002 | .002 | .002 |
| **3.7** | .084 | .033 | .017 | .010 | .007 | .005 | .004 | .003 | .002 | .002 | .002 | .002 |
| **3.8** | .082 | .031 | .016 | .010 | .006 | .004 | .003 | .003 | .002 | .002 | .001 | .001 |
| **3.9** | .080 | .030 | .015 | .009 | .006 | .004 | .003 | .002 | .002 | .001 | .001 | .001 |
| **4.0** | .078 | .029 | .014 | .008 | .005 | .004 | .003 | .002 | .002 | .001 | .001 | .001 |

| | Degrees of Freedom ($\nu$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | **13** | **14** | **15** | **16** | **17** | **18** | **19** | **20** | **21** | **22** | **23** | **24** |
| **0.0** | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 |
| **0.1** | .461 | .461 | .461 | .461 | .461 | .461 | .461 | .461 | .461 | .461 | .461 | .461 |
| **0.2** | .422 | .422 | .422 | .422 | .422 | .422 | .422 | .422 | .422 | .422 | .422 | .422 |
| **0.3** | .384 | .384 | .384 | .384 | .384 | .384 | .384 | .384 | .384 | .383 | .383 | .383 |
| **0.4** | .348 | .347 | .347 | .347 | .347 | .347 | .347 | .347 | .347 | .347 | .346 | .346 |
| **0.5** | .313 | .312 | .312 | .312 | .312 | .312 | .311 | .311 | .311 | .311 | .311 | .311 |
| **0.6** | .279 | .279 | .279 | .278 | .278 | .278 | .278 | .278 | .278 | .277 | .277 | .277 |
| **0.7** | .248 | .247 | .247 | .247 | .247 | .246 | .246 | .246 | .246 | .246 | .245 | .245 |
| **0.8** | .219 | .218 | .218 | .218 | .217 | .217 | .217 | .217 | .216 | .216 | .216 | .216 |
| **0.9** | .192 | .191 | .191 | .191 | .190 | .190 | .190 | .189 | .189 | .189 | .189 | .189 |
| **1.0** | .168 | .167 | .167 | .166 | .166 | .165 | .165 | .165 | .164 | .164 | .164 | .164 |
| **1.1** | .146 | .144 | .144 | .144 | .143 | .143 | .143 | .142 | .142 | .142 | .141 | .141 |
| **1.2** | .126 | .124 | .124 | .124 | .123 | .123 | .122 | .122 | .122 | .121 | .121 | .121 |
| **1.3** | .108 | .107 | .107 | .106 | .105 | .105 | .105 | .104 | .104 | .104 | .103 | .103 |
| **1.4** | .092 | .091 | .091 | .090 | .090 | .089 | .089 | .089 | .088 | .088 | .087 | .087 |
| **1.5** | .079 | .077 | .077 | .077 | .076 | .075 | .075 | .075 | .074 | .074 | .074 | .073 |
| **1.6** | .067 | .065 | .065 | .065 | .064 | .064 | .063 | .063 | .062 | .062 | .062 | .061 |
| **1.7** | .056 | .055 | .055 | .054 | .054 | .053 | .053 | .052 | .052 | .052 | .051 | .051 |
| **1.8** | .048 | .046 | .046 | .045 | .045 | .044 | .044 | .043 | .043 | .043 | .042 | .042 |
| **1.9** | .040 | .038 | .038 | .038 | .037 | .037 | .036 | .036 | .036 | .035 | .035 | .035 |
| **2.0** | .033 | .032 | .032 | .031 | .031 | .030 | .030 | .030 | .029 | .029 | .029 | .028 |
| **2.1** | .028 | .027 | .027 | .026 | .025 | .025 | .025 | .024 | .024 | .024 | .023 | .023 |
| **2.2** | .023 | .022 | .022 | .021 | .021 | .021 | .020 | .020 | .020 | .019 | .019 | .019 |
| **2.3** | .019 | .018 | .018 | .018 | .017 | .017 | .016 | .016 | .016 | .016 | .015 | .015 |
| **2.4** | .016 | .015 | .015 | .014 | .014 | .014 | .013 | .013 | .013 | .013 | .012 | .012 |
| **2.5** | .013 | .012 | .012 | .012 | .011 | .011 | .011 | .011 | .010 | .010 | .010 | .010 |
| **2.6** | .011 | .010 | .010 | .010 | .009 | .009 | .009 | .009 | .008 | .008 | .008 | .008 |
| **2.7** | .009 | .008 | .008 | .008 | .008 | .007 | .007 | .007 | .007 | .007 | .006 | .006 |
| **2.8** | .008 | .007 | .007 | .006 | .006 | .006 | .006 | .006 | .005 | .005 | .005 | .005 |
| **2.9** | .006 | .005 | .005 | .005 | .005 | .005 | .005 | .004 | .004 | .004 | .004 | .004 |
| **3.0** | .005 | .004 | .004 | .004 | .004 | .004 | .004 | .004 | .003 | .003 | .003 | .003 |
| **3.1** | .004 | .004 | .004 | .003 | .003 | .003 | .003 | .003 | .003 | .003 | .003 | .002 |
| **3.2** | .003 | .003 | .003 | .003 | .003 | .002 | .002 | .002 | .002 | .002 | .002 | .002 |
| **3.3** | .003 | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .001 |
| **3.4** | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .001 | .001 | .001 | .001 | .001 |
| **3.5** | .002 | .002 | .002 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 |
| **3.6** | .002 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 |
| **3.7** | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 |
| **3.8** | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .000 | .000 | .000 |
| **3.9** | .001 | .001 | .001 | .001 | .001 | .001 | .000 | .000 | .000 | .000 | .000 | .000 |
| **4.0** | .001 | .001 | .001 | .001 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |

| $t$ | 25 | 26 | 27 | 28 | 29 | 30 | 35 | 40 | 60 | 120 | $\infty(=z)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 | .500 |
| **0.1** | .461 | .461 | .461 | .461 | .461 | .461 | .460 | .460 | .460 | .460 | .460 |
| **0.2** | .422 | .422 | .421 | .421 | .421 | .421 | .421 | .421 | .421 | .421 | .421 |
| **0.3** | .383 | .383 | .383 | .383 | .383 | .383 | .383 | .383 | .383 | .382 | .382 |
| **0.4** | .346 | .346 | .346 | .346 | .346 | .346 | .346 | .346 | .345 | .345 | .345 |
| **0.5** | .311 | .311 | .311 | .310 | .310 | .310 | .310 | .310 | .309 | .309 | .309 |
| **0.6** | .277 | .277 | .277 | .277 | .277 | .277 | .276 | .276 | .275 | .275 | .274 |
| **0.7** | .245 | .245 | .245 | .245 | .245 | .245 | .244 | .244 | .243 | .243 | .242 |
| **0.8** | .216 | .215 | .215 | .215 | .215 | .215 | .215 | .214 | .213 | .213 | .212 |
| **0.9** | .188 | .188 | .188 | .188 | .188 | .188 | .187 | .187 | .186 | .185 | .184 |
| **1.0** | .163 | .163 | .163 | .163 | .163 | .163 | .162 | .162 | .161 | .160 | .159 |
| **1.1** | .141 | .141 | .141 | .140 | .140 | .140 | .139 | .139 | .138 | .137 | .136 |
| **1.2** | .121 | .120 | .120 | .120 | .120 | .120 | .119 | .119 | .117 | .116 | .115 |
| **1.3** | .103 | .103 | .102 | .102 | .102 | .102 | .101 | .101 | .099 | .098 | .097 |
| **1.4** | .087 | .087 | .086 | .086 | .086 | .086 | .085 | .085 | .083 | .082 | .081 |
| **1.5** | .073 | .073 | .073 | .072 | .072 | .072 | .071 | .071 | .069 | .068 | .067 |
| **1.6** | .061 | .061 | .061 | .060 | .060 | .060 | .059 | .059 | .057 | .056 | .055 |
| **1.7** | .051 | .051 | .050 | .050 | .050 | .050 | .049 | .048 | .047 | .046 | .045 |
| **1.8** | .042 | .042 | .042 | .041 | .041 | .041 | .040 | .040 | .038 | .037 | .036 |
| **1.9** | .035 | .034 | .034 | .034 | .034 | .034 | .033 | .032 | .031 | .030 | .029 |
| **2.0** | .028 | .028 | .028 | .028 | .027 | .027 | .027 | .026 | .025 | .024 | .023 |
| **2.1** | .023 | .023 | .023 | .022 | .022 | .022 | .022 | .021 | .020 | .019 | .018 |
| **2.2** | .019 | .018 | .018 | .018 | .018 | .018 | .017 | .017 | .016 | .015 | .014 |
| **2.3** | .015 | .015 | .015 | .015 | .014 | .014 | .014 | .013 | .012 | .012 | .011 |
| **2.4** | .012 | .012 | .012 | .012 | .012 | .011 | .011 | .011 | .010 | .009 | .008 |
| **2.5** | .010 | .010 | .009 | .009 | .009 | .009 | .009 | .008 | .008 | .007 | .006 |
| **2.6** | .008 | .008 | .007 | .007 | .007 | .007 | .007 | .007 | .006 | .005 | .005 |
| **2.7** | .006 | .006 | .006 | .006 | .006 | .006 | .005 | .005 | .004 | .004 | .003 |
| **2.8** | .005 | .005 | .005 | .005 | .005 | .004 | .004 | .004 | .003 | .003 | .003 |
| **2.9** | .004 | .004 | .004 | .004 | .004 | .003 | .003 | .003 | .003 | .002 | .002 |
| **3.0** | .003 | .003 | .003 | .003 | .003 | .003 | .002 | .002 | .002 | .002 | .001 |
| **3.1** | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .002 | .001 | .001 | .001 |
| **3.2** | .002 | .002 | .002 | .002 | .002 | .002 | .001 | .001 | .001 | .001 | .001 |
| **3.3** | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .000 |
| **3.4** | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .000 | .000 |
| **3.5** | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .001 | .000 | .000 | .000 |
| **3.6** | .001 | .001 | .001 | .001 | .001 | .001 | .000 | .000 | .000 | .000 | .000 |
| **3.7** | .001 | .001 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |
| **3.8** | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |
| **3.9** | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |
| **4.0** | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |

## TABLE 5 Percentiles of the $F$ Distribution: $F_{.90}(n_1, n_2)$




$F_{.90}(n_1, n_2)$

$n_1$ = degrees of freedom for numerator

$n_2$ = degrees of freedom for denominator

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 39.86 | 49.50 | 53.59 | 55.83 | 57.24 | 58.20 | 58.91 | 59.44 | 59.86 | 60.19 | 60.71 | 61.22 | 61.74 | 62.00 | 62.26 | 62.53 | 62.79 | 63.06 | 63.33 |
| 2 | 8.53 | 9.00 | 9.16 | 9.24 | 9.29 | 9.33 | 9.35 | 9.37 | 9.38 | 9.39 | 9.41 | 9.42 | 9.44 | 9.45 | 9.46 | 9.47 | 9.47 | 9.48 | 9.49 |
| 3 | 5.54 | 5.46 | 5.39 | 5.34 | 5.31 | 5.28 | 5.27 | 5.25 | 5.24 | 5.23 | 5.22 | 5.20 | 5.18 | 5.18 | 5.17 | 5.16 | 5.15 | 5.14 | 5.13 |
| 4 | 4.54 | 4.32 | 4.19 | 4.11 | 4.05 | 4.01 | 3.98 | 3.95 | 3.94 | 3.92 | 3.90 | 3.87 | 3.84 | 3.83 | 3.82 | 3.80 | 3.79 | 3.78 | 3.76 |
| 5 | 4.06 | 3.78 | 3.62 | 3.52 | 3.45 | 3.40 | 3.37 | 3.34 | 3.32 | 3.30 | 3.27 | 3.24 | 3.21 | 3.19 | 3.17 | 3.16 | 3.14 | 3.12 | 3.10 |
| 6 | 3.78 | 3.46 | 3.29 | 3.18 | 3.11 | 3.05 | 3.01 | 2.98 | 2.96 | 2.94 | 2.90 | 2.87 | 2.84 | 2.82 | 2.80 | 2.78 | 2.76 | 2.74 | 2.72 |
| 7 | 3.59 | 3.26 | 3.07 | 2.96 | 2.88 | 2.83 | 2.78 | 2.75 | 2.72 | 2.70 | 2.67 | 2.63 | 2.59 | 2.58 | 2.56 | 2.54 | 2.51 | 2.49 | 2.47 |
| 8 | 3.46 | 3.11 | 2.92 | 2.81 | 2.73 | 2.67 | 2.62 | 2.59 | 2.56 | 2.50 | 2.50 | 2.46 | 2.42 | 2.40 | 2.38 | 2.36 | 2.34 | 2.32 | 2.29 |
| 9 | 3.36 | 3.01 | 2.81 | 2.69 | 2.61 | 2.55 | 2.51 | 2.47 | 2.44 | 2.42 | 2.38 | 2.34 | 2.30 | 2.28 | 2.25 | 2.23 | 2.21 | 2.18 | 2.16 |
| 10 | 3.29 | 2.92 | 2.73 | 2.61 | 2.52 | 2.46 | 2.41 | 2.38 | 2.35 | 2.32 | 2.28 | 2.24 | 2.20 | 2.18 | 2.16 | 2.13 | 2.11 | 2.08 | 2.06 |
| 11 | 3.23 | 2.86 | 2.66 | 2.54 | 2.45 | 2.39 | 2.34 | 2.30 | 2.27 | 2.25 | 2.21 | 2.17 | 2.12 | 2.10 | 2.08 | 2.05 | 2.03 | 2.00 | 1.97 |
| 12 | 3.18 | 2.81 | 2.61 | 2.48 | 2.39 | 2.33 | 2.28 | 2.24 | 2.21 | 2.19 | 2.15 | 2.10 | 2.06 | 2.04 | 2.01 | 1.99 | 1.96 | 1.93 | 1.90 |
| 13 | 3.14 | 2.76 | 2.56 | 2.43 | 2.35 | 2.28 | 2.23 | 2.20 | 2.16 | 2.14 | 2.10 | 2.05 | 2.01 | 1.98 | 1.96 | 1.93 | 1.90 | 1.88 | 1.85 |
| 14 | 3.10 | 2.73 | 2.52 | 2.39 | 2.31 | 2.24 | 2.19 | 2.15 | 2.12 | 2.10 | 2.05 | 2.01 | 1.96 | 1.94 | 1.91 | 1.89 | 1.86 | 1.83 | 1.80 |
| 15 | 3.07 | 2.70 | 2.49 | 2.36 | 2.27 | 2.21 | 2.16 | 2.12 | 2.09 | 2.06 | 2.02 | 1.97 | 1.92 | 1.90 | 1.87 | 1.85 | 1.82 | 1.79 | 1.76 |
| 16 | 3.05 | 2.67 | 2.46 | 2.33 | 2.24 | 2.18 | 2.13 | 2.09 | 2.06 | 2.03 | 1.99 | 1.94 | 1.89 | 1.87 | 1.84 | 1.81 | 1.78 | 1.75 | 1.72 |
| 17 | 3.03 | 2.64 | 2.44 | 2.31 | 2.22 | 2.15 | 2.10 | 2.06 | 2.03 | 2.00 | 1.96 | 1.91 | 1.86 | 1.84 | 1.81 | 1.78 | 1.75 | 1.72 | 1.69 |
| 18 | 3.01 | 2.62 | 2.42 | 2.29 | 2.20 | 2.13 | 2.08 | 2.04 | 2.00 | 1.98 | 1.93 | 1.89 | 1.84 | 1.81 | 1.78 | 1.75 | 1.72 | 1.69 | 1.66 |
| 19 | 2.99 | 2.61 | 2.40 | 2.27 | 2.18 | 2.11 | 2.06 | 2.02 | 1.98 | 1.96 | 1.91 | 1.86 | 1.81 | 1.79 | 1.76 | 1.73 | 1.70 | 1.67 | 1.63 |
| 20 | 2.97 | 2.59 | 2.38 | 2.25 | 2.16 | 2.09 | 2.04 | 2.00 | 1.96 | 1.94 | 1.89 | 1.84 | 1.79 | 1.77 | 1.74 | 1.71 | 1.68 | 1.64 | 1.61 |
| 21 | 2.96 | 2.57 | 2.36 | 2.23 | 2.14 | 2.08 | 2.02 | 1.98 | 1.95 | 1.92 | 1.87 | 1.83 | 1.78 | 1.75 | 1.72 | 1.69 | 1.66 | 1.62 | 1.59 |
| 22 | 2.95 | 2.56 | 2.35 | 2.22 | 2.13 | 2.06 | 2.01 | 1.97 | 1.93 | 1.90 | 1.86 | 1.81 | 1.76 | 1.73 | 1.70 | 1.67 | 1.64 | 1.60 | 1.57 |
| 23 | 2.94 | 2.55 | 2.34 | 2.21 | 2.11 | 2.05 | 1.99 | 1.95 | 1.92 | 1.89 | 1.84 | 1.80 | 1.74 | 1.72 | 1.69 | 1.66 | 1.62 | 1.59 | 1.55 |
| 24 | 2.93 | 2.54 | 2.33 | 2.19 | 2.10 | 2.04 | 1.98 | 1.94 | 1.91 | 1.88 | 1.83 | 1.78 | 1.73 | 1.70 | 1.67 | 1.64 | 1.61 | 1.57 | 1.53 |
| 25 | 2.92 | 2.53 | 2.32 | 2.18 | 2.09 | 2.02 | 1.97 | 1.93 | 1.89 | 1.87 | 1.82 | 1.77 | 1.72 | 1.69 | 1.66 | 1.63 | 1.59 | 1.56 | 1.52 |
| 26 | 2.91 | 2.52 | 2.31 | 2.17 | 2.08 | 2.01 | 1.96 | 1.92 | 1.88 | 1.86 | 1.81 | 1.76 | 1.71 | 1.68 | 1.65 | 1.61 | 1.58 | 1.54 | 1.50 |
| 27 | 2.90 | 2.51 | 2.30 | 2.17 | 2.07 | 2.00 | 1.95 | 1.91 | 1.87 | 1.85 | 1.80 | 1.75 | 1.70 | 1.67 | 1.64 | 1.60 | 1.57 | 1.53 | 1.49 |
| 28 | 2.89 | 2.50 | 2.29 | 2.16 | 2.06 | 2.00 | 1.94 | 1.90 | 1.87 | 1.84 | 1.79 | 1.74 | 1.69 | 1.66 | 1.63 | 1.59 | 1.56 | 1.52 | 1.48 |
| 29 | 2.89 | 2.50 | 2.28 | 2.15 | 2.06 | 1.99 | 1.93 | 1.89 | 1.86 | 1.83 | 1.78 | 1.73 | 1.68 | 1.65 | 1.62 | 1.58 | 1.55 | 1.51 | 1.47 |
| 30 | 2.88 | 2.49 | 2.28 | 2.14 | 2.05 | 1.98 | 1.93 | 1.88 | 1.85 | 1.82 | 1.77 | 1.72 | 1.67 | 1.64 | 1.61 | 1.57 | 1.54 | 1.50 | 1.46 |
| 40 | 2.84 | 2.44 | 2.23 | 2.09 | 2.00 | 1.93 | 1.87 | 1.83 | 1.79 | 1.76 | 1.71 | 1.66 | 1.61 | 1.57 | 1.54 | 1.51 | 1.47 | 1.42 | 1.38 |
| 60 | 2.79 | 2.39 | 2.18 | 2.04 | 1.95 | 1.87 | 1.82 | 1.77 | 1.74 | 1.71 | 1.66 | 1.60 | 1.54 | 1.51 | 1.48 | 1.44 | 1.40 | 1.35 | 1.29 |
| 120 | 2.75 | 2.35 | 2.13 | 1.99 | 1.90 | 1.82 | 1.77 | 1.72 | 1.68 | 1.65 | 1.60 | 1.55 | 1.48 | 1.45 | 1.41 | 1.37 | 1.32 | 1.26 | 1.19 |
| ∞ | 2.71 | 2.30 | 2.08 | 1.94 | 1.85 | 1.77 | 1.72 | 1.67 | 1.63 | 1.60 | 1.55 | 1.49 | 1.42 | 1.38 | 1.34 | 1.30 | 1.24 | 1.17 | 1.00 |

**TABLE 5  Percentiles of the $F$ Distribution: $F_{.95}(n_1, n_2)$ (Continued)**

$n_1$ = degrees of freedom for numerator

$n_2$ = degrees of freedom for denominator

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 240.5 | 241.9 | 243.9 | 245.9 | 248.0 | 249.1 | 250.1 | 251.1 | 252.2 | 253.3 | 254.3 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 | 4.36 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.83 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 | 2.85 | 2.79 | 2.72 | 2.65 | 2.61 | 2.57 | 2.53 | 2.49 | 2.45 | 2.40 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 | 2.67 | 2.60 | 2.53 | 2.46 | 2.42 | 2.38 | 2.34 | 2.30 | 2.25 | 2.21 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 | 2.60 | 2.53 | 2.46 | 2.39 | 2.35 | 2.31 | 2.27 | 2.22 | 2.18 | 2.13 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 | 2.42 | 2.35 | 2.28 | 2.24 | 2.19 | 2.15 | 2.11 | 2.06 | 2.01 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 | 2.45 | 2.38 | 2.31 | 2.23 | 2.19 | 2.15 | 2.10 | 2.06 | 2.01 | 1.96 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 | 2.34 | 2.27 | 2.19 | 2.15 | 2.11 | 2.06 | 2.02 | 1.97 | 1.92 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.93 | 1.88 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.87 | 1.81 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 | 2.30 | 2.23 | 2.15 | 2.07 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.78 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 | 2.27 | 2.20 | 2.13 | 2.05 | 2.01 | 1.96 | 1.91 | 1.86 | 1.81 | 1.76 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 | 1.73 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.77 | 1.71 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 | 2.22 | 2.15 | 2.07 | 1.99 | 1.95 | 1.90 | 1.85 | 1.80 | 1.75 | 1.69 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 | 2.20 | 2.13 | 2.06 | 1.97 | 1.93 | 1.88 | 1.84 | 1.79 | 1.73 | 1.67 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.71 | 1.65 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 | 2.18 | 2.10 | 2.03 | 1.94 | 1.90 | 1.85 | 1.81 | 1.75 | 1.70 | 1.64 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 | 2.16 | 2.09 | 2.01 | 1.93 | 1.89 | 1.84 | 1.79 | 1.74 | 1.68 | 1.62 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 | 2.08 | 2.00 | 1.92 | 1.84 | 1.79 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 | 1.99 | 1.92 | 1.84 | 1.75 | 1.70 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.17 | 2.09 | 2.02 | 1.96 | 1.91 | 1.83 | 1.75 | 1.66 | 1.61 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| $\infty$ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 | 1.83 | 1.75 | 1.67 | 1.57 | 1.52 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

## TABLE 5 Percentiles of the $F$ Distribution: $F_{.975}(n_1, n_2)$ (Continued)

$n_1$ = degrees of freedom for numerator

$n_2$ = degrees of freedom for denominator

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 647.8 | 799.5 | 864.2 | 899.6 | 921.8 | 937.1 | 948.2 | 956.7 | 963.3 | 968.6 | 976.7 | 984.9 | 993.1 | 997.2 | 1001 | 1006 | 1010 | 1014 | 1018 |
| 2 | 38.51 | 39.00 | 39.17 | 39.25 | 39.30 | 39.33 | 39.36 | 39.37 | 39.39 | 39.40 | 39.41 | 39.43 | 39.45 | 39.46 | 39.46 | 39.47 | 39.48 | 39.49 | 39.50 |
| 3 | 17.44 | 16.04 | 15.44 | 15.10 | 14.88 | 14.73 | 14.62 | 14.54 | 14.47 | 14.42 | 14.34 | 14.25 | 14.17 | 14.12 | 14.08 | 14.04 | 13.99 | 13.95 | 13.90 |
| 4 | 12.22 | 10.65 | 9.98 | 9.60 | 9.36 | 9.20 | 9.07 | 8.98 | 8.90 | 8.84 | 8.75 | 8.66 | 8.56 | 8.51 | 8.46 | 8.41 | 8.36 | 8.31 | 8.26 |
| 5 | 10.01 | 8.43 | 7.76 | 7.39 | 7.15 | 6.98 | 6.85 | 6.76 | 6.68 | 6.62 | 6.52 | 6.43 | 6.33 | 6.28 | 6.23 | 6.18 | 6.12 | 6.07 | 6.02 |
| 6 | 8.81 | 7.26 | 6.60 | 6.23 | 5.99 | 5.82 | 5.70 | 5.60 | 5.52 | 5.46 | 5.37 | 5.27 | 5.17 | 5.12 | 5.07 | 5.01 | 4.96 | 4.90 | 4.85 |
| 7 | 8.07 | 6.54 | 5.89 | 5.52 | 5.29 | 5.12 | 4.99 | 4.90 | 4.82 | 4.76 | 4.67 | 4.57 | 4.47 | 4.42 | 4.36 | 4.31 | 4.25 | 4.20 | 4.14 |
| 8 | 7.57 | 6.06 | 5.42 | 5.05 | 4.82 | 4.65 | 4.53 | 4.43 | 4.36 | 4.30 | 4.20 | 4.10 | 4.00 | 3.95 | 3.89 | 3.84 | 3.78 | 3.73 | 3.67 |
| 9 | 7.21 | 5.71 | 5.08 | 4.72 | 4.48 | 4.32 | 4.20 | 4.10 | 4.03 | 3.96 | 3.87 | 3.77 | 3.67 | 3.61 | 3.56 | 3.51 | 3.45 | 3.39 | 3.33 |
| 10 | 6.94 | 5.46 | 4.83 | 4.47 | 4.24 | 4.07 | 3.95 | 3.85 | 3.78 | 3.72 | 3.62 | 3.52 | 3.42 | 3.37 | 3.31 | 3.26 | 3.20 | 3.14 | 3.08 |
| 11 | 6.72 | 5.26 | 4.63 | 4.28 | 4.04 | 3.88 | 3.76 | 3.66 | 3.59 | 3.53 | 3.43 | 3.33 | 3.23 | 3.17 | 3.12 | 3.06 | 3.00 | 2.94 | 2.88 |
| 12 | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.44 | 3.37 | 3.28 | 3.18 | 3.07 | 3.02 | 2.96 | 2.91 | 2.85 | 2.79 | 2.72 |
| 13 | 6.41 | 4.97 | 4.35 | 4.00 | 3.77 | 3.60 | 3.48 | 3.39 | 3.31 | 3.25 | 3.15 | 3.05 | 2.95 | 2.89 | 2.84 | 2.78 | 2.72 | 2.66 | 2.60 |
| 14 | 6.30 | 4.86 | 4.24 | 3.89 | 3.66 | 3.50 | 3.38 | 3.29 | 3.21 | 3.15 | 3.05 | 2.95 | 2.84 | 2.79 | 2.73 | 2.67 | 2.61 | 2.55 | 2.49 |
| 15 | 6.20 | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.29 | 3.20 | 3.12 | 3.06 | 2.96 | 2.86 | 2.76 | 2.70 | 2.64 | 2.59 | 2.52 | 2.46 | 2.40 |
| 16 | 6.12 | 4.69 | 4.08 | 3.73 | 3.50 | 3.34 | 3.22 | 3.12 | 3.05 | 2.99 | 2.89 | 2.79 | 2.68 | 2.63 | 2.57 | 2.51 | 2.45 | 2.38 | 2.32 |
| 17 | 6.04 | 4.62 | 4.01 | 3.66 | 3.44 | 3.28 | 3.16 | 3.06 | 2.98 | 2.92 | 2.82 | 2.72 | 2.62 | 2.56 | 2.50 | 2.44 | 2.38 | 2.32 | 2.25 |
| 18 | 5.98 | 4.56 | 3.95 | 3.61 | 3.38 | 3.22 | 3.10 | 3.01 | 2.93 | 2.87 | 2.77 | 2.67 | 2.56 | 2.50 | 2.44 | 2.38 | 2.32 | 2.26 | 2.19 |
| 19 | 5.92 | 4.51 | 3.90 | 3.56 | 3.33 | 3.17 | 3.05 | 2.96 | 2.88 | 2.82 | 2.72 | 2.62 | 2.51 | 2.45 | 2.39 | 2.33 | 2.27 | 2.20 | 2.13 |
| 20 | 5.87 | 4.46 | 3.86 | 3.51 | 3.29 | 3.13 | 3.01 | 2.91 | 2.84 | 2.77 | 2.68 | 2.57 | 2.46 | 2.41 | 2.35 | 2.29 | 2.22 | 2.16 | 2.09 |
| 21 | 5.83 | 4.42 | 3.82 | 3.48 | 3.25 | 3.09 | 2.97 | 2.87 | 2.80 | 2.73 | 2.64 | 2.53 | 2.42 | 2.37 | 2.31 | 2.25 | 2.18 | 2.11 | 2.04 |
| 22 | 5.79 | 4.38 | 3.78 | 3.44 | 3.22 | 3.05 | 2.93 | 2.84 | 2.76 | 2.70 | 2.60 | 2.50 | 2.39 | 2.33 | 2.27 | 2.21 | 2.14 | 2.08 | 2.00 |
| 23 | 5.75 | 4.35 | 3.75 | 3.41 | 3.18 | 3.02 | 2.90 | 2.81 | 2.73 | 2.67 | 2.57 | 2.47 | 2.36 | 2.30 | 2.24 | 2.18 | 2.11 | 2.04 | 1.97 |
| 24 | 5.72 | 4.32 | 3.72 | 3.38 | 3.15 | 2.99 | 2.87 | 2.78 | 2.70 | 2.64 | 2.54 | 2.44 | 2.33 | 2.27 | 2.21 | 2.15 | 2.08 | 2.01 | 1.94 |
| 25 | 5.69 | 4.29 | 3.69 | 3.35 | 3.13 | 2.97 | 2.85 | 2.75 | 2.68 | 2.61 | 2.51 | 2.41 | 2.30 | 2.24 | 2.18 | 2.12 | 2.05 | 1.98 | 1.91 |
| 26 | 5.66 | 4.27 | 3.67 | 3.33 | 3.10 | 2.94 | 2.82 | 2.73 | 2.65 | 2.59 | 2.49 | 2.39 | 2.28 | 2.22 | 2.16 | 2.09 | 2.03 | 1.95 | 1.88 |
| 27 | 5.63 | 4.24 | 3.65 | 3.31 | 3.08 | 2.92 | 2.80 | 2.71 | 2.63 | 2.57 | 2.47 | 2.36 | 2.25 | 2.19 | 2.13 | 2.07 | 2.00 | 1.93 | 1.85 |
| 28 | 5.61 | 4.22 | 3.63 | 3.29 | 3.06 | 2.90 | 2.78 | 2.69 | 2.61 | 2.55 | 2.45 | 2.34 | 2.23 | 2.17 | 2.11 | 2.05 | 1.98 | 1.91 | 1.83 |
| 29 | 5.59 | 4.20 | 3.61 | 3.27 | 3.04 | 2.88 | 2.76 | 2.67 | 2.59 | 2.53 | 2.43 | 2.32 | 2.21 | 2.15 | 2.09 | 2.03 | 1.96 | 1.89 | 1.81 |
| 30 | 5.57 | 4.18 | 3.59 | 3.25 | 3.03 | 2.87 | 2.75 | 2.65 | 2.57 | 2.51 | 2.41 | 2.31 | 2.20 | 2.14 | 2.07 | 2.01 | 1.94 | 1.87 | 1.79 |
| 40 | 5.42 | 4.05 | 3.46 | 3.13 | 2.90 | 2.74 | 2.62 | 2.53 | 2.45 | 2.39 | 2.29 | 2.18 | 2.07 | 2.01 | 1.94 | 1.88 | 1.80 | 1.72 | 1.64 |
| 60 | 5.29 | 3.93 | 3.34 | 3.01 | 2.79 | 2.63 | 2.51 | 2.41 | 2.33 | 2.27 | 2.17 | 2.06 | 1.94 | 1.88 | 1.82 | 1.74 | 1.67 | 1.58 | 1.48 |
| 120 | 5.15 | 3.80 | 3.23 | 2.89 | 2.67 | 2.52 | 2.39 | 2.30 | 2.22 | 2.16 | 2.05 | 1.94 | 1.82 | 1.76 | 1.69 | 1.61 | 1.53 | 1.43 | 1.31 |
| $\infty$ | 5.02 | 3.69 | 3.12 | 2.79 | 2.57 | 2.41 | 2.29 | 2.19 | 2.11 | 2.05 | 1.94 | 1.83 | 1.71 | 1.64 | 1.57 | 1.48 | 1.39 | 1.27 | 1.00 |

**TABLE 5 Percentiles of the $F$ Distribution: $F_{.99}(n_1, n_2)$ (Continued)**

$n_l$ = degrees of freedom for numerator

$n_2$ = degrees of freedom for denominator

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4052 | 4999.5 | 5403 | 5625 | 5764 | 5859 | 5928 | 5982 | 6022 | 6056 | 6106 | 6157 | 6209 | 6235 | 6261 | 6287 | 6313 | 6339 | 6366 |
| 2 | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.39 | 99.40 | 99.42 | 99.43 | 99.45 | 99.46 | 99.47 | 99.47 | 99.48 | 99.49 | 99.50 |
| 3 | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.67 | 27.49 | 27.35 | 27.23 | 27.05 | 26.87 | 26.69 | 26.60 | 26.50 | 26.41 | 26.32 | 26.22 | 26.13 |
| 4 | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.98 | 14.80 | 14.66 | 14.55 | 14.37 | 14.20 | 14.02 | 13.93 | 13.84 | 13.75 | 13.65 | 13.56 | 13.46 |
| 5 | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 | 10.05 | 9.89 | 9.72 | 9.55 | 9.47 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| 6 | 13.75 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 | 7.87 | 7.72 | 7.56 | 7.40 | 7.31 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| 7 | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 | 6.62 | 6.47 | 6.31 | 6.16 | 6.07 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| 8 | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 | 5.81 | 5.67 | 5.52 | 5.36 | 5.28 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| 9 | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 | 5.26 | 5.11 | 4.96 | 4.81 | 4.73 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| 10 | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 | 4.85 | 4.71 | 4.56 | 4.41 | 4.33 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| 11 | 9.65 | 7.21 | 6.22 | 5.67 | 5.32 | 5.07 | 4.89 | 4.74 | 4.63 | 4.54 | 4.40 | 4.25 | 4.10 | 4.02 | 3.94 | 3.86 | 3.78 | 3.69 | 3.60 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 | 4.30 | 4.16 | 4.01 | 3.86 | 3.78 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| 13 | 9.07 | 6.70 | 5.74 | 5.21 | 4.86 | 4.62 | 4.44 | 4.30 | 4.19 | 4.10 | 3.96 | 3.82 | 3.66 | 3.59 | 3.51 | 3.43 | 3.34 | 3.25 | 3.17 |
| 14 | 8.86 | 6.51 | 5.56 | 5.04 | 4.69 | 4.46 | 4.28 | 4.14 | 4.03 | 3.94 | 3.80 | 3.66 | 3.51 | 3.43 | 3.35 | 3.27 | 3.18 | 3.09 | 3.00 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 | 3.80 | 3.67 | 3.52 | 3.37 | 3.29 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| 16 | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 4.03 | 3.89 | 3.78 | 3.69 | 3.55 | 3.41 | 3.26 | 3.18 | 3.10 | 3.02 | 2.93 | 2.84 | 2.75 |
| 17 | 8.40 | 6.11 | 5.18 | 4.67 | 4.34 | 4.10 | 3.93 | 3.79 | 3.68 | 3.59 | 3.46 | 3.31 | 3.16 | 3.08 | 3.00 | 2.92 | 2.83 | 2.75 | 2.65 |
| 18 | 8.29 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.84 | 3.71 | 3.60 | 3.51 | 3.37 | 3.23 | 3.08 | 3.00 | 2.92 | 2.84 | 2.75 | 2.66 | 2.57 |
| 19 | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.77 | 3.63 | 3.52 | 3.43 | 3.30 | 3.15 | 3.00 | 2.92 | 2.84 | 2.76 | 2.67 | 2.58 | 2.49 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 | 3.37 | 3.23 | 3.09 | 2.94 | 2.86 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| 21 | 8.02 | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.64 | 3.51 | 3.40 | 3.31 | 3.17 | 3.03 | 2.88 | 2.80 | 2.72 | 2.64 | 2.55 | 2.46 | 2.36 |
| 22 | 7.95 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.59 | 3.45 | 3.35 | 3.26 | 3.12 | 2.98 | 2.83 | 2.75 | 2.67 | 2.58 | 2.50 | 2.40 | 2.31 |
| 23 | 7.88 | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.54 | 3.41 | 3.30 | 3.21 | 3.07 | 2.93 | 2.78 | 2.70 | 2.62 | 2.54 | 2.45 | 2.35 | 2.26 |
| 24 | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 2.50 | 3.36 | 3.26 | 3.17 | 3.03 | 2.89 | 2.74 | 2.66 | 2.58 | 2.49 | 2.40 | 2.31 | 2.21 |
| 25 | 7.77 | 5.57 | 4.68 | 4.18 | 3.85 | 3.63 | 3.46 | 3.32 | 3.22 | 3.13 | 2.99 | 2.85 | 2.70 | 2.62 | 2.54 | 2.45 | 2.36 | 2.27 | 2.17 |
| 26 | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.42 | 3.29 | 3.18 | 3.09 | 2.96 | 2.81 | 2.66 | 2.58 | 2.50 | 2.42 | 2.33 | 2.23 | 2.13 |
| 27 | 7.68 | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.39 | 3.26 | 3.15 | 3.06 | 2.93 | 2.78 | 2.63 | 2.55 | 2.47 | 2.38 | 2.29 | 2.20 | 2.10 |
| 28 | 7.64 | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.36 | 3.23 | 3.12 | 3.03 | 2.90 | 2.75 | 2.60 | 2.52 | 2.44 | 2.35 | 2.26 | 2.17 | 2.06 |
| 29 | 7.60 | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.33 | 3.20 | 3.09 | 3.00 | 2.87 | 2.73 | 2.57 | 2.49 | 2.41 | 2.33 | 2.23 | 2.14 | 2.03 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 | 2.98 | 2.84 | 2.70 | 2.55 | 2.47 | 2.39 | 2.30 | 2.21 | 2.11 | 2.01 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 | 2.80 | 2.66 | 2.52 | 2.37 | 2.29 | 2.20 | 2.11 | 2.02 | 1.92 | 1.80 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 | 2.63 | 2.50 | 2.35 | 2.20 | 2.12 | 2.03 | 1.94 | 1.84 | 1.73 | 1.60 |
| 120 | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 2.79 | 2.66 | 2.56 | 2.47 | 2.34 | 2.19 | 2.03 | 1.95 | 1.86 | 1.76 | 1.66 | 1.53 | 1.38 |
| ∞ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.64 | 2.51 | 2.41 | 2.32 | 2.18 | 2.04 | 1.88 | 1.79 | 1.70 | 1.59 | 1.47 | 1.32 | 1.00 |