ASSIGNMENT 4

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1 Problem 7.4(c)

method of difference						
n	0	1	2	3	4	5
A(n)	1	2	5	10	17	26
A'(n)		1	3	5	7	9
A"(n)			2	2	2	2

This function is 2nd order polynomial.

$$A(n) = b_0 + b_1 n + b_2 n^2 \Rightarrow \begin{cases} b_0 = 1 \\ b_0 + b_1 + b_2 = 2 \\ b_0 + 2b_2 + 4b_2 = 5 \end{cases} \Rightarrow \begin{cases} b_0 = 1 \\ b_1 = 0 \\ b_2 = 1 \end{cases}$$

$A(n) = 1 + n^2, n \geq 0$

Proof. Set $P(n) : A_n = 1 + n^2$, where $n \in \mathbb{N}$.

Base Case $P(0) = 1$, $P(1) = 2$; both true

Induction step Prove $P(n) \rightarrow P(n+1)$

- Assume $P(0) \wedge P(1) ... P(n)$ are true
- $$\begin{aligned} A_{n+1} &= 2A_n - A_{n-1} + 2 \\ &= 2(n^2 + 1) - ((n-1)^2 + 1) + 2 \\ &= 2n^2 + 2 - n^2 + 2n - 2 + 2 \\ &= n^2 + 2n + 2 \end{aligned} \tag{1}$$
- $P(n) \rightarrow P(n+1)$ is true

By induction, $P(n)$ is true for $n \in \mathbb{N}$ ■

2 Problem 7.56 (a),(b),(c)

(a) (1) $M(0, k) = 0$,

(2) $M(n, 1) = n$, since we need to do linear search here.

(3) $M(n, k) = \log_2(n) + 1$ in worst case.
The number of times to do binary search is $\log_2 n$, but in worst case, we need an extra time to identify the exact number when the number of floor is odd.

(b) Let the total number of floor n.

i) $\log_2(n - x) + 1$

ii) $log_2(x) + 1$

(c) i) $M(7, 3) = 3$

ii) $M(8, 3) = 4$

iii) $M(9, 3) = 4$

iv) $M(10, 3) = 4$

3 Problem 8.6

We assume the definition of the set is the following:

$$\textcircled{1}\ 2^0 \in S$$
$$\textcircled{2}\ 2^n \in S \to 2^{n+1} \in S$$

(a) $P(n)$: Every Element in set is non-negative power of 2

Base Case $P(0) = 2^0 = 1$, which is a non-negative power of 2

Induction Step Assume $P(n) = 2^n$, $n \geq 0$ is true

- The constructor $n + 1$ would always larger than 0 as n starts from 0
- 2^{n+1} will always be a non-negative power of 2

By structural induction, every element in set will always a non-negative power of 2 is true. ■

(b) $P(n)$: Every non-negative power of 2 is in the set

Base Case $P(0) = 2^0 = 1$; 1 is in the set, $P(0)$ is true

Induction Step The constructor shows that once we have 2^n, 2^{n+1} will then be in the set. We could prove the relationship use direct proof

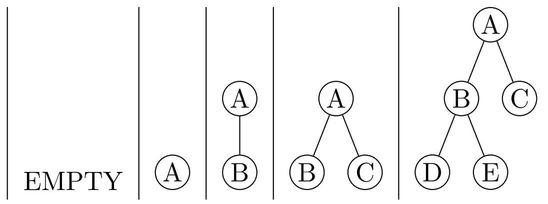
- Assume $P(1) \wedge P(2) \wedge ...P(n-1) \wedge P(n)$ is true
- From constructor, since $P(n)$ is true, we would get 2^{n+1} is in the set
- $P(n+1)$ is true

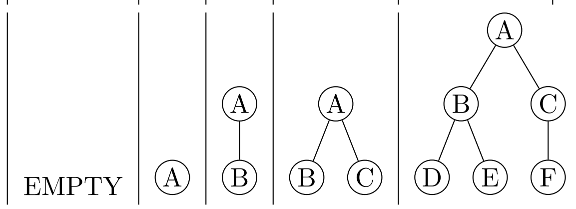
By structural induction, every element non-negative power of 2 is in the set is true. ■

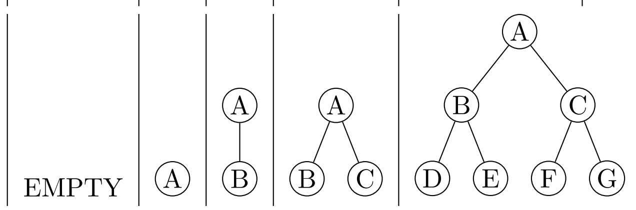
4 Problem 8.18

(a) Graph Example

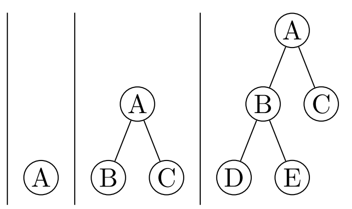
- RBT

5 vertices 

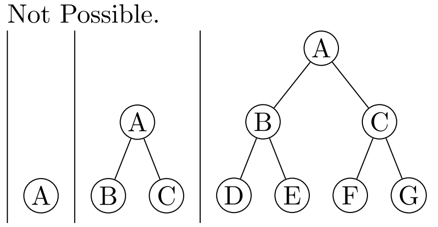
6 vertices 

7 vertices 

- RFBT

5 vertices 

6 vertices Not Possible.

7 vertices 

(b) We define RFBT as the following:

① A single root-node is an RFBT

② If T_1, T_2 are disjoint RFBTs with root r_1, r_2 , then linking r_1, r_2 to a new root r gives a new RFBT with root r

$P(n)$: the nth RFBT has odd number of vertices,

Base Case $P(1)$: RFBT with single node has odd number of vertices.

Induction Steps In the constructor, a single-root node is the parent, and it will accept two new disjoint RFBTs. We use direct proof to prove nth RFBT will always have odd number of vertices.

(a) Assume $P(1) \wedge P(2) \wedge ... \wedge P(n-1) \wedge P(n)$ is true

(b) Any new RFBT will have the sum of two odd number plus 1 vertices. (The root node)

(c) Since odd number + odd number = even number, while even number + odd number = an odd number, the children will always have odd number vertices

(d) $P(n)$ is always true

By structural induction, P is true. ■

5 Problem 9.3 (b),(e)

(b) $\sum_{i=1}^{n}\sum_{j=1}^{i}(i+j)$

Compute the inner side

$$\begin{aligned}\sum_{j=1}^{i}(i+j) &= \sum_{j=1}^{i} i + \sum_{j=1}^{i} j \\ &= \sum_{j=1}^{i} i + \frac{1}{2}i(i+1) \\ &= i\sum_{j=1}^{i} 1 + \frac{1}{2}i(i+1) \\ &= i(i+1-1) + \frac{1}{2}i(i+1) \\ &= i^2 + \frac{i^2+i}{2}\end{aligned} \tag{2}$$

– Combined

$$\begin{aligned}\sum_{i=1}^{n} i^2 + \frac{i^2+i}{2} &= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\frac{i^2+i}{2} \\ &= \sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i^2 + i \\ &= \sum_{i=1}^{n} i^2 + \frac{1}{2}(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i) \\ &= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}(\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)) \\ &= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}n(n+1) \\ &= \frac{1}{2}n(n+1)^2\end{aligned} \tag{3}$$

■

(e) $\sum_{i=0}^{n}\sum_{j=0}^{m} 2^{i+j}$

Compute the inner side

$$\begin{aligned}\sum_{j=0}^{m} 2^{i+j} &= \sum_{j=0}^{m} 2^i \times 2^j \\ &= 2^i\sum_{j=0}^{m} 2^j \\ &= 2^i(2^{m+1}-1)\end{aligned} \tag{4}$$

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$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{m}2^{i+j} &= \sum_{i=0}^{n}2^{i}(2^{m+1}-1)\\ &= (2^{m+1}-1)\sum_{i=0}^{n}2^{i}\\ &= (2^{m+1}-1)(2^{n+1}-1)\end{aligned} \tag{5}$$

■

6 Problem 9.37

(a) n^3

i) $\lim_{n\to\infty} \frac{n^3}{n^2 \log_2^2 n} = \infty$, $g \in O(f)$

ii) $\lim_{n\to\infty} \frac{n^3}{n^3+n^2} = 1$. the result is constant, so both $f \in O(g)$ and $g \in O(f)$

iii) $\lim_{n\to\infty} \frac{n^3}{n^{3.5}} = 0$, $f \in O(g)$

iv) $\lim_{n\to\infty} \frac{n^3}{2^{3\log_2 n+2}} = \frac{n^3}{4n^3} = \frac{1}{4}$. the result is constant, so both $f \in O(g)$ and $g \in O(f)$

v) $\lim_{n\to\infty} \frac{n^3}{2^{\log_2^2 n}} = 0$, $f \in O(g)$

(b) 2^n

i) $\lim_{n\to\infty} \frac{2^n}{3^n} = 0$, $f \in O(g)$

ii) $\lim_{n\to\infty} \frac{2^n}{2\sqrt{n}} = \infty$, $g \in O(f)$

iii) $\lim_{n\to\infty} \frac{2^n}{2^{2n}} = \lim_{n\to\infty} 2^{-n} = 0$, $f \in O(g)$

iv) $\lim_{n\to\infty} \frac{2^n}{2^{n+\log_2 n}} = 0$, $f \in O(g)$

v) $\lim_{n\to\infty} \frac{2^n}{2^{n+4}+2^{\sqrt{n}}} = \frac{1}{16}$. the result is constant, so both $f \in O(g)$ and $g \in O(f)$

(c) $n!$

i) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$, $f \in O(g)$

ii) $\lim_{n\to\infty} \frac{n!}{n^{n/2}} = \infty$, $g \in O(f)$

iii) $\lim_{n\to\infty} \frac{n!}{(n+1)!} = 0$, $f \in O(g)$

iv) $\lim_{n\to\infty} \frac{n!}{2^{nlogn}} = 0$, $f \in O(g)$

v) $\lim_{n\to\infty} \frac{n!}{2^{n^2}} = 0$, $f \in O(g)$

(d) $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{2n^2+3n+1}{6}$

i) $\lim_{n\to\infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^2} = \infty$, $g \in O(f)$

ii) $\lim_{n\to\infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^2 \log n} = \infty$, $g \in O(f)$

iii) $\lim_{n\to\infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} = \frac{1}{3}$. The result is constant, so both $f \in O(g)$ and $g \in O(f)$

iv) $\lim_{n\to\infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{4^{\log_2 n}} = \infty$, $g \in O(f)$

v) $\lim_{n\to\infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{8^{\log_2 n}} = \frac{1}{3}$. The result is constant, so both $f \in O(g)$ and $g \in O(f)$

(e) $\sum_{i=1}^{n} \sum_{j=1}^{n} 2^{i+j} = (2^{n+1} - 2)^2$

i) $\lim_{n\to\infty} \frac{(2^{n+1}-2)^2}{2^n} = \infty$, $g \in O(f)$

ii) $\lim_{n\to\infty} \frac{(2^{n+1}-2)^2}{2^{2n}} = 4$. The result is constant, so both $f \in O(g)$ and $g \in O(f)$

iii) $\lim_{n\to\infty} \frac{(2^{n+1}-2)^2}{2^{3n}} = 0$, $f \in O(g)$

iv) $\lim_{n\to\infty} \frac{(2^{n+1}-2)^2}{2^{n^2}} = 0$, $f \in O(g)$

v) $\sum_{i=1}^{n}\sum_{j=1}^{i} 2^{i+j} = \sum_{i=1}^{n} 2^i(2^{i+1}-2) \approx 2^{2n+1}$
$\lim_{n\to\infty} \frac{(2^{n+1}-1)^2}{2^{2n+1}} = 2$. The result is constant, so both $f \in O(g)$ and $g \in O(f)$

(f) $\sum_{i=1}^{n} i\sqrt{i} \approx n^{\frac{5}{2}}$

i) $\lim_{n\to\infty} \frac{n^{\frac{5}{2}}}{n^2} = \infty$, $g \in O(f)$

ii) $\lim_{n\to\infty} \frac{n^{\frac{5}{2}}}{n^2 \log_2 n} = \infty$, $g \in O(f)$

iii) Since $-1 \leq sin \leq 1$, $n^{-3} \leq n^{3sin(n\pi/2)} \leq n^3$. As it is alternating, neither situation fit in.

iv) $\lim_{n\to\infty} \frac{n^{\frac{5}{2}}}{4^{\log_2 n}} = \infty$, $g \in O(f)$

v) $\lim_{n\to\infty} \frac{n^{\frac{5}{2}}}{8^{\log_2 n}} = 0$, $f \in O(g)$

7 Problem 9.44 (a)

$\sum_{i=1}^{n} \frac{i^2}{i^3+1}$, this function is decreasing over time.

- Integration

$$\begin{aligned} \int \frac{i^2}{i^3+1} di &= \int \frac{1}{3}\frac{1}{a} da, a = i^3+1 \\ &= \frac{1}{3} ln(a) \\ &= \frac{1}{3} ln(i^3+1) \end{aligned} \tag{6}$$

- Upper Bond

$$\begin{aligned} \frac{1}{3} ln(i^3+1)|_0^n &= \frac{1}{3}(ln(n^3+1) - 0) \\ &= \frac{1}{3} ln(n^3+1) \end{aligned} \tag{7}$$

- Lower Bond

$$\begin{aligned} \frac{1}{3} ln(i^3+1)|_1^{n+1} &= \frac{1}{3}(ln((n+1)^3+1) - ln(2)) \\ &= \frac{1}{3} ln((n+1)^3+1) - \frac{1}{3} ln(2) \end{aligned} \tag{8}$$

- Behavior

$$\theta(\frac{1}{3} ln(i^3+1)|_0^n) = \theta(\frac{1}{3} ln(n^3+1)) \approx \theta(\frac{1}{3} ln(n^3)) = \theta(ln(n)) \tag{9}$$