ASSIGNMENT 3

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Tuesday 12^{th} May, 2020

1 Problem 5.10(j)

Claim: $P(n)$ as for $n \in \mathbb{N}$, 3 divides $n^3 + 5n + 6$.

Base Case: Claim that $P(1)$ is true; $1 + 5 + 6 = 12$; 3 divides 12, which is true

Induction Step: Show $P(n) \to P(n+1)$, Using direct proof

i) Assume $P(n)$ is true

ii)

$$\begin{aligned}(n+1)^3 + 5(n+1) + 6 &= n^3 + 3n^2 + 3n + 1 + 5n + 5 + 6 \\ &= n^3 + 3n^2 + 8n + 12 \\ &= (n^3 + 5n + 6) + (3n^2 + 3n + 6) \\ &= (n^3 + 5n + 6) + 3(n^2 + n + 2)\end{aligned} \tag{1}$$

iii) Since 3 divides $n^3 + 5n + 6$ and $3(n^2 + n + 2)$, $P(n+1)$ is true

iv) $P(n) \to P(n+1)$ is true

By Induction, $P(n)$ is true for $n \in \mathbb{N}$ ■

2 Problem 5.12(i)

Claim: $P(n)$ as for $n \geq 1$, $n! \geq n^n e^{-n}$,

Base Case: Claim that $P(1)$ is true; $1 \geq e^{-1}$, $P(1)$ is true

Induction Step: Show $P(n) \to P(n+1)$, Using direct proof

i) Assume $P(n)$ is true

ii)

$$\begin{aligned}
n! &\geq n^n e^{-n} \\
n!(n+1) &\geq n^n e^{-n}(n+1) \\
(n+1)! &\geq n^n e^{-n}(n+1) \\
(n+1)! &\geq n^n e^{-n-1}(n+1)(1+\frac{1}{n})^n \text{(derived from hint)} \\
(n+1)! &\geq (n+1)^n e^{-n-1}(n+1) \\
(n+1)! &\geq (n+1)^{(n+1)} e^{-n-1}
\end{aligned} \tag{2}$$

iii) $P(n) \to P(n+1)$ is true

By Induction, $P(n)$ is true for $n \geq 1$ ■

3 Problem 5.18(a)

Claim: $P(n)$ as for $H_1 + H_2 + ... + H_n = (n+1)H_n - n$,

Base Case: Claim that $P(1)$ is true; $1 = (1+1) * 1 - 1$, $P(1)$ is true

Induction Step: Show $P(n) \to P(n+1)$, Using direct proof

i) Assume $P(n)$ is true

ii)

$$\begin{aligned} H_1 + H_2 + ... + H_n + H_{n+1} &= H_{n+1} + (n+1)H_n - n \\ &= H_{n+1} + (n+1)(H_{n+1} - \frac{1}{n+1}) - n \\ &= H_{n+1} + (n+1)(H_n + 1) - 1 - n \\ &= (n+2)H_{n+1} - 1 - n \end{aligned} \tag{3}$$

iii) $P(n) \to P(n+1)$ is true

By Induction, $P(n)$ is true for $n \geq 1$ ■

4 Problem 5.60

(a) The perimeter highlighted by the thick line is 42

(b) Prove by induction

Claim: Perimeter is even for all $n \geq 1$

Base Case: $P(1)$ is true since 4 is an even number

Induction Step: Show $P(n) \rightarrow P(n+1)$, Using direct proof

i) Assume $P(n)$ is true, the perimeter will be an even number

ii) there are 5 cases when a new square is added

(1) None side is blocked by the existing squares: the perimeter will +4 under such condition. An even number +4 is still an even number

(2) One side of the square is blocked by the existing squares: the perimeter will +2. An even number +2 is still an even number

(3) Two sides of the square are blocked by the existing squares: The perimeter stays the same, which stays an even number

(4) Three sides of the square are blocked by the existing squares: The perimeter will be -2. An even number -2 stays an even number

(5) Four sides of the square are blocked by the existing squares: the perimeter will -4. An even number -4 stays an even number.

(6) $P(n) \rightarrow P(n+1)$ is true

By Induction, $P(n)$ is true ■

5 Problem 6.6

(a) Proof.

Claim: $P(n)$ as for $\frac{H_1}{1} + \frac{H_2}{2} + ... + \frac{H_n}{n} \leq \frac{1}{2}{H_n}^2 + 1$,

Base Case: Claim that $P(1)$ is true; $\frac{1}{1} \leq \frac{1}{2}1^2 + 1$, $P(1)$ is true

Induction Step: Show $P(1) \wedge P(2) \wedge ... \wedge P(n) \to P(n+1)$, Using direct proof

i) Assume $P(1) \wedge P(2) \wedge ... \wedge P(n)$ is true

ii)

$$\begin{aligned}
\frac{H_1}{1} + \frac{H_2}{2} + ... + \frac{H_n}{n} &\leq \frac{1}{2}{H_n}^2 + 1 \\
\frac{H_1}{1} + \frac{H_2}{2} + ... + \frac{H_n}{n} + \frac{H_{n+1}}{n+1} &\leq \frac{1}{2}{H_n}^2 + 1 + \frac{H_{n+1}}{n+1} \\
&= \frac{1}{2}{H_n}^2 + 1 + \frac{H_n}{n+1} + \frac{1}{(n+1)^2} \\
&= \frac{1}{2}{H_n}^2 + \frac{H_n}{n+1} + \frac{1}{(n+1)^2} + 1 \\
&= \frac{1}{2}({H_n}^2 + \frac{2H_n}{n+1} + \frac{1}{(n+1)^2}) + \frac{1}{2(n+1)^2} + 1 \\
&= \frac{1}{2}(H_n + \frac{1}{n+1})^2 + \frac{1}{2(n+1)^2} + 1 \\
&= \frac{1}{2}{H_{n+1}}^2 + \frac{1}{2(n+1)^2} + 1
\end{aligned} \tag{4}$$

iii) $\frac{1}{2}{H_{n+1}}^2 + 1 \leq \frac{1}{2}{H_{n+1}}^2 + \frac{1}{2(n+1)^2} + 1$

iv) $P(1) \wedge P(2) \wedge ... \wedge P(n) \to P(n+1)$ is true

By Induction, $P(n)$ is true ■

(b) Proof.

Claim: $P(n)$ as for $\frac{H_1}{1} + \frac{H_2}{2} + ... + \frac{H_n}{n} \leq \frac{1}{2}{H_n}^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2})$,

Base Case: Claim that $P(1)$ is true; $\frac{1}{1} \leq \frac{1}{2}1^2 + 1$, $P(1)$ is true

Induction Step: Show $P(1) \wedge P(2) \wedge ... \wedge P(n) \to P(n+1)$, Using direct proof

i) Assume $P(1) \wedge P(2) \wedge ... \wedge P(n)$ is true ii)

$$\begin{aligned}
\frac{H_1}{1} + \frac{H_2}{2} + ... + \frac{H_n}{n} &\leq \frac{1}{2}{H_n}^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}) \\
\frac{H_1}{1} + \frac{H_2}{2} + ... + \frac{H_n}{n} + \frac{H_{n+1}}{n+1} &\leq \frac{1}{2}{H_n}^2 + \frac{H_{n+1}}{n+1} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}) \\
&= \frac{1}{2}{H_n}^2 + \frac{H_n}{n+1} + \frac{1}{{(n+1)}^2} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}) \\
&= \frac{1}{2}{H_n}^2 + \frac{H_n}{n+1} + \frac{1}{{(n+1)}^2} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}) \\
&= \frac{1}{2}({H_n}^2 + \frac{2H_n}{n+1} + \frac{1}{{(n+1)}^2}) + \frac{1}{2{(n+1)}^2} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}) \\
&= \frac{1}{2}{(H_n + \frac{1}{n+1})}^2 + \frac{1}{2{(n+1)}^2} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}) \\
&= \frac{1}{2}{H_{n+1}}^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2} + \frac{1}{{(n+1)}^2})
\end{aligned} \tag{5}$$

iii) $P(1) \land P(2) \land ... \land P(n) \to P(n+1)$ is true

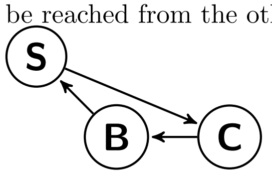
By Induction, $P(n)$ is true ■

(c) The second claim is stronger since it has narrower scope than the first one.

6 Problem 6.45(a)

Claim: $P(n)$ as There is one special city can be reached by every other city either directly or via one stop.

Base Case: There are three city, and each of them has a one way flight, $P(3)$ is true as there is always a city can be reached from the others

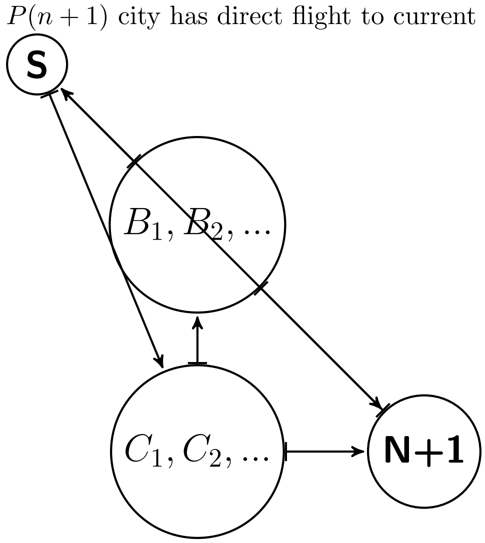


Induction Step: Show $P(3) \wedge P(4) \wedge ...P(n) \rightarrow P(n+1)$, Using direct proof

i) Assume $P(3) \wedge P(4) \wedge ...P(n)$ is true

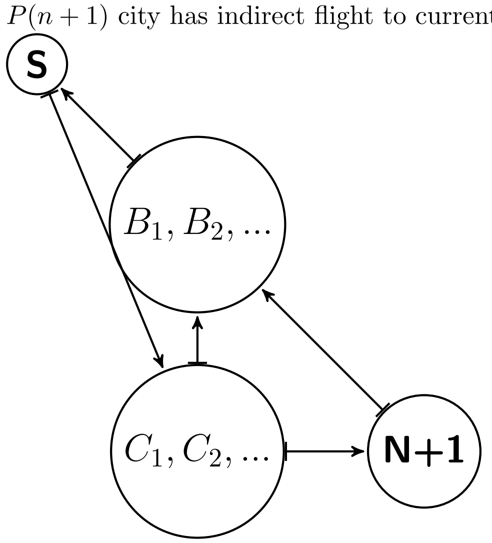
ii) There are three cases. In each cases, $B_1, B_2, ...$ city and $C_1, C_2, ...$ have their own connection between each other other than the special city **S**

Case 1 $P(n+1)$ city has direct flight to current special city.



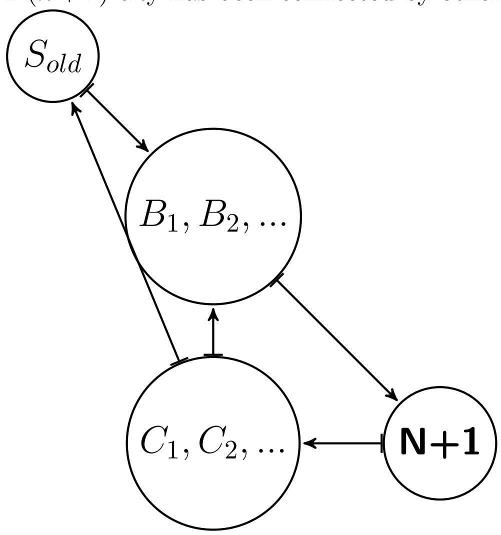
The new city will be the same as **B** series cities.

Case 2 $P(n+1)$ city has indirect flight to current special city.



The new city will be the same as **C** series cities.

Case 3 $P(n+1)$ city has been connected by other cities



The new city will become the special city.

iii) $P(n) \to P(n+1)$ is true

By Induction, $P(n)$ is true ■