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1. 证明:

将AXB的第i列($i=1,2,\dots n$)视作一个整体，记为$\alpha_i$

则 $vec(AXB)=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}$

$$X=\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \qquad AX=\begin{bmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ | & | & & | \end{bmatrix}$$

$$B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

则 $\alpha_i=\sum_{j=1}^{n} b_{ji}Ax_j$

将$(B^T\otimes A)vec(X)$记为 $\begin{bmatrix}e_1\\ e_2\\ \vdots\\ e_n\end{bmatrix}$ 则每个$e_i$($i=1,2,\dots n$)包含m个元素

$$B^T\otimes A=\begin{bmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{2n}A & \cdots & b_{nn}A \end{bmatrix}$$

$b_{ij}A$的尺寸为$m\times m$

$x_i$的尺寸为$m\times 1$

故 $b_{ij}Ax_i$的尺寸为$m\times 1$

($i,j=1,2,\dots,n$)

则 $e_i=\sum_{j=1}^{n} b_{ji}Ax_j$ 由于 $\alpha_i=e_i$ ($i=1,2,\dots n$)

故 $vec(AXB)=(B^T\otimes A)vecX$

3. 解:

$$记\ A=\begin{bmatrix}0&1&&&\\&0&1&&\\&&\ddots&\ddots&\\&&&0&1\\1&&&&0\end{bmatrix}$$

$$\lambda I-A=\begin{bmatrix}\lambda&-1&&&\\&\lambda&-1&&\\&&\ddots&\ddots&\\&&&\lambda&-1\\-1&&&&\lambda\end{bmatrix}$$

$$|\lambda I-A|=\lambda\begin{vmatrix}\lambda&-1&&&\\&\lambda&-1&&\\&&\ddots&\ddots&\\&&&\lambda&-1\\&&&&\lambda\end{vmatrix}_{n-1}+\begin{vmatrix}0&-1&&&\\&\lambda&-1&&\\&&\ddots&\ddots&\\&&&\lambda&-1\\-1&&&&\lambda\end{vmatrix}_{n-1}$$

$$记\ a_{n-1}=\begin{vmatrix}\lambda&-1&&&\\&\lambda&-1&&\\&&\ddots&\ddots&\\&&&\lambda&-1\\&&&&\lambda\end{vmatrix}_{n-1}\qquad b_{n-1}=\begin{vmatrix}0&-1&&&\\&\lambda&-1&&\\&&\ddots&\ddots&\\&&&\lambda&-1\\-1&&&&\lambda\end{vmatrix}_{n-1}$$

$$则\ |\lambda I-A|=\lambda\left(\lambda a_{n-2}+\begin{vmatrix}0&-1&&\\&\lambda&-1&\\&&\ddots&\ddots\\&&&\lambda&-1\\&&&&\lambda\end{vmatrix}_{n-2}\right)+b_{n-2}$$

$$记\ c_{n-2}=\begin{vmatrix}0&-1&&\\&\lambda&-1&\\&&\ddots&\ddots\\&&&\lambda&-1\\&&&&\lambda\end{vmatrix}_{n-2}\qquad 则\ |\lambda I-A|=\lambda(\lambda a_{n-2}+c_{n-2})+b_{n-2}$$

$b_{n-2} = b_{n-3} = \cdots = b_2 = \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} = -1$

$c_{n-2} = c_{n-3} = \cdots = c_2 = 0$

$a_{n-1} = \lambda a_{n-2} + c_{n-2} = \lambda a_{n-2}$

$a_{n-2} = a_{n-3} + c_{n-3} = \lambda a_{n-3}$

故 $a_{n-2} = \lambda a_{n-3} = \cdots = \lambda^{n-4} a_2 = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^{n-2}$

因此 $|\lambda I - A| = \lambda^n - 1$ 令 $|\lambda I - A| = 0$ 得 ~~$\lambda=1$~~

因此 所求特征值为 1

当n为奇数时 $\lambda=1$

当n为偶数时 $\lambda=\pm 1$

$\lambda=1$ 时，所有特征向量为 $k\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$ $k=\pm 1, \pm 2, \cdots$

$\lambda=-1$ 时 所有特征向量为 $k\begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{n-1} \end{bmatrix}_{n\times 1}$ $k=\pm 1, \pm 2, \cdots$

$\lambda = e^{\frac{2ik\pi}{n}}, k=0, \ldots, n-1$

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4. 证明：$z_1=z_2=z_3$时显然成立，下证$z_1$、$z_2$、$z_3$互不相等时的情况

充分性：

$z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_1z_3$

$=(z_1-z_2)^2+(z_2-z_3)^2+z_1z_2+z_2z_3-z_2^2-z_1z_3$

$=(z_2-z_1)^2+(z_3-z_2)^2+(z_3-z_2)(z_2-z_1)=0$

故 $\left(\frac{z_2-z_1}{z_3-z_2}\right)^2+\frac{z_2-z_1}{z_3-z_2}+1=0$

故 $\frac{z_2-z_1}{z_3-z_2}=\frac{-1\pm\sqrt{3}i}{2}$

$=e^{\frac{2\pi}{3}i}$ 或 $e^{\frac{4\pi}{3}i}$

故 $|z_2-z_1|=|z_3-z_2|$

同理可得 $|z_2-z_1|=|z_1-z_3|$

因此 $|z_1-z_2|=|z_2-z_3|=|z_3-z_1|$

必要性：由于 $|z_1-z_2|=|z_2-z_3|=|z_3-z_1|$ 设三者均为k

故 $z_1-z_2$、$z_2-z_3$、$z_3-z_1$ 构成等边三角形

不妨设 $z_1,z_2,z_3$ 按逆时针排列 在复平面上

设 $z_1-z_2=ke^{i\theta}$ 则 $z_2-z_3=ke^{i(\theta+\frac{2}{3}\pi)}$ $z_3-z_1=ke^{i(\theta+\frac{4}{3}\pi)}$

则 $z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_1z_3$

$=\frac{1}{2}[(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2]$

$=\frac{k^2}{2}[e^{i\cdot2\theta}+e^{i\cdot2(\theta+\frac{2}{3}\pi)}+e^{i\cdot2(\theta+\frac{4}{3}\pi)}]$

$=\frac{k^2}{2}e^{i\cdot2\theta}\cdot(1+e^{i\frac{4}{3}\pi}+e^{i\frac{8}{3}\pi})$

$= \frac{k^2}{2} e^{i\cdot 2\theta} \left( 1 + (-\frac{1}{2}) + \frac{\sqrt{3}}{2}i + (-\frac{1}{2}) - \frac{\sqrt{3}}{2}i \right)$

$= 0$

即 $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_1z_3 = 0$

5. 解:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad xx^T = \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_1x_2 & x_2^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1x_n & x_2x_n & \cdots & x_n^2 \end{bmatrix}$$

$xx^T$ 也是一个对称阵

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad a_{ij} = a_{ji}$$

$$A + xx^T = \begin{bmatrix} a_{11}+x_1^2 & a_{12}+x_1x_2 & \cdots & a_{1n}+x_1x_n \\ a_{21}+x_1x_2 & a_{22}+x_2^2 & \cdots & a_{2n}+x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+x_1x_n & a_{n2}+x_2x_n & \cdots & a_{nn}+x_n^2 \end{bmatrix}$$

$(A+xx^T)^T = A^T + xx^T = A + xx^T$

故 $A + xx^T$ 是对称阵

故 $tr((A+xx^T)^2)$ 即 $A+xx^T$ 中所有元素的平方和

$$tr((A+xx^T)^2) = \sum_{1 \le i,j \le n,\ i,j \in Z} (a_{ij} + x_ix_j)^2$$

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$$\frac{\partial f}{\partial x_k} = 4\left[ x_k (x_1^2 + x_2^2 + \dots + x_n^2) + \sum_{i=1}^{n} a_{ki} x_i \right]$$

故 $$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = 4\left( (x^T x) x + A x \right)$$

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