1. 记 $A=\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}$

则记 $D_n=\det(\lambda I-A)$ $\quad D_1=\lambda \quad D_2=\lambda^2-1$

$D_n=\lambda D_{n-1}-D_{n-2}$

设 $D_n-\alpha D_{n-1}=\beta(D_{n-1}-\alpha D_{n-2})$

则 $D_n-\beta D_{n-1}=\alpha(D_{n-1}-\beta D_{n-2})$

其中 $\begin{cases} \alpha+\beta=\lambda \\ \alpha\beta=1 \end{cases}$ $\quad \therefore D_n-\alpha D_{n-1}=\beta^{n-2}(D_2-\alpha D_1)=\beta^n$

$D_n-\beta D_{n-1}=\alpha^n$

① 当 $\alpha\neq\beta$，即 $\lambda^2\neq4$ $\lambda\neq\pm2$ 时

$$D_n=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

令 $D_n=0$ 则 $\alpha^{n+1}=\beta^{n+1}$ $\therefore \alpha^{2n+2}=1$

$\therefore \alpha=e^{i\frac{k\pi}{n+1}}$ $(k=0,1,\dots,2n+1)$

$\lambda=\alpha+\alpha^{-1}=2\cos\frac{k\pi}{n+1}$ $(k=1,2,\cdots n)$

② 当 $\alpha=\beta$ 即 $\lambda^2=4$ 时

$$D_n=(n+1)\left(\frac{\lambda}{2}\right)^n\neq0$$

$\therefore$ ±2 不是 A 的特征值

$\therefore$ A 的所有特征值为 $2\cos\frac{k\pi}{n+1}$ $(k=1,2,3,\dots,n)$

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2. 设 $P = diag\{1,2,4,8\}$ 则 $P^{-1} = diag\{1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\}$.

$$B = PAP^{-1} = \begin{bmatrix} 16 & 8 & 0 & 0 \\ 8 & 16 & 6 & 0 \\ 0 & 6 & 16 & 8 \\ 0 & 0 & 8 & 16 \end{bmatrix}$$ 由圆盘定理可知 B的特征值的实部都 ≥ 2

由于A与B相似 ∴ A的特征值的实部都严格大于0

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3. ∵ $\langle\cdot,\cdot\rangle_1$ 和 $\langle\cdot,\cdot\rangle_2$ 都是 $\mathbb{C}^n$ 上的内积

∴ $\langle x,x\rangle_1 \geq 0$，当且仅当 $x=0$ 时有 $\langle x,x\rangle_1 = 0$

$\langle x,x\rangle_2 \geq 0$，当且仅当 $x=0$ 时有 $\langle x,x\rangle_2 = 0$

$\langle x,y\rangle_1 = \overline{\langle y,x\rangle_1}$ $\quad\langle x,y\rangle_2 = \overline{\langle y,x\rangle_2}$

$\langle x_1+x_2,y\rangle_1 = \langle x_1,y\rangle_1 + \langle x_2,y\rangle_1$

$\langle x_1+x_2,y\rangle_2 = \langle x_1,y\rangle_2 + \langle x_2,y\rangle_2$

$\langle x\alpha,y\rangle_1 = \alpha\langle x,y\rangle_1$ $\quad\langle x\alpha,y\rangle_2 = \alpha\langle x,y\rangle_2$ 对一切 $\alpha\in\mathbb{C}$ 成立

∴ ① $\langle x,x\rangle = \langle x,x\rangle_1 + \langle x,x\rangle_2 \geq 0$，当且仅当 $x=0$ 时有 $\langle x,x\rangle = 0$

② $\langle x,y\rangle = \langle x,y\rangle_1 + \langle x,y\rangle_2 = \overline{\langle y,x\rangle_1} + \overline{\langle y,x\rangle_2}$
$= \overline{\langle y,x\rangle}$

③ $\langle x_1+x_2,y\rangle = \langle x_1+x_2,y\rangle_1 + \langle x_1+x_2,y\rangle_2$
$= \langle x_1,y\rangle_1 + \langle x_1,y\rangle_2 + \langle x_2,y\rangle_1 + \langle x_2,y\rangle_2$
$= \langle x_1,y\rangle + \langle x_2,y\rangle$

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对$\forall \alpha \in \mathbb{C}$

④ $\langle x\alpha, y\rangle = \langle x\alpha, y\rangle_1 + \langle x\alpha, y\rangle_2$

$= \alpha\langle x, y\rangle_1 + \alpha\langle x, y\rangle_2$

$= \alpha\langle x, y\rangle$

$\therefore \langle \cdot,\cdot\rangle$ 也是$\mathbb{C}^n$上的一个内积

4. 证明:

$\langle a-b, c-d\rangle + \langle a-d, b-c\rangle$

$= \langle a, c-d\rangle + \langle -b, c-d\rangle + \langle a, b-c\rangle + \langle -d, b-c\rangle$

$= \langle c-d, a\rangle - \langle c-d, b\rangle + \langle b-c, a\rangle - \langle b-c, d\rangle$

$= \langle c, a\rangle - \langle d, a\rangle - \langle c, b\rangle + \langle d, b\rangle + \langle b, a\rangle - \langle c, a\rangle$
$- \langle b, d\rangle + \langle c, d\rangle$

$= \langle a, b\rangle - \langle c, b\rangle - \langle a, d\rangle + \langle c, d\rangle$

$= \langle a-c, b\rangle + \langle c-a, d\rangle$

$= \langle b, a-c\rangle - \langle d, a-c\rangle$

$= \langle b-d, a-c\rangle$

$= \langle a-c, b-d\rangle$

$\therefore$ 证毕

5. 证明:

$\forall x \in R^n$ 记 $x = [x_1, x_2, \cdots, x_n]^T$

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$$x^T H x = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_i x_j}{i+j-1} = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1 x_i x_j\, y^{i+j-2}\,dy$$

$$= \int_0^1 \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y^{i-1}\cdot x_j y^{j-1}\,dy$$

$$= \int_0^1 \left(\sum_{i=1}^{n} x_i y^{i-1}\right)^2 dy \geq 0$$

令 $x^T H x = 0 \Leftrightarrow \sum_{i=1}^{n} x_i y^{i-1} = 0 \Leftrightarrow x_1 = x_2 = \cdots = x_n = 0$

$\Leftrightarrow x = 0$

$\therefore$ $x \neq 0$时 $x^T H x > 0$ 又$\because H^T = H$

$\therefore$ n阶 Hilbert 矩阵是正定矩阵

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