1. 设 $\operatorname{rank}(B)=r$

由满秩分解 $\exists X\in K^{n_2\times r}$ 和 $Y\in K^{r\times n_3}$，使得

$$B=XY$$，其中 X 列满秩，Y 行满秩

$\therefore$ $\operatorname{rank}(AB)=\operatorname{rank}((AX)Y)=\operatorname{rank}(AX)$

$\operatorname{rank}(BC)=\operatorname{rank}(X(YC))=\operatorname{rank}(YC)$

由 Sylvester 不等式

$$\operatorname{rank}(ABC)=\operatorname{rank}((AX)(YC))$$

$$\geq \operatorname{rank}(AX)+\operatorname{rank}(YC)-r$$

$$=\operatorname{rank}(AB)+\operatorname{rank}(BC)-\operatorname{rank}(B)$$

$\therefore$ 证毕

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2. 证明:

设V_1的一组基为$\{s_1, s_2, \ldots, s_m\}$, V_2的一组基为$\{t_1, t_2, \ldots, t_n\}$

在V_2的基的基础上扩充m个线性无关矢量$p_1, p_2, \ldots, p_m$

构成V的基

设T是V上的线性变换

且满足 $T([t_1, t_2, \ldots, t_n, p_1, p_2, \ldots, p_m]) = [\underbrace{0, 0, \ldots, 0}_{n个}, s_1, s_2, \ldots s_m]$

则对于$\forall v \in V$

$$v = [t_1, t_2, \ldots, t_n, p_1, p_2, \ldots, p_m]\vec{x}$$

令$T(v) = T([t_1, t_2, \ldots, t_n, p_1, p_2, \ldots, p_m])\vec{x} = 0$

则 $x_{n+1}s_1 + x_{n+2}s_2 + \cdots + x_{n+m}s_m = 0$

则 $x_{n+1} = x_{n+2} = \cdots = x_{n+m} = 0$

$\therefore v = x_1t_1 + x_2t_2 + \cdots + x_nt_n$

故 $Ker(T) = V_2$

又$\because T(v) = x_{n+1}S_1 + x_{n+2}S_2 + \cdots + x_{n+m}S_m$

故 $Range(T) = V_1$

4.

$$f(\alpha_1, \alpha_2, \alpha_3) = [\alpha_1, \alpha_2, \alpha_3]\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}[\alpha_1, \alpha_2, \alpha_3]^T$$

$\because [\beta_1, \beta_2, \beta_3]^T = C[\alpha_1, \alpha_2, \alpha_3]^T$

$\therefore [\alpha_1, \alpha_2, \alpha_3]^T = C^{-1}[\beta_1, \beta_2, \beta_3]^T$

$[\alpha_1, \alpha_2, \alpha_3] = [\beta_1, \beta_2, \beta_3](C^{-1})^T$

$\therefore$ 关于$\beta_1, \beta_2, \beta_3$的二次型为

$$[\beta_1, \beta_2, \beta_3](C^{-1})^T\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}C^{-1}[\beta_1, \beta_2, \beta_3]^T$$

经计算得 $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ 的特征值为 $\lambda_1=1$ $\lambda_2=\lambda_3=-\frac{1}{2}$.

对应的特征向量分别为 $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$，$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$，$\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

记 $B=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$ $\Lambda=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

要使 $(C^{-1})^TB\Lambda B^TC^{-1}$ 为对角阵 只需 $B^TC^{-1}=I$

即 $C=B^T=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$

5. 证明:

设A的第i行第j列元素为 a_{ij} $i,j=1,2,3,\dots,n$

$\because f(x)=x^TAx$ 对一切 $x\in R^n$ 的取值恒为0

$\therefore$ 取 $x=e_i=[0,0,\cdots,1,\cdots,0]^T$

则 $e_i^TAe_i=a_{ii}=0$，$i=1,2,\dots,n$

再取 $x=[0,\cdots,1,0,\cdots,1,0,\cdots,0]^T=e_i+e_j$

$(i\neq j, i,j=1,2,3,\dots,n)$

则 $(e_i+e_j)^TA(e_i+e_j)=e_i^TAe_i+e_i^TAe_j+e_j^TAe_i+e_j^TAe_j$

$=a_{ij}+a_{ji}=0$

$\therefore a_{ij}=-a_{ji}$ $i\neq j$，$i,j=1,2,\dots,n$

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又 $\because$ A是对称矩阵

$\therefore a_{ij}=a_{ji}$ $i\neq j$，$i,j=1,2,\dots,n$

$\therefore a_{ij}=0$ $i\neq j$，$i,j=1,2,\dots,n$

又 $\because a_{ii}=0$，$i=1,2,\dots,n$ $\therefore A=0$ 证毕

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