DATA130026.01 Optimization

Assignment 1

Due Time: at the beginning of the class, Mar. 9, 2023

1. Show that both the second order cone, i.e., $\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_2 \leq t\}$, and the semidefinite cone, i.e., $\{Z \in \mathbb{S}^n : Z \succeq 0\}$, are convex cones.

2. Let $a, b \in \mathbb{R}^n (a \neq b)$. For what values of μ ($\mu > 0$) is the set

$$S_\mu = \{x \in \mathbb{R}^n : \|x - a\|_2 \leq \mu \|x - b\|_2\}$$

convex?

3. Let $C \in \mathbb{R}^n$ be a nonempty convex set. For each $x \in C$ define the normal cone of C at x by

$$N_C(x) = \{w \in \mathbb{R}^n : w^T (y - x) \leq 0 \text{ for all } y \in C\},$$

and define $N_C(x) = \emptyset$ when $x \notin C$. Show that $N_C(x)$ is convex and closed. Particularly, when $x \in \text{int}(C)$, we have $N_C(x) = \{0\}$.

4. *Supporting hyperplanes.*
   (a) Express the closed convex set $\{x \in \mathbb{R}^n_+ | x_1 x_2 \geq 1\}$ as an intersection of halfspaces.
   (b) Let $C = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$, the l_∞ norm unit $\mathbb{R}^n$ and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

5. **The following question is only required for DATA130026h.01.**
   (a) *Set distributive characterization of convexity* [Rockafellar]. Show that $C \in \mathbb{R}^n$ is convex if and only if $(\alpha + \beta)C = \alpha C + \beta C$ for all nonnegative α, β.