DATA130026.01 Optimization
Assignment 6
Due Time: at the beginning of the class, Apr. 13, 2023

1. Find a dual problem to the convex problem

$$\begin{array}{ll} \min & x_1^2 + 0.5x_2^2 + x_1x_2 - 2x_1 - 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 1. \end{array}$$

   Find the optimal solutions of both the dual and primal problems.

2. Consider the primal optimization problem

$$\begin{array}{ll} \min & x_1^4 - 2x_2^2 - x_2 \\ \text{s.t.} & x_1^2 + x_2^2 + x_2 \leq 0. \end{array}$$

   (i) Is the problem convex?

   (ii) Does there exist an optimal solution to the problem?

   (iii) Write a dual problem. Solve it.

   (iv) Is the optimal value of the dual problem equal to the optimal value of primal problem? Find the optimal solution of the primal problem.

3. Find a dual problem to the following convex minimization problem:

$$\begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} (a_i x_i^2 + 2b_i x_i + e^{\alpha_i x_i}) \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} x_i = 1, \end{array}$$

   where $\mathbf{a}, \alpha \in \mathbb{R}^{\mathbf{n}}_{++}, \mathbf{b} \in \mathbb{R}^{\mathbf{n}}$.

4. Consider the following optimization problem in the variables $\alpha \in \mathbb{R}$ and $q \in \mathbb{R}^n$:

$$\text{(P)} \qquad \begin{array}{ll} \min & \alpha \\ \text{s.t.} & Aq = \alpha f \\ & \|q\|_2^2 \leq \epsilon, \end{array}$$

   where $A \in \mathbb{R}^{m \times n}$, $f \in \mathbb{R}^m$, $\epsilon \in \mathbb{R}_{++}$. Assume in addition that the rows of A are linearly independent.

   (a) Explain why strong duality holds for problem.

   (b) Find a dual problem to problem (P). (Do not assign a Lagrange multiplier to the quadratic constraint.)

   (c) Solve the dual problem obtained in part (ii) and find the optimal solution of problem (P).

5. Consider the convex optimization problem

$$\begin{array}{ll} \min & \sum_{j=1}^{n} x_j \ln \frac{x_j}{c_j} \\ \text{s.t.} & Ax \geq b \\ & \sum_{j=1}^{n} x_j = 1, \end{array}$$

where $c > 0$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Find a dual problem. Simplify it as much as possible.

6. **[Only required for DATA130026h.01.]** Let $f(x) = x^T Ax + 2b^T x + c$, where $A \in \mathbb{R}^{n \times n}$ is symmetric, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then the following two claims are equivalent:

   - $f(x) \equiv x^T Ax + 2b^T x + c \geq 0$ for all $x \in \mathbb{R}^n$.
   - $\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \succeq 0$.