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Assignment 8

1.(a)

证明: $\forall x \in L^n, \alpha \geq 0$

$\because x_n \geq \sqrt{x_1^2+\cdots+x_{n-1}^2}$

$\therefore \alpha x_n \geq \alpha\sqrt{x_1^2+\cdots+x_{n-1}^2} = \sqrt{(\alpha x_1)^2+\cdots+(\alpha x_{n-1})^2}$

即 $\alpha x \in L^n$ $\therefore L^n$是锥

(b)

证明: ① $\forall x \in L^n$

对$\forall y \in L^n$ $\because x^T y = x_n y_n + \sum_{i=1}^{n-1} x_i y_i$

$\geq \sqrt{x_1^2+\cdots x_{n-1}^2}\cdot\sqrt{y_1^2+\cdots+y_{n-1}^2} + \sum_{i=1}^{n-1} x_i y_i$

≥ 0 (Cauchy-Schwarz不等式)

$\therefore x \in (L^n)^*$ 即 $L^n \subset (L^n)^*$

② $\forall y \in (L^n)^*$

1) 若$y=0$, 则 $y_n = \sqrt{y_1^2+\cdots+y_{n-1}^2} = 0$

$\therefore y \in L^n$

2) 若$y \neq 0$, 下用反证法证明 $y_n \geq \sqrt{y_1^2+\cdots+y_{n-1}^2}$

假设 $y_n < \sqrt{y_1^2+\cdots+y_{n-1}^2}$

i 若$\sqrt{y_1^2+\cdots+y_{n-1}^2}=0$, 即 $y_1=\cdots=y_{n-1}=0$

则 $y_n<0$, 取 $x=(0,0,\ldots,0,-y_n)^T \in L^n$

则 $x^T y = -y_n^2 < 0$, 这与$y\in(L^n)^*$矛盾

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ii. 若 $\sqrt{y_1^2+\cdots+y_{n-1}^2}>0$

则取 $x=(-y_1,-y_2,\cdots,-y_{n-1},\sqrt{y_1^2+\cdots+y_{n-1}^2})^T\in L^n$

则 $x^Ty=-y_1^2-y_2^2-\cdots-y_{n-1}^2+y_n\sqrt{y_1^2+\cdots+y_{n-1}^2}$

$<-y_1^2-y_2^2-\cdots-y_{n-1}^2+y_1^2+\cdots+y_{n-1}^2$

$=0$

这与 $y\in(L^n)^*$ 矛盾

$\therefore$ 假设不成立

$\therefore y_n\geq\sqrt{y_1^2+\cdots+y_{n-1}^2}$，即 $y\in L^n$

综合 i)、ii)，得 $(L^n)^*\subset L^n$

综合①、②，知 $L^n=(L^n)^*$，即 ice cream cone 自对偶

(c) 证明:

① $\forall X\in S_+^n$

对 $\forall Y\in S_+^n$：易知 $X\cdot Y\geq0$ $\therefore X\in(S_+^n)^*$，即 $S_+^n\subset(S_+^n)^*$

② $\forall Y\in(S_+^n)^*$，下用反证法证明 $Y\in S_+^n$

假设 $Y\notin S_+^n$ 则 $\exists\lambda<0$，$x\in R^n\backslash\{0\}$，s.t. $Yx=\lambda x$

$\therefore x^TYx=\lambda x^Tx<0$

$\therefore tr(Yxx^T)=tr(x^TYx)=x^TYx<0$

$\therefore Y\cdot(xx^T)=tr(Yxx^T)<0$

又 $\because xx^T\in S_+^n$

$\therefore$ 这与 $Y\in(S_+^n)^*$ 矛盾，故假设不成立

$\therefore Y\in S_+^n$ $\therefore (S_+^n)^*\subset S_+^n$

综合①.②知，$S_+^n = (S_+^n)^*$，即S_+^n是自对偶的

2. Lagrangian: $L(x,\lambda) = c^Tx + \lambda^T(Ax-b)$

$$= (c+A^T\lambda)^Tx - \lambda^Tb$$

dual function: $g(\lambda) = \inf\limits_{x} L(x,\lambda)$

$$= \begin{cases} -\lambda^Tb, & c+A^T\lambda = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

dual problem:

$$\begin{aligned} \max\quad & -\lambda^Tb \\ \text{s.t.}\quad & A^T\lambda + c = 0 \\ & \lambda \succeq_{K^*} 0 \end{aligned}$$

3.(a) 令 $y_i = A_ix + b_i$ $\quad t_i = c_i^Tx + d_i$

则原问题转化为

$$\begin{aligned} \min\quad & f^Tx \\ \text{s.t.}\quad & \|y_i\|_2 \le t_i, \quad i=1,\dots,m \\ & y_i = A_ix+b_i, \quad i=1,\dots,m \\ & t_i = c_i^Tx + d_i, \quad i=1,\dots,m \end{aligned}$$

Lagrangian: $L(x,y,t,\lambda,\nu,\mu)$

$$= f^Tx + \sum_{i=1}^m (\|y_i\|_2 - t_i)\cdot\lambda_i + \sum_{i=1}^m \nu_i^T(y_i - A_ix - b_i)$$

$$+ \sum_{i=1}^m \mu_i(t_i - c_i^Tx - d_i)$$

$$= \left(f - \sum_{i=1}^m A_i^T\nu_i - \sum_{i=1}^m \mu_i c_i\right)^Tx + \sum_{i=1}^m (\lambda_i\|y_i\|_2 + \nu_i^Ty_i)$$

$$+ \sum_{i=1}^m (\mu_i - \lambda_i)t_i + \sum_{i=1}^m (-\nu_i^Tb_i - \mu_i d_i)$$

$$\inf_{x} \left(f - \sum_{i=1}^{m}(A_i^T\gamma_i + \mu_i c_i)\right)^T x = \begin{cases} 0, & f = \sum_{i=1}^{m}(A_i^T\gamma_i + \mu_i c_i) \\ -\infty, & \text{otherwise} \end{cases}$$

$$\inf_{y_i} \left(\lambda_i \|y_i\|_2 + \gamma_i^T y_i\right) = \begin{cases} 0 & \|\gamma_i\|_2 \le \lambda_i \\ -\infty & \text{otherwise} \end{cases}$$

$$\inf_{t_i} (\mu_i - \lambda_i) t_i = \begin{cases} 0 & \mu_i = \lambda_i \\ -\infty & \text{otherwise} \end{cases}$$

$\therefore$ dual function: $= \inf_{x,y,t} L(x,y,t,\lambda,\gamma,\mu) =$

$$g(\lambda,\gamma,\mu) = \begin{cases} -\sum_{i=1}^{m}(\gamma_i^T b_i + \mu_i d_i), & f = \sum_{i=1}^{m}(A_i^T\gamma_i + \mu_i c_i) \text{ 且 } \|\gamma_i\|_2 \le \lambda_i, i=1,\ldots \text{ 且 } \mu = \lambda \\ -\infty, & \text{otherwise} \end{cases}$$

$\therefore$ dual problem:

$$\begin{aligned} \max \quad & -\sum_{i=1}^{m}(\gamma_i^T b_i + \mu_i d_i) \\ \text{s.t.} \quad & -\sum_{i=1}^{m}(A_i^T\gamma_i + \mu_i c_i) + f = 0 \\ & \|\gamma_i\|_2 \le \cancel{\lambda_i}\ \mu_i, \quad i=1,\ldots,m \end{aligned}$$

将 μ_i 替换为 ν_i，γ_i 替换为 $-u_i$ 即可得到题中给出的形式。

(b) 原问题可转化为 $\min f^T x$

$$\text{s.t.} \quad -(A_i x + b_i,\ c_i^T x + d_i) \preceq_{K_i} 0, \quad i=1,\ldots,m$$

Lagrangian: $L(x,u,v) = f^T x - \sum_{i=1}^{m} u_i^T(A_i x + b_i) - \sum_{i=1}^{m} v_i(c_i^T x + d_i)$

$$= \left(f - \sum_{i=1}^{m} A_i^T u_i - \sum_{i=1}^{m} v_i c_i\right)^T x - \sum_{i=1}^{m}(u_i^T b_i + v_i d_i)$$

dual function: $g(u,v) = \inf_x L(x,u,v)$

$$= \begin{cases} -\sum_{i=1}^{m}(b_i^T u_i + v_i d_i), & f = \sum_{i=1}^{m}(A_i^T u_i + v_i c_i) \\ -\infty, & \text{otherwise} \end{cases}$$

dual problem:
$$\begin{aligned} \max\ & -\sum_{i=1}^{m}(b_i^T u_i + d_i v_i) \\ \text{s.t}\ & -\sum_{i=1}^{m}(A_i^T u_i + v_i c_i) + f = 0 \\ & \|u_i\|_2 \le v_i,\ i=1,\dots,m \end{aligned}$$

将 u_i 替换为 $-u_i$ 可得题中所给形式

注:

对1.(c)的补充说明(两半正定阵的内积非负)

若 $X \in S_+^n$, $Y \in S_+^n$

由谱分解可得 $X = Q\Lambda Q^T$, Λ是对角阵且对角元都非负 $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

$\because$ Q^TYQ的对角元也非负($\forall x \in R^n$, $x^TQ^TYQx = (Qx)^TY(Qx) \ge 0$, 即$Q^TYQ$半正定)

$$\begin{aligned} \therefore X \cdot Y &= tr(XY) = tr(Q\Lambda Q^T Y) \\ &= tr(\Lambda Q^T Y Q) \\ &= \Lambda \cdot (Q^T Y Q) \quad (\text{设}Q^TYQ\text{的对角元为}y_1, \dots, y_n) \\ &= \sum_{i=1}^{n} \lambda_i y_i \ge 0 \end{aligned}$$

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