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Assignment 12

1. $\because h''(t) = -\frac{1}{t^2} < 0$，且 $h(t)$ 的定义域 $(0,+\infty)$ 是凸集

$\therefore h(t)$ ~~在 $(0,+\infty)$ 上~~ 是凹函数

又 $\because h'(t) = -1+\frac{1}{t}$ 且 $h'(1)=0$

$\therefore t^*=1$ 是 $h(t)$ 的全局最大值点

又 $\because h(t^*) = h(1) = 0$

$\therefore h(t) \le 0$ 对所有 $t>0$ 均成立

2. $\because trace(B) = \sum_{i=1}^{n} \lambda_i$　　$det(B) = \prod_{i=1}^{n} \lambda_i > 0$

$$\therefore \psi(B) = trace(B) - \ln(det(B))$$
$$= \sum_{i=1}^{n} \lambda_i - \ln\left(\prod_{i=1}^{n} \lambda_i\right)$$
$$= \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} \ln\lambda_i$$
$$= \sum_{i=1}^{n} (\lambda_i - \ln\lambda_i)$$

由第1题知，$\lambda_i - \ln\lambda_i \ge 1$ 对所有 $\lambda_i > 0$ 均成立，当且仅当 $\lambda_i = 1$ 时取到等号

$$\therefore \psi(B) = \sum_{i=1}^{n} (\lambda_i - \ln\lambda_i) \ge n > 0$$

3. (a) 当 $x=0$ 时，$det(I+xy^T) = det I = 1 = 1+y^Tx$

当 $x \ne 0$ 时，由基扩张定理，存在 $w_1, w_2, \dots, w_{n-1} \in R^n$ 使得 $\{x, w_1, w_2, \dots, w_{n-1}\}$ 为 R^n 的一组基，故 $x, w_1, w_2, \dots, w_{n-1}$ 线性无关，定义 $Q = [x, w_1, w_2, \dots, w_{n-1}]$，则 Q 非奇异

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且 $x=Qe_1$，其中 $e_1=(1,0,0,\cdots,0)^T$

$\therefore e_1=Q^{-1}x$

定义 $y^TQ=(z_1,z_2,\cdots,z_n)$

则 $z_1=y^TQe_1=y^Tx$

$$\begin{aligned}\therefore \det(I+xy^T)&=\det(Q^{-1}(I+xy^T)Q)\\&=\det(I+Q^{-1}x\,y^TQ)\\&=\det(I+e_1y^TQ)\\&=\det\left(\begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix}+\begin{bmatrix}z_1&z_2&\cdots&z_n\\&0&&\\&&\ddots&\\&&&0\end{bmatrix}\right)\\&=\det\left(\begin{bmatrix}1+z_1&z_2&\cdots&z_n\\&1&&\\&&\ddots&\\&&&1\end{bmatrix}\right)\\&=1+z_1=1+y^Tx\end{aligned}$$

（注：矩阵中未标的位置上的元素均为0）

证毕

(b) 证明：

① 若 $x=0$ 且 $u=0$，则 $\det(I+xy^T+uv^T)=1=(1+y^Tx)(1+v^Tu)-(x^Tv)(y^Tu)$

② 若 $x\neq0$ 且 $u=0$，情况同(a)，等式成立

③ 若 $x=0$ 且 $u\neq0$，情况也同(a)，等式成立

④ 若 $x\neq0$ 且 $u\neq0$

i 若 x 和 u 线性无关，则由基扩张定理，存在 $w_2,w_3,\cdots,w_{n-1}\in R^n$ ~~则令(a)中的 $w_1=u$~~ 使得 $x,u,w_2,w_3,\cdots,w_{n-1}$ 线性无关

定义 $Q=[x,u,w_2,\cdots,w_{n-1}]$，易知 Q 非奇异

且 $Qe_1=x$，$Qe_2=u$，其中 $e_1=(1,0,0,\cdots,0)^T$，$e_2=(0,1,0,0,0,\cdots,0)^T$

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$\therefore e_1 = Q^{-1}x \ , \ e_2 = Q^{-1}u$

定义 $y^TQ = (z_1, z_2, \dots, z_n) \quad v^TQ = (b_1, b_2, \dots, b_n)$

则 $z_1 = y^TQe_1 = y^Tx \quad z_2 = y^TQe_2 = y^Tu$

$b_1 = v^TQe_1 = v^Tx \quad b_2 = v^TQe_2 = v^Tu$

$$\begin{aligned}
\therefore \det(I + xy^T + uv^T) &= \det(Q^{-1}(I + xy^T + uv^T)Q) \\
&= \det(I + Q^{-1}xy^TQ + Q^{-1}uv^TQ) \\
&= \det(I + e_1y^TQ + e_2v^TQ) \\
&= \det\left(\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} z_1 & z_2 & \cdots & z_n \\ & 0 & & \vdots \\ & & \ddots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_1 & b_2 & b_3 \cdots & b_n \\ & & \ddots & 0 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 1+z_1 & z_2 & z_3 \cdots & z_n \\ b_1 & 1+b_2 & b_3 \cdots & b_n \\ & & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\right) \\
&= (1+z_1)(1+b_2) - b_1z_2 \\
&= (1+y^Tx)(1+v^Tu) - (x^Tv)(y^Tu)
\end{aligned}$$

成立

ii 若 $u = kx \ (k \neq 0)$

则沿用 (a) 中的 Q ; $Q = [x, w_1, w_2, \dots, w_{n-1}]$

则 $Qe_1 = x \qquad kQe_1 = u$

$$\begin{aligned}
\therefore \det(I + xy^T + uv^T) &= \det(Q^{-1}(I + xy^T + kQe_1v^T)Q) \\
&= \det(I + e_1y^TQ + ke_1v^TQ)
\end{aligned}$$

定义 $y^TQ = (z_1, z_2, \dots, z_n) \quad v^TQ = (b_1, b_2, \dots, b_n)$

则 $z_1 = y^TQe_1 = y^Tx \qquad b_1 = v^TQe_1 = v^Tx$

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$\therefore \det(I + xy^T + uv^T) = \det\left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} z_1 & z_2 & \cdots & z_n \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} + k\begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} 1+z_1+kb_1 & z_2+kb_2 & \cdots & z_n+kb_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\right)$$

$$= 1+z_1+kb_1$$

$$= 1+y^Tx + kv^Tx$$

$$= 1+y^Tx + v^Tu$$

$(1+y^Tx)(1+v^Tu) - (x^Tv)(y^Tu)$

$= 1+v^Tu+y^Tx + (y^Tx)(v^Tu) - (x^Tv)(y^Tu)$

$= 1+y^Tx + v^Tu + k(y^Tx)(v^Tx) - k(v^Tx)(y^Tx)$

$= 1+y^Tx+v^Tu$

$\therefore \det(I+xy^T+uv^T) = (1+y^Tx)(1+v^Tu) - (x^Tv)(y^Tu)$

综上，证毕

(c) 在BFGS算法中，$B_{k+1} = B_k - \dfrac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \dfrac{y_k y_k^T}{y_k^T s_k}$

$$= B_k\left(I - \frac{s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{B_k^{-1} y_k y_k^T}{y_k^T s_k}\right)$$

令 $x = s_k \quad y^T = -\dfrac{s_k^T B_k}{s_k^T B_k s_k} \quad u = \dfrac{B_k^{-1} y_k}{y_k^T s_k} \quad v^T = y_k^T$

则由(b)可知

$\det\left(I - \dfrac{s_k s_k^T B_k}{s_k^T B_k s_k} + \dfrac{B_k^{-1} y_k y_k^T}{y_k^T s_k}\right) = \det(I + xy^T + uv^T)$

$= (1+y^Tx)(1+v^Tu) - (x^Tv)(y^Tu)$

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(接上页) $= (1-1)\left(1+\frac{y_k^T B_k^{-1} y_k}{y_k^T s_k}\right) + (y_k^T s_k)\cdot\frac{s_k^T y_k}{(s_k^T B_k s_k)(y_k^T s_k)}$

$$= \frac{y_k^T s_k}{s_k^T B_k s_k}$$

$\therefore \det(B_{k+1}) = \det(B_k)\frac{y_k^T s_k}{s_k^T B_k s_k}$ 证毕

A