DATA130026.01 Optimization
Assignment 2
Due Time: at the beginning of the class, Mar. 16, 2023

1. Give an example of two closed convex sets that are disjoint but cannot be strictly separated. That is, there exists $a \neq 0$ and $b \in \mathbb{R}$ such that

$$a^T x > b, \forall x \in C \text{ and } a^T x < b, \forall x \in D.$$

2. For each of the following sets determine whether they are convex or not (explaining your choice).

   (a) $C_1 = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$.

   (b) $C_2 = \{x \in \mathbb{R}^n : \max_{i=1,2,\ldots,n} x_i \leq 1\}$.

   (c) $C_3 = \{x \in \mathbb{R}^n : \min_{i=1,2,\ldots,n} x_i \leq 1\}$.

   (d) $C_4 = \{x \in \mathbb{R}^n_{++} : \Pi_{i=1}^n x_i \leq 1\}$, where $\Pi_{i=1}^n x_i = x_1 x_2 \cdots x_n$.

3. Let $C \in \mathbb{R}^n$ be a convex set. Let f be a convex function over C, and let g be a strictly convex function over C. Show that the sum function $f + g$ is strictly convex over C.

4. Show that the following functions are convex over the specified domain C:

   (a) $f(x_1, x_2, x_3) = -\sqrt{x_1 x_2} + 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1 x_2 - 2x_2 x_3$ over $\mathbb{R}^3_{++}$.

   (b) $f(x) = \|x\|^4$ over $\mathbb{R}^n$.

   (c) $f(x) = \sqrt{x^T Q x + 1}$ over $\mathbb{R}^n$, where $Q \succeq 0$ is an $n \times n$ matrix.

5. [**Only required for DATA130026h.01.**] Show that a separating hyperplane exists for two disjoint convex sets C and D. You can use the result proved in the class, i.e., that a separating hyperplane exists when there exist points in the two sets whose distance is equal to the distance between the two sets. Hint. If C and D are disjoint convex sets, then the set $\{x - y \mid x \in C, y \in D\}$ is convex and does not contain the origin.

6. [**Only required for DATA130026h.01.**] Suppose f is a strictly convex function and is differentiable over its domain. Show that

$$f(y) > f(x) + \nabla f(x)^T (y - x).$$