DATA130026.01 Optimization
Assignment 8
Due Time: at the beginning of the class, May 4, 2023

1. (a) Let L^n be the n-dimensional ice-cream cone

$$L^n = \{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\}.$$

Prove that L^n is a cone.

   (b) Prove that the ice-cream cone is self-dual:

$$(L^n)^* = L^n.$$

   (c) Prove that the positive semidefinite cone $S_+^n = \{X : X \succeq 0\}$ is self-dual.

2. Find the Lagrange dual problem of the conic form problem in inequality form

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \preceq_K b \end{array}$$

where $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m$, and K is a proper cone in $\mathbb{R}^m$. Make any implicit equality constraints explicit.

3. Show that the dual of the SOCP

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m, \end{array}$$

with variables $x \in \mathbb{R}^n$, can be expressed as

$$\begin{array}{ll} \max & \sum_{i=1}^m \left(b_i^T u_i - d_i v_i\right) \\ \text{s.t.} & \sum_{i=1}^m \left(A_i^T u_i - c_i v_i\right) + f = 0 \\ & \|u_i\|_2 \leq v_i, \quad i = 1, \ldots, m, \end{array}$$

with variables $u_i \in \mathbb{R}^{n_i},\ v_i \in \mathbb{R},\ i = 1, \ldots, m$. The problem data are $f \in \mathbb{R}^n,\ A_i \in \mathbb{R}^{n_i \times n},\ b_i \in \mathbb{R}^{n_i},\ c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R},\ i = 1, \ldots, m$.
Derive the dual in the following two ways.

   (a) Introduce new variables $y_i \in \mathbb{R}^{n_i}$ and $t_i \in \mathbb{R}$ and equalities $y_i = A_i x + b_i,\ t_i = c_i^T x + d_i$, and derive the Lagrange dual.

   (b) Start from the conic formulation of the SOCP and use the conic dual. Use the fact that the second-order cone is self-dual.