DATA130026.01 Optimization
Assignment 12
Due Time: at the beginning of the class, Jun. 1, 2023

1. Define $h(t) = 1 - t + \ln t$, and note that $h'(t) = -1 + 1/t, h''(t) = -1/t^2 < 0$, $h(1) = 0$, and $h'(1) = 0$. Show that $h(t) \leq 0$ for all $t > 0$.

2. Denote the eigenvalues of the positive definite matrix B by $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Show that the ψ function defined in

$$\psi(B) = \operatorname{trace}(B) - \ln(\det(B))$$

can be written as

$$\psi(B) = \sum_{i=1}^{n} (\lambda_i - \ln \lambda_i).$$

Use this form to show that $\psi(B) > 0$.

3. The most popular quasi-Newton algorithm is the BFGS method. In this method, the new iterate is $x_{k+1} = x_k + \alpha_k p_k$ and the quadratic model of the objective function at the current iterate x_k can be written as

$$m_k(p) = f_k + \nabla f_k^T p + \frac{1}{2} p^T B_k p.$$

Set

$$s_k = x_{k+1} - x_k = \alpha_k p_k, \quad y_k = \nabla f_{k+1} - \nabla f_k.$$

The object of this exercise is to prove

$$\det(B_{k+1}) = \det(B_k) \frac{y_k^T s_k}{s_k^T B_k s_k}. \tag{1}$$

(a) Show that $\det\left(I + xy^T\right) = 1 + y^T x$, where x and y are n-vectors. Hint: Assuming that $x \neq 0$, we can find vectors $w_1, w_2, \ldots, w_{n-1}$ such that the matrix Q defined by

$$Q = [x, w_1, w_2, \ldots, w_{n-1}]$$

is nonsingular and $x = Qe_1$, where $e_1 = (1, 0, 0, \ldots, 0)^T$. If we define

$$y^T Q = (z_1, z_2, \ldots, z_n),$$

then

$$z_1 = y^T Q e_1 = y^T Q \left(Q^{-1} x\right) = y^T x,$$

and

$$\det\left(I + xy^T\right) = \det\left(Q^{-1}\left(I + xy^T\right) Q\right) = \det\left(I + e_1 y^T Q\right).$$

(b) Use a similar technique to prove that

$$\det\left(I + xy^T + uv^T\right) = \left(1 + y^T x\right)\left(1 + v^T u\right) - \left(x^T v\right)\left(y^T u\right).$$

(c) Use this relation to establish (1).