SCUFF-SPECTRUM

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Contents

1 Frequency derivatives of BEM matrix elements

$$\begin{aligned}
M_{\alpha\beta}(\omega) &= M^{\text{ext}}_{\alpha\beta}(\omega) + M^{\text{int}}_{\alpha\beta}(\omega), \qquad \boldsymbol{\mathcal{B}}_\alpha, \boldsymbol{\mathcal{B}}_\beta \quad \text{on same surface} \\
&= M^{\text{ext}}_{\alpha\beta}(\omega) \qquad \boldsymbol{\mathcal{B}}_\alpha, \boldsymbol{\mathcal{B}}_\beta \quad \text{on different surfaces}
\end{aligned}$$

$$M^r_{ab} = i\frac{\omega}{c_0}\left(\begin{array}{cc} \mu_r \mathbb{G}_{ab}(k_r) & -n_r\mathbb{C}_{ab}(k_r) \\ -n_r\mathbb{C}_{ab}(k_r) & -\epsilon_r\mathbb{G}_{ab}(k_r)\end{array}\right)$$
$$n_r = \sqrt{\epsilon_r \mu_r}$$

$$\begin{aligned}
\frac{d}{d\omega}M^r_{ab} &= \frac{1}{\omega}M^r_{ab} + i\frac{\omega}{c_0}\left(\begin{array}{cc} \mu'_r \mathbb{G}_{ab}(k_r) & -n'_r\mathbb{C}_{ab}(k_r) \\ -n'_r\mathbb{C}_{ab}(k_r) & -\epsilon'_r\mathbb{G}_{ab}(k_r)\end{array}\right) \\
&\quad + i\frac{\omega}{c_0}\left(\begin{array}{cc} \mu_r n_r \mathbb{G}'_{ab}(k_r) & -n_r^2\mathbb{C}_{ab}(k_r) \\ -n_r^2\mathbb{C}'_{ab}(k_r) & -\epsilon_r n_r\mathbb{G}_{ab}(k_r)\end{array}\right)
\end{aligned}$$

Here primes on $\{\epsilon_r, \mu_r, n_r\}$ denote differentiation with respect to ω, while primes on $\mathbb{G}$ and $\mathbb{C}$ denote differentiation with respect to k.

The $\mathbb{G}, \mathbb{C}$ matrix elements and their k derivatives are

$$\begin{aligned}
\mathbb{G}_{ab}(k) &= \int \left(\mathbf{b}_a\cdot\mathbf{b}_b - \frac{[\nabla\cdot\mathbf{b}_a]\,[\nabla\cdot\mathbf{b}_b]}{k^2}\right) G_0(k,\mathbf{r}) d^4\mathbf{r} \\
\mathbb{G}'_{ab}(k) &= \frac{2}{k^3}\int [\nabla\cdot\mathbf{b}_a]\,[\nabla\cdot\mathbf{b}_b]\, G_0(k,\mathbf{r})\, d^4\mathbf{r} \\
&\quad + \int \left(\mathbf{b}_a\cdot\mathbf{b}_b - \frac{[\nabla\cdot\mathbf{b}_a]\,[\nabla\cdot\mathbf{b}_b]}{k^2}\right) G'_0(k,\mathbf{r})\, d^4\mathbf{r} \\
\mathbb{C}_{ab}(k) &= \frac{1}{ik}\int (\mathbf{b}_a\times\mathbf{b}_b)\cdot\nabla G_0(k,\mathbf{r})\, d^4\mathbf{r} \\
\mathbb{C}'_{ab}(k) &= -\frac{1}{k}\mathbb{C}_{ab}(k) + \int (\mathbf{b}_a\times\mathbf{b}_b)\cdot\nabla G'_0(k,\mathbf{r})\, d^4\mathbf{r}
\end{aligned}$$

In these equations, I have

$$G_0(k,\mathbf{r}) = \begin{cases} \dfrac{e^{ikr}}{4\pi r}, & \text{non-periodic} \\[2ex] \displaystyle\sum_{\mathbf{L}} e^{i\mathbf{k}_{\text{B}}\cdot\mathbf{L}}\frac{e^{ik|\mathbf{r}+\mathbf{L}|}}{4\pi|\mathbf{r}+\mathbf{L}|}, & \text{Bloch-periodic with Bloch vector } \mathbf{k}_{\text{B}} \end{cases}$$

In either case, k derivatives of G_0 may be related to spatial derivatives according to

$$\frac{\partial}{\partial k}G_0 = -i|\mathbf{r}|^2\left(\frac{\mathbf{r}\cdot\nabla G_0}{|\mathbf{r}|} - ikG_0\right) \tag{1}$$

$$\frac{\partial}{\partial k}\nabla G_0 = -k\mathbf{r}G_0 \tag{2}$$

Importantly, the kernels defined by (2) are both *nonsingular* at $\mathbf{r} = 0$, allowing the use of simple numerical cubature to evaluate matrix elements.