Electromagnetism in the Cylindrical-Wave Basis

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Contents

1 Vector Cylindrical Wave Solutions to Maxwell's Equations

Radial functions

$$R_\nu^{\text{reg}}(k_\rho r) \equiv J_\nu(k_\rho r), \qquad R_\nu^{\text{out}}(k_\rho r) \equiv H_\nu^{(1)}(k_\rho r)$$

Vector spherical wave functions

$$\mathbf{M}_{nk_z}(\rho,\theta,z) = \left\{\begin{array}{c} M_\rho \\ M_\varphi \\ M_z \end{array}\right\} = \left\{\begin{array}{c} \frac{\nu k}{\rho k_\rho^2} R_\nu(k_\rho\rho) \\ \frac{ik}{k_\rho} R_\nu'(k_\rho\rho) \\ 0 \end{array}\right\} e^{i(\nu\theta + k_z z)}$$

$$\mathbf{N}_{nk_z}(\rho,\theta,z) = \left\{\begin{array}{c} N_\rho \\ N_\varphi \\ N_z \end{array}\right\} = \left\{\begin{array}{c} \frac{ik_z}{k_\rho} R_\nu'(k_\rho\rho) \\ -\frac{\nu k_z}{\rho k_\rho^2} R_\nu(k_\rho\rho) \\ R_\nu(k_\rho\rho) \end{array}\right\} e^{i(\nu\theta + k_z z)}$$

$$k_\rho = \sqrt{\epsilon\mu k_0^2 - k_z^2}$$

Vector spherical waves

$$\nabla\times\mathbf{M} = -ik\mathbf{N}, \qquad \nabla\times\mathbf{N} = +ik\mathbf{M}$$

Expansion of plane wave I consider plane waves whose directions of propagation make an angle θ with the cylinder axis ($\theta = 0$ for normal incidence); without loss of generality I orient my coordinate system such that

$$\hat{\mathbf{k}} \equiv \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}.$$

I define polarization vectors transverse to $\hat{\mathbf{k}}$:

$$\begin{aligned} \boldsymbol{\epsilon}^{\text{TE}} &\equiv \hat{\mathbf{y}} \\ \boldsymbol{\epsilon}^{\text{TM}} &\equiv \Big(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}\Big) \end{aligned}$$

Then TE and TM plane waves have the expansion

$$\begin{aligned} \mathbf{E}^{\text{TE}}(k_0,\theta;\mathbf{x}) \equiv \boldsymbol{\epsilon}^{\text{TE}} e^{i(k_\rho x + k_z z)} &= \sum_\nu \Big\{ P_{\nu k_z}^{\text{TE}} \mathbf{M}_{\nu k_z}^{\text{reg}}(\mathbf{x}) + Q_{\nu k_z}^{\text{TE}} \mathbf{N}_{\nu k_z}^{\text{reg}}(\mathbf{x}) \Big\} \\ \mathbf{E}^{\text{TM}}(k_0,\theta;\mathbf{x}) \equiv \boldsymbol{\epsilon}^{\text{TM}} e^{i(k_\rho x + k_z z)} &= \sum_\nu \Big\{ P_\nu^{\text{TM}} \mathbf{M}_{\nu k_z}^{\text{reg}}(\mathbf{x}) + Q_\nu^{\text{TM}} \mathbf{N}_{\nu k_z}^{\text{reg}}(\mathbf{x}) \Big\} \end{aligned}$$

where $k_z \equiv k_0 \sin\theta$. The expansion coefficients are given by

$$\begin{array}{llllll} P_\nu^{\text{TE}} & = & & Q_\nu^{\text{TE}} & = & \\ P_\nu^{\text{TM}} & = & & Q_\nu^{\text{TM}} & = & \cos\theta i^\nu \end{array}$$

Scattering from dielectric cylinder

Incident field

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum_\alpha P_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) + \sum_\alpha Q_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x})$$
$$\mathbf{H}^{\text{inc}}(\mathbf{x}) = \frac{1}{Z_0} \sum_\alpha \{Q_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) - P_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x})\}$$

Fields inside

$$\mathbf{E}^{\text{inside}}(\mathbf{x}) = \sum_\alpha A_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) + \sum_\alpha B_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x})$$
$$\mathbf{H}^{\text{inside}}(\mathbf{x}) = \frac{1}{Z_0 Z_r} \sum_\alpha \{B_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) - A_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x})\}$$

Scattered fields outside

$$\mathbf{E}^{\text{outside}}(\mathbf{x}) = \sum_\alpha C_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) + \sum_\alpha D_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x})$$
$$\mathbf{H}^{\text{outside}}(\mathbf{x}) = \frac{1}{Z_0} \sum_\alpha \left\{D_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) - C_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x})\right\}$$

Scattering coefficients