Self-study guide for numerical modeling of electromagnetic fluctuation-induced phenomena

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1 Notation

$\mathbf{r}^{\text{in}}$	a point inside a material body
$\mathbf{r}^{\text{out}}$	a point out the material body
J_x	x-directed electric current
E_y	y-directed electric field
H_z	z-directed magnetic field
P_x	x-component of Poynting vector
T_{yz}	yz-component of Maxwell stress tensor
$Q(\mathbf{r},t)$	instantaneous value of a physical quantity at position $\mathbf{r}$ and time t
$\big\langle Q(\mathbf{r})\big\rangle$	time-average value of $Q(\mathbf{r},t)$
$\big\langle Q(\mathbf{r})\big\rangle_\omega$	contribution of fluctuations at just the single frequency ω to $\big\langle Q(\mathbf{r})\big\rangle$, defined such that $$\big\langle Q(\mathbf{r})\big\rangle = \int_0^\infty \big\langle Q(\mathbf{r})\big\rangle_\omega d\omega$$

2 Classical Electromagnetism

A1. Consider a frequency-ω point dipole radiator at a point $\mathbf{r}$ in vacuum. This source may be described by a time-dependent dipole moment $\mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t}$ localized at $\mathbf{r}$ (here $\mathbf{p}_0$ is a three-vector with units of charge $\cdot$ length).

(a) Write down the electric and magnetic fields at a point $\mathbf{r}'$ due to this source.

(b) Write an expression for the current density distribution $\mathbf{J}(\mathbf{x})$ (for all points $\mathbf{x}$, i.e. you are writing down a function of a three-dimensional variable $\mathbf{x}$ that ranges over all space) corresponding to this dipole source. Your expression will involve $\mathbf{p}_0$, $\mathbf{r}$, and ω.

A2. How is your answer to problem **A1(a)** modified if, instead of vacuum, the point dipole radiator is embedded in an infinite medium with relative permittivity ϵ?

A3. Now suppose that your point dipole radiator exists in the presence of a compact homogeneous material body with relative dielectric constant $\epsilon(\omega)$. (If it helps to be specific, you can imagine that the body is a sphere centered at the origin.) Suppose that both the points $\mathbf{r}$ (source location) and $\mathbf{r}'$ (evaluation point) lie *outside* the body.

(a) Explain qualitatively why the electric and magnetic fields at $\mathbf{r}'$ are no longer given by the expressions you wrote down in problem **A1(a)**.

(b) Now describe explicitly (using equations) how you would use each of the following techniques to compute the electric and magnetic fields at $\mathbf{r}'$.

- **(i)** Special-function expansions (such as Mie scattering for spheres, etc.)
- **(ii)** The volume-integral-equation (VIE) method
- **(iii)** The surface-integral-equation (SIE) method
- **(iv)** The finite-difference method

Note that your answer to item **(ii)** here will involve your answer to problem **A1(a)**, while your answer to item **(iii)** will involve your answers to both problems **A1(a)** and **A2**. For this reason, you are encouraged to adopt appropriate shorthand notation for your answers to those problems.

A4. How are your answers to part **(b)** of the previous problem modified if

(a) $\mathbf{r}$ lies inside the body, while $\mathbf{r}'$ remains outside the body?

(b) both $\mathbf{r}, \mathbf{r}'$ lie inside the body?

A5. Now return to the situation of problem **A1** (a point source at $\mathbf{r}$ in vacuum). Using your answer to problem **A1(a)**,

(a) Write an expression for the time-averaged flux of energy radiated by this dipole in the ith cartesian direction at an arbitrary point $\mathbf{r}'$.

(b) Write an expression for *total* time-average flux of energy radiated by this dipole into a closed surface $\mathcal{S}$.

(c) Explicitly evaluate your expression in part **(b)** for

- **(i)** a closed surface $\mathcal{S}$ containing $\mathbf{r}$,
- **(ii)** a closed surface $\mathcal{S}$ not containing $\mathbf{r}$.

Express your answers as closed-form expressions involving $\mathbf{p}_0$ and ω. (For case **(i)** you may take $\mathbf{r} = 0$ and $\mathcal{S}$=a sphere centered at the origin.)

A5. Now return to the situation of problem **A3** (a point source lying outside a material body). In terms of the total electric and magnetic fields $\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r})$, write expressions for

(a) The energy per unit time absorbed by the body (that is, the power transfer from the source to the body).

(b) The i-directed linear momentum per unit time absorbed by the body (that is, the i-directed force on the body).

(c) The i-directed angular momentum per unit time absorbed by the body (that is, the i-directed torque on the body).

A6. Now suppose that the fields in the previous problem $\mathbf{E}, \mathbf{H}$ arise from a point source lying *inside* the material body.

3 Statistical Treatment of Electrodynamic Fluctuations

B1. Consider a homogeneous material body in vacuum. Let the body have a frequency-dependent (that is, dispersive, i.e. lossy) dielectric function $\epsilon(\omega)$, and suppose the body is in internal thermodynamic equilibrium at temperature T, while the external environment is at temperature $T^{\text{env}} = 0$. For each of the following measurable physical quantities, state whether you expect the results of a measurement to yield zero or non-zero values. (Refer to the table in Section **??**) for definitions of symbols.)

1. $J_x(\mathbf{r}^{\text{in}}, t)$	2. $\Big\langle J_x(\mathbf{r}^{\text{in}})\Big\rangle$
3. $J_x(\mathbf{r}^{\text{out}}, t)$	4. $\Big\langle J_x(\mathbf{r}^{\text{out}})\Big\rangle$
5. $E_x(\mathbf{r}^{\text{in}}, t)$	6. $\Big\langle E_x(\mathbf{r}^{\text{in}})\Big\rangle$
7. $E_x(\mathbf{r}^{\text{out}}, t)$	8. $\Big\langle E_x(\mathbf{r}^{\text{out}})\Big\rangle$
9. $J_x(\mathbf{r}^{\text{in}}, t)J_x(\mathbf{r}^{\text{in}}, t)$	10. $\Big\langle J_x(\mathbf{r}^{\text{in}})\, J_x(\mathbf{r}^{\text{in}})\Big\rangle$
9. $J_x(\mathbf{r}^{\text{out}}, t)J_x(\mathbf{r}^{\text{out}}, t)$	10. $\Big\langle J_x(\mathbf{r}^{\text{out}})\, J_x(\mathbf{r}^{\text{out}})\Big\rangle$
11. $J_x(\mathbf{r}^{\text{in}}, t)J_y(\mathbf{r}^{\text{in}}, t)$	12. $\Big\langle J_x(\mathbf{r}^{\text{in}})\, J_y(\mathbf{r}^{\text{in}})\Big\rangle$
13. $J_x(\mathbf{r}^{\text{out}}, t)J_y(\mathbf{r}^{\text{out}}, t)$	14. $\Big\langle J_x(\mathbf{r}^{\text{out}})\, J_y(\mathbf{r}^{\text{out}})\Big\rangle$
15. $E_x(\mathbf{r}^{\text{in}}, t)E_x(\mathbf{r}^{\text{in}}, t)$	16. $\Big\langle E_x(\mathbf{r}^{\text{in}})\, E_x(\mathbf{r}^{\text{in}})\Big\rangle$
17. $E_x(\mathbf{r}^{\text{out}}, t)E_x(\mathbf{r}^{\text{out}}, t)$	18. $\Big\langle E_x(\mathbf{r}^{\text{out}})\, E_x(\mathbf{r}^{\text{out}})\Big\rangle$
19. $E_x(\mathbf{r}^{\text{in}}, t)E_y(\mathbf{r}^{\text{in}}, t)$	20. $\Big\langle E_x(\mathbf{r}^{\text{in}})\, E_y(\mathbf{r}^{\text{in}})\Big\rangle$
21. $E_x(\mathbf{r}^{\text{out}}, t)E_y(\mathbf{r}^{\text{out}}, t)$	22. $\Big\langle E_x(\mathbf{r}^{\text{out}})\, E_y(\mathbf{r}^{\text{out}})\Big\rangle$