Surface Impedance Boundary Conditions in SCUFF-EM

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Contents

1 IBCs for PEC surfaces

The usual boundary condition imposed at the surface of a perfectly electrically conducting (PEC) scatterer is that the total tangential electric field vanish:

$$\mathbf{E}_{\parallel}^{\text{tot}}(\mathbf{x}) = 0. \tag{1}$$

At the surface of an *imperfectly* electrically conducting (IPEC) scatterer with dimensionless relative surface impedance ζ, the boundary condition (1) is modified to read

$$\mathbf{E}_{\parallel}^{\text{tot}}(\mathbf{x}) = \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{H}^{\text{tot}}(\mathbf{x}) \tag{2}$$

where $Z_0 \approx 377\,\Omega$ is the impedance of vacuum.

I will refer to (2) as the "impedance boundary condition" (IBC).

2 Two SIE formulations for IPEC bodies

2.1 Review: SIE formulation for PEC bodies

I will consider two distinct SIE formulations for IPEC bodies. These are both variants of the usual SIE procedure for PEC bodies, which—by way of review—I summarize thusly:

1. We introduce an electric surface current $\mathbf{K}(\mathbf{x})$ on the surface of a PEC scatterer. This current is related to the total tangential $\mathbf{H}$-field according to
$$\mathbf{K}(\mathbf{x}) = \hat{\mathbf{n}} \times \mathbf{H}^{\text{tot}}(\mathbf{x}). \tag{3}$$

2. We do *not* need to introduce a magnetic surface current; such a current would be proportional to the total tangential $\mathbf{E}$ field, but this vanishes in view of the boundary condition (1):
$$\mathbf{N}(\mathbf{x}) = -\hat{\mathbf{n}} \times \mathbf{E}^{\text{tot}}(\mathbf{x}) \equiv 0. \tag{4}$$

3. $\mathbf{K}$ gives rise to scattered $\mathbf{E}$ and $\mathbf{H}$ fields according to
$$\mathbf{E}^{\text{scat}} = \int \mathbf{\Gamma}_{\parallel}^{\text{EE}}(\mathbf{x},\mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') d\mathbf{x}' \tag{5}$$
$$\mathbf{H}^{\text{scat}} = \int \mathbf{\Gamma}_{\parallel}^{\text{ME}}(\mathbf{x},\mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') d\mathbf{x}'. \tag{6}$$

4. We solve for $\mathbf{K}$ by demanding that the scattered field to which it gives rise satisfy the boundary condition (1):
$$\mathbf{E}_{\parallel}^{\text{scat}}(\mathbf{x}) = -\mathbf{E}_{\parallel}^{\text{inc}}(\mathbf{x}) \tag{7}$$
or
$$\underbrace{\int \mathbf{\Gamma}_{\parallel}^{\text{EE}}(\mathbf{x},\mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') d\mathbf{x}'}_{\mathbf{\Gamma}_{\parallel}^{\text{EE}}\star\mathbf{K}} = -\mathbf{E}^{\text{inc}}(\mathbf{x}). \tag{8}$$

2.2 First SIE formulation for IPEC bodies

My first SIE formulation for IPEC bodies

1. As in the PEC case, to each IPEC I continue to assign an electric surface current $\mathbf{K}$ related to the total $\mathbf{H}$ field by equation (3).

2. As in the PEC case, I continue to assign *no* magnetic surface current $\mathbf{N}$ to IPEC surfaces.

3. To determine $\mathbf{K}$ from $\mathbf{E}^{\text{inc}}$, I use (2) in place of (1) and (7). This has the effect of replacing (8) with
$$\mathbf{\Gamma}^{\text{EE}} \star \mathbf{K} - \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{\Gamma}^{\text{ME}} \star \mathbf{K} = -\mathbf{E}^{\text{inc}} + \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{H}^{\text{inc}} \tag{9}$$

2.3 Second SIE formulation for IPEC bodies

My alternative SIE formulation for IPEC bodies goes like this:

1. As before, to each IPEC surface I continue to assign an electric surface current $\mathbf{K}$ related to the total $\mathbf{H}$ field by equation (3).

2. Unlike before, I now assign to each IPEC surface a nonvanishing magnetic current, which is not an independent unknown but is instead determined by $\mathbf{K}$ via the (2):

$$\mathbf{N}(\mathbf{x}) = -\hat{\mathbf{n}} \times \mathbf{E}^{\text{tot}}(\mathbf{x}) = -\zeta Z_0 \hat{\mathbf{n}} \times \mathbf{K}(\mathbf{x}).$$

3. $\mathbf{K}$ and $\mathbf{N} \equiv \mathbf{N}[\mathbf{K}]$ give rise to scattered $\mathbf{E}$ and $\mathbf{H}$ fields according to

$$\mathbf{E}^{\text{scat}} = \mathbf{\Gamma}^{\text{EE}} \star \mathbf{K} + \mathbf{\Gamma}^{\text{EM}} \star \mathbf{N}, \qquad \mathbf{H}^{\text{scat}} = \mathbf{\Gamma}^{\text{ME}} \star \mathbf{K} + \mathbf{\Gamma}^{\text{MM}} \star \mathbf{N} \tag{10}$$

4. I solve for $\mathbf{K}$ by demanding that the scattered fields (10) satisfy the boundary condition (2):

$$\mathbf{E}_{\parallel}^{\text{scat}}(\mathbf{x}) - \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{H}_{\parallel}^{\text{scat}}(\mathbf{x}) = -\mathbf{E}_{\parallel}^{\text{inc}}(\mathbf{x}) + \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{H}_{\parallel}^{\text{inc}}(\mathbf{x}).$$

or

$$\begin{aligned}
\int \Big\{ & \mathbf{\Gamma}_{\parallel}^{\text{EE}}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') \\
& - \zeta Z_0 \mathbf{\Gamma}_{\parallel}^{\text{ME}}(\mathbf{x}, \mathbf{x}') \cdot \Big[\hat{\mathbf{n}} \times \mathbf{K}(\mathbf{x}') \Big] \\
& - \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{\Gamma}_{\parallel}^{\text{EM}}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') \\
& + \zeta^2 Z_0^2 \hat{\mathbf{n}} \times \mathbf{\Gamma}_{\parallel}^{\text{MM}}(\mathbf{x}, \mathbf{x}') \cdot \Big[\hat{\mathbf{n}} \times \mathbf{K}(\mathbf{x}') \Big] \Big\} = -\mathbf{E}_{\parallel}^{\text{inc}}(\mathbf{x}) + \zeta Z_0 \hat{\mathbf{n}} \times \mathbf{H}_{\parallel}^{\text{inc}}(\mathbf{x}).
\end{aligned}$$

3 Example 1: Dielectric film with thin conductive coating layer

3.1 Exact solution

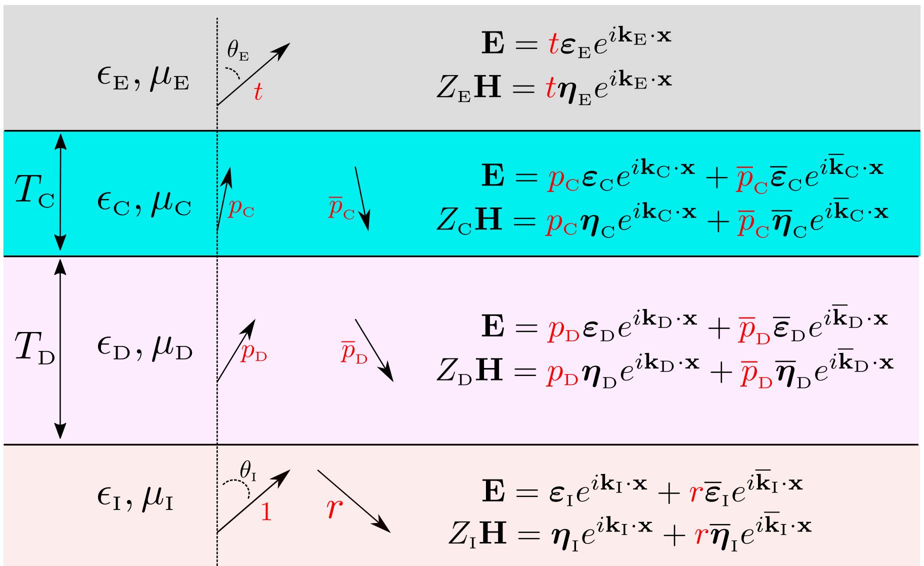


Figure 1: Transmission through a two-layer film. Eventually I will take layer C to be thin and highly conductive and will try to model it by an infinitesimally thin surface-impedance layer. Subscripts: I=initial/incident/interior, D=dielectric, C=conducting, E=exit/exterior.

Figure 1 shows a transmission problem in which a plane wave emanating from infinite dielectric medium I (for "initial" or "incident") impinges on a two-layer film (consisting of a dielectric layer D and a conductive layer C with thicknesses $T_{\mathrm{D,C}}$ and relative material properties $\{\epsilon, \mu\}_{\mathrm{D,C}}$) and is transmitted into region E (for "exit" or "exterior"). The input parameters are the frequency ω, incidence angle θ_{A}, and incident **E**-field polarization $\boldsymbol{\varepsilon}_{\mathrm{A}}$. I define coordinates so that the incident wavevector lives in the xz plane. The propagation and polarization vectors in each region ($R = \{I, D, C, E\}$) are then determined by

$$\begin{aligned}
\mathbf{k}_{\mathrm{R}} &= n_{\mathrm{R}} k_0 \Big(\sin\theta_{\mathrm{R}} \hat{\mathbf{x}} + \cos\theta_{\mathrm{R}} \hat{\mathbf{z}} \Big) & \overline{\mathbf{k}}_{\mathrm{R}} &= n_{\mathrm{R}} k_0 \Big(\sin\theta_{\mathrm{R}} \hat{\mathbf{x}} - \cos\theta_{\mathrm{R}} \hat{\mathbf{z}} \Big) \\
\boldsymbol{\varepsilon}^{\mathrm{TE}}_{\mathrm{R}} &= \hat{\mathbf{y}} & \overline{\boldsymbol{\varepsilon}}^{\mathrm{TE}}_{\mathrm{R}} &= \hat{\mathbf{y}} \\
\boldsymbol{\varepsilon}^{\mathrm{TM}}_{\mathrm{R}} &= \cos\theta_{\mathrm{R}} \hat{\mathbf{x}} - \sin\theta_{\mathrm{R}} \hat{\mathbf{z}} & \overline{\boldsymbol{\varepsilon}}^{\mathrm{TM}}_{\mathrm{R}} &= -\cos\theta_{\mathrm{R}} \hat{\mathbf{x}} + \sin\theta_{\mathrm{R}} \hat{\mathbf{z}} \\
\boldsymbol{\eta}_{\mathrm{R}} &= \hat{\mathbf{k}}_{\mathrm{R}} \times \boldsymbol{\varepsilon}_{\mathrm{R}} & \overline{\boldsymbol{\eta}}_{\mathrm{R}} &= \widehat{\overline{\mathbf{k}}}_{\mathrm{R}} \times \overline{\boldsymbol{\varepsilon}}_{\mathrm{R}}
\end{aligned}$$

$$\theta_{\mathrm{R}}=\sin^{-1}\left[\frac{n_{\mathrm{I}}}{n_{\mathrm{R}}}\sin\theta_{\mathrm{I}}\right]$$

with

$$k_0=\frac{\omega}{c},\qquad n_{\mathrm{R}}=\sqrt{\epsilon_{\mathrm{R}}\mu_{\mathrm{R}}},\qquad Z_{\mathrm{R}}=Z_0\sqrt{\frac{\mu_{\mathrm{R}}}{\epsilon_{\mathrm{R}}}}.$$

For pure-TE or pure-TM polarization, each of the **E** and **H** fields has precisely one nonzero tangential component.

Matching tangential **E** and **H** fields at each of the three interfaces then yields a linear system of 6 equations in the six unknown field-expansion coefficients. For example, the two equations contributed by the lowermost interface read

$$\begin{array}{cccccc} -r\overline{\boldsymbol{\varepsilon}}_{\mathrm{I}\parallel} & +p_{\mathrm{D}}\boldsymbol{\varepsilon}_{\mathrm{D}\parallel} & +\overline{p}_{\mathrm{D}}\overline{\boldsymbol{\varepsilon}}_{\mathrm{D}\parallel} & = & \boldsymbol{\varepsilon}_{\mathrm{I}\parallel} \\ -\dfrac{r}{Z_{\mathrm{I}}}\overline{\boldsymbol{\eta}}_{\mathrm{I}\parallel} & +\dfrac{p}{Z_{\mathrm{D}}}\boldsymbol{\eta}_{\mathrm{D}\parallel} & +\dfrac{\overline{p}_{\mathrm{D}}}{Z_{\mathrm{D}}}\overline{\boldsymbol{\eta}}_{\mathrm{D}\parallel} & = & \dfrac{1}{Z_{\mathrm{I}}}\boldsymbol{\eta}_{\mathrm{I}\parallel} \end{array}$$

Assembling the equations for all interfaces gives a 6×6 linear system of the form

$$\left(\begin{array}{c} \\ \\ \mathbf{M} \\ \\ \\ \end{array}\right)\left(\begin{array}{c} r \\ p_{\mathrm{D}} \\ \overline{p}_{\mathrm{D}} \\ p_{\mathrm{C}} \\ \overline{p}_{\mathrm{C}} \\ t \end{array}\right)=\left(\begin{array}{c} \boldsymbol{\varepsilon}_{\mathrm{I}\parallel} \\ \frac{1}{Z_I}\boldsymbol{\eta}_{\mathrm{I}\parallel} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right) \tag{11}$$

that we solve for the field-expansion coefficients.

Results: Transmission through dielectric layer with thin gold film

Figure 2 shows the tangential **E** and **H** fields for the geometry of Figure 1 with parameters

$$\epsilon_{\mathrm{I}}=\epsilon_{\mathrm{E}}=1,\qquad \epsilon_{\mathrm{D}}=10,\qquad \epsilon_{\mathrm{C}}=\epsilon_{\mathrm{gold}},\qquad T_{\mathrm{D}}=1\,\mu\mathrm{m}\qquad T_{\mathrm{C}}=10\text{ nm},$$

i.e. a 1-μm-thick dielectric layer coated by a thin (10 nm) layer of gold, with vacuum above and below. The point of these figures is that the tangential electric field is continuous across the gold layer, while the magnetic field appears to suffer a discontinuous jump upon traversing the gold layer.

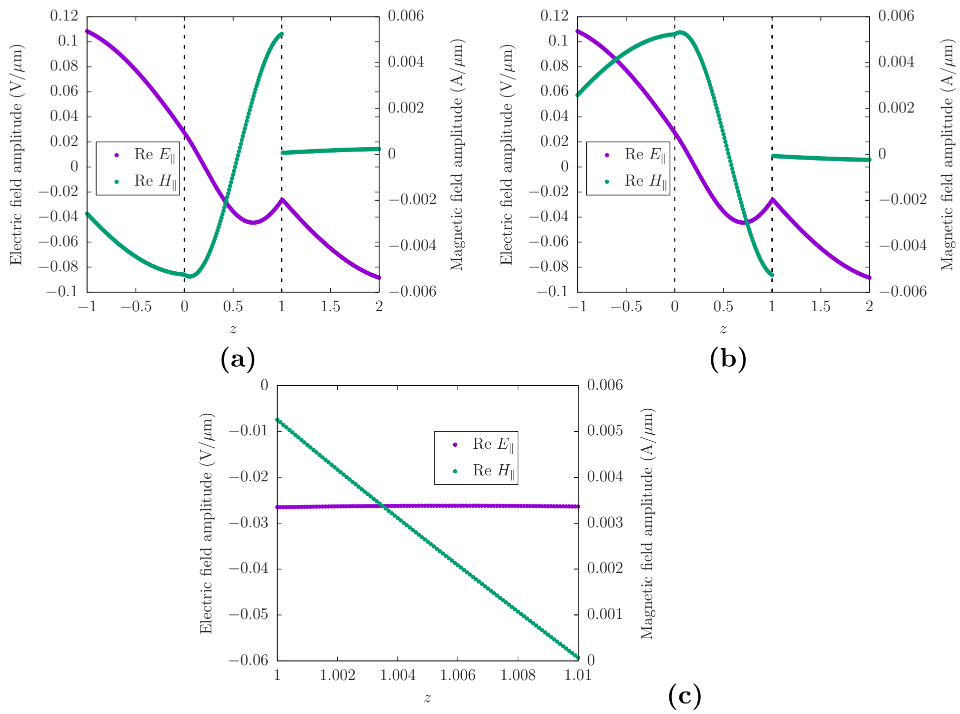


Figure 2: Tangential electric (purple) and magnetic (green) fields for the geometry of Figure (1) irradiated by TE **(a)** and TM **(b)** plane waves at incident angle $\theta = \frac{\pi}{4}$. The region between the dashed lines is the dielectric layer ($\epsilon_{\rm D} = 10$). The gold layer lives at the right dashed line. In both cases the tangential **E**-field is continuous across the gold layer, while the tangential **H**-field appears to suffer a discontinuous jump across the layer. Zooming in to just the gold layer [plot **(c)**] we see that the **E**-field is nearly constant in the layer, while the **H**-field varies rapidly, giving the impression of the discontinuous jumps in Figures **(a,b)**.

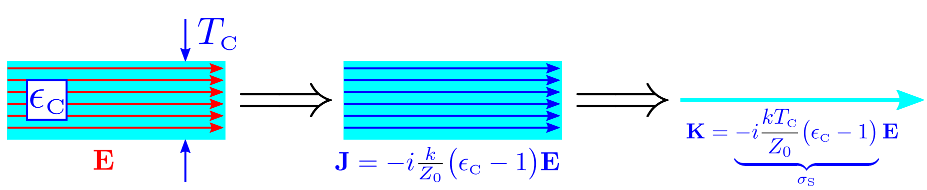


Figure 3: A constant tangential **E**-field in a thin layer of thickness $T_{\rm C}$ and relative permittivity ϵ produces a constant volume-current density $\mathbf{J} = -\frac{ik}{Z_0}(\epsilon - 1)\mathbf{E}$, which we model as an infinitesimally thin sheet with surface-current density $\mathbf{K} = T_{\rm C}\mathbf{J}$.

The fact that the **E**-field is constant throughout the gold layer [Figure 2**(c)**] suggests the scenario cartooned in Figure 3: A constant tangential **E**-field in a thin layer of thickness $T_{\rm C}$ and relative permittivity ϵ produces a constant volume-current density $\mathbf{J} = -\frac{ik}{Z_0}(\epsilon - 1)\mathbf{E}$, which we model as an infinitesimally thin sheet with surface-current density $\mathbf{K} = T_{\rm C}\mathbf{J}$. The effective surface conductivity of the layer is related to its thickness and bulk conductivity by

$$\sigma_{\rm S} = \frac{|\mathbf{K}|}{|\mathbf{E}_{\parallel}|} = -\frac{ikT^{\rm C}}{Z_0}(\epsilon - 1). \tag{12}$$

3.2 IBC solution

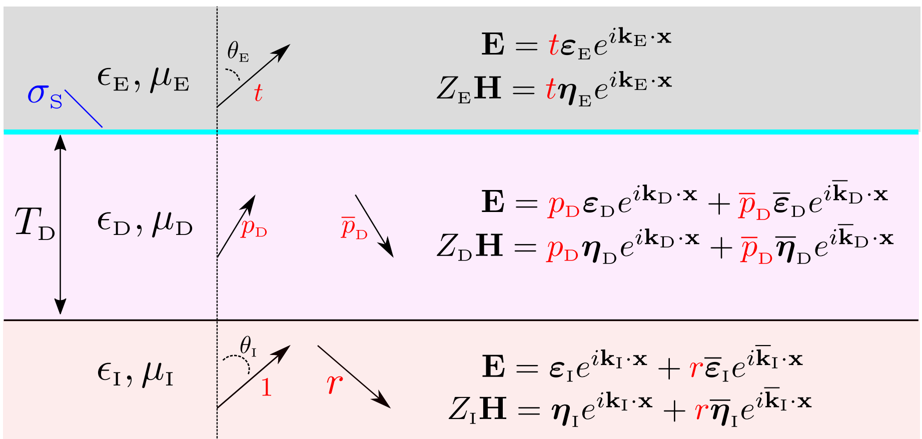


Figure 4: The two-layer film of the previous section with the upper layer modeled as an IBC layer.

I now consider modeling layer C in Figure 1 as an infinitesimally thin sheet with surface impedance $\sigma_{\rm S} \equiv \frac{1}{Z_0 Z_{\rm S}}$ (with $Z_{\rm S}$ the dimensionless relative surface impedance), as shown in Figure 4. The impedance boundary conditions relate the discontinuity in the tangential **H** field across the sheet to the average **E**-field above and below the sheet:

$$\mathbf{E}\left(z = T_{\rm D}^{+}\right) - \mathbf{E}\left(z = T_{\rm D}^{-}\right) = 0 \tag{13}$$

$$\mathbf{H}\left(z = T_{\rm D}^{+}\right) - \mathbf{H}\left(z = T_{\rm D}^{-}\right) = -\frac{\sigma_{\rm S}}{2}\hat{\mathbf{z}} \times \Big[\mathbf{E}\left(z = T_{\rm D}^{+}\right) + \mathbf{E}\left(z = T_{\rm D}^{-}\right)\Big], \tag{14}$$

or

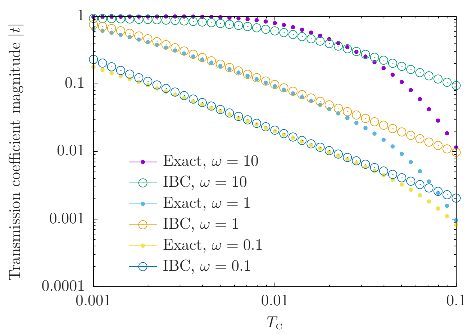
$$\begin{array}{cccccc}
-p_{\rm D}\boldsymbol{\varepsilon}_{{\rm D}\parallel}\zeta_{\rm D} & -\overline{p}_{\rm D}\overline{\boldsymbol{\varepsilon}}_{{\rm D}\parallel}\overline{\zeta}_{\rm D} & +t\boldsymbol{\varepsilon}_{{\rm E}\parallel}\zeta_{\rm E} & = & 0 \\
-\dfrac{p_{\rm D}}{Z_{\rm D}}\boldsymbol{\eta}_{{\rm D}\parallel}\zeta_{\rm D} & -\dfrac{\overline{p}_{\rm D}}{Z_{\rm D}}\overline{\boldsymbol{\eta}}_{{\rm D}\parallel}\overline{\zeta}_{\rm D} & +\dfrac{t}{Z_{\rm E}}\boldsymbol{\eta}_{{\rm E}\parallel}\zeta_{\rm E} & = & -\sigma_{\rm S}t\big(\hat{\mathbf{z}} \times \boldsymbol{\varepsilon}_{\rm e}\big)\zeta_{\rm E}
\end{array}$$

$$\zeta_{\rm R} = e^{i\mathbf{k}_{\rm R}\cdot T_{\rm D}\hat{\mathbf{z}}}, \qquad \overline{\zeta}_{\rm R} = e^{i\overline{\mathbf{k}}_{\rm R}\cdot T_{\rm D}\hat{\mathbf{z}}}$$

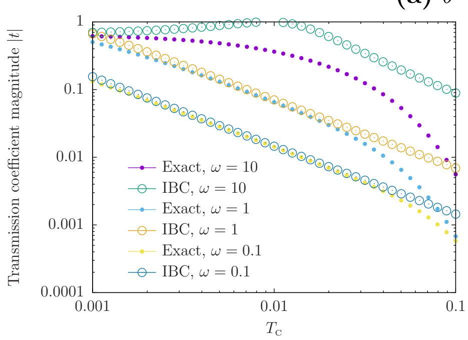
These two equations replace the last 4 lines of the system (11).

Results: Transmission coefficient in the full calculation and in the IBC approximation

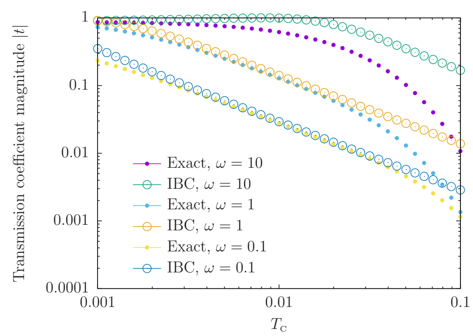
For the parameters considered above (gold film of thickness $T_{\rm C}$ on $\epsilon_{\rm D} = 10$ dielectric layer of thickness 1 μm), Figure 3.2 plots the transmission coefficient magnitude $|t|$ as computed using the full calculation of Section 3.1 (filled circles) and the IBC approximation of Section 3.2 (hollow circles) vs. film thickness $T_{\rm C}$.



(a) $\theta = 0$



(b) $\theta = \frac{\pi}{4}$, TE



(c) $\theta = \frac{\pi}{4}$, TE

4 Example 2: Dielectric sphere with thin conductive coating layer

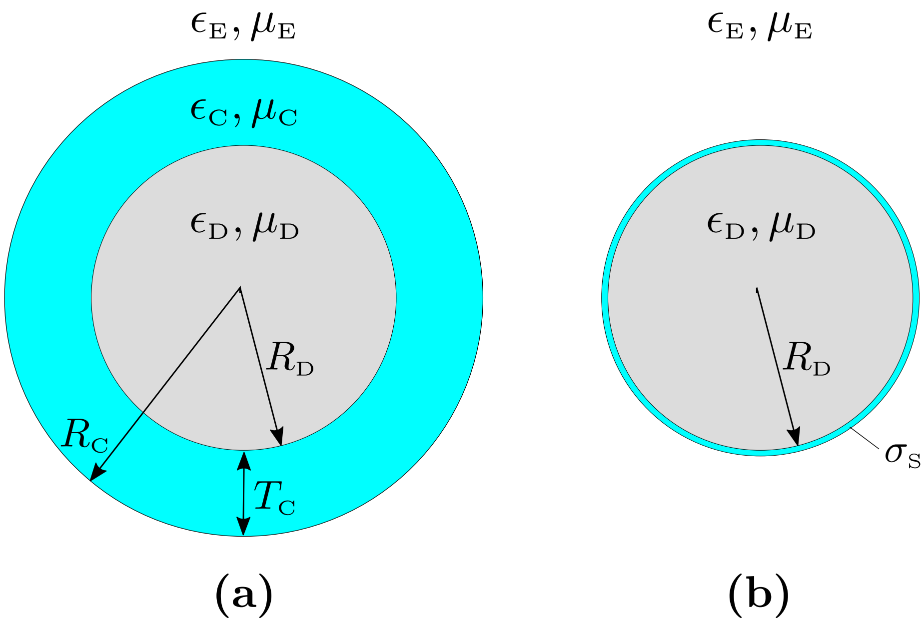


Figure 5: Dielectric sphere covered by finite-thickness coating layer. **(a)** Full geometry. **(b)** IBC-layer approximation.

I next consider a spherical version of the geometry of Figure 1: a dielectric sphere (permittivity $\epsilon_{\rm D}$, radius $R_{\rm D}$) coated by a conducting layer (permittivity $\epsilon_{\rm C}$, thickness $T_{\rm C}$) illuminated by a linear combination of regular spherical waves:[1]

$$\mathbf{E}^{\rm inc}(\mathbf{x}) = \sum_\alpha \Big\{ P_\alpha \mathbf{M}_\alpha^{\rm reg}(\mathbf{x}) Q_\alpha \mathbf{N}_\alpha^{\rm reg}(\mathbf{x}) \Big\}$$

4.1 Exact solution

Fields outside:

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}^{\rm inc}(\mathbf{x}) + \sum_\alpha \Big\{ A_\alpha \mathbf{M}_\alpha^{\rm out}(\mathbf{x}) + B_\alpha \mathbf{N}_\alpha^{\rm out}(\mathbf{x}) \Big\} \tag{15a}$$

$$\mathbf{H}^{\rm scat}(\mathbf{x}) = \mathbf{H}^{\rm inc}(\mathbf{x}) + \frac{1}{Z_0} \sum_\alpha \Big\{ B_\alpha \mathbf{M}_\alpha^{\rm out}(\mathbf{x}) - A_\alpha \mathbf{N}_\alpha^{\rm out}(\mathbf{x}) \Big\} \tag{15b}$$

$$\tag{15c}$$

[1] My conventions for vector spherical waves are detailed in the companion memo, "Electromagnetism in the Spherical-Wave Basis" (`http://homerreid.github.io/scuff-em-documentation/tex/scuffSpherical.pdf`).

Fields in conducting layer:

$$\mathbf{E} = \sum_\alpha \Big\{ C_\alpha \mathbf{M}^{\text{out}}_\alpha(\mathbf{x}) + D_\alpha \mathbf{N}^{\text{out}}_\alpha(\mathbf{x}) + E_\alpha \mathbf{M}^{\text{reg}}_\alpha(\mathbf{x}) + F_\alpha \mathbf{N}^{\text{reg}}_\alpha(\mathbf{x}) \Big\} \tag{16}$$

$$\mathbf{H} = \frac{1}{Z_0 Z_{\text{C}}} \sum_\alpha \Big\{ D_\alpha \mathbf{M}^{\text{out}}_\alpha(\mathbf{x}) - C_\alpha \mathbf{N}^{\text{out}}_\alpha(\mathbf{x}) + F_\alpha \mathbf{M}^{\text{reg}}_\alpha(\mathbf{x}) - E_\alpha \mathbf{N}^{\text{reg}}_\alpha(\mathbf{x}) \Big\} \tag{17}$$

Fields inside dielectric sphere:

$$\mathbf{E} = \sum_\alpha \Big\{ G_\alpha \mathbf{M}^{\text{reg}}_\alpha(\mathbf{x}) + H_\alpha \mathbf{N}^{\text{reg}}_\alpha(\mathbf{x}) \Big\} \tag{18a}$$

$$\mathbf{H} = \frac{1}{Z_0 Z_{\text{D}}} \sum_\alpha \Big\{ H_\alpha \mathbf{M}^{\text{reg}}_\alpha(\mathbf{x}) - G_\alpha \mathbf{N}^{\text{reg}}_\alpha(\mathbf{x}) \Big\} \tag{18b}$$

Match tangential fields:

$$\left(\begin{array}{cccc} R^{\text{out}}(R_{\text{C}}) & -R^{\text{out}}(R_{\text{C}}) & -R^{\text{reg}}(R_{\text{C}}) & 0 \\ \overline{R}^{\text{out}}(R_{\text{C}}) & -\frac{1}{Z_{\text{C}}}\overline{R}^{\text{out}}(R_{\text{C}}) & -\frac{1}{Z_{\text{C}}}\overline{R}^{\text{reg}}(R_{\text{C}}) & 0 \\ 0 & R^{\text{out}}(R_{\text{D}}) & R^{\text{reg}}(R_{\text{D}}) & -R^{\text{reg}}(R_{\text{D}}) \\ 0 & \frac{1}{Z_{\text{C}}}\overline{R}^{\text{out}}(R_{\text{D}}) & \frac{1}{Z_{\text{C}}}\overline{R}^{\text{reg}}(R_{\text{D}}) & -\frac{1}{Z_{\text{D}}}\overline{R}^{\text{reg}}(R_{\text{D}}) \end{array}\right) \left(\begin{array}{c} A \\ C \\ E \\ G \end{array}\right) = -P \left(\begin{array}{c} R^{\text{reg}}(R_{\text{C}}) \\ \overline{R}^{\text{reg}}(R_{\text{C}}) \\ 0 \\ 0 \end{array}\right)$$

The equation for the vector $\left(\begin{array}{c} B \\ D \\ F \\ H \end{array}\right)$ is similar with $R \leftrightarrow \overline{R}$ everywhere.

4.2 IBC solution

In the IBC version of the problem we have only the fields (15) and (18). The IBC condition reads

$$\mathbf{H}\big(R_{\text{D}}^+\big) - \mathbf{H}\big(R_{\text{D}}^-\big) = -\sigma_{\text{S}} \hat{\mathbf{r}} \times E\big(R_{\text{D}}^-\big)$$

or (using an obvious shorthand)

$$Q\mathbf{M}^{\text{reg}} - P\mathbf{N}^{\text{reg}} + B\mathbf{M}^{\text{out}} - A\mathbf{N}^{\text{out}} - \frac{1}{Z^{\text{D}}} H\mathbf{M}^{\text{reg}} + \frac{1}{Z^{\text{D}}} G\mathbf{N}^{\text{reg}} = -\sigma_{\text{S}} \left(G\hat{\mathbf{r}} \times \mathbf{M}^{\text{reg}} + H\hat{\mathbf{r}} \times \mathbf{N}^{\text{reg}} \right)$$

Noting that

$$\hat{\mathbf{r}} \times \mathbf{M}^{\text{reg}} = R^{\text{reg}}\mathbf{Z}, \qquad \hat{\mathbf{r}} \times \mathbf{N}^{\text{reg}} = -i\overline{R}^{\text{reg}}\mathbf{X}$$

and working through the algebra yields

$$\left(\begin{array}{cc} R^{\text{out}}(a) & -R^{\text{reg}}(na) \\ \overline{R}^{\text{out}}(a) & -\frac{1}{Z_{\text{D}}}\overline{R}^{\text{reg}}(na) + iZ_0\sigma_{\text{S}} R^{\text{reg}}(na) \end{array}\right) \left(\begin{array}{c} A \\ G \end{array}\right) = -P \left(\begin{array}{c} R^{\text{reg}}(a) \\ \overline{R}^{\text{reg}}(a) \end{array}\right)$$

$$\left(\begin{array}{cc} \overline{R}^{\text{out}}(a) & -\overline{R}^{\text{reg}}(na) \\ R^{\text{out}}(a) & -\frac{1}{Z_{\text{D}}} R^{\text{reg}}(na) - iZ_0\sigma_{\text{S}}\overline{R}^{\text{reg}}(na) \end{array}\right) = -Q \left(\begin{array}{c} \overline{R}^{\text{reg}}(a) \\ R^{\text{reg}}(a) \end{array}\right)$$