# Electromagnetism in the Spherical-Wave Basis:

A (Somewhat Random) Compendium of Reference Formulas

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August 1, 2016

**Abstract**

This memo consolidates and collects for reference a somewhat random hodgepodge of formulas and results in the spherical-wave approach to electromagnetism that I have found useful over the years in developing and testing SCUFF-EM and BUFF-EM.

## Contents

# 1 Vector Spherical Wave Solutions to Maxwell's Equations

Many authors define pairs of three-vector-valued functions $\{\mathbf{M}_{\ell m}(\mathbf{x}), \mathbf{N}_{\ell m}(\mathbf{x})\}$ describing exact solutions of Maxwell's equations in spherical coordinates for a homogeneous medium with wavenumber $k$, i.e.

$$\Big[\nabla\times\nabla\times -k^2\Big]\left\{\begin{array}{c}\mathbf{M}\\ \mathbf{N}\end{array}\right\}=0.$$

These functions always involve spherical Bessel functions and spherical harmonics, but the precise definitions (including sign conventions and normalization factors) vary from author to author. In this section I set down the particular conventions that I use. In the next section I give explicit closed-form expressions for small $\ell$.

### Vector spherical harmonics

$$\begin{aligned}\mathbf{X}_{\ell m}(\theta,\varphi) &= \frac{i}{\ell(\ell+1)}\nabla\times\Big\{Y_{\ell m}(\theta,\varphi)\hat{\mathbf{r}}\Big\}\\ \mathbf{Z}_{\ell m}(\theta,\varphi) &= \hat{\mathbf{r}}\times\mathbf{X}_{\ell m}(\theta,\varphi)\end{aligned}$$

More explicitly, the components of $\mathbf{X}$ and $\mathbf{Z}$ are

$$\begin{aligned}\mathbf{X}_{\ell m}(\theta,\phi) &= \frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{im}{\sin\theta}Y_{\ell m}\widehat{\boldsymbol{\theta}} - \frac{\partial Y_{\ell m}}{\partial\theta}\widehat{\boldsymbol{\varphi}}\right]\\ \mathbf{Z}_{\ell m}(\theta,\phi) &= \frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{\partial Y_{\ell m}}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{im}{\sin\theta}Y_{\ell m}\widehat{\boldsymbol{\varphi}}\right].\end{aligned}$$

Their divergences are:

$$\nabla\cdot\mathbf{X}_{\ell m} = -\frac{m\cot\theta\csc\theta Y_{\ell m}(\theta,\varphi)}{r\sqrt{\ell(\ell+1)}} \tag{1a}$$

$$\nabla\cdot\mathbf{Z}_{\ell m} = \frac{i\cot\theta}{r\sqrt{\ell(\ell+1)}}\Big[m\cot\theta Y_{\ell m}(\theta,\varphi) + \xi_{\ell m}e^{-i\varphi}Y_{\ell,m+1}(\theta,\varphi)\Big] \tag{1b}$$

$$\xi_{\ell m} \equiv \sqrt{\frac{(\ell-m)!(\ell+m+1)!}{(\ell-m-1)!(\ell+m)!}} \tag{1c}$$

### Radial functions

$$\begin{aligned}R_\ell^{\text{outgoing}}(kr) &= h_\ell^{(1)}(kr)\\ R_\ell^{\text{incoming}}(kr) &= h_\ell^{(2)}(kr)\\ R_\ell^{\text{regular}}(kr) &= j_\ell(kr).\end{aligned}$$

I also define the shorthand symbols

$$\overline{R}_\ell(kr) \equiv \frac{1}{kr}\left|R_\ell(x) + \frac{d}{dx}R_\ell(x)\right|_{x=kr} \qquad \not{R}_\ell(kr) = -\frac{\sqrt{l(l+1)}}{kr}R_\ell(kr).$$

**Vector spherical wave functions**

$$\mathbf{M}_{\ell m}(k;\mathbf{r}) \equiv R_\ell(kr)\mathbf{X}_{\ell m}(\Omega) \tag{2a}$$

$$\mathbf{N}_{\ell m}(k;\mathbf{r}) \equiv i\overline{R}_\ell(kr)\mathbf{Z}_{\ell m}(\Omega) + \not{R}_\ell(kr)Y_{\ell m}(\Omega)\hat{\mathbf{r}} \tag{2b}$$

**Curl Identities**

$$\nabla\times\mathbf{M} = -ik\mathbf{N}, \qquad \nabla\times\mathbf{N} = +ik\mathbf{M}.$$

**General solution of source-free Maxwell equations** The general solution of Maxwell's equations in a source-free medium with relative material properties $\epsilon^r, \mu^r$ then reads

$$\mathbf{E}(\mathbf{x}) = \sum_\alpha \Big\{\mathbf{A}_\alpha\mathbf{M}_\alpha(k;\mathbf{r}) + \mathbf{B}_\alpha\mathbf{N}_\alpha(k;\mathbf{r})\Big\} \tag{3a}$$

$$\mathbf{H}(\mathbf{x}) = \frac{1}{Z_0Z^r}\sum_\alpha \Big\{\mathbf{B}_\alpha\mathbf{M}_\alpha(k;\mathbf{r}) - \mathbf{A}_\alpha\mathbf{N}_\alpha(k;\mathbf{r})\Big\} \tag{3b}$$

where $k = \sqrt{\epsilon_0\epsilon^r\mu_0\mu^r}\cdot\omega$ is the photon wavenumber in the medium, $Z_0 = \sqrt{\mu_0/\epsilon_0} \sim 377\,\Omega$ is the impedance of vacuum, $Z^r = \sqrt{\mu^r/\epsilon^r}$ is the relative wave impedance of the medium, and we must choose the $\mathbf{M},\mathbf{N}$ functions to be regular, incoming, or outgoing depending on the physical conditions of the problem.

## 2 Explicit expression for small $\ell$

**The first few radial functions**

$$R_1^{\text{regular}}(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} \qquad \overline{R}_1^{\text{regular}}(x) = \frac{\cos x}{x^2} + \frac{(x^2-1)\sin x}{x^3}$$

$$R_1^{\text{outgoing}}(x) = -i\frac{(1-ix)e^{ix}}{x^2} \qquad \overline{R}_1^{\text{outgoing}}(x) = i\frac{(1-ix+x^2)e^{ix}}{x^3}$$

**The first few regular functions** In what follows, the $\zeta_n$ are dimensionless sinusoidal functions:

$$\zeta_1(x) = \sin x - x\cos x$$
$$\zeta_2(x) = (1-x^2)\sin x - x\cos x$$

$$\mathbf{M}_{1,\pm 1}^{\text{regular}}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left[\frac{\zeta_1(kr)}{(kr)^2}\right] e^{\pm i\varphi}\begin{pmatrix} 0 \\ 1 \\ \pm i\cos\theta \end{pmatrix}$$

$$\mathbf{M}_{1,0}^{\text{regular}}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left[\frac{\zeta_1(kr)}{(kr)^2}\right]\begin{pmatrix} 0 \\ 0 \\ i\sin\theta \end{pmatrix}$$

$$\mathbf{N}_{1,\pm 1}^{\text{regular}}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left[\frac{1}{(kr)^3}\right] e^{\pm i\varphi}\begin{pmatrix} \pm 2\zeta_1(kr)\sin\theta \\ \mp\zeta_2(kr)\cos\theta \\ -i\zeta_2(kr) \end{pmatrix}$$

$$\mathbf{N}_{1,0}^{\text{regular}}(\mathbf{r}) = \frac{1}{2\sqrt{2}(kr)^3}\begin{pmatrix} -2\zeta_1(kr)\cos\theta \\ -\zeta_2(kr)\sin\theta \\ 0 \end{pmatrix}$$

**The first few outgoing functions** In what follows, the $Q_n$ are dimensionless polynomial factors:

$$Q_1(x) = 1 - x$$
$$Q_{2a}(x) = 1 - x + x^2$$
$$Q_{2b}(x) = 3 - 3x + x^2$$
$$Q_3(x) = 6 - 6x + 3x^2 - x^3$$

$$\mathbf{M}^{\text{outgoing}}_{1,\pm 1}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left(\frac{e^{ikr}}{k^2r^2}\right)e^{\pm i\phi}\left(\begin{array}{c} 0 \\ -iQ_1(ikr) \\ \pm Q_1(ikr)\cos\theta \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{1,0}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left(\frac{e^{ikr}}{k^2r^2}\right)\left(\begin{array}{c} 0 \\ 0 \\ Q_1(ikr)\sin\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{1,\pm 1}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm i\phi}\left(\begin{array}{c} \mp - 2(ikr)Q_1(ikr)\sin\theta \\ \pm iQ_{2a}(ikr)\cos\theta \\ -Q_{2a}(ikr) \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{1,0}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)\left(\begin{array}{c} 2iQ_1(ikr)\cos\theta \\ +iQ_{2a}(ikr)\sin\theta \\ 0 \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{2,\pm 2}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm 2i\phi}\left(\begin{array}{c} 0 \\ \pm iQ_{2b}(ikr)\sin\theta \\ -Q_{2b}(ikr)\cos\theta\sin\theta \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{2,\pm 1}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm i\phi}\left(\begin{array}{c} 0 \\ -iQ_{2b}(ikr)\cos\theta \\ \pm Q_{2b}(ikr)\cos 2\theta \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{2,0}(\mathbf{r}) = \sqrt{\frac{15}{8\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)\left(\begin{array}{c} 0 \\ 0 \\ -Q_{2b}(ikr)\cos\theta\sin\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{2,\pm 2}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)e^{\pm 2i\phi}\left(\begin{array}{c} 3iQ_{2b}(ikr)\sin^2\theta \\ -iQ_3(ikr)\cos\theta\sin\theta \\ \pm Q_3(ikr)\sin\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{2,\pm 1}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)e^{\pm i\phi}\left(\begin{array}{c} \mp 3iQ_{2b}(ikr)\sin 2\theta \\ \pm iQ_3(ikr)\cos 2\theta \\ -Q_3(ikr)\cos\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{2,0}(\mathbf{r}) = \sqrt{\frac{15}{8\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)\left(\begin{array}{c} iQ_{2b}(ikr)(3\cos^2\theta - 1) \\ iQ_3(ikr)\cos\theta\sin\theta \\ 0 \end{array}\right).$$

# 3 Translation matrices

Translation matrices arise when we want to evaluate the fields produced by sources not located at the origin.

**Scalar case** Although we don't need it for electromagnetism problems, the scalar-wave analog of (2) is

$$\psi_{\ell m}(\mathbf{x}) = R_\ell(kr) Y_{\ell m}(\theta,\phi)$$

or, more specifically,

$$\psi^{\text{out}}_{\ell m}(\mathbf{x}) = R^{\text{out}}_\ell(kr) Y_{\ell m}(\theta,\phi), \qquad \psi^{\text{reg}}_{\ell m}(\mathbf{x}) = R^{\text{reg}}_\ell(kr) Y_{\ell m}(\theta,\phi)$$

Now consider a point source at $\mathbf{x}^{\text{S}}$ whose fields we wish to evaluate at an evaluation ("destination") point $\mathbf{x}^{\text{D}}$, using a basis of spherical waves centered at an origin $\mathbf{x}^{\text{O}}$. Then waves emitted by the source, which appear to be outgoing in a coordinate system centered at $\mathbf{x}^{\text{S}}$, can be described as superpositions of regular waves in a coordinate system centered at $\mathbf{x}^{\text{O}}$:

$$\psi^{\text{out}}_\alpha\big(\mathbf{x}^{\text{D}} - \mathbf{x}^{\text{S}}\big) = \sum_\beta A_{\alpha\beta}\big(k; \mathbf{x}^{\text{S}} - \mathbf{x}^{\text{O}}\big) \psi^{\text{reg}}_\beta\big(\mathbf{x}^{\text{D}} - \mathbf{x}^{\text{O}}\big) \tag{4}$$

where $\alpha, \beta$ are compound indices (i.e. $\alpha = \{\ell_\alpha, m_\alpha\}$) and

$$\begin{aligned}
A_{\alpha\beta}(k, \mathbf{L}) &= 4\pi \sum_\gamma i^{(\ell_\alpha - \ell_\beta + \ell_\gamma)} a_{\alpha\gamma\beta} \psi^{\text{out}}_\gamma(\mathbf{L}) \\
a_{\alpha\beta\gamma} &= \int Y_\alpha(\Omega) Y^*_\beta(\Omega) Y^*_\gamma(\Omega)\, d\Omega \\
&= (-1)^{m_\alpha} \sqrt{\frac{(2\ell_\alpha+1)(2\ell_\beta+1)(2\ell_\gamma+1)}{4\pi}} \left(\begin{array}{ccc} \ell_\alpha & \ell_\beta & \ell_\gamma \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} \ell_\alpha & \ell_\beta & \ell_\gamma \\ -m_\alpha & m_\beta & m_\gamma \end{array}\right).
\end{aligned}$$

**Vector case**

$$\left(\begin{array}{c} \mathbf{M} \\ \mathbf{N} \end{array}\right)^{\text{out}}_\alpha = \sum_\beta \left(\begin{array}{cc} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{array}\right)_{\alpha\beta} \left(\begin{array}{c} \mathbf{M} \\ \mathbf{N} \end{array}\right)^{\text{reg}}_\beta$$

$$\begin{aligned}
B_{\alpha\beta}(k, \mathbf{L}) &= 4\pi \sum_\gamma i^{(\ell_\alpha - \ell_\beta + \ell_\gamma)} \left[\frac{\ell_\alpha(\ell_\alpha+1) + \ell_\beta(\ell_\beta+1) - \ell_\gamma(\ell_\gamma+1)}{2\sqrt{\ell_\alpha(\ell_\alpha+1)\ell_\beta(\ell_\beta+1)}}\right] a_{\alpha\gamma\beta} \psi^{\text{out}}_\gamma(\mathbf{L}) \\
C_{\alpha\beta}(k, \mathbf{L}) &= -\frac{k}{\sqrt{\ell_\alpha(\ell_\alpha+1)\ell_\beta(\ell_\beta+1)}} \left[\frac{\lambda_+}{2}\big(L_x - iL_y\big) A_{\alpha_+,\beta} + \frac{\lambda_-}{2}\big(L_x + iL_y\big) A_{\alpha_-,\beta} + m_\alpha L_z A_{\alpha,\beta}\right]
\end{aligned}$$

$$\lambda_\pm = \sqrt{(\ell_\alpha \mp \ell_\beta)(\ell_\alpha \pm \ell_\beta + 1)}, \qquad \alpha_\pm = \{\ell_\alpha, m_\alpha \pm 1\}$$

# 4 Scattering from a homogeneous dielectric sphere

I consider scattering from a single homogeneous sphere with relative permittivity and permeability $\epsilon^r, \mu^r$ in vacuum irradiated by spherical waves emanating from within our outside the sphere.

Irrespective of the origin of the incident fields, the scattered fields inside and outside the sphere take the form ($n = \sqrt{\epsilon^r \mu^r}$, $Z^r = \sqrt{\mu^r/\epsilon^r}$)

**Inside the sphere:**

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_\alpha \Big\{ A_\alpha \mathbf{M}_\alpha^{\text{reg}}(nk_0; \mathbf{r}) + B_\alpha \mathbf{N}_\alpha^{\text{reg}}(nk_0; \mathbf{r}) \Big\} \tag{5a}$$

$$\mathbf{H}^{\text{scat}}(\mathbf{x}) = \frac{1}{Z_0 Z^r} \sum_\alpha \Big\{ B_\alpha \mathbf{M}_\alpha^{\text{reg}}(nk_0; \mathbf{r}) - A_\alpha \mathbf{N}_\alpha^{\text{reg}}(nk_0; \mathbf{r}) \Big\} \tag{5b}$$

**Outside the sphere:**

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_\alpha \Big\{ C_\alpha \mathbf{M}_\alpha^{\text{out}}(k_0; \mathbf{r}) + D_\alpha \mathbf{N}_\alpha^{\text{out}}(k_0; \mathbf{r}) \Big\} \tag{6a}$$

$$\mathbf{H}^{\text{scat}}(\mathbf{x}) = \frac{1}{Z_0} \sum_\alpha \Big\{ D_\alpha \mathbf{M}_\alpha^{\text{out}}(k_0; \mathbf{r}) - C_\alpha \mathbf{N}_\alpha^{\text{out}}(k_0; \mathbf{r}) \Big\} \tag{6b}$$

The $\{A, B, C, D\}$ coefficients are proportional to the spherical-wave expansion coefficients of the incident fields, with the proportionality constants determined by enforcing continuity of the tangential components of the total fields $\{\mathbf{E}, \mathbf{H}\}^{\text{tot}} = \{\mathbf{E}, \mathbf{H}\}^{\text{inc}} + \{\mathbf{E}, \mathbf{H}\}^{\text{scat}}$ at $r = r_0$,

$$\Big| \hat{\mathbf{r}} \times \mathbf{E}^{\text{tot}} \Big|_{r \to r_0^+} = \Big| \hat{\mathbf{r}} \times \mathbf{E}^{\text{tot}} \Big|_{r \to r_0^-} \tag{7a}$$

$$\Big| \hat{\mathbf{r}} \times \mathbf{H}^{\text{tot}} \Big|_{r \to r_0^+} = \Big| \hat{\mathbf{r}} \times \mathbf{H}^{\text{tot}} \Big|_{r \to r_0^-} \tag{7b}$$

## 4.1 Sources outside the sphere

If the sources of the incident field lie outside the sphere (the usual Mie scattering problem), then I can expand the incident fields in the form

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum_\alpha \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{regular}}(k_0; \mathbf{x}) + Q_\alpha \mathbf{N}_\alpha^{\text{regular}}(k_0; \mathbf{x}) \Big\} \tag{8a}$$

$$\mathbf{H}^{\text{inc}}(\mathbf{x}) = \frac{1}{Z_0} \sum_\alpha \Big\{ Q_\alpha \mathbf{M}_\alpha^{\text{regular}}(k_0; \mathbf{x}) - P_\alpha \mathbf{N}_\alpha^{\text{regular}}(k_0; \mathbf{x}) \Big\}. \tag{8b}$$

Matching tangential fields at the sphere surface then determines the scattered-field expansion coefficients in terms of the incident-field expansion coefficients ($a = k_0 r_0$):

$$A_\alpha = \left[\frac{R^{\text{reg}}(a)\overline{R}^{\text{out}}(a) - \overline{R}^{\text{reg}}(a)R^{\text{out}}(a)}{R^{\text{reg}}(na)\overline{R}^{\text{out}}(a) - \frac{1}{Z^r}\overline{R}^{\text{reg}}(na)R^{\text{out}}(a)}\right]P_\alpha \tag{9a}$$

$$B_\alpha = \left[\frac{\overline{R}^{\text{reg}}(a)R^{\text{out}}(a) - R^{\text{reg}}(a)\overline{R}^{\text{out}}(a)}{\overline{R}^{\text{reg}}(na)R^{\text{out}}(a) - \frac{1}{Z^r}R^{\text{reg}}(na)\overline{R}^{\text{out}}(a)}\right]Q_\alpha \tag{9b}$$

$$C_\alpha = \underbrace{\left[\frac{R^{\text{reg}}(a)\overline{R}^{\text{reg}}(na) - Z^r\overline{R}^{\text{reg}}(a)R^{\text{reg}}(na)}{Z^r R^{\text{reg}}(na)\overline{R}^{\text{out}}(a) - \overline{R}^{\text{reg}}(na)R^{\text{out}}(a)}\right]}_{\mathbb{T}^{\text{M}}_\alpha}P_\alpha \tag{9c}$$

$$D_\alpha = \underbrace{\left[\frac{\overline{R}^{\text{reg}}(a)R^{\text{reg}}(na) - Z^r R^{\text{reg}}(a)\overline{R}^{\text{reg}}(na)}{Z^r \overline{R}^{\text{reg}}(na)R^{\text{out}}(a) - R^{\text{reg}}(na)\overline{R}^{\text{out}}(a)}\right]}_{\mathbb{T}^{\text{N}}_\alpha}Q_\alpha \tag{9d}$$

In (9c,d) I have identified the quantities $C_\alpha/P_\alpha$ and $D_\alpha/Q_\alpha$ as elements of the $\mathbb{T}$-matrix for the $M$- and $N$- polarizations.[1]

The coefficients here may be expressed in closed form, e.g.

$$\frac{A_1}{P_1} = \frac{2a^3e^{-ia}n^3Z}{((1-ia)\left(a^2n^2-1\right) - (-1+a(a+i))nZ)\sin(an) + an((-1+a(a+i))nZ - ia + 1)\cos(an)}$$

$$\frac{B_1}{Q_1} = \frac{2a^3e^{-ia}n^3Z}{((1-ia)Z\left(a^2n^2-1\right) - (-1+a(a+i))n)\sin(an) + an((-1+a(a+i))n - iaZ + Z)\cos(an)}$$

where $a = (k_0 r_0)$ is the dimensionless Mie size parameter. The low-frequency limiting forms (assuming $\mu = 1$:) are

$$\frac{A_1}{P_1} = \frac{2}{\sqrt{\epsilon}} + \left[\frac{(\epsilon-1)}{3\sqrt{\epsilon}}\right]a^2 + O(a^3) \qquad \frac{B_1}{Q_1} = \frac{6}{\epsilon+2} + \left[\frac{3\left(\epsilon^2+9\epsilon-10\right)}{5(\epsilon+2)^2}\right]a^2 + O(a^3)$$

$$\frac{A_2}{P_2} = \frac{2}{\epsilon} + \left[\frac{(\epsilon-1)}{5\epsilon}\right]a^2 + O(a^3) \qquad \frac{B_2}{Q_2} = \frac{10}{\sqrt{\epsilon}(2\epsilon+3)} + \left[\frac{5a^2\left(2\epsilon^2+5\epsilon-7\right)}{7\sqrt{\epsilon}(2\epsilon+3)^2}\right]a^2 + O(a^3)$$

[1] The $\mathbb{T}$-matrix multiplies a vector of regular-wave incident-field coefficients to yield a vector of outgoing-wave scattered-field coefficients. If, instead of the regular-wave incident field (8), I irradiated the sphere with a superposition of *incoming* waves as the incident field, then the resulting modified versions of equations (9c,d) would instead define elements of the $\mathbb{S}$-matrix (scattering matrix).

### 4.2 Sources inside the sphere

If the sources of the incident field lie inside the sphere, then I can expand the incident field in the form

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum_\alpha \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{out}}(nk_0;\mathbf{x}) + Q_\alpha \mathbf{N}_\alpha^{\text{out}}(nk_0;\mathbf{x}) \Big\} \tag{10}$$

The total fields inside and outside then read

$$\mathbf{E}^{\text{in}}(\mathbf{x}) = \sum \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) + Q_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) \Big\} + \sum \Big\{ A_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) + B_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x}) \Big\} \tag{11}$$

$$\begin{aligned}\mathbf{H}^{\text{in}}(\mathbf{x}) = -\frac{1}{Z_0 Z^r} \sum \Big\{ P_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) - Q_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) \Big\} \\ - \frac{1}{Z_0 Z^r} \sum \Big\{ C_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x}) - D_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) \Big\}\end{aligned} \tag{12}$$

$$\mathbf{E}^{\text{out}}(\mathbf{x}) = \sum \Big\{ C_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) + D_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) \Big\} \tag{13}$$

$$\mathbf{H}^{\text{out}}(\mathbf{x}) = -\frac{1}{Z_0} \sum \Big\{ C_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) - D_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) \Big\} \tag{14}$$

Now equate tangential components of $\mathbf{E}^{\text{in,out}}$ and $\mathbf{H}^{\text{in,out}}$ at the sphere surface ($r = r_0$), take inner products with $\mathbf{M}$ and $\mathbf{N}$, and use the orthogonality relations to find

$$\begin{array}{ccccc} R_\ell^{\text{out}}(nkr_0)P_\alpha & + & R_\ell^{\text{reg}}(nkr_0)A_\alpha & = & R_\ell^{\text{out}}(kr_0)C_\alpha \\ \overline{R}_\ell^{\text{out}}(nkr_0)P_\alpha & + & \overline{R}_\ell^{\text{reg}}(nkr_0)A_\alpha & = & Z^r\overline{R}_\ell^{\text{out}}(kr_0)C_\alpha \\ \overline{R}_\ell^{\text{out}}(nkr_0)Q_\alpha & + & \overline{R}_\ell^{\text{reg}}(nkr_0)B_\alpha & = & \overline{R}_\ell^{\text{out}}(kr_0)D_\alpha \\ R_\ell^{\text{out}}(nkr_0)Q_\alpha & + & R_\ell^{\text{reg}}(nkr_0)B_\alpha & = & Z^r R_\ell^{\text{out}}(kr_0)D_\alpha \end{array}$$

which we solve to obtain the coefficients of the scattered field outside the sphere in terms of the incident-field coefficients:

$$C_\alpha = \left[ \frac{R_\ell^{\text{out}}(na)\overline{R}_\ell^{\text{reg}}(na) - \overline{R}_\ell^{\text{out}}(na)R_\ell^{\text{reg}}(na)}{R_\ell^{\text{out}}(a)\overline{R}_\ell^{\text{reg}}(na) - Z^r\overline{R}_\ell^{\text{out}}(a)R_\ell^{\text{reg}}(na)} \right] P_\alpha \tag{15a}$$

$$D_\alpha = \left[ \frac{\overline{R}_\ell^{\text{out}}(na)R_\ell^{\text{reg}}(na) - R_\ell^{\text{out}}(na)\overline{R}_\ell^{\text{reg}}(na)}{\overline{R}_\ell^{\text{out}}(a)R_\ell^{\text{reg}}(na) - Z^r R_\ell^{\text{out}}(a)\overline{R}_\ell^{\text{reg}}(na)} \right] Q_\alpha. \tag{15b}$$

# 5 Scattering from a sphere with impedance boundary conditions

For a sphere characterized by a surface-impedance boundary condition with relative surface impedance[2] $\eta$, the continuity condition (7) is replaced by a relationship between the tangential **E** and **H** fields at the sphere surface:

$$\mathbf{E}_{\|} = \eta Z_0 \Big( \hat{\mathbf{r}} \times \mathbf{H} \Big) \quad \text{at} \quad r = R.$$

Equations (9c,d) for the $\mathbb{T}$-matrix elements are replaced by

$$C_\alpha = \underbrace{\left[ \frac{R^{\text{reg}}(a)}{i\eta \overline{R}^{\text{out}}(a) - R^{\text{out}}(a)} \right]}_{\mathbb{T}^{\text{M}}_\alpha} P_\alpha, \qquad D_\alpha = \underbrace{\left[ \frac{\overline{R}^{\text{reg}}(a)}{i\eta R^{\text{out}}(a) + \overline{R}^{\text{out}}(a)} \right]}_{\mathbb{T}^{\text{N}}_\alpha} Q_\alpha, \quad (16)$$

In particular, taking $\eta \to 0$ yields the $\mathbb{T}$-matrix elements for a perfectly electrically conducting (PEC) sphere.

[2] Note that $\eta$ is dimensionless; the absolute surface impedance is $\eta Z_0$ where $Z_0 \approx 377\,\Omega$ is the impedance of vacuum.

# 6 Spherical-wave expansion of plane waves

For a scattering problem in which the incident field is a $z$-directed plane wave, i.e.

$$\mathbf{E}^{\text{inc}} = \mathbf{E}_0 e^{ikz}, \qquad \mathbf{H}^{\text{inc}} = \frac{1}{Z_0}\hat{\mathbf{z}} \times \mathbf{E}_0 e^{ikz}$$

the spherical-wave expansion coefficients in (8) take the following forms for various possible polarizations:

$$\begin{array}{lcll}
\mathbf{E}_0 & = & \hat{\mathbf{x}} + i\hat{\mathbf{y}} & \text{(right circular polarization)}: \quad P_{\ell m} = \delta_{m,+1} P_\ell \\
\mathbf{E}_0 & = & \hat{\mathbf{x}} - i\hat{\mathbf{y}} & \text{(left circular polarization)}: \quad P_{\ell m} = \delta_{m,-1} P_\ell \\
\mathbf{E}_0 & = & \hat{\mathbf{x}} & \text{(linear polarization)}: \quad P_{\ell m} = \frac{1}{2}\Big(\delta_{m,+1} + \delta_{m,-1}\Big) P_\ell
\end{array}$$

where in all cases I have

$$P_\ell = i^\ell \sqrt{4\pi(2\ell+1)}, \qquad Q_{\ell,\pm 1} = \mp i P_\ell.$$

# 7 Spherical-wave expansion of point-source fields

In this section I obtain the coefficients $P_\alpha, N_\alpha$ in the spherical-wave expansion (8) for the case in which the incident field arises from a point dipole. My strategy is first to write down the spherical-wave expansion of the fields in a coordinate system centered at the dipole, then use the spherical-wave translation formulas to write these in terms of coordinates centered on the sphere.

## 7.1 Fields of a point source at the origin

Let $\mathbf{E}(\mathbf{x};\mathbf{p})$ be the electric field at evaluation point $\mathbf{x}$ due to an electric dipole $\mathbf{p}$ at the origin. The spherical-wave expansion of this field involves only $\mathbf{N}$-functions with $\ell = 1$, i.e.

$$\mathbf{E}(\mathbf{x};\mathbf{p}) = \sum_{m=-1}^{1} \xi_{1m}(\mathbf{p})\mathbf{N}_{1m}^{\text{outgoing}}(\mathbf{x})$$

where the $\xi$ coefficients are

$$\begin{array}{llll}
\mathbf{p} = p_x\hat{\mathbf{x}} & \longrightarrow & \xi_{1,1} = -\xi_{1,-1} = \dfrac{i}{2}\dfrac{k^3}{\sqrt{3\pi}}\dfrac{p_x}{\epsilon}, & \xi_{1,0} = 0 \\
\mathbf{p} = p_y\hat{\mathbf{y}} & \longrightarrow & \xi_{1,1} = +\xi_{1,-1} = \dfrac{1}{2}\dfrac{k^3}{\sqrt{3\pi}}\dfrac{p_y}{\epsilon} & \xi_{1,0} = 0 \\
\mathbf{p} = p_z\hat{\mathbf{z}} & \longrightarrow & \xi_{1,1} = \xi_{1,-1} = 0, & \xi_{1,0} = -\dfrac{ik^3}{\sqrt{6\pi}}\dfrac{p_z}{\epsilon}
\end{array}$$

Here $\epsilon = \epsilon_0\epsilon^r$ is the absolute permittivity of the medium.

Similarly, the magnetic fields of a magnetic dipole $\mathbf{m}$ at the origin are

$$\mathbf{H}(\mathbf{x};\mathbf{m}) = \sum_{\alpha} \widehat{\xi}_\alpha(\mathbf{m})\mathbf{N}_{\alpha}^{\text{outgoing}}(\mathbf{x})$$

where the $\widehat{\xi}$ coefficients are the same as the $\xi$ coefficients above with the replacement $\frac{p}{\epsilon} \to \frac{m}{\mu}$.

## 7.2 Fields of a point source away from the origin

If the point source lies at $\mathbf{x}^{\mathrm{S}} \neq 0$ (here "S" stands for "source"), then I can use the vector-wave translation formulas of Section 3 to write the incident fields as a sum of spherical waves centered at the origin.

$$\mathbf{E}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{p}) = \sum_{\alpha\beta}\Big\{-\xi_\alpha C_{\alpha\beta}\mathbf{M}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}}) + \xi_\alpha B_{\alpha\beta}\mathbf{N}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}})\Big\} \tag{17a}$$

$$\mathbf{H}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{p}) = \frac{1}{Z}\sum_{\alpha\beta}\Big\{\xi_\alpha C_{\alpha\beta}\mathbf{N}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}}) + \xi_\alpha B_{\alpha\beta}\mathbf{M}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}})\Big\} \tag{17b}$$

$$\mathbf{E}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{m}) = -Z\sum_{\alpha\beta}\Big\{\widehat{\xi}_\alpha B_{\alpha\beta}\mathbf{M}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}}) + \widehat{\xi}_\alpha C_{\alpha\beta}\mathbf{N}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}})\Big\} \tag{17c}$$

$$\mathbf{H}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{m}) = \sum_{\alpha\beta}\Big\{-\widehat{\xi}_\alpha C_{\alpha\beta}\mathbf{M}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}}) + \widehat{\xi}_\alpha B_{\alpha\beta}\mathbf{N}_\beta^{\text{regular}}(\mathbf{x}^{\text{D}})\Big\} \tag{17d}$$

# 8 Dyadic Green's functions

The scattering part of the electric dyadic Green's function $\boldsymbol{\mathcal{G}}^{\text{EE}}(\mathbf{x}^{\text{D}},\mathbf{x}^{\text{S}})$ is a $3\times 3$ matrix whose $i,j$ component $\mathcal{G}^{\text{EE}}_{ij}(\mathbf{x}^{\text{D}},\mathbf{x}^{\text{S}})$ is the (appropriately normalized)[3] $i$ component of the scattered electric field at $\mathbf{x}^{\text{D}}$ due to a $j$-directed point electric dipole source at $\mathbf{x}^{\text{S}}$. (The superscripts on $\mathbf{x}$ stand for "destination" and "source").

If I take the electic-dipole fields Section 7 [equations (17a,b)] to be the incident fields in the externally-sourced scattering problem of Section 4.1 [so that, for example, the coefficient of $\mathbf{M}^{\text{reg}}_{\alpha}$ in the incident-field expansion (8) is $P_\alpha = -\sum_\beta \xi_\beta C_{\beta\alpha}$], then I need only multiply by $\mathbb{T}$-matrix elements [equation (9)] to get the outgoing-wave coefficients in the scattered-field expansion (6).

Thus the **E**- and **H**-fields at $\mathbf{x}^{\text{D}}$ due to an electric dipole source $\mathbf{p}$ at $\mathbf{x}^{\text{S}}$ are

$$\begin{aligned}
\mathbf{E}^{\text{scat}}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{p}) &= \sum_{\alpha\beta}\xi_\alpha(\mathbf{p})\Big\{-C_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{M}}_{\beta}\mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + B_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{N}}_{\beta}\mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}})\Big\}\\
\mathbf{H}^{\text{scat}}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{p}) &= \frac{1}{Z}\sum_{\alpha\beta}\xi_\alpha(\mathbf{p})\Big\{+C_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{M}}_{\gamma\beta}\mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + B_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{N}}_{\gamma\beta}\mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}})\Big\}
\end{aligned}$$

The **E**- and **H**-fields at $\mathbf{x}^{\text{D}}$ due to a magnetic dipole source $\mathbf{m}$ at $\mathbf{x}^{\text{S}}$ are

$$\begin{aligned}
\mathbf{E}^{\text{scat}}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{m}) &= -Z\sum_{\alpha\beta}\widehat{\xi}_\alpha(\mathbf{m})\Big\{+C_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{M}}_{\beta}\mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + B_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{N}}_{\beta}\mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}})\Big\}\\
\mathbf{H}^{\text{scat}}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{m}) &= \sum_{\alpha\beta}\widehat{\xi}_\alpha(\mathbf{m})\Big\{-B_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{N}}_{\beta}\mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + C_{\alpha\beta}(\mathbf{x}^{\text{S}})\mathbb{T}^{\text{M}}_{\beta}\mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}})\Big\}.
\end{aligned}$$

In writing out these equations, I have used the fact that the $\mathbb{T}$-matrix of a homogeneous sphere is diagonal. However, similar equations could be written down for the DGFs of any *arbitrary*-shaped object; in this case the $\mathbb{T}$-matrix would not be diagonal and the double sums would become triple sums, but such a representation might nonetheless be useful in some cases.

[3] The normalization just involves dividing by dimensionful prefactors to ensure that the components of $\mathcal{G}$ have units of inverse length and are independent of the point-source magnitude.

# 9 Computation of power, force, and torque

## Power

The power radiated away from (or, the negative of the power absorbed by) the sphere is obtained by integrating the outward-pointing normally-directed Poynting vector over any bounding surface containing the sphere. For convenience we will take the bounding surface to be a sphere of radius $r_b > r_0$ (denote this sphere by $\mathcal{S}_b$). Then the power is

$$\begin{aligned}
P &= \frac{1}{2}\text{Re}\,\oint_{\mathcal{S}_b}\hat{\mathbf{r}}\cdot\Big[\mathbf{E}^*(\mathbf{r})\times\mathbf{H}(\mathbf{r})\Big]dA \\
&= \frac{r_b^2}{2}\text{Re}\,\oint\hat{\mathbf{H}}^*(r_b,\Omega)\cdot\Big[\hat{\mathbf{r}}\times\mathbf{E}(r_b,\Omega)\Big]d\Omega \\
&= \frac{r_b^2}{4}\oint\Big[\mathbf{E}^*\cdot\big(\mathbf{H}\times\hat{\mathbf{r}}\big)+\mathbf{H}^*\cdot\big(\hat{\mathbf{r}}\times\mathbf{E}\big)\Big]d\Omega
\end{aligned} \tag{18}$$

The integrand here may be expressed as a 6-dimensional vector-matrix-vector product:

$$= \frac{r_b^2}{4}\oint\left(\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right)^\dagger\left(\begin{array}{cc}0 & \boldsymbol{\mathcal{P}}\\ -\boldsymbol{\mathcal{P}} & 0\end{array}\right)\left(\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right)d\Omega \tag{19}$$

where, in our shorthand,

$$\left(\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right)^\dagger\left(\begin{array}{cc}0 & \boldsymbol{\mathcal{P}}\\ -\boldsymbol{\mathcal{P}} & 0\end{array}\right)\left(\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right)\equiv\left(\begin{array}{c}E_r\\ E_\theta\\ E_\varphi\\ H_r\\ H_\theta\\ H_\varphi\end{array}\right)^\dagger\left(\begin{array}{cccccc}0&0&0&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ -1&0&0&0&0&0\end{array}\right)\left(\begin{array}{c}E_r\\ E_\theta\\ E_\varphi\\ H_r\\ H_\theta\\ H_\varphi\end{array}\right)$$

with

$$\boldsymbol{\mathcal{P}} = \left(\begin{array}{ccc}0&0&0\\ 0&0&-1\\ 0&1&0\end{array}\right).$$

If we now insert the expansions (13) and (14) into (19), we obtain the total radiated power as a bilinear form in the $\{C,D\}_{\ell m}$ coefficients:

## Force

The $i$th Cartesian component of the time-average force experienced by the sphere is obtained by integrating the time-average Maxwell stress tensor over a sphere with radius $r_b > r_0$ (call this sphere $\mathcal{S}_b$:)

$$F_x = \frac{1}{2}\text{Re}\;r_b^2\int T_{ij}(r_b,\Omega)\hat{n}_j(\Omega)d\Omega. \tag{20}$$

For definiteness I will consider the $x$-component of the force, $i = x$. The relevant quantity involving the stress tensor is

$$T_{xj}n_j = \epsilon_0 \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right)^\dagger \left(\begin{array}{ccc} \frac{1}{2}n_x & n_y & n_z \\ 0 & -\frac{1}{2}n_x & 0 \\ 0 & 0 & -\frac{1}{2}n_x \end{array}\right) \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right) \quad + \quad \mu_0(\mathbf{E} \to \mathbf{H})$$

where all fields are to be evaluated just outside the sphere surface. The time-average $x$-directed force per unit area is

$$\begin{aligned} f_x &= \frac{1}{2}\text{Re } T_{xj}n_j \\ &= \frac{1}{4}\Big[T_{xj}n_j + (T_{xj}n_j)^*\Big] \\ &= \frac{\epsilon_0}{4} \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right)^\dagger \left(\begin{array}{ccc} n_x & n_y & n_z \\ n_y & -n_x & 0 \\ n_z & 0 & -n_x \end{array}\right) \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right) \quad + \quad \mu_0(\mathbf{E} \to \mathbf{H}) \end{aligned}$$

which we may write in the shorthand form

$$f_x = \frac{\epsilon}{4}\mathbf{E}^{\text{C}\dagger}\boldsymbol{\mathcal{N}}_x\mathbf{E}^{\text{C}} + \frac{\mu}{4}\mathbf{H}^{\text{C}\dagger}\boldsymbol{\mathcal{N}}_x\mathbf{H}^{\text{C}} \tag{21}$$

where $\{\mathbf{E}, \mathbf{H}\}^{\text{C}}$ are three-vectors of cartesian field components (the superscript $C$ stands for "cartesian") and the $3 \times 3$ matrix $\boldsymbol{\mathcal{N}}_x$ is

$$\boldsymbol{\mathcal{N}}_x = \left(\begin{array}{ccc} n_x & n_y & n_z \\ n_y & -n_x & 0 \\ n_z & 0 & -n_x \end{array}\right) = \left(\begin{array}{ccc} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & 0 \\ \cos\theta & 0 & -\sin\theta\cos\phi \end{array}\right) \tag{22}$$

where the latter form is appropriate for points on the surface of a spherical bounding surface.

On the other hand, the Cartesian and spherical components of the field are related by

$$\left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right) = \left(\begin{array}{ccc} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{array}\right) \left(\begin{array}{c} E_r \\ E_\theta \\ E_\phi \end{array}\right) \tag{23}$$

or, in shorthand,

$$\mathbf{E}^{\text{C}} = \boldsymbol{\Lambda}\,\mathbf{E}^{\text{S}} \tag{24}$$

where $\boldsymbol{\Lambda}$ is the $3 \times 3$ matrix in equation (24). Inserting (23) into (21) yields

$$f_x = \frac{\epsilon_0}{4}\mathbf{E}^{\text{S}\dagger}\boldsymbol{\mathcal{F}}_x\mathbf{E}^{\text{S}} + \frac{\mu_0}{4}\mathbf{H}^{\text{S}\dagger}\boldsymbol{\mathcal{F}}_x\mathbf{H}^{\text{S}} \tag{25}$$

where $\{\mathbf{E}, \mathbf{H}\}^{\text{S}}$ are 3-dimensional vectors of spherical field components and $\boldsymbol{\mathcal{F}}_x$ is a product of three matrices:

$$\boldsymbol{\mathcal{F}}_x = \boldsymbol{\Lambda}^\dagger\,\boldsymbol{\mathcal{N}}_x\,\boldsymbol{\Lambda}.$$

Working out the matrix multiplications, one finds

$$\boldsymbol{\mathcal{F}}_x = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 \\ -\sin\phi & 0 & -\sin\theta\cos\phi \end{pmatrix} \tag{26}$$

and, proceeding similarly for the $y$- and $z$-directed force,

$$\boldsymbol{\mathcal{F}}_y = \begin{pmatrix} \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta\sin\phi & -\sin\theta\sin\phi & 0 \\ \cos\phi & 0 & -\sin\theta\sin\phi \end{pmatrix} \tag{27}$$

$$\boldsymbol{\mathcal{F}}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & -\cos\theta \end{pmatrix}. \tag{28}$$

If I now insert expressions (13) and (14) into equation (25), I obtain the $x$-directed force per unit area as a bilinear form in the $C, D$ coefficients:

$$\begin{aligned} f_x(\mathbf{x}) = \frac{\epsilon}{4}\sum_{\alpha\beta}\Bigg\{ &\Big(C_\alpha^* C_\beta + D_\alpha^* D_\beta\Big)\Big[\mathbf{M}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{M}_\beta(\mathbf{x}) + \mathbf{N}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{N}_\beta(\mathbf{x})\Big] \\ &+ \Big(C_\alpha^* D_\beta - D_\alpha^* C_\beta\Big)\Big[\mathbf{M}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{N}_\beta(\mathbf{x}) - \mathbf{N}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{M}_\beta(\mathbf{x})\Big]\Bigg\} \end{aligned} \tag{29}$$

The total $x$-directed force on the sphere is the surface integral of (29) over the full sphere $\mathcal{S}_b$:

$$\begin{aligned} F_x &= \oint_{\mathcal{S}_b} f_x(\mathbf{x})\,d\mathbf{x} \\ &= \frac{\epsilon}{4}\sum_{\alpha\beta}\Bigg\{\Big(C_\alpha^* C_\beta + D_\alpha^* D_\beta\Big)\Big[\big\langle\mathbf{M}_\alpha\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{M}_\beta\big\rangle + \big\langle\mathbf{N}_\alpha\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{N}_\beta\big\rangle\Big] \\ &\qquad + \Big(C_\alpha^* D_\beta - D_\alpha^* C_\beta\Big)\Big[\big\langle\mathbf{M}_\alpha\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{N}_\beta\big\rangle - \big\langle\mathbf{N}_\alpha\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{M}_\beta\big\rangle\Big]\Bigg\} \end{aligned} \tag{30}$$

where the inner products involve integrals over the radius-$r_b$ spherical bounding surface, i.e.

$$\big\langle\mathbf{M}_\alpha\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{M}_\beta\big\rangle = r_b^2\int \mathbf{M}_\alpha^\dagger(r_b,\Omega)\boldsymbol{\mathcal{F}}_x\mathbf{M}_\beta(r_b,\Omega)\,d\Omega. \tag{31}$$

# 10 A specific power example

As a specific example of a radiated-power computation, let's consider a pointlike dipole source at the center of a lossy sphere and ask for the total power radiated away from the sphere.

Assuming the dipole is $z$-directed, i.e. $\mathbf{p} = p_z\hat{\mathbf{z}}$, the only nonvanishing spherical multipole coefficient of the incident field [equation (10)] is

$$Q_{1,0} = \frac{-icZ_0(nk_0)^3}{\epsilon\sqrt{6\pi}}p_z$$

where $k_0 = \omega/c$ is the free-space wavenumber.

The total fields outside the sphere are

$$\mathbf{E}^{\rm tot}(\mathbf{x}) = D_{1,0}\mathbf{N}^{\rm out}_{1,0}(\mathbf{x}), \qquad \mathbf{H}^{\rm tot}(\mathbf{x}) = \frac{1}{Z_0}D_{1,0}\mathbf{M}^{\rm out}_{1,0}(\mathbf{x})$$

where $D_{1,0}$ is given by (15):

$$D_{1,0} = \frac{1}{\sqrt{6\pi}}\frac{cZ_0e^{-ia}n^3a^6p_z}{[n^2\left(a^3+a+i\right)-a-i]\sin(na)+na[in^2(-1+a(a+i))+a+i]\cos(na)} \tag{32}$$

where the dimensionless "size parameter" is

$$a = k_0R.$$

The radiated power is

$$\begin{aligned} P^{\rm rad} &= \frac{1}{2}\text{Re}\oint(\mathbf{E}^*\times\mathbf{H})\cdot\hat{\mathbf{r}}\,dA \\ &= \frac{|D_{1,0}|^2}{2Z_0}\cdot r^2\underbrace{\text{Re}\int\Big(\mathbf{N}^*_{1,0}(r,\Omega)\times\mathbf{M}_{1,0}(r,\Omega)\Big)\cdot\hat{\mathbf{r}}\,d\Omega}_{=1/(k_0r)^2} \\ &= \frac{|D_{1,0}|^2}{2k_0^2Z_0} \end{aligned} \tag{33}$$

## Sanity check

As a sanity check, let's first try putting $\epsilon = 1$. Then equation (32) reads

$$D_{1,0}(\epsilon=1) = \frac{-icZ_0k^3}{\sqrt{6\pi}}p_z$$

and equation (33) reads

$$P^{\rm rad} = \frac{c^2Z_0k^4}{12\pi}p_z^2$$

in agreement with Jackson equation (9.24).

## Nontrivial examples

As less trivial examples, consider putting **(a)** $\epsilon = 3$ and **(b)** $\epsilon = 3 + 6i$.

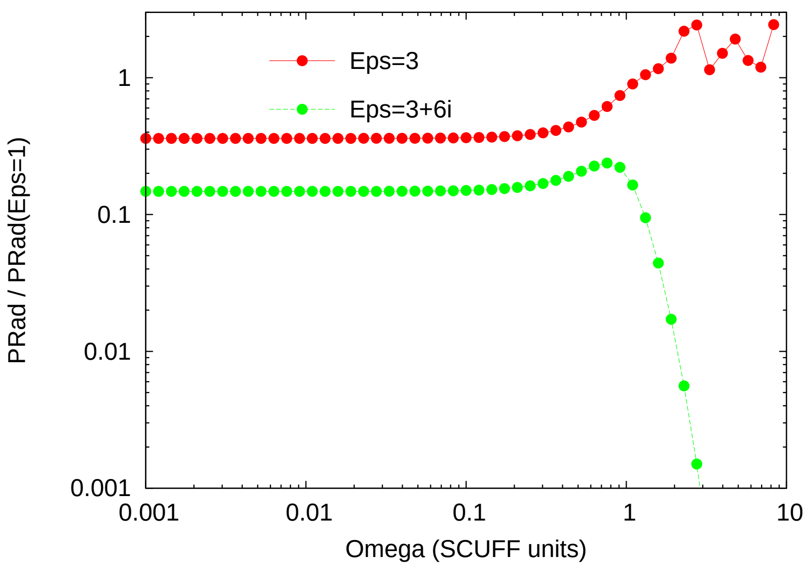


Figure 1: Power radiated by a dipole at the center of a dielectric sphere, normalized by the power radiated by a dipole in free space.

# 11 A specific force example

The simplest incident-field configuration that gives rise to a nonvanishing total force on the sphere is a superposition of $(1,0)$ and $(2,1)$ spherical waves, corresponding to coherent dipole and quadrupole sources at the origin. Thus, in the incident-field expansion (10) we take

$$P_{(1,0)} = P_{(2,1)} = 1, \qquad P_\alpha = 0 \text{ for all other } \alpha, \qquad Q_\alpha = 0 \text{ for all } \alpha. \tag{34}$$

The coefficients in expansions (13, 14) for the fields outside the sphere are then similarly given by

$$C_{(1,0)} = C_{(2,1)} = \text{nonzero}, \qquad C_\alpha = 0 \text{ for all other } \alpha, \qquad D_\alpha = 0 \text{ for all } \alpha. \tag{35}$$

The actual values of $C_{(1,0)}$ and $C_{(2,1)}$, which are less important for our immediate goals, are determined by equation (15) for a specific frequency, dielectric constant, and sphere radius. For example, for the particular case $\{\omega, r_0, \epsilon\} = \{3\cdot 10^{14} \text{ rad/sec}, 1\ \mu\text{m}, 10\}$ we find

$$C_{(1,0)} = -0.558 + 0.760i, \qquad C_{(2,1)} = 0.100 + 0.001i.$$

## 11.1 $x$-directed force density

At a point $\mathbf{x} = (r_b, \Omega)$ on the surface of the bounding sphere of radius $r_b$, the $x$-directed force per unit area is, from (25),

$$\begin{aligned} f_x(\mathbf{x}) = \frac{\epsilon}{4}\Big\{ & C_{10}^* C_{10}\Big[\mathbf{M}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{10}(\mathbf{x}) + \mathbf{N}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{10}(\mathbf{x})\Big] \\ & + C_{10}^* C_{21}\Big[\mathbf{M}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{21}(\mathbf{x}) + \mathbf{N}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{21}(\mathbf{x})\Big] \\ & + C_{21}^* C_{10}\Big[\mathbf{M}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{10}(\mathbf{x}) + \mathbf{N}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{10}(\mathbf{x})\Big] \\ & + C_{21}^* C_{21}\Big[\mathbf{M}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{21}(\mathbf{x}) + \mathbf{N}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{21}(\mathbf{x})\Big]\Big\} \end{aligned}$$

## 11.2 Total $x$-directed force

The *total* force is obtained from equation (2):

$$F_x = \frac{\epsilon_0}{4}\Big\{ C_{10}^* C_{21}\Big[\big\langle \mathbf{M}_{10}\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{M}_{21}\big\rangle + \big\langle \mathbf{N}_{10}\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{N}_{21}\big\rangle\Big] + \text{CC}\Big\} \tag{36}$$

where CC stands for "complex conjugate." [The inner product here is defined by equation (31).] With some effort, we compute

$$\big\langle \mathbf{M}_{10}\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{M}_{21}\big\rangle + \big\langle \mathbf{N}_{10}\big|\boldsymbol{\mathcal{F}}_x\big|\mathbf{N}_{21}\big\rangle = -i\sqrt{\frac{3}{10}}\frac{1}{k^2}$$

and thus the total force (36) reads

$$F_x = -\frac{\epsilon_0}{2k^2}\sqrt{\frac{3}{10}}\,\text{Im}\,\Big(C_{10}^{*}C_{21}\Big) \tag{37}$$

To make sense of the units here, suppose that field-strength coefficients like $P, Q, C, D$ in (10) and (13) are measured in typical SCUFF-EM units of V/$\mu$m, while $k$ is measured in units of inverse $\mu m$. Then the units of (37) are

$$\text{units of (37)} = \frac{\epsilon_0 \cdot \text{V}^2 \cdot \mu\text{m}^{-2}}{\mu\text{m}^{-2}}$$

Use $\epsilon_0 = \frac{1}{Z_0 c}$ where $c$ is the vacuum speed of light:

$$= \frac{1}{Z_0 c} \cdot \text{V}^2$$

Use $Z_0 = 376.7$ V/A:

$$= \frac{376.7 \text{ V} \cdot \text{A}}{3 \cdot 10^{14}\,\mu\text{m} \cdot \text{s}^{-1}}$$

Now use 1 V· A=1 watt, 1 watt · 1 s = 1 joule, 1 joule / 1 $\mu$m = $10^6$ Newtons:

$$= 1.26 \cdot 10^{-6} \text{ Newtons.}$$

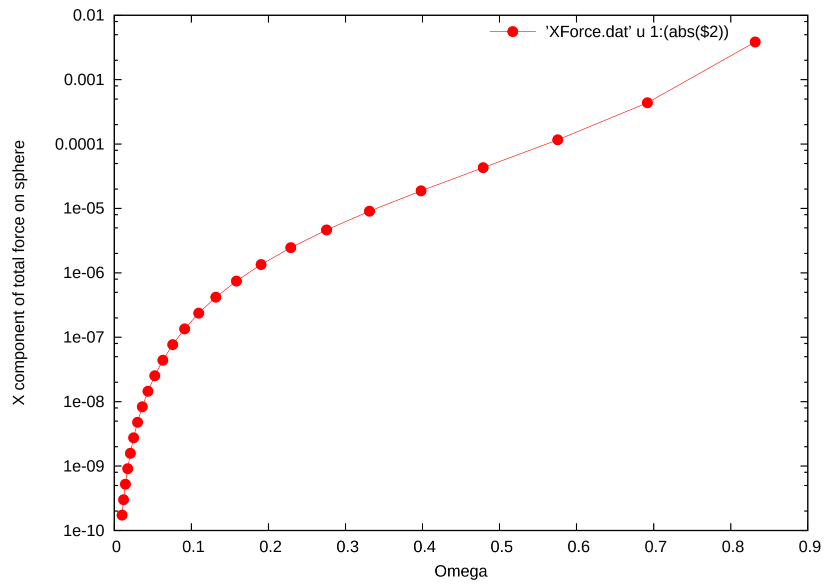


Figure 2: $x$-component of total force on sphere irradiated from within by an incident field of the form (10) with coefficients (35).

# 12 Surface-current approach to Mie scattering

It is interesting to ask how the well-known theory of Mie scattering emerges from the surface-integral-equation approach to scattering, in which the fundamental objects of interest are the equivalent electric and magnetic surface currents $\mathbf{K}(\mathbf{x}), \mathbf{N}(\mathbf{x})$ flowing on the sphere surface. These are defined in terms of the total fields at the object surface by

$$\mathbf{K} = \hat{\mathbf{r}} \times \mathbf{H}^{\text{tot}}, \qquad \mathbf{N} = -\hat{\mathbf{r}} \times \mathbf{E}^{\text{tot}}.$$

## 12.1 Expressions for interior and exterior fields

**Incident fields** In Mie scattering, the incident field is a $z$-traveling plane wave,

$$\mathbf{E}^{\text{inc}} = \mathbf{E}_0 e^{ik_0 z}, \qquad \mathbf{H}^{\text{inc}} = \frac{1}{Z_0}\hat{\mathbf{z}} \times \mathbf{E}^{\text{inc}}$$

with spherical-wave expansion

$$\begin{aligned}
\mathbf{E}^{\text{inc}} &= \sum_\alpha \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{r}) + Q_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{r}) \Big\} \\
\mathbf{H}^{\text{inc}} &= \frac{1}{Z_0} \sum_\alpha \Big\{ -P_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{r}) + Q_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{r}) \Big\}
\end{aligned}$$

where $\mathbf{E}_0$ and the $\{P, Q\}_\alpha$ coefficients depend on the polarization:

**Right-circular polarization**

| | | |
|---|---|---|
| **Right circular polarization** | $\mathbf{E}_0 = \hat{\mathbf{x}} + i\hat{\mathbf{y}}$, | $P_\alpha = \delta_{m,+1} P_\ell$ |
| **Left circular polarization** | $\mathbf{E}_0 = \hat{\mathbf{x}} - i\hat{\mathbf{y}}$, | $P_\alpha = \delta_{m,-1} P_\ell$ |
| **Linear polarization** | $\mathbf{E}_0 = \hat{\mathbf{x}}$ | $P_\alpha = \frac{1}{2}\Big(\delta_{m+1} + \delta_{m-1}\Big) P_\ell$ |

In all cases I have

$$P_\ell = i^l \sqrt{4\pi(2\ell+1)}, \qquad Q_{\ell,\pm 1} = \mp i P_\ell.$$

**Matching equations** Using the expansions (5) and (6) for the scattered fields, the equations matching tangential field components at the surface read

$$\left(\sum_\alpha P_\alpha \mathbf{M}_\alpha^{\text{reg}}(k_0;\mathbf{r}) + C_\alpha \mathbf{M}_\alpha^{\text{out}}(k_0;\mathbf{r}) + Q_\alpha \mathbf{N}_\alpha^{\text{reg}}(k_0;\mathbf{r}) + D_\alpha \mathbf{N}_\alpha^{\text{out}}(k_0;\mathbf{r})\right)_\parallel$$
$$= \left(\sum_\alpha \Big\{A_\alpha \mathbf{M}_\alpha^{\text{reg}}(nk_0;\mathbf{r}) + B_\alpha \mathbf{N}_\alpha^{\text{reg}}(nk_0;\mathbf{r})\Big\}\right)_\parallel \tag{38a}$$

$$\left(\sum_\alpha P_\alpha \mathbf{N}_\alpha^{\text{reg}}(k_0;\mathbf{r}) + C_\alpha \mathbf{N}_\alpha^{\text{out}}(k_0;\mathbf{r}) - Q_\alpha \mathbf{M}_\alpha^{\text{reg}}(k_0;\mathbf{r}) - D_\alpha \mathbf{M}_\alpha^{\text{out}}(k_0;\mathbf{r})\right)_\parallel$$
$$= \frac{1}{Z^r}\left(\sum_\alpha \Big\{A_\alpha \mathbf{N}_\alpha^{\text{reg}}(nk_0;\mathbf{r}) - B_\alpha \mathbf{M}_\alpha^{\text{reg}}(nk_0;\mathbf{r})\Big\}\right)_\parallel \tag{38b}$$

Testing equations (38) with $\mathbf{X}(\Omega)$ and $\mathbf{Z}(\Omega)$ yields

$$\begin{aligned} P_\alpha R_\alpha^{\text{reg}}(a) + C_\alpha R_\alpha^{\text{out}}(a) &= A_\alpha R_\alpha^{\text{reg}}(na) \\ P_\alpha \overline{R}_\alpha^{\text{reg}}(a) + C_\alpha \overline{R}_\alpha^{\text{out}}(a) &= \frac{1}{Z^r} A_\alpha \overline{R}_\alpha^{\text{reg}}(na) \end{aligned}$$

$$\begin{aligned} Q_\alpha \overline{R}_\alpha^{\text{reg}}(a) + D_\alpha \overline{R}_\alpha^{\text{out}}(a) &= B_\alpha \overline{R}_\alpha^{\text{reg}}(na) \\ Q_\alpha R_\alpha^{\text{reg}}(a) + D_\alpha R_\alpha^{\text{out}}(a) &= \frac{1}{Z^r} B_\alpha R_\alpha^{\text{reg}}(na) \end{aligned}$$

## 12.2 Exterior fields

Solving (39) for the $C$ and $D$ coefficients yields

$$\frac{C_\alpha}{P_\alpha} = -\frac{R^{\text{reg}}(a)\overline{R}^{\text{reg}}(na) - Z^r\overline{R}^{\text{reg}}(a)R^{\text{reg}}(na)}{R^{\text{out}}(a)\overline{R}^{\text{reg}}(na) - Z^r\overline{R}^{\text{out}}(a)R^{\text{reg}}(na)}$$

$$\frac{D_\alpha}{Q_\alpha} = -\frac{\overline{R}^{\text{reg}}(a)R^{\text{reg}}(na) - Z^r R^{\text{reg}}(a)\overline{R}^{\text{reg}}(na)}{\overline{R}^{\text{out}}(a)R^{\text{reg}}(na) - Z^r R^{\text{out}}(a)\overline{R}^{\text{reg}}(na)}$$

Including all terms up to second order in $a$, the total fields outside the body are

$$\frac{E_x}{E_0} =$$

## 12.3 Interior fields

Including all terms up to second order in $a$, the total fields inside the body are

$$
\begin{aligned}
\frac{E_x}{E_0} &= \frac{3}{2+\epsilon} + \left[\frac{\epsilon+4}{3+2\epsilon}\right] ikz + \left[\frac{(\epsilon-1)(35\epsilon+46)}{5(\epsilon+2)^2(3\epsilon+4)}\right] k^2x^2 \\
&\qquad + \left[\frac{(\epsilon-1)(-2\epsilon^2+29\epsilon+42)}{5(\epsilon+2)^2(3\epsilon+4)}\right] k^2y^2 - \left[\frac{14\epsilon^3+3\epsilon^2+114\epsilon+184}{10(\epsilon+2)^2(3\epsilon+4)}\right] k^2z^2 \\
\frac{E_y}{E_0} &= \left[\frac{2(\epsilon^2-1)}{5(2+\epsilon)(4+3\epsilon)}\right] k^2xy \\
\frac{E_z}{E_0} &= -\left[\frac{\epsilon-1}{2\epsilon+3}\right] ikx + \left[\frac{(\epsilon-1)(7\epsilon+12)}{5(2+\epsilon)(4+3\epsilon)}\right] k^2xz \\
\frac{H_x}{Z_0E_0} &= \left[\frac{(\epsilon-1)^2}{5(2\epsilon+3)}\right] k^2xy \\
\frac{H_y}{Z_0E_0} &= 1 + \left[\frac{2\epsilon+1}{2+\epsilon}\right] ikz \\
&\qquad - \left[\frac{(\epsilon-1)(\epsilon-6)}{15(3+2\epsilon)}\right] k^2x^2 + \left[\frac{\epsilon-1}{15}\right] k^2y^2 - \left[\frac{2\epsilon^2+46\epsilon+27}{30(3+2\epsilon)}\right] k^2z^2 \\
\frac{H_z}{Z_0E_0} &= -\left[\frac{\epsilon-1}{\epsilon+2}\right] iky + \left[\frac{(\epsilon-1)(\epsilon+4)}{5(3+2\epsilon)}\right] k^2yz
\end{aligned}
$$

Field derivatives:

$$
\begin{aligned}
\partial_z\mathbf{E} &= \Big(ikC_1 - 2k^2C_2z\Big)\hat{\mathbf{x}} + k^2C_3x\,\hat{\mathbf{z}} \\
\partial_z\mathbf{H} &= \Big(ikC_4 - 2k^2C_5z\Big)\hat{\mathbf{y}} + k^2C_6y\,\hat{\mathbf{z}}
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \frac{\epsilon+4}{3+2\epsilon}, \\
C_2 &= \frac{14\epsilon^3+3\epsilon^2+114\epsilon+184}{10(\epsilon+2)^2(3\epsilon+4)} \\
C_3 &= \frac{(\epsilon-1)(7\epsilon+12)}{5(2+\epsilon)(4+3\epsilon)} \\
C_4 &= \frac{2\epsilon+1}{2+\epsilon} \\
C_5 &= \frac{2\epsilon^2+46\epsilon+27}{30(3+2\epsilon)} \\
C_6 &= \frac{(\epsilon-1)(\epsilon+4)}{5(3+2\epsilon)}
\end{aligned}
$$

**Volume integral for the absorbed power:**

$$\begin{aligned} P^{\mathrm{abs}} &= \frac{1}{2}\,\mathrm{Re}\,\int \mathbf{J}^*\cdot\mathbf{E}\,dV \\ &= -\frac{\omega\,\mathrm{Im}\,\epsilon}{2}\int |\mathbf{E}|^2 dV \end{aligned}$$

Retaining terms up to 1st order in $k$ and neglecting terms whose volume integral vanishes, I find

$$|\mathbf{E}|^2 = \frac{9}{|2+\epsilon|^2} + O(k^2)$$

The volume integral reads

$$P^{\mathrm{abs}} = \frac{12\pi\omega\mathrm{Im}\,\epsilon}{|2+\epsilon|^2} + O(k^3)$$

**Volume integral for the $z$-directed force:**

$$\begin{aligned} F_z &= \frac{1}{2\omega}\,\mathrm{Im}\,\int \mathbf{J}^*\cdot\partial_z\mathbf{E}\,dV \\ &= -\frac{\mathrm{Im}\,\epsilon}{2}\int_V \mathrm{Im}\,\Big\{\mathbf{E}^*\cdot\partial_z\mathbf{E}\Big\}dV \end{aligned}$$

Neglecting terms of odd total order in $xyz$, from the above I have

$$\mathbf{E}^*\cdot\partial_z\mathbf{E} = \Big[\zeta_{x0} + \zeta_{xz}ikz + \zeta_{kx2}k^2x^2 + \zeta_{ky2}k^2y^2 + \zeta_{kz2}k^2z^2\Big]^*\Big[\zeta_{xz}ik + 2k^2\zeta_{kz2}z\Big]$$

so

Im $\mathbf{E}^*\cdot\partial_z\mathbf{E} =$

Whoops! This turns out to be too complicated too work out by hand.