Electromagnetism in the Spherical-Wave Basis:

A (Somewhat Random) Compendium of Reference Formulas

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August 1, 2016

Abstract

This memo consolidates and collects for reference a somewhat random hodgepodge of formulas and results in the spherical-wave approach to electromagnetism that I have found useful over the years in developing and testing SCUFF-EM and BUFF-EM.

Contents

1 Vector Spherical Wave Solutions to Maxwell's Equations

Many authors define pairs of three-vector-valued functions $\{\mathbf{M}_{\ell m}(\mathbf{x}), \mathbf{N}_{\ell m}(\mathbf{x})\}$ describing exact solutions of the source-free Maxwell's equations—namely, the vector Helmholtz equation plus the divergence-free condition—in spherical coordinates for a homogeneous medium with wavenumber k, i.e.

$$\Big[\nabla\times\nabla\times -k^2\Big]\left\{\begin{array}{c}\mathbf{M}_{\ell m}\\ \mathbf{N}_{\ell m}\end{array}\right\}=0, \qquad \nabla\cdot\left\{\begin{array}{c}\mathbf{M}_{\ell m}\\ \mathbf{N}_{\ell m}\end{array}\right\}=0. \tag{1}$$

In some cases, the set $\{\mathbf{M},\mathbf{N}\}_{\ell m}$ is augmented to include a third function $\{\mathbf{L}_{\ell m}\}$ that satisfies the vector Helmholtz equation but is now *curl*-free (and not divergenceless):

$$\Big[\nabla\times\nabla\times -k^2\Big]\mathbf{L}_{\ell m}=0, \qquad \nabla\times\mathbf{L}_{\ell m}=0. \tag{2}$$

The function $\mathbf{L}_{\ell m}$ is not a solution of Maxwell's equations and is never needed in a basis of expansion functions for *fields*, but must be retained in a basis for expanding *currents* in inhomogeneous and/or anisotropic media.

In all cases, the $\{\mathbf{M},\mathbf{N},\mathbf{L}\}$ functions involve spherical Bessel functions and spherical harmonics, but the precise definitions (including sign conventions and normalization factors) vary from author to author. In this section I set down the particular conventions that I use. In the next section I give explicit closed-form expressions for small ℓ.

Vector spherical harmonics

$$\begin{aligned}\mathbf{X}_{\ell m}(\theta,\varphi) &\equiv \frac{i}{\sqrt{\ell(\ell+1)}}\nabla\times\Big\{Y_{\ell m}(\theta,\varphi)\mathbf{r}\Big\}\\ \mathbf{Z}_{\ell m}(\theta,\varphi) &\equiv \hat{\mathbf{r}}\times\mathbf{X}_{\ell m}(\theta,\varphi)\end{aligned}$$

More explicitly, the components of $\mathbf{X}$ and $\mathbf{Z}$ are

$$\begin{aligned}\mathbf{X}_{\ell m}(\theta,\phi) &= \frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{im}{\sin\theta}Y_{\ell m}\widehat{\boldsymbol{\theta}} - \frac{\partial Y_{\ell m}}{\partial\theta}\widehat{\boldsymbol{\varphi}}\right]\\ \mathbf{Z}_{\ell m}(\theta,\phi) &= \frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{\partial Y_{\ell m}}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{im}{\sin\theta}Y_{\ell m}\widehat{\boldsymbol{\varphi}}\right].\end{aligned}$$

These are orthonormal in the sense that

$$\big\langle\mathbf{X}\big|\mathbf{X}\big\rangle=\big\langle\mathbf{Z}\big|\mathbf{Z}\big\rangle=1, \quad \big\langle\mathbf{X}\big|\mathbf{Z}\big\rangle=0$$

where the inner product is

$$\big\langle\mathbf{F}\big|\mathbf{G}\big\rangle \equiv \int \mathbf{F}^*\cdot\mathbf{G}\,d\Omega = \int_0^\pi\int_0^{2\pi}\mathbf{F}^*(\theta,\varphi)\cdot\mathbf{G}(\theta,\varphi)\,\sin\theta\,d\varphi\,d\theta$$

Their divergences are:

$$\nabla \cdot \mathbf{X}_{\ell m} = -\frac{m \cot\theta \csc\theta Y_{\ell m}(\theta,\varphi)}{r\sqrt{\ell(\ell+1)}} \tag{3a}$$

$$\nabla \cdot \mathbf{Z}_{\ell m} = \frac{i \cot\theta}{r\sqrt{\ell(\ell+1)}}\Big[m \cot\theta Y_{\ell m}(\theta,\varphi) + \xi_{\ell m} e^{-i\varphi} Y_{\ell,m+1}(\theta,\varphi)\Big] \tag{3b}$$

$$\xi_{\ell m} \equiv \sqrt{\frac{(\ell-m)!(\ell+m+1)!}{(\ell-m-1)!(\ell+m)!}} \tag{3c}$$

Radial functions

$$\begin{aligned} R_\ell^{\text{outgoing}}(kr) &\equiv h_\ell^{(1)}(kr) \\ R_\ell^{\text{incoming}}(kr) &\equiv h_\ell^{(2)}(kr) \\ R_\ell^{\text{regular}}(kr) &\equiv j_\ell(kr). \end{aligned}$$

I also define the shorthand symbols

$$\overline{R}_\ell(kr) \equiv \frac{1}{kr} R_\ell(kr) + R'_\ell(kr) \qquad \not{R}_\ell(kr) \equiv -\frac{\sqrt{l(l+1)}}{kr} R_\ell(kr)$$

where $R'_\ell(kr) = \left|\frac{d}{dz} R_\ell(z)\right|_{z=kr}$.

Scalar Helmholtz solutions

$$\Big[\nabla^2 + k^2\Big]\psi_{\ell m}(\mathbf{r}) = 0 \qquad \Longrightarrow \qquad \psi_{\ell m}(r,\theta,\varphi) = R_\ell(kr) Y_{\ell m}(\theta,\varphi)$$

where R_ℓ is one of the radial functions defined above.

Vector spherical wave functions

$$\begin{aligned} \mathbf{M}_{\ell m}(k;\mathbf{r}) &\equiv \frac{i}{\sqrt{\ell(\ell+1)}}\nabla\Big\{\psi_{\ell m}\mathbf{r}\Big\} &&= R_\ell(kr)\mathbf{X}_{\ell m}(\Omega) \\ \mathbf{N}_{\ell m}(k;\mathbf{r}) &\equiv -\frac{1}{ik}\nabla\times\mathbf{M}_{\ell m} &&= i\overline{R}_\ell(kr)\mathbf{Z}_{\ell m}(\Omega) + \not{R}_\ell(kr) Y_{\ell m}(\Omega)\hat{\mathbf{r}} \\ \mathbf{L}_{\ell m}(k;\mathbf{r}) &\equiv \frac{1}{k\sqrt{\ell(\ell+1)}}\nabla\psi_{\ell m} &&= -i\frac{R_\ell(kr)}{kr}\mathbf{Z}_{\ell m}(\Omega) + \frac{1}{\sqrt{\ell(\ell+1)}} R'_\ell(kr) Y_{\ell m}(\Omega)\hat{\mathbf{r}} \end{aligned} \tag{4}$$

$$\mathbf{L}_{00}(k;\mathbf{r}) = \frac{R'_0(kr)}{\sqrt{4\pi}}\hat{\mathbf{r}}$$

Curl Identities

$$\nabla\times\mathbf{M} = -ik\mathbf{N}, \qquad \nabla\times\mathbf{N} = +ik\mathbf{M}.$$

General solution of source-free Maxwell equations The general solution of Maxwell's equations in a source-free medium with relative material properties ϵ^r, μ^r then reads

$$\mathbf{E}(\mathbf{x}) = \sum_\alpha \Big\{ \mathbf{A}_\alpha \mathbf{M}_\alpha(k;\mathbf{r}) + \mathbf{B}_\alpha \mathbf{N}_\alpha(k;\mathbf{r}) \Big\} \tag{5a}$$

$$\mathbf{H}(\mathbf{x}) = \frac{1}{Z_0 Z^r} \sum_\alpha \Big\{ \mathbf{B}_\alpha \mathbf{M}_\alpha(k;\mathbf{r}) - \mathbf{A}_\alpha \mathbf{N}_\alpha(k;\mathbf{r}) \Big\} \tag{5b}$$

where $k = \sqrt{\epsilon_0 \epsilon^r \mu_0 \mu^r} \cdot \omega$ is the photon wavenumber in the medium, $Z_0 = \sqrt{\mu_0/\epsilon_0} \sim 377\,\Omega$ is the impedance of vacuum, $Z^r = \sqrt{\mu^r/\epsilon^r}$ is the relative wave impedance of the medium, and we must choose the $\mathbf{M}, \mathbf{N}$ functions to be regular, incoming, or outgoing depending on the physical conditions of the problem.

Unified notation for $\mathbf{M}, \mathbf{N}$ waves I will use the symbol $\boldsymbol{\mathcal{W}}_\alpha$ to refer collectively to $\mathbf{M}$ and $\mathbf{N}$ waves; here[1] $\alpha = (\ell m P)$ is a compound index with $P = \{M, N\}$ identifying the polarization. With this notation, expansions such as (5) read

$$\mathbf{E} = \sum_\alpha C_\alpha \boldsymbol{\mathcal{W}}_\alpha, \qquad \mathbf{H} = \frac{1}{Z_0 Z^r} \sum_\alpha C_\alpha \sigma_\alpha \boldsymbol{\mathcal{W}}_{\overline{\alpha}} \tag{6}$$

where

$$\sum_\alpha \{\cdots\} = \sum_\ell \sum_{m=-\ell}^{\ell} \sum_{P \in \{M,N\}} \{\cdots\}$$

and

$$\overline{\ell m M} = \ell m N, \qquad \overline{\ell m N} = \ell m M, \qquad \sigma_{\ell m M} = +1, \qquad \sigma_{\ell m N} = -1.$$

Spherical-wave expansion of dyadic Green's function Let $\mathbb{G}(k;\mathbf{r})$ and $\mathbb{C}(k;\mathbf{r})$ be the usual homogeneous dyadic Green's functions, with Cartesian components

$$\mathbb{G}_{ij}(k;\mathbf{r}) = \Big(\delta_{ij} + \frac{1}{k^2}\partial_i \partial_j\Big) \frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|}, \qquad \mathbb{C}_{ij}(k,\mathbf{r}) = +\frac{1}{ik}\varepsilon_{ijl}\partial_l \mathbb{G}(k,\mathbf{r}) \tag{7}$$

Then have the spherical-wave expansion

$$\mathbb{G}(\mathbf{x},\mathbf{x}') = -\frac{1}{k^2}\delta(\mathbf{x}-\mathbf{x}')\hat{\mathbf{r}}\hat{\mathbf{r}}' + ik \sum_\alpha \boldsymbol{\mathcal{W}}^{\text{out}}_\alpha(\mathbf{x}_>) \boldsymbol{\mathcal{W}}^{\text{reg}}_\alpha(\mathbf{x}_<), \tag{8a}$$

$$\mathbb{C}(\mathbf{x},\mathbf{x}') = ik \sum_\alpha \sigma_\alpha \begin{cases} \boldsymbol{\mathcal{W}}^{\text{out}}_{\overline{\alpha}}(\mathbf{x}) \boldsymbol{\mathcal{W}}^{\text{reg}}_\alpha(\mathbf{x}'), & |\mathbf{x}| > |\mathbf{x}'|, \\ \boldsymbol{\mathcal{W}}^{\text{out}}_{\overline{\alpha}}(\mathbf{x}') \boldsymbol{\mathcal{W}}^{\text{reg}}_\alpha(\mathbf{x}), & |\mathbf{x}| < |\mathbf{x}'| \end{cases} \tag{8b}$$

[1] With this notation I am committing the faux pas of using the same symbol α for the two-fold compound index (ℓm) in (5) and the three-fold compound index $(\ell m P)$ in (6), but *whaddya gonna do.*

2 Explicit expression for small ℓ

The first few radial functions

$$\begin{aligned}
R_0^{\text{regular}}(x) &= \frac{\sin x}{x} & \overline{R}_0^{\text{regular}}(x) &= \frac{\cos x}{x} \\
R_0^{\text{outgoing}}(x) &= -i\frac{e^{ix}}{x} & \overline{R}_0^{\text{outgoing}}(x) &= \frac{e^{ix}}{x} \\
R_1^{\text{regular}}(x) &= \frac{\sin x - x\cos x}{x^2} & \overline{R}_1^{\text{regular}}(x) &= \frac{x\cos x + (x^2-1)\sin x}{x^3} \\
R_1^{\text{outgoing}}(x) &= -\frac{ie^{ix}}{x^2}\big(1 - ix\big) & \overline{R}_1^{\text{outgoing}}(x) &= \frac{ie^{ix}}{x^3}\big(1 - ix - x^2\big)
\end{aligned}$$

The first few regular functions In what follows, the ζ_n are dimensionless sinusoidal functions:

$$\zeta_1(x) = \sin x - x\cos x$$
$$\zeta_2(x) = (1 - x^2)\sin x - x\cos x$$

$$\mathbf{L}_{00}^{\text{regular}}(\mathbf{r}) = -\frac{\zeta_1(kr)}{4\pi(kr)^2}\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right)$$

$$\xrightarrow{k\to 0} \frac{kr}{6\sqrt{\pi}}\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right)$$

$$\mathbf{M}_{1,\pm 1}^{\text{regular}}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left[\frac{\zeta_1(kr)}{(kr)^2}\right]e^{\pm i\varphi}\left(\begin{array}{c} 0 \\ 1 \\ \pm i\cos\theta \end{array}\right)$$

$$\xrightarrow{k\to 0} \frac{kr}{4\sqrt{3\pi}}e^{\pm i\varphi}\left(\begin{array}{c} 0 \\ 1 \\ \pm i\cos\theta \end{array}\right)$$

$$\mathbf{M}_{1,0}^{\text{regular}}(\mathbf{r}) = i\sqrt{\frac{3}{2\pi}}\left(\frac{\sin kr - kr\cos kr}{(kr)^2}\right)\left(\begin{array}{c} 0 \\ 0 \\ \sin\theta \end{array}\right)$$

$$\xrightarrow{k\to 0} i\sqrt{\frac{3}{8\pi}}\left[\frac{\zeta(kr)}{(kr)^2}\right]\left(\begin{array}{c} 0 \\ 0 \\ \sin\theta \end{array}\right)$$

$$\mathbf{N}_{1,\pm 1}^{\text{regular}}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left[\frac{1}{(kr)^3}\right]e^{\pm i\varphi}\left(\begin{array}{c} \pm 2\zeta_1(kr)\sin\theta \\ \mp\zeta_2(kr)\cos\theta \\ -i\zeta_2(kr) \end{array}\right)$$

$$\xrightarrow{k\to 0} \sqrt{\frac{e^{\pm i\varphi}}{12\pi}}\left(\begin{array}{c} \pm\sin\theta \\ \pm\cos\theta \\ i \end{array}\right)$$

$$\mathbf{N}_{1,0}^{\text{regular}}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left[\frac{1}{(kr)^3}\right]\left(\begin{array}{c} -2\zeta_1(kr)\cos\theta \\ -\zeta_2(kr)\sin\theta \\ 0 \end{array}\right)$$

$$\xrightarrow{k\to 0} \frac{1}{\sqrt{6\pi}}\left(\begin{array}{c} -\cos\theta \\ \sin\theta \\ 0 \end{array}\right) = -\frac{1}{\sqrt{6\pi}}\hat{\mathbf{z}}$$

The first few outgoing functions In what follows, the Q_n are dimensionless polynomial factors:

$$\begin{aligned}
Q_1(x) &= 1 - x \\
Q_{2a}(x) &= 1 - x + x^2 \\
Q_{2b}(x) &= 3 - 3x + x^2 \\
Q_3(x) &= 6 - 6x + 3x^2 - x^3
\end{aligned}$$

$$\mathbf{M}^{\text{outgoing}}_{1,\pm1}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left(\frac{e^{ikr}}{k^2r^2}\right)e^{\pm i\phi}\begin{pmatrix} 0 \\ -iQ_1(ikr) \\ \pm Q_1(ikr)\cos\theta \end{pmatrix}$$

$$\mathbf{M}^{\text{outgoing}}_{1,0}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left(\frac{e^{ikr}}{k^2r^2}\right)\begin{pmatrix} 0 \\ 0 \\ Q_1(ikr)\sin\theta \end{pmatrix}$$

$$\mathbf{N}^{\text{outgoing}}_{1,\pm1}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm i\phi}\begin{pmatrix} \mp -2(ikr)Q_1(ikr)\sin\theta \\ \pm iQ_{2a}(ikr)\cos\theta \\ -Q_{2a}(ikr) \end{pmatrix}$$

$$\mathbf{N}^{\text{outgoing}}_{1,0}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)\begin{pmatrix} 2iQ_1(ikr)\cos\theta \\ +iQ_{2a}(ikr)\sin\theta \\ 0 \end{pmatrix}$$

$$\mathbf{M}^{\text{outgoing}}_{2,\pm2}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm 2i\phi}\begin{pmatrix} 0 \\ \pm iQ_{2b}(ikr)\sin\theta \\ -Q_{2b}(ikr)\cos\theta\sin\theta \end{pmatrix}$$

$$\mathbf{M}^{\text{outgoing}}_{2,\pm1}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm i\phi}\begin{pmatrix} 0 \\ -iQ_{2b}(ikr)\cos\theta \\ \pm Q_{2b}(ikr)\cos 2\theta \end{pmatrix}$$

$$\mathbf{M}^{\text{outgoing}}_{2,0}(\mathbf{r}) = \sqrt{\frac{15}{8\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)\begin{pmatrix} 0 \\ 0 \\ -Q_{2b}(ikr)\cos\theta\sin\theta \end{pmatrix}$$

$$\mathbf{N}^{\text{outgoing}}_{2,\pm2}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)e^{\pm 2i\phi}\begin{pmatrix} 3iQ_{2b}(ikr)\sin^2\theta \\ -iQ_3(ikr)\cos\theta\sin\theta \\ \pm Q_3(ikr)\sin\theta \end{pmatrix}$$

$$\mathbf{N}^{\text{outgoing}}_{2,\pm1}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)e^{\pm i\phi}\begin{pmatrix} \mp 3iQ_{2b}(ikr)\sin 2\theta \\ \pm iQ_3(ikr)\cos 2\theta \\ -Q_3(ikr)\cos\theta \end{pmatrix}$$

$$\mathbf{N}^{\text{outgoing}}_{2,0}(\mathbf{r}) = \sqrt{\frac{15}{8\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)\begin{pmatrix} iQ_{2b}(ikr)(3\cos^2\theta - 1) \\ iQ_3(ikr)\cos\theta\sin\theta \\ 0 \end{pmatrix}.$$

3 Translation matrices

Translation matrices arise when we want to evaluate the fields produced by sources not located at the origin.

Scalar case Although we don't need it for electromagnetism problems, the scalar-wave analog of (4) is

$$\psi_{\ell m}(\mathbf{x}) = R_\ell(kr)Y_{\ell m}(\theta,\phi)$$

or, more specifically,

$$\psi^{\text{out}}_{\ell m}(\mathbf{x}) = R^{\text{out}}_\ell(kr)Y_{\ell m}(\theta,\phi), \qquad \psi^{\text{reg}}_{\ell m}(\mathbf{x}) = R^{\text{reg}}_\ell(kr)Y_{\ell m}(\theta,\phi)$$

Now consider a point source at $\mathbf{x}^{\text{S}}$ whose fields we wish to evaluate at an evaluation ("destination") point $\mathbf{x}^{\text{D}}$, using a basis of spherical waves centered at an origin $\mathbf{x}^{\text{O}}$. Then waves emitted by the source, which appear to be outgoing in a coordinate system centered at $\mathbf{x}^{\text{S}}$, can be described as superpositions of regular waves in a coordinate system centered at $\mathbf{x}^{\text{O}}$:

$$\psi^{\text{out}}_\alpha\big(\mathbf{x}^{\text{D}} - \mathbf{x}^{\text{S}}\big) = \sum_\beta A_{\alpha\beta}\big(k;\mathbf{x}^{\text{S}} - \mathbf{x}^{\text{O}}\big)\psi^{\text{reg}}_\beta\big(\mathbf{x}^{\text{D}} - \mathbf{x}^{\text{O}}\big) \tag{9}$$

where α,β are compound indices (i.e. $\alpha = \{\ell_\alpha, m_\alpha\}$) and

$$\begin{aligned}
A_{\alpha\beta}(k,\mathbf{L}) &= 4\pi\sum_\gamma i^{(\ell_\alpha-\ell_\beta+\ell_\gamma)}a_{\alpha\gamma\beta}\psi^{\text{out}}_\gamma(\mathbf{L})\\
a_{\alpha\beta\gamma} &= \int Y_\alpha(\Omega)Y^*_\beta(\Omega)Y^*_\gamma(\Omega)\,d\Omega\\
&= (-1)^{m_\alpha}\sqrt{\frac{(2\ell_\alpha+1)(2\ell_\beta+1)(2\ell_\gamma+1)}{4\pi}}\begin{pmatrix}\ell_\alpha & \ell_\beta & \ell_\gamma\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell_\alpha & \ell_\beta & \ell_\gamma\\ -m_\alpha & m_\beta & m_\gamma\end{pmatrix}.
\end{aligned}$$

Vector case

$$\begin{pmatrix}\mathbf{M}\\ \mathbf{N}\end{pmatrix}^{\text{out}}_\alpha = \sum_\beta \begin{pmatrix}\mathbf{B} & \mathbf{C}\\ -\mathbf{C} & \mathbf{B}\end{pmatrix}_{\alpha\beta}\begin{pmatrix}\mathbf{M}\\ \mathbf{N}\end{pmatrix}^{\text{reg}}_\beta$$

$$\begin{aligned}
B_{\alpha\beta}(k,\mathbf{L}) &= 4\pi\sum_\gamma i^{(\ell_\alpha-\ell_\beta+\ell_\gamma)}\left[\frac{\ell_\alpha(\ell_\alpha+1)+\ell_\beta(\ell_\beta+1)-\ell_\gamma(\ell_\gamma+1)}{2\sqrt{\ell_\alpha(\ell_\alpha+1)\ell_\beta(\ell_\beta+1)}}\right]a_{\alpha\gamma\beta}\psi^{\text{out}}_\gamma(\mathbf{L})\\
C_{\alpha\beta}(k,\mathbf{L}) &= -\frac{k}{\sqrt{\ell_\alpha(\ell_\alpha+1)\ell_\beta(\ell_\beta+1)}}\left[\frac{\lambda_+}{2}\big(L_x - iL_y\big)A_{\alpha_+,\beta} + \frac{\lambda_-}{2}\big(L_x + iL_y\big)A_{\alpha_-,\beta} + m_\alpha L_z A_{\alpha,\beta}\right]
\end{aligned}$$

$$\lambda_\pm = \sqrt{(\ell_\alpha \mp \ell_\beta)(\ell_\alpha \pm \ell_\beta + 1)}, \qquad \alpha_\pm = \{\ell_\alpha, m_\alpha \pm 1\}$$

4 Spherical-wave expansion of incident fields

4.1 Plane waves

For a scattering problem in which the incident field is a z-directed plane wave, i.e.

$$\mathbf{E}^{\text{inc}} = \mathbf{E}_0 e^{ikz}, \qquad \mathbf{H}^{\text{inc}} = \frac{1}{Z_0}\hat{\mathbf{z}} \times \mathbf{E}_0 e^{ikz}$$

the spherical-wave expansion coefficients in (14) take the following forms for various possible polarizations:

$$\begin{array}{rclll}
\mathbf{E}_0 & = & \hat{\mathbf{x}} + i\hat{\mathbf{y}} & \text{(right circular polarization)}: & P_{\ell m} = \delta_{m,+1} P_\ell \\
\mathbf{E}_0 & = & \hat{\mathbf{x}} - i\hat{\mathbf{y}} & \text{(left circular polarization)}: & P_{\ell m} = \delta_{m,-1} P_\ell \\
\mathbf{E}_0 & = & \hat{\mathbf{x}} & \text{(linear polarization)}: & P_{\ell m} = \frac{1}{2}\Big(\delta_{m,+1} + \delta_{m,-1}\Big) P_\ell
\end{array}$$

where in all cases I have

$$P_\ell = i^\ell \sqrt{4\pi(2\ell+1)}, \qquad Q_{\ell,\pm 1} = \mp i P_\ell.$$

4.2 Point sources at the origin

Let $\mathbf{E}(\mathbf{x};\mathbf{p})$ be the electric field at evaluation point $\mathbf{x}$ due to an electric dipole $\mathbf{p}$ at the origin. The spherical-wave expansion of this field involves only $\mathbf{N}$-functions with $\ell = 1$, i.e.

$$\mathbf{E}(\mathbf{x};\mathbf{p}) = \sum_{m=-1}^{1} \xi_{1m}(\mathbf{p}) \mathbf{N}_{1m}^{\text{outgoing}}(\mathbf{x})$$

where the ξ coefficients are

$$\begin{array}{lcll}
\mathbf{p} = p_x \hat{\mathbf{x}} & \longrightarrow & \xi_{1,1} = -\xi_{1,-1} = \dfrac{i}{2} \dfrac{k^3}{\sqrt{3\pi}} \dfrac{p_x}{\epsilon}, & \xi_{1,0} = 0 \\
\mathbf{p} = p_y \hat{\mathbf{y}} & \longrightarrow & \xi_{1,1} = +\xi_{1,-1} = \dfrac{1}{2} \dfrac{k^3}{\sqrt{3\pi}} \dfrac{p_y}{\epsilon} & \xi_{1,0} = 0 \\
\mathbf{p} = p_z \hat{\mathbf{z}} & \longrightarrow & \xi_{1,1} = \xi_{1,-1} = 0, & \xi_{1,0} = -\dfrac{ik^3}{\sqrt{6\pi}} \dfrac{p_z}{\epsilon}
\end{array}$$

Here $\epsilon = \epsilon_0 \epsilon^r$ is the absolute permittivity of the medium.

Similarly, the magnetic fields of a magnetic dipole $\mathbf{m}$ at the origin are

$$\mathbf{H}(\mathbf{x};\mathbf{m}) = \sum_\alpha \widehat{\xi}_\alpha(\mathbf{m}) \mathbf{N}_\alpha^{\text{outgoing}}(\mathbf{x})$$

where the $\widehat{\xi}$ coefficients are the same as the ξ coefficients above with the replacement $\frac{p}{\epsilon} \to \frac{m}{\mu}$.

4.3 Point sources not at the origin

The fields of point sources *not* at the origin may be obtained by applying the translation matrices of Section to the fields of Section 4.2. If the point source lies at $\mathbf{x}^{\text{S}} \neq 0$ (here "S" stands for "source"), then its fields at destination point $\mathbf{x}^{\text{D}}$ read

$$\mathbf{E}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{p}) = \sum_{\alpha\beta}\Big\{ -\xi_\alpha C_{\alpha\beta}\mathbf{M}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}}) + \xi_\alpha B_{\alpha\beta}\mathbf{N}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}})\Big\} \tag{10a}$$

$$\mathbf{H}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{p}) = \frac{1}{Z}\sum_{\alpha\beta}\Big\{ \xi_\alpha C_{\alpha\beta}\mathbf{N}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}}) + \xi_\alpha B_{\alpha\beta}\mathbf{M}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}})\Big\} \tag{10b}$$

$$\mathbf{E}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{m}) = -Z\sum_{\alpha\beta}\Big\{ \widehat{\xi}_\alpha B_{\alpha\beta}\mathbf{M}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}}) + \widehat{\xi}_\alpha C_{\alpha\beta}\mathbf{N}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}})\Big\} \tag{10c}$$

$$\mathbf{H}(\mathbf{x}^{\text{D}};\mathbf{x}^{\text{S}},\mathbf{m}) = \sum_{\alpha\beta}\Big\{ -\widehat{\xi}_\alpha C_{\alpha\beta}\mathbf{M}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}}) + \widehat{\xi}_\alpha B_{\alpha\beta}\mathbf{N}^{\text{regular}}_\beta(\mathbf{x}^{\text{D}})\Big\} \tag{10d}$$

Here $\{B, C\}_{\alpha\beta}$ are elements of the translation matrices $\mathbf{B}(k,\mathbf{x}^{\text{S}}), \mathbf{C}(k,\mathbf{x}^{\text{S}})$.

Note that, for a given source point $\mathbf{x}^{\text{S}}$, I only have to assemble the translation matrices $\mathbf{B}$ and $\mathbf{C}$ once (at a given frequency), after which I can get the fields at any number of destination points $\mathbf{x}^{\text{D}}$ from equation (10).

5 Scattering from a homogeneous dielectric sphere

I consider scattering from a single homogeneous sphere with relative permittivity and permeability ϵ^r, μ^r in vacuum irradiated by spherical waves emanating from within our outside the sphere.

Irrespective of the origin of the incident fields, the scattered fields inside and outside the sphere take the form ($n = \sqrt{\epsilon^r \mu^r}$, $Z^r = \sqrt{\mu^r/\epsilon^r}$)

Inside the sphere:

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_\alpha \Big\{ A_\alpha \mathbf{M}_\alpha^{\text{reg}}(nk_0;\mathbf{r}) + B_\alpha \mathbf{N}_\alpha^{\text{reg}}(nk_0;\mathbf{r}) \Big\} \tag{11a}$$

$$\mathbf{H}^{\text{scat}}(\mathbf{x}) = \frac{1}{Z_0 Z^r}\sum_\alpha \Big\{ B_\alpha \mathbf{M}_\alpha^{\text{reg}}(nk_0;\mathbf{r}) - A_\alpha \mathbf{N}_\alpha^{\text{reg}}(nk_0;\mathbf{r}) \Big\} \tag{11b}$$

Outside the sphere:

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_\alpha \Big\{ C_\alpha \mathbf{M}_\alpha^{\text{out}}(k_0;\mathbf{r}) + D_\alpha \mathbf{N}_\alpha^{\text{out}}(k_0;\mathbf{r}) \Big\} \tag{12a}$$

$$\mathbf{H}^{\text{scat}}(\mathbf{x}) = \frac{1}{Z_0}\sum_\alpha \Big\{ D_\alpha \mathbf{M}_\alpha^{\text{out}}(k_0;\mathbf{r}) - C_\alpha \mathbf{N}_\alpha^{\text{out}}(k_0;\mathbf{r}) \Big\} \tag{12b}$$

The $\{A, B, C, D\}$ coefficients are proportional to the spherical-wave expansion coefficients of the incident fields, with the proportionality constants determined by enforcing continuity of the tangential components of the total fields $\{\mathbf{E}, \mathbf{H}\}^{\text{tot}} = \{\mathbf{E}, \mathbf{H}\}^{\text{inc}} + \{\mathbf{E}, \mathbf{H}\}^{\text{scat}}$ at $r = r_0$,

$$\Big|\hat{\mathbf{r}} \times \mathbf{E}^{\text{tot}}\Big|_{r\to r_0^+} = \Big|\hat{\mathbf{r}} \times \mathbf{E}^{\text{tot}}\Big|_{r\to r_0^-} \tag{13a}$$

$$\Big|\hat{\mathbf{r}} \times \mathbf{H}^{\text{tot}}\Big|_{r\to r_0^+} = \Big|\hat{\mathbf{r}} \times \mathbf{H}^{\text{tot}}\Big|_{r\to r_0^-} \tag{13b}$$

5.1 Sources outside the sphere

If the sources of the incident field lie outside the sphere (the usual Mie scattering problem), then I can expand the incident fields in the form

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum_\alpha \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{regular}}(k_0;\mathbf{x}) + Q_\alpha \mathbf{N}_\alpha^{\text{regular}}(k_0;\mathbf{x}) \Big\} \tag{14a}$$

$$\mathbf{H}^{\text{inc}}(\mathbf{x}) = \frac{1}{Z_0}\sum_\alpha \Big\{ Q_\alpha \mathbf{M}_\alpha^{\text{regular}}(k_0;\mathbf{x}) - P_\alpha \mathbf{N}_\alpha^{\text{regular}}(k_0;\mathbf{x}) \Big\}. \tag{14b}$$

Matching tangential fields at the sphere surface then determines the scattered-field expansion coefficients in terms of the incident-field expansion coefficients ($a = k_0 r_0$):

$$A_\alpha = \left[\frac{R^{\rm reg}(a)\overline{R}^{\rm out}(a) - \overline{R}^{\rm reg}(a)R^{\rm out}(a)}{R^{\rm reg}(na)\overline{R}^{\rm out}(a) - \frac{1}{Z^r}\overline{R}^{\rm reg}(na)R^{\rm out}(a)} \right] P_\alpha \tag{15a}$$

$$B_\alpha = \left[\frac{\overline{R}^{\rm reg}(a)R^{\rm out}(a) - R^{\rm reg}(a)\overline{R}^{\rm out}(a)}{\overline{R}^{\rm reg}(na)R^{\rm out}(a) - \frac{1}{Z^r}R^{\rm reg}(na)\overline{R}^{\rm out}(a)} \right] Q_\alpha \tag{15b}$$

$$C_\alpha = \underbrace{\left[\frac{R^{\rm reg}(a)\overline{R}^{\rm reg}(na) - Z^r\overline{R}^{\rm reg}(a)R^{\rm reg}(na)}{Z^r R^{\rm reg}(na)\overline{R}^{\rm out}(a) - \overline{R}^{\rm reg}(na)R^{\rm out}(a)} \right]}_{\mathbb{T}^{\rm M}_\alpha} P_\alpha \tag{15c}$$

$$D_\alpha = \underbrace{\left[\frac{\overline{R}^{\rm reg}(a)R^{\rm reg}(na) - Z^r R^{\rm reg}(a)\overline{R}^{\rm reg}(na)}{Z^r \overline{R}^{\rm reg}(na)R^{\rm out}(a) - R^{\rm reg}(na)\overline{R}^{\rm out}(a)} \right]}_{\mathbb{T}^{\rm N}_\alpha} Q_\alpha \tag{15d}$$

In (15c,d) I have identified the quantities C_α/P_α and D_α/Q_α as elements of the $\mathbb{T}$-matrix for the M- and N- polarizations.[2]

5.1.1 Analytical results in the low-frequency limit

The coefficients (15) may be expressed in closed form, e.g.

$$\frac{A_1}{P_1} = \frac{2a^3 e^{-ia} n^3 Z}{((1-ia)(a^2n^2-1) - (-1+a(a+i))nZ)\sin(an) + an((-1+a(a+i))nZ - ia + 1)\cos(an)}$$

$$\frac{B_1}{Q_1} = \frac{2a^3 e^{-ia} n^3 Z}{((1-ia)Z(a^2n^2-1) - (-1+a(a+i))n)\sin(an) + an((-1+a(a+i))n - iaZ + Z)\cos(an)}$$

where $a = (k_0 r_0)$ is the dimensionless Mie size parameter. The low-frequency limiting forms (assuming $\mu = 1$:) are

$$\frac{A_1}{P_1} = \frac{2}{\sqrt{\epsilon}} + \left[\frac{(\epsilon-1)}{3\sqrt{\epsilon}}\right]a^2 + O(a^3) \qquad \frac{B_1}{Q_1} = \frac{6}{\epsilon+2} + \left[\frac{3\left(\epsilon^2+9\epsilon-10\right)}{5(\epsilon+2)^2}\right]a^2 + O(a^3)$$

$$\frac{A_2}{P_2} = \frac{2}{\epsilon} + \left[\frac{(\epsilon-1)}{5\epsilon}\right]a^2 + O(a^3) \qquad \frac{B_2}{Q_2} = \frac{10}{\sqrt{\epsilon}(2\epsilon+3)} + \left[\frac{5a^2\left(2\epsilon^2+5\epsilon-7\right)}{7\sqrt{\epsilon}(2\epsilon+3)^2}\right]a^2 + O(a^3)$$

[2] The $\mathbb{T}$-matrix multiplies a vector of regular-wave incident-field coefficients to yield a vector of outgoing-wave scattered-field coefficients. If, instead of the regular-wave incident field (14), I irradiated the sphere with a superposition of *incoming* waves as the incident field, then the resulting modified versions of equations (15c,d) would instead define elements of the $\mathbb{S}$-matrix (scattering matrix).

Interior fields

For a sphere irradiated by a linearly-polarized plane wave, the fields inside the body to second order in $a = kR$ read

$$\begin{aligned}
\frac{E_x}{E_0} &= \frac{3}{2+\epsilon} + \left[\frac{\epsilon+4}{3+2\epsilon}\right] ikz + \left[\frac{(\epsilon-1)(35\epsilon+46)}{5(\epsilon+2)^2(3\epsilon+4)}\right] k^2x^2 \\
&\qquad + \left[\frac{(\epsilon-1)(-2\epsilon^2+29\epsilon+42)}{5(\epsilon+2)^2(3\epsilon+4)}\right] k^2y^2 - \left[\frac{14\epsilon^3+3\epsilon^2+114\epsilon+184}{10(\epsilon+2)^2(3\epsilon+4)}\right] k^2z^2 \\
\frac{E_y}{E_0} &= \left[\frac{2(\epsilon^2-1)}{5(2+\epsilon)(4+3\epsilon)}\right] k^2xy \\
\frac{E_z}{E_0} &= -\left[\frac{\epsilon-1}{2\epsilon+3}\right] ikx + \left[\frac{(\epsilon-1)(7\epsilon+12)}{5(2+\epsilon)(4+3\epsilon)}\right] k^2xz \\
\frac{H_x}{Z_0E_0} &= \left[\frac{(\epsilon-1)^2}{5(2\epsilon+3)}\right] k^2xy \\
\frac{H_y}{Z_0E_0} &= 1 + \left[\frac{2\epsilon+1}{2+\epsilon}\right] ikz \\
&\qquad - \left[\frac{(\epsilon-1)(\epsilon-6)}{15(3+2\epsilon)}\right] k^2x^2 + \left[\frac{\epsilon-1}{15}\right] k^2y^2 - \left[\frac{2\epsilon^2+46\epsilon+27}{30(3+2\epsilon)}\right] k^2z^2 \\
\frac{H_z}{Z_0E_0} &= -\left[\frac{\epsilon-1}{\epsilon+2}\right] iky + \left[\frac{(\epsilon-1)(\epsilon+4)}{5(3+2\epsilon)}\right] k^2yz
\end{aligned}$$

Field derivatives:

$$\begin{aligned}
\partial_z\mathbf{E} &= \big(ikC_1 - 2k^2C_2z\big)\hat{\mathbf{x}} + k^2C_3x\,\hat{\mathbf{z}} \\
\partial_z\mathbf{H} &= \big(ikC_4 - 2k^2C_5z\big)\hat{\mathbf{y}} + k^2C_6y\,\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{\epsilon+4}{3+2\epsilon}, \\
C_2 &= \frac{14\epsilon^3+3\epsilon^2+114\epsilon+184}{10(\epsilon+2)^2(3\epsilon+4)} \\
C_3 &= \frac{(\epsilon-1)(7\epsilon+12)}{5(2+\epsilon)(4+3\epsilon)} \\
C_4 &= \frac{2\epsilon+1}{2+\epsilon} \\
C_5 &= \frac{2\epsilon^2+46\epsilon+27}{30(3+2\epsilon)} \\
C_6 &= \frac{(\epsilon-1)(\epsilon+4)}{5(3+2\epsilon)}
\end{aligned}$$

5.2 Sources inside the sphere

If the sources of the incident field lie inside the sphere, then I can expand the incident field in the form

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum_\alpha \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{out}}(nk_0;\mathbf{x}) + Q_\alpha \mathbf{N}_\alpha^{\text{out}}(nk_0;\mathbf{x}) \Big\} \tag{16}$$

The total fields inside and outside then read

$$\mathbf{E}^{\text{in}}(\mathbf{x}) = \sum \Big\{ P_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) + Q_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) \Big\} + \sum \Big\{ A_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) + B_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x}) \Big\} \tag{17}$$

$$\begin{aligned}\mathbf{H}^{\text{in}}(\mathbf{x}) = -\frac{1}{Z_0 Z^r} & \sum \Big\{ P_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) - Q_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) \Big\} \\ & - \frac{1}{Z_0 Z^r} \sum \Big\{ C_\alpha \mathbf{N}_\alpha^{\text{reg}}(\mathbf{x}) - D_\alpha \mathbf{M}_\alpha^{\text{reg}}(\mathbf{x}) \Big\}\end{aligned} \tag{18}$$

$$\mathbf{E}^{\text{out}}(\mathbf{x}) = \sum \Big\{ C_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) + D_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) \Big\} \tag{19}$$

$$\mathbf{H}^{\text{out}}(\mathbf{x}) = -\frac{1}{Z_0} \sum \Big\{ C_\alpha \mathbf{N}_\alpha^{\text{out}}(\mathbf{x}) - D_\alpha \mathbf{M}_\alpha^{\text{out}}(\mathbf{x}) \Big\} \tag{20}$$

Now equate tangential components of $\mathbf{E}^{\text{in,out}}$ and $\mathbf{H}^{\text{in,out}}$ at the sphere surface ($r = r_0$), take inner products with $\mathbf{M}$ and $\mathbf{N}$, and use the orthogonality relations to find

$$\begin{array}{rcrcl}
R_\ell^{\text{out}}(nkr_0)P_\alpha & + & R_\ell^{\text{reg}}(nkr_0)A_\alpha & = & R_\ell^{\text{out}}(kr_0)C_\alpha \\
\overline{R}_\ell^{\text{out}}(nkr_0)P_\alpha & + & \overline{R}_\ell^{\text{reg}}(nkr_0)A_\alpha & = & Z^r\overline{R}_\ell^{\text{out}}(kr_0)C_\alpha \\
\overline{R}_\ell^{\text{out}}(nkr_0)Q_\alpha & + & \overline{R}_\ell^{\text{reg}}(nkr_0)B_\alpha & = & \overline{R}_\ell^{\text{out}}(kr_0)D_\alpha \\
R_\ell^{\text{out}}(nkr_0)Q_\alpha & + & R_\ell^{\text{reg}}(nkr_0)B_\alpha & = & Z^r R_\ell^{\text{out}}(kr_0)D_\alpha
\end{array}$$

which we solve to obtain the coefficients of the scattered field outside the sphere in terms of the incident-field coefficients:

$$C_\alpha = \left[\frac{R_\ell^{\text{out}}(na)\overline{R}_\ell^{\text{reg}}(na) - \overline{R}_\ell^{\text{out}}(na)R_\ell^{\text{reg}}(na)}{R_\ell^{\text{out}}(a)\overline{R}_\ell^{\text{reg}}(na) - Z^r\overline{R}_\ell^{\text{out}}(a)R_\ell^{\text{reg}}(na)} \right] P_\alpha \tag{21a}$$

$$D_\alpha = \left[\frac{\overline{R}_\ell^{\text{out}}(na)R_\ell^{\text{reg}}(na) - R_\ell^{\text{out}}(na)\overline{R}_\ell^{\text{reg}}(na)}{\overline{R}_\ell^{\text{out}}(a)R_\ell^{\text{reg}}(na) - Z^r R_\ell^{\text{out}}(a)\overline{R}_\ell^{\text{reg}}(na)} \right] Q_\alpha. \tag{21b}$$

6 Scattering from a sphere with impedance boundary conditions

For a sphere characterized by a surface-impedance boundary condition with relative surface impedance[3] η, the continuity condition (13) is replaced by a relationship between the tangential $\mathbf{E}$ and $\mathbf{H}$ fields at the sphere surface:

$$\mathbf{E}_{\parallel} = \eta Z_0 \Big(\hat{\mathbf{r}} \times \mathbf{H} \Big) \quad \text{at} \quad r = R.$$

Equations (15c,d) for the $\mathbb{T}$-matrix elements are replaced by

$$C_\alpha = \underbrace{\left[\frac{R^{\text{reg}}(a)}{i\eta \overline{R}^{\text{out}}(a) - R^{\text{out}}(a)} \right]}_{\mathbb{T}^{\text{M}}_\alpha} P_\alpha, \qquad D_\alpha = \underbrace{\left[\frac{\overline{R}^{\text{reg}}(a)}{i\eta R^{\text{out}}(a) + \overline{R}^{\text{out}}(a)} \right]}_{\mathbb{T}^{\text{N}}_\alpha} Q_\alpha, \quad (22)$$

In particular, taking $\eta \to 0$ yields the $\mathbb{T}$-matrix elements for a perfectly electrically conducting (PEC) sphere.

[3] Note that η is dimensionless; the absolute surface impedance is ηZ_0 where $Z_0 \approx 377\,\Omega$ is the impedance of vacuum.

7 Dyadic Green's functions

The scattering part of the electric dyadic Green's function $\boldsymbol{\mathcal{G}}^{\text{EE}}(\mathbf{x}^{\text{D}}, \mathbf{x}^{\text{S}})$ is a 3×3 matrix whose i, j component $\mathcal{G}^{\text{EE}}_{ij}(\mathbf{x}^{\text{D}}, \mathbf{x}^{\text{S}})$ is the (appropriately normalized)[4] i component of the scattered electric field at $\mathbf{x}^{\text{D}}$ due to a j-directed point electric dipole source at $\mathbf{x}^{\text{S}}$. (The superscripts on $\mathbf{x}$ stand for "destination" and "source").

If I take the electic-dipole fields Section 4.3 [equations (10a,b)] to be the incident fields in the externally-sourced scattering problem of Section 5.1 [so that, for example, the coefficient of $\mathbf{M}^{\text{reg}}_{\alpha}$ in the incident-field expansion (14) is $P_\alpha = -\sum_\beta \xi_\beta C_{\beta\alpha}$], then I need only multiply by $\mathbb{T}$-matrix elements [equation (15)] to get the outgoing-wave coefficients in the scattered-field expansion (12).

Thus the $\mathbf{E}$- and $\mathbf{H}$-fields at $\mathbf{x}^{\text{D}}$ due to an electric dipole source $\mathbf{p}$ at $\mathbf{x}^{\text{S}}$ are

$$\begin{aligned}
\mathbf{E}^{\text{scat}}(\mathbf{x}^{\text{D}}; \mathbf{x}^{\text{S}}, \mathbf{p}) &= \sum_{\alpha\beta} \xi_\alpha(\mathbf{p}) \Big\{ -C_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{M}}_{\beta} \mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + B_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{N}}_{\beta} \mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) \Big\} \\
\mathbf{H}^{\text{scat}}(\mathbf{x}^{\text{D}}; \mathbf{x}^{\text{S}}, \mathbf{p}) &= \frac{1}{Z} \sum_{\alpha\beta} \xi_\alpha(\mathbf{p}) \Big\{ +C_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{M}}_{\gamma\beta} \mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + B_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{N}}_{\gamma\beta} \mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) \Big\}
\end{aligned}$$

The $\mathbf{E}$- and $\mathbf{H}$-fields at $\mathbf{x}^{\text{D}}$ due to a magnetic dipole source $\mathbf{m}$ at $\mathbf{x}^{\text{S}}$ are

$$\begin{aligned}
\mathbf{E}^{\text{scat}}(\mathbf{x}^{\text{D}}; \mathbf{x}^{\text{S}}, \mathbf{m}) &= -Z \sum_{\alpha\beta} \widehat{\xi}_\alpha(\mathbf{m}) \Big\{ +C_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{M}}_{\beta} \mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + B_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{N}}_{\beta} \mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) \Big\} \\
\mathbf{H}^{\text{scat}}(\mathbf{x}^{\text{D}}; \mathbf{x}^{\text{S}}, \mathbf{m}) &= \sum_{\alpha\beta} \widehat{\xi}_\alpha(\mathbf{m}) \Big\{ -B_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{N}}_{\beta} \mathbf{M}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) + C_{\alpha\beta}(\mathbf{x}^{\text{S}}) \mathbb{T}^{\text{M}}_{\beta} \mathbf{N}^{\text{out}}_{\beta}(\mathbf{x}^{\text{D}}) \Big\}.
\end{aligned}$$

In writing out these equations, I have used the fact that the $\mathbb{T}$-matrix of a homogeneous sphere is diagonal. However, similar equations could be written down for the DGFs of any *arbitrary*-shaped object; in this case the $\mathbb{T}$-matrix would not be diagonal and the double sums would become triple sums, but such a representation might nonetheless be useful in some cases.

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[4] The normalization just involves dividing by dimensionful prefactors to ensure that the components of $\mathcal{G}$ have units of inverse length and are independent of the point-source magnitude.

8 VSWVIE: Volume-integral-equation approach to scattering with vector spherical waves as volume-current basis functions

8.1 Review of the standard VIE formalism

Consider a material body with spatially-varying relative permittivity tensor $\boldsymbol{\epsilon}(\mathbf{x})$ illuminated by incident radiation with electric field $\mathbf{E}^{\text{inc}}(\mathbf{x})$ at frequency $\omega = kc$ in vacuum. In the usual volume-integral-equation formulation of the scattering problem, the scattered field $\mathbf{E}^{\text{scat}}(\mathbf{x})$ is understood to arise from an induced volume current distribution $\mathbf{J}(\mathbf{x})$, which is itself proportional to the local total (incident+scattered) field at each point:

$$\mathbf{E}^{\text{scat}} = ikZ_0\mathbb{G}_0 \star \mathbf{J}, \qquad \mathbf{J} = -\frac{ik}{Z_0}(\boldsymbol{\epsilon}-\mathbf{1})\Big(\mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{scat}}\Big)$$

Combining these yields

$$\Big[\mathbf{1} + \mathbb{V}\mathbb{G}\Big] \star \mathbf{J}(\mathbf{x}) = \frac{i}{kZ_0}\mathbb{V}\mathbf{E}^{\text{inc}} \tag{23}$$

where the diagonal operator $\mathbb{V}(\mathbf{x},\mathbf{x}') \equiv -k^2\big(\boldsymbol{\epsilon}-\mathbf{1}\big)\delta(\mathbf{x},\mathbf{x}')$ is the "potential." Another way to write this is

$$\Big[\mathbb{A} + \mathbb{G}\Big] \star \mathbf{J}(\mathbf{x}) = \frac{i}{kZ_0}\mathbf{E}^{\text{inc}} \tag{24}$$

where $\mathbb{A}(\mathbf{x},\mathbf{x}') \equiv -\frac{1}{k^2}\big(\boldsymbol{\epsilon}-\mathbf{1}\big)^{-1}\delta(\mathbf{x}-\mathbf{x}')$.

Upon approximating the induced current as an expansion in a finite set of N_{BF} vector-valued basis functions,

$$\mathbf{J}(\mathbf{x}) \approx \sum_{\alpha=1}^{N_{\text{BF}}} j_\alpha \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x}) \tag{25}$$

the integral equation (24) becomes an $N_{\text{BF}} \times N_{\text{BF}}$ linear system:

$$[\boldsymbol{\Lambda} + \mathbf{G}]\,\mathbf{j} = \frac{i}{kZ_0}\mathbf{e}$$

where $\mathbf{j}_\alpha$ is the vector of expansion coefficients in (25) and

$$\begin{aligned} e_\alpha &\equiv \frac{i}{kZ_0}\Big\langle \boldsymbol{\mathcal{B}}_\alpha \Big| \mathbf{E}^{\text{inc}}\Big\rangle, \\ \boldsymbol{\Lambda}_{ab} &= -\frac{1}{k^2}\Big\langle \mathbf{B}_\alpha \Big| \left(\boldsymbol{\epsilon}-\mathbf{1}\right)^{-1} \Big| \mathbf{B}_\beta \Big\rangle \\ \mathbf{G}_{ab} &= -\frac{1}{k^2}\Big\langle \mathbf{B}_\alpha \Big| \mathbb{G} \Big| \mathbf{B}_\beta \Big\rangle. \end{aligned}$$

8.2 Vector spherical waves as volume-current basis functions

For a compact material body confined within a sphere of radius R, I use the regular vector spherical waves to define a basis of volume-current basis functions:

$$\boldsymbol{\mathcal{B}}_{P\ell m}(\mathbf{x}) = \begin{cases} 0, & |\mathbf{x}| > R \\ \mathbf{M}^{\text{reg}}_{\ell m}(\mathbf{x}), & |\mathbf{x}| \le R, P = M \\ \mathbf{N}^{\text{reg}}_{\ell m}(\mathbf{x}), & |\mathbf{x}| \le R, P = N \\ \mathbf{L}^{\text{reg}}_{\ell m}(\mathbf{x}), & |\mathbf{x}| \le R, P = L. \end{cases} \tag{26}$$

I use these basis functions to expand currents and fields in (23):

$$\mathbf{J}(\mathbf{x}) = \sum j_\alpha \boldsymbol{\mathcal{B}}_\alpha, \qquad \mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum e_\alpha \boldsymbol{\mathcal{B}}_\alpha. \tag{27}$$

Note that the e_α coefficients have dimensions of electric field strength.

8.3 Solution of VIE equation in VSW basis

Inserting (27) into (23) yields a relation between the vectors of volume-current expansion coefficients and incident-field expansion coefficients:

$$\mathbf{j} = \frac{i}{kZ_0}\Big[\mathbf{S} + \mathbf{VG}\Big]^{-1}\mathbf{Ve} \tag{28}$$

where the elements of the $\mathbf{SV}, \mathbf{G}$ matrices are

$$S_{\alpha\beta} = \int_{|\mathbf{x}|<R} \boldsymbol{\mathcal{B}}^*_\alpha(\mathbf{x})\boldsymbol{\mathcal{B}}_\beta(\mathbf{x})\,d\mathbf{x}, \qquad V_{\alpha\beta} = -k^2\int_0^R \boldsymbol{\mathcal{B}}^*_\alpha(\mathbf{x})\Big[\boldsymbol{\epsilon}(\mathbf{x}) - 1\Big]\boldsymbol{\mathcal{B}}_\beta(\mathbf{x})\,d\mathbf{x},$$

$$G_{\alpha\beta} = \int_0^R\int_0^R \boldsymbol{\mathcal{B}}^*_\alpha(\mathbf{x})\mathbb{G}(\mathbf{x},\mathbf{x}')\boldsymbol{\mathcal{B}}_\beta(\mathbf{x}')\,d\mathbf{x}\,d\mathbf{x}'$$

and $\mathbf{e}$ is the vector of incident-field projections onto the $\{\mathbb{B}_\alpha\}$ basis. If $\mathbf{e}$ is expressed as an expansion in the usual (non-normalized) regular VSWs, as in equation (16), then the entries of $\mathbf{e}$ are just the P, Q coefficients in that equation:

$$\mathbf{e}_{M\ell m} = P_{\ell m}, \qquad \mathbf{e}_{N\ell m} = Q_{\ell m}.$$

8.4 Overlap integrals

The overlap matrix is diagonal in the ℓ and m indices and independent of m:

$$S_{P\ell m;P'\ell' m'} \equiv S_{PP'\ell}\delta_{\ell,\ell'}\delta_{m,m'}$$

It is also "partially diagonal' in the P index, in the sense that the $\mathbf{M}$ functions are orthogonal to the $\mathbf{N}$ and $\mathbf{L}$ functions, but the $\mathbf{N}$ and $\mathbf{L}$ functions have nonvanishing overlap.

The overlap integrals are

$$S_{PP'\ell}(R) \equiv \int_0^R \mathcal{R}_{PP'\ell}(r)\,dr \quad \text{where} \quad R_{PP'\ell}(r) \equiv r^2 \int \boldsymbol{\mathcal{B}}^*_{P\ell m}(r,\Omega)\boldsymbol{\mathcal{B}}_{P'\ell m}(r,\Omega)\,d\Omega.$$

Explicit forms of the integrand function are

$$\mathcal{R}_{MM\ell} \equiv r^2\big[j_\ell(kr)\big]^2$$

$$\mathcal{R}_{NN\ell} \equiv \frac{\ell(\ell+1)}{k^2}\big[j'_\ell(kr)\big]^2 + 2\frac{r}{k}j_\ell(kr)j'_\ell(kr) + \frac{\ell^2+\ell+1}{k^2}\big[j_\ell(kr)\big]^2$$

which may be simplified [**?**] to read

$$\mathcal{R}_{NN\ell} \equiv \left(\frac{\ell+1}{2\ell+1}\right) r^2 j^2_{\ell-1}(kr) + \left(\frac{\ell}{2\ell+1}\right) r^2 j^2_{\ell+1}(kr)$$

$$\mathcal{R}_{LL\ell} \equiv \frac{1}{k^2}\big[j_\ell(kr)\big]^2 + \frac{[rj'_\ell(kr)]^2}{\ell(\ell+1)}$$

$$\mathcal{R}_{NL\ell} \equiv -\frac{2r}{k}j_\ell(kr)j'_\ell(kr) - \frac{1}{k^2}j^2_\ell(u)$$

Explicit forms for the overlap integrals are

$$S_{MM\ell}(R) = \frac{j^2_\ell(kR) - j_{\ell-1}(kR)j_{\ell+1}(kR)}{2} \tag{29a}$$

$$\begin{aligned} S_{NN\ell}(R) = \frac{1}{2(2\ell+1)}\Big\{&(\ell+1)\big[j_{\ell-1}(kR)\big]^2 - (\ell+1)j_\ell(kR)j_{\ell-2}(kR) \\ &- \ell j_\ell(kR)j_{\ell+2}(kR) + \ell\big[j_{\ell+1}(kR)\big]^2\Big\} \end{aligned} \tag{29b}$$

I also need another type of overlap integral:

$$\widehat{S}_{PP'\ell}(R) \equiv \int_0^R \widehat{\mathcal{R}}_{PP'\ell}(r)\,dr \quad \text{where} \quad \widehat{\mathcal{R}}_{PP'\ell} \equiv \int \boldsymbol{\mathcal{B}}^*_{P\ell m}(r,\Omega)\widehat{\boldsymbol{\mathcal{B}}}_{P'\ell m}(r,\Omega)\,d\Omega.$$

The $\widehat{\mathcal{R}}$ integrands are similar to the $\mathcal{R}$ integrands given above, except that one of the two j_ℓ factors in each term is replaced by $h^{(1)}_\ell$:

$$\widehat{\mathcal{R}}_{MM\ell} \equiv j_\ell(kr)h^{(1)}_\ell(kr)$$

$$\widehat{\mathcal{R}}_{NN\ell} \equiv r^2\overline{j_\ell}(kr)\overline{h^{(1)}_\ell}(kr) + \frac{\ell(\ell+1)}{k^2}j_\ell(kr)h^{(1)}_\ell(kr)$$

$$\widehat{\mathcal{R}}_{LN\ell} \equiv -\frac{r}{k}j'_\ell(kr)h^{(1)}_\ell(kr) - \frac{r}{k}j_\ell(kr)\overline{h^{(1)}_\ell}(kr)$$

8.5 Fields produced by VSW basis functions

I will use the symbol $\mathbb{E}_\alpha(\mathbf{x})$ to denote $1/(i\omega\epsilon_0)$ times the **E**-field at $\mathbf{x}$ due to a current distribution produced by basis function $\boldsymbol{\mathcal{B}}_\alpha$ populated with unit strength. The quantity $\mathbb{E}$ has dimensions of length^2.

Evaluation using dyadic Green's function

$$\mathbb{E}_\alpha(\mathbf{x}) \equiv \int_{|\mathbf{x}'|<R} \mathbb{G}(\mathbf{x},\mathbf{x}')\boldsymbol{\mathcal{B}}_\alpha(\mathbf{x}')\,d\mathbf{x}'$$

The integral here may be easily evaluated using the standard eigenexpansion of $\mathbb{G}$ [**? ?**]:

$$\mathbb{G}(\mathbf{x},\mathbf{x}') = -\frac{\hat{\mathbf{r}}\hat{\mathbf{r}}}{k^2}\delta(\mathbf{x}-\mathbf{x}') + ik\sum_\alpha \Big\{ \widehat{\mathbf{M}}_\alpha(\mathbf{r}_>)\mathbf{M}^*_\alpha(\mathbf{r}_<) + \widehat{\mathbf{N}}_\alpha(\mathbf{r}_>)\mathbf{N}^*_\alpha(\mathbf{r}_<) \Big\}$$

The result is:

Evaluation point outside sphere ($|\mathbf{x}| > R$):

$$\begin{aligned}
\mathbb{E}_{M\ell m}(\mathbf{r}) &= ikS_{MM\ell}(R)\widehat{\mathbf{M}}(\mathbf{r}) \\
\mathbb{E}_{N\ell m}(\mathbf{r}) &= ikS_{NN\ell}(R)\widehat{\mathbf{N}}(\mathbf{r}) \\
\mathbb{E}_{L\ell m}(\mathbf{r}) &= ikS_{NL\ell}(R)\widehat{\mathbf{N}}(\mathbf{r})
\end{aligned}$$

Evaluation point inside sphere ($|\mathbf{x}| < R$):

$$\mathbb{E}_{M\ell m}(\mathbf{r}) = ikS_{MM\ell}(r)\widehat{\mathbf{M}}(\mathbf{r}) + ik\widehat{S}_{MM\ell}(r,R)\mathbf{M}(\mathbf{r}) \tag{30a}$$

$$\mathbb{E}_{N\ell m}(\mathbf{r}) = -\frac{1}{k^2}\Big[\hat{\mathbf{r}}\cdot\mathbf{N}_{\ell m}\Big]\hat{\mathbf{r}} + ikS_{NN\ell}(r)\widehat{\mathbf{N}}(\mathbf{r}) + ik\widehat{S}_{NN\ell}(r,R)\mathbf{N}(\mathbf{r}) \tag{30b}$$

$$\mathbb{E}_{L\ell m}(\mathbf{r}) = -\frac{1}{k^2}\Big[\hat{\mathbf{r}}\cdot\mathbf{L}_{\ell m}\Big]\hat{\mathbf{r}} + ikS_{NL\ell}(r)\widehat{\mathbf{N}}(\mathbf{r}) + ik\widehat{S}_{NL\ell}(r,R)\mathbf{N}(\mathbf{r}) \tag{30c}$$

Evaluation using scalar Green's function

I can also evaluate the fields using the scalar Green's function, whose eigenexpansion reads

$$G_0(\mathbf{r},\mathbf{r}') = ik\sum_\alpha R^{\text{reg}}_\alpha(kr_<)R^{\text{out}}_\alpha(kr_>)Y_\alpha(\theta,\phi)Y^*_\alpha(\theta',\phi')$$

In general, the $\mathbb{E}$-field produced by a current distribution $\mathbf{J}$ reads

$$\mathbb{E}(\mathbf{x}) = \underbrace{\int_V G_0(\mathbf{x}-\mathbf{x}')\mathbf{J}(\mathbf{x}')\,dV}_{\mathbb{E}^1} + \underbrace{\frac{1}{k^2}\nabla_{\mathbf{x}}\int_V G_0(\mathbf{x}-\mathbf{x}')\Big[\nabla\cdot\mathbf{J}(\mathbf{x}')\Big]\,dV}_{\mathbb{E}^2}$$
$$\underbrace{-\frac{1}{k^2}\nabla_{\mathbf{x}}\oint_V G_0(\mathbf{x}-\mathbf{x}')J_r(\mathbf{x}')dA}_{\mathbb{E}^3}$$

The third integral here captures the effect of the surface charge layer due to the discontinuous dropoff of the currents at the sphere surface.

In evaluating $\mathbb{E}^1$ integrals I need to use the 3x3 matrices that convert spherical vector components to cartesian vector components and vice versa:

$$\left(\begin{array}{c} V_x \\ V_y \\ V_z \end{array}\right) = \underbrace{\left(\begin{array}{ccc} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{array}\right)}_{\mathbf{\Lambda}^{\text{S2C}}(\theta,\varphi)} \left(\begin{array}{c} V_r \\ V_\theta \\ V_\varphi \end{array}\right)$$

$$\left(\begin{array}{c} V_r \\ V_\theta \\ V_\varphi \end{array}\right) = \underbrace{\left(\begin{array}{ccc} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{array}\right)}_{\mathbf{\Lambda}^{\text{C2S}}(\theta,\varphi)} \left(\begin{array}{c} V_x \\ V_y \\ V_z \end{array}\right)$$

Relevant integrals include:

$$\int Y^*_\alpha(\theta,\phi)\mathbf{\Lambda}^{\text{C2S}}(\theta,\phi)\cdot\{Y_\beta(\theta,\phi)\hat{\mathbf{r}}\}\,d\Omega =$$

I then find:

$$\begin{aligned}
&\mathbb{E}^1_{M\ell m}(\mathbf{x}) \\
&\mathbb{E}^2_{M\ell m}(\mathbf{x}) = 0 \quad \text{because} \quad \nabla\cdot\mathbf{M} \\
&\mathbb{E}^3_{M\ell m}(\mathbf{x}) = 0 \quad \text{because} \quad M_r = 0
\end{aligned}$$

8.6 Matrix elements of $\mathbb{G}$

$$\begin{aligned}
G_{PP'\ell} &\equiv \big\langle \boldsymbol{\mathcal{B}}_{P\ell m}\big|\mathbb{G}\big|\boldsymbol{\mathcal{B}}_{P'\ell m}\big\rangle \\
&\equiv \int_{|\mathbf{x}|<R} \boldsymbol{\mathcal{B}}^*_{P\ell m}(\mathbf{x})\cdot\mathbb{E}_{P'\ell(\mathbf{x})}\,d\mathbf{x}
\end{aligned}$$

Using equation (30), one finds

$$\begin{aligned}
G_{MM\ell} &= 2ik\int_0^R\int_r^R \mathcal{R}_{MM\ell}(r)\widehat{\mathcal{R}_{MM\ell}}(r')\,dr\,dr' \\
&= 2ik\int_0^R\int_r^R \Big[j_\ell(kr)\Big]^2 j_\ell(kr')h_\ell^{(1)}(kr')\,r^2r'^2\,dr'\,dr \qquad (31a)
\end{aligned}$$

$$\begin{aligned}
G_{NN\ell} &= -\frac{1}{k^2}\int_{|\mathbf{x}|<R}\Big|N_{\ell m r}(\mathbf{x})\Big|^2 d\mathbf{x} + 2ik\int_0^R\int_r^R \mathcal{R}_{NN\ell}(r)\widehat{\mathcal{R}}_{NN\ell}(r')\,dr\,dr' \\
&= -\frac{\ell(\ell+1)}{k^4}\int_0^R\Big[j_\ell(kr)\Big]^2 dr + 2ik\int_0^R\int_r^R \mathcal{R}_{NN\ell}(r)\widehat{\mathcal{R}}_{NN\ell}(r')\,dr\,dr' \qquad (31b)
\end{aligned}$$

$$\begin{aligned}
G_{LN\ell} &= +\frac{1}{k^3}\int_0^R rj_\ell(kr)j_\ell'(kr)\,dr + ik\int_0^R\int_r^R \mathcal{R}_{NN\ell}(r)\widehat{\mathcal{R}}_{LN\ell}(r')\,dr\,dr' \\
&\quad + ik\int_0^R\int_r^R \widehat{\mathcal{R}}_{NN\ell}(r)\mathcal{R}_{LN\ell}(r')\,dr\,dr' \qquad (31c)
\end{aligned}$$

$$G_{LL\ell} = -\frac{1}{\ell(\ell+1)k^2}\int_0^R\Big[rj_\ell(kr)\Big]^2 dr + 2ik\int_0^R\int_r^R \mathcal{R}_{LN\ell}(r)\widehat{\mathcal{R}}_{LN\ell}(r')\,dr\,dr' \qquad (31d)$$

The convenient thing about the VSW basis is that the $\mathbf{G}$ matrix is diagonal (and m-independent) in this basis, with elements that may be computed in closed form:

$$\begin{aligned}
G_{P\ell m;P'\ell'm'} &= \delta_{PP'}\delta_{\ell\ell'}\delta_{mm'}\left[-\frac{\delta_{PN}}{3k^2} + i\mathcal{G}_{P\ell}\right] \\
\mathcal{G}_{P\ell} &= \frac{2k}{\mathcal{S}_{P\ell}(R)}\int_0^R \mathcal{S}_{P\ell}(r)\mathcal{R}_{P\ell}(r)\,r^2\,dr \\
&= \begin{cases} \dfrac{2k}{\mathcal{S}_{M\ell}(R)}\displaystyle\int_0^R r^2\Big[j_\ell(kr)^2\Big]\,dr, & P=M \\ \dfrac{2k}{\mathcal{S}_{N\ell}(R)}\displaystyle\int_0^R r^2\Big[\overline{j}_\ell(kr)^2 + \jmath_\ell(kr)^2\Big]\,dr, & P=N. \end{cases}
\end{aligned}$$

In the $\ell=1$ sector one finds

$$\begin{aligned}
k^2G_{M1m,M1m} &= -\frac{1}{3} + \frac{i}{4a}\Big[-2+2\cos(2a)+2a^2+a\sin(2a)\Big] \\
&= -\frac{1}{3} + \frac{i}{45}a^5 - \frac{i}{315}a^7 + O\big(a^9\big) \\
k^2G_{N1m,N1m} &= -\frac{1}{3} + \frac{i}{4a^3}\Big[2\big(a^4-a^2-1\big) - a\big(a^2-4\big)\sin(2a) - 2\big(a^2-1\big)\cos(2a)\Big] \\
&= -\frac{1}{3} + \frac{2i}{9}a^3 - \frac{2i}{45}a^5 + O(a^7)\cdots
\end{aligned}$$

8.7 Alternative expression for total fields inside the body

$$\begin{aligned}\mathbf{E}^{\text{tot}}(\mathbf{x}) &= ikZ_0\mathbb{V}^{-1}\mathbf{J}(\mathbf{x})\\ &= ikZ_0\mathbb{V}^{-1}\sum j_\alpha \mathbb{B}_\alpha(\mathbf{x}).\end{aligned}$$

8.8 Homogeneous dielectric sphere

For a homogeneous dielectric sphere irradiated by a incident field of the form (16), the solution of (28) is

$$\mathbf{j}_\alpha = -\left(\frac{i}{kZ_0}\right)\left[\frac{-k^2(\epsilon-1)}{1-k^2(\epsilon-1)G_{\alpha\alpha}}\right]\frac{1}{\mathcal{N}_\alpha}\{PQ\}_\alpha$$

Total fields inside:

$$\mathbf{E}^{\text{tot}}(\mathbf{x}) = \sum_\alpha \left[\frac{1}{1-k^2(\epsilon-1)G_{\alpha\alpha}}\right]\{P,Q\}_\alpha\{\mathbf{M},\mathbf{N}\}^{\text{reg}}_\alpha(\mathbf{x})$$

Scattered fields outside:

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_\alpha \left[\frac{ik^3(\epsilon-1)}{1-k^2(\epsilon-1)G_{\alpha\alpha}}\right]S_\alpha\{P,Q\}_\alpha\{\mathbf{M},\mathbf{N}\}^{\text{out}}_\alpha(\mathbf{x})$$

Mode-matching solution

From the discussion of Section 5.1 with $P_\alpha = 1$, the total fields in the two regions are

$$\mathbf{E}^{\text{tot}} = \begin{cases} A_\alpha M^{\text{reg}}_\alpha(nk_0;\mathbf{r}), & |\mathbf{r}| < R\\ M^{\text{reg}}_\alpha(k_0;\mathbf{r}) + C_\alpha M^{\text{out}}_\alpha(k_0;\mathbf{r}), & |\mathbf{r}| > R\end{cases}$$

VSWVIE solution

Now I solve the same problem using the VSWVIE formalism of equation (28). The RHS vector $\mathbf{e}_\alpha$ in (28) contains a single nonzero entry:

$$e_\alpha = \sqrt{S_\alpha}$$

For a homogeneous sphere, the $\mathbf{V}$ matrix is proportional to the identity matrix, $\mathbf{V} = -k^2(\epsilon-1)\mathbf{1}$, and since $\mathbf{G}$ is also always diagonal in the VSW basis it follows that the entirety of equation (28) is diagonal, so we have only a single nonzero volume-current coefficient,

$$j_\alpha = \frac{ik}{Z_0}\frac{(\epsilon-1)\sqrt{S_\alpha}}{1-k^2(\epsilon-1)G_{\alpha\alpha}}$$

The scattered field is

$$\begin{aligned}
\mathbf{E}^{\text{scat}} &= ikZ_0\mathbb{G}\star\mathbf{J}\\
&= ikZ_0 j_\alpha \mathbb{G}\star\mathbb{B}_\alpha\\
&= -\left[\frac{k^2(\epsilon-1)}{1-k^2(\epsilon-1)G_{\alpha\alpha}}\right]\Big[C_\alpha^{\text{reg}}\mathbf{M}^{\text{reg}}(k_0;\mathbf{r}) + D_\alpha^{\text{out}}\mathbf{M}^{\text{out}}(k_0;\mathbf{r})\Big]\\
C_\alpha &= \begin{cases} -\dfrac{1}{3k^2} + ik\displaystyle\int_r^R R_\alpha^{\text{out}}(kr')R^{\text{reg}}(kr')r'^2\,dr', & r<R\\ 0, & r>R\end{cases}\\
D_\alpha &= ik\int_0^{\min(r,R)}\Big[R_\alpha^{\text{reg}}(kr')\Big]^2 r'^2\,dr'
\end{aligned}$$

9 Stress-tensor approach to power, force, and torque computation

Power

The power radiated away from (or, the negative of the power absorbed by) the sphere is obtained by integrating the outward-pointing normally-directed Poynting vector over any bounding surface containing the sphere. For convenience we will take the bounding surface to be a sphere of radius $r_b > r_0$ (denote this sphere by $\mathcal{S}_b$). Then the power is

$$\begin{aligned}
P &= \frac{1}{2}\text{Re} \oint_{\mathcal{S}_b} \hat{\mathbf{r}} \cdot \Big[\mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r})\Big] dA \\
&= \frac{r_b^2}{2}\text{Re} \oint \hat{\mathbf{H}}^*(r_b, \Omega) \cdot \Big[\hat{\mathbf{r}} \times \mathbf{E}(r_b, \Omega)\Big] d\Omega \\
&= \frac{r_b^2}{4} \oint \Big[\mathbf{E}^* \cdot \big(\mathbf{H} \times \hat{\mathbf{r}}\big) + \mathbf{H}^* \cdot \big(\hat{\mathbf{r}} \times \mathbf{E}\big)\Big] d\Omega
\end{aligned} \tag{32}$$

The integrand here may be expressed as a 6-dimensional vector-matrix-vector product:

$$= \frac{r_b^2}{4} \oint \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \boldsymbol{\mathcal{P}} \\ -\boldsymbol{\mathcal{P}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} d\Omega \tag{33}$$

where, in our shorthand,

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \boldsymbol{\mathcal{P}} \\ -\boldsymbol{\mathcal{P}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \equiv \begin{pmatrix} E_r \\ E_\theta \\ E_\varphi \\ H_r \\ H_\theta \\ H_\varphi \end{pmatrix}^\dagger \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_r \\ E_\theta \\ E_\varphi \\ H_r \\ H_\theta \\ H_\varphi \end{pmatrix}$$

with

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

If we now insert the expansions (19) and (20) into (33), we obtain the total radiated power as a bilinear form in the $\{C, D\}_{\ell m}$ coefficients:

Force

The ith Cartesian component of the time-average force experienced by the sphere is obtained by integrating the time-average Maxwell stress tensor over a sphere with radius $r_b > r_0$ (call this sphere $\mathcal{S}_b$:)

$$F_x = \frac{1}{2}\text{Re } r_b^2 \int T_{ij}(r_b, \Omega)\hat{n}_j(\Omega) d\Omega. \tag{34}$$

For definiteness I will consider the x-component of the force, $i = x$. The relevant quantity involving the stress tensor is

$$T_{xj}n_j = \epsilon_0 \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right)^\dagger \left(\begin{array}{ccc} \frac{1}{2}n_x & n_y & n_z \\ 0 & -\frac{1}{2}n_x & 0 \\ 0 & 0 & -\frac{1}{2}n_x \end{array}\right) \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right) \quad + \quad \mu_0(\mathbf{E} \to \mathbf{H})$$

where all fields are to be evaluated just outside the sphere surface. The time-average x-directed force per unit area is

$$\begin{aligned} f_x &= \frac{1}{2}\text{Re } T_{xj}n_j \\ &= \frac{1}{4}\Big[T_{xj}n_j + (T_{xj}n_j)^*\Big] \\ &= \frac{\epsilon_0}{4} \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right)^\dagger \left(\begin{array}{ccc} n_x & n_y & n_z \\ n_y & -n_x & 0 \\ n_z & 0 & -n_x \end{array}\right) \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right) \quad + \quad \mu_0(\mathbf{E} \to \mathbf{H}) \end{aligned}$$

which we may write in the shorthand form

$$f_x = \frac{\epsilon}{4}\mathbf{E}^{\text{C}\dagger}\boldsymbol{\mathcal{N}}_x\mathbf{E}^{\text{C}} + \frac{\mu}{4}\mathbf{H}^{\text{C}\dagger}\boldsymbol{\mathcal{N}}_x\mathbf{H}^{\text{C}} \tag{35}$$

where $\{\mathbf{E}, \mathbf{H}\}^{\text{C}}$ are three-vectors of cartesian field components (the superscript C stands for "cartesian") and the 3×3 matrix $\boldsymbol{\mathcal{N}}_x$ is

$$\boldsymbol{\mathcal{N}}_x = \left(\begin{array}{ccc} n_x & n_y & n_z \\ n_y & -n_x & 0 \\ n_z & 0 & -n_x \end{array}\right) = \left(\begin{array}{ccc} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & 0 \\ \cos\theta & 0 & -\sin\theta\cos\phi \end{array}\right) \tag{36}$$

where the latter form is appropriate for points on the surface of a spherical bounding surface.

On the other hand, the Cartesian and spherical components of the field are related by

$$\left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array}\right) = \left(\begin{array}{ccc} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{array}\right) \left(\begin{array}{c} E_r \\ E_\theta \\ E_\phi \end{array}\right) \tag{37}$$

or, in shorthand,

$$\mathbf{E}^{\text{C}} = \boldsymbol{\Lambda}\,\mathbf{E}^{\text{S}} \tag{38}$$

where $\boldsymbol{\Lambda}$ is the 3×3 matrix in equation (38). Inserting (37) into (35) yields

$$f_x = \frac{\epsilon_0}{4}\mathbf{E}^{\text{S}\dagger}\boldsymbol{\mathcal{F}}_x\mathbf{E}^{\text{S}} + \frac{\mu_0}{4}\mathbf{H}^{\text{S}\dagger}\boldsymbol{\mathcal{F}}_x\mathbf{H}^{\text{S}} \tag{39}$$

where $\{\mathbf{E}, \mathbf{H}\}^{\text{S}}$ are 3-dimensional vectors of spherical field components and $\boldsymbol{\mathcal{F}}_x$ is a product of three matrices:

$$\boldsymbol{\mathcal{F}}_x = \boldsymbol{\Lambda}^\dagger\,\boldsymbol{\mathcal{N}}_x\,\boldsymbol{\Lambda}.$$

Working out the matrix multiplications, one finds

$$\boldsymbol{\mathcal{F}}_x = \left(\begin{array}{ccc} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 \\ -\sin\phi & 0 & -\sin\theta\cos\phi \end{array}\right) \tag{40}$$

and, proceeding similarly for the y- and z-directed force,

$$\boldsymbol{\mathcal{F}}_y = \left(\begin{array}{ccc} \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta\sin\phi & -\sin\theta\sin\phi & 0 \\ \cos\phi & 0 & -\sin\theta\sin\phi \end{array}\right) \tag{41}$$

$$\boldsymbol{\mathcal{F}}_z = \left(\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & -\cos\theta \end{array}\right). \tag{42}$$

If I now insert expressions (19) and (20) into equation (39), I obtain the x-directed force per unit area as a bilinear form in the C, D coefficients:

$$\begin{aligned} f_x(\mathbf{x}) = \frac{\epsilon}{4}\sum_{\alpha\beta} \Bigg\{ &\Big(C_\alpha^* C_\beta + D_\alpha^* D_\beta\Big)\Big[\mathbf{M}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{M}_\beta(\mathbf{x}) + \mathbf{N}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{N}_\beta(\mathbf{x})\Big] \\ &+ \Big(C_\alpha^* D_\beta - D_\alpha^* C_\beta\Big)\Big[\mathbf{M}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{N}_\beta(\mathbf{x}) - \mathbf{N}_\alpha^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x\mathbf{M}_\beta(\mathbf{x})\Big]\Bigg\} \end{aligned} \tag{43}$$

The total x-directed force on the sphere is the surface integral of (43) over the full sphere $\mathcal{S}_b$:

$$\begin{aligned} F_x &= \oint_{\mathcal{S}_b} f_x(\mathbf{x})\, d\mathbf{x} \\ &= \frac{\epsilon}{4}\sum_{\alpha\beta}\Bigg\{\Big(C_\alpha^* C_\beta + D_\alpha^* D_\beta\Big)\Big[\Big\langle \mathbf{M}_\alpha \Big| \boldsymbol{\mathcal{F}}_x \Big| \mathbf{M}_\beta\Big\rangle + \Big\langle \mathbf{N}_\alpha \Big| \boldsymbol{\mathcal{F}}_x \Big| \mathbf{N}_\beta\Big\rangle\Big] \\ &\qquad + \Big(C_\alpha^* D_\beta - D_\alpha^* C_\beta\Big)\Big[\Big\langle \mathbf{M}_\alpha \Big| \boldsymbol{\mathcal{F}}_x \Big| \mathbf{N}_\beta\Big\rangle - \Big\langle \mathbf{N}_\alpha \Big| \boldsymbol{\mathcal{F}}_x \Big| \mathbf{M}_\beta\Big\rangle\Big]\Bigg\} \end{aligned} \tag{44}$$

where the inner products involve integrals over the radius-r_b spherical bounding surface, i.e.

$$\Big\langle \mathbf{M}_\alpha \Big| \boldsymbol{\mathcal{F}}_x \Big| \mathbf{M}_\beta\Big\rangle = r_b^2 \int \mathbf{M}_\alpha^\dagger(r_b,\Omega)\boldsymbol{\mathcal{F}}_x\mathbf{M}_\beta(r_b,\Omega)\, d\Omega. \tag{45}$$

9.1 Sample stress-tensor power calculation

As a specific example of a radiated-power computation, let's consider a pointlike dipole source at the center of a lossy sphere and ask for the total power radiated away from the sphere.

Assuming the dipole is z-directed, i.e. $\mathbf{p} = p_z\hat{\mathbf{z}}$, the only nonvanishing spherical multipole coefficient of the incident field [equation (16)] is

$$Q_{1,0} = \frac{-icZ_0(nk_0)^3}{\epsilon\sqrt{6\pi}}p_z$$

where $k_0 = \omega/c$ is the free-space wavenumber.

The total fields outside the sphere are

$$\mathbf{E}^{\text{tot}}(\mathbf{x}) = D_{1,0}\mathbf{N}^{\text{out}}_{1,0}(\mathbf{x}), \qquad \mathbf{H}^{\text{tot}}(\mathbf{x}) = \frac{1}{Z_0}D_{1,0}\mathbf{M}^{\text{out}}_{1,0}(\mathbf{x})$$

where $D_{1,0}$ is given by (21):

$$D_{1,0} = \frac{1}{\sqrt{6\pi}}\frac{cZ_0e^{-ia}n^3a^6p_z}{[n^2\left(a^3+a+i\right)-a-i]\sin(na)+na[in^2(-1+a(a+i))+a+i]\cos(na)} \tag{46}$$

where the dimensionless "size parameter" is

$$a = k_0R.$$

The radiated power is

$$\begin{aligned}
P^{\text{rad}} &= \frac{1}{2}\text{Re}\oint(\mathbf{E}^*\times\mathbf{H})\cdot\hat{\mathbf{r}}\,dA \\
&= \frac{|D_{1,0}|^2}{2Z_0}\cdot r^2\underbrace{\text{Re}\int\Big(\mathbf{N}^*_{1,0}(r,\Omega)\times\mathbf{M}_{1,0}(r,\Omega)\Big)\cdot\hat{\mathbf{r}}\,d\Omega}_{=1/(k_0r)^2} \\
&= \frac{|D_{1,0}|^2}{2k_0^2Z_0}
\end{aligned} \tag{47}$$

Sanity check

As a sanity check, let's first try putting $\epsilon = 1$. Then equation (46) reads

$$D_{1,0}(\epsilon=1) = \frac{-icZ_0k^3}{\sqrt{6\pi}}p_z$$

and equation (47) reads

$$P^{\text{rad}} = \frac{c^2Z_0k^4}{12\pi}p_z^2$$

in agreement with Jackson equation (9.24).

Nontrivial examples

As less trivial examples, consider putting **(a)** $\epsilon = 3$ and **(b)** $\epsilon = 3+6i$.

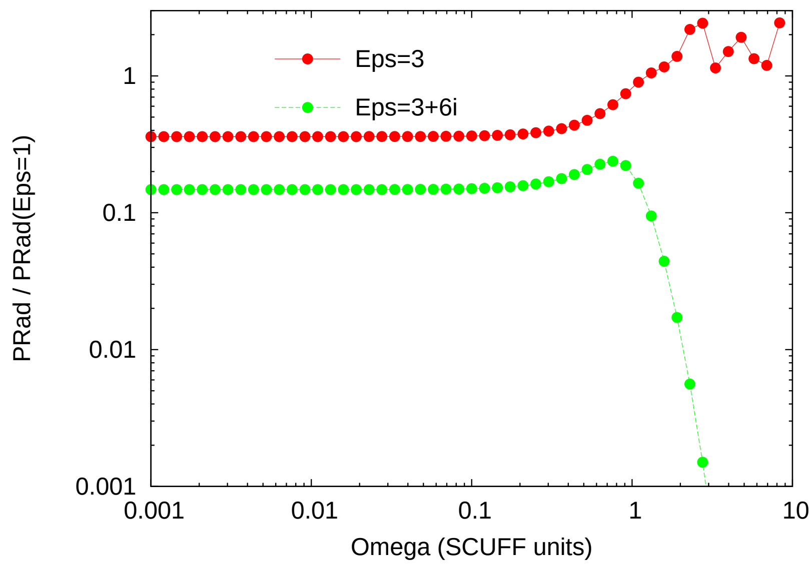

Figure 1: Power radiated by a dipole at the center of a dielectric sphere, normalized by the power radiated by a dipole in free space.

9.2 Sample stress-tensor force calculation

The simplest incident-field configuration that gives rise to a nonvanishing total force on the sphere is a superposition of $(1,0)$ and $(2,1)$ spherical waves, corresponding to coherent dipole and quadrupole sources at the origin. Thus, in the incident-field expansion (16) we take

$$P_{(1,0)} = P_{(2,1)} = 1, \qquad P_\alpha = 0 \text{ for all other } \alpha, \qquad Q_\alpha = 0 \text{ for all } \alpha. \tag{48}$$

The coefficients in expansions (19, 20) for the fields outside the sphere are then similarly given by

$$C_{(1,0)} = C_{(2,1)} = \text{nonzero}, \qquad C_\alpha = 0 \text{ for all other } \alpha, \qquad D_\alpha = 0 \text{ for all } \alpha. \tag{49}$$

The actual values of $C_{(1,0)}$ and $C_{(2,1)}$, which are less important for our immediate goals, are determined by equation (21) for a specific frequency, dielectric constant, and sphere radius. For example, for the particular case $\{\omega, r_0, \epsilon\} = \{3\cdot 10^{14} \text{ rad/sec}, 1\ \mu\text{m}, 10\}$ we find

$$C_{(1,0)} = -0.558 + 0.760i, \qquad C_{(2,1)} = 0.100 + 0.001i.$$

9.3 x-directed force density

At a point $\mathbf{x} = (r_b, \Omega)$ on the surface of the bounding sphere of radius r_b, the x-directed force per unit area is, from (39),

$$\begin{aligned} f_x(\mathbf{x}) = \frac{\epsilon}{4}\Big\{ & C_{10}^* C_{10}\Big[\mathbf{M}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{10}(\mathbf{x}) + \mathbf{N}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{10}(\mathbf{x})\Big] \\ & + C_{10}^* C_{21}\Big[\mathbf{M}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{21}(\mathbf{x}) + \mathbf{N}_{10}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{21}(\mathbf{x})\Big] \\ & + C_{21}^* C_{10}\Big[\mathbf{M}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{10}(\mathbf{x}) + \mathbf{N}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{10}(\mathbf{x})\Big] \\ & + C_{21}^* C_{21}\Big[\mathbf{M}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{M}_{21}(\mathbf{x}) + \mathbf{N}_{21}^*(\mathbf{x})\boldsymbol{\mathcal{F}}_x(\Omega)\mathbf{N}_{21}(\mathbf{x})\Big]\Big\} \end{aligned}$$

9.4 Total x-directed force

The *total* force is obtained from equation (2):

$$F_x = \frac{\epsilon_0}{4}\left\{ C_{10}^* C_{21}\left[\left\langle \mathbf{M}_{10}\middle|\boldsymbol{\mathcal{F}}_x\middle|\mathbf{M}_{21}\right\rangle + \left\langle \mathbf{N}_{10}\middle|\boldsymbol{\mathcal{F}}_x\middle|\mathbf{N}_{21}\right\rangle\right] + \mathrm{CC}\right\} \tag{50}$$

where CC stands for "complex conjugate." [The inner product here is defined by equation (45).] With some effort, we compute

$$\left\langle \mathbf{M}_{10}\middle|\boldsymbol{\mathcal{F}}_x\middle|\mathbf{M}_{21}\right\rangle + \left\langle \mathbf{N}_{10}\middle|\boldsymbol{\mathcal{F}}_x\middle|\mathbf{N}_{21}\right\rangle = -i\sqrt{\frac{3}{10}}\frac{1}{k^2}$$

and thus the total force (50) reads

$$F_x = -\frac{\epsilon_0}{2k^2}\sqrt{\frac{3}{10}}\ \mathrm{Im}\ \Big(C_{10}^* C_{21}\Big) \tag{51}$$

To make sense of the units here, suppose that field-strength coefficients like P, Q, C, D in (16) and (19) are measured in typical SCUFF-EM units of V/μm, while k is measured in units of inverse μm. Then the units of (51) are

$$\text{units of (51)} = \frac{\epsilon_0 \cdot \mathrm{V}^2 \cdot \mu\mathrm{m}^{-2}}{\mu\mathrm{m}^{-2}}$$

Use $\epsilon_0 = \frac{1}{Z_0 c}$ where c is the vacuum speed of light:

$$= \frac{1}{Z_0 c} \cdot \mathrm{V}^2$$

Use $Z_0 = 376.7$ V/A:

$$= \frac{376.7\ \mathrm{V}\cdot\mathrm{A}}{3\cdot 10^{14}\ \mu\mathrm{m}\cdot\mathrm{s}^{-1}}$$

Now use 1 V$\cdot$ A=1 watt, 1 watt $\cdot$ 1 s = 1 joule, 1 joule / 1 μm = 10^6 Newtons:

$$= 1.26 \cdot 10^{-6} \text{ Newtons.}$$

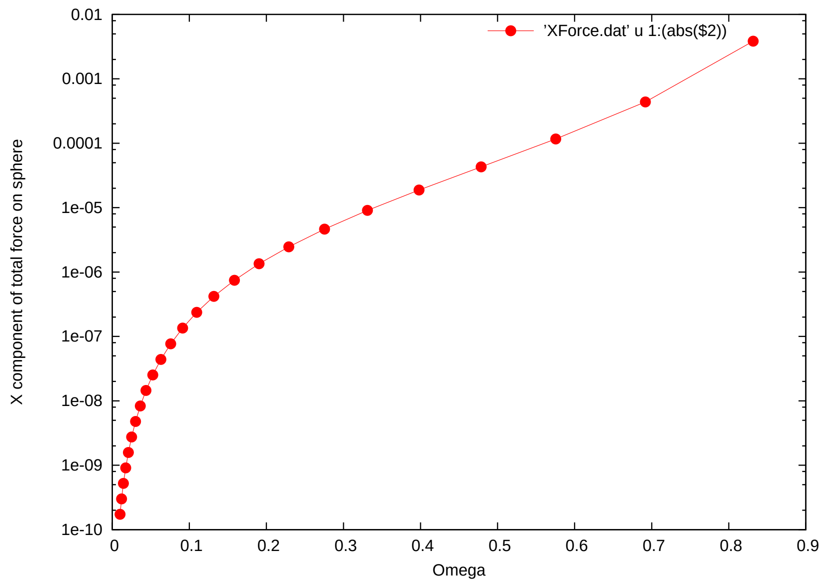

Figure 2: x-component of total force on sphere irradiated from within by an incident field of the form (16) with coefficients (49).

10 $\mathbb{T}$-matrix elements and surface currents

The $\mathbb{T}$-matrix for a body relates the coefficients of the outgoing spherical-wave expansion of the scattered field to the coefficients of the regular spherical-wave expansion of the incident field; thus, if we write

$$\mathbf{E}^{\text{inc}} = \sum c_\alpha^{\text{inc}} \boldsymbol{\mathcal{W}}_\alpha^{\text{regular}}, \qquad \mathbf{E}^{\text{scat}} = \sum c_\alpha^{\text{scat}} \boldsymbol{\mathcal{W}}_\alpha^{\text{outgoing}}$$

then the coefficient vectors are related by

$$\mathbf{c}^{\text{scat}} = \mathbb{T}\mathbf{c}^{\text{inc}}.$$

Individual $\mathbb{T}$-matrix elements have the significance

$$\mathbb{T}_{\alpha\beta} = \left\{ \begin{array}{l} \text{coefficient of outgoing } \alpha\text{-type wave} \\ \text{due to irradiation by unit-amplitude} \\ \text{incoming } \beta\text{-type wave.} \end{array} \right\}$$

Now consider a scattering geometry irradiated by a unit-amplitude regular wave, and let $\mathbf{K}$ and $\mathbf{N}$ be the electric and magnetic surface currents induced by this excitation. The scattered field is

$$\mathbf{E}^{\text{scat}} = ikZ\mathbb{G} \star \mathbf{K} + ik\mathbb{C} \star \mathbf{N}$$

with k and Z the wavevector and (absolute) wave impedance of the exterior medium. Insert the expansions (8):

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = -k^2 \sum_\alpha \Big\{ Z\Big\langle \boldsymbol{\mathcal{W}}_\alpha^{\text{reg}} \Big| \mathbf{K} \Big\rangle \boldsymbol{\mathcal{W}}_\alpha^{\text{out}}(\mathbf{x}) + \sigma_\alpha \Big\langle \boldsymbol{\mathcal{W}}_\alpha^{\text{reg}} \Big| \mathbf{N} \Big\rangle \boldsymbol{\mathcal{W}}_{\overline{\alpha}}^{\text{out}}(\mathbf{x}) \Big\}$$

or, rearranging slightly,

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_\alpha \underbrace{-k^2 \Big[Z\big\langle \boldsymbol{\mathcal{W}}_\alpha^{\text{reg}} \big| \mathbf{K} \big\rangle - \sigma_{\overline{\alpha}} \big\langle \boldsymbol{\mathcal{W}}_{\overline{\alpha}}^{\text{reg}} \big| \mathbf{N} \big\rangle \Big]}_{\mathbb{T}_{\alpha\beta}} \boldsymbol{\mathcal{W}}_\alpha^{\text{out}}(\mathbf{x}). \tag{52}$$

This yields a prescription for computing $\mathbb{T}$-matrix elements directly from surface currents.