SCUFF-SCATTER Implementation Notes

Homer Reid

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1 Concise Formulas for Scattered and Absorbed Power

In many scattering problems we will want to compute the total power scattered from, and the total power absorbed by, the object(s) in a scattering geometry. The naïve way to do this would be to integrate the normal Poynting vector over some fictitious bounding surface surrounding the object(s); this integral could be evaluated by numerical quadrature, with the values of the Poynting vector at each quadrature point computed by the usual LIBSCUFF methods for calculating scattered fields. SCUFF-SCATTER uses a more efficient approach, obtaining the scattered and absorbed power *directly* from matrix-vector and vector-vector (dot) products involving the BEM matrices and vectors.[1]

Throughout this section, $\mathcal{S}$ denotes the boundary of the exterior medium (that is, the union of the outer surfaces of all objects contained in the exterior medium), and $\hat{\mathbf{n}}$ denotes the normal vector at a point on $\mathcal{S}$; $\hat{\mathbf{n}}$ is taken positive pointing into the exterior medium (away from the object). We work at a single frequency and suppress ω arguments.

1.1 Absorbed Power

The absorbed power is the integral of the inward-directed normal component of the total Poynting vector over the full surface $\mathcal{S}$:

$$P^{\text{abs}} = -\oint \mathbf{P}^{\text{tot}}(\mathbf{x}) \cdot \hat{\mathbf{n}}\, dA \tag{1}$$

where the minus sign arises because by convention we define $\hat{\mathbf{n}}$ to be the outward-directed surface normal.

At a point $\mathbf{x}$ on $\mathcal{S}$, the (outward-directed) normal component of the total Poynting vector is

$$\mathbf{P}^{\text{tot}}(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = \frac{1}{2}\text{Re}\left\{ \hat{\mathbf{n}}(\mathbf{x}) \cdot \Big[\mathbf{E}^{\text{tot}*}(\mathbf{x}) \times \mathbf{H}^{\text{tot}}(\mathbf{x}) \Big] \right\} \tag{2}$$

Using the fact that $|\hat{\mathbf{n}}| = 1$, we may rewrite this in the form (temporarily suppressing $\mathbf{x}$ arguments)

$$= -\frac{1}{2}\text{Re}\left\{ \big(\hat{\mathbf{n}} \times \mathbf{H}^{\text{tot}*}\big) \cdot \big(\hat{\mathbf{n}} \times \big(- \hat{\mathbf{n}} \times \mathbf{E}^{\text{tot}}\big)\big) \right\}.$$

[1] The possibility of deriving compact expressions like these seems to have been first noticed by Steven Johnson.

But the quantities in parentheses here are just the electric and magnetic surface currents that enter into the SIE formulation of scattering problems, so we find simply

$$\mathbf{P}^{\text{tot}}(\mathbf{x})\cdot\hat{\mathbf{n}}(\mathbf{x}) = -\frac{1}{2}\text{Re}\bigg\{\mathbf{K}^*\cdot\Big[\hat{\mathbf{n}}\times\mathbf{N}\Big]\bigg\}. \tag{3}$$

Inserting (**??**) into (**??**), we have

$$P^{\text{abs}} = \frac{1}{2}\text{Re}\,\oint\bigg\{\mathbf{K}^*(\mathbf{x})\cdot\Big[\hat{\mathbf{n}}\times\mathbf{N}(\mathbf{x})\Big]\bigg\}d\mathbf{x} \tag{4}$$

Insert the surface-current expansions $\mathbf{K}(\mathbf{x}) = \sum_\alpha k_\alpha \mathbf{f}_\alpha(\mathbf{x})$, $\mathbf{N}(\mathbf{x}) = -Z_0\sum_\alpha n_\alpha \mathbf{f}_\alpha(\mathbf{x})$:

$$= -\frac{Z_0}{2}\text{Re}\,\sum_{\alpha\beta} k_\alpha^* n_\beta \oint \mathbf{f}_\alpha(\mathbf{x})\cdot\Big[\hat{\mathbf{n}}\times\mathbf{f}_\beta(\mathbf{x})\Big]d\mathbf{x} \tag{5}$$

$$= -\frac{Z_0}{2}\text{Re}\,\sum_{\alpha\beta} k_\alpha^* n_\beta \oint \mathbf{f}_\alpha(\mathbf{x})\cdot\Big[\hat{\mathbf{n}}\times\mathbf{f}_\beta(\mathbf{x})\Big]d\mathbf{x} \tag{6}$$

$$= -\frac{Z_0}{2}\text{Re}\,\sum_{\alpha\beta} k_\alpha^* O^{(\times)}_{\alpha\beta} n_\beta \tag{7}$$

where I have introduced the "crossed overlap matrix" $\mathbf{O}^{(\times)}$ for the RWG basis, with matrix elements

$$O^{(\times)}_{\alpha\beta} = \oint \mathbf{f}_\alpha(\mathbf{x})\cdot\Big[\hat{\mathbf{n}}\times\mathbf{f}_\beta(\mathbf{x})\Big]d\mathbf{x}.$$

This is an extremely sparse matrix, with vanishing diagonals and precisely 4 nonzero entries per row, which may be computed in closed form (cf. LIBSCUFF technical memo, Section 12.2). Thus the operation count for evaluating (**??**) is $\mathcal{O}(N)$, as compared to $\mathcal{O}(N^2)$ for the equivalent matrix-vector product formulas.

Units

The crossed-overlap matrix element $O^{(\times)}_{\alpha\beta}$ has units of area. The electric and magnetic surface-current expansion coefficients $\{k_\alpha, n_\alpha\}$ both have units of current/length. (The magnetic current $\mathbf{N}$ has units of voltage/length, but in LIBSCUFF conventions the magnetic-current expansion coefficients $\{n_\alpha\}$ are defined with the Z_0 prefactor that ensures they have the same units as $\{k_\alpha\}$.) Then the units of (**??**) are

$$\left[\frac{\text{voltage}}{\text{current}}\right]\cdot\left[\frac{\text{current}}{\text{length}}\right]\cdot\Big[\text{length}^2\Big]\cdot\left[\frac{\text{current}}{\text{length}}\right] = [\text{voltage}\ \cdot\ \text{current}] = [\text{power}]\ \checkmark$$

Nested Surfaces

Expression **??** remains valid as-is when nested surfaces are present, as long as the summation is understood to run only over basis functions defined on the surfaces of objects that border the exterior medium.

1.2 Scattered Power

The scattered power is the integral of the outward-directed normal component of the *scattered* Poynting vector over $\mathcal{S}$:

$$P^{\text{scat}} = +\oint \mathbf{P}^{\text{scat}}(\mathbf{x}) \cdot \hat{\mathbf{n}}\, dA \tag{8}$$

where the scattered Poynting vector is the Poynting vector as computed using only the scattered fields. In analogy to equation (**??**), we write

$$\mathbf{P}^{\text{scat}}(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = \frac{1}{2}\text{Re}\left\{\hat{\mathbf{n}}(\mathbf{x}) \cdot \Big[\mathbf{E}^{\text{scat}*}(\mathbf{x}) \times \mathbf{H}^{\text{scat}}(\mathbf{x})\Big]\right\}$$

Noting that scattered fields are the differences between total and incident fields and again suppressing $\mathbf{x}$ arguments, we find

$$= \frac{1}{2}\text{Re}\left\{\hat{\mathbf{n}} \cdot \Big[\big(\mathbf{E}^{\text{tot}*} - \mathbf{E}^{\text{inc}*}\big) \times \big(\mathbf{H}^{\text{tot}} - \mathbf{H}^{\text{inc}}\big)\Big]\right\}$$

which we write as a sum of three terms:

$$\begin{aligned} = \quad & \mathbf{P}^{\text{tot}} \cdot \hat{\mathbf{n}} && \text{(9a)}\\ & + \frac{1}{2}\text{Re}\left\{\hat{\mathbf{n}} \cdot \Big[\mathbf{E}^{\text{inc}*} \times \mathbf{H}^{\text{inc}}\Big]\right\} && \text{(9b)}\\ & - \frac{1}{2}\text{Re}\left\{\hat{\mathbf{n}} \cdot \Big[\mathbf{E}^{\text{inc}*} \times \mathbf{H}^{\text{tot}}\Big] + \hat{\mathbf{n}} \cdot \Big[\mathbf{E}^{\text{tot}*} \times \mathbf{H}^{\text{inc}}\Big]\right\}. && \text{(9c)} \end{aligned}$$

First Term

The first term here (**??**a) is the normal component of the total Poynting vector, as considered in the previous section; the surface integral of this term yields $-P^{\text{abs}}$.

Second Term

The second term (**??**b) is the normal Poynting flux due to the incident field sources alone. The surface integral of this term yields the net power delivered to the volume of the scatterer in the absence of the scatterer. But in the absence of the scatterer there is nowhere for this power to go; there is nothing to absorb it, so any power that flows in must flow back out. Hence the surface integral of this term vanishes.

Third Term

The third term (**??**b) is

$$
\begin{aligned}
-\frac{1}{2}\text{Re}&\left\{\hat{\mathbf{n}}\cdot\Big[\mathbf{E}^{\text{inc}*}\times\mathbf{H}^{\text{tot}}\Big] + \hat{\mathbf{n}}\cdot\Big[\mathbf{E}^{\text{tot}*}\times\mathbf{H}^{\text{inc}}\Big]\right\}\\
&= +\frac{1}{2}\text{Re}\left\{\mathbf{E}^{\text{inc}*}\cdot\Big[\hat{\mathbf{n}}\times\mathbf{H}^{\text{tot}}\Big] + \mathbf{H}^{\text{inc}*}\cdot\Big[-\hat{\mathbf{n}}\times\mathbf{E}^{\text{tot}}\Big]\right\}\\
&= +\frac{1}{2}\text{Re}\left\{\mathbf{E}^{\text{inc}*}\cdot\mathbf{K} + \mathbf{H}^{\text{inc}*}\cdot\mathbf{N}\right\}
\end{aligned}
$$

Insert the surface-current expansions $\mathbf{K}=\sum k_\alpha \mathbf{f}_\alpha, \mathbf{N}=-Z_0\sum n_\alpha \mathbf{f}_\alpha$:

$$
= +\frac{1}{2}\text{Re}\ \sum_\alpha \left\{k_\alpha \mathbf{E}^{\text{inc}*}\cdot\mathbf{f}_\alpha - Z_0 n_\alpha \mathbf{H}^{\text{inc}*}\cdot\mathbf{f}_\alpha\right\}
$$

The surface integral of this is

$$
\begin{aligned}
\oint\{\} &= \frac{Z_0}{2}\text{Re}\ \sum_\alpha\left\{k^*_\alpha \underbrace{\Big[\oint \mathbf{E}^{\text{inc}*}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})d\mathbf{x}\Big]/Z_0}_{v^{\text{E}}_\alpha} - n^*_\alpha \underbrace{\Big[\oint \mathbf{H}^{\text{inc}*}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})d\mathbf{x}\Big]}_{v^{\text{H}}_\alpha}\right\}\\
&= \frac{Z_0}{2}\sum_\alpha\Big[k^*_\alpha v^{\text{E}}_\alpha - n^*_\alpha v^{\text{H}}_\alpha\Big]
\end{aligned}
$$

where $\{v^{\text{E}}_\alpha, v^{\text{H}}_\alpha\}$ are the elements of the RHS vector of the BEM system as computed by LIBSCUFF.

Total Scattered Power

Combining all of the above, the total scattered power is

$$
P^{\text{scat}} = -P^{\text{abs}} + \frac{1}{2}\texttt{KN}^*\cdot\overline{\texttt{RHS}}
$$

where `KN` and `RHS` are respectively the vector of surface-current expansion coefficients and the RHS vector, as computed by LIBSCUFF, and the bar on the second term in the dot product indicates that the magnetic contributions enter with a minus sign.

2 Plane-wave scattering from a dielectric sphere

To demonstrate the validity of the concise power formulas, we'll use them here to investigate the scattering and absorption cross-sections for a dielectric sphere illuminated by a plane wave.

Exact Results

Calculations using SCUFF-SCATTER

Converting power data to cross-sections

The quantities that SCUFF-SCATTER writes to the `Sphere.power` file are the total scattered and absorbed power. To obtain the scattering and absorption cross-sections, we divide the power by the incident power per unit area, which in this case is

$$\text{incident flux of a plane wave in vacuum} = \frac{|\mathbf{E}_0|^2}{2Z_0}.$$