SIE Approach to Fresnel Scattering

Homer Reid

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Abstract

I revisit the theory of Fresnel scattering (that is, the scattering of a plane wave from an infinite planar dielectric interface) from the perspective of the surface-integral-equation approach to electromagnetic scattering. The idea is to develop some intuition for the physical significance of effective electric and magnetic surface currents by demonstrating how they arise in the simplest analytically-solvable scattering problem.

Contents

1 Scattering from a Planar Surface: Exact Solution

1.1 Normal Incidence

Consider an infinite dielectric half-space filling the region $z < 0$, and a plane wave normally incident from above.

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = E_0 e^{-ik_0 z}\hat{\mathbf{x}} \qquad \mathbf{H}^{\text{inc}}(\mathbf{x}) = -\frac{1}{Z_0}E_0 e^{-ik_0 z}\hat{\mathbf{y}} \tag{1}$$

The scattered fields above the xy plane, and the total fields below the xy plane, are

$$\begin{aligned} \mathbf{E}^{\text{scat,above}}(\mathbf{x}) &= Ae^{+ik_0 z}\hat{\mathbf{x}}, & \mathbf{H}^{\text{scat,above}}(\mathbf{x}) &= \frac{1}{Z_0}Ae^{+ik_0 z}\hat{\mathbf{y}}, \\ \mathbf{E}^{\text{tot,below}}(\mathbf{x}) &= Be^{-ik_1 z}\hat{\mathbf{x}}, & \mathbf{H}^{\text{tot,below}}(\mathbf{x}) &= -\frac{1}{Z_0 Z^1}Be^{-ik_1 z}\hat{\mathbf{y}} \end{aligned}$$

where Z_0 is the impedance of free space, Z^1 is the dimensionless relative wave impedance of the half-space, and A and B are unknown coefficients to be determined.

Matching tangential fields at $z = 0$ yields

$$\begin{aligned} E_0 + A &= B, \\ E_0 - A &= \frac{1}{Z^1}B, \end{aligned}$$

with solution

$$A = \frac{Z^1 - 1}{Z^1 + 1}E_0, \qquad B = \frac{2Z^1}{Z^1 + 1}E_0.$$

For $Z^1 = 1$ (the dielectric half-space isn't there) we have $A = 0, B = 1$, so there is no reflected wave and the full wave is transmitted.

Surface Currents

$$\begin{aligned} \mathbf{K} &= \hat{\mathbf{z}} \times \mathbf{H}^{\text{tot}} &&= +\frac{2E_0}{Z_0(Z^1+1)}\hat{\mathbf{x}} \\ \mathbf{N} &= -\hat{\mathbf{z}} \times \mathbf{E}^{\text{tot}} &&= -\frac{2Z^1 E_0}{(Z^1+1)}\hat{\mathbf{y}} \end{aligned} \tag{2}$$

1.2 Non-Normal Incidence, E perpendicular to plane of incidence

I now rotate the propagation vector through an angle θ around the y axis. (For $\theta = 0$ I recover the results of the previous section.) The wavevectors of the incident, reflected, and transmitted waves are

$$\begin{aligned}\hat{\mathbf{k}}^i &= k_0\Big(\sin\theta\hat{\mathbf{y}} - \cos\theta\hat{\mathbf{z}}\Big)\\ \hat{\mathbf{k}}^r &= k_0\Big(\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\Big)\\ \hat{\mathbf{k}}^t &= k_1\Big(\sin\theta'\hat{\mathbf{y}} + \cos\theta'\hat{\mathbf{z}}\Big)\end{aligned}$$

where θ' is related to the incident angle by

$$k_0\sin\theta = k_1\sin\theta'.$$

Incident wave:

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = E_0 e^{ik_0[y\sin\theta - z\cos\theta]}\,\hat{\mathbf{x}} \tag{3}$$

$$\mathbf{H}^{\text{inc}}(\mathbf{x}) = \frac{E_0}{Z_0}e^{ik_0[y\sin\theta - z\cos\theta]}\Big(-\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}\Big) \tag{4}$$

Reflected wave:

$$\mathbf{E}^{\text{refl}}(\mathbf{x}) = rE_0 e^{ik_0[y\sin\theta + z\cos\theta]}\,\hat{\mathbf{x}} \tag{5}$$

$$\mathbf{H}^{\text{refl}}(\mathbf{x}) = \frac{rE_0}{Z_0}e^{ik_0[y\sin\theta + z\cos\theta]}\Big(+\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}\Big) \tag{6}$$

Transmitted wave:

$$\begin{aligned}\mathbf{E}^{\text{trans}}(\mathbf{x}) &= tE_0 e^{ik_1[y\sin\theta' + z\cos\theta']}\,\hat{\mathbf{x}}\\ \mathbf{H}^{\text{trans}}(\mathbf{x}) &= \frac{tE_0}{Z_1}e^{ik_1[y\sin\theta' + z\cos\theta']}\Big(-\cos\theta'\,\hat{\mathbf{y}} - \sin\theta'\,\hat{\mathbf{z}}\Big)\end{aligned}$$

Surface currents

$$\begin{aligned}\mathbf{K} &= \frac{E_0}{Z_0}(1-r)\cos\theta e^{ik_0 y\sin\theta}\,\hat{\mathbf{x}}\\ \mathbf{N} &= E_0(1+r)e^{ik_0 y\sin\theta}\,\hat{\mathbf{y}}\end{aligned}$$

Let's check that the normal fields at the surface are correctly recovered from the divergence-of-surface-current prescription:

$$E_z = \frac{\nabla\cdot\mathbf{K}}{i\epsilon\omega} = 0 \qquad \checkmark$$

$$H_z = -\frac{\nabla\cdot\mathbf{N}}{i\mu\omega} = -\frac{iE_0k_0\sin\theta(1+r)}{i\mu\omega}e^{ik_0 y\sin\theta} = -\frac{E_0}{Z_0}(1+r)e^{ik_0 y\sin\theta} \qquad \checkmark$$

1.3 Non-Normal Incidence, E parallel to plane of incidence

Incident wave:

$$\begin{aligned}\mathbf{E}^{\text{inc}}(\mathbf{x}) &= E_0 e^{ik_0[y\sin\theta - z\cos\theta]}\Big(\cos\theta\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}\Big)\\ \mathbf{H}^{\text{inc}}(\mathbf{x}) &= \frac{E_0}{Z_0}e^{ik_0[y\sin\theta - z\cos\theta]}\,\hat{\mathbf{x}}\end{aligned}$$

Reflected wave:

$$\begin{aligned}
\mathbf{E}^{\text{refl}} &= rE_0 e^{ik_0[y\sin\theta + z\cos\theta]}\Big(-\cos\theta\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}\Big)\\
\mathbf{H}^{\text{refl}}(\mathbf{x}) &= r\frac{E_0}{Z_0}e^{ik_0[y\sin\theta + z\cos\theta]}\,\hat{\mathbf{x}}
\end{aligned}$$

Transmitted wave:

$$\begin{aligned}
\mathbf{E}^{\text{trans}}(\mathbf{x}) &= tE_0 e^{ik_1[y\sin\theta' - z\cos\theta']}\Big(\cos\theta'\,\hat{\mathbf{y}} + \sin\theta'\,\hat{\mathbf{z}}\Big)\\
\mathbf{H}^{\text{trans}}(\mathbf{x}) &= \frac{tE_0}{Z_1}e^{ik_0[y\sin\theta' - z\cos\theta']}\,\hat{\mathbf{x}}
\end{aligned}$$

Surface currents

$$\begin{aligned}
\mathbf{K} &= \frac{E_0}{Z_0}(1+r)e^{ik_0 y\sin\theta}\,\hat{\mathbf{y}}\\
\mathbf{N} &= E_0(1-r)\cos\theta e^{ik_0 y\sin\theta}\,\hat{\mathbf{x}}
\end{aligned}$$

Let's check that the normal fields at the surface are correctly recovered from the divergence-of-surface-current prescription:

$$\begin{aligned}
E_z &= \frac{\nabla\cdot\mathbf{K}}{i\epsilon\omega} = E_0(1+r)\sin\theta \qquad \checkmark\\
H_z &= -\frac{\nabla\cdot\mathbf{N}}{i\mu\omega} = 0 \qquad \checkmark
\end{aligned}$$

2 Scattering from a Planar Surface: SIE solution

I now re-solve the same problem using the SIE method.

2.1 Surface-Current Basis Functions

Notation:

- $\mathbf{x} = (x, y)$ is a two-component coordinate vector.
- $\mathbf{r} = (x, y, z) = (\mathbf{x}, z)$ is a three-component coordinate vector.
- $\mathbf{p}$ is a two-component Fourier vector.

There are two types of surface-current basis functions for surface currents at $z = 0$:

$$\mathbf{f}_{x\mathbf{p}}(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}}\hat{\mathbf{x}}, \qquad \mathbf{f}_{y\mathbf{p}}(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}}\hat{\mathbf{y}}$$

The electric and magnetic surface currents at $z = 0$ are expanded in the $\mathbf{f}_\mathbf{p}$ basis:

$$\mathbf{K}(\mathbf{x}) = \sum_\mathbf{p} \Big[K_{x\mathbf{p}}\mathbf{f}_{x\mathbf{p}}(\mathbf{x}) + K_{y\mathbf{p}}\mathbf{f}_{y\mathbf{p}}(\mathbf{x})\Big]$$

$$\mathbf{N}(\mathbf{x}) = \sum_\mathbf{p} \Big[N_{x\mathbf{p}}\mathbf{f}_{x\mathbf{p}}(\mathbf{x}) + N_{y\mathbf{p}}\mathbf{f}_{y\mathbf{p}}(\mathbf{x})10\Big]$$

where

$$\sum_\mathbf{p} = \int \frac{d^2\mathbf{p}}{(2\pi)^2}.$$

2.2 Convolutions of surface currents with G and C dyadics:

I first recall my conventions for the dyadic green's functions: In homogeneous medium r, the fields arising from surface currents are

$$\mathbf{E}(\mathbf{x}) = \int \Big\{\mathbf{\Gamma}^{\text{EE},r}(\mathbf{x},\mathbf{x}')\cdot\mathbf{K}(\mathbf{x}') + \mathbf{\Gamma}^{\text{EM},r}(\mathbf{x},\mathbf{x}')\cdot\mathbf{N}(\mathbf{x}')\Big\}d\mathbf{x}'$$

$$\mathbf{H}(\mathbf{x}) = \int \Big\{\mathbf{\Gamma}^{\text{ME},r}(\mathbf{x},\mathbf{x}')\cdot\mathbf{K}(\mathbf{x}') + \mathbf{\Gamma}^{\text{MM},r}(\mathbf{x},\mathbf{x}')\cdot\mathbf{N}(\mathbf{x}')\Big\}d\mathbf{x}'$$

where

$$\begin{array}{rclcrcl}
\mathbf{\Gamma}^{\text{EE},r} & = & ik_r Z_0 Z^r \mathbf{G}, & \qquad & \mathbf{\Gamma}^{\text{EM},r} & = & +ik_r\mathbf{C}, \\
\mathbf{\Gamma}^{\text{ME},r} & = & -ik_r\mathbf{C}, & \qquad & \mathbf{\Gamma}^{\text{MM},r} & = & +\frac{ik_r}{Z_0 Z^r}\mathbf{G},
\end{array}$$

In the following, I put

$$Q \equiv Q(k,\mathbf{p}) = \sqrt{k^2 - |\mathbf{p}|^2}.$$

Note that Q is imaginary for $|\mathbf{p}| > k$. At a temporal frequency of $\omega = ck$, the fields arising from surface currents with spatial frequency $|\mathbf{p}| > k$ are evanescent waves.

The inner products I need are

$$\int \mathbf{G}(k;\mathbf{r},\mathbf{r}')\cdot\mathbf{f}_{x\mathbf{p}}(\mathbf{r}')d\mathbf{r}' = \frac{ie^{i\mathbf{p}\cdot\mathbf{x}}e^{iQ|z|}}{2Q}\left[\left(1-\frac{p_x^2}{k^2}\right)\hat{\mathbf{x}} - \frac{p_xp_y}{k^2}\hat{\mathbf{y}} - \frac{iQp_x}{k^2}\hat{\mathbf{z}}\right]$$

$$\int \mathbf{G}(k;\mathbf{r},\mathbf{r}')\cdot\mathbf{f}_{y\mathbf{p}}(\mathbf{r}')d\mathbf{r}' = \frac{ie^{i\mathbf{p}\cdot\mathbf{x}}e^{iQ|z|}}{2Q}\left[-\frac{p_xp_y}{k^2}\hat{\mathbf{x}} + \left(1-\frac{p_y^2}{k^2}\right)\hat{\mathbf{y}} - \frac{iQp_y}{k^2}\hat{\mathbf{z}}\right]$$

$$\int \mathbf{C}(k;\mathbf{r},\mathbf{r}')\cdot\mathbf{f}_{x\mathbf{p}}(\mathbf{r}')d\mathbf{r}' = \text{sign }(z)\frac{ie^{i\mathbf{p}\cdot\mathbf{x}}e^{iQ|z|}}{2k}\left[\hat{\mathbf{y}} + \frac{ip_y}{Q}\hat{\mathbf{z}}\right]$$

$$\int \mathbf{C}(k;\mathbf{r},\mathbf{r}')\cdot\mathbf{f}_{y\mathbf{p}}(\mathbf{r}')d\mathbf{r}' = -\text{sign }(z)\frac{ie^{i\mathbf{p}\cdot\mathbf{x}}e^{iQ|z|}}{2k}\left[\hat{\mathbf{x}} - \frac{ip_x}{Q}\hat{\mathbf{z}}\right]$$

Note that the tangential component of the $\mathbf{C}$ integrals (that is, the magnetic field due to an electric surface current, or vice versa) is discontinuous at $z = z'$.

2.3 Inner Products of $\mathbf{f}_{\mathbf{p}}$ functions with dyadic GFs

Note: In what follows I am being a little cavalier. The four-dimensional integrals in a quantity like $\left\langle \mathbf{f}_{\mathbf{p}}\middle|\mathbf{G}\middle|\mathbf{f}_{\mathbf{p}}\right\rangle$ are actually infinite: if I restricted each surface integral to a finite area A, then the matrix element would be proportional to A and would diverge with the area of the planar interface. The proper way to write this is to say

$$\left\langle \mathbf{f}_{\mathbf{p}}\middle|\mathbf{G}\middle|\mathbf{f}_{\mathbf{p}'}\right\rangle = \overline{\left\langle \mathbf{f}_{\mathbf{p}}\middle|\mathbf{G}\middle|\mathbf{f}_{\mathbf{p}}\right\rangle}\cdot(2\pi)^2\delta(\mathbf{p}-\mathbf{p}')$$

where the δ function factor has units of area, and the barred inner product is finite. In what follows I will not bother to write out the overbar or the δ function factor, but it's good to keep them in mind.

I have verified the following calculations numerically:

$$\left\langle \mathbf{f}_{x\mathbf{p}}(\mathbf{x})\middle|\mathbf{G}(\mathbf{x},z;\mathbf{x}',z')\middle|\mathbf{f}_{x\mathbf{p}}(\mathbf{x}')\right\rangle = \frac{ie^{iQ|z-z'|}}{2Q}\left(1-\frac{p_x^2}{k^2}\right)$$

$$\left\langle \mathbf{f}_{x\mathbf{p}}(\mathbf{x})\middle|\mathbf{G}(\mathbf{x},z;\mathbf{x}',z')\middle|\mathbf{f}_{y\mathbf{p}}(\mathbf{x}')\right\rangle = -\frac{ie^{iQ|z-z'|}}{2Q}\cdot\frac{p_xp_y}{k^2}$$

$$\left\langle \mathbf{f}_{y\mathbf{p}}(\mathbf{x})\middle|\mathbf{G}(\mathbf{x},z;\mathbf{x}',z')\middle|\mathbf{f}_{x\mathbf{p}}(\mathbf{x}')\right\rangle = -\frac{ie^{iQ|z-z'|}}{2Q}\cdot\frac{p_xp_y}{k^2}$$

$$\left\langle \mathbf{f}_{y\mathbf{p}}(\mathbf{x})\middle|\mathbf{G}(\mathbf{x},z;\mathbf{x}',z')\middle|\mathbf{f}_{y\mathbf{p}}(\mathbf{x}')\right\rangle = \frac{ie^{iQ|z-z'|}}{2Q}\left(1-\frac{p_y^2}{k^2}\right)$$

$$\Big\langle \mathbf{f}_{x\mathbf{p}}(\mathbf{x}) \Big| \mathbf{C}(\mathbf{x}, z; \mathbf{x}', z') \Big| \mathbf{f}_{y\mathbf{p}}(\mathbf{x}') \Big\rangle = \text{sign}(z - z') \frac{ie^{iQ|z-z'|}}{2k}$$
$$\Big\langle \mathbf{f}_{y\mathbf{p}}(\mathbf{x}) \Big| \mathbf{C}(\mathbf{x}, z; \mathbf{x}', z') \Big| \mathbf{f}_{x\mathbf{p}}(\mathbf{x}') \Big\rangle = -\text{sign}(z - z') \frac{ie^{iQ|z-z'|}}{2k}$$

Note that, at $z \to z'$, the $\Big\langle \mathbf{f} \Big| \mathbf{C} \Big| \mathbf{f} \Big\rangle$ inner products approach a nonzero constant whose sign depends on whether $z \to z'^{+}$ or $z \to z'^{-}$. This subtlety turns out to be unimportant for setting up and solving scattering problems using the SIE formalism, but crucial for understanding the compact power formulas discussed in the next section.

2.4 SIE Matrices

In this section I omit the $\mathbf{p}$ subscript.

I order the expansion coefficients and the incident-field inner products into 4-dimensional vectors thusly:

$$\begin{pmatrix} K_x \\ K_y \\ N_x \\ N_y \end{pmatrix}, \qquad \begin{pmatrix} \big\langle \mathbf{f}_x \big| \mathbf{E}^{\rm inc} \big\rangle \\ \big\langle \mathbf{f}_y \big| \mathbf{E}^{\rm inc} \big\rangle \\ \big\langle \mathbf{f}_x \big| \mathbf{H}^{\rm inc} \big\rangle \\ \big\langle \mathbf{f}_y \big| \mathbf{H}^{\rm inc} \big\rangle \end{pmatrix}. \tag{7}$$

Then the SIE matrix has the following structure:

$$\mathbf{M} = \begin{pmatrix}
\big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm EE} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm EE} \big| \mathbf{f}_y \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm EM} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm EM} \big| \mathbf{f}_y \big\rangle \\
\big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm EE} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm EE} \big| \mathbf{f}_y \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm EM} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm EM} \big| \mathbf{f}_y \big\rangle \\
\big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm ME} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm ME} \big| \mathbf{f}_y \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm MM} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{\Gamma}^{\rm MM} \big| \mathbf{f}_y \big\rangle \\
\big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm ME} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm ME} \big| \mathbf{f}_y \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm MM} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{\Gamma}^{\rm MM} \big| \mathbf{f}_y \big\rangle
\end{pmatrix}.$$

The contribution of medium m to the SIE matrix is

$$\mathbf{M}^{(m)} = ik_m \begin{pmatrix}
Z_0 Z^m \big\langle \mathbf{f}_x \big| \mathbf{G} \big| \mathbf{f}_x \big\rangle & Z_0 Z^m \big\langle \mathbf{f}_x \big| \mathbf{G} \big| \mathbf{f}_y \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{C} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_x \big| \mathbf{C} \big| \mathbf{f}_y \big\rangle \\
Z_0 Z^m \big\langle \mathbf{f}_y \big| \mathbf{G} \big| \mathbf{f}_x \big\rangle & Z_0 Z^m \big\langle \mathbf{f}_y \big| \mathbf{G} \big| \mathbf{f}_y \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{C} \big| \mathbf{f}_x \big\rangle & \big\langle \mathbf{f}_y \big| \mathbf{C} \big| \mathbf{f}_y \big\rangle \\
-\big\langle \mathbf{f}_x \big| \mathbf{C} \big| \mathbf{f}_x \big\rangle & -\big\langle \mathbf{f}_x \big| \mathbf{C} \big| \mathbf{f}_y \big\rangle & \frac{1}{Z_0 Z^m} \big\langle \mathbf{f}_x \big| \mathbf{G} \big| \mathbf{f}_x \big\rangle & \frac{1}{Z_0 Z^m} \big\langle \mathbf{f}_x \big| \mathbf{G} \big| \mathbf{f}_y \big\rangle \\
-\big\langle \mathbf{f}_y \big| \mathbf{C} \big| \mathbf{f}_x \big\rangle & -\big\langle \mathbf{f}_y \big| \mathbf{C} \big| \mathbf{f}_y \big\rangle & \frac{1}{Z_0 Z^m} \big\langle \mathbf{f}_y \big| \mathbf{G} \big| \mathbf{f}_x \big\rangle & \frac{1}{Z_0 Z^m} \big\langle \mathbf{f}_y \big| \mathbf{G} \big| \mathbf{f}_y \big\rangle
\end{pmatrix}.$$

Inserting the results of the last section, the exterior and interior contributions to the SIE matrix are

$$\mathbf{M}^{(0)} = -\frac{k_0}{2} \begin{pmatrix}
\frac{Z_0}{Q_0}\left(1 - \frac{p_x^2}{k_0^2}\right) & \frac{Z_0}{Q_0}\left(-\frac{p_x p_y}{k_0^2}\right) & 0 & \frac{1}{k_0} \\
\frac{Z_0}{Q_0}\left(-\frac{p_x p_y}{k_0^2}\right) & \frac{Z_0}{Q_0}\left(1 - \frac{p_y^2}{k_0^2}\right) & -\frac{1}{k_0} & 0 \\
0 & -\frac{1}{k_0} & \frac{1}{Q_0 Z_0}\left(1 - \frac{p_x^2}{k_0^2}\right) & \frac{1}{Q_0 Z_0}\left(-\frac{p_x p_y}{k_0^2}\right) \\
\frac{1}{k_0} & 0 & \frac{1}{Q_0 Z_0}\left(-\frac{p_x p_y}{k_0^2}\right) & \frac{1}{Q_0 Z_0}\left(1 - \frac{p_y^2}{k_0^2}\right)
\end{pmatrix}.$$

$$\mathbf{M}^{(1)} = -\frac{k_1}{2} \begin{pmatrix}
\frac{Z_0 Z^1}{Q_1}\left(1 - \frac{p_x^2}{k_1^2}\right) & \frac{Z_0 Z^1}{Q_1}\left(-\frac{p_x p_y}{k_1^2}\right) & 0 & -\frac{1}{k_1} \\
\frac{Z_0 Z^1}{Q_1}\left(-\frac{p_x^2}{k_1^2}\right) & \frac{Z_0 Z^1}{Q_1}\left(1 - \frac{p_y^2}{k_1^2}\right) & +\frac{1}{k_1} & 0 \\
0 & +\frac{1}{k_1} & \frac{1}{Q_1 Z_0 Z^1}\left(1 - \frac{p_x^2}{k_1^2}\right) & \frac{1}{Q_1 Z_0 Z^1}\left(-\frac{p_x p_y}{k_1^2}\right) \\
-\frac{1}{k_1} & 0 & \frac{1}{Q_1 Z_0 Z^1}\left(-\frac{p_x p_y}{k_1^2}\right) & \frac{1}{Q_1 Z_0 Z^1}\left(1 - \frac{p_y^2}{k_1^2}\right)
\end{pmatrix}.$$

2.5 SIE Scattering Solution: Normal Incidence

For the normally incident plane wave of equation (1), the RHS vector of the SIE system [again being somewhat cavalier about neglecting factors of $\delta(\mathbf{p}-\mathbf{p}')$] is

$$-\begin{pmatrix} \big\langle \mathbf{f}_x \big| \mathbf{E}^{\text{inc}} \big\rangle \\ \big\langle \mathbf{f}_y \big| \mathbf{E}^{\text{inc}} \big\rangle \\ \big\langle \mathbf{f}_x \big| \mathbf{H}^{\text{inc}} \big\rangle \\ \big\langle \mathbf{f}_y \big| \mathbf{H}^{\text{inc}} \big\rangle \end{pmatrix} = -E_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{Z_0} \end{pmatrix}.$$

Out of the four unknown coefficients in the unknown vector (7), only K_x and N_y are nonzero, and then only for $\mathbf{p}=0$, in which case I have simply $Q=k$. The SIE system reduces to

$$-\frac{1}{2}\begin{pmatrix} Z_0(Z^1+1) & 0 \\ 0 & \dfrac{1}{Z_0(\frac{1}{Z^1}+1)} \end{pmatrix} \begin{pmatrix} K_x \\ N_y \end{pmatrix} = -E_0 \begin{pmatrix} 1 \\ -\dfrac{1}{Z_0} \end{pmatrix}$$

with solution

$$K_x = \frac{2E_0}{Z_0(Z^1+1)}, \qquad N_y = -\frac{2Z^1 E_0}{(Z^1+1)}.$$

This reproduces the exact result (2).

3 An Important Subtlety in the SIE matrices

Before proceeding, I pause to note the following important subtlety in the SIE matrices. The subtlety is easiest to see for the case $\mathbf{p}=0$, although it is present for $\mathbf{p}\neq 0$ as well.

By the results of the previous section, the exterior and interior contributions to the SIE matrices at $\mathbf{p}=0$ are as follows:

$$\mathbf{M}^{(0)} = \begin{pmatrix} -\frac{Z_0}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{Z_0}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2Z_0} & -\frac{1}{2Z_0} \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2Z_0} \end{pmatrix}.$$

$$\mathbf{M}^{(1)} = \begin{pmatrix} -\frac{Z_0 Z^1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{Z_0 Z^1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2Z_0 Z^1} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2Z_0 Z^1} \end{pmatrix}.$$

The total SIE matrix is the sum of the exterior and interior contributions:

$$\mathbf{M} = \begin{pmatrix} -\frac{Z_0}{2}(1+Z^1) & 0 & 0 & 0 \\ 0 & -\frac{Z_0}{2}(1+Z^1) & 0 & 0 \\ 0 & 0 & -\frac{1}{2Z_0}(1+\frac{1}{Z^1}) & \\ 0 & 0 & 0 & -\frac{1}{2Z_0}(1+\frac{1}{Z^1}) \end{pmatrix}.$$

Note something interesting here: The off-diagonal elements of $\pm\frac{1}{2}$, which are present in the *individual* contributions to the SIE matrix, cancel when we form the sum and are *absent* from the final SIE matrix.

Origin of the $\pm\frac{1}{2}$ factors

The factors of $\pm\frac{1}{2}$ come from a δ function in the $\mathbf{C}$ dyadic Green's function. The proper way to write inner products involving the $\mathbf{C}$ function is something like this:

$$\Big\langle \mathbf{f}\Big|\mathbf{C}\Big|\mathbf{g}\Big\rangle = \Big\langle \mathbf{f}\Big|\mathbf{C}^{\text{finite}}\Big|\mathbf{g}\Big\rangle \pm \frac{1}{2ik}\cdot\Big\langle \mathbf{f}\Big|\hat{\mathbf{n}}\times\Big|\mathbf{g}\Big\rangle \tag{8}$$

The first term here gives zero inner for coplanar surface-current basis functions $\mathbf{f}$ and $\mathbf{g}$, but the second term is nonvanishing whenever $\mathbf{f}$ and $\mathbf{g}$ have crossed overlap (in fact, this term is proportional to what Steven calls the "T matrix.") The choice of $\pm$ sign is determined by whether we are taking the limit as $\mathbf{f}$ approaches $\mathbf{g}$ from the *exterior* or the *interior*.

Implications for concise power formulas in SCUFF-EM

The additional δ function term here is **not taken into account by** SCUFF-EM. Indeed, as noted above, it is perfectly acceptable to ignore this term as long as we are only ever computing the *total* SIE matrices. However, when we need to compute *individual* contributions to the SIE matrices, we need to account for the extra δ function term.

Computationally, this is actually easy, because the crossed-overlap integrals $\Big\langle \mathbf{f} \Big| \hat{\mathbf{n}} \times \Big| \mathbf{g} \Big\rangle$ can be computed in closed form for the RWG basis functions.

4 Comparison of Power Formulas

4.1 Exact Results for Transmitted, Scattered, and Total Power

It is easy to read off expressions for the various power quantities from the exact solution of Section 1.

Transmitted (Absorbed) Power

The total Poynting vector at $z = 0$ is

$$\mathbf{P}^{\text{tot}} = \frac{1}{2}\,\text{Re}\,\Big[\mathbf{E}^{\text{tot}*} \times \mathbf{H}^{\text{tot}}\Big] = -\text{Re}\,\frac{|B|^2}{2Z_0 Z^1}\hat{\mathbf{z}}$$

so the power per unit area transmitted through the infinite planar surface is

$$P^{\text{trans}} = \frac{4\cdot(\text{Re}\ Z_1)}{|Z_1+1|^2}\cdot\frac{|E_0|^2}{2Z_0}. \tag{9}$$

This quantity is the analogue of what we normally compute as the absorbed power for a finite scatterer; I call it P^{trans} instead of P^{abs} because it contains both the power absorbed by the dielectric (if it is lossy) and the flux of the plane wave travelling through the dielectric toward $z \to -\infty$. (In particular, it is nonzero even for a lossless dielectric, which would not be the case for a finite scatterer.)

Scattered Power

On the other hand, the scattered Poynting vector at $z = 0$ is

$$\mathbf{P}^{\text{scat}} = \frac{1}{2}\,\text{Re}\,\Big[\mathbf{E}^{\text{scat}*} \times \mathbf{H}^{\text{scat}}\Big] = +\text{Re}\,\frac{|A|^2}{2Z_0}\hat{\mathbf{z}}$$

so the power scattered per unit area by the infinite planar surface is

$$P^{\text{scat}} = \left|\frac{Z^1-1}{Z^1+1}\right|^2\cdot\frac{|E_0|^2}{2Z_0}. \tag{10}$$

Total Power

The analogue of what we normally call the "total power" (or the "extinction") is the sum of the transmitted and scattered powers:

$$P^{\text{tot}} = P^{\text{scat}} + P^{\text{abs}} = \frac{|E_0|^2}{2Z_0}. \tag{11}$$

Note that this is actually equal to the entire flux of the incident plane wave, i.e. 100% of its power its "extinguished" by the interaction with the dielectric half-space. This just reflects the fact that, in contrast to the case of a finite scatterer, in this case there is no way for a portion of the incident power to flow "around" the scatterer; all power is either reflected or transmitted.

4.2 Concise SIE Formulas for Transmitted, Scattered, and Total Power

4.2.1 Transmitted power by HR formula

The HR formula for the transmitted power is, in SGJ notation,

$$P^{\text{trans,HR}} = \frac{1}{4} x^* T x$$

where the elements of the T matrix are the crossed-overlaps of the basis functions:

$$T_{\alpha\beta} = \Big\langle \mathbf{f}_\alpha \Big| \hat{\mathbf{n}} \times \Big| \mathbf{f}_\beta \Big\rangle.$$

In the present case this works out to

$$\begin{aligned} P^{\text{trans,HR}} &= -\frac{1}{2} \text{Re } K_x^* N_y \Big\langle \mathbf{f}_x \Big| \hat{\mathbf{z}} \times \Big| \mathbf{f}_y \Big\rangle \\ &= \frac{4 \cdot (\text{Re } Z^1)}{|Z^1+1|^2} \cdot \frac{|E_0|^2}{2Z_0} \end{aligned}$$

in agreement with the exact solution (9).

4.2.2 Absorbed power by SGJ formula

The SGJ formula for the absorbed power reads (in SGJ notation)

$$P^{\text{abs,SGJ}} = \frac{1}{4} \text{Re} \Big[x^* G^0 x + x^* s^0 \Big] \tag{12}$$

The first term here is

$$\begin{aligned} \frac{1}{4} \text{ Re } x^* G^0 x &= -\frac{1}{8} \text{ Re } \left(\begin{array}{c} \frac{2E_0}{Z_0(Z^1+1)} \\ -\frac{2Z^1 E_0}{Z^1+1} \end{array} \right)^\dagger \left(\begin{array}{cc} Z_0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{Z_0} \end{array} \right) \left(\begin{array}{c} \frac{2E_0}{Z_0(Z^1+1)} \\ -\frac{2Z^1 E_0}{Z^1+1} \end{array} \right) \\ &= -\frac{|E_0|^2}{8Z_0} \text{ Re } \Bigg\{ \underbrace{\frac{4}{|Z^1+1|^2}}_{\text{EE}} + \underbrace{\frac{2Z^1}{|Z^1+1|^2}}_{\text{EM}} + \underbrace{\frac{2Z^{1*}}{|Z^1+1|^2}}_{\text{ME}} + \underbrace{\frac{4|Z^1|^2}{|Z^1+1|^2}}_{\text{MM}} \Bigg\} \\ &= -\frac{|E_0|^2}{2Z_0} \left\{ 1 - \frac{4(\text{Re}Z^1)}{|Z^1+1|^2} \right\} \end{aligned} \tag{13}$$

The second term is

$$\begin{aligned} &= \frac{1}{4} \text{Re} \left(\begin{array}{c} \frac{2E_0}{Z_0(Z^1+1)} \\ -\frac{2Z^1 E_0}{Z^1+1} \end{array} \right)^\dagger \cdot \left(\begin{array}{c} E_0 \\ -\frac{E_0}{Z_0} \end{array} \right) \\ &= \frac{|E_0|^2}{2Z_0} \end{aligned} \tag{14}$$

Inserting (13) and (14) into (12), I find

$$\begin{aligned}
P^{\text{abs,SGJ}} &= \frac{1}{4}\text{Re}\left[x^* G^0 x\right] + \frac{1}{4}\text{Re}\left[x^* s^0\right] \\
&= +\frac{4\cdot(\text{Re } Z^1)}{|Z^1+1|^2}\cdot\frac{|E_0|^2}{2Z_0}
\end{aligned}$$

again reproducing the correct result.

4.2.3 Contribution of δ-function terms to SGJ formula

Let's now investigate what would happen if the weird off-diagonal factors of $\frac{1}{2}$ were missing from the individual SIE matrices (as is the case in SCUFF-EM). This would correspond to eliminating the terms marked "EM" and "ME" from the sum in equation (13). The contribution made by these terms to the sum is

$$\begin{aligned}
\Delta P^{\text{abs,SGJ}} &= -\frac{\text{Re } Z^1}{|Z^1+1|^2}\cdot\frac{|E_0|^2}{2Z_0} \\
&= \frac{1}{4}P^{\text{abs,SGJ}}.
\end{aligned}$$

So eliminating these terms from the sum would result in an answer that is off by a factor of $\frac{3}{4}$, not $\frac{1}{2}$ as I seem to observe in practice.

5 Comparison of Force Formulas

Exact force for E−field perpendicular to plane

In general, the time average of the i-directed force per unit area at a point $(\mathbf{x})$ on a surface is related to the Maxwell stress tensor at $\mathbf{x}$ according to

$$F_i = -\frac{1}{2}\text{Re } T_{ij}(\mathbf{x})\hat{n}_j(\mathbf{x})$$

(where $\hat{\mathbf{n}}$ is the *outward*-pointing surface normal)

$$= -\frac{1}{2}\text{Re}\left\{\epsilon E_i^*\Big[\mathbf{E}\cdot\hat{\mathbf{n}}\Big] + \mu H_i^*\Big[\mathbf{H}\cdot\hat{\mathbf{n}}\Big] - \frac{\hat{n}_i}{2}\Big[\epsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\Big]\right\}. \qquad (15)$$

For the particular case of the non-normally-incident plane wave of (4), we have $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$ and the fields at $z = 0$ are

$$\left(\begin{array}{c} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{array}\right) = E_0 e^{ik_0 y \sin\theta}\left(\begin{array}{c} (1+r) \\ 0 \\ 0 \\ 0 \\ -\frac{1}{Z_0}(1-r)\cos\theta \\ -\frac{1}{Z_0}(1+r)\sin\theta \end{array}\right)$$

with

$$\mathbf{E}\cdot\hat{\mathbf{n}} = 0, \qquad \mathbf{H}\cdot\hat{\mathbf{n}} = -\frac{E_0}{Z_0}(1+r)\sin\theta e^{ik_0 y\sin\theta},$$

$$|\mathbf{E}|^2 = |E_0|^2|1+r|^2, \qquad |\mathbf{H}|^2 = \frac{|E_0|^2}{Z_0^2}\Big[|1-r|^2\cos^2\theta + |1+r|^2\sin^2\theta\Big]$$

so from (15) we find

$$F_x = 0$$

$$F_y = -\frac{\epsilon_0|E_0|^2}{2}(1-|r|^2)\cos\theta\sin\theta$$

$$\begin{aligned} F_z &= -\frac{\epsilon_0|E_0|^2}{2}\text{Re}\left\{|1+r|^2\sin^2\theta - \frac{1}{2}|1+r|^2 - \frac{1}{2}\Big[|1-r|^2\cos^2\theta + |1+r|^2\sin^2\theta\Big]\right\} \\ &= -\frac{\epsilon_0|E_0|^2}{4}\cos^2\theta\,\text{Re}\,\Big[|1+r|^2 - |1-r|^2\Big] \\ &= -\epsilon_0|E_0|^2\cos^2\theta\cdot\Big(\text{Re } r\Big). \end{aligned}$$

The force is proportional to the real part of the reflection coefficient, and it vanishes for an incident field propagating in the direction parallel to the interface.

FSC Force for E-field perpendicular to plane

The formula for the normally-directed stress tensor in terms of the surface currents is

$$
\begin{aligned}
T_{ij}n_j = &\frac{1}{i\omega}\Big\{(\hat{\mathbf{n}}\times\mathbf{N}^*)_i(\nabla\cdot\mathbf{K}) + (\hat{\mathbf{n}}\times\mathbf{K}^*)_i(\nabla\cdot\mathbf{N})\Big\} \\
&-\frac{\hat{n}_i}{2}\Big\{\mu\Big[|\mathbf{K}|^2 - \frac{|\nabla\cdot\mathbf{K}|^2}{k^2}\Big] + \epsilon\Big[|\mathbf{N}|^2 - \frac{|\nabla\cdot\mathbf{N}|^2}{k^2}\Big]\Big\} \qquad (16)
\end{aligned}
$$

In this case we have

$$
K_x = \frac{E_0}{Z_0}(1-r)\cos\theta e^{ik_0 y\sin\theta}, \qquad N_y = E_0(1+r)e^{ik_0 y\sin\theta}
$$

with all other components of the surface currents vanishing. Also,

$$
\nabla\cdot\mathbf{K} = 0, \qquad |\mathbf{K}|^2 = \frac{|E_0|^2}{Z_0^2}|1-r|^2\cos^2\theta,
$$

$$
\nabla\cdot\mathbf{N} = ik_0\sin\theta E_0(1+r)e^{ik_0 y\sin\theta}, \qquad |\mathbf{N}|^2 = |E_0|^2|1+r|^2.
$$

The components of equation (16) then read

$$
\begin{aligned}
T_{xj}n_j &= 0 \\
T_{yj}n_j &= \frac{1}{i\omega}\Big\{\Big[\frac{E_0^*}{Z_0}(1-r)^*\cos\theta\Big]\Big[ik_0E_0(1+r)\sin\theta\Big]\Big\} \\
&= \epsilon_0|E_0|^2(1-r)^*(1+r)\cos\theta\sin\theta \\
T_{zj}n_j &= -\frac{\epsilon_0|E_0|^2}{2}\Big\{|1-r|^2\cos^2\theta + |1+r|^2 - |1+r|^2\sin^2\theta\Big\} \\
&= -\frac{\epsilon_0|E_0|^2}{2}\cos^2\theta\Big\{|1-r|^2 + +|1+r|^2 - |1+r|^2\sin^2\theta\Big\}
\end{aligned}
$$