Electromagnetism in the Spherical-Wave Basis:

A (Somewhat Random) Compendium of Reference Formulas

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Abstract

This memo consolidates and collects for reference a somewhat random hodgepodge of formulas and results in the spherical-wave approach to electromagnetism that I have found useful over the years in developing and testing SCUFF-EM and BUFF-EM.

Contents

1 Vector Spherical Wave Solutions to Maxwell's Equations

Many authors define pairs of three-vector-valued functions $\{\mathbf{M}_{\ell m}(\mathbf{x}), \mathbf{N}_{\ell m}(\mathbf{x})\}$ describing exact solutions of Maxwell's equations in spherical coordinates for a homogeneous medium with wavenumber k, i.e.

$$\Big[\nabla\times\nabla\times +k^2\Big]\left\{\begin{array}{c}\mathbf{M}\\ \mathbf{N}\end{array}\right\}=0.$$

These functions always involve spherical Bessel functions and spherical harmonics, but the precise definitions (including sign conventions and normalization factors) vary from author to author. In this section I set down the particular conventions that I use. In the next section I give explicit closed-form expressions for small ℓ.

Vector spherical harmonics

$$\begin{aligned}\mathbf{X}_{\ell m}(\theta,\varphi)&=\frac{i}{\ell(\ell+1)}\nabla\times\Big\{Y_{\ell m}(\theta,\varphi)\hat{\mathbf{r}}\Big\}\\ \mathbf{Z}_{\ell m}(\theta,\varphi)&=\hat{\mathbf{r}}\times\mathbf{X}_{\ell m}(\theta,\varphi)\end{aligned}$$

More explicitly, the components of $\mathbf{X}$ and $\mathbf{Z}$ are

$$\begin{aligned}\mathbf{X}_{\ell m}(\theta,\phi)&=\frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{im}{\sin\theta}Y_{\ell m}\widehat{\boldsymbol{\theta}}-\frac{\partial Y_{\ell m}}{\partial\theta}\widehat{\boldsymbol{\varphi}}\right]\\ \mathbf{Z}_{\ell m}(\theta,\phi)&=\frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{\partial Y_{\ell m}}{\partial\theta}\widehat{\boldsymbol{\theta}}+\frac{im}{\sin\theta}Y_{\ell m}\widehat{\boldsymbol{\varphi}}\right].\end{aligned}$$

Their divergences are:

Radial functions

$$\begin{aligned}R_\ell^{\text{outgoing}}(kr)&=h_\ell^{(1)}(kr)\\ R_\ell^{\text{incoming}}(kr)&=h_\ell^{(2)}(kr)\\ R_\ell^{\text{regular}}(kr)&=j_\ell(kr).\end{aligned}$$

I also define the shorthand symbols

$$\overline{R}_\ell(kr)\equiv\frac{1}{kr}\left|R_\ell(x)+\frac{d}{dx}R_\ell(x)\right|_{x=kr}\qquad \not{R}_\ell(kr)=-\frac{\sqrt{l(l+1)}}{kr}R_\ell(kr).$$

Vector spherical wave functions

$$\begin{aligned}\mathbf{M}_{\ell m}(k;\mathbf{r})&\equiv R_\ell(kr)\mathbf{X}_{\ell m}(\Omega)\\ \mathbf{N}_{\ell m}(k;\mathbf{r})&\equiv i\overline{R}_\ell(kr)\mathbf{Z}_{\ell m}(\Omega)+\not{R}_\ell(kr)Y_{\ell m}(\Omega)\hat{\mathbf{r}}\end{aligned}$$

2 Explicit expression for small ℓ

The first few radial functions

$$
\begin{aligned}
R_1^{\text{regular}}(kr) &= -\frac{i(ikr)\cos(kr)+\sin(kr)}{(ikr)^2} & \overline{R}_1^{\text{regular}}(kr) &= \frac{(ikr)\cos(kr)-\Big[-1-(ikr)^2\Big]\sin(kr)}{k^3r^3} \\
R_1^{\text{outgoing}}(kr) &= \Big[-1+ikr-(ikr)^2\Big]\frac{e^{ikr}}{k^3r^3} & \overline{R}_1^{\text{outgoing}}(kr) &= \Big[-ikr+(ikr)^2\Big]\frac{e^{ikr}}{k^3r^3}
\end{aligned}
$$

The first few outgoing functions In what follows, the Q_n are dimensionless polynomial factors:

$$Q_1(x) = 1 - x \tag{1a}$$

$$Q_{2a}(x) = 1 - x + x^2 \tag{1b}$$

$$Q_{2b}(x) = 3 - 3x + x^2 \tag{1c}$$

$$Q_3(x) = 6 - 6x + 3x^2 - x^3 \tag{1d}$$

$$\mathbf{M}^{\text{outgoing}}_{1,\pm 1}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left(\frac{e^{ikr}}{k^2r^2}\right)e^{\pm i\phi}\left(\begin{array}{c} 0 \\ -iQ_1(ikr) \\ \pm Q_1(ikr)\cos\theta \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{1,0}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left(\frac{e^{ikr}}{k^2r^2}\right)\left(\begin{array}{c} 0 \\ 0 \\ Q_1(ikr)\sin\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{1,\pm 1}(\mathbf{r}) = \sqrt{\frac{3}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm i\phi}\left(\begin{array}{c} \mp 2iQ_1(ikr)\sin\theta \\ \pm iQ_{2a}(ikr)\cos\theta \\ -Q_{2a}(ikr) \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{1,0}(\mathbf{r}) = \sqrt{\frac{3}{8\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)\left(\begin{array}{c} 2iQ_1(ikr)\cos\theta \\ +iQ_{2a}(ikr)\sin\theta \\ 0 \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{2,\pm 2}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm 2i\phi}\left(\begin{array}{c} 0 \\ \pm iQ_{2b}(ikr)\sin\theta \\ -Q_{2b}(ikr)\cos\theta\sin\theta \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{2,\pm 1}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)e^{\pm i\phi}\left(\begin{array}{c} 0 \\ -iQ_{2b}(ikr)\cos\theta \\ \pm Q_{2b}(ikr)\cos 2\theta \end{array}\right)$$

$$\mathbf{M}^{\text{outgoing}}_{2,0}(\mathbf{r}) = \sqrt{\frac{15}{8\pi}}\left(\frac{e^{ikr}}{k^3r^3}\right)\left(\begin{array}{c} 0 \\ 0 \\ -Q_{2b}(ikr)\cos\theta\sin\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{2,\pm 2}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)e^{\pm 2i\phi}\left(\begin{array}{c} 3iQ_{2b}(ikr)\sin^2\theta \\ -iQ_3(ikr)\cos\theta\sin\theta \\ \pm Q_3(ikr)\sin\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{2,\pm 1}(\mathbf{r}) = \sqrt{\frac{5}{16\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)e^{\pm i\phi}\left(\begin{array}{c} \mp 3iQ_{2b}(ikr)\sin 2\theta \\ \pm iQ_3(ikr)\cos 2\theta \\ -Q_3(ikr)\cos\theta \end{array}\right)$$

$$\mathbf{N}^{\text{outgoing}}_{2,0}(\mathbf{r}) = \sqrt{\frac{15}{8\pi}}\left(\frac{e^{ikr}}{k^4r^4}\right)\left(\begin{array}{c} iQ_{2b}(ikr)(3\cos^2\theta - 1) \\ iQ_3(ikr)\cos\theta\sin\theta \\ 0 \end{array}\right).$$