Subproblem Basics

The subproblem attempts to find the most expensive scenario that could develop for any given arrangement of lines built, "x^*", assuming producers are still making optimal decisions to reduce costs. This "x^*" is a fixed parameter taken as input from a previous master problem solve. As output, the subproblem provides the worst-case realizations uncertain parameters, demand and generation capacity as well as a potential upper bound to the optimization. However, each subproblem solve does not necessarily improve the upper bound.

The subproblem combines the two-stage inner problem

$$\max_{d\in D}\left(\min_{y\in\Omega(\mathrm{x},\mathrm{d})} b^t y\right)$$

into a single linear maximization problem. "*d*" are the uncertain parameters in the model, e.g. external events such as changes in supply, demand, disruptions in generation and so on within an uncertainty set "*D*". $\Omega(\mathrm{x},\mathrm{d})$ is the scenario realization for a given *x* and *d*. The subproblem accomplishes this by using a Fortuny-Amat transformation and KKT optimality conditions on the innermost minimization of hourly decision costs. This is reasonable since KKT conditions are sufficient for optimality for any linear problem. The reformation into a mixed integer linear problem (MILP) allows for efficient algorithms for solving linear systems to be applied, such as a branch and bound..

Subproblem Detailed Formulation

The full formulation, using the choice of parameters as used in [1] is below. Also, See [2] for a more detailed explanation. Note, that this model assumes only one connection can be made between nodes, but that multiple generators "g" and demands "d" can exist at a single node. The objective is:

$$\mathbf{max}\ \ \sigma * \left[\sum_g \left(\mathrm{C}_g^{\mathrm{E}} * Gen_g\right) - \sum_d \left(\mathrm{C}_d^{\mathrm{D}} * Shed_d\right)\right]$$

ϕ is a constant of 8760 to convert hourly costs to yearly costs. "C" is a vector of fixed costs for both generation "Gen" and load shedding "Shed".

Second, are the uncertainty budget constraints on generation and demand (discussed in detail in the section below). Different regions "r" can have different uncertainty budgets for either supply or demand:

$$Gen_g^{cap} \in \left[Gen_g^{min}, Gen_g^{max}\right] \qquad \forall g$$

$$\frac{\sum_g\left(Gen_g^{max} - Gen_g^{cap}\right)}{\sum_g\left(Gen_g^{max} - Gen_g^{min}\right)} \geq \Gamma_r^{\mathrm{Gen}} \qquad \forall r$$

$$Dem_d \in \left[Dem_d^{min}, Dem_d^{max}\right] \qquad \forall d$$

$$\frac{\sum_d\left(Dem_d - Dem_d^{min}\right)}{\sum_d\left(Dem_d^{max} - Dem_d^{min}\right)} \leq \Gamma_r^{\mathrm{Dem}} \quad \forall r$$

Then, there are the primal feasibility conditions. A line goes from "$s(l)$" to "$r(l)$" and can only transport as much as the capacity. Also, this includes the DC power constraints to model the physical limits on electrical flow. One of the voltage angels at a chosen reference node, "ref", is set to zero:

$$\sum_{g} Gen_g - \sum_{l|s(l)=n} flow_l + \sum_{l|r(l)=n} flow_l = \sum_{d} (Dem_d - Shed_d) \ \forall n$$

$$0 \le Gen_g \le Gen_g^{max} \qquad \forall g$$

$$0 \le Shed_d \le Dem_d^{max} \qquad \forall d$$

$$Flow_l = x(Sus_l)(\theta_{s(l)} - \theta_{r(l)}) \qquad \forall l$$

$$-Cap_l \le Flow_l \le Cap_l \qquad \forall l$$

$$-\pi \le \theta_n \le \pi \qquad \forall n$$

$$\theta_n = 0 \ n{:}\, ref$$

Next, are the stationary conditions of the KKT found by differentiating with respect to the variables in the inner problem. λ are the duals for the equality constraints on electrical flow and ϕ are the other duals for:

min/max generation: ϕ^{Emin}/ϕ^{Emax};
min/max demand: ϕ^{Dmin}/ϕ^{Dmax};
min/max line capacity: ϕ^{Lmin}/ϕ^{Lmax};
min/max voltage angle: ϕ^{Nmin}/ϕ^{Nmax};
and reactance limits on flow: ϕ^{L}

χ^{ref} is a variable to allow the reference node to be set to zero without affecting the duals. Thus:

$$\sigma C_g^G - \lambda_{n(g)} + \phi_g^{\mathrm{E,max}} - \phi_g^{\mathrm{E,min}} = 0 \qquad \forall g$$

$$\sigma C_d^D - \lambda_{n(d)} + \phi_d^{\mathrm{D,max}} - \phi_d^{\mathrm{D,min}} = 0 \qquad \forall d$$

$$\lambda_{s(l)} - \lambda_{r(l)} - \phi_l^{\mathrm{L}} + \phi_l^{\mathrm{L,max}} - \phi_l^{\mathrm{L,min}} = 0 \quad \forall l$$

$$\sum_{l|s(l)=n} (\mathrm{x_l} * \mathrm{B_l} * \phi_l^{\mathrm{L}}) - \sum_{l|r(l)=n} \left(x_l * \mathrm{B_l} * \phi_l^{\mathrm{L}}\right) + \phi_n^{\mathrm{N,max}} + \phi_n^{\mathrm{N,min}} - \chi_{n=ref}^{ref} = 0 \quad \forall n$$

Finally, there are the complementary slackness and dual feasibility conditions of the KKT:

$$0 \le \phi_l^{\mathrm{L,max}} \perp Cap_l - Flow_l \ge 0 \qquad \forall l$$

$$0 \le \phi_l^{\mathrm{L,min}} \perp Cap_l + Flow_l \ge 0 \qquad \forall l$$

$$0 \le \phi_g^{\mathrm{E,max}} \perp Gen_g^{cap} - Gen_g \ge 0 \quad \forall g$$

$$0 \le \phi_g^{\mathrm{E,min}} \perp Gen_g \ge 0 \qquad \forall g$$

$$0 \le \phi_d^{\mathrm{D,max}} \perp Dem_d - Shed_d \ge 0 \quad \forall d$$

$$0 \le \phi_d^{\mathrm{D,min}} \perp Shed_d \ge 0 \qquad \forall d$$

$$0 \le \phi_n^{\mathrm{N,max}} \perp \pi - \theta_n \ge 0 \qquad \forall n$$

$$0 \le \phi_n^{\mathrm{N,min}} \perp \pi + \theta_n \ge 0 \qquad \forall n$$

As mentioned above, since these complementary slackness and dual feasibility constraints are non-linear, they should be converted to linear constraints with the use of a chosen "big M". As such, the linear version of each constraint above:

$$0 \le a \ \perp b \ge 0$$

would be translated to four linear constraints with the addition of a binary variable "*z*".

$$a \ge 0$$

$$b \ge 0$$

$$a \le Mz$$

$$b \le M(1 - z)$$

Note that the "*M*" for each complementary slackness constraint could be a different constant, and if the "*M*" chosen is too large, there is risk of introducing numerical and rounding errors. Of course, an "M" too small risks artificially constraining the problem space and finding an inferior solution. Luckily, information about the model can be used to intelligently choose candidates. To illustrate, notice "*M*" serves as an upper bound on dual variables, which are shadow prices representing how much relaxing the restraint could affect the objective value. So, for the ϕ_l^L, the line capacity duals, the most adding one unit of capacity to a line could affect the total price, is the most expensive load shedding penalty. So, an appropriate "*M*" to use for capacity would be the maximum element of:

$$\sigma * b_{shed}$$

Similar logic can be used to show that the most increasing generation or reducing demand could affect their related duals is also $\sigma * b_{shed}$. Finally, the dual associated with the $-\pi \leq \theta_n \leq \pi$ constraint is the largest possible value of:

$$\sigma * Sus * b_{Shed}^t$$

This is since increasing/decreasing θ by for any node by one could increase transmission between that node and any other by, at most, the max susceptance of any line connected to that node. And then that transmission could lower the total optimal by the amount times maximum load shedding.

Uncertainty Budget

A feature of the column-and-constraint method is the inclusion of an uncertainty budget, Γ, that allows planners to consider realizations that are less than absolute worst-cases. It is represented as:

$$d_k \in \left[d_k^{max}, d_k^{min}\right] \qquad \forall k$$

$$\frac{\sum_{k \in \gamma_r} |d_k^{ref} - d_k|}{\sum_{k \in \gamma_r} (d_k^{max} - d_k^{min})} \leq \Gamma_r \quad \forall r$$

Here, d_k^{max} and $d_k^{min}$are the upper and lower bounds of the realizations of "d_k" within the uncertainty set "*D*" for each "k" elements of "d". At the simplest, "k" could just be two elements, supply and demand, but these can be further subdivided into short-term and long-term versions of those elements, or by splitting generation into different types. Next, "r" is a group of nodes grouped together in a geographical region. Regions can have different uncertainty budgets, allowing different parts of the network to be modeled with different reliability or to model variability in forecasting future supply and demand. Lastly, d_k^{ref}, is a reference value, which is d_k^{max} for supply and d_k^{min} for supply. When Γ is 0, $d_k^{ref} = d_k$ so there is no uncertainty, but with $\Gamma = 1$, d_k can take any value within the feasible range, including the worst-case. Thus, $\Gamma = 1$ is the traditional worst-case robust optimization, and $0 < \Gamma < 1$ represents some moderate level of uncertainty. In this scheme, the uncertainty budget is included by adding the following two constraints to general formulation:

$$\frac{\sum_{i \in \gamma_r} (Gen_i^{max} - Gen_i^{cap})}{\sum_{i \in \gamma_r} (Gen_i^{max} - Gen_i^{min})} \geq \Gamma_r^{\mathrm{Gen}} \qquad \forall r$$

$$\frac{\sum_{i \in \gamma_r} (Dem_i - Dem_i^{min})}{\sum_{i \in \gamma_r} (Dem_i^{max} - Dem_i^{min})} \leq \Gamma_r^{\mathrm{Dem}} \quad \forall r$$

A special note is Gen_i^{cap} represents possible generation capacity, of which $Gen_i \leq Gen_i^{cap}$ since a producer need not use full production even if available.

Example Uncertainty Budget

To give an example of uncertainty budgets, take 3 nodes $i = 1 \dots 3$, each with the same bounds

$$Dem_i^{min} = 10, \qquad Dem_i^{max} = 20, \qquad \forall i$$

Then, adding an uncertainty budget of 0.4 is equivalent to the constraints:

$$10 \leq Dem_i \leq 20 \quad \forall i$$

$$Dem_1 + Dem_2 + Dem_3 = \sum Dem_i^{min} + (0.4) * (\sum Dem_i^{max} - \sum Dem_i^{min})$$

Thus, $Dem_1 + Dem_2 + Dem_3 = 42$, and as one example, a vector of demands [10,13,19] is feasible, as would be [20,10,12]. It is difficult to express what value Γ should be, except that as it approaches zero it relaxes the problem but increases the likelihood that a possible realization will fall outside the decision space.

References

[1] Antonio J. Conejo, Luis Baringo Morales, S. Jalal Kazempour, and Afzal S. Siddiqui. 2016. Investment in Electricity Generation and Transmission: Decision Making Under Uncertainty (1st ed.). Springer Publishing Company, Incorporated.

[2] Ruiz, C, and A.J Conejo. "Robust Transmission Expansion Planning." *European Journal of Operational Research*, vol. 242, no. 2, 2015, pp. 390–401.