Probability 易错汇总

- $F_X(x)$ 要写出 $x<0$, $x>1$ 的函数. $f_X(x)$ 不用.
- 通过 transformation 求 $F_Y(y)$ 和 $f_Y(y)$ 时，写区间之外时，$F_Y(y)$ 的取值 (即 $F_Y(y)=0$, $F_Y(y)=1$)
- 用中心极限定理找不到 μ 和 σ^2 的情况下可能是用二项分布，一个参数就够了.
- 独立 $\Rightarrow$ $cov(X,Y)=0$. (如果只告诉了 $cov(X,Y)=0$ 不一定独立，需用题中具体例子检验).
- 二项分布是离散的. 算 cdf 的时候注意取等问题.

Summary

5 . .

★ Probability.

两题型作一个对比.

(106) An engineering company uses rivets which arrive in boxes from three factories. Factory 1 sends twice as many boxes as Factory 2, and Factory 2 sends twice as many boxes as Factory 3. Rivets produced by Factory 1 are defective independently with probability 0.02. Rivets produced by Factory 2 are defective independently with probability 0.02 and Rivets produced by Factory 3 are defective independently with probability 0.05.

(i) A box is chosen but its label has come off and it is not known which factory it comes from. Assuming that the chosen box is equally likely to be any box that arrives at the company, show that the probability that box comes from Factory 1, 2 and 3 are $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$ respectively.

(ii) A rivet is taken out of the box and found to be defective, What is the probability that the box comes from Factory 1.

这里的审题要仔细
这个box并未说明
来自哪个工厂. 暗示了
贝叶斯条件下的
三种可能

(iii) A second rivet is taken from the box and also found to be defective. What is the probability that a third rivet taken from the box is defective?

(i) $P(F_1)=2P(F_2)$ $P(F_2)=2P(F_3)$, $P(F_1)+P(F_2)+P(F_3)=1$

$P(F_1)=\frac{4}{7}$ $P(F_2)=\frac{2}{7}$ $P(F_3)=\frac{1}{7}$

Summary

5 . .

(ii) Let D_i be the event that i^{th} rivet is defective.

$$P(F_1|D_1) = \frac{P(D_1|F_1)P(F_1)}{P(D_1|F_1)P(F_1)+P(D_1|F_2)P(F_2)+P(D_1|F_3)P(F_3)}$$

$$= \frac{0.02\times\frac{4}{7}}{0.02\times\frac{4}{7}+0.02\times\frac{2}{7}+0.05\times\frac{1}{7}} = \frac{8}{17}$$

(iii) $$P(F_i|D_1\cap D_2) = \frac{P(D_1\cap D_2|F_i)P(F_i)}{P(D_1\cap D_2|F_1)P(F_1)+P(D_1\cap D_2|F_2)P(F_2)+P(D_1\cap D_2|F_3)P(F_3)}$$

$$\boxed{P(D_1\cap D_2|F_2)} = \boxed{P(D_1|F_2)}\,\boxed{P(D_2|F_2)}$$

$$P(F_1|D_1\cap D_2) = \frac{\frac{2}{100}\times\frac{2}{100}\times\frac{4}{7}}{\frac{2}{100}\cdot\frac{2}{100}\cdot\frac{4}{7}+\frac{2}{100}\cdot\frac{2}{100}\cdot\frac{2}{7}+\frac{5}{100}\cdot\frac{5}{100}\cdot\frac{1}{7}} = \frac{16}{49}$$

F_1, F_2, F_3

$$P(F_2|D_1\cap D_2) = \frac{8}{49} \qquad P(F_3|D_1\cap D_2) = \frac{25}{49}$$

$$P(D_3|D_1\cap D_2) = P(D_3|F_1)P(F_1|D_1\cap D_2)+P(D_3|F_2)P(F_2|D_1\cap D_2)+P(D_3|F_3)P(F_3|D_1\cap D_2)$$

$$= \frac{173}{4900}.$$

第二题：
做到这里，模型的轮廓在脑海中越来越清晰了。

Summary

☆ Probability

(125) Three student, A. B. C, catch the same bus to get to an early morning tutorial. The students oversleep, and hence miss the tutorial, independently with probability $\frac{1}{3}$. The bus fails to turn up with probability $\frac{1}{4}$, independently of which student oversleep. If the bus turns up, each students that did not oversleep attend the tuturial; If the bus fails to turn up, each student who did not oversleep makes their own way to the tutorial independently with probability $\frac{1}{2}$, otherwise they do not attend the tutorial.

Calculate the probability:

(i) that student A attends the tutorial.

(ii) that the bus turned up given that student A did not attend the tutorial.

(iii) that student C attends the tutorial given that students A and B do not attend.

Let A{A attend tutorial} B{B attend tutorial}

C{C attend tutorial} T{bus turn up} $p(\bar{T}) = \frac{1}{4}$

$p(T) = \frac{3}{4}$ $\quad p(A|T) = p(B|T) = p(C|T) = \frac{2}{3}$

这一步容易错. $p(A^c|T) = p(B^c|T) = p(C^c|T) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$

因为在 Turns up 基础上考虑, 所以不用乘 Turns up 的概率 $\frac{3}{4}$

(i) $p(A) = p(A|T)\,p(T) + p(A|\bar{T})\cdot p(\bar{T}) = \frac{3}{4}\cdot\frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{7}{12}$

(ii) $p(T|A^c) = \frac{p(A^c \cap T)}{p(A^c)} = \frac{p(T)\cdot p(A^c|T)}{1-p(A)} = \frac{\frac{3}{4}\times(1-2/3)}{1-7/12} = \frac{3}{5}$

Summary

(其实里面包含了一个多重的条件. 就是说到达的话必须同时不睡过头也要 Make own way. $\frac{1}{2}\times\frac{2}{3}$.)

而且本来这个 $p(T)$ $p(T^c)$ 是用来算贝叶斯的. 不能这么快就被用了.

5

(iii) We want $P(C^c \mid A^c \cap B^c)$

C have two senarios. $\begin{cases} \text{bus turns up} \\ \text{bus not turns up} \end{cases}$

$P(C \mid A^c \cap B^c) = P(C \mid T)\,P(T \mid A^c \cap B^c) + P(C \mid T^c)\,P(T^c \mid A^c \cap B^c)$

$P(T \mid A^c \cap B^c) = \dfrac{P(A^c \cap B^c \mid T)\,P(T)}{P(A^c \cap B^c)}$ → 有两种情况

$= \dfrac{P(A^c \cap B^c \mid T)\,P(T)}{P(A^c \cap B^c \mid T)\,P(T) + P(A^c \cap B^c \mid T^c)\,P(T^c)}$

$= \dfrac{P(A^c \mid T)\,P(B^c \mid T)\cdot P(T)}{P(A^c \mid T)\,P(B^c \mid T)\cdot P(T) + P(A^c \mid T^c)\,P(B^c \mid T^c)\,P(T^c)}$

$= \dfrac{\frac{1}{3} \times \frac{1}{3} \times \frac{3}{4}}{\frac{1}{3} \times \frac{1}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{2}{3} \times \frac{1}{4}} = \dfrac{3}{7}$

同理 $P(T^c \mid A^c \cap B^c) = \dfrac{4}{7}$

$\therefore P(C \mid A^c \cap B^c) = P(C \mid T)\cdot P(T \mid A^c \cap B^c) + P(C \mid T^c)\,P(T^c \mid A^c \cap B^c)$

$= \frac{2}{3} \cdot \frac{3}{7} + \frac{1}{3} \times \frac{4}{7}$

$= \frac{10}{21}$

· A. B. C. 相互独立

· 每一种 A. B. C 都含附加考虑两种情况即

bus $\begin{cases} \text{turns up} \\ \text{doesn't turns up.} \end{cases}$

Summary

5

☆ Probability

(100) Two dice are available, one coloured red on four faces and blue on two faces, the other coloured blue on four faces and red on two faces.

(i) One of these dice is chosen uniformly at random and rolled. The uppermost face is red. What is the probability that the die with four red faces are was chosen?

(ii) The same die is rolled again and its uppermost face is still red. The same die is then rolled for a third time. What is the probability that its uppermost face is still red?

die 1 $\begin{cases} R: 4 \\ B: 2 \end{cases}$ die 2 $\begin{cases} R: 2 \\ B: 4 \end{cases}$

Let D_1 {Dice with 4 Reds} D_2 {Die with 2 Reds}

R {Uppermost is Red}

$P(D_1) = P(D_2) = 1/2 \quad P(R|D_1) = \frac{2}{3} \quad P(R|D_2) = \frac{1}{3}$

$$P(D_1|R) = \frac{P(D_1 \cap R)}{P(R)} = \frac{P(R|D_1)P(D_1)}{P(R|D_1)P(D_1) + P(R|D_2)P(D_2)} = \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

(ii) Let R_i be the event that i^{th} roll is Red.

from (i) $P(R_1) = P(R_2) = P(R_3) = P(R) = P(R|D_1)P(D_1) + P(R|D_2)P(D_2)$
$= \frac{2}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = \frac{1}{2}$

$P(R_3|D_1) = P(R_2|D_1) = P(R_1|D_1) = P(R|D_1) = \frac{2}{3}$

$P(R_3|D_2) = P(R_2|D_2) = P(R_1|D_2) = P(R|D_2) = \frac{1}{3}$

We want $P(R_3 | R_1 \cap R_2)$ { from D_1 or from D_2

Summary

$P(R_3|R_1 \cap R_2) = P(R_3|D_1)P(D_1|R_1 \cap R_2) + P(R_3|D_2)P(D_2|R_1 \cap R_2)$

$$P(D_1|R_1 \cap R_2) = \frac{P(R_1 \cap R_2|D_1)P(D_1)}{P(R_1 \cap R_2)} = \frac{P(R_1 \cap R_2|D_1)P(D_1)}{P(R_1 \cap R_2|D_1)P(D_1) + P(R_1 \cap R_2|D_2)P(D_2)}$$

$$= \frac{P(R_1|D_1)P(R_2|D_1)P(D_1)}{P(R_1|D_1)P(R_2|D_1)P(D_1) + P(R_1|D_2)P(R_2|D_2)P(D_2)}$$

$$= \frac{\frac{2}{3}\times\frac{2}{3}\times\frac{1}{2}}{\frac{2}{3}\times\frac{2}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{3}\times\frac{1}{2}} = \frac{4}{5}$$

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同理:

~~$P(D_1 \cap R$~~

$P(D_2 | R_1 \cap R_2) = \frac{1}{5}$

$\therefore P(R_3 | R_1 \cap R_2) = P(R_3 | D_1) P(D_1 | R_1 \cap R_2) + P(R_3 | D_2) P(D_2 | R_1 \cap R_2)$
$= \frac{2}{3} \times \frac{4}{5} + \frac{1}{3} \times \frac{1}{5} = \frac{9}{15} = \frac{3}{5}$

☆ Probability.

(10) A Professor of Probability always gets his son to answer incoming telephone calls on his home landline. Of these calls, 20% are nuisance calls and the remaining 80% are genuine. If his son considereds a given call to be genuine the he passes to his father, otherwise. put the phone down. The probability that his son considers a genuine call to be a nuisance call is 0.05 and the probability that his son considers a nuisance call to be genuine is 0.3.

a) Calculate the following probabilities:

i) that the son puts the phone down on a given call.

ii) that a call passed to his father is in fact a nuisance call.

iii) that the son makes the correct decision on a given call.

b) The professor is unhappy with the number of nuisance calls getting through to him. Thus now if his son deems a call to be genuine.

Summary

he passes to the professor's daughter, who then passes the phone to her father if she also considers the call to be genuine., otherwise she puts the phone down. Suppose that his son's and daughter's assessments of each call are independent but with the same misclassification probabilities. What is the probability that a call put down by his daughter (after being passed to her by his son) is in fact genuine.?

要搞清楚谁是结果
谁是方法.
求结果 ⇒ 全概率
求方法 ⇒ 贝叶斯

(a): ~~list.~~ $G\{$Genuine call$\}$ $N\{$Nuisance call$\}$

$A\{$son passes to father$\}$

$p(G)=0.8 \quad p(N)=0.2 \quad p(A|G)=1-0.05=\frac{19}{20}$

$p(A|N)=\frac{3}{10}$.

(i) $p(A^c)=p(A^c|G)p(G)+p(A^c|N)p(N)$

$=\frac{1}{20}\times 0.8+0.7\times\frac{1}{5}=\frac{9}{50}$

(ii) $p(N|A)=\dfrac{p(A|N)p(N)}{\underset{1-p(A^c)}{p(A)}}=\dfrac{p(A|N)p(N)}{p(A|N)p(N)+p(A|G)p(G)}$

$=\dfrac{0.3\times 0.2}{0.3\times 0.2+\frac{19}{20}\times 0.8}=\frac{3}{41}$

(iii) $p($correct decision$)$

$=p((G\cap A)\cup(N\cap A^c))$

$=p(G\cap A)+p(N\cap A^c)$

$=p(A|G)p(G)+p(A^c|N)p(N)$

$=\frac{19}{20}\times 0.8+\frac{7}{10}\times 0.2=9/10$

Summary

(b) let B {daughter passes to father}

$$P(G \mid A\cap B^c) = \frac{P(A\cap B^c \mid G)P(G)}{P(A\cap B^c\mid G)P(G) + P(A\cap B^c\mid N)P(N)}$$

$$= \frac{P(A\mid G)P(B^c\mid G)P(G)}{P(A\mid G)P(B^c\mid G)P(G) + P(A\mid N)P(B^c\mid N)P(N)}$$

$$= \frac{\frac{19}{20}\cdot\frac{1}{20}\times\frac{4}{5}}{\frac{19}{20}\times\frac{1}{20}\times\frac{4}{5}+\frac{3}{10}\times\frac{2}{10}\times\frac{1}{5}}$$

$$= \frac{19}{40}.$$

到这里，四类题的这种特殊贝叶斯就讨论完了，里面都涉及到了三种互相独立的过程，并且要求解出在前两种情况条件之下的第三种过程；而每一种过程又暗含了三种或两种情况，所以这种特殊的贝叶斯公式会呈现出这样一个情况

$$P(D_3\mid D_1\cap D_2) = P(D_3\mid F)P(F\mid D_1\cap D_2) + P(D_3\mid F^c)P(F^c\mid D_1\cap D_2)$$

而在求 $P(F\mid D_1\cap D_2)$ 形式时，又会牵连到独立性，写到 $P(D_1\cap D_2\mid F)P(F)$

$$= \underbrace{P(D_1\mid F)}\,\underbrace{P(D_2\mid F)}\,P(F)$$

通过这四种相同类型的题目，对这种含多条件、相互独立的贝叶斯模型应该会越来越熟悉。

Summary

和 86 是同一种题型

Probability

这个题的三种分布都非常经典

(108) A fair die is rolled repeatedly. Let X be the number of even numbers obtained in the first 18 rolls. Let Y be the number of rolls required until the third 6 appears. Let $Z = W_1 + W_2 + \cdots + W_6$, where

$$W_i = \begin{cases} 1 & \text{if the number on the } i^{th} \text{ roll divides } i \text{ exactly.} \\ 0 & \text{otherwise.} \end{cases}$$

16. Find $P(|X-9| \le 2)$ to 3 decimal places.

(a) 0.762 (b) 0.712 (c) 0.661 (d) 0.519 (e) None

17. Find $E(Y)$

(a) 18 (b) 17 (c) 12 (d) 11 (e) None

18. Find $E(Z)$

(a) 2 (b) $\frac{13}{6}$ (c) $\frac{7}{6}$ (d) $\frac{7}{3}$ (e) None.

16. $X \sim B(18, \frac{1}{2})$

$P(|X-9| \le 2) = P(7 \le X \le 11) = F_X(11) - F_X(6) = 0.8811 - 0.1189 = 0.7622$ (a).

实然冒出来个离散<=的分布. 如果写成7 7本身就被减掉了.

17. $Y \sim NB$. sum of $Geo(\frac{1}{6})$

$E(Y) = \frac{r}{p}$ $r=3$

$E(Y) = \frac{3}{\frac{1}{6}} = 18$ (a)

一定要和那个抓小动物的模型联系在一起 这种是1, 是0的问题其实并不难.

18. 类似于抓小动物. 本质上Z是用来计数的, 当然只考虑1即可. 首先label这些结果.

$P(W_1 = 1) = 1$ 第一次 $P(W_2 = 1) = \frac{1}{2}$ $P(W_3 = 1) = \frac{1}{3}$

$P(W_4 = 1) = \frac{1}{6}$ $P(W_5 = 1) = \frac{1}{6}$ $P(W_6 = 1) = \frac{1}{6}$

$\therefore E(Z) = 1 \times 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{3} + 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + 1 \times \frac{1}{6} = \frac{7}{3}$

Summary

2020.1.5. 考前2天把这学期所有错题已经过了一遍了. 收获挺多. 不过sample上的应用题还未攻破. 其实好多问题都是自己解决出来的. 尤其是通过不同题对比发现. 目前基础应该很扎实了.

George Willsons

Probability.

好像没怎么研究 pgf 家。。。。。。

(109) Find the probability generating function of random variables having the following distribution.

(i) $P(X=k) = p(1-p)^k \quad k = 0, 1, \cdots \quad 0<p<1$

(ii) $P(Y=k) = \alpha p^k / k, \quad k = 1, 2, \cdots; \quad 0<p<1$

$\alpha = \log(1-p)!$

Use pgf to determine $E(X)$. $Var(X)$, $E(Y)$ and $var(Y)$ Deduce the mean and variance. of the Geometric distribution given in the notes.

Let's deduce pgf first.

(i) $p(X=k) = p(1-p)^k$

$$P_X(s) = E(s^X) = \sum_{k=0}^{\infty} s^k p(1-p)^k$$

$$= p \sum_{k=0}^{\infty} [s(1-p)]^k$$

$$= \frac{p}{1-s(1-p)} \sum_{k=0}^{\infty} [s(1-p)]^k [1-s(1-p)]$$

$$= \frac{p}{1-s(1-p)}.$$

~~$E(X) = p$~~

求导的过程注意点的问题是 s 才是变量

$$P_X'(s) = \frac{p(1-p)}{[1-s(1-p)]^2} \qquad P_X''(s) = \frac{2p(1-p)^2}{[1-(1-p)s]^3}$$

$$E(X) = P_X'(1) = \frac{1-p}{p} \qquad Var(X) = P_X''(1) + P_X'(1) - [P_X'(1)]^2$$

$$= \frac{1-p}{p^2}$$

(ii) $p(Y=k) = \alpha p^k / k.$

$$P_Y(s) = E[s^Y]$$

$$= \sum_{k=1}^{\infty} \frac{\log(1-p)\, s^k \cdot p^k}{k}$$

Summary

$$= \sum_{k=1}^{\infty} \frac{\log(1-sp)\,(sp)^k}{k} \cdot \frac{\log(1-p)}{\log(1-sp)}$$

$$= \frac{\log(1-p)}{\log(1-sp)}$$

$P_Y'(s) = \frac{\lambda p}{1-ps}$ $\quad P_Y''(s) = \frac{\lambda p^2}{(1-ps)^2}$

$E[Y] = P_Y'(1) = \frac{\lambda p}{1-p}$

$var(Y) = P_Y''(1) + P_Y'(1) - [P_Y'(1)]^2 = \frac{\lambda p^2}{(1-p)^2} + \frac{\lambda p}{1-p} - \frac{\lambda^2 p^2}{(1-p)^2}$

$= \frac{\lambda p(1-\lambda p)}{(1-p)^2}$

Note that $X = Z - 1$ $\quad Z \sim Geo(p)$.

$E[Z] = E[X+1] = E[X]+1 = \frac{1-p}{p} + 1 = \frac{1}{p}$

对比一下: $P(X=k) = p(1-p)^k$ $\quad k=0,1,2,\dots$

$P(Z=k) = p(1-p)^{k-1}$ $\quad k=1,2,3,\dots$ $\quad Z \sim Geo(p)$

Z 始终比 X 多试验一次, X 可以不试验 即 $k=0$.

Z 必须试验.

Summary

Probability.

第一问可联系(104)拨号问题。核心是居然每次概率都是1/6. 因为用过了钥匙之后，将不重复使用.并且每一次实验都建立在前一次实验基础之上.

难题总是最后一个被攻克的.

(110).

(a). Let X be a random variable with probability mass function $P_X(x) = \frac{1}{n}$ $(x = 1, 2, 3, \ldots n)$ where n is a fixed strictly positive integer. Show that $E(X) = \frac{1}{2}(n+1)$ and $Var(X) = \frac{1}{12}(n^2-1)$

(b) A new porter has to lock 6 doors in a building and has a bunch of 6 different keys. one for each lock. He visits the doors in turn. At each door, he tries the keys on the bunch at random, without repetition, and removes the key from the bunch when it has locked the door. Let X be the number of attempts he requires to lock the first door. and let T be the total number of attempts until he has locked all the doors.

(i) Determine the mean and variance of X.

(ii) Determine the mean and variance of T.

(iii) Determine the mean and variance of the total number of failed attempts made by the porter.

(a). 是 Uniform 分布.

$$E(X) = \frac{n+1}{2}$$

$$Var(X) = \frac{n^2-1}{12}$$

Summary

(i). Let X_i be the event that i^{th} trial he lock the door. $i=1,\cdots 6$. $X=1,2,3,4,5,6$.

$p(X=1)=p(X_1)=\frac{1}{6}$ $p(X=2)=p(\bar{X_1}X_2)=\frac{5}{6}\times\frac{1}{5}=\frac{1}{6}$

$p(X=3)=p(\bar{X_1}\bar{X_2}X_3)=\frac{5}{6}\times\frac{4}{5}\times\frac{1}{4}=\frac{1}{6}$ ...

$p(X=j)=\frac{1}{6}$ $j=1,2,3,4,5,6$.

$E(X)=\frac{7}{2}$ $var(X)=\frac{1}{12}(6^2-1)=\frac{35}{12}$

(ii) T是总次数，我们细化到每一次分析，开第一扇门 $n=6$. $p(x)=\frac{1}{6}$，但当第一扇门被锁好，就只剩下5把钥匙了，即 $n=5$. $p(x)=\frac{1}{5}$，因此概率递减，且每一次的期望和方差都不一样. Let T_i be the event that the number of trials to lock i^{th} door.

整个计算过程中，i是一个变量.

即 $p(T_1)=\frac{1}{6}$ $p(T_2)=\frac{1}{5}$ $p(T_3)=\frac{1}{4}$ $p(T_4)=\frac{1}{3}$ $p(T_5)=\frac{1}{2}$

$p(T_6)=1$ $\boxed{\text{发现规律：} p(T_i)=\frac{1}{7-i}}$ $(i=1,2,\cdots 6)$

$\therefore 7-i=n$. $E(T_i)=\frac{8-i}{2}$ $var(T_i)=\frac{(7-i)^2-1}{12}$

$T=T_1+T_2+\cdots+T_6$ $T=\sum_{i=1}^{6}(T_i)$

$E(T)=E(\sum_{i=1}^{6}T_i)=\sum_{i=1}^{6}E(T_i)=\frac{7}{2}+\frac{6}{2}+\cdots+\frac{2}{2}=\frac{27}{2}$

$var(T)=var(\sum_{i=1}^{6}T_i)=\sum_{i=1}^{6}var(T_i)=\frac{1}{12}(35+24+15+\cdots+0)=\frac{85}{12}$.

第(iii)问列个表深入对比一下成功和失败次数

	A	B	C	D	E	F		A	B	C	D	E	F	
T 第i次成功次数	1	1	1	1	1	1	$\to 6$	5	4	4	1	1	1	$\to 16$
Z 失败次数	0	0	0	0	0	0	$\to 0$	4	3	3	0	0	0	$\to 10$

对比一下很容易发现第n次成功说明试了n次，而对应的便是前i-1次失败，所以成功次数始终比失败次数多1，总共成功次数就比总的失败多6

Summary

$Z=T-6$ Let Z { fail attempts

$E[Z]=E(T)-6=\frac{27}{2}-6=\frac{15}{2}$

$var(Z)=var(T)=85/12$.

这道困扰我又困多的题目终于了结了。事实上，这种有几何分布影子的题会给人一种"表象难"的感觉，因为几何分布本身就是一种简化了的思维（即你第i次成功，那么你前i-1次注定失败）如果按平常的思维，很可能陷入一一列举的无穷尽泥潭之中，这时，找到临界点，换种思考方式就显得尤为重要。

几何分布本身探讨的就是实验次数问题本身无穷尽，但在研究过程中我们提前列好了分布。相当于在已知结果的状态下从中发现规律。

这种求 Total Number 的题，总会先从每一项的研究开始，Total 即每一项求和而得到。

Summary

5

AcF

(111) Determine $\lim_{x \to \infty} f(x)$, if it exists.

$f(x) = \sqrt{x} \sin(1/x)$

Lemma: for all $t \in R$, $|\sin t| \leq |t|$

proof: $\sin t = \sin t - \sin 0$

$$= \int_0^t \cos x \, dx \leq \int_0^t 1 \cdot dx = [x]_0^t = t$$

~~$\therefore \sin t \leq t$~~ $\therefore |\sin t| \leq |t|$

核心. $\sin x \leq x$ ← 是极限→∞时情况

$|f(x)| = |\sqrt{x} \sin(1/x)| \leq \sqrt{x} \cdot \frac{1}{x} = \left|\frac{1}{\sqrt{x}}\right|$

Let $x_n \to \infty$, then $\frac{1}{\sqrt{x_n}} \to 0$

$\Rightarrow -\frac{1}{\sqrt{x}} \leq f(x_n) \leq \frac{1}{\sqrt{x_n}}$

$\downarrow \qquad\qquad \downarrow$

$0 \qquad\qquad 0$

By Sandwich Theorem: $f(x) \to 0$

Therefore, $\lim_{x \to 0} f(x) = 0$

AcF

(112) Show $f(x) = |x|$ is not differentiable at 0

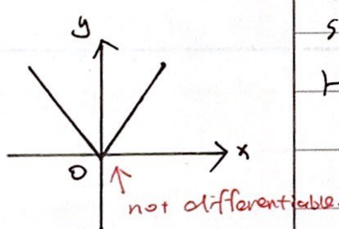
Basic idea: right ≠ left.

$$\lim_{x \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{x \to 0^-} \frac{-h}{h} = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{x \to 0^+} \frac{|h|}{|h|} = \lim_{x \to 0^+} (1) = 1$$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence $|x|$ is not differentiable at 0



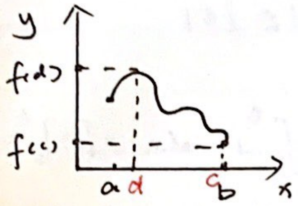
Summary

Derivative of a function

$f: R \to R$ at a real number a is defined to be the limit.

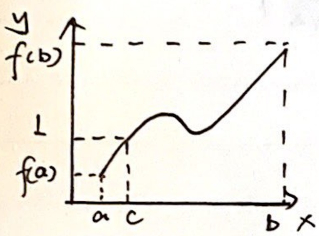
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ if it exists.}$$

⑪③ Some Theorems.



Extreme Value Theorem

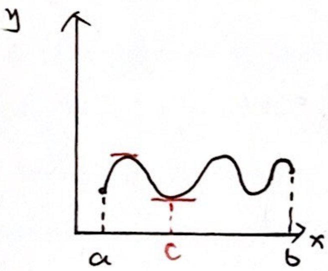
Let $f: [a, b] = I \to R$ be continuous. Then there exist $c, d \in I$ s.t. $f(c) \le f(x) \le f(d)$ for all $x \in I$.



Intermediate Value Theorem

Let $I = [a, b]$ and let $f: I \to R$ be continuous. If $f(a) < L < f(b)$ or if $f(b) < L < f(a)$, then there exists $c \in (a, b)$ with $f(c) = L$

Special case: $f: [a, b] \to R$ is a continuous with $f(a) < 0 < f(b)$. f must have a root $x \in (a, b)$ that is, a solution of $f(x) = 0$.



Rolle's Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then for some $c \in (a, b)$ we must have $f'(c) = 0$

写定义：

① 先写区间

② 连续 + 可导

③ If, ··· exist.

Summary

ACF

(114) Find the limit by L'Hôpital's rule, type "0/0"

$$L = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$$

1. since $|x^2 \sin(1/x)| \le |x^2|$

$-x^2 \le x^2 \sin(1/x) \le x^2$, $\lim_{x\to 0} x^2 = 0$ $\lim_{x\to 0} (-x^2) = 0$

By Sandwich Theorem. $x^2 \sin(1/x) \to 0$ and $\sin x \to 0$

By L'Hôpital's rule:

$$L = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x} = \lim_{x\to 0} \frac{2x\sin(1/x) + x^2 \cos\frac{1}{x}(-\frac{1}{x^2})}{\cos x}$$

$$= \lim \frac{\overset{\to 0}{2x\sin(1/x)}}{\underset{\to 1}{\cos x}} + \lim \frac{\overset{\text{oscillating}}{-\cos(1/x)}}{\underset{\to 1}{\cos x}}$$

we fail by using this way.

2. $\left|\frac{x^2 \sin(1/x)}{\sin x}\right| \le \left|\frac{x^2}{\sin x}\right|$ so we conside the limit of $\frac{x^2}{\sin x}$

$x^2 \to 0$, $\sin x \to 0$ as $x \to 0$

By L'Hôpital's rule. $\lim_{x\to 0} \frac{x^2}{\sin x} = \lim_{x\to 0} \frac{2x}{\cos x} = 0$

By Sandwich Theorem,

$$\cancel{\frac{x^2\sin(1/x)}{x^2}}$$

$$-\frac{x^2}{\sin x} \le \frac{x^2 \sin(1/x)}{\sin x} \le \frac{x^2}{\sin x} \xrightarrow{\text{s.t.}}$$

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{x^2} = 0$$

Sandwich

↓

L'Hôpital's rule

Fail

L'Hôpital's rule

↓

Sandwich.

Summary

5 . .

ACT

(115) $L = \lim_{x\to\infty} x \ln(1+1/x)$

Analysis: $\frac{1}{x} \to 0 \to \ln(1+\frac{1}{x}) \to 0$

$x\to\infty \to \infty \cdot 0$ type

cannot apply L'Hôpital's rule directly

$$L = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot (-\frac{1}{x^2})}{(-\frac{1}{x^2})}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$$

$$= \frac{1}{1+0} = 1$$

Actually: $\lim_{x\to\infty} x \ln(1+\frac{1}{x})$

$$= \lim_{x\to\infty} \ln(1+\frac{1}{x})^x$$

$$= \ln \lim (1+\frac{1}{x})^x$$

$$= \ln e$$

$$= 1.$$

ACT. (116) From (115) we can calculate the limit of the sequence $x_n = (1+1/n)^n$.

consider: $f(x) = (1+\frac{1}{x})^x$

Rearrange it as $f(x) = (1+\frac{1}{x})^x = e^{\ln(1+\frac{1}{x})^x}$

Summary

$\boxed{\text{L'Hôpital's rule}}$ version I : type "0/0"

Let f and g be real-valued functions s.t. $\lim f(x) = \lim g(x) = 0$

Suppose that $\lim \frac{f'(x)}{g'(x)}$ exists. $= L$, then $\lim \frac{f(x)}{g(x)}$ exists $= L$

(finite or infinite)

f.g. differentiable

$$\lim_{x\to\infty}(1+\tfrac{1}{x})^x = \lim_{x\to\infty} e^{\ln(1+\frac{1}{x})^x}$$

$$= e^{\lim_{x\to\infty}\ln(1+\frac{1}{x})^x}$$

$$= e^1$$

$$= e$$

Hence: $f(x_n) = (1+\frac{1}{n})^n \xrightarrow[n\to\infty]{} e$

ACF

(11) $L = \lim_{x\to 0^+}(e^x+x)^{1/x}$

$$\lim_{x\to\infty}\frac{\ln(1+e^x)}{x} = \lim_{x\to\infty}\frac{\ln(e^x(1+e^{-x}))}{x}$$

$$= \lim_{x\to\infty}\frac{\ln e^x + \ln(1+e^{-x})}{x}$$

$$= \lim \frac{x}{x} + \lim_{x\to\infty}\frac{\overset{\to 0}{\ln(1+e^{-x})}}{\underset{\to\infty}{x}}$$

$$= 1+0$$

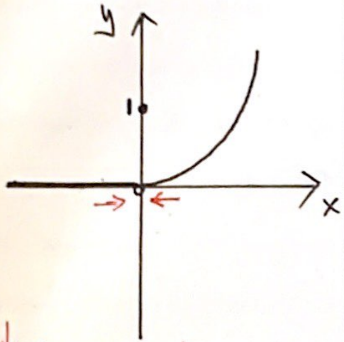
$$= 1$$

Summary

$\boxed{\text{L'Hôpital's rule}}$ version II type "∞/∞":

Let f and g be real-valued functions s.t. $\lim f(x)$ and $\lim g(x)$ both exist, and are each ∞ or $-\infty$ (they are same). Suppose $\lim\frac{f'(x)}{g'(x)}$ exists $= L$ (finite or infinite), then $\lim\frac{f(x)}{g(x)}$ exists $= L$

A-F



↓ Although, skip to 1 but from both sides we can get the limit.

(118) Define a function $h(x)$ by

$h(x)=0 \quad (x<0) \qquad h(x)=1 \quad (x=0) \quad , \; h(x)=x^{\alpha} \; (x>0)$

Sketh the graph of $h(x)$ and determine the following limits. if they exists.

(i) $\lim_{x\to 0} h(x)$ (ii) $\lim_{x\to 0} h(h(x))$

$\lim_{x\to 0} h(x) = 0$ clearly.

For $x<0$ $\quad h(x)=0 \Rightarrow h(h(x)) = \underline{\underline{1}}$

For $x>0$ $\quad h(x)=x^{\alpha}>0 \Rightarrow h(h(x)) = x^{\alpha^2} \to \underline{\underline{0}}$ as $x \to 0^+$

since $\lim_{x\to 0^-} h(h(x)) \neq \lim_{x\to 0^+} h(h(x))$

$\lim_{x\to 0} h(h(x))$ $\boxed{\text{does not exist.}}$

Intermediate forms A-F

(119) Intermediate form: an expressing involving two functions whose limit cannot be determined solely from the limits of the individual functions.

i) $\lim_{x\to \pi^-} \frac{\sec x}{\sin x}$

as $x \to \pi^-$ we have $\sec x \to -1$, $\sin x \to 0$

so this is not an intermediate forms.

Thus the limit is $-\infty$

ii) $\lim_{x\to 2} \frac{(x+2)^2}{(x-2)^2}$. as $x \to 2$, $(x+2)^{\alpha} - 4^2$, $(x-2)^2 \to 0$ with positive

thus the limit exists and is ∞.

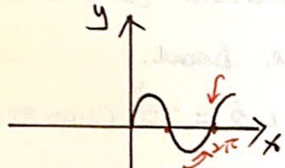
Summary

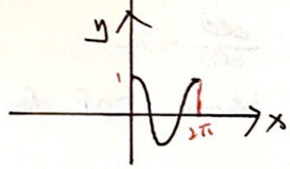
iii) $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$ intermediate form, of type ∞/∞

iv) $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$ intermediate form, of type $-\infty/\infty$

ACF

(120) Let $n \in \mathbb{Z}$ and $f(x) = x - \cot \pi x$. Determine $\lim_{x\to n^-} f(x)$ and $\lim_{x\to n^+} f(x)$. Hence show that the equation $f(x)=0$ has a solution $x^* \in (n, n+1)$. By considering f'. show that x^* is the only solution of $f(x)=0$ in $(n, n+1)$.





Note $\sin n\pi = 0$, since $n \in \mathbb{Z}$

Suppose n is even, then

$\lim_{x\to n} \cos \pi x = \cos \pi n = 1$

$\lim_{x\to n^-} \sin x\pi < 0$ $\quad \lim_{x\to n^+} \sin x\pi > 0$

$\lim_{x\to n^-} x - \cot \pi x = \infty$ $\quad \lim_{x\to n^+} x - \cot \pi x = -\infty$

(the same as n is even). ↗ sign of $\sin n\pi$ reverse

↘ $\cos \pi n = -1$ → not change

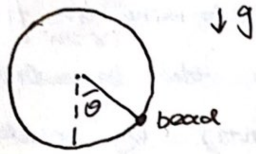
since $\lim_{x\to n^+} f(x) = -\infty$ and $\lim_{x\to (n+1)^-} f(x) = \infty$

By Intermediate Value theorem. $f(x^*) = 0$ $x^* \in (n, n+1)$

$f'(x) = 1 + \pi \operatorname{cosec}^2 \pi x \geq 1 > 0$, then x^* is unique.

Summary

APP



批注：这类题的易错点为 $\theta(0)=0$. 但 $\dot\theta(0)$ 不一定为0. $\dot\theta$ 要通过初速度和 $v=r\omega$ 角速度关系来决定,

(121) A bead of mass m moves inside a fixed rough circular cylinder of radius a, in a vertical plane. The coefficient of sliding friction between the bead and the cylinder is μ (so that when the bead is in motion the magnitude of the frictional force is μ times that of the normal reaction force.

The bead is projected from the lowest point of the circle with speed U.

(a) Write down the radial and transverse components of the equations of motion of the bead.

(b) Show that θ satisfied $\ddot\theta + \mu\dot\theta^2 = -\frac{g}{a}(\mu\cos\theta + \sin\theta)$

(c) Let $\phi = \dot\theta^2$. Show that $\dot\phi = 2\dot\theta\ddot\theta$ and hence use the chain rule to show that $\frac{d\phi}{d\theta} = 2\ddot\theta$.

(d) Hence deduce a first-order linear ODE for ϕ as a function of θ.

(e) When $\mu = 1/\sqrt2$, show that $\phi = \frac{U^2}{a^2}\left\{e^{-\sqrt2\theta} - \frac{\sqrt2 ga}{U^2}\sin\theta\right\}$

(f) Show that the bead rises beyond $\theta = \pi/2$ if $U^2 > \sqrt2 e^{\pi/\sqrt2} ga$.

[Hint: in second part of (c), use $\frac{d\phi}{d\theta} = \frac{d\phi}{dt}\cdot\frac{dt}{d\theta} = \frac{1}{\dot\theta}\cdot\frac{d\phi}{dt}$

part (d) combines (b) and (c)]

Summary

Polar coordinates

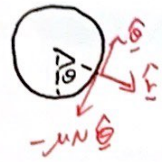
position: $x = r\hat{r}$

velocity: $v = \dot r\hat{r} + r\dot\theta\hat{\theta}$

acceleration: $a = (\ddot r - r\dot\theta^2)\hat{r} + (2\dot r\dot\theta + r\ddot\theta)\hat{\theta}$

$= (\ddot r - r\dot\theta^2)\hat{r} + \frac{1}{r}\left(\frac{d}{dt}(r^2\dot\theta)\right)\hat{\theta}$

5



$r = a$. $\dot{r} = \ddot{r} = 0$

(a) $N = -N\hat{r}$ $\quad mg = mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}$ $\quad T = -\mu N\hat{\theta}$

NII) $\Rightarrow$ $f = ma$

$$-N\hat{r} + mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta} - \mu N\hat{\theta} = m\left\{(\ddot{r} - r\dot{\theta}^2)\,\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{\theta}\right\}$$

$\hat{r}$: $N = mg\cos\theta + ma\dot{\theta}^2$

$\hat{\theta}$: $-\mu N - mg\sin\theta = ma\ddot{\theta}$

b) plug in N.

$$\mu(mg\cos\theta + ma\dot{\theta}^2) - mg\sin\theta = ma\ddot{\theta}$$

$$\ddot{\theta} + \mu\dot{\theta}^2 = \frac{-g}{a}(\mu\cos\theta + \sin\theta)$$

c) let $\phi = \dot{\theta}^2$ $\quad \dot{\phi} = 2\dot{\theta}\cdot\ddot{\theta}$

that is $\frac{d\phi}{dt} = 2\cdot\frac{d\theta}{dt}\cdot\ddot{\theta}$ $\quad \times\frac{dt}{d\theta}$

$\frac{d\phi}{d\theta} = 2\ddot{\theta}$ $\quad (\phi(\theta))$

d) $\ddot{\theta} + \mu\dot{\theta}^2 = \frac{1}{2}\frac{d\phi}{d\theta} + \mu\phi = -\frac{g}{a}(\mu\cos(\theta) + \sin(\theta))$

$\Rightarrow$ $\frac{d\phi}{d\theta} + 2\mu\phi = -\frac{2g}{a}(\mu\cos\theta + \sin\theta)$

$\boxed{e)}$ $\mu = 1/\sqrt{2}$ $\quad \boxed{\frac{d\phi}{d\theta} + \sqrt{2}\phi = -\frac{2g}{a}\left(\frac{1}{\sqrt{2}}\cos\theta + \sin\theta\right)}$

Integrating Factor: $IF = e^{\int\sqrt{2}\,d\theta} = e^{\sqrt{2}\theta}$

use this ⇐ ~~By Euler's formula.~~

$$\int e^{\sqrt{2}\theta}(\cos\theta + i\sin\theta)\,d\theta = \int e^{(\sqrt{2}+i)\theta}\,d\theta = \frac{e^{(\sqrt{2}+i)\theta}}{\sqrt{2}+i}\cdot\frac{\sqrt{2}-i}{\sqrt{2}-i}$$

$$= \frac{1}{3}(\sqrt{2}-i)\,e^{\sqrt{2}\theta}(\cos\theta + i\sin\theta)$$

?? $\quad = e^{\sqrt{2}\theta}\left\{\frac{\sqrt{2}}{3}\cos\theta + \frac{1}{3}\sin\theta + i\left(\frac{\sqrt{2}}{3}\sin\theta - \frac{1}{3}\cos\theta\right)\right\}$

K提前加上. 不然 ⇐
移项时 K全被除

$$e^{\sqrt{2}\theta}\phi = -\frac{2g}{a}e^{\sqrt{2}\theta}\left\{\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{3}\cos\theta + \frac{1}{3}\sin\theta\right) + \frac{\sqrt{2}}{3}\sin\theta - \frac{1}{3}\cos\theta\right\} + K$$

$\phi = -\frac{\sqrt{2}g}{a}\sin\theta + Ke^{-\sqrt{2}\theta}$. when $\theta = 0$ $\phi = K = \dot{\theta}^2 = \frac{U^2}{a^2}$

Summary

$\boxed{\text{Euler's Formula}}$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$