Mathematical Method

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Contents

1 Introduction

This is my reading notes based on *Mathematical Methods, University of Cambridge Part IB Mathematical Tripos, David Skinner*. This notes is largely imcomplete one, I only write down some essential parts based on my own learning path.

2   Fourier Series

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Definition 2.1 (Innner product)**.** Inner product is defined as a mapping $(,) : V \times V \to F$ that obeys:

1. $(u, v) = (v, u)*$ (Conjugate symmetry)

2. $(u, \lambda v) = \lambda(u, v)$ (linearity)

3. $(u, v + w) = (u, v) + (u, w)$ (additivity)

4. $(u, u) \geq 0, \forall u \in V$, and with equality $\iff u = 0$

If $F = \mathbb{C}$, the map is called sesquilinear.

2.1   Motivation

If we're given an orthonormal basis, we can use the inner product to explicitly decompose a general into. this basis.

For any element $u \in V$, it can be uniquely written as

$$u = \sum_{i=1}^{n} \lambda_i v_i \tag{1}$$

we can use inner product to find the coefficient λ_i. Please note that $(v_i, v_j) = \delta_{ij}$.

$$(v_j, u) = (v_j, \sum_{i=1}^{n} \lambda_i v_i) = \sum_{i=1}^{n} (v_i, \lambda_i v_i) = \sum_{i=1}^{n} \lambda_i (v_j, v_i) = \lambda_j \tag{2}$$

Thus, we find that $\lambda_i = (v_i, u)$, for real vectors, λ_j is just the projection of u onto v_j.

Definition 2.2 (Inner product for complex valued function)**.** Consider a function $f : \Omega \to \mathbb{C}$. The inner product is defined as:

$$(f, g) = \int_{\Omega} f(x)^* g(x) d\mu \tag{3}$$

This is an generalization of the inner product between two finite dimensional vectors.

If $\Omega = [a, b]$, then $(f, g) = \int_a^b f(x)^* g(x) dx$, with measure dx If Ω is disc D_2, then $(f, g) = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} f(r, \theta)^* g(r, \theta) r dr d\theta$, with. measure $d\mu = r dr d\theta$.

Let's applied above results to construct fourier series.

Consider inner product:

$$(e^{im\theta}, e^{in\theta}) = \int_{\pi}^{\pi} e^{-im\theta} e^{in\theta} d\theta = 2\pi \delta_{m,n} \tag{4}$$

this function $e^{in\theta}$ can be used as orthonormal basis after some scaling.

Fourier claimed that

$$f(\theta) \stackrel{?}{=} \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta} \tag{5}$$

Since our basis is orthonormal, we can use inner product to get the cofficients $\hat{f}_n$.

$$\hat{f}_n = \frac{1}{2\pi}(e^{in\theta}, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta \tag{6}$$

Let's rewrite the function:

$$f = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta} \tag{7}$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} (\hat{f}_n e^{in\theta} + \hat{f}_{-n} e^{-in\theta}) \tag{8}$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} (\hat{f}_n e^{in\theta} + \hat{f}_n^* e^{-in\theta}) \tag{9}$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \tag{10}$$

In the final line, we set $\hat{f}_n = (a_n - ib_n)/2$, then we get

$$a = 2\Re \hat{f}_n, b = -2\Im \hat{f}_n \tag{11}$$

$$a = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n\theta f(\theta) d\theta, b = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta f(\theta) d\theta \tag{12}$$

2.2 Some results

Definition 2.3. The finite sum is defined as

$$S_n f = \sum_{k=-n}^{n} \hat{f}_k e^{in\theta} \tag{13}$$

Theorem 2.1 (Parseval's identity)**.**

$$(f, f) = 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2 \tag{14}$$

Parseval's indentity can be view as an infinite dimensional version of Phythagoras'theorem: the norm squared f is equal to the sum of the (mod)-squares of its coefficients in the (Fourier) basis of orthogonal functions. If we view the Fourier series as a map from a function to a sequence, Parseval's identity tells us this map is an isometry, which means it preserves lengths.

3 Sturm-Liouville Theory

3.1 Matices foundation: Self-adjoint matrices

Definition 3.1 (Linear Map)**.** Let V and W be finite dimensional vector spaces (over complex numbers), with $\dim V = n, \dim W = m$, a linear map is defined by $M : V \to M$

We can. represent the map M in terms of an $m \times n$ matrix M, with components:

$$M_{ai} = (w_a, Mv_i), a = 1, \ldots, m, i = 1, \ldots, n \tag{15}$$

Definition 3.2 (Hermitian conjugate)**.** Given a matirx M, its Hermitian conjugate $M^\dagger$ is defined to be the complex conjugate of the transpose matrix, $M^\dagger = (M^T)^*$, where the complex conjugation acts on each entry of M^T.

A matrix is said to be Hermitian or self-adjoint if $M^\dagger = M$. This is also another neaty way of this definition. since for two vectors we have, $(u, v) = u^\dagger \cdot v$, (if both vector are real, this is same as inner product as usual, also, $(Bu)^\dagger = u^\dagger B^\dagger$, we have the definition that: matrix B is the adjoint of a matrix $A \iff (Bu, v) = (u, Av)$.

There some important properties below.

1. Since $\lambda_i(v_i, v_i) = (v_i, Mv_i) = (Mv_i, v_i) = \lambda_i^*(v_i, v_i)$, the eigenvalues of self-adjoint matrix are always real (because $\lambda_i = \lambda_i^*$).

2. We have $\lambda_i(v_j, v_i) = (v_j, Mv_i) = (Mv_j, v_i) = \lambda_i(v_i, v_i)$, then $(\lambda_i - \lambda_j)(v_j, v_i) = 0$, the eigenvectors corresponding to distinct eigenvalues are orthogonal w.r.t. the inner product.

3.2 Solving linear system

A self-adjoint matrix M is non-singular $\iff$ all its eigenvalues are non-zero. In this case, we can solve the linear equation $\mathbf{Mu} = \mathbf{f}$. The solution is $\mathbf{u} = \mathbf{M^{-1}f}$.

Suppose $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of eigenvectors of $\mathbf{M}$. Then we can write:

$$\mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{v_i}, \tag{16}$$

$$\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{v_i} \tag{17}$$

where $f_i = (\mathbf{v_i}, \mathbf{f})$ (see previous inner product). In order to solve $\mathbf{u}$, we need to find the coefficients u_i in the $\{v_i\}$ basis.

By linearity:

$$\mathbf{Mu} = \sum_{i=1}^{n} u_i \mathbf{Mv_i} = \sum_{i=1}^{n} u_i \lambda_i \mathbf{v_i} = \mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{v_i} \tag{18}$$

Take the inner product of this equation with $\mathbf{v_j}$ gives

$$\sum_{i=1}^{n} u_i \lambda_i (v_j, v_i) = u_j \lambda_j = \sum_{i=1}^{n} f_i (v_j, v_i) = \sum_{i=1}^{n} f_i \delta_{ij} = f_j \tag{19}$$

Thus,

$$\mathbf{u} = \sum_{i=1}^{n} \frac{f_i}{\lambda_i} \mathbf{v_i} \tag{20}$$

If $\mathbf{M}$ is singular then either $\mathbf{Mu} = \mathbf{f}$ has no solution or else has a non-unique solution.

3.3 Differential Operator

This section is highly related to the previous discussion about self-adjoint matrices.

Definition 3.3 (Linear Operator).

$$L = A_p(x)\frac{d^p}{dx^p} + A_{p-1}\frac{d^{p-1}}{dx^{p-1}} + \ldots + A_1(x)\frac{d}{dx} + A_0(x) \tag{21}$$

This is a linear map between spaces of functions beacause for two function $y_{1,2}(x)$ and constants $c_{1,2}$, we have $L(c_1 y_1 + c_2 y_2) = c_1 Ly_1 + c_2 Ly_2$.

We will be interested in second order linear differential operators

$$L = P(x)\frac{d^2}{dx^2} + R(x)\frac{d}{dx} - Q(x) \tag{22}$$

The homogenous equation $Ly(x) = 0$ has precisely two non-trivial linearly independent solutions, say $y = y_1(x), y = y_2(x)$, and the general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$, which is called complementary function. For inhomogenous equation $Ly(x) = f$, we seek any single solution $y(x) = y_p(x)$, which is called partial integral, and the final general solution is the a linear combination

$$y(x) = c_p y_p(x) + c_1 y_1(x) + c_2 y_2(x) \tag{23}$$

Sturm-Liouville theory provides a more systematic approach, analogous to solving the matrix equation $\mathbf{Mu} = \mathbf{f}$.

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3.4 Self-adjoint differential operators

The second order differential operator considered by Sturm and Liouville take the form

$$Ly = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y \tag{24}$$

Provided $P(x) \neq 0$, divided through by $P(x)$ to obtain

$$\frac{d^2}{dx^2} + \frac{R(x)}{P(x)}\frac{d}{dx} - \frac{Q(x)}{P(x)} = e^{-\int_0^x R(t)/P(t)dt}\frac{d}{dx}\left(e^{\int_0^x R(t)/P(t)dt}\frac{d}{dx}\right) - \frac{Q(x)}{P(x)} \tag{25}$$

with integrating factor $e^{\int_0^x R(t)/P(t)dt}$.

The beautiful feature of Sturm-Liouville operators is that they are self-adjoint w.r.t inner product

$$(f,g) = \int_a^b f(x)^* g(x)dx \tag{26}$$

provided the functions on which they act obey appropriate boundary conditions. Let's illustrate this fact.

Recall the neaty definition: matrx B is the adjoint of a matrix $A \iff (Bu, v) = (u, Av)$. Thus, let take inner product (Lf, g)

$$(Lf, g) = \int_a^b \left[\frac{d}{dx}(p(x)\frac{df^*}{dx}) - q(x)f^*(x)\right]g(x)dx \tag{27}$$

$$= \left[p\frac{df^*}{dx}g\right]_a^b - \int_a^b p(x)\frac{df^*}{dx}\frac{dg}{dx} - q(x)f(x)^*g(x)dx \tag{28}$$

$$= \left[p\frac{df^*}{dx}g - pf^*\frac{dg}{dx}\right]_a^b + \int_a^b f(x)^*\left[\frac{d}{dx}\left(p(x)\frac{dg}{dx}\right) - q(x)g(x)\right]dx \tag{29}$$

$$= \underbrace{\left[p(x)\left(\frac{df^*}{dx}g - f^*\frac{dg}{dx}\right)\right]_a^b}_{boundary\ condition\ applied=0} + (f, Lg) \tag{30}$$

Exampls of such boundary conditions are to require that all our functions satisfy

$$b_1 f'(a) + b_2 f(a) = 0 \tag{31}$$
$$c_1 f'(b) + c_2 f(b) = 0 \tag{32}$$

There are some cases about this boundary conditions

1. If the function $p(x)$ obeys $p(a) = p(b)$, then all our functions are periodic, so that $f(a) = f(b), f'(a) = f'(b)$

2. If $p(a) = p(b) = 0$, the endpoints of the interval $[a, b]$ are singular points of the differential equation.

This section need to be illustrated later.

3.5 Eigenfunctions and weight functions

Definition 3.4 (Eigenfunction). A function $y(x)$ is said to be an eigenfunction of L with eigenvalue λ and weight $w(x)$ if

$$Ly(x) = \lambda w(x)y(x) \tag{33}$$

Definition 3.5 (Inner product with weight).

$$(f,g)_\omega = \int_a^b f(x)^* g(x)\omega(x)dx \tag{34}$$

Since omega(x) is real, so $(f, g)_\omega = (f, \omega g) = (\omega f, g)$

Definition 3.6 (Eigenvalues of SL operator are always real). If $Lf = \lambda \omega f$, then

$$\lambda(f, f)_\omega = (f, \lambda \omega f) = (f, Lf) = (Lf, f) = \lambda^*(f, f)_\omega \tag{35}$$

Theorem 3.1. Eigenfunctions f_1, f_2 with distinct eigenvalues, but the same weight function, are orthogonal w.r.t the inner product with weight ω.

Proof. Since:

$$\lambda_i(f_j, f_i)_\omega = (f_j, Lf_i) = (Lf_j, f_i) = \lambda_j(f_j, f_i)_\omega \tag{36}$$

so that if $\lambda_i \neq \lambda_j$,then:

$$(f_j, f_i)_\omega = \int_a^b f_j(x)^* f_i(x)\omega(x)dx = 0 \tag{37}$$

$\square$

Application of this theorem is that: given a self-adjoint operator L, we can form an orthonormal set $\{Y_1(x), Y_2(x), \ldots\}$ of its eigenfunctions by setting:

$$Y_n(x) = y_n(x)/\sqrt{\int_a^b |y_n|^2 \omega dx} \tag{38}$$

Finally, making particular choice of boundary conditions, any function $f(x)$ in $[a, b]$ that obeys the chosen boundary conditions may be expanded as:

$$f(x) = \sum_{n=1}^\infty f_n Y_n(x), \ where \ f_n = (Y_n, f)_\omega = \int_a^b Y_n^*(x)f(x)\omega(x)dx \tag{39}$$

Let's take a look at some example: Taking the domain $[-L, L]$ and impose the homogeneous boundary conditions that all our functions are periodic i.e. $f(-L) = f(L), f'(-L) = f'(L)$. Choosing $p(x) = 1$ and $q(x) = 0$, thus SL operator is

$$L = \frac{d^2}{dx^2} \tag{40}$$

Choosing weight function to be 1: The eigenfunction equation becomes:

$$Ly(x) = -\lambda y(x) \tag{41}$$

If $\lambda < 0$, the only solution that obeys the periodic boundary conditions is the trivial case $y(x) = 0$; If $\lambda \geq 0$, then a basis of solution is given by

$$y_n(x) = \exp(i\frac{n\pi x}{L}), for \ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{42}$$

For another example:

$$\frac{1}{2}H'' - xH' = -\lambda H(x) \tag{43}$$

subject to the condition that $H(x)$ behaves like a polynomial as $|x| \to \infty$. This equation is not yet in SL form, so we first compute the integrating factor:

$$\frac{d}{dx}\left(e^{-x^2}\frac{dH}{dx}\right) = -2\lambda e^{-x^2}H(x) \tag{44}$$

This equation is known as Hermite's equation. The solution is,

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \tag{45}$$

Thus, $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x$, which is known as Hermite polynomial with the following property:

$$(H_m, H_n)_{e^{-x^2}} = \int_{-\infty}^\infty H_m(x)H_n(x)e^{-x^2}dx = \delta_{m,n}2^n\sqrt{\pi}n! \tag{46}$$

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3.6 Inhomogeneous equations and Green's functions

We have discussed the problem of solving $\mathbf{Mu} = \mathbf{f}$, where $\mathbf{M}$ is a self-adjoint matrix.

We now consider solving problem,

$$L\phi(x) = \omega(x)F(x) \tag{47}$$

The idea is that the function $\phi(x), F(x)$ can be expanded in a complete set of eigenfunctions of L. So we suppose that the set $\{Y_1(x), Y_2(x), \ldots\}$ form a complete set of such eigenfunctions with

$$LY_n(x) = \lambda_n \omega(x)Y_n(x), and \ (Y_m, Y_n)_\omega = \delta_{m,n} \tag{48}$$

and we expand:

$$\phi(x) = \sum_{n=1}^{\infty} Y_n(x), \ F(x) = \sum_{n=1}^{\infty} F_n Y_n(x) \tag{49}$$

In matrix case, we assume that the function $F(x)$ is known, as well as the coefficient $F_n(x) = (Y_n, F(x))$. Since L is a linear operator, we have,

$$LHS = L\phi = \sum_{n=1}^{\infty} \phi_n L Y_n = \omega \sum_{n=1}^{\infty} \phi_n \lambda_n Y_n \tag{50}$$

$$RHS = \omega F = \omega \sum_{n=1}^{\infty} F_n Y_n \tag{51}$$

Thus, if we take the inner product with Y_m, by orthogonality, we can get $\phi_m \lambda_m = F_m$. So the solution (particular integral) is,

$$\phi_p(x) = \sum_{n=1}^{\infty} \frac{F_n}{\lambda_n} Y_n(x) \tag{52}$$

The complementary function $\phi_c(x)$ satisfying $L\phi_c(x) = 0$.

Let's study the solution in more details.

$$\phi_p(x) = \sum_{n=1}^{\infty} \frac{(Y_n, F)_\omega}{\lambda_n} Y_n(x) \tag{53}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_a^b Y_n(x)Y_n^*(t)F(t)\omega(t)dt \tag{54}$$

$$= \int_a^b \sum_{n=1}^{\infty} \frac{Y_n(x)Y_n^*(t)}{\lambda_n} \omega(t)F(t)dt \tag{55}$$

$$= \int_a^b G(x,t)f(t)dt \tag{56}$$

where $G(x,t) = \sum_{n=1}^{\infty} \frac{Y_n(x)Y_n^*(x)}{\lambda_n}, f(t) = \omega(t)F(t), (x,t) \in [a,b] \times [a,b]$.

Green function:

1. depends on operator L both through its eigenfunctions and through the boundary conditions we choose to ensure self-adjoint.

2. do not depend on the forcing function f.

For matrix case:

$$\mathbf{Mu} = \mathbf{f} \implies \mathbf{u} = \mathbf{M}^{-1}\mathbf{f} \tag{57}$$

For this case:

$$Ly(x) = f \implies y(x) = \int_a^b G(x,t)f(t)dt \tag{58}$$

Thus, green's function provides a formal inverse to the differential operator L

3.6.1 Parseval's identity II

Let $\{Y_1(x), Y_2(x), \ldots\}$ be a complete set of functions that are orthonormal with respect to some weight function $\omega(x)$, so that $(Y_m, Y_n) = \delta_{m,n}$. Then expanding,

$$f(x) = \sum_{n=1}^{\infty} f_n Y_n(x), \, with \, f_n = (Y_n, f)_\omega \tag{59}$$

we have,

$$(f, f)_\omega = \int_a^b \sum_{m=1}^{\infty} f_m^* Y^*(x) \sum_{n=1}^{\infty} f_m Y(x) \omega(x) dx \tag{60}$$

$$= \sum_{n,m} f_m^* f_n (Y_m, Y_n)_\omega \tag{61}$$

$$= \sum_{n=1}^{\infty} |f_n|^2 \tag{62}$$

This is the Parseval's identity in for the case of the inner product with weight ω.

Recall that the Parseval's indentity:

$$(f, f) = 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2 \tag{63}$$

Parseval's identity can be viewed as an infinite dimensional version of Pythagoras's theorem: it says that the norm squared of f is equal to the sum of the (mod-) squares of its coefficients in the (Fourier) basis of orthogonal functions.

3.6.2 Least square approximation

We have already introduced the matrix and algebraic form of solving equations. In real world, our computers have finite power and memory, and we typically don't have the resources to handle a very large number of eigenfunctions. So in practical applications, it's important to know how accurately we can represent a function by expanding it in just a limited **incomplete** set of eigenfunctions. Thus, considering a finite sum:

$$g(x) = \sum_{i=1}^{n} c_i Y_i(x) \tag{64}$$

One notation of what we mean by 'closely' is the distance, which should be minimized.

$$(f - g, f - g)_\omega = \int_a^b |f - g|^2 \omega(x) dx \tag{65}$$

and

$$\frac{\partial}{\partial c_k}(f - g, f - g)_\omega = \frac{\partial}{\partial c_k}[(f, f) - (g, f) - (f, g) + (g, g)] \tag{66}$$

$$= -\frac{\partial}{\partial c_k}(\sum_{i=1}^{n} c_i^* f_i + f_i^* - |c_i|^2) \tag{67}$$

$$= -f_k^* + c_k^* \tag{68}$$

and likewise,

$$\frac{\partial}{\partial c_k^*}(f - g, f - g)_\omega = -f_k + c_k \tag{69}$$

These derivatives vanish iff $c_k = f_k$. Thus, the best approximation is to choose the coefficients $c_k = (Y_k, f)_\omega$ exactly as its true expansion.

4   Laplace's Equation

Let's introduce Laplace's eqution:

$$\nabla^2 \psi = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \psi = 0 \tag{70}$$

Definition 4.1 (Dirichlet boundary condition). Consider a bounded domain $\Omega \subset \mathbb{R}^d$ and a function $f : \partial\Omega \to \mathbb{R}$

$$\psi(\mathbf{x}) = f(\mathbf{x}) \text{ at each point } \mathbf{x} \in \Omega \tag{71}$$

Theorem 4.1. In the case of Dirichlet boundary conditions, if we can find a solution, then it must be unique.

Proof. Suppose there are two solutions ψ_1, ψ_2, both obey Laplace's equations in Ω and $\psi_1|_{\partial\Omega} = \psi_2|_{\partial\Omega} = f$, then setting $\delta\psi = \psi_1 - \psi_2$ and integrating by parts we have,

$$0 = \int_\Omega \delta\psi \nabla^2 \delta\psi dV = \underbrace{\int_{\partial\Omega} \delta\psi \mathbf{n} \cdot (\nabla\delta\psi) dS}_{\delta\psi|_{\partial\Omega}=0} - \int_\Omega (\nabla\delta\psi) \cdot (\nabla\delta\psi) dV \tag{72}$$

The only way that the integral can be zero is for $\nabla\delta\psi = 0$ throughout Ω. Hence $\delta\psi = const$ and need to vanishes on the boundary, so $\delta\psi = 0$. Therefore, $\psi_1 = \psi_2$ everywhere and our solution is unique. $\qquad\square$

Definition 4.2 (Neumann boundary condition). Consider a bounded domain $\Omega \subset \mathbb{R}^d$ and a function $g : \partial\Omega \to \mathbb{R}$

$$\mathbf{n} \cdot \nabla\psi(\mathbf{x}) = g(\mathbf{x}) \text{ at each point } \mathbf{x} \in \partial\Omega \tag{73}$$

where $\mathbf{n}$ is the outward normal. The argument of uniqueness is similar that $\delta\psi = const$, but now the conditions no longer force this constant to vanish. Thus, Neumann boundary conditions determines our solution only up to the addition of a constant.

4.1   Laplace's equation on a disc

One powerful method for solving Laplace's equation is based on the fact that we can think of $\mathbb{R}^2$ as the complex plane $\mathbb{C}$. For $(x, y) \in \mathbb{R}^2$, we introduce $z = x = iy, \bar{z} = x - iy$. The solution of this is

$$\psi(x, y) = \underbrace{\phi(z)}_{holomorphic, \frac{\partial\phi}{\partial\bar{z}}=0} + \underbrace{\chi(\bar{z})}_{antiholomorphic, \frac{\partial\chi}{\partial z}=0} \tag{74}$$

4.2   Separation of variables

Suppose Ω is the three dimensional region:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le a, 0 \le y \le b, z \ge 0\} \tag{75}$$

Solving Laplace's equation $\nabla^2 \psi = 0$, with the following Dirichtlet boundary conditions in Ω:

$$\psi(0, y, z) = 0, \psi(a, y, z) = 0 \tag{76}$$
$$\psi(x, 0, z) = 0, \psi(x, b, z) = 0 \tag{77}$$
$$\psi(x, y, 0) = f(x, y), \psi(x, y, z) \to 0 \text{ as } z \to \infty \tag{78}$$

Assuming the solution takes the form:

$$\psi(x, y) = X(x)Y(y)Z(z) \tag{79}$$

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After substitution, we will get three ODEs,

$$X'' = -\lambda X, \ Y'' = -\mu Y, \ Z'' = (\lambda + \mu)Z \tag{80}$$

The solution is:

$$\psi(x, y, z) = A(\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x))(\sin(\sqrt{\mu}y) + C\cos(\mu y))(e^{-z\sqrt{\lambda+\mu}} + De^{+z\sqrt{\lambda+\mu}}) \tag{81}$$

Imposing bounary conditions:

$$\psi(x, y)_{n,m}(x, y, z) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\exp\left(-\pi z\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\right) \tag{82}$$

The general solution is a linear combination:

$$\psi(x, y) = \sum_{n,m=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\exp\left(-\pi z\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\right) \tag{83}$$

For inhomogenous solution $f(x, y)$, it must satisfy,

$$f(x, y) = \sum_{n,m=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \tag{84}$$

As usual, the coefficient could be obtained by inner prodct,

$$A_{n,m} = \left(\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right), f(x, y)\right) \tag{85}$$

$$= \frac{4}{ab}\int_{[0,a]\times[0,b]}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dxf(x, y)dxdy \tag{86}$$

where we use the orthogonality to adjust the factor,

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx = \frac{a}{2}\delta_{m,n} \tag{87}$$

Summary of method of separation of variables:

1. Solution form

2. PDE $\rightarrow$ ODE (always Sturm-Liouville type)

3. Solving SL equations and impose boundary conditons

4. Linear combination of eigenfunctions

5. Using inhomogeneous boundary conditions

4.3   The Laplacian in shperical polar coordinates

Spherical polar coordinate: (r, θ, ϕ), $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, r > 0$

$$dV = dxdydz = r^2\sin\theta drd\theta d\phi \tag{88}$$

The Laplacian operator becomes:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{89}$$

Now, seeking solution for $\nabla^2 \psi(r,\theta) = 0$ since we restrict attention to axisymmetric situations thus the final term of Laplacian is omitted.

Using separation of variables by writing $\psi(r,\theta) = R(r)\Theta(\theta)$, we could get two ODEs,

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\sin\theta\Theta = 0 \tag{90}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = 0 \tag{91}$$

Each of equations is of Sturm-Liouville type.

4.3.1 Legendre polynomials

We can simplity the angular equation with the substitution $x = \cos(\theta)$

$$-\sin\theta\frac{d}{dx}\left[\sin\theta\left(-\sin\theta\frac{d\Theta}{dx}\right)\right] + \lambda\sin\theta\Theta = 0 \tag{92}$$

which is equivalent to

$$\frac{d}{dx}\left[(1-x^2)\frac{\Theta}{dx}\right] = -\lambda\Theta \tag{93}$$

which is known as Legendre's equation. This is a standard SL eigenfunction problem on $\Omega = [-1, 1]$, with coefficient functions $p(x) = (1-x^2), q(x) = 0, \omega(x) = 1$

The self-adjointness of SL operator holds since,

$$(f, Lg) = (Lf, g) + \underbrace{[p(x)(f *' g - f * g')]_{-1}^{1}}_{p(x)=0 \text{ at } x=\pm1} \tag{94}$$

Solving legendre's equation with the form $\Theta(x) = \sum_{n=0}^{\infty} a_n x^n$, the recurrence relationhsip is:

$$a_{n+2} = \left[\frac{n(n+1) - \lambda}{(n+1)(n+2)}\right] \tag{95}$$

Picking a_0 and a_1 freely,

$$\Theta(x) = a_0\Theta_0(x) + a_1\Theta(x) \tag{96}$$

where

$$\Theta_0(x) = 1 + \frac{-\lambda}{2!}x^2 + \frac{(-\lambda)(6-\lambda)}{4!}x^4 + \ldots \tag{97}$$

$$\Theta_1(x) = x + \frac{2-\lambda}{3!}x^3 + \frac{(2-\lambda)(12-\lambda)}{5!}x^5 + \ldots \tag{98}$$

However, the series diverges at $x = \pm1$, violating boundary condition. To avoid this, we could terminate the power series. Thus

$$\lambda = l(l+1), \text{for } l \in \mathbb{Z}_{\geq 0} \tag{99}$$

The polynomial is know as Legendre polynomial, which could be represented as (Rodrigues' formula),

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l \tag{100}$$

Orthogonal property of legendre polynomial,

$$\int_{-1}^{1} P_m(x)P_n(x) = \frac{2}{2l+1}\delta_{m,l} \tag{101}$$

In summary, the Legendre polynomials are the solutions of Legendre's equation with eigenvalue $\lambda = l(l+1)$ for $l \in \mathbb{Z}_{\geq 0}$

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4.4   Laplace's equation in cylindrical polar coordinates

Cylindrical polar coordinate: (r, θ, z). The Laplacian operator becomes:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{102}$$

Separation of variables by taking form $\psi = R(r)\Theta(\theta)Z(z)$ gives three ODEs,

$$\Theta = -\lambda\Theta, Z'' = \mu Z, r^2R'' + rR' + (\mu r^2 - \lambda)R \tag{103}$$

4.4.1   Bessel functions

We solve the radian equation by multiplying this equation through by r and noting that $rR'' + R' = (rR')'$. We obtain the SL form,

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R = -\mu rR \tag{104}$$

where $p(r) = r, q(r) = -\lambda_n/r = -n^2/r, \omega(r) = r$, we can eliminate μ by introducing $x = r\sqrt{\mu}$,

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0 \tag{105}$$

which is known as Bessel's equation of order n.

5   The Heat Equation

The equation is,

$$\frac{\partial \Phi}{\partial t} = K\nabla^2\Phi \tag{106}$$

5.1   Properties of heat equation

The first property is that the total amount of heat is conserved.

$$\frac{d}{dt}\left(\int_\Omega \Phi dV\right) = \int_\Omega \frac{\partial \Phi}{\partial t}dV = K\int_\Omega \nabla^2\Phi dV = K\int_\Omega \mathbf{n}\cdot\nabla\Phi dS \tag{107}$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$ of the spatial region. In particula, if $\mathbf{n}\cdot\nabla\Phi|_\Omega = 0$, which says that no heat flows out of our region, then,

$$\frac{d}{dt}\left(\int_\Omega \Phi dV\right) = 0 \tag{108}$$

and the total amount of heat in Ω is conserved.

The second property is that if $\Phi(x, t)$ solves the heat equation for $(x,, t) \in \mathbb{R}^n \times [0, \infty]$, then so do the translated function and rescaled function.

$$\Phi_1(x, t) = \Phi(x - x_0, t - t_0) \tag{109}$$
$$\Phi_2(x, t) = A\Phi(\lambda x, \lambda^2 t) \tag{110}$$

Let's try to choose the constant A so that the total amount of heat in the solution Φ_2 is the same as in our original solution Φ. We have

$$\int_{\mathbb{R}^n} \Phi_2 d^n x = A\int_{\mathbb{R}^n} \Phi(\lambda x, \lambda^2 t)d^n x = A\lambda^{-n}\int_{\mathbb{R}^n} \Phi(y, \lambda^2 t)d^n y \tag{111}$$

wherer $y = \lambda x$. If we choose $A = \lambda^n$, the the total heat in Φ_2 at time t will be the same as the total heat in Φ at time $\lambda^2 t$. Since the total heat is conserved, so they are the same at all times. What does the rescaling property tell us? It says that there is nothing really new about the time variable t compared to the spatial variables. In other words, although $\Phi(x, t)$ looks as though it depends on n spatial variables and one time variable, we can always rescale so as to eliminate one of these variables.

5.2 The fundamental solution

Choosing $\lambda = 1/\sqrt{Kt}$, and we consider the solutions of the form with similarity variable,

$$\eta = \frac{x}{\sqrt{Kt}} \tag{112}$$

$$\Phi(x,t) = \frac{1}{\sqrt{Kt}} F(x/\sqrt{Kt}, 1) = \frac{1}{\sqrt{Kt}} F(\eta) \tag{113}$$

Therefore, our solution is,

$$\Phi(x,t) = G(x,t) = \frac{1}{\sqrt{4\pi Kt}} \exp\left(-\frac{x^2}{4Kt}\right) \tag{114}$$

This class of solution is known as the heat knernel.

5.3 Brownian motion and the existence of atoms

Let $p(y)$ be the probability that the dust particle moves through a displacement y at each time step, with $\int_{-\infty}^{\infty} p(y)dy = 1$. We assume that $p(y)$ is an even function of y so that the mean value $\langle y \rangle$.

Let $P(x,t)$ be the probability density that the dust particle is located at some position $x \in \mathbf{R}$ at time t, then we have

$$P(x; t + \Delta t) = \int_{-\infty}^{\infty} p(y) P(x-y; t) dy \tag{115}$$

Physical meaning: this equation says that the probability of the particle is located at x at time $t + \Delta t$ is the product of the probability it was some amount y away at the previous time interval, times the probability it stepped through exactly y, for any choice of y. Expanding $P(x-y;t)$ as a Taylor series in y we have,

$$P(x; t + \Delta t) = \int_{-\infty}^{\infty} p(y)\left(P(x;t) - y\frac{\partial P}{\partial x}(x;t) + \frac{1}{2}y^2\frac{\partial^2 P}{\partial x^2}(x;t) + \ldots\right) \tag{116}$$

$$= \sum_{r=0}^{\infty} \frac{\langle y^r \rangle}{r!} \frac{\partial^r P}{\partial x^r}(x;t) \tag{117}$$

where $\langle y^r \rangle = \int_{-\infty}^{\infty} y^r p(y) dy$

We have assumed that $\langle y \rangle = 0$, and if the motion of the particle is small so that $p(y)$ is concentrated near $y = 0$, then $\langle y^r \rangle$ will drop rapidly as r increases. Keeping only the learding non-trivial term we have,

$$P(x; t + \Delta t) - P(x; t) = \frac{1}{2}\langle y^2 \rangle \frac{\partial^2 P}{\partial x^2}(x;t) \tag{118}$$

As $\Delta t \to 0$, we find that $P(x,t)$ satifies the heat equation,

$$\frac{\partial P}{\partial t} = K\nabla^2 P \tag{119}$$

where $K = \langle y^2 \rangle / 2\Delta t$

5.4 Boundary conditions and uniqueness

Let $M = \Omega \times [0, T]$ be a compact domain with boundary $\partial\Omega$, we'd like to check the uniqueness of the solution of the heat equation with Dirichlet condition,

$$\psi|_{\Omega \times \{0\}} = f(x), \text{(initial condition)} \tag{120}$$

$$\psi|_{\partial\Omega \times [0,T]} = g(x,t), \text{(boundary condition)} \tag{121}$$

for some function $f : \Omega \to \mathbb{R}, g : \partial\Omega \times [0, T] \to \mathbb{R}$.

Suppose ψ_1, ψ_2 are two solutions of heat equation, obeying the boundary conditions. Then if setting $\delta\psi = \psi_1 - \psi_2$, we have,

$$0 = \int_\Omega \delta\psi \left(\frac{\partial \delta\psi}{\partial t} - K\nabla^2\psi \right) dV \tag{122}$$

Differentiating the intergral with respect to time we have,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \delta\psi^2 dV = \int_\Omega \delta\psi \frac{\partial \delta\psi}{\partial t} dV = K \int \Omega\delta\psi\nabla^2\delta\psi dV \tag{123}$$

$$= -K\int_\Omega (\nabla\delta\psi)\cdot(\nabla\delta\psi)dV + \underbrace{K\int_\Omega \delta\psi\nabla\delta\psi\cdot dS}_{=0} \tag{124}$$

Therefore,

$$\frac{d}{dt}\left(\int_\Omega \delta\psi^2 dV \right) \leq 0 \tag{125}$$

The quantity $E(t) = \int_\Omega \delta\psi^2 dV$ is a non-negative function, which can never increase. Since $E(0) = 0$ by initial condition, we must have $E(t) = 0$ at all times. This is only possible if $\delta\psi = 0$ everywhere in Ω at all times in $t \in [0, T]$. Thus, $\psi_1 = \psi_2$ everywhere in M.

6 The Wave Equation

The wave equation is,

$$\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi \tag{126}$$

with the boundary conditions,

$$\phi(0,t) = \phi(L,t) = 0 \tag{127}$$

and the intial conditions,

$$\phi(x,0) = f(x), \frac{\partial \phi}{\partial t}(x,0) = g(x) \tag{128}$$

Using separation of variable, seeking the solution of the form $\phi(x,t) = X(x)T(t)$, and the general solution is,

$$\phi(x,t) = \sum_{n=1}^\infty \sin\frac{n\pi x}{L}\left[A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L} \right] \tag{129}$$

where the constants A_n, B_n can by determined by the inhomogenous initial conditions,

$$A_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}f(x)dx, \ B_n = \frac{2}{n\pi c}\int_0^L \sin\frac{n\pi x}{L}g(x)dx \tag{130}$$

6.1 Energetics and uniqueness

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