5

LM

(187) Consider $v_1 = (0,1,2)$ $v_2 = (2,0,1)$ $v_3 = (6,1,5) \in \mathbb{R}^3$

(i) Determined whether $v_1\ v_2\ v_3$ are linear independent find an equation

(ii) Show that the span $\langle v_1\ v_2\ v_3 \rangle$ is the plane $x + 4y - 2z = 0$

(i) $av_1 + bv_2 + cv_3 = 0$

$$\begin{cases} 2b + 6c = 0 \\ a + c = 0 \\ 2a + b + 5c = 0 \end{cases} \quad \begin{pmatrix} 0 & 2 & 6 \\ 1 & 0 & 1 \\ 2 & 1 & 5 \end{pmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

或 det = 0 ⇔ dependent.
相当于解出 a. b. c

free variable: c. 把所有free移右边

$$\begin{cases} a = -c \\ b = -3c \\ c = c \end{cases} \Rightarrow (a.b.c) = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \cdot c$$

so they are linearly independent with equation

$v_1 + 3v_2 - v_3 = 0$

(ii) since they are dependent

so: $\boxed{v = k_1' v_1 + k_2' v_2}$

或者写成 $k_1v_1 + k_2v_2 + k_3v_3$
将 $v_3 = v_1 + 3v_2$ 换掉

$$v = k_1'\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + k_2'\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2k_2' \\ k_1' \\ 2k_1' + k_2' \end{pmatrix}$$

$v \in \langle v_1\ v_2\ v_3 \rangle$ and
v lies in one plane
$x + 4y - 2z = 0$
$\dim(\langle v_1\ v_2\ v_3 \rangle) = 2$.

$$\text{Since } r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2k_2' \\ k_1' \\ 2k_1' + k_2' \end{pmatrix}$$

$z = 2 \cdot y + \frac{1}{2}x$

$x + 4y - 2z = 0$.

Summary

LM

repeated roots

eigenvalues.

(188) $A=\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}$ Show that the eigenvector associated with the repeated eigenvalue is of the form.

$v_2 = t_2\begin{pmatrix}-1\\0\\1\end{pmatrix} + s_2\begin{pmatrix}0\\1\\0\end{pmatrix}$ $(s_2, t_2 \in \mathbb{R})$

s_2, t_1 are not both zero. Show that the eigenspace corresponding to the repeated eigenvalue is ~~$x=y$~~ $x+z=0$ $y\in\mathbb{R}$.

$\det(A-\lambda I) = -(\lambda-2)^2(\lambda-1)$

Eigenvalue: $\lambda=2$ (repeated) $\lambda_0=1$

注意这一步的格式 ⇐

① $\lambda=1$. $v_1=\begin{pmatrix}\alpha_1\\\beta_1\\\gamma_1\end{pmatrix}$ $\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}\begin{pmatrix}\alpha_1\\\beta_1\\\gamma_1\end{pmatrix}=1\cdot\begin{pmatrix}\alpha_1\\\beta_1\\\gamma_1\end{pmatrix}$

$Av=\lambda v$ $\boxed{格式}$!

$\Rightarrow \left(\begin{array}{ccc|c}-1&0&-2&0\\1&1&1&0\\1&0&2&0\end{array}\right)$ $(A-\lambda I)v=0$

$\left(\begin{array}{ccc|c}-1&0&-2&0\\0&1&-1&0\\0&0&0&0\end{array}\right)$

$v_1 = t_1\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $t_1\neq 0$

② $\lambda=2$ $v_2=\begin{pmatrix}\alpha_2\\\beta_2\\\gamma_2\end{pmatrix}$ $(A-\lambda I)v_2=0$

$\left(\begin{array}{ccc|c}-2&0&-2&0\\1&0&1&0\\1&0&1&0\end{array}\right)\sim\left(\begin{array}{ccc|c}-2&0&-2&0\\0&0&0&0\\0&0&0&0\end{array}\right)$

$-2\alpha_2-2\gamma_2=0$ β_2 completely arbitrary

set $\gamma_2=t_2$ $\alpha_2=-t_2$ $\beta_2=s_2$

$\therefore v_2 = t_2\begin{pmatrix}-1\\0\\1\end{pmatrix}+s_2\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\left(=\begin{pmatrix}-t_2\\s_2\\t_2\end{pmatrix}\right)$

Eigenspace: S, $S=\left\{t_2\begin{pmatrix}-1\\0\\1\end{pmatrix}+s_2\begin{pmatrix}0\\1\\0\end{pmatrix} \middle| t_2, s_2\in\mathbb{R}\right\}$

Summary

[illegible]

Parametric form

$r = p + \lambda v + \mu u$

$n = v \times u$

2M

S is the plane containing vectors $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

A normal vector

$$n = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$r \cdot n = 0$. $\quad x + z = 0$.

(189) $A = \begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix}$

Show that the eigenspace associated with repeated eigenvalue is the plane

$x - 3y + 2z = 0$

$\det(A - \lambda I) = (2-\lambda)^2(4-\lambda)$

$\lambda_1 = 4$. $\quad \lambda_2 = 2$ (repeated)

① $\lambda_1 = 4 \quad v_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \quad Av_1 = 4v_1 \Rightarrow \left(\begin{array}{ccc|c} -1 & -3 & 2 & 0 \\ -1 & 1 & -2 & 0 \\ -1 & 3 & -4 & 0 \end{array} \right)$

$\sim \left(\begin{array}{ccc|c} -1 & -3 & 2 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad 4\beta_1 - 4\gamma_1 = 0. \quad \beta_1 = \gamma_1 \quad -\alpha_1 - 3\beta_1 + 2\gamma_1 = 0$

γ_1 is free variable. $\gamma_1 = \beta_1 = t_1$, $\alpha_1 = -t_1$

$v_1 = t_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad t_1 \neq 0$

② $\lambda_2 = 2 \quad v_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \quad Av_2 = 2v_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 3 & -2 & 0 \end{array} \right)$

$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \alpha_2 - 3\beta_2 + 2\gamma_2 = 0$

$$v_2 = \begin{pmatrix} 3\beta_2 - 2\gamma_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \beta_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \gamma_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Summary

$v_2 = \beta_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \gamma_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ $(\beta_2, \gamma_2 \neq 0)$

Eigenspace is spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

A normal:

$$n = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$r \cdot n = 0$. ie. $x - 3y + 2z = 0$

LM

(19a) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ show that

$$A^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \quad A^3 = \begin{pmatrix} \cos(3\theta) & -\sin(3\theta) \\ \sin(3\theta) & \cos(3\theta) \end{pmatrix}$$

(i) What about A^n?

(ii) Show that the eigenvalues of A are $\lambda_1 = e^{i\theta}$ $\lambda_2 = e^{-i\theta}$ with corresponding eigenvectors

$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

(iii) write down an invertible matrix s.t

$$P^{-1}AP = D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

show:

$$P^{-1}A^nP = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} \quad n \in N$$

show:

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \quad n \in N$$

Summary

$\cos\theta - \lambda)^2 = -1 \cdot \sin^2\theta$

$\cos\theta - \lambda = \pm\sqrt{-1}\sin\theta$

$i = \sqrt{-1}$

$\lambda = \cos\theta \pm i\sin\theta$

$A\bar{v}_1 = \bar{\lambda}_1\bar{v}_1$

$\bar{\lambda}_1 = e^{-i\theta}$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

suggest $A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$(\cos\theta - \lambda)^2 + \sin^2\theta = 0 \Rightarrow \lambda = \cos\theta \pm i\sin\theta = \boxed{e^{\pm i\theta}}$

① $\lambda_1 = e^{i\theta}$: $\begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{i\theta} \end{pmatrix} = \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix}$

$R_2 - iR_1$ $\begin{pmatrix} -i\sin\theta & -\sin\theta \\ 0 & 0 \end{pmatrix}$ $\alpha_1(-\sin\theta) + \beta_1(-\sin\theta) = 0$

$\beta_1 = -i\alpha_1$

$v_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \beta_1\begin{pmatrix} i \\ 1 \end{pmatrix}$ $\beta_1 \neq 0$

$\lambda_2 = e^{-i\theta}$ $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

set $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ $P^{-1} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$

$P^{-1}A^nP = (P^{-1}AP)^n = D^n = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix}$

$A^n = P\begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix}P^{-1}$

$= \frac{1}{2i}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{in\theta} \end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2}(e^{in\theta} + e^{-in\theta}) & -\frac{1}{2i}(e^{in\theta} - e^{-in\theta}) \\ \frac{1}{2i}(e^{in\theta} - e^{-in\theta}) & \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \end{pmatrix}$

$= \begin{pmatrix} \cos n\theta & -\sin(n\theta) \\ \sin n\theta & \cos(n\theta) \end{pmatrix}$

Summary

5

LM

Quadratic form

Orthogonal

(19) $A = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$ determine the eigenvecles and corresponding eigenvectors.

- Use normalised eigenvectors. write down an invertible matrix P s.t. $P^{-1}AP = D$
- Confirm that $P^TP = I$ for further use $P^{-1} = P^T$.

The function: $Q(x, y, z) = 6x^2 + 6y^2 + 5z^2 - 4xy - 2xz - 2yz$

- is an example of a quadatic form,

If $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ check that $Q = x^TAx$

- Writing $x = PX$ show $Q = X^TP^{-1}APX$, where $X = P^{-1}x$

If $X = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$, find $X\ Y\ Z$. in terms of $x\ y\ z$

prove that Q can be expressed as a sum of perfect squares in the form.

$$Q(x, y, z) = 4(x-y)^2 + (x+y-2z)^2 + (x+y+z)^2$$

- $\det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 8 \quad \lambda_2 = 6 \quad \lambda_3 = 3$.

① $\lambda_1 = 8 \quad v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$

② $\lambda_2 = 6 \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$

③ $\lambda_3 = 3 \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad e_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

Normalised them $\Longrightarrow$

row vector. inner product $= 0$

column vector inner product $= 0$.

$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \quad P^{-1}AP = D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P^TP = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Summary

$\boxed{\text{Ortogonal Matrix}}$

A square matrix A is said to be orthogoral if its transpose is the same as its inverse

$A^{-1} = A^T$

$\Longleftrightarrow AA^T = A^TA = I$.

已经是 orthogonal 了.

$\therefore P^{-1} = P^T$

$Q(x, y, z) = 6x^2 + 6y^2 + 5z^2 - 4xy - 2xz - 2yz$

$$x^T A x = (x, y, z) \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$X^T P^T A \; PX$

$x = PX \qquad x^T = X^T P^T$

$$Q = X^T \boxed{P^T} A P X = X^T \boxed{P^{-1}} A P X = X^T D X$$

$$= (X \; Y \; Z) \begin{pmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$= 8X^2 + 6Y^2 + 3Z^2$$

$$X = P^{-1} x = P^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$X = \frac{1}{\sqrt{2}}(x - y) \quad Y = \frac{1}{\sqrt{6}}(x + y - 2z), \quad Z = \frac{1}{\sqrt{3}}(x + y + z)$$

$$\therefore Q = 4(x-y)^2 + (x+y-2z)^2 + (x+y+z)^2$$

Summary

Equivalent $n \times n$ Matix A

- A is orthogonal
- The row vectors of A form an orthonormal set in R^n with Euclidean inner product
- The column vectors of A form an orthonormal set in R^n with Euclidean inner product.

LM

Change of Basis

(19) Find relative to the standard bases for R^2 R^3, the matrix representation of the linear transformations $T: R^3 \to R^2$ where

$T(x,y,z) = (2x-4y+9,\ 5x+3y-z)$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-4y+9 \\ 5x+3y-z \end{pmatrix} = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$M_T = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & -1 \end{pmatrix}$ with respect to standard basis.

Applying T to the basis.

也可以将其他 ← 的基向量放进去.

$$T(e_1) = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$$

gives the 3 columns of required Matrix.

G. Strang ←

The Matrix of a Linear Tranformation.

$T(v) = Av$

Key idea:

- input basis vectors $v_1 \cdots v_n$
- Column 1 to n of the Matrix will contains those outputs $T(v_1)$ to $T(v_n)$

Summary

原来的写成线性组合

$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \boxed{v_1} + c_2 \boxed{v_2} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$T\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \boxed{T(v_1)} + c_2 \boxed{T(v_2)}$

$= \begin{pmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\boxed{T(x) = Ax.}$

Example1: Suppose T transforms $v_1 = (1, 0)$ to $T(v_1) = (2, 3, 4)$

$v_2 = (0, 1)$ goes to $T(v_2) = (5, 5, 5)$

$v_1 = (1, 0) \xrightarrow{T} T(v_1) = (2, 3, 4)$

$v_2 = (0, 1) \xrightarrow{T} T(v_2) = (5, 5, 5)$

Column 1 to n will contain those outputs

$A = \begin{pmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{pmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = T(v_1 + v_2) = \begin{pmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

Example 2: $T: V = R^2 \to W = R^2$

$v_1 = (3, 3) \to w_1 = (3, 1)$

$v_2 = (6, 8) \to w_2 = (0, 2)$

$\Uparrow$ input basis $\quad$ $\Uparrow$ output basis

$\begin{cases} v_1 = 1w_1 + 1w_2 \\ v_2 = 2w_1 + 3w_2 \end{cases}$

$\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} (B) = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$

$\uparrow w_1 \quad \uparrow w_2 \qquad \uparrow v_1 \quad \uparrow v_2$

$WB = V$

$B = W^{-1}V = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

Formula: $B = W^{-1}V$ when $V = I$ (Standard basis),

$B = W^{-1}.$

$w_1 = 3v_1 - v_2$

$w_2 = -2v_1 + v_2$

$A: V \to W \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$B: W \to V \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \leftarrow$

$A \cdot B = I.$ 互逆.

Summary

LM

(193) The linear transformation $T: R^3 \to R^3$ is defined by $T((x, y, z)) = (y+z,\ x-z,\ -x-y)$

Write down the matrix representation for T with respect to the standard basis $e_1\ e_2\ e_3$ of R^3.

Let $v_1 = (0, 1, -1)$ $v_2 = (1, -1, 1)$ $v_3 = (-1, 1, 0)$

verify that each v_i is an eigenvector of T and the v_i are linearly independent.

What is the matrix of T with respect to the basis $v_1\ v_2\ v_3$?

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ x-z \\ -x-y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\therefore A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ with respect to standard basis.

Compute.

$T(v_1) = v_1 \quad T(v_2) = 0 \quad T(v_3) = -v_3$

$T(v) = \boxed{A}v = \boxed{\lambda}v.$

$\lambda = 1.\ 0.\ -1$

~~Output~~:

form $C = \begin{array}{c} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \\ \ \ v_1 \ \ \ v_2 \ \ \ v_3 \end{array}$

eigenvalues.

$$C^{-1}A \cdot C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

from standard basis to v.

Summary

Dimension Anaylsis.

MKS system : (Standard set of units)

Meters , Kilograms , Seconds.

↓ form

SI System of Base Unit

[Length] = Meter [Time] = second [Mass] = kilogram

[Temperature] = Kelvin [Electric current] = ampere

[light intensity] = candela [Material Quantity] = Mole.

seven foundamental dimensional units

Key points:

- The dimension of any derived physical quantity is a product of powers of the base quantity dimensions
  $[Q] = [Length]^{\alpha} [Time]^{\beta} [Mass]^{\gamma} \cdots$
- A derived quantity is dimensionless if its numerical value remains invariant when the base unit are change.
- Special function, Logarithm Exponential trigonometric etc. with dimensionless arguments are therefore derived quantities with dimension unity.

Acceleration: $L \cdot T^{-2}$

Force: $ML \cdot T^{-2}$

Density: ML^{-3}

Pressure: $ML^{-1}T^{-2}$

Energy: $ML^{2}T^{-2}$

Power: $ML^{2}T^{-3}$

Summary

from Hung-yi Lee.

Coordinate System.

Let vector set $B=\{u_1 \cdots u_n\}$ be a basis for a subspace

For any v in R^n, there are unique scalars $c_1\ c_2 \cdots c_n$ s.t. $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in R^n.$$

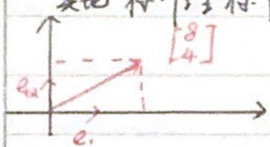
Other Coordinates $\longrightarrow$ Cartesian.

v在B下的坐标是 $[v]_B = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$
现在我想知道它在 Cartesian下的坐标是多少

$$B=\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right\} \quad [v]_B = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$$

$$v = 3\times\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 5 \end{bmatrix}$$

类比标准坐标下的情况



$\{e_1\ e_2\}$ is a coordinate system

这个时候是8个e_1加上4个e_2.

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 8e_1 + 4e_2.$$

就说现在已知在B下的vector. 现在想知道这个vector在直角笛卡尔坐标下的vector是多少.

How to find v given $[v]_B$?

$v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$

$= B[v]_B$.

$\left([v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad B=[u_1\ u_2 \cdots u_n]\right)$

Summary

现在已知笛卡尔系下的坐标为 $\begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$，现在想知道 B 下的坐标是多少，当然都是同一个向量。

Cartesian → Other system

$v = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$ $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$ find $[v]_B$.

设 $[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ 表示在 B 下有 c_1 个 $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ c_3 个 $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$v = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\underline{v} = \underline{B} \cdot [v]_B$$

$$[v]_B = B^{-1} v = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

Summary:

$$v \xrightarrow{[v]_B = B^{-1} v} [v]_B$$

$$v \xleftarrow{v = B[v]_B} [v]_B$$

基变换与坐标变换

两个基 $\in V^m$ $\{\alpha_1, \alpha_2 \cdots \alpha_m\}$ $\{\beta_1, \cdots \beta_m\}$

⇒将 β 表示为 $\alpha_1 \cdots \alpha_m$ 的线性组合

$$\begin{cases} \beta_1 = a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{m1}\alpha_m \\ \vdots \\ \beta_m = a_{1m}\alpha_1 + a_{2m}\alpha_2 + \cdots + a_{mm}\alpha_m \end{cases}$$

矩阵形式： $(\beta_1, \beta_2, \cdots \beta_m) = \cancel{AB}\ (\alpha_1, \alpha_2 \cdots \alpha_m) A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & & \cdots & a_{mm} \end{pmatrix} \to \alpha \to \beta \text{ 过渡矩阵.}$$

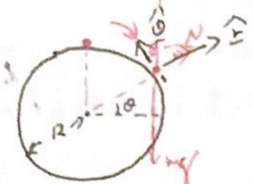
线性组合中每行的系数

Summary

APP additional Model

① A Ball is displaced from the top of a cylinder lying flat.

(Assume surface is smooth)



支持力沿法方向

$\underline{N} = N\hat{r}$

$m\underline{g} = -mg\cos\theta\,\hat{\theta} - mg\sin\theta\,\hat{r}$

$\underline{f} = m\underline{a}$

$N\hat{r} - mg\cos\theta\,\hat{\theta} - mg\sin\theta\,\hat{r} = m\{(\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}\}$

$\hat{r}$-component: $N - mg\sin\theta = m(\ddot{r} - r\dot{\theta}^2)$

$\boxed{r = R \quad \dot{r} = \ddot{r} = 0}$ $\underline{N = mg\sin\theta - mR\dot{\theta}^2}$

$\hat{\theta}$-component: $\underline{-mg\cos\theta = mR\ddot{\theta}}$

$R\ddot{\theta} + g\cos\theta = 0$

$\ddot{\theta} + \frac{g}{R}\cos\theta = 0$

Trick $\Rightarrow$ $\times\dot{\theta}$

$\ddot{\theta}\cdot\dot{\theta} + \frac{g}{R}\cos\theta\cdot\dot{\theta} = 0$

$\frac{d}{dt}(\frac{1}{2}(\dot{\theta}^2)) + \frac{g}{R}\cos\theta\cdot\frac{d\theta}{dt} = 0$ 故此积分常数不要忘了

integrate

$\frac{1}{2}\dot{\theta}^2 + \frac{g}{R}\int\cos\theta\cdot\frac{d\theta}{dt}\cdot dt = const.$

$\frac{1}{2}\dot{\theta}^2 + \frac{g}{R}\sin\theta = K$

如果有初始速度 ← initial: $\theta(0) = \frac{\pi}{2}$ $\quad \dot{\theta}(0) = 0 \quad K = \frac{g}{R}$

$\dot{\theta}(0) = \frac{v}{a}$ a is radius.

$\frac{1}{2}\dot{\theta}^2 = \frac{g}{R} - \frac{g}{R}\sin\theta$

求N的关键步骤 ← $\boxed{R\dot{\theta}^2 = 2g(1 - \sin\theta)}$

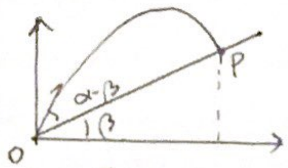
$N = mg\sin\theta - mR\dot{\theta}^2 = mg\sin\theta - m(2g(1-\sin\theta))$

$N = mg(3\sin\theta - 2)$

fall if $N = 0$ $\quad 3\sin\theta = 2 \quad \theta = \sin^{-1}(\frac{2}{3})$

Summary

5



$\alpha = \frac{\pi}{3}$, $g = (0, -g)$

$\beta = \frac{\pi}{6}$

② Air resistance is negligible

$$\begin{cases} x(t) = v\cos\alpha \cdot t \\ y(t) = v\sin\alpha\, t - \frac{1}{2}gt^2 \end{cases}$$

In general case with $\beta \in (0, \frac{\pi}{2})$ find the angle $\alpha \in (\beta, \frac{\pi}{2})$ which give the maximum range OP.

$$\tan\beta = \frac{y}{x} = \frac{v\sin\alpha \cdot t - \frac{1}{2}gt^2}{v\cos\alpha \cdot t} = \frac{v\sin\alpha - \frac{1}{2}gt}{v\cos\alpha} \quad \text{get } t$$

这种很有意味可化简.

$$|OP| = \frac{1}{\cos\beta} x(T) = \frac{1}{\cos\beta}\frac{2v}{g}\boxed{(\sin\alpha - \cos\alpha\tan\beta)}$$

$$= \frac{1}{\cos\beta}\frac{2v}{g} \cdot \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\beta}$$

$$= \frac{1}{\cos\beta^2}\frac{2v\cos\alpha}{g}\sin(\alpha-\beta)$$

和差积公式用了两次.

$h(\alpha) = \cos\alpha\sin(\alpha-\beta)$ to maximize it.

$h'(\alpha) = -\sin\alpha\sin(\alpha-\beta) + \cos\alpha\cos(\alpha-\beta) = \cos(2\alpha-\beta) = 0$

$\alpha_0 = \frac{\pi}{4} + \frac{\beta}{2} < \frac{\pi}{2}$.

at α_0 reached.

Summary

MF Chapter 1.

$\mathbb{Z}$: integers. 0, ±1, ±2 ...

自然数里无0 ←

$\mathbb{N}$: nature number 1, 2. $0 \notin \mathbb{N}$

$\mathbb{Q}$: rational numbers.

$\mathbb{R}$: real number

$\mathbb{C}$: complex number

$\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

Mathematical Statement: is a meaningful assertion about mathematical objects which can be either true or false.

Implication

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

你考100，我给你100块

守信 T 不守信 F.

你考100，我不给你100块

我失信 F.

只要你没考到100，无论我给不给你，我都没失信，所以都是T.

$A \Rightarrow B$: A only if B or A is a sufficient condition for B

$A \Leftarrow B$: A if B, or A is nesessary condition for B.

Summary

5 . .

~~Converse: $A \Rightarrow B$ = $B \Rightarrow B$.~~

Converse: $B \Rightarrow A$ is called converse of the statenent $A \Rightarrow B$

Negation: $\neg A$.

A: $\boxed{\forall x \in C}$, p holds

$\neg A$; $\boxed{\exists x \in C}$, $\neg p$ holds

只要看到∃.∀后面条件不变

Contrapositive: $\neg B \Rightarrow \neg A$ is called the contrapositive of $A \Rightarrow B$

eg: negation: A or B. is neither A nor B $\Leftrightarrow$ $\neg A$ and $\neg B$

and<->or

negation: A and B is <u>not</u> both A and B $\Leftrightarrow$ $\neg A$ or $\neg B$

Chapter 2.

Prove by Mathematical Induction:

Anchor step: Prove $p(1)$? 自然数不包括 0

Induction step: Prove $p(n) \Rightarrow p(n+1)$ for all $n \in N$

不一定都从 $n=1$ 开始. 事实上.

$p(N)$ is true

$p(n) \Rightarrow p(n+1)$ for all $n \geq N$

Then $p(n)$ is true for all $n \geq N$

Chapter 3.

Cauchy - Schwarz Inequality

Let $n \in N$ and let $a_1, \ldots a_n$ and $b_1 \ldots b_n$ be real numbers. Then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

proof

$$f(x) = \sum_{k=1}^{n} (x a_k - b_k)^2$$

$f(x) \geq 0$.

Summary

5

Arithmetic mean $A = \frac{1}{n}\sum a_i$

Geometric mean $G = (a_1 \cdots a_n)^{\frac{1}{n}}$

if all a_i equal $a_i = A = G$

$A \geq G$.

Chapter 4.

Bounded above and below

$\exists M \in \mathbb{R}$ s.t. $\forall x \in A$, $x \leq M$

$\boxed{\text{Definition}}$ An element $M \in \mathbb{R}$ is an upper bound for A if $x \leq M$ for all x in A; If an upper bound for A exist, we say A is bounded above

$\boxed{\text{Definition}}$ An element $m \in \mathbb{R}$ is a lower bound for A if $m \leq x$ for all x in A. If a lower bound for A exist, we say that A is bounded below

A is called bounded if it is bounded above and below

不需要在集合里.

Maxima and minima $A \subseteq \mathbb{R}$ be a set of real numbers.

we say X is a maximum element of A, and write $X = \max A$ if (1) $\boxed{X \in A}$ (2) $x \leq X$ for all x in A

we say Y is a minimum element of A, and write $Y = \min A$ if (1) $\boxed{Y \in A}$ (2) $Y \leq x$ for all x in A

Summary

Proposition: Any subset $A \subseteq \mathbb{R}$ has at most one maximum and at most one minimum

Least Upper bound & Greatest lower bound.

~~An~~ Let $A \subseteq \mathbb{R}$ be a set of real number.

$\boxed{\text{Definition:}}$ An element $s \in \mathbb{R}$ is called a least upper bound for A, or supremum of A if

(1) $x \leq s$ for all $x \in A$ (2) for every real number $t < s$ there exists some element x of A s.t. $t < x$

Upper bound 前提 ←

(2) $\Rightarrow$ no real numbers less than s is an upper bound for A

$\boxed{\text{Definition}}$: (1) $x \leq s$ for all $x \in A$ (2) for every $\varepsilon > 0$. there exists $x \in A$ s.t. $x > s - \varepsilon$

least upper bound property of real number

If A is a non-empty subset of $\mathbb{R}$ which is bounded above, then A has a unique least upper bound in $\mathbb{R}$

The rational numbers $\mathbb{Q}$ do not have the least upper bound property.

eg: $S = \{ x \in \mathbb{Q} \mid x^2 < 2 \}$. bounded above by $2 \in \mathbb{Q}$

However, $\sqrt{2} \notin \mathbb{Q}$, so doesn't have the least upper bound. in $\mathbb{Q}$.

Summary

Proposition: Let $s = \sup A$

① if $s \in A$, then s is maximum of A

② if $s \notin A$, then A doesn't have a maximum element.

Greatest lower bound Let $A \subseteq \mathbb{R}$ be a set of real numbers

Definition. An element $g \in \mathbb{R}$ is called a greatest lower bound for A, or infimum of A if

(1) g is a lower bound for A (2) no real numbers greater than g is a lower bound for A.

Greatest lower bound property of the real number

If A is a non-empty subsets of $\mathbb{R}$ ~~and~~ which is bounded below, then A has a unique greatest lower bound in $\mathbb{R}$.

Proposition: Let $g = \inf A$

① if $g \in A$, g is a minimum element of A

② if $g \notin A$, A does not have a minimum element.

Summary

下界和下确界都不一定存在，如果都存在，下界不一定唯一，但下确界唯一

确界原理：S为非空数集，若S有上界，则必有上确界

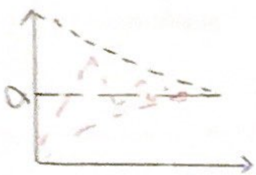
若S有下界，则必有下确界

Chapter 5

Converges:

For every positive real number ε, there exists an integer $n_0(\varepsilon)$ s.t. $|x_n - \alpha| < \varepsilon$ for all integers $n \geq n_0(\varepsilon)$.

$\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon)$ s.t. $|x_n - \alpha| < \varepsilon$ for $\forall\, n \geq n_0(\varepsilon)$



Sandwich Therom

Suppose (a_n) (b_n) (c_n) are real sequences s.t. $a_n \leq b_n \leq c_n$ for each n. Suppose further that (a_n) and (c_n) both converges to $\alpha \in R$. Then (b_n) converges to α also.

Lemma: for every $s > 0$ and $n \in N$ $\ln n \leq \frac{n^s}{s}$

~~这个ratio test里 针对数列的所以 不用加 limit~~

Theorem (Ratio test):

Let (x_n) be a real sequence s.t. each x_n is nonzero. Suppose: $\lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = L < 1$

then (x_n) $\boxed{\text{converges to zero.}}$

lemma: $\ln n = \ln n - \ln 1 = \int_1^n \frac{1}{x}\, dx$

$1 \leq x \leq n$. $\frac{1}{x} \leq \frac{x^s}{x} = x^{s-1}$

$\int_1^n \frac{1}{x}\, dx \leq \int_1^n x^{s-1}\, dx = \frac{n^s}{s} - \frac{1}{s} \leq \frac{n^s}{s}$ $\quad n \leq \frac{n^s}{s}$

Summary

Monotone Sequence Theorem

(a) let (x_n) be Non-decreasing for $n \geq N$. If the set

$$A = \{x_n \mid n \geq N\}$$

bounded above, (x_n) converges to $\sup A$

not bounded above, (x_n) diverges to ∞

(b) let (x_n) be Non-increasing for $n \geq N$. If the set $A = \{x_n \mid n \geq N\}$

bounded below, (x_n) converges to $\inf A$

not bounded below, (x_n) diverges to $-\infty$

Chapter 6

let $f: [a, \infty) \to \mathbb{R}$ be a function. We say

$$\lim_{x \to \infty} f(x) = L$$

if for every real sequence (x_n) with $\lim_{n \to \infty} x_n = \infty$

we have $\lim_{n \to \infty} f(x_n) = L$

MST. for function

① let $f: [a, +\infty) \to \mathbb{R}$ a non-decreasing function. If the set

$A = \{f(x) \mid x \in [a, \infty)\}$ is bounded above

then $\lim_{x \to \infty} f(x) = \sup A$.; not bound above

$\lim_{x \to \infty} f(x) = \infty$

② let $f: [a, \infty) \to \mathbb{R}$ be a non-increasing. if the set

$A = \{f(x) \mid x \in [a, \infty)\}$ is bounded below

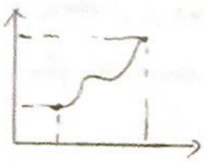
Summary

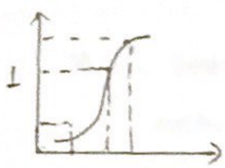
then $\lim_{x \to \infty} f(x) = \inf(A)$; not bounded ~~above~~ below

$\lim_{x \to \infty} f(x) = -\infty$.

$\lim_{x\to a} f(x) = L$ if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$.





Continuity: The function $f: (a,b) \to R$ is called continous at $c \in (a,b)$ if $\lim_{x\to c} f(x) = f(c)$
which means $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

Extreme Value Theorem: Let $f: I \to R$ be continous
Then there exists $c, d \in I$ s.t.

$$f(c) \le f(x) \le f(d) \quad \text{for all } x \in I.$$

Intermediate Value Theorem: Let $I = [a, b]$ and let $f: I \to R$ be continous. If $f(a) < L < f(b)$ or if $f(b) < L < f(a)$, there exists $c \in (a,b)$ with $f(c) = L$

L'Hopital's rule

① Let f and g be real-valued functions s.t. $\boxed{\lim f(x) = \lim g(x) = 0}$. Suppose that $\lim \frac{f'(x)}{g'(x)}$ exists and equals L (finite or infinite), then $\lim \frac{f(x)}{g(x)}$ exists and equals L.

② $\lim f(x)$ and $\lim g(x)$ exist and are each ∞ or $-\infty$. Suppose that $\lim \frac{f'(x)}{g'(x)}$ exists, and equals L (finite or infinite) $\lim \frac{f(x)}{g(x)}$ exists and equals L.

Summary

用 Mean Value Theorem
证明 Contraction.

Roll's Theorem: Suppose f is continous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then for some $c \in (a, b)$ we must have $f'(c) = 0$.

Chapter 7:

In general, if a sequence is given by $x_{n+1} = f(x_n)$ and if x_0 is a fixpoint ($x_0 = f(x_0)$), then the sequence (x_n) is constant: $x_n = x_0$ for all n.

$\boxed{\text{Theorem}}$ let $I = [a, b]$ be an interval in R. let $f: I \to I$ be a continuous function.

$\boxed{\text{Contraction}}$: let $I = [a, b]$ and let $f: I \to R$ be a function. Suppose there is a exist number $0 < k < 1$ s.t. $|f(x) - f(y)| \le k|x - y|$ for all $x, y \in I$. then f is called a contraction on I.

Preposition: let $f: I \to I$ be a function. $\longrightarrow$

(i) If f is a contraction, then f has exactly one fixpoint L in I.

(ii) If f is a contraction, then for any choice of $x_0 \in I$, the sequence given by $x_{n+1} = f(x_n)$

Summary