5 . .

converges to the unique fixpoint $L \in I$ of f.

Fact: If $f: I \to \mathbb{R}$ is differentiable and we can find $K < 1$ s.t. $|f'(t)| \le K$ for all $t \in I$ then f is a contraction.

Newton-Raphson Method

$$F(x) := x - \frac{f(x)}{f'(x)}$$

Chapter 8

Definition (Improper Integral): Let $a \in \mathbb{R}$ and $f: [a, +\infty) \to \mathbb{R}$ and consider $\int_a^{\infty} f(x)\,dx$, an improper integral "of first kind". Suppose

$\int_a^L f(x)\,dx$ exists for every real $L > a$ (f continuous) define

$$\int_a^{\infty} f(x)\,dx := \lim_{L \to \infty} \int_a^L f(x)\,dx$$ if the limit

exists and is finite. 非常重要.

最后带值 $[\ ]_x^L$

converges.

Comparison test

Theorem: Suppose that $a \in \mathbb{R}$, and $f: [a, \infty) \to \mathbb{R}$ and $g: [a, \infty) \to \mathbb{R}$ are continuous functions with $|f(x)| \le g(x)$ for all x.

Suppose $\int_a^{\infty} g(x)\,dx$ converges. Then $\int_a^{\infty} f(x)\,dx$ converges

take $x, y \in I \times I$]

→ $t \in (x, y)$ MVT.

$g(y) - g(x) = (y - x)\, g'(t)$

关键求出 $g'(t)$

并判断范围

代回原等式可得

Summary 方十钱.

Rule: If $f: I \to \mathbb{R}$ is differentiable and we can find $K < 1$ s.t. $f'(t) \le K$ for all $t \in I$. then f is a contraction

take $x, y \in I$. By mean value theorem

$t \in (x, y)$ $f'(t) = \frac{f(y) - f(x)}{y - x}$ Hence.

$|f(x) - f(y)| = |f'(t)|\,|x - y| \le K|x - y|$

$|f'(t)| \le K$

The series converges if the sequence of partial sum: $S_n = \sum_{k=1}^{n} a_K$ converges to a finite limit as $n \to \infty$

注意 harmonic series $\sum \frac{1}{k}$

Chapter 9.

Definite: Let $p \in \mathbb{Z}$ and let $a_p, a_{p+1}, a_{p+2} \cdots \in R$ for $p \le n \in N$, we define the partial sums by

$S_n := \sum_{k=p}^{n} a_K$ we say series $\sum_{k=p}^{\infty} a_K$ converges

to $S \in R$, if the sequence of partial sum converges to S. i.e. if $\lim_{n\to\infty} S_n = S$.

$\boxed{\text{Divergence test.}}$

If the series $\sum_{k=p}^{\infty} a_K$ converges, then $a_n \to 0$ as $n \to \infty$.

proof: $S_n := \sum a_K \to L$

$S_{n-1} \to L$

$a_n = S_n - S_{n-1} = L - L = 0$

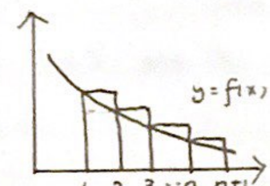
Contrapositive:

If $a_n \not\to 0$, then $\sum a_K$ diverges

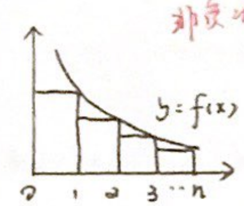
$\boxed{\text{Integral Test}}$

Let $p \in \mathbb{Z}$ and let $f: [p, \infty) \to R$ be ① a continous ② non-negative and ③ non-increasing function

$\int_p^{\infty} f(x)\,dx$ $\sum_{k=p}^{\infty} f(k)$ either both converges or both diverges.



非负减函数



$$\begin{cases} \int_1^{n+1} f(x) \le a_1 + a_2 + \dots + a_n \\ a_2 + \dots + a_n \le \int_1^{n} f(x) \end{cases}$$

$$\boxed{\int_1^{n+1} \le \sum_{k=1}^{n} a_K \le \int_1^{n} + a_1}$$

if $\int_1^{\infty} f(x)$ is finite, RHS shows $\sum a_n$ finite

Summary

if $\int_1^{\infty} f(x)$ is infinite, LHS shows $\sum a_n$ infinite.

Absolute Converges

If $\sum_{k=p}^{\infty} |a_k|$ converges, Then $\sum_{k=p}^{\infty} a_k$ converges.

Contradition

Algebra of Limits.

Comparison test

Let $\sum_{k=p}^{\infty} a_k$ be a series, Suppose we have an integer $q \geq p$, a real number $M > 0$. and a convergent series $\sum_{k=q}^{\infty} b_k$ s.t.

$$|a_k| \leq M b_k, \text{ for } k \geq q.$$

Then $\sum_{k=p}^{\infty} a_k$ converges absolutely.

choose $p=q$:

$$\sum_{p}^{\infty} a_k \leq \sum_{p}^{\infty} |a_k| \leq M \sum_{p}^{\infty} b_k$$

Limit Comparison Test

Let $(a_k)_{k \geq p}$ and $(b_k)_{k \geq p}$ be sequences s.t. $a_k \ b_k > 0$ for all k and such that $\lim_{k \to \infty} a_k / b_k = L \in (0, \infty)$ exists. Then $\sum_{k=p}^{\infty} a_k$ and $\sum_{k=p}^{\infty} b_k$ either both converges or both diverges

Summary

The Ratio Test

let $(a_k)_{k \geq p}$ be a sequence s.t. $a_k \neq 0$ for all k and s.t. $\lim_{k \to \infty} |a_{k+1}/a_k| = L$ exists

Then $\sum_{k=p}^{\infty} a_k$

(i) diverges if $L > 1$

(ii) converges absolutely if $L < 1$

no info. (iii) may converges or diverges if $L = 1$

Alternating Series Test

let $(b_k)_{k \geq p}$ be a real sequence satisfying

non-negative (i) $b_k \geq 0$ for all $k \geq p$;

non-increasing (ii) $b_{k+1} \leq b_k$ for all $k \geq p$;

$b_k \to 0$ (iii) $b_k \to 0$ as $k \to \infty$

Then $\sum_{k=p}^{\infty} (-1)^{k+1} b_k$

Power Series

let a and $c_0, c_1, \cdots$ be real numbers. The series

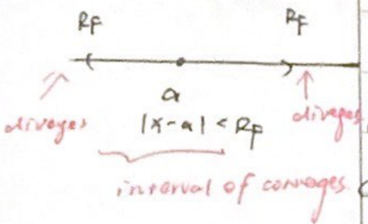
$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$$

is called power series with centre a and coefficient c_k.

Summary

$\boxed{\text{Radius of Convergence.}}$

let $F(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ be a power series.
then is R_F with $0 \le R_F \le \infty$. the radius of convergence with the

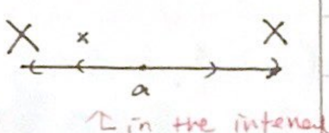


(i) if $|x-a| > R_F$, $F(x)$ diverges
(ii) if $|x-a| < R_F$, $F(x)$ converges absolutely
(iii) if $0 < R_F < \infty$ $|x-a| = R_F$. $F(x)$ may di or co.

if $R_F = 0$ $F(x)$ only converges for $x = a$

if $R_F = \infty$ $F(x)$ converges absolutely for all $x \in \mathbb{R}$,

if $0 < R_F < \infty$, we call $\{x : |x-a| < R_F\}$ the interval of converges



Lemma: Suppose that $|x-a| < |X-a|$ and that $F(X) = \sum_{k=0}^{\infty} c_k (X-a)^k$ converges. Then $F(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ converges absolutely.

Theorem let $F(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$
if R_F, F is continuous and can be differentied term by term. $x = a$.

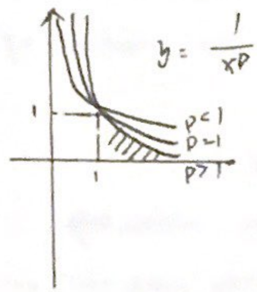
$F(a) = c_0$ $F'(a) = 1 \cdot c_1$ $F''(a) = 2 \cdot 1 \cdot c_2$

$F^{(n)}(a) = n! \cdot c_n$

$c_n = \frac{F^{(n)}(a)}{n!}$ for all $n \in N$

Summary

$\boxed{p\text{-test.}}$



$$\int_1^{+\infty} \frac{dx}{x^p}$$

$$\int_1^{u} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p}(u^{1-p}-1) & p\neq 1 \\ \ln u & p=1 \end{cases}$$

$p\uparrow$, $y=\frac{1}{x^p}$ (when $x>1$) 越靠近x轴.
曲线下方的阴影区域存在有限面积的可能性也越大

$$\lim_{u\to+\infty} \int_1^{u} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & p>1 \\ +\infty, & p\leq 1 \end{cases}$$

瑕积分

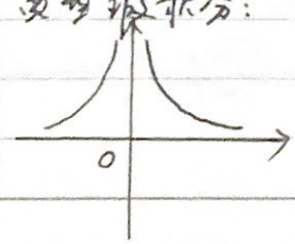
设f定义在$(a,b]$，在a的任一右邻域上无界，但在任何闭区间$[u,b]\subset(a,b]$上有界且可积。若存在极限

$$\lim_{u\to a^+} \int_u^b f(x)\,dx = J$$

则称此函数为无界函数f在$(a,b]$上的反常积分，记作 $J=\int_a^b f(x)\,dx$. 并称反常积分$\int_a^b f(x)\,dx$收敛

f在a近旁无界，a称为f的瑕点，无界反常积分称为瑕积分

典型瑕积分: $\int_{-1}^{1} \frac{dx}{x^2}$. 奇点为0.



Summary

柯西判别法则：

设 $f:[a,+\infty)\to R$

① $0\le f(x)\le \frac{1}{x^p}$，$x\in[a,+\infty)$，且 $p>1$ 时 $\int_a^{+\infty} f(x)dx$ 收敛

② $f(x)\ge \frac{1}{x^p}$，$x\in[a,+\infty)$，且 $p\le 1$ 时，$\int_a^{+\infty} f(x)dx$ 发散

设 $f:[a,+\infty)\to R$

$$\lim_{x\to+\infty} x^p f(x)=\lambda$$

① 当 $p>1$，$0\le\lambda<+\infty$，$\int_a^{+\infty} f(x)dx$ 收敛

② 当 $p\le1$，$0<\lambda\le+\infty$，$\int_a^{+\infty} f(x)dx$ 发散

狄利克雷判别法

若 $F(u)=\int_a^u f(x)dx$ 在 $[a,+\infty)$ 上有界，$g(x)$ 在 $[a,+\infty)$ 上当 $x\to+\infty$ 时单调趋于 0，则 $\int_a^{+\infty} f(x)g(x)dx$ 收敛

阿贝尔判别法

若 $\int_a^{+\infty} f(x)$ 收敛，$g(x)$ 在 $[a,+\infty)$ 上单调有界

则 $\int_a^{+\infty} f(x)g(x)dx$ 收敛.

Summary

级数阿贝尔判别法：

若$\{a_n\}$为单调有界数列，且级数$\sum b_n$收敛

则级数$\sum a_n b_n$收敛

级数狄利克雷判别法：

若数列$\{a_n\}$单调递减，且$\lim\limits_{n\to\infty} a_n = 0$，又级数

$\sum b_n$的部分和数列有界，则$\sum a_n b_n$收敛

幂级数

$\sum a_n(x-x_0)^n = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n$

阿贝尔定理，

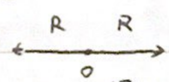
若幂级数$\sum a_n x^n$在$x=\bar{x}\neq 0$处收敛

则对满足不等式$|x|<|\bar{x}|$的任何x，幂级数

收敛且绝对收敛；若幂级数在$x=\bar{x}$处发散

则对满足不等式$|x|>|\bar{x}|$的任何x，幂级数

发散。



收敛域

$R=0$ 幂级数$\sum a_n x^n$仅在$x=0$处收敛

$R=+\infty$ 幂级数$\sum a_n x^n$在$(-\infty,+\infty)$上收敛

$0<R<+\infty$ 幂级数$\sum a_n x^n$在$(-R,R)$上收敛 $|x|>R$发散

$x=\pm R$，可能收敛可能发散。

$(-R,R)$ 收敛区间.

Summary

柯西-阿达马（Cauchy-Hadamard 定理），幂级数

$$\rho = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$$

① $0<\rho<+\infty$，收敛半径 $R=\frac{1}{\rho}$

② $\rho=0$，$R=+\infty$

③ $\rho=+\infty$，$R=0$

Nondimensionlization

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega_0 t)$$

variables: x t $\to$ $\tilde{x}$ $\tilde{t}$

- reduce the parameters
- identify relative size of terms

$$\frac{dA}{dt} = rA$$

Define a new variable

pick existing parameters

to multiply or divide

to create

a dimensionless

variable.

Steps:

variables: A, t

parameters: r

dimension: N, T

constant: N_0, T_0

nondim variables: $x = \frac{A}{N_0}$, $\tau = \frac{t}{T_0}$ dimenless.

Rearrange: $A = xN_0$, $t = \tau T_0$

substitute: $\frac{d(xN_0)}{d(\tau T_0)} = r\,x\,N_0$

$$\frac{dx}{d\tau} = rT_0\,x$$

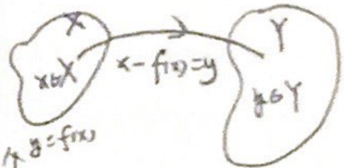
ID

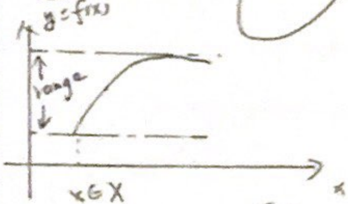
dimensionless = dimensionless

Summary

choose constant: $T_0 = 1/r$

$$\frac{dx}{d\tau} = x$$





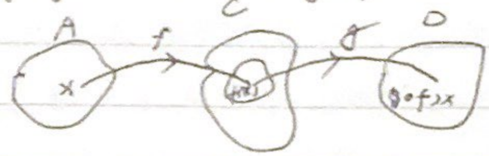
domain of f

Calculus Summary

Composition of function

$f: A \to B \quad g: C \to D \quad B \subseteq C$

$g \circ f \ (A \to D) \quad g(f(x))$



Inverse function

Definition: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

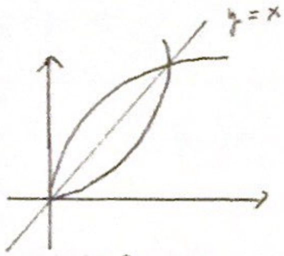
f is injective. f is invertible.

there exist a inverse function f^{-1}

$x = f^{-1}(y)$

$f^{-1}(f(x)) = x \quad (f^{-1} \circ f)(x) = x$

$f(f^{-1}(y)) = y \quad (f \circ f^{-1})(y) = y$



reflection.
$y = x$.

Remark: strictly increasing: $f(x_1) < f(x_2)$

strictly decreasing: $g(x_2) < g(x_1)$

Every strictly increasing function and every strictly decreasing function is injective and so have an inverse.

Summary

Polynomial function.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

$(f \in R)$

n: degree of the polynomial

Polynomial function has finite terms.

$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + x^6 + \cdots$ is not a polynomial function.

Rational function

$f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials and q is not the identically vanishing polynomial $(q \neq 0)$

$f \in \{x \in R \mid q(x) \neq 0\}$

a polynomial is a special case of a rational function, $q(x) = 1$

Alegbra function.

$a_n(x) y^n + a_{n-1}(x) y^{n-1} + \cdots + a_1(x) y + a_0(x) = 0$

with polynomial function $a_0(x)$, $a_1(x)$ $a_2(x)$ ...

a rational function is a special case of an algebra function, a_k. $k \geq 2$. are identically vanishing.

$f(x) = \sqrt{1-x^2}$ ✓

(satisfied $x^2 + y^2 - 1 = 0$)

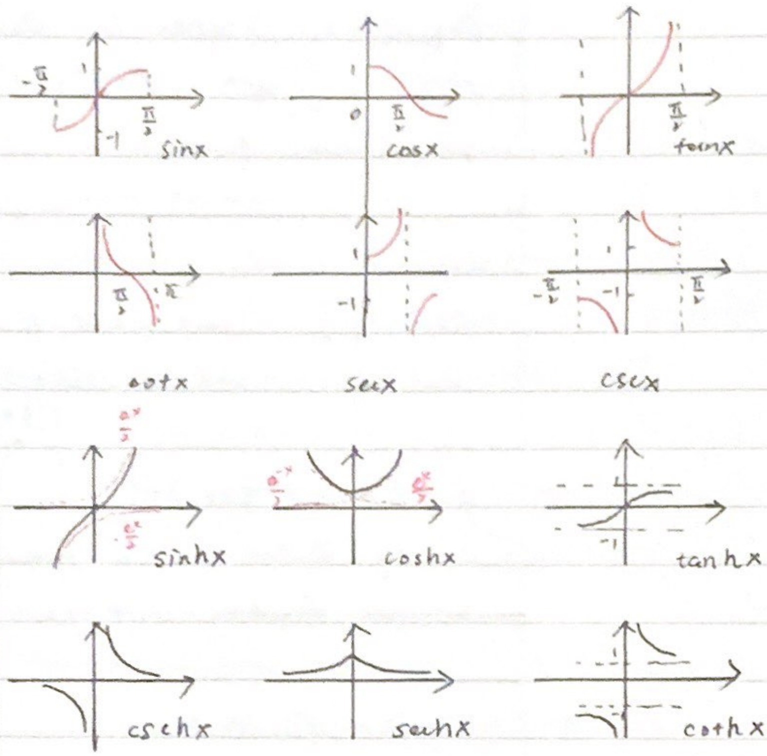
Summary

Transcendetal function

A function that is not algebra

Trigometric, exponential, logarithmic

and hyperbolic function



Hyperbolic function.

$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

Even function: $f(-x) = f(x)$ for all x

Odd function: $f(-x) = -f(x)$ for all x

Periodic: $f(x+p) = f(x)$

$f = \frac{1}{p}$ frequency.

Summary

limit:

Definition: Let f be a function defined on an interval containing $x = x_0$, but not necessarily at $x = x_0$, and let $c \in R$. Suppose that for any positive number ϵ there exists a positive number δ such that $|f(x) - c| < \epsilon$ for all $0 < |x - x_0| < \delta$, We say that f has a limit c as x tend to x_0

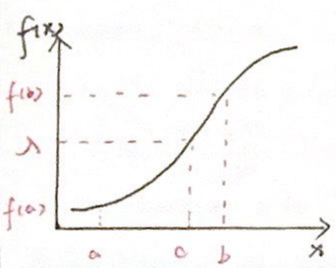
$\exists$ [illegible] > 0 $\forall$ [illegible] 0

Continuity.

Definition: A function f is continuous at a point $x = x_0$ iff $\lim_{x \to x_0} f(x) = f(x_0)$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

- If f is known to be continuous, its limit as $x \to x_0$ can be found by just ~~[illegible]~~ evaluate $f(x_0)$



Intermediate Value Theorem

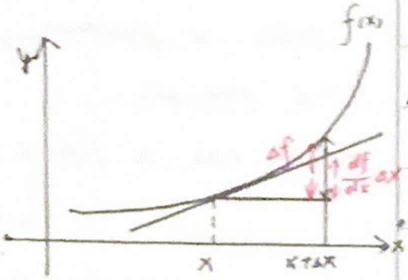
Theorem: Suppose f is $\boxed{\text{continuous}}$ on $[a, b]$ and $\boxed{f(a) \neq f(b)}$. Let λ be a number in the open interval betwen the end-point values of f, i.e. $f(a) < \lambda < f(b)$ or $f(b) < \lambda < f(a)$. There exist some $c \in (a, b)$ S.t. $\lambda = f(c)$

Summary

Differentiation.

Definition: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Theorem: Differentiable $\Rightarrow$ continuous



$\Delta f \approx \frac{df}{dx} \Delta x$

• $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$

• $\Delta f \approx \frac{df}{dx} \Delta x$

$f(x) \approx f(x_0) + \frac{df}{dx} \Delta x \qquad \Delta x = x - x_0$

Differention rule

• $(u+v)' = u' + v'$

• $(uv)' = u'v + uv'$

• $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

---

chain rule $h = f \circ g$

$\frac{dh}{dx} = \frac{df}{dy} \frac{dg}{dx} \qquad y = g(x)$

or

$h'(x_0) = f'(g(x_0))\, g'(x_0)$

$f \circ f^{-1}: \quad f(f^{-1}(x)) = x$

$f'(f^{-1}(x)) \boxed{\frac{d}{dx} f^{-1}(x)} = 1$

chain rule.

Inverse function

$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \quad$ (write $y = f^{-1}(x)$ $\quad x = f(y)$)

Summary

5

Leibniz's Theorem (Higher Order)

Theorem: Suppose u and v are functions that are n times differentiable at a point x and $f = uv$; then f is n times differentiable and

$$f^{(n)} = uv^{(n)} + {}^{n}C_1 u'v^{(n-1)} + {}^{n}C_2 u''v^{(n-2)} + \cdots + {}^{n}C_{n-1} u^{(n-1)}v' + u^{(n)}v$$
$$= \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^{n}C_r \qquad f^{(0)} = f$$

Differentiability

Theorem: Suppose f_1 and f_2 are differentiable and one-side limits $\lim_{x \to x_0^-} f_1'(x)$ and $\lim_{x \to x_0^+} f_2'(x)$ exists and are finite. Then f is differentiable at x_0 iff:

(i) f is continuous at x_0

(ii) $\lim_{x \to x_0^-} f_1'(x) = \lim_{x \to x_0^+} f_2'(x)$

$$f'(x_0) = \lim_{x \to x_0^-} f_1'(x) = \lim_{x \to x_0^+} f_2'(x)$$

Example:

$$f(x) = \begin{cases} x & \text{for } x<0 \\ a & \text{for } x=0 \\ bx & \text{for } x>0 \end{cases}$$

a) continuous at $x=0$?

b) differentiable at $x=0$?

a) $\lim_{x \to 0^-} f(x) = 0$, $\lim_{x \to 0^+} f(x) = 0$

$f(0) = a$

Hence, f is continous at $x=0$ iff $a=0$

Summary

b) $\lim_{x \to 0^-} f'(x) = 1$ $\lim_{x \to 0^+} f'(x) = b$

differentiable at $x=0$ iff

$a=0$ $b=1$

5 . .

Local Extrema

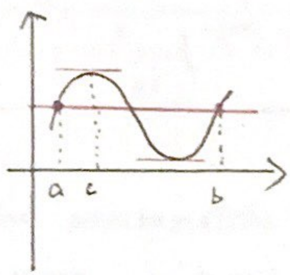
Definition: f be a real function on interval around x_0

1. f is said to be local maximum at x_0 if $\exists\ \varepsilon>0$

   $f(x) \le f(x_0)$ whenever $|x-x_0|<\varepsilon$

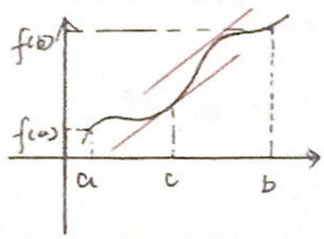
2. f is said to have a local minimum at x_0 if $\exists\ \varepsilon>0$

   $f(x) \ge f(x_0)$ whenever $|x-x_0|<\varepsilon$



Rolle's Theorem:

Theorem: Suppose f is ① continuous on $[a,b]$ and ② differentiable on (a,b). If $f(a)=f(b)$ there exists a point $c \in (a,b)$ for which $f'(c)=0$.



Mean Value Theorem:

Theorem: Suppose f is ① continuous on $[a,b]$ and ② differentiable on (a,b). Then for some $c \in (a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Summary

Taylor's Theorem

Theorem: Suppose f is $n+1$ times differentiable on $[x_0, x]$. Then

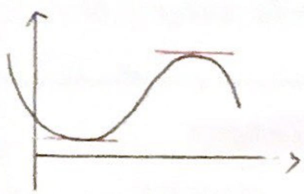
$$f(x) = \underbrace{f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0)}_{\text{Taylor polynomial}} + E_n(x)$$

$$E_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \quad \xi \in (x_0, x)$$

Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$



Stationary point of a function

Definition: For a differentiable function f, a point at which f' vanishes is called a stationary point of f. $f'=0$.

Theorem: Suppose f'' exists and is continuous in some interval containing the stationary point x_0.

(i) $f''(x_0) < 0 \Rightarrow x_0$ is a maximum

(ii) $f''(x_0) > 0 \Rightarrow x_0$ is a minimum

Summary

5

{ lowest nonvanishing

Theorem: Suppose that the lowest nonvanishing derivative of f at the stationary point x_0 is $\boxed{f^{(k)}(x_0)}$ $k \geq 2$ and suppose $f^{(k+1)}$ exists and is continuous in some interval containing x_0. Then:

(i) K is even and $f^{(k)}(x_0) < 0 \Rightarrow x_0$ maximum

(ii) K is even and $f^{(k)}(x_0) > 0 \Rightarrow x_0$ minimum

(iii) K is odd $\Rightarrow x_0$ is point of inflection

Integration

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\delta x_i$$

Fundamental theorem of calculus

Theorem:

变限积分

1. If f is continuous on $[a,b]$, $x \in [a,b]$

$$A(x) = \int_a^{x} f(t)\,dt$$ 自变量

$$\frac{dA(x)}{dx} = f(x)$$

2. If f is continuous on $[a,b]$ and F is any anti-derivative of f. then

$$\int_a^b f(x)dx = F(b) - F(a) = \left[F(x)\right]_{x=a}^{x=b}$$

Summary

Integration by part.

$(fg)' = f'g + fg'$

$\int f'g = fg - \int fg'$

How to choose:

f' is simpler than f

g is simpler than g'

Reduction formula

$$I_n = \int_0^\infty x^n e^{-ax}\, dx$$

Involve n times

- Find explicit formula for I_n

$$I_0 = \int_0^\infty e^{-ax}\, dx = \left[-\frac{1}{a} e^{-ax}\right]_0^\infty = \frac{1}{a}$$

$$I_n = \int_0^\infty x^n e^{-ax}\, dx = \left[x^n\left(-\frac{1}{a} e^{-ax}\right)\right]_0^\infty - \int_0^\infty n x^{n-1}\left(-\frac{1}{a} e^{-ax}\right) dx$$

$$= 0 - 0 + \frac{n}{a} \int_0^\infty x^{n-1} e^{-ax}\, dx$$

$$= \frac{n}{a} I_{n-1}$$

integration by part.

recursive:

$$I_n = \frac{n}{a} I_{n-1} = \left(\frac{n}{a}\right)\left(\frac{n-1}{a}\right) I_{n-2}$$

$$= \left(\frac{n}{a}\right)\left(\frac{n-1}{a}\right)\left(\frac{n-2}{a}\right) \cdots \left(\frac{2}{a}\right)\left(\frac{1}{a}\right) I_0$$

$I_0 = 1/a$

$I_n = \cancel{\frac{n!}{a^{n+1}}}\ n!/a^{n+1}$

Summary

5

Integration by trigonometric substitution

$t = \tan(\frac{x}{2})$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

Partial fraction

1. $\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$

2. $\frac{1}{(x+a)^2(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}$

3. $\frac{1}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$

Numerical integration.

$I = \int_a^b f(x)\,dx$

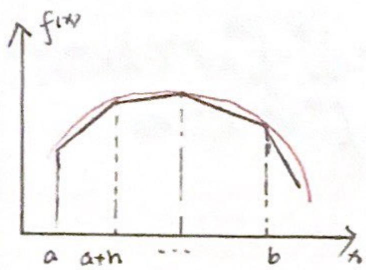
Trapezium rule for a single strip

$I = \int_a^{a+h} f(x)\,dx$

rectangle: $hf(a)$

triangle: $\frac{1}{2}h[f(a+h) - f(a)]$

$I = \frac{1}{2}h[f(a) + f(a+h)]$



$h = \frac{b-a}{N}$

$A_1 = \frac{h}{2}(f_0 + f_1)$

$A_2 = \frac{h}{2}(f_1 + f_2)$

$A_3 = \frac{h}{2}(f_2 + f_3)$

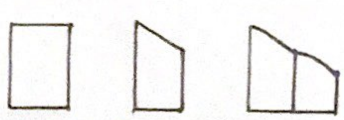
$\vdots$

$A_{n-1} = \frac{h}{2}(f_{n-2} + f_{n-1})$

$A_n = \frac{h}{2}(f_{n-1} + f_n)$

Summary

$\int_a^b f(x) \approx \frac{h}{2}[f_0 + 2(f_1 + f_2 + \dots + f_{n-1}) + f_n]$



Rieman Sums　Trapezoidal rule　Simpson's rule.

check.
(Foundation Calculus)

$\frac{(b-a)^3 M}{12 n^2}$

Error:

$$\varepsilon(h) = \int_a^{a+h} f(x) - \frac{1}{2} h [f(a) + f(a+h)]$$

$$|\varepsilon(h)| \leq \frac{1}{12} h^3 (\max_{a \leq x \leq a+h} |f''(x)|)$$

Trapezium rule for multiple strip.

$$\int_a^b f(x)dx = \frac{1}{2} h [f(a) + f(b)] + h \sum_{n=1}^{N-1} f(a+nh)$$

$a + Nh = b.$

$$|\varepsilon_{total}| \leq \frac{1}{12} N h^3 (\max_{a \leq x \leq b} |f''(x)|) = \frac{1}{12}(b-a)h^2(\max_{a \leq x \leq b}|f''(x)|)$$

Ordinary differential Equation

Order: highest derivative

Degree: degree of ODE is the power to which the highest-order derivative is raised.

Linearity

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

$f(x) = 0.$

homongerous.

Summary

Euler homogenity

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substitution $y = xv$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx} = F(v) \text{ or}$$

$$x\frac{dv}{dx} = F(v) - v$$

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x} + C$$

Second-order

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$am^2 + bm + c = 0$$

Case 1: real distinct $y = Ae^{m_1x} + Be^{m_2x}$

Case 2: real repeated roots $y = Ae^{m_1x} + Bxe^{m_1x}$

Case 3: complex conjugate root $y = e^{\alpha x}[C\cos\beta x + D\sin\beta x]$

$m = \alpha \pm \beta i$

Inhomogeneous

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$y(x) = y_{cf} + y_{pi}.$$

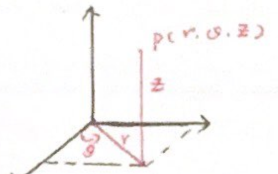
System of ODE

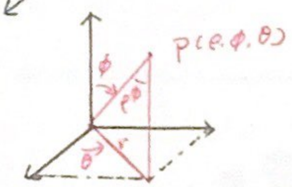
Summary

Two variables:

$z = f(x, y)$

$f(x,y) = \alpha$ level curve.





Cartesion coordinate

Cylindenrizal coordinate (r, θ, z)

Sphereical coordinate. (ρ, ϕ, θ) $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$

Parameteric form $r = (x, y, z) = (x(t), y(t), z(t))$

Tagent vector: $\frac{dr}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ is a tangent vector to the curve at $r(t)$

Surfaces in parametric form:

$r(u, v) = (x(u,v), y(u,v), z(u,v))$

$r_u \times r_v$ → normal.

Limit

Definition: $f(x,y)$ has a limit L as (x,y) approaches (a,b), given any $\varepsilon > 0$ we can find $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ when $0 < |(x,y) - (a,b)| < \delta$

Continuity

Definition: f is continuous at (a,b) if $(a,b) \in$ the domain of f and $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$

Summary

Partial Derivatives

$z = f(x, y)$

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

Higher Order

$\frac{\partial^2 z}{\partial y \partial x}$ $\frac{\partial^2 z}{\partial x^2}$ $\frac{\partial^2 z}{\partial y^2}$

Implicit differention

$z = z(x, y)$ $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Small changes and differentials

$y = y(x)$ $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ $\Delta y \simeq \frac{dy}{dx} \Delta x$

$z = z(x, y)$ $\Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$

Differentials

$dy = \frac{dy}{dx} dx$

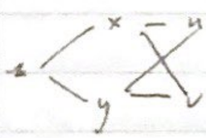
~~dy~~

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

(dy dx are called differentials)

Chain rule.

$z = z(x, y)$ $x = x(u, v)$ $y = y(u, v)$



Summary