Mathematical Method

Mingdong He
The University of Nottingham

April 26, 2022

Contents

1 Introduction

This is my reading notes based on *Mathematical Methods, University of Cambridge Part IB Mathematical Tripos, David Skinner*. This notes is largely imcomplete one, I only write down some essential parts based on my own learning path.

2 Fourier Series

.

Definition 2.1 (Innner product)**.** Inner product is defined as a mapping $(,) : V \times V \to F$ that obeys:

1. $(u, v) = (v, u)*$ (Conjugate symmetry)
2. $(u, \lambda v) = \lambda(u, v)$ (linearity)
3. $(u, v + w) = (u, v) + (u, w)$ (additivity)
4. $(u, u) \geq 0, \forall u \in V$, and with equality $\iff u = 0$

If $F = \mathbb{C}$, the map is called sesquilinear.

2.1 Motivation

If we're given an orthonormal basis, we can use the inner product to explicitly decompose a general into. this basis.

For any element $u \in V$, it can be uniquely written as

$$u = \sum_{i=1}^{n} \lambda_i v_i \tag{1}$$

we can use inner product to find the coefficient λ_i. Please note that $(v_i, v_j) = \delta_{ij}$.

$$(v_j, u) = (v_j, \sum_{i=1}^{n} \lambda_i v_i) = \sum_{i=1}^{n} (v_i, \lambda_i v_i) = \sum_{i=1}^{n} \lambda_i (v_j, v_i) = \lambda_j \tag{2}$$

Thus, we find that $\lambda_i = (v_i, u)$, for real vectors, λ_j is just the projection of u onto v_j.

Definition 2.2 (Inner product for complex valued function)**.** Consider a function $f : \Omega \to \mathbb{C}$. The inner product is defined as:

$$(f, g) = \int_{\Omega} f(x)^* g(x) d\mu \tag{3}$$

This is an generalization of the inner product between two finite dimensional vectors.

If $\Omega = [a, b]$, then $(f, g) = \int_a^b f(x)^* g(x) dx$, with measure dx If Ω is disc D_2, then $(f, g) = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} f(r, \theta)^* g(r, \theta) r dr d\theta$, with. measure $d\mu = r dr d\theta$.

Let's applied above results to construct fourier series.

Consider inner product:

$$(e^{im\theta}, e^{in\theta}) = \int_{\pi}^{\pi} e^{-im\theta} e^{in\theta} d\theta = 2\pi \delta_{m,n} \tag{4}$$

this function $e^{in\theta}$ can be used as orthonormal basis after some scaling.

Fourier claimed that

$$f(\theta) \stackrel{?}{=} \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta} \tag{5}$$

Since our basis is orthonormal, we can use inner product to get the cofficients $\hat{f}_n$.

$$\hat{f}_n = \frac{1}{2\pi} (e^{in\theta}, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta \tag{6}$$

Let's rewrite the function:

$$f = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta} \tag{7}$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} (\hat{f}_n e^{in\theta} + \hat{f}_{-n} e^{-in\theta}) \tag{8}$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} (\hat{f}_n e^{in\theta} + \hat{f}_n^* e^{-in\theta}) \tag{9}$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \tag{10}$$

In the final line, we set $\hat{f}_n = (a_n - ib_n)/2$, then we get

$$a = 2\Re \hat{f}_n, b = -2\Im \hat{f}_n \tag{11}$$

$$a = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n\theta f(\theta) d\theta, b = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta f(\theta) d\theta \tag{12}$$

2.2 Some results

Definition 2.3. The finite sum is defined as

$$S_n f = \sum_{k=-n}^{n} \hat{f}_k e^{in\theta} \tag{13}$$

Theorem 2.1 (Parseval's identity)**.**

$$(f, f) = 2\pi \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2 \tag{14}$$

Parseval's indentity can be view as an infinite dimensional version of Phythagoras'theorem: the norm squared f is equal to the sum of the (mod)-squares of its coefficients in the (Fourier) basis of orthogonal functions. If we view the Fourier series as a map from a function to a sequence, Parseval's identity tells us this map is an isometry, which means it preserves lengths.

3 Sturm-Liouville Theory

3.1 Matices foundation: Self-adjoint matrices

Definition 3.1 (Linear Map)**.** Let V and W be finite dimensional vector spaces (over complex numbers), with $\dim V = n, \dim W = m$, a linear map is defined by $M : V \to M$

We can. represent the map M in terms of an $m \times n$ matrix M, with components:

$$M_{ai} = (w_a, Mv_i), a = 1, \ldots, m, i = 1, \ldots, n \tag{15}$$

Definition 3.2 (Hermitian conjugate)**.** Given a matirx M, its Hermitian conjugate $M^\dagger$ is defined to be the complex conjugate of the transpose matrix, $M^\dagger = (M^T)^*$, where the complex conjugation acts on each entry of M^T.

A matrix is said to be Hermitian or self-adjoint if $M^\dagger = M$. This is also another neaty way of this definition. since for two vectors we have, $(u, v) = u^\dagger \cdot v$, (if both vector are real, this is same as inner product as usual, also, $(Bu)^\dagger = u^\dagger B^\dagger$, we have the definition that: matrix B is the adjoint of a matrix $A \iff (Bu, v) = (u, Av)$.

There some important properties below.

1. Since $\lambda_i(v_i, v_i) = (v_i, Mv_i) = (Mv_i, v_i) = \lambda_i^*(v_i, v_i)$, the eigenvalues of self-adjoint matrix are always real (because $\lambda_i = \lambda_i^*$).
2. We have $\lambda_i(v_j, v_i) = (v_j, Mv_i) = (Mv_j, v_i) = \lambda_i(v_i, v_i)$, then $(\lambda_i - \lambda_j)(v_j, v_i) = 0$, the eigenvectors corresponding to distinct eigenvalues are orthogonal w.r.t. the inner product.

3.2 Solving linear system

A self-adjoint matrix M is non-singular $\iff$ all its eigenvalues are non-zero. In this case, we can solve the linear equation $\mathbf{Mu} = \mathbf{f}$. The solution is $\mathbf{u} = \mathbf{M^{-1}f}$.

Suppose $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of eigenvectors of $\mathbf{M}$. Then we can write:

$$\mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{v_i}, \tag{16}$$

$$\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{v_i} \tag{17}$$

where $f_i = (\mathbf{v_i}, \mathbf{f})$ (see previous inner product). In order to solve $\mathbf{u}$, we need to find the coefficients u_i in the $\{v_i\}$ basis.

By linearity:

$$\mathbf{Mu} = \sum_{i=1}^{n} u_i \mathbf{M} \mathbf{v_i} = \sum_{i=1}^{n} u_i \lambda_i \mathbf{v_i} = \mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{v_i} \tag{18}$$

Take the inner product of this equation with $\mathbf{v_j}$ gives

$$\sum_{i=1}^{n} u_i \lambda_i (v_j, v_i) = u_j \lambda_j = \sum_{i=1}^{n} f_i (v_j, v_i) = \sum_{i=1}^{n} f_i \delta_{ij} = f_j \tag{19}$$

Thus,

$$\mathbf{u} = \sum_{i=1}^{n} \frac{f_i}{\lambda_i} \mathbf{v_i} \tag{20}$$

If $\mathbf{M}$ is singular then either $\mathbf{Mu} = \mathbf{f}$ has no solution or else has a non-unique solution.

3.3 Differential Operator

This section is highly related to the previous discussion about self-adjoint matrices.

Definition 3.3 (Linear Operator)**.**

$$L = A_p(x) \frac{d^p}{dx^p} + A_{p-1} \frac{d^{p-1}}{dx^{p-1}} + \ldots + A_1(x) \frac{d}{dx} + A_0(x) \tag{21}$$

This is a linear map between spaces of functions beacause for two function $y_{1,2}(x)$ and constants $c_{1,2}$, we have $L(c_1 y_1 + c_2 y_2) = c_1 L y_1 + c_2 L y_2$.

We will be interested in second order linear differential operators

$$L = P(x) \frac{d^2}{dx^2} + R(x) \frac{d}{dx} - Q(x) \tag{22}$$

The homogenous equation $Ly(x) = 0$ has precisely two non-trivial linearly independent solutions, say $y = y_1(x), y = y_2(x)$, and the general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$, which is called complementary function. For inhomogenous equation $Ly(x) = f$, we seek any single solution $y(x) = y_p(x)$, which is called partial integral, and the final general solution is the a linear combination

$$y(x) = c_p y_p(x) + c_1 y_1(x) + c_2 y_2(x) \tag{23}$$

Sturm-Liouville theory provides a more systematic approach, analogous to solving the matrix equation $\mathbf{Mu} = \mathbf{f}$.

3.4 Self-adjoint differential operators

The second order differential operator considered by Sturm and Liouville take the form

$$Ly = \frac{d}{dx}\left(p(x) \frac{dy}{dx}\right) - q(x) y \tag{24}$$

Provided $P(x) \neq 0$, divided through by $P(x)$ to obtain

$$\frac{d^2}{dx^2} + \frac{R(x)}{P(x)} \frac{d}{dx} - \frac{Q(x)}{P(x)} = e^{-\int_0^x R(t)/P(t) dt} \frac{d}{dx}\left(e^{\int_0^x R(t)/P(t) dt} \frac{d}{dx}\right) - \frac{Q(x)}{P(x)} \tag{25}$$

with integrating factor $e^{\int_0^x R(t)/P(t) dt}$.

The beautiful feature of Sturm-Liouville operators is that they are self-adjoint w.r.t inner product

$$(f, g) = \int_a^b f(x)^* g(x) dx \tag{26}$$

provided the functions on which they act obey appropriate boundary conditions. Let' s illustrate this fact.

Recall the neaty definition: matrx B is the adjoint of a matrix $A \iff (Bu, v) = (u, Av)$. Thus, let take inner product (Lf, g)

$$
\begin{aligned}
(Lf, g) &= \int_a^b \left[\frac{d}{dx}(p(x)\frac{df^*}{dx}) - q(x)f^*(x) \right] g(x)dx && (27)\\
&= \left[p\frac{df^*}{dx}g \right]_a^b - \int_a^b p(x)\frac{df^*}{dx}\frac{dg}{dx} - q(x)f(x)^*g(x)dx && (28)\\
&= \left[p\frac{df^*}{dx}g - pf^*\frac{dg}{dx} \right]_a^b + \int_a^b f(x)^* \left[\frac{d}{dx}\left(p(x)\frac{dg}{dx} \right) - q(x)g(x) \right] dx && (29)\\
&= \underbrace{\left[p(x)\left(\frac{df^*}{dx}g - f^*\frac{dg}{dx} \right) \right]_a^b}_{boundary\ condition\ applied=0} + (f, Lg) && (30)
\end{aligned}
$$

Exampls of such boundary conditions are to require that all our functions satisfy

$$b_1 f'(a) + b_2 f(a) = 0 \tag{31}$$
$$c_1 f'(b) + c_2 f(b) = 0 \tag{32}$$

There are some cases about this boundary conditions

1. If the function $p(x)$ obeys $p(a) = p(b)$, then all our functions are periodic, so that $f(a) = f(b), f'(a) = f'(b)$
2. If $p(a) = p(b) = 0$, the endpoints of the interval $[a, b]$ are singular points of the differential equation.

This section need to be illustrated later.

3.5 Eigenfunctions and weight functions

Definition 3.4 (Eigenfunction)**.** A function $y(x)$ is said to be an eigenfunction of L with eigenvalue λ and weight $w(x)$ if

$$Ly(x) = \lambda w(x)y(x) \tag{33}$$

Definition 3.5 (Inner product with weight)**.**

$$(f, g)_\omega = \int_a^b f(x)^* g(x)\omega(x)dx \tag{34}$$

Since omega(x) is real, so $(f, g)_\omega = (f, \omega g) = (\omega f, g)$

Definition 3.6 (Eigenvalues of SL operator are always real)**.** If $Lf = \lambda\omega f$, then

$$\lambda(f, f)_\omega = (f, \lambda\omega f) = (f, Lf) = (Lf, f) = \lambda^*(f, f)_\omega \tag{35}$$

Theorem 3.1. Eigenfunctions f_1, f_2 with distinct eigenvalues, but the same weight function, are orthogonal w.r.t the inner product with weight ω.

Proof. Since:

$$\lambda_i(f_j, f_i)_\omega = (f_j, Lf_i) = (Lf_j, f_i) = \lambda_j(f_j, f_i)_\omega \tag{36}$$

so that if $\lambda_i \neq \lambda_j$,then:

$$(f_j, f_i)_\omega = \int_a^b f_j(x)^* f_i(x)\omega(x)dx = 0 \tag{37}$$

□

Application of this theorem is that: given a self-adjoint operator L, we can form an orthonormal set $\{Y_1(x), Y_2(x), \ldots\}$ of its eigenfunctions by setting:

$$Y_n(x) = y_n(x) / \sqrt{\int_a^b |y_n|^2 \omega dx} \tag{38}$$

Finally, making particular choice of boundary conditions, any function $f(x)$ in $[a, b]$ that obeys the chosen boundary conditions may be expanded as:

$$f(x) = \sum_{n=1}^{\infty} f_n Y_n(x), \ where \ f_n = (Y_n, f)_\omega = \int_a^b Y_n^*(x) f(x) \omega(x) dx \tag{39}$$

Let's take a look at some example: Taking the domain $[-L, L]$ and impose the homogeneous boundary conditions that all our functions are periodic i.e. $f(-L) = f(L), f'(-L) = f'(L)$. Choosing $p(x) = 1$ and $q(x) = 0$, thus SL operator is

$$L = \frac{d^2}{dx^2} \tag{40}$$

Choosing weight function to be 1: The eigenfunction equation becomes:

$$Ly(x) = -\lambda y(x) \tag{41}$$

If $\lambda < 0$, the only solution that obeys the periodic boundary conditions is the trivial case $y(x) = 0$; If $\lambda \geq 0$, then a basis of solution is given by

$$y_n(x) = \exp(i \frac{n\pi x}{L}), for \ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{42}$$

For another example:

$$\frac{1}{2} H'' - xH' = -\lambda H(x) \tag{43}$$

subject to the condition that $H(x)$ behaves like a polynomial as $|x| \to \infty$. This equation is not yet in SL form, so we first compute the integrating factor:

$$\frac{d}{dx} \left(e^{-x^2} \frac{dH}{dx} \right) = -2\lambda e^{-x^2} H(x) \tag{44}$$

This equation is known as Hermite's equation. The solution is,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{45}$$

Thus, $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x$, which is known as Hermite polynomial with the following property:

$$(H_m, H_n)_{e^{-x^2}} = \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \delta_{m,n} 2^n \sqrt{\pi} n! \tag{46}$$