Rank and nullity case study

(18.4) $V = \mathbb{R}^3 \quad W = \mathbb{R}^3$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 3y + z \\ 2x - y + 3z \\ x + 2y + 4z \end{bmatrix}$$

1. Check that T is a Linear Transformation

$$\boxed{\text{i.e. } T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + k\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + k\begin{bmatrix} x_1 \\ y_2 \\ z_2 \end{bmatrix}}$$

Since: $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 3y + z \\ 2x - y + 3z \\ x + 2y + 4z \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Express in Matrix form

$T(\underline{v}) = \underline{A}\underline{v} \qquad A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

批注：$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ 全为0也成立 但是核空间是一个向量，$t=0$ 时也就是全为0的情况。

$\ker(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\uparrow \underline{v} \in V \qquad \uparrow \underline{0} \in W = \mathbb{R}^3$

If $\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker(T)$, then

$A\underline{v} = \underline{0} \Rightarrow \begin{bmatrix} -1 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $\left(\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ 1 & 2 & 4 & 0 \end{array}\right) \rightarrow v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$

$\therefore \ker(T) = \left\{ t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

$\Rightarrow \quad \ker(T) = \left\langle \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\rangle$ Straight line through origin

$$\boxed{\begin{aligned} &\ker(T) = \left\{ t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} \\ &\dim(\ker(T)) = \dim\left(\left\langle \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\rangle\right) = 1 \end{aligned}}$$

Summary

Kernel and Range

$\ker(T) = \{ \underline{v} \in \underline{V} \mid T(\underline{v}) = 0 \} \qquad \ker(T) \leq V$

$\text{Range}(T) = \{ \underline{w} \in \underline{W} \mid \underline{w} = T(\underline{v}) \text{ for some } \underline{v} \in W \} \qquad \text{Range}(T) \leq W.$

Recall $T: V\ \mathbb{R}^3 \to \mathbb{R}^3$
$\qquad\quad\ \ \uparrow \qquad \uparrow$
$\qquad\quad\ \ V \qquad W.$

$\text{Range}(T) = \{ \underline{w} \in W = \mathbb{R}^3 \mid w = T(\underline{v}) \text{ for some } \underline{v} \in V = \mathbb{R}^3 \}$

Let $\boxed{\underline{w} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}} \in \text{Range}(T)$

$\exists \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Upper case X Y Z lower case x y z

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \underline{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\in \left\langle \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\rangle$$

$$\text{Range}(T) = \left\langle \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\rangle$$

HERE! Need to ⇒ Check if they are independent.

Use determinant to check:

$\det \begin{pmatrix} -1 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{pmatrix} = 0$. so it is no independent.

Since: $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

$\therefore \left\langle \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\rangle = \text{Range}(T).$

$\dim(\text{Range}(T)) = 2$

$\text{Rank}(T) = \dim(\text{Range}(T)) = 2$

$\text{NULLITY}(T) = \dim(\ker(T)) = 1$

$\text{Rank} + \text{nullity}(T) = 2 + 1 = 3 = \dim(\mathbb{R}^3) = \dim(V)$

$\text{Range}(T) = \left\langle \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\rangle = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{Range}(T) \Rightarrow$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{cases} x = -k_1 + 3k_2 \\ y = 2k_1 - k_2 \\ z = k_1 + 2k_2 \end{cases}$$

$x + y - z = 0.$

Range is a 2-D plance

特征子空间一样做法
用向量组合表示. 然后消去未知数.

Summary

$\boxed{\text{Rank and nullity}}$

$T: V \to W$. T is a L.T.

$\dim(\ker(T)) = \text{Nullity}(T)$

$\dim(\text{Range}(T)) = \text{Rank}(T)$

$\boxed{\text{Theorem:}}$

$\text{Rank}(T) + \text{nullity}(T) = \dim(V)$

$\dim\{\underline{0}\} = 0$

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Linear Transformation.

- Matrix Transformation: $T_A = R^n \to R^m \quad V = R^n \quad W = R^m$
- Zero Transformation: $T(v) = 0$ since:

  $T(u+v) = 0 \quad T(u) = 0 \quad T(v) = 0 \ \& \ T(ku) = 0$

  $T(u+v) = T(u) + T(v) \ \& \ T(ku) = kT(u)$.

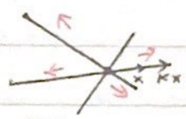
- Dilation and Contraction Operator: k is scalar: $T: V \to V \quad T(x) = kx$.

  $T(cu) = k(cu) = c(ku) = cT(u)$

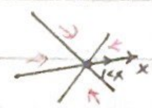
  $T(u+v) = k(u+v) = ku + kv = T(u) + T(v)$

  if $0<k<1$, T is called the contraction of V

  if $k>1$, T is called the dilation of V



Dilation



Contraction

- P_n to P_{n+1}: $p = p(x) = c_0 + c_1x + \cdots + c_nx^n$ is a polynomial.

  $T: P_n \to P_{n+1} \quad T(p) = T(p(x)) = x\,p(x) = c_0x + c_1x^2 + \cdots + c_nx^{n+1}$

  $T(kp) = T(kp(x)) = x(kp(x)) = k(xp(x)) = kT(p)$.

  $T(p_1 + p_2) = T(p_1(x) + p_2(x)) = x(p_1(x) + p_2(x))$

  $= xp_1(x) + xp_2(x) = T(p_1) + T(p_2)$

- Inner product: V is a inner product space: v_0 be any fixed vector in V. let $T: V \to R \quad T(x) = \langle x, x_0 \rangle$

  $T(ku) = \langle ku, v_0 \rangle = k\langle u, v_0 \rangle = kT(u)$.

  $T(u+v) = \langle u+v, v_0 \rangle = \langle u, v_0 \rangle + \langle v, v_0 \rangle = T(u) + T(v)$

Summary

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LM.

(184) 的逆过程.

(185) Let M_{nn} be the vector space of $n \times n$ matrices. In each part determine whether the transformation is linear.

(a) $T_1(A) = A^T$ (b) $T_2(A) = \det(A)$.

(a) $T_1(kA) = (kA)^T = kA^T = kT_1(A)$

$T_1(A+B) = (A+B)^T = A^T + B^T = T_1(A) + T_2(B)$

so T_1 is linear.

(b) $T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A)$

$\det(A+B) \neq \det(A) + \det(B)$

so. it is not linear.

LM. (186) Consider the basis $S = \{v_1\ v_2\ v_3\}$ for R^3. where. $v_1 = (1, 1, 1)$ $v_2 = (1, 1, 0)$. $v_3 = (1, 0, 0)$

Let $T: R^3 \to R^2$ be the linear transformation.

$T(v_1) = (1, 0)$ $T(v_2) = (2, -1)$ $T(v_3) = (4, 3)$

Find a formula for $\boxed{T(x_1, x_2, x_3)}$ and then use that formula to compute $T(2, -3, 5)$.

Express $x = (x_1, x_2, x_3)$ as a linear combination of $v_1\ v_2\ v_3$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + c_2 + c_3 = x_1$$
$$c_1 + c_2 = x_2$$
$$c_1 = x_3$$

最重要一部

$c_1 = x_3$. $c_2 = x_2 - x_3$ $c_3 = x_1 - x_2$.

$(x_1\ x_2\ x_3) = x_3(1, 1, 0) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0)$

对每个基v_i做了变换.

$= x_3 \underline{v_1} + (x_2 - x_3) v_2 + (x_1 - x_2) v_3$

Summary

$\boxed{\text{Theorem}}$: Let $T: V \to W$ be a L.T. where V is finite dimensional. If $S = \{v_1, v_2 \cdots v_n\}$ is a basis for V. then the image of any vector v in V can be expressed as $\boxed{T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)}$

Required to express v as a L.C. of the vectors in S.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \boxed{v_1} + k_2 \boxed{v_2} + k_3 \boxed{v_3}$$

input

$$\downarrow$$

$$\underline{T\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = k_1 \underline{T(v_1)} + k_2 \underline{T(v_2)} + k_3 \underline{(v_3)}$$

output

某变换

Thus:

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 T(v_1) + (x_2 - x_3) T(v_2) + (x_1 - x_2) T(v_3)$$

$$= x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_2 - x_3) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + (x_1 - x_2) \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 - 2x_2 - x_3 \\ 3x_1 - 4x_2 + x_3 \end{bmatrix}$$

$$\therefore T\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{pmatrix} 9 \\ 23 \end{pmatrix}$$

Special case:

Standard basis: $e_1\ e_2 \cdots e_n$

$T_A(e_1) = Ae_1 \quad T_A(e_2) = Ae_2 \cdots T_A(e_n) = Ae_n$.

Ae_j is a linear combination of the column of A

But all entries of e_j are zero except j^{th}.

so: $A = [\, T_A(e_1) \mid T_A(e_2) \mid \cdots \mid T_A(e_n) \,]$.

so: the image of any vector $V = (c_1 \cdots c_n)$

$T(v) = c_1 T(e_1) + c_2 T(e_2) + \cdots + c_n T(e_n)$

$\Downarrow$ General form

$T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$.

Summary

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Linear Mathematics Summary

Vector Space

Definition: A vector space is a set V along with an addition on V and a scalar multiplication on V s.t.

Axioms for addition

(A1) $v+w=w+v$

(A2) $u+(v+w)=(u+v)+w$

(A3) $v+0=v$

(A4) $v+(-v)=0$

Axioms for scalar Multiplication

(M1) $k(v+w)=kw+kv$

(M2) $(k+k')v=kv+k'v$

(M3) $(kk')v=k(k'v)$

(M4) $1v=v$.

· 包含向量

· 向量的运动仍在空间中

A vector space over $\mathbb{R}$ is called real vector space

A vector space over $\mathbb{C}$ is called complex vector space.

Sub space

Definition: Let V be a vector space over the field $\mathbb{F}$. A subset W of V is itself a vector space over $\mathbb{F}$:

0

(i) $0\in W$

closed under addition

(ii) $v\in W,\ w\in W,\ v+w\in W$

closed under scalar multiplication

(iii) $v\in W,\ kv\in W,\ k\in\mathbb{F}$.

$W\leq V$.

Summary

1. $V\leq V$.

2. $\{0\}$ is a subspace of every vector space.

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Direct Sum.

Definition: Let W_1 W_2 be two subspaces of V s.t. $W_1 \cap W_2 = \{0\}$. we say W_1 W_2 are in direct sum. iff $W_1 \cap W_2 = \{0\}$ for $W_1 + W_2$.

$W_1 \oplus W_2$

Linear Combination and Span

Linear combination

A linear combination of a list $v_1 \cdots v_m$ of vectors in V is a vector of the form

$a_1 v_1 + \cdots + a_m v_m$. where $a_1 \cdots a_m \in F$

Span

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called span of S. we say the vectors in S span that subspace. If $S = \{w_1, w_2 \cdots w_r\}$, span$\{w_1 \cdots w_r\}$ or span(S)

spanning set for P_n

$p = a_0 + a_1 x + \cdots + a_n x^n$

$P_n = \text{span}\{1, x, x^2, \cdots, x^n\}$

Summary

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$v_1 = (1,1,2)$
$v_2 = (1,0,1)$
$v_3 = (2,1,3)$ does not span the vector space R^3.

Test for spanning

must determined whether an arbitrary vector $b = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination.

[the system is consistent iff its coefficient matrix has nonzero determinants]

Linear Independence.

Definition: A set S of vectors in a vector space V is linearly dependent if there is a linear dependence relation

$$\sum_{i=1}^{n} k_i v_i = 0 \quad (k_i \in F,\ v_i \in S)$$

where k_i are not all zero. Otherwise, independent.

Remark:

- Any set S with zero vector 0 is linearly dept.
- Set containing just one element, S is independent. $(v \neq 0)$
- Set containing two elements, is indept, provided that $v \neq 0$ & v_1 is not a scalar multiple of v_2.

向量组里有0向量，一定线性相关 因为 $\det(A)=0$.

Test for independence.

1. write as linear combination form check $k_1 \cdots k_n$
2. $\det(A) \neq 0$. A is $n \times n$ matrix with v_j in column j.

Coordinates and Basis.

Definition: If V is any vector space and $S = \{v_1, v_2 \cdots v_n\}$ is a finite set of vectors in V, then S is called a basis for V if: (i) S is linearly independent (ii) S spans V.

Summary

结论: 线性无关的向量组，升维后仍无关
线性相关的向量组，降维后仍相关

2D (线性无关) $\xrightarrow{升维}$ 3D (线性无关)

反之因为投影可能会在一条直线上.

Standard Basis for R^n

$e_1 = (1, 0, \cdots 0, 0)$ $e_2 = (0, 1, \cdots 0)$ $e_n = (0, 0, \cdots 0, 1)$

Standard Basis for P_n

$S = \{1, x, \cdots x^n\}$

Another Basis for R^3

Must show these vectors are linearly independent and span R^3. i.e.

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ has only the trivial solution and every vector $b = (b_1, b_2, b_3)$ in R^3 can be expressed as

$c_1 v_1 + c_2 v_2 + c_3 v_3 = b$.

$v_1 = (1, 2, 1)$

$v_2 = (2, 9, 0)$

$v_3 = (3, 3, 4)$

form a basis for R^3

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 + 4c_3 = 0 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = b_1 \\ 2c_1 + 9c_2 + 3c_3 = b_2 \\ c_1 + 4c_3 = b_3 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

$\det(A) \neq 0$

- linearly independent
- Consistent

- Uniqueness of Basis.
- Spanning list contains a basis

[Every spanning list in a vector space can be reduced to a basis of the vector space]

- Every finite-dimensional vector space has a basis
- Linearly independent list extends to a basis.

Dimension.

Definition: the dimension of a finite-dimensional vector space is the length of any basis of the vector space. $\dim(V)$. (n elements).

Summary

Coordinates: Let $b_1, b_2 \cdots b_n$ be a basis for the vector space V of dimension n over the field of scalar F. Then every $v \in V$ has a unique expression as a L.C.

$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n$ with $x_i \in F$.

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$\dim F^n = n$ $\quad \dim P_m(F) = m+1$ $\quad \dim(M_{mn}) = mn$

- $U \leq V$. $\quad \dim U \leq \dim V$
- Dimension Theorem:

  $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
- If W_1, W_2 are in direct Sum.

  $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$
- Every subspace of V is part of a direct sum equal to V.

  Let V be a finite-dimensional space and $W \leq V$, $\exists\, Z$ s.t. $W \oplus Z = V$

Change of Basis.

批注：已知前后的基底. $\quad$ [New basis | old basis] $\rightarrow$ [I | transition from old to new]

$M_B^{B'}$

$\quad\quad$ NO $\rightarrow$ it

Summary

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Linear Transformation

Row space column space null space.

A : m×n matrix.

Row space — The subspace of R^n spanned by row vectors of A

column space. — The subspace of R^m spanned by the column vectors of A

Null space. — The solution space of homogenous system of equation

$Ax=0$, which is a subspace of R^n.

向量组的最大无关组并不唯一，但最大无关组包含的数目相同此不度的数目为秩

(反应了复杂度)

Rank & nullity.

Dim(Row space) = Dim(column space)

dim(row space) = rank

dim(null space) = nullity.

rank ≤ min(m, n) for non-square matrix A m×n

Dimension Therom for Matrics

If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

$$\begin{bmatrix}\text{number of leading}\\ \text{variables}\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{free}\\ \text{variables}\end{bmatrix} = n$$

(echelon form)

Linear Transformation:

- W=V is possible
- Usually described as Matrices.

$T: V \to W$ is a linear transformation if for all $v_1, v_2 \in V$, $k \in R$.

(i) $T(v_1+v_2) = T(v_1) + T(v_2)$

(ii) $T(kv_1) = kT(v_1)$

Summary

OR Equivalently:

$T(v_1 + kv_2) = T(v_1) + kT(v_2)$

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$\text{Kernel} = \{ v \in V \mid T(v) = 0 \}$

$\text{Range} = \{ w \in W \mid w = \underset{\uparrow}{T(v)} \}$ for some $v \in V$

Summary

Linear System

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

Solution:

$Ax = b$

consistent / inconsistent

unique solution / infinite solution

Therom: A linear system is consistent if and only if the row echelon form of its augmented matrix contains no row of the form $[0, 0, \cdots 0 | b]$. $b \neq 0$

Echelon form

(a) if there are any rows which are full of zeros, they must be grouped together at the bottom

(b) for any two of the other row (i.e. rows which are not full of zeros), the first non-zero entry in the lower row must be (strictly) further to the right than the first non-zero entry in the upper row.

Equivolent statement

- A is invertible
- $\det(A) \neq 0$
- $Ax=0$ has only trivial solution
- $Ax=b$ has one solution for $n\times 1$ matrix b.
- The reduced row form of is I_n.

唯一解: $Ax=b$

$\det(A) \neq 0$

等式个数=未知数.

无穷解 $Ax=b$ $A_{m\times n}$.

~~rank~~ $< n$.

~~$m > n$~~

①非零行在零行上面

②某一行先导元素所在列位于前一行先导元素右边

Summary

③某一先导元素所在列下方元素都是零.

Leading ~~variables~~ entries: non-zero entries (Pivot)

Pivot column: column in which these pivots occur.

leading variables: corresponding variable

free variable: remaining variable.

Reduced echelon form.

all of the leading (pivot) entries are 1.

Cramer's rule:

$Ax = y \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$x_1 = \frac{\begin{vmatrix} y_1 & b \\ y_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a & y_1 \\ c & y_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$x_j = \frac{\det(A_j)}{\det(A)}.$$

Gauss-Jordan inversion of a matrix

$(A \mid I) \to (I \mid C)$

$C = A^{-1}.$

Summary

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invertible – nonsingular

not invertible – singular

Matrix Algebra

Transpose:

$(A+B)^T = A^T + B^T$

$(kA)^T = kA^T$

$(AB)^T = B^T A^T$

$(A^T)^T = A$

Inverse:

$(kA)^{-1} = k^{-1}A^{-1}$ $\quad A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$ $\quad = \frac{\mathrm{Adj}(A)}{\det(A)}$

$(A^{-1})^{-1} = A$

$AA^{-1} = I$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}$ $a, b \neq 0$

Conjugate $\bar{A}$

Hermitian conjugate or **adjoint** $A^* = (\bar{A})^T = \overline{A^T}$

Adjugate $\mathrm{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

In general: Adjugate:

$A_{ij} = (-1)^{i+j} M_{ij}$

Summary

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Trace and Determinant

Definition: Given a square matrix $A \in M_{n \cdot n} F$

The trace of A is the sum of the entries of the leading diagnol.

the trace is not defined for the matrix of size $n \times m$ with $n \neq m$

$tr(A) = \sum_{i=1}^{n} [A]_{ii}$

$tr(A+B) = tr(A) + tr(B)$

$tr(kA) = k\,tr(A)$

$tr(AB) = tr(BA)$

$tr(A) = tr(A^T)$

Determinate of an operator det T.

Suppose $T \in \mathcal{L}(V)$.

- If $F = C$, then the determinant of T is the product of the eigenvalues of T

- If $F = R$, then the determinant of T is the product of the eigenvalues of T_C

方阵才有行列式.

- Invertible $\Leftrightarrow$ det $\neq 0$ $\Leftrightarrow$ $\lambda = 0$ is not an eigenvalue of A.
- determinant of Matrix.

$\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(M_{ik}) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(M_{kj})$.

Summary

对换两行变号
倍数加另一行,不变
两行成比例,行列式=0

$\det(A+B) \neq \det(A)+\det(B)$

$A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ $B=\begin{pmatrix}-2&2\\1&-2\end{pmatrix}$

$\det(A^T B^{-1} A^{-1})$

$=\det(A^T)\cdot\det(B^{-1})\det(A^{-1})$

$=\det(A)\det(B^{-1})\det(A^{-1})$

$=\underbrace{\det(A)\det(A^{-1})}_{\det(AA^{-1})}\det(B^{-1})$

$=1\times\det(B)^{-1}$

$=\left(\begin{vmatrix}-2&2\\1&-2\end{vmatrix}\right)^{-1}$

$=\frac{1}{2}$

$\det(kA)=k^n\det(A)$

$\det(AB)=\det(A)\cdot\det(B)$

$\det(AB)=\det(BA)$

$\det(A)=\det(A^T)$

$\det(A)=\prod_{i=1}^{n}a_{ii}=a_{11}\cdot a_{22}\cdots a_{nn}$ (Triangular Matrix)

$\det(B)^{-1}=\det(B^{-1})$

$\det(A^T)=\det(A)$

Eigenvector and Eigenvalue.

$\boxed{\text{Definition}}$: A vector $v\neq 0_n$ is an eigenvector of the matrix A if $Av=\lambda v$, for λ, called eigenvalue.

- v must be non-zero
- λ is scalar

Characteristic Equation:

$\boxed{\text{Definition}}$: if A is an $n\times n$ matrix, then λ is an eigenvalue of A iff it satisfied equation

$$\det(A-\lambda I)=0$$

$\boxed{\text{Theorem}}$: if A is an $n\times n$ triangular Matrix

Upper. low. diagonal

, then the eigenvalues of A are the entries on the main diagonal of A.

$\boxed{\text{Theorem}}$: if K is a positive integer, λ is an eigenvale of Matrix A, x is eigenvector, then λ^K is an eigenvalue of A^K and x is a corresponding eigenvector.

Summary

Cayley Hamilton Theorem: Matrix A in $M_{n,n}(F)$

$$P(A)=O_{n\times n}.$$

P is characteristic polynomial.

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Eigenspace

Definition: The subspace $E(A;\lambda) \leq F^n$ is called eigenspace of A relative to λ.

Diagonalization.

Similarity transformation

Definition: If A and B are square matrices then we say that B is similar to A if there is an invertible matrix P s.t.

$$B = P^{-1}AP.$$

Similar properties:

- Determinant
- Invertibility
- Rank
- Nullity
- Trace
- Characteristic polynomial
- Eigenvalues
- Eigenspace dimension

Theorem: A is diagonalizable if there exists an invertible matix P s.t $P^{-1}AP$ is diagonal matrix.

Summary

A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

(A is $n \times n$ matrix).

$D = P^{-1} A P$

$A = P D P^{-1}$

$A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$

$= PD^k P^{-1}$

Inner product Space.

Inner product

An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in F$ and:

- $\langle u, v \rangle \geq 0$ for all $v \in V$
- $\langle u, v \rangle = 0$ iff $v = 0$
- $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$
- $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in F$, $u, v \in V$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

Summary

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Basic properties:

- each fixed $u \in V$, the function that takes v to $\langle v, u\rangle$ is a linear map from V to $\mathbf{F}$.
- $\langle 0, u\rangle = 0$ for every $u \in V$
- $\langle u, 0\rangle = 0$ for every $u \in V$
- $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$
- $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$. for all $\lambda \in \mathbf{F}$. $u, v \in V$.

$$\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \bar{\lambda}\overline{\langle v, u\rangle} = \bar{\lambda}\langle u, v\rangle$$

norm $\|v\|$

$\boxed{\text{Definition}}$: For $v \in V$, the norm of V. denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}$$

- $\|v\| = 0$ iff $v = 0$
- $\|\lambda v\| = |\lambda|\,\|v\|$ for all $\lambda \in \mathbf{F}$

$$\|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle = \lambda\langle v, \lambda v\rangle = \lambda\bar{\lambda}\langle v, v\rangle = |\lambda|^2\|v\|^2$$

$$\|\lambda v\| = |\lambda|\,\|v\|$$

orthogonal

perpendicular.

$\boxed{\text{Definition}}$: two vector $u, v \in V$ are called orthogonal if $\langle u, v\rangle = 0$.

- 0 is orthogonal to every vector in V
- 0 is the only vector in V that is orthogonal to itself.

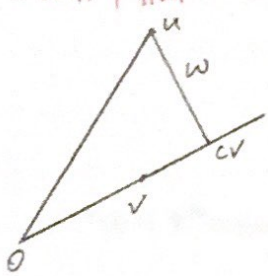
Summary

$\|u+v\| = \langle u+v, u+v\rangle$

$= \langle u,u\rangle + \underbrace{\langle u,v\rangle}_{0}$

$+ \underbrace{\langle v,u\rangle}_{0} + \langle v,v\rangle$

$= \|u\|^2 + \|v\|^2$



to discover how to write u as a scalar multiple of v plus a vector orthogonal to v.

$u = cv + (u - cv)$

$0 = \langle u - cv, v\rangle$

$= \langle u,v\rangle - c\|v\|^2$

$u = \frac{\langle u,v\rangle}{\|v\|^2} v + (u - \frac{\langle u,v\rangle}{\|v\|^2} v)$

Pythagorean Theorem

Suppose u and v are orthogonal vectors in V

Then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Orthogonal Decomposition

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u,v\rangle}{\|v\|^2}$

and $w = u - \frac{\langle u,v\rangle}{\|v\|^2} v$, Then

$\langle w, v\rangle = 0$ and $u = cv + w$.

Cauchy-Schwarz Inequality

Suppose $u, v \in V$ Then

$$|\langle u,v\rangle| \leq \|u\| \|v\|$$

proof: $\boxed{u = \frac{\langle u,v\rangle}{\|v\|^2} v + w}$

$$\|u\|^2 = \left\|\frac{\langle u,v\rangle}{\|v\|^2} v\right\|^2 + \|w\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2$$

$$\geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

multiply by $\|v\|^2$

$$|\langle u,v\rangle| \leq \|u\| \|v\|.$$

Triangular Inequality

Suppose $u, v \in V$.

$$\|u+v\| \leq \|u\| + \|v\|.$$

Summary

proof: $\|u+v\|^2 = \langle u+v, u+v\rangle$

$= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle$

$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u,v\rangle$

$\le \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|$

$\le \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$

$= (\|u\| + \|v\|)^2$

Parallelogram Equality.

Suppose $u, v \in V$.

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Proof: $\|u+v\|^2 + \|u-v\|^2$

$= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

$= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle$

$+ \|u\|^2 + \|v\|^2 - \langle u,v\rangle - \langle v,u\rangle$

$= 2(\|u\|^2 + \|v\|^2)$

Orthonormal Basis

Orthonormal

A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.

$$\langle e_j, e_k\rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Summary

5 . .

Gram-Schmidt Procedure

To convert a basis $\{u_1, u_2 \cdots u_r\}$ into an orthogonal basis $\{v_1, v_2 \cdots v_r\}$, perform the following computations;

Step 1: $v_1 = u_1$

Step 2: $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Step 3: $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$\vdots$

Normalize them.

QR decomposition.

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column column vectors, and R is an $n \times n$ invertible upper triangular matrix.

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Gram: $q_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ $q_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ $q_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R}$$

Summary

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Orthogonal Diagonalization

Definition: If A and B are square matrix then we say that A B are orthogonally similar if there is an orthogonal matrix P s.t. $P^TAP = B$.

$D = P^TAP$

$A = PDP^T$

Quadratic form.

$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \quad x^TAx = [x_1 \;\; x_2]\begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

A is symmetric

$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3$

$x^TAx = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

no cross product terms $\lambda_1x_1^2 + \lambda_2x_2^2 + \cdots + \lambda_nx_n^2 \quad x^TAx = [x_1 \cdots x_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$x = Py$ (substitution)

if P is orthonormal

$x^TAx = (Py)^TA(Py) = y^TP^TAPy = y^T\boxed{(P^TAP)}y$

$P^TP = I$

$x^TAx = y^TDy = [y_1 \cdots y_n]\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1y_1^2 + \lambda_2y_2^2 \cdots \lambda_ny_n^2$

$P^T = P^{-1}$

$\therefore P^TAP = P^{-1}AP = D$

$\downarrow$ eigenvalues in diagonal.

Summary