Cauchy - Schwarz

$| \langle u, v \rangle | < \|u\| \|v\|$

Prove: Sum of the eigenvalues is equal to the trace

A is diagonable. there exists p

$$p^{-1}Ap = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$tr(p^{-1}Ap) = tr\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i$$

$$tr(p^{-1}Ap) = tr(App^{-1}) = tr(AI_n) = tr(A)$$

$tr(AB) = tr(BA)$.

交换顺序.

$tr(A) = \sum_i \lambda_i$

$tr(A^k) = \sum_i \lambda_i^k$

$det(A) = \prod_i \lambda_i$

同理

Prove: If A is diagonable, the product of the eigenvalues is equal to the determinant

$$det(p^{-1}Ap) = det\begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{vmatrix} = \prod_{i=1}^{n} \lambda_i$$

$$det(p^{-1}Ap) = det(App^{-1}) = det(AI_n) = det(A)$$

$det(A - \lambda I_n) = p_A(\lambda) \quad det(A) = p_A(0) = a_0$

$p_A = x^n - a_{n-1}x^{n-1} + \cdots + a_0 \quad a_{n-1} = tr(A)$

$2 \times 2: \quad t^2 - tr(A)t + det(A)$

$$p_2(x) = (a_{11}a_{22} - a_{12}a_{21}) - x(a_{11} + a_{22}) + x^2$$
$$= det(A) - x\,tr(A) + x^2$$

Summary

$p_A(t)$ is characteristic polynomial

$p_A(0)$ is $\det(-A) = (-1)^n \det(A)$

coefficient of t^n is 1

coefficient of t^{n-1} is $tr(-A) = -tr(A)$

Dimension Theorem

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

A is conjugate to B if there exist P invertible s.t.

$A = P^{-1} B P$

same :

- trace

$tr(A) = tr(P^{-1}BP) = tr(BPP^{-1}) = tr(BI_n) = tr(B)$

- determinant

$\det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P)$

$= \det(P)\det(P^{-1})\det(B) = 1 \cdot \det(B) = \det(B)$

- Characteristic polynomial: $A = P^{-1}BP$ $\quad p_A = p_B$

$p_A(t) = \det(A - tI_n) = \det(P^{-1}BP - tI_n)$

$= \det(P^{-1}BP - P^{-1}(tI_n)P)$

$= \det(P^{-1}(BP - tI_nP))$

$= \det(P^{-1}(B - tI_n)P)$

$= \det(P^{-1})\det(B - tI_n)\det(P)$

$= p_B(t)$

- rank

Summary

$Ax=b$ $\quad A: m\times n$

$\text{rank}(A) = \text{rank}(A|b) \Leftrightarrow$ consistent $\Leftrightarrow$ $\text{rank}(A)=m$

$Ax=b$ consistent $\Leftrightarrow$ b in the column space of A

$\text{rank}(A) = \text{rank}(A^T)$

$\text{rank}(A) + \text{nullity}(A) = n.$ (A has n columns)

$A: m\times n$

$\text{rank}(A) = \#$ leading variables in the solution $Ax=0$

$\text{nullity}(A) = \#$ parameters in the general solution $Ax=0$

V: finite - dimensional vector space $\{v_1 \cdots v_n\}$ basis.

(a) If a set has more than n vectors, then it is linear dependent

(b) If a set has fewer than n vectors, then it does not span V

Basis: ① linearly independent ② S span V

A square matrix A is invertible iff $\lambda=0$ is not an eigenvalue of A

proof: $\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$

$\lambda=0$: $\det(-A) = c_n$ or $(-1)^n \det(A) = c_n$

Summary

If $n \times n$: A has n distinct eigenvalues, then A is diagonalizable

Diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors

$AA^T = I$. orthogonal matrix

if A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

co plane: $a \cdot (b \times c) = 0$

co-plane: $a = \lambda b + \mu c$

可以解出 λ μ

Summary