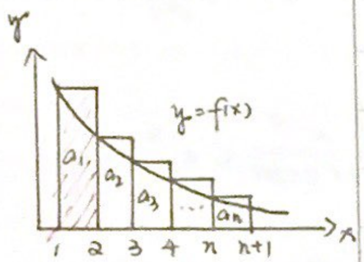
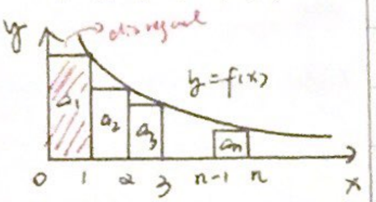
5 . .

AcF

The integral test 说明反常积分和级数可互换

Theorem: let $p \in \mathbb{Z}$ and let $f : [p, +\infty) \Rightarrow \mathbb{R}$ be a ① continuous ② non-negative and ③ non-increasing function. Then $\int_p^\infty f(x)\,dx$ and $\sum_{k=p}^{\infty} f(k)$ either both convergenet or both divergent.





proof: THE LINK BETWEEN $\sum_{k=p}^{\infty} f(k)$ and $\int_p^\infty f(x)\,dx$.

f is decreasing and $f(n) = a_n$

From $x=1$, to $x=n+1$

$$\int_1^{n+1} f(x) \le a_1 + a_2 + \cdots + a_n \quad ①$$

If we disregard a_1.

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$$

If we include a_1

$$a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx \quad ②$$

Combining ① ②, gives:

$$\boxed{\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx}$$

if $\int_1^\infty f(x)$ is finite, the RHS inequalities shows that $\sum a_n$ is finite. If $\int_1^\infty f(x)\,dx$ is infinite. the LHS inequality shows that $\sum a_n$ is infinite. so the series $\sum_{k=p}^{\infty} f(k)$ and the integral are either both finite or both infinite

Summary

5

Ref.

(158) Consider the real sequence defined iteratively by $x_{n+1} = f(x_n)$, where $f(x) = \sqrt{\frac{1}{2}x^2+1}$.

(i) What are the possible finite limits of the sequence $\langle x_n \rangle$? Check your answer

(ii) Does f have any attracting fixpoints?

(iii) Show that if $0 < x_n < \sqrt{2}$, then $x_n < x_{n+1} < \sqrt{2}$ (Hint Consider x_{n+1}/x_n)

(iv) Determined the behavior of the sequence when the starting value x_0 satisfies $0 < x_0 < \sqrt{2}$

(v) What happens when $-1 \le x_0 < 0$? (Hint: consider x_1)

(i) Assume $x_n \to L$ $x_{n+1} \to L$ as $n \to \infty$

By the continuity of f.

$L = f(L)$ $L = \sqrt{\frac{L^2}{2}+1}$ $L = \pm\sqrt{2}$.

Since $f(x) > 0$. $\therefore$ possible limit is $L = \sqrt{2}$.

(ii) $f'(x) = \dfrac{x}{2\sqrt{\frac{x^2}{2}+1}}$ $f'(L) = \dfrac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} < 1$

$\therefore \sqrt{2}$ is a attracting fixpoint.

Considering two sides: x_n and x_{n+1} ⇐

(iii) ① since $0 < x_n < \sqrt{2}$, $x_{n+1} = \sqrt{\frac{x_n^2}{2}+1} < \sqrt{\frac{(\sqrt{2})^2}{2}+1} = \sqrt{2}$

$\therefore x_{n+1} < \sqrt{2}$

② since $0 < x_n < \sqrt{2}$, $\dfrac{x_{n+1}}{x_n} = \dfrac{\sqrt{x_n^2/2+1}}{x_n} = \sqrt{\frac{1}{2}+\frac{1}{x_n^2}} > \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

$\therefore 0 < x_n < x_{n+1} < \sqrt{2}$

still satisfies the condition we derive from (iv) ←

(iv) from (iii) $0 < x_n < x_{n+1} < \sqrt{2}$ $x_n \to \sqrt{2}$ (MST)

(v) If $-1 \le x_0 < 0 \Rightarrow$ $0 < x_1 = \sqrt{\frac{x_0^2}{2}+1} \le \sqrt{\frac{3}{2}} < \sqrt{2}$

By (iv). $x_n \to \sqrt{2}$.

Summary

5 . .

just like the RHS of the inequality in the proof of integral test.

$> \infty . \to$ diverge

(159) $\int_1^\infty \frac{e^{\cos x}}{x}\,dx$ 这个 comparison test 是 给反常积分用的

Note $\int_1^\infty \frac{e^{\cos x}}{x} \geq \int_1^\infty \frac{\frac{1}{e}}{x}$

sin $\frac{1}{e}\int_1^\infty \frac{dx}{x}$ diverges

By $\boxed{\text{contrapositive}}$ of C.T.

$\int_1^\infty \frac{e^{\cos x}}{x}\,dx$ diverges.

$\boxed{\text{Contrapositive of Comparison Test.}}$

Theorem: Let $f[a,+\infty) \to R$ and $g:[a,+\infty) \to R$ be continuous function with $|f(x)| \leq g(x)$

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

$> \infty$

(160) Define a $f: R \to R$ $f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$

Let $a \in R$, $\exists$ rational sequence $x_n \to a$

$x_n \in Q$, $x_n \to a$ $f(x_n) = 1 \to 1$

$y_n = (\underset{\in Q}{x_n} + \underset{\notin Q}{\frac{\sqrt{2}}{n}}) \notin Q$ $\left.\begin{matrix} x_n \to a \\ \frac{\sqrt{2}}{n} \to 0 \end{matrix}\right\} \to a$

$y_n \to a$. $y_n \notin Q$ $f(y_n) = 0 \to 0$

since $0 \neq 1$ No limit $f(x)$

Summary

5

L'Hôpital's Rule

get the limit

$L_1 = \lim_{x\to 1} \frac{x^7-2x+1}{\sin(\pi x)}$

$f(x) = x^7-2x+1 \to 0 \quad g(x) = \sin\pi x \to 0 \quad 0/0$ type.

$\lim_{x\to 1}\frac{f'(x)}{g'(x)} = \lim_{x\to 1}\frac{7x^6-2}{\pi\cos\pi x} = \frac{7-2}{\pi(-1)} = -5/\pi$

$L_1 = \lim_{x\to 1}\frac{x^7-2x+1}{\sin\pi x} = -5/\pi.$

L'Hôpital Twice

$L_2 = \lim_{x\to 0}\frac{x\tan x \to 0}{1-\cos x \to 0}$

By. L'Hôpital's Rule:

$L_2 = \lim_{x\to 0}\frac{\sin x\cos x + x}{\cos^2 x\sin x} = \lim_{x\to 0}\frac{\frac{1}{2}\sin 2x + x \to 0}{\sin x - \sin^3 x \to 0}$ L'H. again

$= \lim_{x\to 0}\frac{\cos 2x + 1}{\cos x - 3\sin^2 x\cos x}$

$= 2.$

Squeeze Theorem + L'Hôpital's Rule

$L_3 = \lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin x}$

$\sin(1/x) \le 1$

since $\left|\frac{x^2\sin(1/x)}{\sin x}\right| \le \left|\frac{x^2}{\sin x}\right|$

$-\left|\frac{x^2}{\sin x}\right| \le \frac{x^2\sin(1/x)}{\sin x} \le \left|\frac{x^2}{\sin x}\right|$

$\lim_{x\to 0}\frac{x^2 \to 0}{\sin x \to 0} = \frac{2x}{\cos x} = 0$ (L'Hôpital)

By Sandwich Theorem.

$L_3 \lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin x} = 0$

Summary

Methodology.

$L = \lim f(x)^{g(x)}$

$= \lim e^{\ln f(x)^{g(x)}}$

$= e^{\lim \ln f(x)^{g(x)}}$

since e^{y} is continuous on R

$L_4 = \lim_{x\to 0} (\sin x)^{\frac{1}{\ln x}}$

$= \lim_{x\to 0} e^{\ln(\sin x)^{\frac{1}{\ln x}}} = e^{\lim_{x\to 0}\ln(\sin x)^{\frac{1}{\ln x}}}$ since e^x is continuous on R

$= e^{\lim_{x\to 0}\frac{\ln \sin x \to 0}{\ln x \to 0}}$ (L'Hôpital's Rule)

$= e^{\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{\frac{1}{x}}}$

$= e^{\lim_{x\to 0}\frac{x\cos x}{\sin x \to 0}}$ (L'Hôpital's Rule Again)

$= e^{\lim_{x\to 0}\frac{\cos x - x\sin x}{\cos x}}$

$= e^{\lim_{x\to 0} 1}$

$= e$

$L_5 = \lim_{x\to 0^+} (e^x+x)^{1/x}$

$= \lim_{x\to 0^+} e^{\ln(e^x+x)^{1/x}}$

$= e^{\lim_{x\to 0^+}\frac{\ln(e^x+x)\to 0}{x \to 0}}$ (L'Hôpital's Rule)

$= e^{\lim_{x\to 0^+}\frac{e^x+1}{e^x+x}}$

$= e^2$

$L_6 = \lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{x}{x-1}\right)$

$= \lim_{x\to 1^+} \frac{x-1-x\ln x \to 0}{\ln x \cdot (x-1) \to 0}$ L'Hop

$= \lim_{x\to 1^+} \frac{-\ln x \to 0}{\ln x + 1 - \frac{1}{x} \to 0}$ L'Hop again

$= \frac{-\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{-x}{x+1} = -\frac{1}{2}$

Summary

Ack

(161) Show that, the integral $\int_1^{\infty} x e^{-x} dx$ converges and determined it's value.

Integrating by part

$$\int_1^L x e^{-x} dx = [x \cdot e^{-x}]_1^L - \int_1^L (-e^{-x}) dx$$
$$= -L e^{-L} + e^{-1} - [e^{-x}]_1^L$$
$$= 2e^{-1} - L e^{-L} - e^{-L}$$

$$\lim_{L\to\infty} \int_1^L x e^{-x} dx = \lim_{L\to\infty} (2e^{-1} - Le^{-L} - e^{-L}) = 2e^{-1}$$

Ack

(162) Use the Comparison Test to show that $\int_1^{\infty} \frac{\cos x + \sin x}{2x^4} dx$ converges

$$\int_1^{\infty} \frac{\cos x + \sin x}{2x^4} \le \int_1^{\infty} \frac{|\cos x| + |\sin x|}{2x^4} \le \int_1^{\infty} \frac{2}{2x^4} = \int_1^{\infty} \frac{1}{x^4}$$

since $\int_1^{\infty} \frac{dx}{x^4}$ converges.

Ack

(163) Does $\int_1^{\infty} \frac{x}{\sqrt{x-1+x^6}} dx$ diverges ?

$$\int_1^{\infty} \frac{x}{\sqrt{x-1+x^6}} \le \int_1^{\infty} \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$$

when $x \ge 1$

$x - 1 + x^6 \ge x^6$

since $\int_1^{\infty} \frac{dx}{x^2}$ converges

By C.T.

$\int_1^{\infty} \frac{x}{\sqrt{x-1+x^6}}$ diverges.

Summary

5

This is based on the contraction's definition.

(164) Let $I=[a,b]$ be a closed interval. Show that if the function $F: I \to I$ and $G: I \to I$ are both contraction, then the composition function $H = F \circ G$ is also a contraction.

- Since F is contraction: there exist $k \in (0,1)$ s.t. $\boxed{|F(x) - F(y)| \le k|x-y|}$ for all $x, y \in I$.
- Since G is a contraction, there exist $L \in (0,1)$ s.t. $\boxed{|G(x) - G(y)| \le L|x-y|}$ for all $x, y \in I$.

So:
$$
\begin{aligned}
|H(x) - H(y)| &= |F(G(x)) - F(G(y))| \\
&\le k\,|G(x) - G(y)| \\
&\le kL\,|x-y|
\end{aligned}
$$
for all $x, y \in I$. So H is a contration beause $0 < kL < 1$.

(165) Does the integral $\int_0^\infty \frac{\cos x}{2+\sin x}\,dx$ converges or diverge?

$$
\begin{aligned}
\int_0^\infty \frac{\cos x}{2+\sin x}\,dx &= \Big[\ln(2+\sin x)\Big]_0^L \\
&= \ln(2+\sin L) - \ln 2
\end{aligned}
$$

if $L = 2n\pi \; (n \in N) \;\to\; 0$

if $L = (2n+\frac{1}{2})\;(n \in N) \;\to\; \ln(3/2)$

Thus the ~~limit~~ integral $\boxed{\text{diverges.}}$

Summary

Contraction : Let $I=[a,b]$ and $f: I \to R$ be a function. Suppose there is number $0 < k < 1$ s.t.

$|f(x) - f(y)| \le k|x-y|$ for all $x, y \in I$.

f is called a contraction on I.

5

Let $S_n \to L$ as $n \to \infty$.

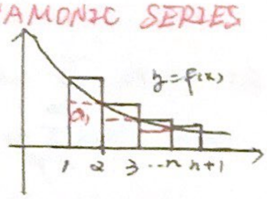
$a_n = S_n - S_{n-1}$

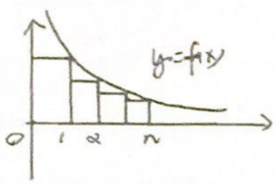
$= L - L$

$= 0$ as $n \to \infty$

PAY ATTATION TO

HAMONIC SERIES





Convergence Test. Summary

$\boxed{\text{1. The divergence test}}$

If the series $\sum_{k=p}^{\infty} a_k$ converges, then $a_n \to 0$ as $n \to \infty$.

Contrapositive: If a_k does not tend to 0, the $\sum a_k$ must diverge.

$\boxed{\text{2. The Integral test}}$

Let $p \in \mathbb{Z}$ and let $f: [p, \infty) \to \mathbb{R}$ be continuous non-negative and non-increasing function.

$\int_p^{\infty} f(x)\,dx$ $\sum_{k=p}^{\infty} f(k)$ either both convergent or both divergent.

$$\begin{cases} \int_1^{n+1} f(x) \le a_1 + a_2 + \cdots + a_n \\ a_2 + a_3 + \cdots + a_n \le \int_1^n f(x) \end{cases}$$

$$\int_1^{n+1} \le \sum_{k=1}^{n} a_k \le \int_1^{n} + a_1$$

$\boxed{\text{3. Absolute Convergence}}$

If $\sum_{k=p}^{\infty} |a_k|$ converges, then $\sum_{k=p}^{\infty} a_k$ converges.

$\boxed{\text{4. The Comparison Test}}$

Let $\sum_{k=p}^{\infty} a_k$ be a series. Suppose we have an integer $q \geq p$. a real number $M > 0$, and a convergent series $\sum_{k=q}^{\infty} b_k$ s.t. $|a_k| \le M b_k$ for $k \geq q$

Then $\sum_{k=p}^{\infty} a_k$ converges absolutely.

Summary

Choose $q = p$: $\sum_{p}^{\infty} a_k \le \sum_{p}^{\infty} |a_k| \le M \sum_{p}^{\infty} b_k$.

5 . .

5 The Limit Comparison Test

Let $(a_k)_{k\geq p}$ and $(b_k)_{k\geq p}$ be sequences s.t. a_k $b_k > 0$ for all k and s.t. $\lim_{k\to\infty} a_k/b_k = L \in (0,\infty)$ exists. The $\sum_{k=p}^{\infty} a_k$ and $\sum_{k=p}^{\infty} b_k$ either both convergent or both divergen.

6. The Ratio Test

Let $(a_k)_{k\geq p}$ be a sequence s.t. $a_k \neq 0$ for all k and s.t. $\lim_{k\to\infty} |a_{k+1}/a_k| = L$ exists. Then $\sum_{k=p}^{\infty} a_k$

(i) Diverges if $L > 1$

(ii) Converges absolutely if $L < 1$

(iii) no conclusion $L = 1$.

7. The Alternating series test

Let $(b_k)_{k\geq p}$ be a real sequence satisfying

(i) $b_k \geq 0$ for all $k \geq p$ non-negative

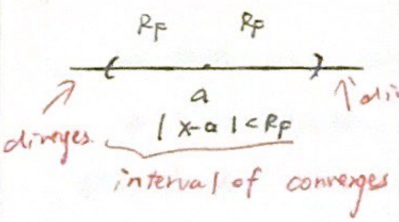
(ii) $b_{k+1} \leq b_k$ for all $k \geq p$ non-increasing

(iii) $b_k \to 0$ as $k \to \infty$ $b_k \to 0$.

Then $\sum_{k=p}^{\infty} (-1)^{k+1} b_k$ converges.

8. Radius of Convergence

combine with ratio test



Let $F(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ be a power series

Then there is R_F with $0 \leq R_F \leq \infty$, the radius of convergence, with:

(i) if $|x-a| > R_F$, then $F(x)$ diverges

(ii) if $|x-a| < R_F$, then $F(x)$ converges absolutely

(iii) if $0 < R_F < \infty$ and $|x-a| = R_F$. the $F(x)$ may converge or diverge.

Summary

5 . .

$\boxed{\text{Lemma}}$: Suppose that $|x-a|<|X-a|$ and $F(X)=\sum_{k=0}^{\infty} c_k(X-a)^k$ converges. Then $F(x)=\sum_{k=0}^{\infty} c_k(x-a)^k$ converges absolutely.

Example: $G(x)=\sum_{k=1}^{\infty} \frac{x^k}{k}$

Use ratio test to find Radius:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{x^{k+1}}{k+1}\right)\left(\frac{k}{x^k}\right)\right| = |x|\frac{k}{k+1} = |x|\left(1-\frac{1}{k+1}\right) \xrightarrow{k\to\infty} |x|$$

$$\begin{cases} \text{Diverges if } |x|>1 \\ \text{Converges if } |x|<1 \end{cases} \Rightarrow R_G = 1.$$

$x=1$ $G(1)=\sum_{k=1}^{\infty}\frac{1^k}{k}$ divergent

$x=-1$ $G(-1)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ convergens. (Lebniz Series)

Thus $G(x)$ converges precisely for $-1\le x<1$

$$G'(x)=\sum_{k=1}^{\infty}\frac{d}{dx}\left(\frac{x^k}{k}\right)=\sum_{k=1}^{\infty}x^{k-1}=\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$$

Integrating $G(x) = G(x)-G(0) = \int_0^x G'(t)\,dt = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x)$

$x=-1$. $\sum_{k=1}^{\infty}\frac{(-1)^k}{k} = -\ln 2$.

$\therefore \ln 2 = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$

Example: $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int \sum_{k=0}^{\infty}(-t^2)^k\,dt$ swapped the integral and sum.

$$= \sum_{k=0}^{\infty}\int_0^x (-t^2)^k\,dt$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

$\pi/4 = \arctan 1$

$\pi/4 = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$

Summary

(166) For which of the parameter b does

$$F(x)=\begin{cases} b\cos \pi x & \text{if } x\le -2 \\ \dfrac{3}{b-|x|} & \text{if } x>-2. \end{cases}$$

define a continuous function on R

(a) Only when $b=-1$.

(b) Only when $b=3$.

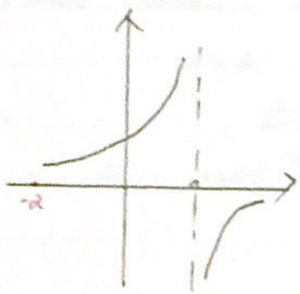
(c) Only when $b=1$ or $b=3$

(d) Only when $b=-3$ or $b=1$

(e) None of the above.

$b\cos(-2\pi) = b = \frac{3}{b-2}$ $\quad b=3$ or $b=-1$

However. $b=3$.



$$F(x)=\frac{3}{3-|x|}=\begin{cases} \dfrac{3}{x+3} & -2<x<0 \\ \dfrac{3}{3-x} & x\geq 0 \end{cases}$$

has asymptote at $x=3$. not continous.

Summary

5

ACF.

Sequence.

Definition: Index list of mathematical Object.

We are mostly interested in the behaviour of the sequence as n gets large.

Convergent Sequence.

For every positive real number ε, there exist an integer $n_0(\varepsilon)$ s.t. $|x_n - \alpha| < \varepsilon$ for all integers $n \geq n_0(\varepsilon)$.

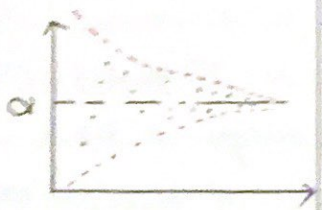
$\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon)$, s.t. $|x_n - \alpha| < \varepsilon$ $\boxed{\forall\, n \geq n_0(\varepsilon)}$.

Algebra of Limit

$x_n \to \alpha \quad y_n \to \beta$

(i) $(x_n + y_n) \to \alpha + \beta$ (ii) $(x_n y_n) \to \alpha\beta$ (iii) $(|x_n|) \to |\alpha|$

(iv) $\beta \neq 0$, $(1/y_n) \to (1/\beta)$.



The Sandwich Theorem (Squeeze Theorem)

Suppose a_n b_n c_n are real sequences s.t.

$a_n \leq b_n \leq c_n$ for each n.

Suppose a_n, c_n both $\to \alpha \in R$, then $b_n \to \alpha \in R$

$\boxed{\text{Lemma}}$: For every $s > 0$ and $n \in N$, $\ln n < \frac{n^s}{s}$.

proof: $\ln n = \ln n - \ln 1 = \int_1^n \frac{1}{x} dx \leq \int_1^n \frac{x^s}{x} \leq \int_1^n x^{s-1} dx$

$\boxed{\ln n} \leq \int_1^n x^{s-1} dx = \frac{n^s}{s} - \frac{1}{s} < \boxed{\frac{n^s}{s}}$

$\ln n \leq \frac{n^s}{s} \quad s > 0$

Summary

Ratio Test for sequence.

Let (x_n) be a real sequence s.t. each x_n is nonzero. Suppose that

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = L < 1$$

Then (x_n) converges to 0.

proof: let $s<1$ & $L<s<1$, For large n. $n \geq Q$

$$\left|\frac{x_{n+1}}{x_n}\right| \leq s$$

$$|x_{Q+1}| \leq s|x_Q|, \quad |x_{Q+2}| \leq s|x_{Q+1}| \leq s^2|x_Q|$$

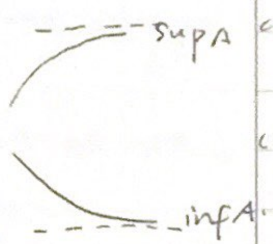
$\therefore \quad |x_{Q+k}| \leq s^k|x_Q|$

$\therefore \quad \overset{\to 0}{0} \leq |x_n| \leq s^{n-Q}|x_Q| = s^n \cdot \frac{|x_Q|}{s^Q} \overset{\to 0}{}$

Since $|s|<1$. $\lim_{n\to\infty} s^n \frac{|x_Q|}{s^Q} = 0$.

By S. T. $x_n \to 0$.

Monotone Sequence Theorem.



(a) (x_n) non-decreasing for $n \geq N$. If set $A = \{x_n \mid n \geq N\}$ is bounded above, then (x_n) converges to sup A.

(b) (x_n) non-increasing for $n \geq N$. If set $A = \{x_n \mid n \geq N\}$ is bounded below, then (x_n) converges to inf A.

Summary

5 . . .

A&F

Very classical example.

(167) Prove $y_n = n^{1/n}$ converges to 1.

let $n^{1/n} = 1+\alpha_n > 1$ for $n \geq 2$. $\alpha_n > 0$

$n = (1+\alpha_n)^n \geq 1 + n\alpha_n + \frac{n(n-1)}{2}\alpha_n^2 \geq \frac{n(n-1)}{2}\alpha_n^2$

for $n \geq 2$. $0 < \alpha_n \leq \sqrt{\frac{2}{n-1}} \leq \frac{2}{\sqrt{n}}$.

By S.T. $\lim_{n\to\infty} \alpha_n \Rightarrow 0$ $n^{1/n} \to 1$.

$\boxed{\text{Series Convergence}}$

A&F

(168) (i) $\sum_{k=1}^{\infty} \frac{2^k+4}{2^k+2012}$ (ii) $\sum_{k=1}^{\infty} k^{2/k}$.

(i) $\lim_{k\to\infty} \frac{2^k+4}{2^k+2012} \to 1 \neq 0$ as $k\to\infty$. Divergent.

Divergent Test.

(ii) $\lim_{k\to\infty} k^{2/k} = \lim_{k\to\infty} (k^{1/k})^2 \to 1 \neq 0$ Divergent

$\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow a_k \to 0$

We used the contrapositive of divergent Test.

(169) $\sum_{k=1}^{\infty} k^{-1/2} + k^{-4}$

Analysis: $\sum_{k=1}^{\infty} k^{-1/2}$ is divergent $\sum_{k=1}^{\infty} k^{-4}$ convergent. (P-test)

$\therefore$ Suppose $\sum_{k=1}^{\infty} (k^{-1/2} + k^{-4})$ is Convergent.

Prove by Contradiction

Algebra of limits

$\sum_{k=1}^{\infty} [(k^{-1/2} + k^{-4}) - k^{-4}]$ must be convergent.

However. $\sum_{k=1}^{\infty} k^{-1/2}$ is divergent. , Contradition

$\therefore$ The given is divergent.

Extra Example from lecture notes:

$\sum_{k=1}^{\infty} (2^{-k} + 1/k)$ divergent. because otherwise we would conclude that $\sum_{k=1}^{\infty} 1/k = \sum_{k=1}^{\infty} (2^{-k}+1/k) - \sum_{k=1}^{\infty} 2^{-k}$ convergent.

Summary

which is false.

5 · A<P

Alternating test

Three Conditions.

(17a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1/4)}$

let $b_k = \frac{1}{k-1/4}$

since: 1. $b_k = \frac{1}{k-1/4}$ is non-negative

2. non-increasing

3. $b_k \to 0$ as $k \to \infty$ for all $k \geq 1$

By Alternating series test. the given is convergent

Ratio Test

(17) (i) $\sum \frac{k!}{k^k}$ (ii) $\sum \frac{\ln k}{k^2}$

(i) $\lim_{k\to\infty} \left| \frac{(k+1)!}{(k+1)^{(k+1)}} \cdot \frac{k^k}{k!} \right| = \lim_{k\to\infty} \frac{(k+1)\, k^k}{(k+1)^{k+1}}$

$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k$

$= \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}}\right)^k$

$= \frac{1}{e} < 1$

Facts: $e = \lim_{n\to\infty} (1+\frac{1}{n})^n$

$\frac{1}{e} = \lim_{n\to\infty} \left(\frac{1}{(1+\frac{1}{n})}\right)^n$

(ii) $\lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0$ as $k \to \infty$ can be used as fact.

since $\left| \frac{\ln k}{k^2} \right| \leq \frac{1}{k^{3/2}}$ $\sum \frac{1}{k^{3/2}}$ converges p-test

$\Rightarrow \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges.

Summary

ACF

Ratio test

Radius and Convergent Series

172 (i) $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ (ii) $\sum_{k=1}^{\infty} \frac{(\ln x)^k}{k^2}$ (iii) $\sum_{k=0}^{\infty} \frac{x^k}{2^k(k^2+1)}$

(i) $\lim_{x\to\infty} \left| \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} \right| = \lim_{x\to\infty} |x| \left(\frac{k}{k+1}\right)^2$

$= \lim_{x\to\infty} |x| \cdot \left(1 - \frac{1}{k+1}\right)^2$

$= |x|$

- Converges if $|x| < 1$
- diverges if $|x| > 1$

Specifically, $x = 1$. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. (p-test)

$x = -1$ $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ 1. non negative, 2. non-increasing

3. $\frac{1}{k^2} \to 0$ Converges (Alternating)

$\therefore$ $|x| \le 1$ Converges.

(ii) $\lim_{k\to\infty} \left| \frac{(\ln x)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(\ln x)^k} \right| = \left(\frac{k}{k+1}\right)^2 |\ln x|$

$= |\ln x|$.

- Converges if $|\ln x| < 1$
- Diverges if $|\ln x| > 1$

Specially, $x = e$. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

$x = \frac{1}{e}$ $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges (Alternating Series Test)

$\therefore$ Converges if $|\ln x| \le 1 \iff e^{-1} \le x \le e$

(iii) Centre is 0; For $x \ne 0$ we set $a_k = x^k / 2^k(k^2+1)$

$\lim_{k\to\infty} \left| \frac{x^{k+1}}{2^{k+1}((k+1)^2+1)} \cdot \frac{2^k(k^2+1)}{x^k} \right| = \lim_{k\to\infty} |x| \frac{k^2+1}{2[(k+1)^2+1]} \to \frac{|x|}{2}$

- Converges if $\frac{|x|}{2} < 1 \iff |x| < 2$
- diverges if $\frac{|x|}{2} > 1 \iff |x| > 2$

Summary

$|x| = 2$ $\begin{cases} x = 2 & \sum \frac{1}{k^2+1} \text{ converges. (C.T.)} \\ x = -2 & \sum \frac{(-1)^k}{k^2+1} \text{ converges (A.T.)} \end{cases}$

$\therefore$ Converges for $|x| \le 2$.

5

Extra Example: $\sum_{k=0}^{\infty} \frac{(x+1)^k}{2^k(k^2+1)}$

Comparing ~~them~~: $\sum_{k=0}^{\infty} \frac{1}{2^k(k^2+1)} \cdot x^k$ $\quad a=0$

$\sum_{k=0}^{\infty} \frac{1}{2^k(k^2+1)} (x+1)^k$ $\quad a=-1$

$$\boxed{\sum_{k=0}^{\infty} c_k (x-a)^k}$$

Converges if $|x+1| \le 2$ $\quad -3 \le x \le 1$.

171 <-> 173 $\sum_{k=1}^{\infty} \frac{\ln kx}{k^2}$

$\sum_{k=1}^{\infty} \frac{\ln k + \ln x}{k^2} = \underbrace{\sum_{k=1}^{\infty} \frac{\ln k}{k^2}}_{\text{con.}} + \ln x \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\text{con.}}$

$$\boxed{\frac{\ln k}{k^2} \le \frac{1}{k^{3/2}}}$$

174 $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k}$

since $k \ge 1$ $\quad$ Since $\frac{e^{1/k}}{k} \ge \frac{1}{k}$, the $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

$\therefore \sum_{k=1}^{\infty} \frac{e^{1/k}}{k}$ divergent (C.T.).

Summary

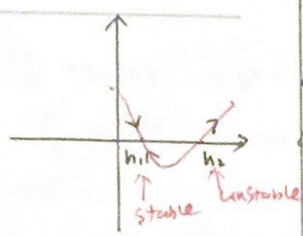
APP Seminar Summary.

Q2.2. $\frac{dh}{dt} = f(h)$

(a) $f(h) = k(h-h_1)(h-h_2)$ $\quad \frac{L}{T} = [k]\cdot L^2 \quad [k] = \frac{1}{LT} = L^{-1}T^{-1}$

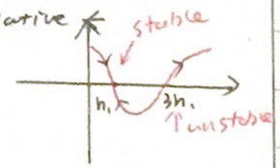
(b) $f(h) = k\cos(\underbrace{\pi h / 2h_1}_{\text{dimensionless}=1})$ $\quad [k] = \frac{L}{T}$

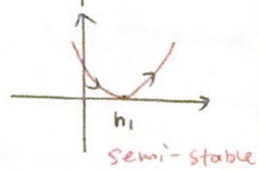
(c) $f(h) = k(\underbrace{\cosh(1-\frac{h}{h_1})-1}_{\text{dimensionless}})$ $\quad h=h_1,\ f(h_1)=0$ $\quad [k] = \frac{L}{T} = LT^{-1}$



positive →

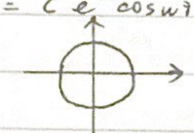
negative





Q.3.19. $x(t) = (e^{kt}\cos wt,\ e^{kt}\sin wt,\ 0)$

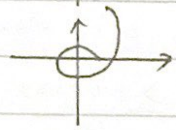
① $k=0$



$x = (\cos wt, \sin wt, 0)$

$x^2+y^2=1$

② $k>0$



spiral that expands

③ $k<0$

spiral that shrinks.

Q3.22 A particle moves in 3 dimensions at constant (nonzero) speed and nonzero acceleration. Show that its velocity and acceleration vector must be perpendicular to one another.

Constant Speed: $s = |\underline{v}| = \text{constant}$

$\underline{v}\cdot\underline{v} = \text{constant}$

$\frac{d}{dt}(\underline{v}\cdot\underline{v}) = 0$

$\underline{a}\cdot\underline{v} + \underline{v}\cdot\underline{a} = 0$

$\Rightarrow 2\underline{a}\cdot\underline{v} = 0$

$\underline{a}\cdot\underline{v} = 0 \qquad a \perp v.$

Summary

The Auxiliary Angle formula.

$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\varphi)$ $\quad \tan\varphi = \frac{b}{a}$

$a\sin x + b\cos x = \sqrt{a^2+b^2}\cos(x-\varphi)$ $\quad \tan\varphi = \frac{a}{b}$

$a\sin x + b\cos x = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$

let $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}$

$\sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$

$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x + \arctan\frac{b}{a})$

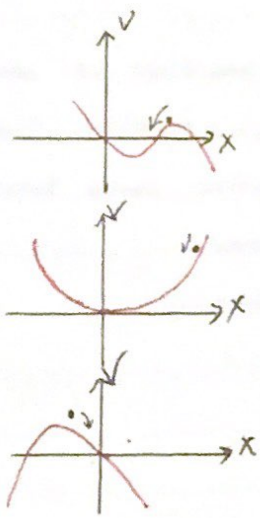
Hooke's law:

$T = \frac{\lambda(l-l_0)}{l_0}$

Conservative force:

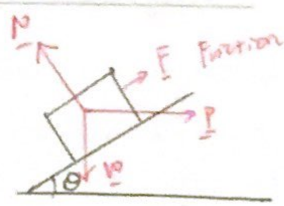
f is a function of x

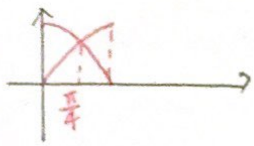
Stable point in Potential Energy



Summary

5





$F \le \mu |N|$

$w(\sin\theta - \cos\theta) \le \mu w(\sin\theta + \cos\theta)$

$\boxed{\sin\theta - \cos\theta} \le \mu(\sin\theta + \cos\theta)$

Case 1: $0 \le \theta \le \frac{\pi}{4}$ $\cos\theta \ge \sin\theta$

$\Rightarrow \cos\theta - \sin\theta \le \mu(\sin\theta + \cos\theta)$

$\Rightarrow (1-\mu)\cos\theta \le (1+\mu)\sin\theta$

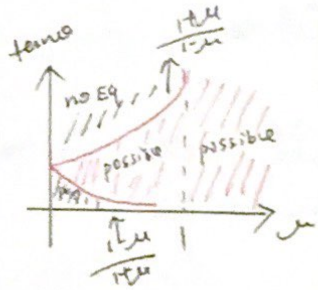
$\tan\theta \ge \frac{1-\mu}{1+\mu}$ for $\mu \ge 1$

Case 2: $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$ $\sin\theta \ge \cos\theta$

$\sin\theta - \cos\theta \le \mu(\sin\theta + \cos\theta)$

$(1-\mu)\sin\theta \le (1+\mu)\cos\theta$ Always true for $\mu \ge 1$

$\tan\theta \le \frac{1+\mu}{1-\mu}$ for $0 \le \mu < 1$

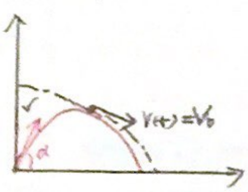


Summary:

if $\mu \ge 1$: equilibrium is always possible.

if $0 \le \mu < 1$ $\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}$

Enveloping parabola: $y = \frac{V^2}{2g} - g \cdot \frac{x^2}{2V^2}$



$V \cdot v_0 = 0$

$$\begin{cases} x = V\cos(\alpha)\, t \\ y = V\sin(\alpha)\, t - \frac{1}{2} g t^2 \end{cases}$$

Path meets: $\boxed{V\sin(\alpha)\, t - \frac{1}{2} g t^2 = \frac{V^2}{2g} - \frac{g(V\cos(\alpha)\, t)^2}{2V^2}}$

$V\sin(\alpha)\cdot t - \frac{1}{2} g \cdot t^2 = \frac{V^2}{2g} - \frac{g}{2}\cos^2\alpha \cdot t^2$

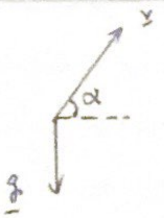
提 $\frac{1}{2}g$

$\frac{1}{2} g \left\{ \sin^2(\alpha)\, t^2 - \frac{2V}{g}\sin(\alpha)\, t + \frac{V^2}{g^2} \right\} = 0$

$\sin\alpha \cdot t - \frac{V}{g} = 0$ $t = \frac{V}{g\sin\alpha}$ (repeated roots for path at the boundary)

Summary

5



Velocity of projectile is:

$v(t) = \underline{V} + \underline{g}t \qquad t = \frac{V}{g\sin\alpha}$

$= \underline{V} + \underline{g} \cdot \frac{V}{g\sin\alpha}$

$\underline{V} \cdot \underline{g} = gV\cos(\alpha + \frac{\pi}{2}) = -gV\sin(\alpha)$

$\underline{v} \cdot \underline{v}_0 = V^2 + (-gV\sin\alpha) \cdot \frac{V}{g\sin\alpha}$

$= V^2 - V^2 = 0 \qquad v \perp v_0.$

Conservation of momentum: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ ①

Collision law: $v_2 - v_1 = e(u_1 - u_2)$ ②

除以 m.

① ÷ m_1 $\quad u_1 + \frac{m_2}{m_1}u_2 = v_1 + \frac{m_2}{m_1}v_2$

$v_2 - v_1 = eu_1 - eu_2$

① + ② $\quad (1+e)u_1 + (\frac{m_2}{m_1} - e)u_2 = (\frac{m_2}{m_1} + 1)v_2 \qquad \times m_1$

$m_1(1+e)u_1 + (m_2 - m_1e)u_2 = (m_2 + m_1)v_2$

$v_2 = \frac{m_1(1+e)u_1 + (m_2 - m_1e)u_2}{m_1 + m_2}$

$v_1 = v_2 - e(u_1 - u_2)$

$= \frac{m_1(1+e)u_1 + (m_2 - m_1e)u_2}{m_1 + m_2} - \frac{e(u_1 - u_2)(m_1 + m_2)}{m_1 + m_2}$

$= \frac{(m_1 - em_2)u_1 + (1+e)m_2u_2}{m_1 + m_2}$

Special case: if $e = 0$ Perfect plastic collision.

$v_1 = v_2 = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$ (Average Velocity)

Summary

5

Special Case: Equal mass. $m_1 = m_2 = m$.

$$v_1 = \frac{(m - em)u_1 + (1+e)mu_2}{m_1 + m_2} = \frac{1}{2}\{(1-e)u_1 + (1+e)u_2\}$$

Similarly: $v_2 = \frac{1}{2}\{(1+e)u_1 + (1-e)u_2\}$

Special Case: Perfectly elastic collision $(e=1)$ and $m_1 = m_2$

$V_1 = u_2 \quad V_2 = u_1$ exchange velocities

Special Case: $m_2 >> m_1$

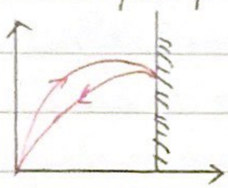
$$v_1 = \frac{(\frac{m_1}{m_2} - e)u_1 + (1+e)u_2}{\frac{m_1}{m_2} + 1}$$

$$v_2 = \frac{(1+e)\frac{m_1}{m_2}u_1 + (1 - \frac{m_2}{m_1}e)u_2}{\frac{m_1}{m_2} + 1}$$

$\underrightarrow{u_1} \quad u_2 = 0$

$v_1 = -eu_1 \quad v_2 = 0$

Example of throwing a ball at a wall



时间相同

the key step is to let the time equal. (x direction & y di.) remember to take the absolute value of the velocity coming back.

~~Osc~~ ~~ttation~~ spring.

if I use the total length, I got a inhomogenous equation ; If I use displacement from the equilibrium point, I got a homogenous equation. and $l = x(t) + l_1$, l_1 is the equilibrium length.

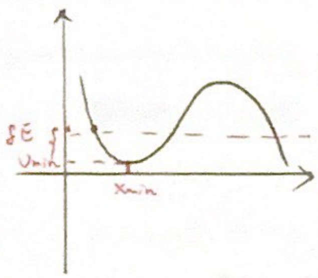
Summary

For Bouyancy Model. it's very important to find the equilibrium situation, since we need to use the equation of equilibrium to cancel the remaining terms in our ODE, as well as the term contains $\rho_i e_w$.

$$\ddot{x}' + \frac{Ag}{V_1} x = 0.$$

Definition of potential Energy:

$$f(x) = -\frac{dV(x)}{dX}.$$



$$x = x_{min} + \bar{x} \qquad |\bar{x}| << 1$$

$$E = V_{min} + \delta E$$

$$\frac{1}{2} m \dot{x}^2 + V(x) = E$$

$$\frac{1}{2} m \left(\frac{d\bar{x}}{dt}\right)^2 + V(x_{min} + \bar{x}) = V_{min} + \delta E$$

Taylor Expansion at $x = x_{min}$

$$V(x_{min} + \bar{x}) = V(x_{min}) + V'(x_{min})\bar{x} + \frac{1}{2} V''(x_{min})\bar{x}^2$$

$$\frac{1}{2} m \left(\frac{d\bar{x}}{dt}\right)^2 + \cancel{V(x_{min})} + \underbrace{V'(x_{min})}_{=0}\bar{x} + \frac{1}{2} V''(x_{min})\bar{x}^2 = \cancel{V_{min}} + \delta E$$

$$\frac{1}{2} m \left(\frac{d\bar{x}}{dt}\right)^2 + \frac{1}{2} V''(x_{min})\bar{x}^2 = \delta E$$

Differentiate it against $\bar{x}$

$$m \frac{d^2\bar{x}}{dt^2} + V''(x_{min})\bar{x} = 0$$

$$\frac{d^2\bar{x}}{dt^2} + \frac{V''(x_{min})}{m}\bar{x} = 0$$

$$T = 2\pi\sqrt{m / V''(x_{min})} \qquad A = \sqrt{\delta E / V''(x_{min})}$$

Summary

Amplitude. $\dot{x} = 0$

5

Dimension Analysis:

when you get the answer, do not forget to multiplied by a $\boxed{\text{constant } K}$

Energy: $[E] = ML^2T^{-2}$

Power: $[P] = ML^2T^{-3}$ $\quad P = \frac{W}{T}$

Logistic Model and Harvest Model.

(17) A small lake in Miyabo can support a school of 1000 fish whose population, represented by $f(t)$, has a linear growth rate of 10% per year. Fishing occurs at a rate of $h(t)$ per year. An appropriate logistic growth model for the fish population is:

$$\frac{df}{dt} = \frac{1}{10} f \left(1 - \frac{f}{1000}\right) - h$$

1) what is the maximum number of fish that can be harvested by fishermen each year without eventually leading to the extinction of the fish population. ⇐ real solution

2) How many fish can be harvested each year to maintain the population size at 100? ⇐ the equilibrium point is 100

3) What is the minimum initial fish population if 21 fish are harvested each year and the long-term fish population is maintained at a constant level? ⇐ remain constant level hens starting at the fixpoint

Summary