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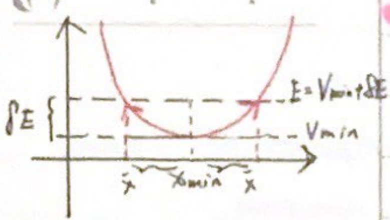

Small Amplitude Motion in a potential well

Equation of the motion:

$\frac{1}{2} m \dot{x}^2 + V(x) = E$

$\frac{dx}{dt} = \frac{d}{dt}(x_{min} + \bar{x}) = \frac{d\bar{x}}{dt}$

$x = x_{min} + \bar{x} \quad |\bar{x}| \ll 1$

$\frac{1}{2} m (\frac{d\bar{x}}{dt})^2 + V(x_{min} + \bar{x}) = V_{min} + \delta E$

Taylor Expansion. close to $x = x_{min}$.

$V(x) = V(x_{min} + \bar{x}) \approx V(x_{min}) + V'(x_{min})\bar{x} + \frac{1}{2} V''(x_{min})\bar{x}^2$

Since $V(x_{min}) = V_{min} \quad V'(x_{min}) = 0$

$V(x) \approx V_{min} + \frac{1}{2} V''(x_{min})\bar{x}^2$

$m(\frac{dx}{dt})^2 + V''(x_{min})\bar{x}^2 = \delta E$

differentiate t $\Rightarrow$ $m \frac{d^2\bar{x}}{dt^2} + V''(x_{min})\bar{x} = 0$

$(\dot{x}^2)' = 2\dot{x} \cdot \ddot{x}$

$(\bar{x}^2)' = 2\bar{x} \cdot \dot{\bar{x}}$

$\omega = \sqrt{\frac{1}{m} \frac{d^2 V(x_{min})}{d\bar{x}^2}}$

Drag - Nonconservative Force.

该模型的研究对象为 Kinetic Energy, K.

$f = -\alpha\dot{x}, \ \alpha > 0$ not a function of x.

$f = ma = -\alpha\dot{x}$

$m\frac{dv}{dt} = -\alpha v \quad (\cdot v)$

$\frac{d}{dt}(\frac{1}{2} m v^2) = -\alpha v^2$

$\boxed{\frac{dk}{dt} = -\alpha v^2 < 0}$

Kinetic Energy $\downarrow \Rightarrow$ heat.

Heat = Kinetic energy of Molecules

In the long term:

$\frac{dk}{dt} = -\alpha v^2 = -\alpha v^2 \cdot \frac{2m}{2m} = -\frac{2\alpha}{m} \cdot \frac{1}{2} m v^2 = -\frac{2\alpha}{m} K$

$\boxed{k(t) = k(0) \cdot e^{-\frac{2\alpha}{m}t}}$

$k \to 0$ as $t \to \infty$

Summary

$\boxed{\text{Taylor Expansion}}$: Generated by f at $x = a$

$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots +$

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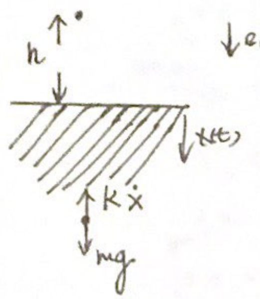


Motion with Energy

A ball bearing of mass m is dropping rest into a bucket of treacle a distance h below. Describe its motion.

- Air resistance is negligible
- In the treacle, Drag $\propto |\underline{v}|$
- Ball bearing = Point particle

$\boxed{\text{Stage 1}}$: $E = 0 + \frac{1}{2} m S^2 = mgh + 0$

$S = \sqrt{2gh}$ (At impact)

$\boxed{\text{Stage 2}}$: $mg - K\dot{x} = m\ddot{x}$ $\quad x(0) = 0 \quad \dot{x}(0) = S$

let $v(t) = \dot{x}(t)$

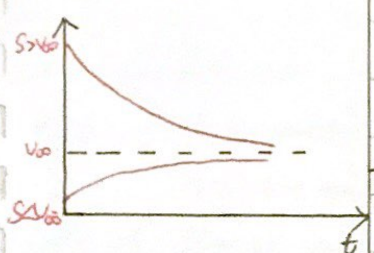
$m\dot{v} = mg - Kv \qquad v(0) = S$

By Integrating factor:

$$v = \frac{mg}{K} + \left(S - \frac{mg}{K}\right) e^{-\frac{K}{m}t} \rightarrow 0$$

$$v(t) = v_\infty + (S - v_\infty) e^{-Kt/m}$$

v_∞ is terminal velocity $v_\infty = mg/K$.



if $S > v_\infty$, the ball will slow down

if $S < v_\infty$, it will speed up.

Write: $\exp(-Kt/m) = \exp(-t/\tau)$ $\quad \tau = \frac{m}{K}$ time scale

More precisely, the difference between v and v_∞ falls by a factor of $e \approx 2.7$ in a time τ.

Summary

$\dot{x} = v(t)$,

$x = V_\infty t + \tau(S - V_\infty)(1 - e^{-t/\tau})$

$x(t) = V_\infty t + (S - V_\infty)\frac{m}{k}(1 - e^{-t/\tau})$

$\boxed{\text{At a very large time } (t >> \tau)}$

$x(t) \approx V_\infty t + (m/k)(S - V_\infty)$

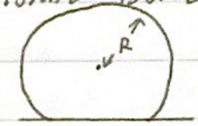
$\boxed{\text{At a very small time } (t << \tau)}$

Taylor series $e^{-t/\tau} = 1 - t/\tau$

$x(t) \approx V_\infty t + (S - V_\infty)\frac{m}{k}\left(\frac{kt}{m}\right) = St$

Dimension Analysis for solving equation.

Atomic Bomb Blast, what's the expression for the radius of the blast.



- Assume a point release of energy E
- Density of air. ρ.

Known Quantities:

Radius: $[R] = L$

Time: $[t] = T$

Density: $[\rho] = \frac{M}{L^3}$

can derivate it $E = \frac{1}{2}mv^2$

Energy: $[E] = ML^2T^{-2}$ (Joules $kg\ m^2/s^2$)

Assume: $R \propto t^a \rho^b E^c$

$[R] = [t]^a[\rho]^b[E]^c$

$L = T^a(ML^{-3})^b \cdot (ML^2T^{-2})^c$

$L = M^{b+c}L^{-3b+2c}T^{a-2c}$

I finally find that the statement like "k is very small or t is very small" can be understood that we can use Taylor series or more precisely, Maclaurin series to expand it at a=0 and we can get the result amazingly.

- Projectiles with Air resistance
- Small Amplitude with potential well.
- Motion with Energy.

Summary

Maclaurin Series: $P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \cdots$

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$M: \ 0 = b + c$
$L: \ 1 = -3b + 2c$
$T: \ 0 = a - 2c$

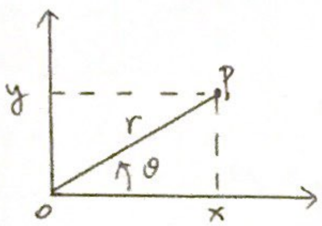
$a = \frac{2}{5}$, $b = -\frac{1}{5}$, $c = \frac{1}{5}$

$R \propto t^{\frac{2}{5}} \rho^{-\frac{1}{5}} E^{\frac{1}{5}}$

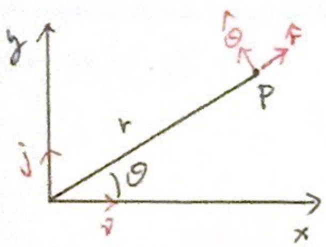
$R = A t^{\frac{2}{5}} \rho^{-\frac{1}{5}} E^{\frac{1}{5}}$ A is const.

Kinematics in polar coordinates

Cartesian coordinates x and y.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x$$



$\hat{r} = i\cos\theta + j\sin\theta \qquad \hat{\theta} = -i\sin\theta + j\cos\theta$

$$\boxed{\begin{aligned} \frac{d\hat{r}}{dt} &= -i\sin\theta\,\dot{\theta} + j\cos\theta\,\dot{\theta} = \dot{\theta}\,\hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -i\cos\theta\,\dot{\theta} - j\sin\theta\,\dot{\theta} = -\dot{\theta}\,\hat{r} \end{aligned}}$$

$\hat{r}$ $\hat{\theta}$ depend on θ

$x = r \cdot \hat{r}$

$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ 径向速度等于矢模长导数，角向速度为位矢模长乘角速度

$a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}$

$\hat{r}$: radial component

$\hat{\theta}$: azimuthal component

$\omega = \dot{\theta}$

$v = r \cdot \omega$

$\cos\phi = \frac{v \cdot a}{|v| \cdot |a|}$

The motion is subject to force F

$F = F_r\hat{r} + F_\theta\hat{\theta}$

$F_r = m(\ddot{r} - r\dot{\theta}^2)$

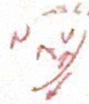
$F_\theta = \cancel{ma_\theta}\ \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = F_\theta$

Summary

圆周运动关键，半径为const. $\dot{r} = \ddot{r} = 0$. $r = R$. radial: $-R\dot{\theta}$

Azimuth: $R\ddot{\theta}$

N→(Normal to the surface)



T→(Frictional retarding force)

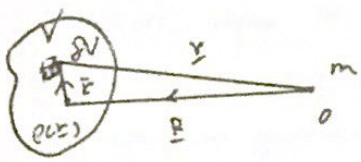
Gravity

- Gravity acts along the lines joining two particles.
- Gravity is attractive
- Obeys inverse square law
- $\frac{GMm}{R^2}$
- $\left[\frac{GMm}{R^2}\right] = [Force] = MLT^{-2}$

$[G] = L^3M^{-1}T^{-2}$ $\quad m^3\,kg^{-1}\,s^{-2}$

$G \approx 6.67\times10^{-11}\ m^3\cdot kg\cdot s^{-2}$

General central force.



$\frac{Gm\rho(r)}{|r|^2}\hat{r}\,\delta V$

$F = F_1 = \int_V \frac{Gm\rho(r)}{|r|^2}\hat{r}\,dV$

If we treat the mass as a point $r=R$

$F = F_2 = \frac{GmM}{|R|^2}\hat{R}$ $\quad , M = \int_V \rho(r)\,dV$

When V is far from the origin: $r = R + \tilde{r}$ $\quad R>>1$

$r \approx R$

$\boxed{\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0}$

conservation of angular momentum.

$r^2\dot{\theta} = h$, a constant.

mh called angular momentum.

$F = \int_V \frac{Gm\rho(r)}{|r|^2}\hat{r}\,dV \approx \frac{Gm}{|R|^2}\hat{R}\int_V \rho(r)\,dV = F_2$

$F = -\gamma m f(r)\hat{r}$.

- $\gamma = GM$
- $f(r) = r^{-2}$

Summary

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$\frac{m}{r}\frac{d}{dt}(r^2\dot\theta) = 0$ Conservation of angular momentum.

Azimuthal component ⇐ $r^2\dot\theta = h$ (const.) mh is angular momentum.

Radial Component ⇐ $\ddot r - r\dot\theta^2 = -\gamma f(r)$

$\ddot r - \frac{h^2}{r^3} + \gamma f(r) = 0$

TRICK 1 $\ddot r \cdot \dot r - \frac{h^2}{r^3}\cdot \dot r + \gamma f(r)\dot r = 0$ → chain rule

$\frac{d}{dt}(\frac{1}{2}\dot r^2) + (\frac{h^2}{2r^2})' + \int \gamma f(r)\,dr = const.$

$\frac{1}{2}m\dot r + \underbrace{m\frac{h^2}{2r^2}}_{v_\theta^2} + m\int \gamma f(r)\,dr = const$

$v = \dot r\hat r + r\dot\theta\hat\theta = \dot r\hat r + \frac{h}{r}\hat\theta$

$|v|^2 = \dot r^2 + \frac{h^2}{r^2}$

$\frac{1}{2}m(\dot r^2 + \frac{h^2}{r^2}) + m\gamma\int f(r)\,dr = const.$

$\frac{1}{2}m|v|^2 + m\gamma\int f(r)\,dr = const.$

$K + V = E$

TRICK 2 effective potential

$V^*(r) = \frac{mh^2}{2r^2} + V(r) \equiv \frac{mh^2}{2r^2} + m\gamma\int f(r)\,dr$

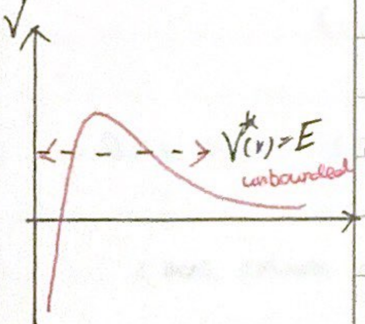
$\frac{1}{2}m\dot r^2 + V^*(r) = E$

Inverse Power Laws.

Consider $f(r) = r^{-n}$ $n = 2$

$V^*(r) = \frac{1}{2}m(h^2 r^{-2} + \frac{2\gamma}{1-n}r^{1-n})$

$\frac{dV^*}{dr} = -\frac{mh^2}{r^3} + m\gamma r^{-n} = 0$ $r = (\frac{h^2}{\gamma})^{\frac{1}{3-n}}$



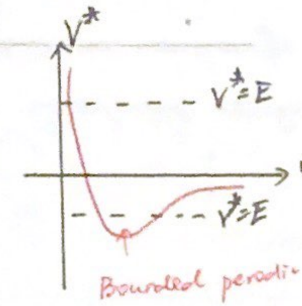
Case (i) $n \geq 3$. $V^* = \underbrace{\frac{mh^2}{2}}_{+}r^{-2} + \underbrace{\frac{m\gamma r^{1-n}}{1-n}}_{-}$

Motion always has either $r \to 0$ or $r \to +\infty$

Moon will either escape to infinite or crash into Earth in finite time

Summary

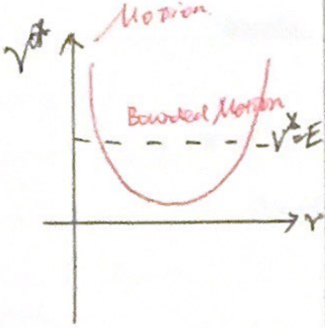
Bounded periodic is not possible



Case (ii) $1<n<3$ $\quad V^* = \underbrace{\frac{mh^2}{2}r^{-2}}_{+} + \underbrace{\frac{m\gamma r^{1-n}}{1-n}}_{-}$

$E<0$. bounded Motion $\Rightarrow$ Particle orbits

$E>0$. Unbounded Motion $\Rightarrow$ Particle escape to infinity



Case (iii) $n<1$ $\quad V^* = \underbrace{\frac{mh^2}{2r^2}}_{+} + \underbrace{\frac{m\gamma r^{1-n}}{1-n}}_{+}$

Only orbit bounded orbits is possible.

Orbit under Gravity

$f(r) = r^{-2}$ $\quad \gamma = GM$

$\ddot{r} - \frac{h^2}{r^3} + \frac{GM}{r^2} = 0$

$u(\theta) = 1/r(\theta)$

$\dot{\theta} = hu^2$

$\dot{r} = \frac{d}{dt}\left(\frac{1}{u(\theta(t))}\right) = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\boxed{\frac{d\theta}{dt}}$

$\dot{\theta} = h/r^2 = hu^2 \qquad \dot{r} = -h\frac{du}{d\theta} \qquad \ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}}$$ SHM

Solution:

$$u = \frac{GM}{h^2}(1 + e\cos(\theta - \theta_0))$$

$$u \equiv \frac{1}{r} \equiv \frac{GM}{h^2}(1 + e\cos\theta) \quad \text{with } e \geq 0$$

- If $e=0$. circle
- If $0<e<1$ ellipse. (u never reaches zero)
- If $e \geq 1$ hyperbola.

Summary

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In Cartesion coordinates

$$(1-e^2)x^2 + \frac{2h^2 e}{GM} x + y^2 = \frac{h^4}{GM}$$

$$\left(x + \frac{h^2 e}{GM(1-e^2)}\right)^2 + \frac{y^2}{1-e^2} = \frac{h^4}{G^2M^2(1-e^2)^2}$$

close to circle orbit.

$$\ddot{r} - \frac{h^2}{r^3} + \gamma r^{-n} = 0$$

The unique circular orbit is $r = r_0 = \left(\frac{h^2}{\gamma}\right)^{\frac{1}{3-n}}$

Consider orbits close to this orbit

$r = r_0 + \bar{r}$ $\quad |\bar{r}| \ll r_0$

Bionomial Expansion

If $\epsilon \ll 1$ then

$(1+\epsilon)^m = 1 + m\epsilon$

$$\frac{d^2\bar{r}}{dt^2} - h^2(r_0+\bar{r})^{-3} + \gamma(r_0+\bar{r})^{-n} = 0$$

$$\frac{d^2\bar{r}}{dt^2} - h^2 r_0^{-3}\left(1+\frac{\bar{r}}{r_0}\right)^{-3} + \gamma r_0^{-n}\left(1+\frac{\bar{r}}{r_0}\right)^{-n} = 0$$

$$\frac{d^2\bar{r}}{dt^2} - h^2 r_0^{-3}\left(1-\frac{3\bar{r}}{r_0}\right) + \gamma r_0^{-n}\left(1-\frac{n\bar{r}}{r_0}\right) \approx 0$$

Ignore the Non-Linear term

$$\ddot{\bar{r}} + \left(\frac{3h^2}{r_0^4} - \frac{\gamma n}{r_0^{n+1}}\right)\bar{r} = 0 \qquad \omega = \sqrt{3h^2 r_0^{-4} - n\gamma r_0^{-n-1}}$$

$\ddot{\bar{r}} + \omega^2 \bar{r} = 0$

Period: $P = \frac{2\pi r_0^2}{h}$ $\left(\frac{2\pi}{\dot{\theta}}\right)$

Need time to be an integer

$$\frac{2\pi r_0^2}{h} = \frac{2\pi N}{\omega} = \frac{2\pi N}{\sqrt{3h^2 r_0^{-4} - n\gamma r_0^{-n-1}}}$$

$$N = \sqrt{3-n}$$

$n = 2, -1, -6, \ldots$

Summary

In fact only $f(r) = r^{-2}$ and $f(r) = r$ have closed orbits in the full solution Betrand's Theorem 1876).

5

L'Hôpital's Rule Proof

Cauchy MVT.

Theorem: Assume f g are continuous on $[a,b]$, g is monotone. $f'g'$ exist in (a,b) $g'(x) \neq 0$ for $x \in (a,b)$ Then there exist $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

proof: Construct a function

$h(x) = [g(b)-g(a)][f(x)-f(a)] - [f(b)-f(a)][g(x)-g(a)]$

$h(x)$ is continuous in $[a,b]$ and $h'(x)$ exists (a,b)

$h(a) = h(b) = 0$. Thus $\exists c \in (a,b)$ s.t.

$h'(c) = 0$ Rolle's theorem.

$h'(c) = [g(b)-g(a)]f'(c) - [f(b)-g(a)]g'(c)$

Since $g(b)-g(a) \neq 0$ implies

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

L'Hôpital's Rules

I VERVISON Theorem: Assume f, g continuous on (a,b). $f'g'$ exist in (a,b) $g'(x) \neq 0$ in (a,b) $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ and $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$ exist

Then: $\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$

Summary

proof: $\lim \frac{f'(x)}{g'(x)} = L$ (finite)

$\forall \varepsilon > 0.\ \exists \delta.$ s.t. $x \in (a, a+\delta) \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$

Define $f(a) = g(a) = 0$, f g will be continous in $[a, x]$ for any $x \in (a, b)$

$\exists c_x \in (a, x)$ s.t. $\frac{f'(c_x)}{g'(c_x)} = \frac{f(x)}{g(x)}$ Thus.

$\forall \varepsilon > 0.\ \exists \delta$ s.t. $x \in (a, a+\delta) \Rightarrow c_x \in (a, a+\delta)$

$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(x)}{g'(x)} - L \right| \le \varepsilon$

II VERVISION | Theorem: Assume f, g continuous in (a, b)

$f' g'$ exist in (a, b) $g'(x) \neq 0$ in (a, b)

$\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^+} g(x) = \infty$ $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists

$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lim_{x \to a^+} \frac{f(x)}{g(x)}$

proof: $\forall \varepsilon > 0.\ \exists \delta x$ s.t. $x \in (a, a+\varepsilon)$

$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$

Apply the theorem on $[y, x]$ $y \in [a, x)$

$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi_y)}{g'(\xi_y)}$

Summary

Definition of Limit: $\lim_{x \to a} f(x) = L$ For $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$| f(x) - L | < \varepsilon$, whenever $0 < |x - a| < \delta$

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Observe that:

$$\frac{f(y)}{g(y)} - L = \frac{f(x) - Lg(x)}{g(y)} + \left[1 - \frac{g(x)}{g(y)}\right] \times \left[\frac{f(y) - f(x)}{g(y) - g(x)} - L\right]$$

Thus if $x, y \in (a, a + \delta_{\varepsilon/2})$

$$\left| \frac{f(y) - f(x)}{g(y) - g(x)} - L \right| = \left| \frac{f'(c_y)}{g'(c_y)} - L \right| < \frac{\varepsilon}{2}$$

Since we obtained.

$$\left| \frac{f(y)}{g(y)} - L \right| \le \left| \frac{f(x) - Lg(x)}{g(y)} \right| + \left| 1 - \frac{g(x)}{g(y)} \right| \cdot \frac{\varepsilon}{2}$$

choose $\tilde{\delta}_\varepsilon$ s.t. $y \in (a, a + \tilde{\delta}_\varepsilon) \Rightarrow$

$$\left| \frac{f(x) - Lg(x)}{g(y)} \right| < \frac{\varepsilon}{2}$$

$$\left| 1 - \frac{g(x)}{g(y)} \right| < 1$$

$$\Rightarrow \left| \frac{f(y)}{g(y)} - L \right| < \varepsilon$$

Summary

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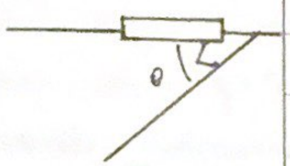
App



(132) A fire door is highed with a spring and a dashpot. Without the dashpot, the door would simply swing to-and-fro indefiretly after someone worked through it; the presence of the dashpot ensures that the door eventually closes. With the dashpot in place, the angle θ of the door opening satisfies

$m\ddot{\theta} + p\dot{\theta} + Q^2\theta = 0$ where $p>0$ represents the effects of the dashpot, and $Q>0$ the effects of the spring.

The door is initially ajar and is pushed closed (so that $\theta(0) = \theta_0 > 0$ and $\dot{\theta}(0) = -\omega_0 < 0$). Fire regulations state that the door should close without ever "opening outwords", in ~~other~~ other words without θ become regative.

(*) Under what conditions does the door close this way? Conversely, under what conditions does θ become regative at least once before the door eventually closes?

Do your answer agree with common sense?

Summary

$m\ddot{\theta} + p\dot{\theta} + Q^2\theta = 0$ $\quad \theta(0) = \theta_0 > 0$ $\quad \dot{\theta}(0) = -w_0 < 0$

let $\theta = e^{\mu t}$ $\quad \dot{\theta} = \mu e^{\mu t}$ $\quad \ddot{\theta} = \mu^2 e^{\mu t}$

$m\mu^2 + p\mu + Q^2 = 0$

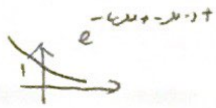
$$\mu_\pm = \frac{-p \pm \sqrt{p^2 - 4mQ^2}}{2m}$$

- $p^2 - 4mQ^2 < 0$ light damping
- $p^2 - 4mQ^2 > 0$ Heavy damping

$\mu_- < \mu_+ < 0$ since $\sqrt{p^2 - 4mQ^2} < p$

$$\begin{cases} \theta = Ae^{\mu_+ t} + Be^{\mu_- t} \\ \dot{\theta} = \mu_+ A e^{\mu_+ t} + \mu_- B e^{\mu_- t} \end{cases} \begin{cases} \theta(0) = \theta_0 \\ \dot{\theta}(0) = -w_0 \end{cases} \Rightarrow \begin{cases} A + B = \theta_0 \\ \mu_+ A + \mu_- B = -w_0 \end{cases}$$

$$B = \frac{\mu_+\theta_0 + w_0}{\mu_+ - \mu_-} \qquad A = \frac{-(\mu_+\theta_0 + w_0)}{\mu_+ - \mu_-}$$



At some finite time

$A + Be^{(\mu_- - \mu_+)t} = 0 \qquad e^{-(\mu_+ - \mu_-)t} = -\frac{A}{B} = \frac{\mu_-\theta_0 + w_0}{\mu_+\theta_0 + w_0}$

since $\mu_+ - \mu_- > 0$ so $0 < e^{-(\mu_+ - \mu_-)t} < 1$

$\therefore \quad 0 < \frac{\mu_-\theta_0 + w_0}{\mu_+\theta_0 + w_0} < 1$

$\mu_-\theta_0 + w_0 > 0 \Rightarrow w_0 > -\mu_-\theta_0$, $\mu_-\theta_0 + w_0 < \mu_+\theta_0 + w_0$ ✓

OR

$\mu_+\theta_0 + w_0 < 0 \Rightarrow w_0 < -\mu_+\theta_0$, $\mu_-\theta_0 + w_0 > \mu_+\theta_0 + w_0$ ✗

so only one senario satisfies this inequality.

if $AB<0$ if $w_0 > -\mu_-\theta_0$, door swings through $\theta = 0$

Avoid this require $w_0 < -\mu_-\theta_0$

Summary

App

Review how to solve non-homogeneous 2^{nd} ODE.

Initial condition must be used after we get the $x_{cf} + x_{p2}$.

very classical Model.

(133) A particle undergoes forced, damped simple harmonic motion, with displacement $x(t)$ satisfying

$$\ddot{x} + 6\dot{x} + 10x = \cos t$$

(a) Is the damping light or heavy?

(b) Find $x(t)$ if the particle starts from rest at the origin ($x=0$)

(c) What is $x(t)$ after long time.

$$\ddot{x} + 6\dot{x} + 10x = \cos t.$$

XCF: Let $x = e^{\mu t}$

$$\mu_\pm = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i$$

$$x_{cf} = e^{-3t}(A\cos t + B\sin t) \Rightarrow \text{Light damping}$$

Xp2: try: $x_{p2} = a\cos t + b\sin t$

$$\dot{x}_{p2} = -a\sin t + b\cos t$$

$$\ddot{x}_{p2} = -a\cos t - b\sin t \quad \text{substitute in to ODE}$$

$$\begin{cases} 9a + 6b = 1 \\ 9b - 6a = 0 \end{cases} \quad \begin{cases} a = \frac{1}{13} \\ b = \frac{2}{39} \end{cases}$$

General solution:

$$x(t) = x_{cf} + x_{p2}$$

$$x(t) = e^{-3t}(A\cos t + B\sin t) + \frac{1}{13}\cos t + \frac{2}{39}\sin t.$$

initial Condition $\quad x(0) = 0 \Rightarrow A = -\frac{1}{13}$

$\quad \dot{x}(0) = 0 \Rightarrow B = -\frac{11}{39}$

Particular solution:

$$x(t) = \underbrace{e^{-3t}\left(-\frac{1}{13}\cos t - \frac{11}{39}\sin t\right)}_{\to 0} + \frac{1}{13}\cos t + \frac{2}{39}\sin t$$

as $t \to \infty$

Summary

$$x \to \frac{1}{13}\cos t + \frac{2}{39}\sin t$$

APP

(134) A particle of mass m has position vector $x(t)\,i$ and moves subject to a force f. Which of the following forces are conservative?

(a) $f = -ma(x-X)\,i$ ✓ conservative

(b) $f = -b\dot{x}\,i$ ✗ non-conservative

(c) $f = (cx - dx^3)\,i$ ✓ conservative.

Conservative forces: f is function of $\underline{x}$.

Existence of Fixpoints

AUF

(135) Show that $p(x) = \frac{x^2}{10} + 1$ is a contraction on $[-1, 4]$ in two different ways.

Method 1: By factorising $p(x) - p(y)$

$|p(x) - p(y)| \le \frac{4}{5}|x-y|$ for all $x, y \in [-1, 4]$

Method 2: Use MVT. (How big can $p'(t)$ be?)

Contraction. (3 main points)

- $|f(x) - f(y)| \le K|x-y|$
- $K \in (0,1)$
- $f: I \to R$

1: $p(x) - p(y) = (\frac{x^2}{10}+1) - (\frac{y^2}{10}+1) = \frac{(x+y)(x-y)}{10}$

since $x, y \in [-1, 4]$. so. $x+y$ at most $|4+4| = 8$

Therefore $|p(x) - p(y)| = \frac{|x+y|}{10}(x-y)$

$\le \frac{8}{10}|x-y| = \frac{4}{5}|x-y|$

2: since $p(x)$ is differentiable. Mean Value theorem tells us if $-1 \le x < y \le 4$, then exists $t \in (x, y)$ s.t. $p'(t) = \frac{p(x)-p(y)}{x-y}$ $p'(t) = \frac{t}{5} \le \frac{4}{5}$

(since t is at most 4)

so $|p(x) - p(y)| = p'(t)|x-y| \le \frac{4}{5}|x-y|$

since $\frac{4}{5} < 1$, $p(x)$ is a contraction on $[-1, 4]$

Summary

Contraction:

Definition: Let $I = [a, b]$ and let $f: I \to R$ be a function. Suppose there is number $0 < K < 1$ s.t. $|f(x) - f(y)| \le K|x-y|$ for all $x, y \in I$. Then. f is called contraction on I.

Calculus Revision.

Function of several variables

- 2D: $y = f(x)$
- 3D: $z = f(x, y)$
- 4D: $u = f(x, y, z)$

Level curve of $f(x,y)$ $z = f(x, y)$

Continuity

f is continuous at (a, b) if $(a, b) \in$ the domain of f and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Remark: continuous $\Rightarrow$ any path $\Rightarrow$ limit same.

Differentials:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

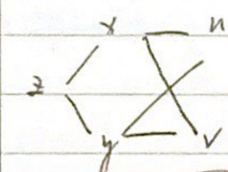
Clairut's theorem: (mixed derivative theorems)

$f_{xy}(a, b) = f_{yx}(a, b)$.

Chain rule

Given $z = z(x, y)$ $x = x(u, v)$ $y = y(u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$



Implicit Differentiation

Given $F(x, y) = 0$ y is a differentiable function of x, $F_y \neq 0$

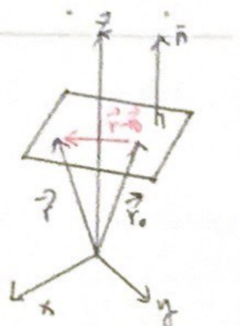
$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

Summary

Rules. (Contraction)

If $f: I \to R$ is differentiable and we can find $K < 1$ s.t. $|f'(t)| \leq K$ for all $t \in I$, then f is a contraction

(proof by MVT)



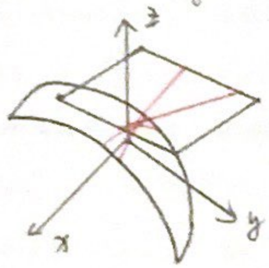
Equation for Plane

$P_0=(x_0, y_0, z_0)$ $P=(x, y, z)$ $\vec{n}(a, b, c)$

$(P-P_0)\cdot\vec{n}=0 \Rightarrow (x-x_0, y-y_0, z-z_0)(a, b, c)=0$

$ax+by+cz-(ax_0+by_0+cz_0)=0$

$ax+by+cz+d=0$



Tagent Plane

Gradient

$\nabla F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$

Tayler Theorem $f(x)$

At $x=a$.

$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots$

At $x=a=0$. Maclaurin.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Taylor Theorem $f(x, y)$

$f(a+h, b+k) = f(a, b) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) + \dots$

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)$$

Linear Approximation $f(x, y)$ near (a, b)

$f(x,y) \simeq f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) = L(x,y)$

$z = L(x,y)$ is the tegent plane

Quadratic Approximation $f(x,y)$ near (a,b)

$f(x,y) \simeq f + (x-a)f_x + (y-b)f_y + \frac{1}{2}[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}]$

Summary

5

Stationary Point

Definition: A point (a,b) at which $f_x = f_y = 0$ is a stationary point of $f(x,y)$.

Local Maximum and Local Minimum

Suppose (a,b) is a stationary point of f.

$f(a,b)$ is

Local Maximum if $f(x,y) \le f(a,b)$ for all (x,y) in D

Local Minimum if $f(x,y) \geq f(a,b)$ for all (x,y) in D

Saddle Point

Suppose (a,b) is a stationary point of f

(a,b) is saddle point if

a point in (x_1,y_1) in D $f(x_1,y_1) > f(a,b)$ and

a point in (x_2,y_2) in D $f(x_2,y_2) < f(a,b)$.

A saddle point is a stationary point which is <u>not a local maximum</u> <u>nor</u> a <u>local minimum</u>.

Hessian : $\boxed{H = f_{xx}f_{yy} - f_{xy}^2}$

- $H < 0$, f has saddle point at (a,b)
- $H > 0$
  - $\boxed{f_{xx}} > 0$, minimum at (a,b)
  - $\boxed{f_{xx}} < 0$, maximum at (a,b)
- $H = 0$, no conclusion

Summary

6 . .

Lagrange Method for solving optimization problems

Lagrange Multiplier λ

$F(x_1, \cdots \lambda_1 \cdots \lambda_m) = f - \sum_{i=1}^{m} \lambda_i g_i$

(m+n variables)

$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$F_x = F_y = F_\lambda = 0.$

(f is subject to constrains g)

$\nabla f = \lambda \nabla g$ two normals parallel to each other.

Double Integral

$$\iint_R f(x, y)\, dA = \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

If f is continuous on rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

Fubini's Theorem

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Centorids

$(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iint_R x\, dx\, dy}{\iint_R dx\, dy}$$

$$\bar{y} = \frac{\iint_R y\, dx\, dy}{\iint_R dx\, dy}$$

$$\text{Area} = \iint_D dA$$

Summary

5

Jacobian

$x = g(u,v) \quad y = h(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\Delta A = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$\boxed{\text{Triple Integral}}$

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

Fubini's Theorem. If f is continuous on a rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Volumn $= V = \iiint_D dV.$

Summary

APP ?

(36) A particle moves in an inverse cube central force field, i.e. $f(r) = r^{-3}$ in equation.
By finding effective potential, $V^*(r)$, determine the qualitative nature of the orbit if (i) $h^2 < \gamma$ (ii) $h^2 > \gamma$

If $r = 1$, $\theta = 0$, $\dot{\theta} = 1$ and $\dot{r} = 0$ when $t = 0$, determine the value of h.

Show that $r = \operatorname{sech}(\sqrt{\gamma - 1}\,\theta)$, $\tanh(\sqrt{\gamma-1}\,\theta) = \sqrt{\gamma-1}\,t$ when $\gamma > 1$

$$r = \operatorname{sech}(\sqrt{1-\gamma}\,\theta) \quad \tanh(\sqrt{1-\gamma}\,\theta) = \sqrt{1-\gamma}\,t$$

when $\gamma < 1$.

What's $\gamma = 1$?

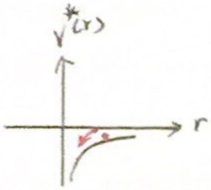
General Central force: $f(r) = r^{-3}$

$$\ddot{r} - \frac{h^2}{r^3} + \gamma r^{-3} = 0$$

$$\ddot{r} + \frac{(\gamma - h^2)}{r^3} = 0$$

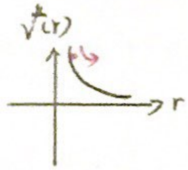
$$V^*(r) = \frac{mh^2}{2r^2} + m\gamma \int r^{-3}\,dr$$

$$V^*(r) = \frac{m}{2r^2}(h^2 - \gamma)$$



i) $h^2 < \gamma$ Bounded Motion $r \to 0$

ii) $h^2 > \gamma$ Unbounded Motion $r \to +\infty$



~~$\dot{\theta} = \frac{h}{r^2} \Rightarrow h = r^2\dot{\theta} = 1$~~

~~Let $r = \frac{1}{u(\theta)}$ $\dot{\theta} = \frac{d\theta}{dt} = 1$~~

~~$\dot{r} = -\frac{1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = -\frac{du}{d\theta}$~~

~~$\ddot{r} = -u''$~~ ~~$[u(\theta)]$~~

Summary

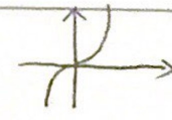
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 - \operatorname{sech}^2 = 1$$

$$(\tanh x)' = \operatorname{sech}^2 = 1 - \tanh^2 x$$



sinh θ

tanh θ

cosh θ.

$\ddot{r} + \frac{(\gamma - h^2)}{r^3} = 0$ Let $r = \frac{1}{u(\theta)}$ $\quad \theta' = \frac{h}{(\frac{1}{u})^2} = hu^2$

$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) = h\frac{du}{d\theta}$

$\ddot{r} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}(hu^2) = -h^2u^2\frac{d^2u}{d\theta^2}$

$-h^2u^2\frac{d^2u}{d\theta^2} + (\gamma - h^2)u^3 = 0 \Rightarrow \frac{d^2u}{d\theta^2} + \frac{(\gamma - h^2)}{(-h^2)}u = 0$

$u'' + (1 - \frac{\gamma}{h^2})u = 0$ define $h=1$

$u'' + (1-\gamma)u = 0$

• If $\gamma > 1$: $u = \cosh(\sqrt{\gamma - 1}\,\theta)$

$r = \frac{1}{u} = \operatorname{sech}(\sqrt{\gamma-1}\,\theta)$

• If $\gamma < 1$: $u = \cos(\sqrt{1-\gamma}\,\theta)$

$r = \frac{1}{u} = \sec(\sqrt{1-\gamma}\,\theta)$

Auxiliary Equation

$am^2 + bm + cm = 0$

$a=1 \quad b=0 \quad c=\sqrt{1-\gamma}$

$m = \pm\sqrt{\gamma - 1}$

• If $\gamma > 1$. $\sqrt{\gamma-1} > 0$

$u(\theta) = Ae^{\sqrt{\gamma-1}\theta} + Be^{\sqrt{\gamma-1}\theta}$

$\frac{du}{d\theta} = \sqrt{\gamma-1}Ae^{\sqrt{\gamma-1}\theta} - \sqrt{\gamma-1}Be^{\sqrt{\gamma-1}\theta}$

Initial: $\theta(0) = 0$. $u=1$

$u' = 0$.

$A = B = \frac{1}{2}$.

$u = \frac{1}{2}e^{\sqrt{\gamma-1}\theta} + \frac{1}{2}e^{-\sqrt{\gamma-1}\theta}$

• If $\gamma < 1$ $m = \pm\sqrt{1-\gamma}\,i$

$u = A\sin(\sqrt{1-\gamma}\,\theta) + B\cos(\sqrt{1-\gamma}\,\theta)$

$u' = \sqrt{1-\gamma}A\cos(\sqrt{1-\gamma}\,\theta) - \sqrt{1-\gamma}B\sin(\sqrt{1-\gamma}\,\theta)$

$u=1$ when $\theta=0$. $\quad B=1$

$u'=0$ when $\theta=0$ $\quad A=0$

Summary

Calculus

(137) f is continuous on R^2 and defined for $(x,y) \neq (0,0)$. Find $f(0,0)$

$$f(x,y) = \cos\left(\frac{x^3-y^3}{x^2+y^2}\right).$$

From any path.

For $y=x$.

$$f(x,y) = \cos\left(\frac{x^3-x^3}{x^2+x^2}\right) = \cos(0) = 1$$

$x \to 0$. $f(0,0) = 1$.

(138) Show Neither of the following functions defined for $(x,y) \neq (0,0)$ has a limit as $(x,y) \to (0,0)$.

(a) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ (b) $f(x,y) = \dfrac{2x^2}{x^4+x^2+y^2}$.

Common Method. $\Leftarrow$ (a). Approach $(0,0)$ through $\boxed{y=mx}$ $x \neq 0$.

$$\lim_{x \to 0} f(x, mx) = \frac{1-m^2}{1+m^2} \quad \text{depends on } m.$$

so. doesn't exist.

(b). On positive and negative y-axis. $x=0$.

$$\lim_{y \to 0} f(0,y) = 0$$

On positive and negative x-axis $y=0$

$$\lim_{x \to 0} f(x,0) = \frac{2}{1+x^2} = 2$$

these limit are different. so

Limit doesn't exist.

Summary

5

Calculus

(13) Let P be the point with coordinates $(x,y,z)=(1,1,2)$. In some neighbourhood of P, the equation $2z^3+3yz^2-x=0$ implicitly defines z as a function of x and y. Express the partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x \partial y}$ of this function as a rational functions of x, y and z. Find the value of these partial derivatives of P.

$2z^3+3yz^2-x=0$. $z=f(x,y)$.

pay attention to product rule.

respect to x: $6z^2 \cdot \frac{\partial z}{\partial x} + 6yz\frac{\partial z}{\partial x} - 1 = 0$

$$\frac{\partial z}{\partial x} = \cancel{\frac{y}{6z(y+z)}} \quad \frac{1}{6z(y+z)}$$

respect to y: $6z^2\frac{\partial z}{\partial y} + \underbrace{3z^2 + 6yz}_{\text{product rules.}}\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial y} = \frac{-z}{2(y+z)}.$$

Mixed derivative:

chain rule.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-z}{2(y+z)}\right) = \frac{-y}{2(y+z)^2}\boxed{\cdot\frac{\partial z}{\partial x}}$$

chain rule

$$= \frac{-y}{2(y+z)^2}\cdot\frac{1}{6z(y+z)} = -\frac{y}{12z(y+z)^3}$$

At P:

$$\frac{\partial z}{\partial x} = \frac{1}{12} \quad \frac{\partial z}{\partial y} = -1 \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{216}.$$

Summary