Calculus

(14.0) Show $u(x,t) = f(x-ct)$, is a solution of wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

c. const. If $f(y) = \operatorname{sech}(y)$, sketch the graph of $u(x,t)$ for a given time t. and explain how this graph changes with time.

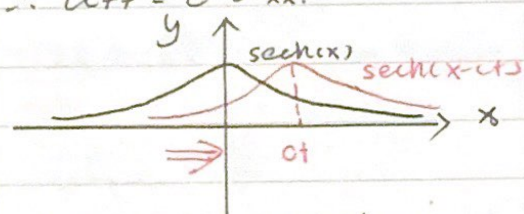
By chain rule:

$u_t = f'(x-ct)\cdot(-c) \qquad u_{tt} = c^2 f''(x-ct)$

Similarly:

$u_x = f'(x-ct)\cdot 1 \qquad u_{xx} = f''(x-ct)$

$\therefore u_{tt} = c^2 u_{xx}$.




$t=0$. $u(x,0) = \operatorname{sech}(x)$.

$t=ct$. $u(x, ct) = \operatorname{sech}(x-ct)$

Summary

5

Calculus

(14.1) Let $f$ be a function of two variables

$z = f(x, y)$ can be solved for $x$ as $x = g(y, z)$

Using diffentials.

(a) Express $\frac{\partial g}{\partial z}$ in terms of $f$.

(b) Express $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of derivatives of $g$.

(Manipulations of this sort are much used in thermodynamics)

the first step is: not use total differentials not the chain rule.

(a): Diffentials:
$$\boxed{\begin{cases} dz = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} dy & ① \\ dx = \frac{\partial g}{\partial y} \cdot dy + \frac{\partial g}{\partial z} dz & ② \end{cases}}$$

Rewrite ① $dx = (\frac{\partial f}{\partial x})^{-1} (-\frac{\partial f}{\partial y} dy + dz)$

Compare the coeffirient:

compare the coeffirient

$$\boxed{\frac{\partial g}{\partial z} = (\frac{\partial f}{\partial x})^{-1}}$$

(b) Rewrite ② $dz = (\frac{\partial g}{\partial z})^{-1} (dx - \frac{\partial g}{\partial y} dy)$

$$\boxed{\frac{\partial f}{\partial x} = (\frac{\partial g}{\partial z})^{-1}}$$

$$\boxed{\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial y} (\frac{\partial g}{\partial z})^{-1}}$$

Summary

## Partial derivatives

- $z = f(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} \cdot dy$$

- $u = f(x, y, z)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Compound Functions:

1 • $u = \varphi(t) \quad v = \psi(t) \quad , z = f(u, v)$

$$z = f[\varphi(t), \psi(t)]$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

- $u = \varphi(t) \quad , v = \psi(t) \quad w = \omega(t) \quad z = f(u, v, w)$

$$z = f[\varphi(t), \psi(t), \omega(t)]$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dt}.$$

2 $u = \varphi(x, y) \quad v = \psi(x, y) \quad z = f(u, v)$

$$z = f[\varphi(x, y), \psi(x, y)]$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$ $d \to t$. set $y$ const.

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}.$$ $d \to t$ . set $x$ const.

$u = \varphi(x, y) . v = \psi(x, y) \quad w = \omega(x, y) \quad z = f(u, v, w)$

$$z = f[\varphi(x, y), \psi(x, y), \omega(x, y)]$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial y}$$

Summary

3.

$u=\varphi(x,y) \quad v=\psi(y) \quad z=f(u,v)$

$z=f[\varphi(x,y),\psi(y)]$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\boxed{\frac{dv}{dy}}$$

$u=\varphi(x,y)$, $z=f(u,x,y)$ $\quad z=f[\varphi(x,y),x,y]$

like 2. set $v=x$, $w=y$

$$\frac{\partial v}{\partial x}=1,\ \frac{\partial w}{\partial x}=0$$

$$\frac{\partial v}{\partial y}=0,\ \frac{\partial w}{\partial y}=1$$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}$$

$\frac{\partial z}{\partial x} \to z=f[\varphi(x,y),x,y]$

$\frac{\partial f}{\partial x} \Rightarrow f(u,x,y)$

**Implicit function:**

- $F(x,f(x))=0$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

- $F(x,y,f(x,y))=0$

$$F_x+F_z\frac{\partial z}{\partial x}=0 \qquad F_y+F_z\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

Summary

**Implicit function:**

$F(x_0,y_0)=0 \qquad \frac{dy}{dx}=-\frac{F_x}{F_y} \quad (F_y\neq 0)$

$F(x_0,y_0,z_0)=0 \qquad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z} \quad (F_z\neq 0)$

eg: $z = e^u \sin v \quad u = xy \quad v = x+y \qquad \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = e^u \sin v \cdot y + e^u \cos v \cdot 1$$
$$= e^{xy}[y\sin(x+y) + \cos(x+y)]$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = e^u \sin v \cdot x + e^u \cos v \cdot 1$$
$$= e^{xy}[x\sin(x+y) + \cos(x+y)]$$

eg: $u = f(x,y,z) = e^{x^2+y^2+z^2} \quad z = x^2 \sin y. \quad \frac{\partial u}{\partial x} \; \frac{\partial u}{\partial y}$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}} = 2x e^{x^2+y^2+z^2} + 2z e^{x^2+y^2+z^2} \cdot 2x \sin y$$
$$= 2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

eg: $z = f(u,v,t) = uv + \sin t \qquad u = e^t \quad v = \cos t \qquad \frac{dz}{dt}$

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial u}\cdot\frac{du}{dt} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial f}{\partial t}}$$
$$= ve^t - u\sin t + \cos t$$
$$= e^t \cos t - e^t \sin t + \cos t$$
$$= e^t(\cos t - \sin t) + \cos t.$$

Summary

(15') ⟺

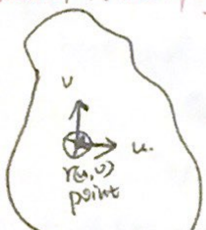
Calculus.

(1e) The surface $S$ is defined by $r = v\cos u\, i + v\sin u\, j + v^2 \sin 2u\, k$
where $0 \le u \le 2\pi$ and $0 < v < \infty$

(i) Find a normal vector at a general point $r(u,v)$

(ii) Find a vector equation of the tangent plane at the point $r(\pi/4, 1)$ in the form $r = a + \lambda b + \mu c$ for a constant vectors $a, b, c$ and parameters $\lambda, \mu$.

参数方程形式的 normal vector, 只能用 $r_u \times r_v$.




(i) Normal vector at $r(u,v)$. $r_u \times r_v$.

$r_u = -v\sin u\, i + v\cos u\, j + 2v\cos 2u\, k$

$r_v = \cos u\, i + \sin u\, j + 2v\sin 2u\, k$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -v\sin u & v\cos u & 2v\cos 2u \\ \cos u & \sin u & 2v\sin 2u \end{vmatrix} = 2v\sin u\, i + 2v\cos u\, j - k$$

(ii) At $r(\pi/4, 1)$ $r_u = (-i+j)/\sqrt{2}$ $r_v = (i+j)/\sqrt{2} + 2k$

The plane

$r = r(\pi/4, 1) + \lambda r_u(\pi/4, 1) + \mu r_v(\pi/4, 1)$

$r = \frac{1}{\sqrt{2}}(i + j + \sqrt{2}k + \lambda(-i+j) + \mu(i + j + 2\sqrt{2}k))$.

$r = a + \lambda b + \mu c$

b. c. are vectors in the plane. just by simply use the $\boxed{r_u \text{ and } r_v}$, which were used to create normal vectors.

Summary

Plane:

One point and a normal: $(r - p)\cdot n = 0$

$ax + by + cz = d$ ((a, b, c) normal vector)

One point and two in-plane vector: $r = p + \lambda v + \mu u$.

$(n = v \cdot u,\ (r-p)\cdot(v \times u) = 0)$

Calculus

(143) Expand $f(x,y)=x^2+3y-2$ as Taylor series in power of $(x-1)$ and $(y+2)$. find linear and quadractic approximations to $f(x,y)$ at $(1,-2)$. Deduce an equation for the tangent plane to the surface $z=f(x,y)$ at $(1,-2)$

$f_x=2x \quad f_{xx}=2 \quad f_y=3 \quad f_{yy}=0 \quad f_{xy}=0$

$$f(x,y)=f+(x-1)f_x+(y+2)f_y+\frac{1}{2}[f_{xx}(x-1)^2+2(x-1)(y+2)f_{xy}+(y+2)^2f_{yy}]+\cdots$$

Linear: $f(x,y)=-7+2(x-1)+3(y+2)$

Quadratic: $f(x,y)=-7+2(x-1)+3(y+2)+(x-1)^2$

tangent plane: $z=-7+2(x-1)+3(y+2)=2x+3y-3$

Summary

Linear Approximation: $f(x,y)\simeq f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)=L(x,y)$

$z=L(x,y)$ is a plane. This is the tangent plane to the surface $z=f(x,y)$ at $(a,b)$.

5 . .

Calculus.

(22) Use Taylor's formula to find a linear and a quadratic approximation to $e^x \sin y$ at the origin. Estimate the error in the quadratic approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

$$f(h,k) = \sum_0^2 \frac{1}{n!} (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^n f(x,y)\Big|_{0,0} + R_2$$

$$R_2 = \frac{1}{3!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^3 f(x,y)\Big|_{th,tk} \quad t\in[0,1]$$

$f_x = e^x \sin y = f$ $\quad f_y = e^x \cos y$ $\quad f_{xx} = f$ $\quad f_{xy} = f_y$ $\quad f_{yy} = -f$

$f_{xxx} = f$ $\quad f_{xyy} = -f$ $\quad f_{yyy} = -f_y$ $\quad f_{xxy} = f_y$

$$f(h,k) = f(0,0) + (hf_x + kf_y)|_{0,0} + \frac{1}{2}(h^2 f_{xx} + 2khf_{xy} + k^2 f_{yy})|_{0,0} + R_2$$

$$= 0 + k + \frac{1}{2}(2hk) + R_2$$

$$= k + hk + R_2$$

$$R_2 = \boxed{\frac{1}{6}(h^3 f_{xxx} + 3h^2 f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})}\Big|_{th,tk} \quad t\in[0,1]$$

want to find an estimate of form $|R_2| \leq M$.

$|f_{xxx}(tx,ty)| = e^{tx}\sin y \leq e^{t|x|} \leq e^{1\times 0.1} = e^{0.1}$

$|f_{xxy}(tx,ty)| = |e^{tx}\cos y| \leq e^{t|x|} \leq e^{1\times 0.1} = e^{0.1}$

$|f_{yxx}(tx,ty)| \leq e^{0.1}$

The difference between the $L(x,y)$ and $f(x,y)$ $\Leftarrow$ $|R_2| \leq \frac{1}{6}(|x|^3 + 3|x|^2|y| + 3|y||x|^2 + |y|^3)e^{0.1}$

$$= \frac{1}{6}(0.1^3 + 3\,0.1^2\,0.1 + 3\,0.1\,0.1^2 + 0.1^3)\,e^{0.1}$$

$$= 0.0014736.$$

$f(x,y) \simeq y + xy$

Summary

Remainders in Taylor Theorem:

1. Single: $R_n(x) \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

2. Two: $E(x,y) \leq \frac{1}{2}M(|x-x_0| + |y-y_0|)^2$ (Only for linear approximation)

5

Calculus

(145) For a closed cylindrical tin of given volumn, use a Lagrange multiplier to show that the total surface area (consisting of the curved part and the top and bottom disks) is least when the diameter of the tin equals the height.

$z=f(x,y)$ is subject to $\boxed{g(x,y)=0}$ 移项.

$g(r,h)=2\pi r^2+2\pi rh.$

$V(r,h)=\boxed{\pi r^2 h = C}$ don't forget C.

$F(r,h,\lambda)=2\pi r^2+2\pi rh-\lambda(\pi r^2 h-C)$

Stationary point equations.

$F_r=4\pi r+2\pi h-2\pi\lambda rh=2\pi[2r+h(1-\lambda r)]=0$

$F_h=2\pi r-\lambda\pi r^2=\pi r(2-\lambda r)=0$

$F_\lambda=-\pi r^2 h+C=0$

$\lambda r=2.$ $\boxed{h=2r.}$ Height = diameter.

Calculus

(146) Maximise $x^2y^2z^2$ subject to $\boxed{x^2+y^2+z^2=r^2}$ ($g(x,y,z)=0$).

where r is a positive constant. Deduce the inequality

$(x^2y^2z^2)^{\frac{1}{3}}\le\frac{1}{3}(x^2+y^2+z^2)$

Lagrange Multiplier:

$F(x,y,z,\lambda)=x^2y^2z^2-\lambda(x^2+y^2+z^2-r^2)$

$$\begin{cases} F_x=2xy^2z^2-2\lambda x=0 \\ F_y=2yx^2z^2-2\lambda y=0 \\ F_z=2zx^2y^2-2\lambda z=0 \\ F_\lambda=-(x^2+y^2+z^2-r^2)=0 \end{cases} \Rightarrow \begin{cases} y^2z^2=\lambda \\ x^2z^2=\lambda \\ x^2y^2=\lambda \\ x^2+y^2+z^2=r^2 \end{cases}$$

Summary

5

$\lambda = y^2z^2 = x^2z^2 = x^2y^2$

Trick.

$\begin{cases} (y^2-x^2)z^2=0 \\ z^2(x^2-y^2)=0 \end{cases}$

$\Rightarrow x^2=y^2=z^2$

$\Rightarrow x^2=y^2=z^2=r^2/3$

$(\pm r/\sqrt{3}, \pm r/\sqrt{3}, \pm r/\sqrt{3})$ 8 solution, are all possible.

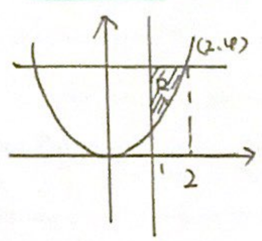
$x^2y^2z^2 = r^6/27$, which is the maximux value.

$\therefore$ Combining $\begin{cases} x^2y^2z^2 = r^6/27 \\ x^2+y^2+z^2 = r^2 \end{cases}$

$(x^2y^2z^2)^{\frac{1}{3}} \le \frac{1}{3}(x^2+y^2+z^2)$

The boundary is not constant.

Calculus.




(4) $I = \int_{x=1}^{x=2}\int_{y=x^2}^{y=4} \frac{1}{y^2}\exp(\frac{x}{\sqrt{y}})\,dy\,dx$, reverse the order.

We can figure out the $R: \{(x,y): x^2 \le y \le 4,\ 1 \le x \le 2\}$

$\Leftrightarrow R: \{(x,y): 1 \le x \le \sqrt{y},\ 1 \le y \le 4\}$

$I = \int_1^4 \{\int_1^{\sqrt{y}} \frac{1}{y^2}\exp(\frac{x}{\sqrt{y}})\,dx\}\,dy$

$= \int_1^4 [y^{-3/2}\exp(xy^{-1/2})]_{x=1}^{x=\sqrt{y}}\,dy$

$= \int_1^4 y^{-3/2}(e - \exp(y^{-1/2}))\,dy$

$= [-2ey^{-1/2} - 2\exp(y^{-1/2})]_1^4$

$= (-e + 2\sqrt{e}) - (-2e + 2e)$

$= 2\sqrt{e} - e$

when boundary is a function, changing the order should be careful.

Although, it doesn't give us the Region. We could figure out it from the bounds. ATTENTION!: it's not as easy as you thought to reverse the order of x y. You need to do more work.

Summary

Centroid:

$A = \iint_R dx\,dy$

$\bar{x} = \frac{\iint x\,dx\,dy}{A}$

$\bar{y} = \frac{\iint y\,dx\,dy}{A}$

5

Caculus.

(148) A region. R area A. in the upper half of the x-y plane (y>0) is rotated through 360° about the x-axis to form in xyz-space. volumn V. By expressing V. A. and $\bar{y}$ ( centorid of R) as double integrals show that $V = 2\pi\bar{y}A$

Use this result to find the volumn of a solid tones

are element: $\delta A = \delta x\,\delta y$

Volume element: $\delta V = 2\pi y\,\delta A = 2\pi y\,\delta x\,\delta y$

~~$A = \iint$~~

$A = \sum_R \delta x\delta y \qquad V = \sum_R 2\pi y\,\delta x\delta y.$

$A = \iint_R dx\,dy \qquad V = 2\pi\iint_R y\,dx\,dy.$

$\bar{y}: \qquad A\bar{y} = \iint_R y\,dx\,dy$

$\therefore V = 2\pi\bar{y}A.$

(149) Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$. by transforming to polar coordinates.

Show that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Use this result to confirm that the integral of the probability density function of the standard normal distribution $N(0,1)$ over $(-\infty, \infty)$ is 1.

This question shows me the importance of writing the Region since it is closed releated to my boundary when I write the double integral

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

Summary

5 . .

Region $\boxed{R = \{(\rho, \varphi) : 0 \le \rho < \infty, \ 0 \le \varphi < 2\pi\}}$

Jacobian. $\frac{\partial(x, y)}{\partial(\rho, \varphi)} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho$

$$I_2 = \int_0^\infty \left\{ \int_0^{2\pi} e^{-\rho^2} \rho \, d\varphi \right\} d\rho$$
$$= 2\pi \int_0^\infty e^{-\rho^2} \rho \, d\rho$$
$$= 2\pi \left[ -\frac{1}{2} e^{-\rho^2} \right]_0^\infty$$
$$= 2\pi \cdot \frac{1}{2}$$
$$= \pi$$

Let $I_1 = \int_{-\infty}^\infty e^{-x^2} dx.$

$$I_2 = \int_{-\infty}^\infty e^{-y^2} \int_{-\infty}^\infty e^{-x^2} dy\,dx = \left\{ \int_{-\infty}^\infty e^{-x^2} dx \right\} \left\{ \int_{-\infty}^\infty e^{-y^2} dy \right\}$$
$$= I_1^2.$$

$$I_1 = \sqrt{I_2} = \sqrt{\pi}$$

So. the standard normal distribution $N(0,1)$. Integral.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} dx$$

Let $x = \sqrt{2} y$ $\quad dx = \sqrt{2} dy$

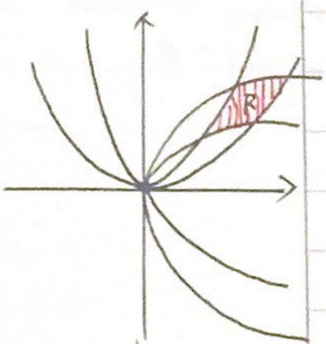
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2} \sqrt{2} \, dy = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2} \cdot \sqrt{\pi} = 1.$$

Thus, the probability is 1.

Summary

(150) Find the area of the region R defined by $x \le y^2 \le 2x$ $y \le x^2 \le 2y$ Sketch R.

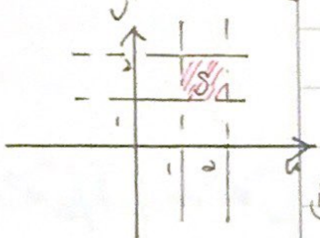
I Hint use transformation $u = \frac{x^2}{y}$, $u = \frac{y^2}{x}$.




$R: \{(x,y),\ x \le y^2 \le 2x,\ y \le x^2 \le 2y\}$

$S = \{(u,v),\ 1 \le u \le 2,\ 1 \le v \le 2\}$ in the u-v plane

$\frac{\partial(u,v)}{\partial(x,y)} = 3 \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{3}$



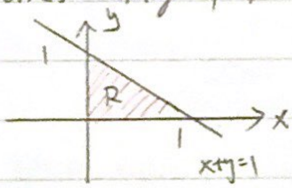

$A = \iint_S \frac{1}{3}\,du\,dv.$

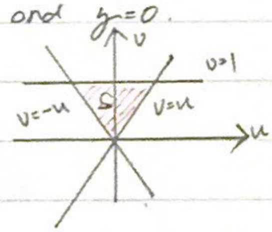
$A = \iint_S \frac{1}{3}\,du\,dv = \frac{1}{3} \times (\text{area of } S) = \frac{1}{3}.$

(151) Using change of variables $u = x-y$. $v = x+y$

Show $\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy = \frac{1}{2}\sin 1$

where R is the finite region bounded by the lines $x+y=1$, $x=0$ and $y=0$.




the key step is to find the relationship between u and v

$$\begin{cases} x+y \le 1 \Rightarrow \text{so } v \le 1. \\ x=0 \Rightarrow u+v=0 \\ y=0 \Rightarrow u-v=0 \end{cases}$$

R and S are not equal. but we know their scalar unit is Jacobian. which means that $\boxed{J}S = R$, we use Jacobian to adjust the area.

补充:

$J = \frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$

## Summary

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad J = \frac{1}{2}$$

$$I = \iint_S \cos\left(\frac{u}{v}\right)\frac{1}{2}\, du\, dv$$
$$= \frac{1}{2}\int_0^1\left\{\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\right\} dv$$
$$= \frac{1}{2}\int_0^1 \left[v\sin\left(\frac{u}{v}\right)\right]_{u=-v}^{u=v} dv$$
$$= \frac{1}{2}\int_0^1 2v\sin 1\, dv = \frac{1}{2}\sin 1.$$

Calculus.

(13.2) The interval $R_1 = \{x\in R : x^2 \le 1\}$, the disk $R_2 = \{(x,y)\in R^2 : x^2+y^2 \le 1\}$ and the ball $R_3 = \{(x,y,z)\in R^3 : x^2+y^2+z^2 \le 1\}$ may be regarded as the unit balls in respectively $R$ $R^2$, $R^3$. Their volumes are defined respectively as the integrals $\int_{R_1} dx$ $\iint_{R_2} dx\,dy$ $\iiint_{R_3} dx\,dy\,dz$ with the well-known value $2, \pi$ and $\frac{4}{3}\pi$.

Extend these to $R^4$, Show that unit ball in $R^4$ $V = \frac{1}{2}\pi^2$. [Hint Denote Cartesian $R^4$ $(x, y, z, w)$ introduce polar coordinate ]

$R_4 : \{(x, y, z, w) \in R^4 : x^2+y^2+z^2+w^2 \le 1\}$

$$V_4 = \iiiint_{R_4} dx\,dy\,dz\,dw$$

$S_4 : \{(\rho, \varphi, r, \theta)\in S^4 : 0\le\varphi<2\pi,\ 0\le\theta<2\pi,\ \rho\ge 0,\ r\ge 0,\ r^2+\rho^2\le 1\}$

$x = \rho\cos\varphi \qquad z = r\cos\theta$

$y = \rho\sin\varphi \qquad w = r\sin\theta$

$$\frac{\partial(x,y,z,w)}{\partial(\rho,\varphi,r,\theta)} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 & 0 \\ \sin\varphi & \rho\cos\varphi & 0 & 0 \\ 0 & 0 & \cos\theta & -r\sin\theta \\ 0 & 0 & \sin\theta & r\cos\theta \end{vmatrix} = \rho r$$

$$\det\begin{vmatrix} B_1 & 0 \\ 0 & B_2 \end{vmatrix} = \det(B_1)\det(B_2) = \rho r$$

$$V_4 = \iiiint_{S_4} \rho r\, dr\, dr\, d\varphi\, d\theta = \iint_{\rho^2+r^2\le 1} \rho r\, d\rho\, dr$$

Summary

5

$$= 4\pi^{\alpha} \int_0^1 \left\{ \int_0^{\sqrt{1-r^2}} \varrho r \, d\varrho \right\} dr$$
$$= 2\pi^{\alpha} \int_0^1 r(1-r^2)\, dr$$
$$= 2\pi^{\alpha} \left[ \frac{1}{2} r^2 - \frac{1}{4} r^4 \right]_0^1 = \frac{1}{2}\pi^2$$

Trick to find p.D. Calculus.

(153) Find $\int_0^1 \frac{x^{100}-1}{\ln x} \, dx$

Let $I(b) = \int_0^1 \frac{x^b - 1}{\ln x} \, dx$. so we need to find $I(100)$.

$$\frac{\partial I}{\partial b} = \frac{\partial}{\partial b} \left( \int_0^1 \frac{x^b-1}{\ln x} \, dx \right)$$
$$= \int_0^1 \frac{\partial}{\partial b} \left( \frac{x^b - 1}{\ln x} \right) dx$$

swapp $\int$ and $\frac{\partial}{\partial b}$.

$(a^x)' = a^x \ln a$ ⇐

$$= \int_0^1 x^b \, dx$$
$$= \frac{x^{b+1}}{b+1} \Big|_0^1$$
$$= \frac{1}{b+1}$$

Like we change the variables. from $x \to b$.

$$I = \int_0^{100} \frac{1}{b+1} \, db$$
$$= \left[ \ln(b+1) \right]_0^{100}$$
$$= \ln 101$$

Gradient. (142) ⟺ (154) For the surface $z = 2xy$. find

(i) a normal vector at a general point $(x, y)$

(ii) a Cartesian equation of the tangent plane at the point where $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

when do the calculus of Gradient, it's necessary to make $F = 0$.

$\boxed{F(x, y, z) = z - 2xy}$

$\boxed{F(x, y, z) = 0}$. $\nabla F = (-2y, -2x, 1)$ is a normal

Summary

(ii) $z = 2xy = 1$ $P_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$ lies over the given surface.

$\vec{n} = \nabla F(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1) = (-\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 1)$

$(P - P_0) \cdot \vec{n} = 0$.

$(x - \frac{1}{\sqrt{2}}, y - \frac{1}{\sqrt{2}}, z - 1) \cdot (-\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 1) = 0$

~~$2x + 2y = \sqrt{5}$~~

~~$x + y = \sqrt{2}$~~ $\sqrt{2}x + \sqrt{2}y - z = 1$

Calculus.

$\Delta z$ is the increment for the whole function, which could be calculated by $dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$.

$\Delta z \approx dz$.

(155) $\boxed{\text{Approximation By Differentials.}}$

(i) $\sqrt{(1.02)^3 + (1.97)^3}$ (ii) $(1.97)^{1.05}$ ($\ln 2 \approx 0.693$)

(i). let $z = \sqrt{x^3 + y^3}$

$z = \sqrt{(x+\Delta x)^3 + (y+\Delta y)^3} = \sqrt{x^3+y^3} + \Delta z$

$\approx \sqrt{x^3+y^3} + dz$

$= \sqrt{x^3+y^3} + \frac{3x^2\Delta x + 3y^2\Delta y}{2\sqrt{x^3+y^3}}$

let $x = 1$, $y = 2$.

$\Delta x = 0.02$ $\Delta y = -0.03$

$z = 2.95$.

(ii) let $z = x^y$

$z = (x + \Delta x)^{y + \Delta y} = x^y + \Delta z \approx x^y + dz$

$= x^y + yx^{y-1}\Delta x + x^y \ln x \Delta y$

$x = 2$, $y = 1$ $\Delta x = -0.03$ $\Delta y = 0.05$

$z = 2.039$

Summary

5 . .

APP.

Second-order ODE

Auxiliary equation

$\downarrow$

C.F

Homogeneous ODE $\rightarrow$ $y = y_{cf}$

Inhomogenous ODE $\rightarrow$ Try $y_{pI}$ $\rightarrow$ $y = y_{cf} + y_{pI}$

General Solution.

Examples:

| | |
|---|---|
| $e^{-x}$ | $ae^{-x}$ |
| $e^{-2x}+3$ | $ae^{-2x}+b$ |
| $3-2x^2$ | $ax^2+bx+c$ |
| $\sin x$ | $a\cos x+b\sin x$ |
| $e^{-2x}\cos x$ | $e^{-2x}(a\cos x+b\sin x)$ |
| $xe^{2x}$ | $(ax+b)e^{2x}$ |

System of ODE. Transformaction

- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x.$

Let $u = \frac{dy}{dx}$, then

$$\begin{cases} \frac{du}{dx} + 3u + 2y = x \\ \frac{dy}{dx} = u \end{cases}$$

Summary

Key step:
Differentiate the $\frac{dy}{dx}$

- $\begin{cases} \frac{dy}{dx} + u = 3 \\ \frac{du}{dx} + y = 2 \end{cases} \Rightarrow \frac{d^2y}{dx^2} + \frac{du}{dx} = 0$

$\frac{du}{dx} = 2 - y$

$\Rightarrow \frac{d^2y}{dx^2} - y = -2.$

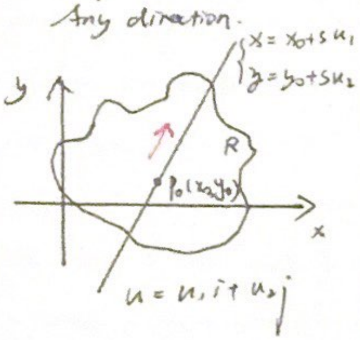
Summary

## Calculus.

x. y direction

↓

Any direction.




## Partial Derivatives Revisiting

$\boxed{x \text{ direction.}}$

$$\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$\boxed{y \text{ direction}}$

$$\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

$\boxed{\text{Directional Derivatives}}$

$u = u_1 i + u_2 j$ direction

$$\left(\frac{df}{ds}\right)_{u, p_0} = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+u_2 s) - f(x_0,y_0)}{s}$$

so. $u = i$ , $x$ direction $\frac{\partial f}{\partial x}$ at $(x_0, y_0)$

$u = j$ , $y$ direction $\frac{\partial f}{\partial y}$ at $(x_0, y_0)$

$$\left(\frac{df}{ds}\right)_{u, p_0} = \frac{\partial f}{\partial x}\bigg|_{p_0}\frac{dx}{ds} + \frac{\partial f}{\partial y}\bigg|_{p_0}\frac{dy}{ds}$$

$$= \frac{\partial f}{\partial x}\bigg|_{p_0} u_1 + \frac{\partial f}{\partial y}\bigg|_{p_0} u_2$$

$$= \underbrace{\left[\frac{\partial f}{\partial x}\bigg|_{p_0} i + \frac{\partial f}{\partial y}\bigg|_{p_0} j\right]}_{\text{Gradient of } f \text{ at } P_0} \times \underbrace{\left[u_1 i + u_2 j\right]}_{\text{Direction } u.}$$

Gradient: $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

$\therefore \left(\frac{df}{ds}\right)_{u, p_0} = \nabla f\,|_{p_0} \cdot u.$

The directional Derivative is a dot Product.

## Summary

5 . .

u must be a unit vector.

Properties.

$D_u f = \nabla f \cdot u = |\nabla f| \cos\theta$.

- $D_u f = |\nabla f| \cos 0 = |\nabla f|$

  f increases most rapidly in the direction of the $\nabla f$ at P.

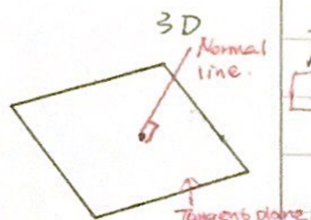
- $D_u f = |\nabla f| \cos(\pi) = -|\nabla f|$

  f decreases most rapidly in the direction of $-\nabla f$ at P

- $D_u f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0$

  $u \perp \nabla f$.

- At every point $(x_0, y_0)$ in the domain of a differentiable function $f(x, y)$. the $\nabla f$ is normal to the level curve through $(x_0, y_0)$

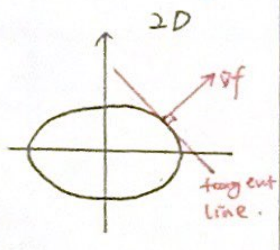



$\boxed{\text{Tangent Plane}}$ to $f(x,y,z) = C$ at $P_0(x_0, y_0, z_0)$

$f_x(P_0)(x - x_0) + f_y(y - y_0) + f_z(P_0)(z - z_0) = 0$

$\boxed{\text{Normal Line}}$ to $f(x, y, z) = C$ at $P_0(x_0, y_0, z_0)$

$x = x_0 + f_x(P_0)t \quad y = y_0 + f_y(P_0)t \quad z = z_0 + f_z(P_0)t$.




$\boxed{\text{Tangent Line}}$ to a level curve.

$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$

Summary

5

If I want to linearize sin x, I will start from the tangent line at (0,0). Similarly, if I want to draw a surface, I will start from tangent plane.

$\boxed{\text{Linearization}}$ of $f(x,y)$ at a point $(x_0, y_0)$

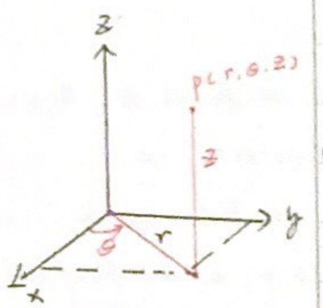
$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$z = L(x,y)$ is $\boxed{\text{tangent plane}}$ to the surface $z = f(x,y)$ at $(x_0, y_0)$.

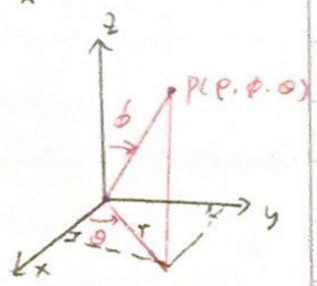
Error: $E(x,y) \le \frac{1}{2} M(|x-x_0| + |y-y_0|)^2$

$\boxed{\text{Differentials}}$

- $df = f'(a)\,\Delta x$
- $df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$




Cylindrical Coordinate $(r, \theta, z)$
( Cartesian + z )




Spherical Coordinates $(\rho, \phi, \theta)$
$0 \le \phi \le \pi \qquad 0 \le \theta \le 2\pi$

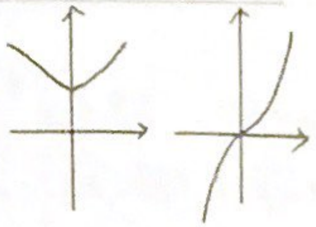
Summary

Trigonometric function.

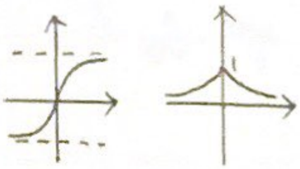
Hyperbolic Trigonometric Function

$\sinh = \dfrac{e^x - e^{-x}}{2}$ $\cosh = \dfrac{e^x + e^{-x}}{2}$

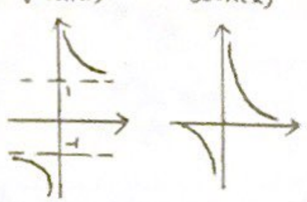
$\tanh = \dfrac{\sinh}{\cosh}$



cosh(x) sinh(x)



tanh(x) sech(x)



coth(x) csch(x)

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

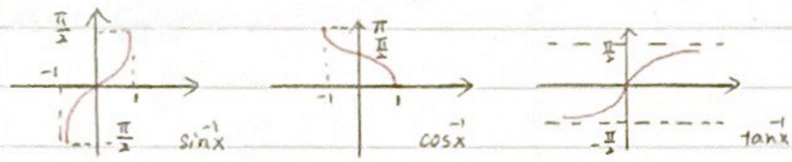
$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

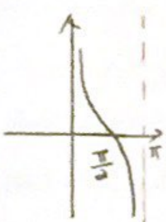
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

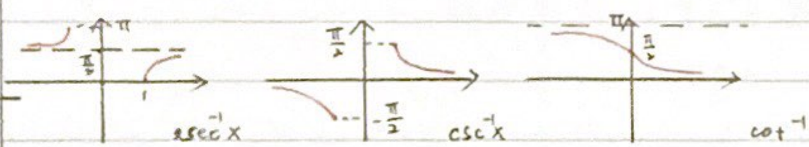
Inverse trigonometric function



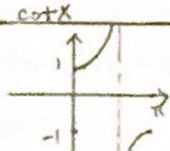

$[-\frac{\pi}{2}, \frac{\pi}{2}]$ $[0, \pi]$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$



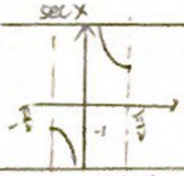
cot x




$[0, \pi], y \neq \frac{\pi}{2}$ $[-\frac{\pi}{2}, \frac{\pi}{2}]\ y \neq 0$ $[0, \pi]$



sec x



cosec x

Summary

Chain rule:

Assume:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\cdot\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x} = \cos\varphi\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} = \sin\varphi\frac{\partial u}{\partial \rho} + \frac{\cos\varphi}{\rho}\cdot\frac{\partial u}{\partial \varphi}.$$

if I want to get $\frac{\partial^2 u}{\partial x^2}$.

Identity: $\frac{\partial}{\partial x} = \cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\cdot\frac{\partial}{\partial \varphi}$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\cdot\frac{\partial}{\partial \varphi}\right)\left(\cos\varphi\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial u}{\partial \varphi}\right)$$

$$= \cos\varphi\left(\cos\varphi\frac{\partial^2 u}{\partial \rho^2} + \frac{\sin\varphi}{\rho^2}\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial^2 u}{\partial \rho\partial \varphi}\right)$$

$$- \frac{\sin\varphi}{\rho}\left(-\sin\varphi\frac{\partial u}{\partial \rho} + \cos\frac{\partial^2 u}{\partial \varphi\partial \rho} - \frac{\cos\varphi}{\rho}\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial^2 u}{\partial \varphi^2}\right.$$

Summary

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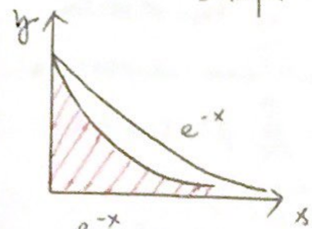
Acf.




$\frac{e^{-x}}{1+x+x^2}$

Acf

Acf

Basic idea of Improper Integral

Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\boxed{\text{The comparison test}}$

Theorem: Suppose that $a \in R$, and $f: [a, \infty) \to R$ and $g: [a, \infty) \to R$ are continuous functions with $\boxed{|f(x)| \le g(x)}$ for all x.

Suppose $\int_a^\infty g(x)\,dx$ converges. Then, $\int_a^\infty f(x)\,dx$ converges.

(156) $\int_0^x \frac{e^{-x}}{1+x+x^2}\,dx$

For $x \ge 0$, take $0 \le f(x) = \frac{e^{-x}}{1+x+x^2} \le e^{-x} = g(x)$

Then:

$$\int_0^\infty g(x)\,dx = \lim_{L\to\infty} \int_0^L e^{-x}\,dx$$
$$= \lim_{L\to\infty} (-e^{-L}+1)$$
$$= 1$$

$|f(x)| \le g(x)$.

$\therefore \int_0^x f(x)\,dx$ converges.

(157)

(i) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[\int_1^t \frac{1}{x^2}\,dx\right] = \lim_{t\to\infty}\left[\frac{-1}{x}\Big|_1^t\right] = \lim\left[1-\frac{1}{t}\right] = 1$

converges.

(ii) $\int_1^\infty \frac{1}{x}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx = \lim \ln(x)\Big|_1^n = \lim_{n\to\infty}(\ln n) = \infty$

diverges.

Summary

$\boxed{\text{Improper Integral}}$ Let $a \in R$ and $f: [a, \infty) \to R$ and consider $\int_a^\infty f(x)\,dx$ an improper integral, $\int_a^L f(x)\,dx$. exists for every real $L > a$

$\boxed{\int_a^\infty f(x)\,dx = \lim_{L\to\infty}\int_a^L f(x)\,dx}$, if the limit exists and finite.