5 . .

LM

(22) if $z = (3+4i)^{1/2}$ and $Re(z) > 0$. then

(a) $Im(z) = i$

(b) $Im(z) = -\frac{1}{2} Re(z)$

(c) $Im(z) = 2 Re(z)$

(d) $Im(z) = 1$

(e) None of the bove.

Actually, I make it complex by using the Moivre's Theorem. This kind of equation can be dealt with by make the corresponding item equal. simplify.

let $z = x + yi$

$(3+4i) = (x+yi)^2 \Rightarrow (x^2 - y^2) + 2xyi = 3 + 4i$

$$\begin{cases} x^2 - y^2 = 3 \\ 4 = 2xy \end{cases} \Rightarrow y = \frac{2}{x}, \quad x^4 - 3x^2 - 4 = 0.$$

let $x^2 = t$, $(t-4)(t+1) = 0$, $(x^2-4)(x^2+1) = 0$.

$\therefore x^2 = 4$. $x = \pm 2$. $\because Re(z) > 0$. $\therefore x = 2$, $y = 1$

$\therefore z = 2 + i$

LM

(23) For general vectors a. b. c. the vector product $(a-b) \times (a-c)$ simplifes to.

(a) $a \times c + b \times a + c \times b$

(b) $c \times a + b \times a + b \times c$

(c) $c \times a + a \times b + c \times b$

(d) $a \times b + b \times c + c \times a$

(e) None of the above.

$(a-b) \times (a-c) = \overbrace{a \times a}^{=0 \cdot \sin\theta = 0,\ \theta = 0.} - a \times c - b \times a + b \times c$

$= c \times a + a \times b + b \times c$

Summary

Remember. $a \times a = 0$.

LM.

(14) The multiple vector product $a\times[b\times(c\times a)]$ is, for general non-zero vectors, a, b, c.

(a) Perpendicular to both a and c.

(b) Perpendicular to b

(c) Perpendicular to c and b.

(d) Parallel to b

(e) None of the above.

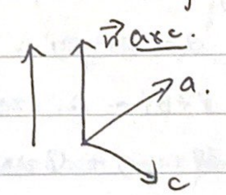
$b\times(c\times a) = c\cdot(b\cdot a) - a(b\cdot c)$.

$a\times[b\times(c\times a)] = a\times[c\cdot(b\cdot a) - a\cdot(b\cdot c)]$

$= (a\times c)\cdot(b\cdot a) - (a\times a)\cdot(b\cdot c)$

$= (a\times c)\boxed{(b\cdot a)} \to \boxed{\text{scalar}}$

$\therefore$ $a\times[b\times(c\times a)]$ is parallel to $\underline{a\times c}$.



$\therefore$ $\perp$ a and c.

$\therefore$ choose (a).

因为是后面的一起运算.

$a\times(b\times c)$

$= b(a\cdot c) - c(a\cdot b)$

BACK - CAB

后面的先乘.

$(a\times b)\times c$

$= (a\cdot c)b - (b\cdot c)a$

先写第一个，然后第二个公式前一项字母和第一项第一个字母一样，倒一下顺序，然后，对称写后面，最后加上括号.

Summary

LM.

(15) If $z = 8+6i$ and $w = 1-i$ then $z - \alpha \cdot (\overline{w^2})$, $\alpha \in \mathbb{C}$. has principle argument $\pi/4$. if:

(a) $Re(\alpha) + Im(\alpha) = 1$ and $Re(\alpha) < 3$

(b) $Re(\alpha) + Im(\alpha) = -1$ and $Re(\alpha) > 3$

(c) $Re(\alpha) + Im(\alpha) = 1$ and $Re(\alpha) > 3$

(d) $Re(\alpha) + Im(\alpha) = -1$ and $Re(\alpha) < 3$

(e) None of the above.

错解: $z = 8+6i - \alpha \cdot (2i)$

$= 8 + (6-2\alpha)i$

$8 = 6-2\alpha \quad \alpha = \frac{1}{2}$ ∴ 选 d.

没有用到 $\alpha \in C$ 是隐藏的条件. 不可以直接代入计算

正解. let $\alpha = x + yi$.

$8+6i - 2i(x+yi)$

$= (8+2y) + (6-2x)i$

$8+2y = 6-2x > 0.$

$$\begin{cases} 8+2y = 6-2x \Rightarrow x+y=-1 \\ 6-2x>0 \Rightarrow x<3. \end{cases}$$

选 d.

Summary

I find that it's very easy to treat complex number as real number when doing some calculations, especially the question that which requires you to get a conclusion from a given equation

PLZ Read question carefuly!

George Willsons

LM. (26) For which values of $t \in R$ are the vetors $\vec{a} = \vec{k}$ $\vec{b} = \vec{i}$ and $\vec{c} = t\vec{i} + \vec{j}$ co-planar?

(a) 1

(b) −1

(c) ~~R~~ 2

(d) −2

(e) None of the above.

By using the cross product, we can check the relationship between three vectors.

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$= K \cdot (\vec{i} \times (t\vec{i} + \vec{j}))$

$= K \cdot (i \times i \cdot t + i \times j) \quad \rightarrow \perp \cdot \sin\frac{\pi}{2} = 1$

$= K \cdot (0 + 1)$

$= K \neq 0.$

$\Leftrightarrow$ they are not co-planar.

LM. (27) If $\vec{a} = \alpha\vec{b} + \beta\vec{c}$, α and β are non-zero real numbers, then, the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$ is:

(a) 0 (c) β (e) None.

(b) α (d) $\alpha\beta$

take $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\alpha\vec{b} + \beta\vec{c}) \cdot (\vec{b} \times \vec{c}) \quad \perp$

$= \alpha\vec{b} \cdot (\vec{b} \times \vec{c}) + \beta\vec{c} \cdot (\vec{b} \times \vec{c})$

$= 0 + 0$

$= 0.$ 选 (a)

Summary

Linear mathematic extra lecture note.

SCALAR TRIPLE PRODUCTS.

$[a, b, c] = a\cdot(b\times c) = (b\times c)\cdot a$

$a\cdot(b\times c) = (a\times b)\cdot c$

a, b, c REMAIN CYCLICALLY ORDERED,

BUT INTERCHANGE "$\cdot$" And "$\times$"

$\therefore a\cdot(b\times c) = (a\times b)\cdot c = (b\times c)\cdot a = b\cdot(c\times a)$

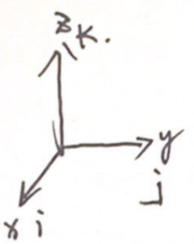
Since ORDER OF VECTORS IS IRRELEVANT IN '$\cdot$' PRODUCT.

SCALAR TRIPLE PRODUCTS ctd.

THE VECTORS a b and c All LIE IN THE SAME PLANE. i.e. ARE CO-PLANAR.

IF AND ONLY IF

$a\cdot(b\times c) = [a, b, c] = 0$



结合右手螺旋定则.

eg: $a = i + j - 2k \quad b = -2i + 3j + k \quad c = 4i - j - 5k$

$b\times c = (-2i + 3j + k)\times(4i - j - 5k)$

$= -14i - 6j - 10k$

CHECK: $b\cdot(b\times c) = 0$ ✓ $\quad c\cdot(b\times c) = 0$ ✓

$[a, b, c] = 0$

SO, THEY ARE CO-PLANLAR

$\therefore a = \lambda b + \mu c$

$(1, 1, -2) = \lambda(-2, 3, 1) + \mu(4, -1, -5)$

Summary

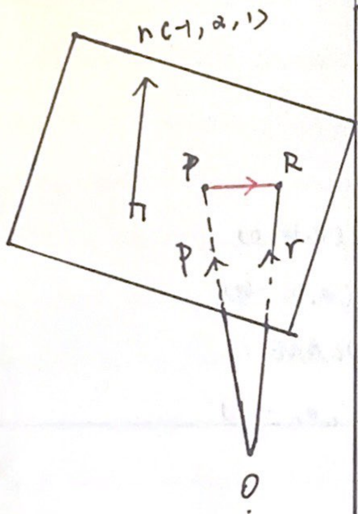
George Willsons

$(a\times b)\times c$

过c的与由a、b平面垂直的法向量

$a\times(b\times c)$

过a的与由bc平面垂直的法向量



$1 = -2\lambda + 4\mu$ ①

$1 = 3\lambda - \mu$ ②

$-2 = \lambda - 5\mu$ ③

$a = \frac{1}{2}b + \frac{1}{2}c$

$\boxed{\text{可解出来}}$

VECTOR TRIPLE PRODUCTS

$a\times(b\times c) = b(a\cdot c) - c(a\cdot b)$

PLANE THROUGH A GIVEN POINT WITH PRESCRIBED NORMAL

$p(1,2,3)$: GIVEN POINT ON PLANE

$R(x,y,z)$: GENERAL POINT ON PLANE

$n=(-1,2,1)$: ~~SEEN~~ GIVEN NORMAL TO PLANE.

$r-p = (x,y,z)-(1,2,3) = (x-1,\ y-2,\ z-3)$

LIES WITHIN THE PLANE, SO $(r-p)$ is PERPENDICULAR TO n.

$\Rightarrow (r-p)\cdot n = 0$

$\therefore (x-1,\ y-2,\ z-3)\cdot(-1,2,1) = 0$

OR $-(x-1) + 2(y-2) + (z-3) = 0$

$x - 2y - z = -6$

$\therefore 1\cdot x + (-2)\cdot y + (-1)z = -6$

$\Rightarrow n \propto (1,-2,-1)$

$r\cdot n = n\cdot p$. $\quad p=(a,b,c)$, $n=(n_1,n_2,n_3)$. given.

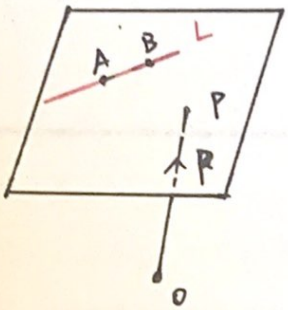
$r\cdot p = (x,y,z)\cdot(a,b,c) = ax+by+cz$

$n\cdot p = (n_1,n_2,n_3)\cdot(a,b,c) = an_1 + bn_2 + cn_3 = d$

$ax + by + cz = d$

Summary

PLANE CONTAINING A POINT AND A GIVEN LINE.



- ASSUME THE POINT DOES NOT LIE ON THE LINE

KNOWING L $\Rightarrow$ KNOWING TWO POINTS. A AND B

$\Rightarrow$ THREE GIVEN POINTS (A, B, P).

$$\begin{cases} L: \dfrac{x-1}{-1} = \dfrac{y+2}{2} = \dfrac{z-3}{2} \\ p = (1, 2, 3) \end{cases}$$

$$\begin{cases} A = (1, -2, 3) \\ B = (-1, 2, 7) \end{cases}$$

$\vec{AP} = p - a = (1, 2, 3) - (1, -2, 3) = (0, 4, 0)$

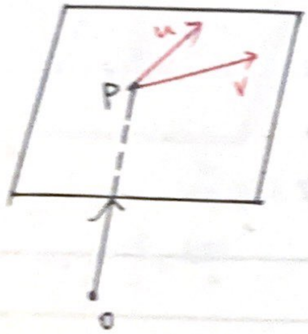
$\vec{BP} = p - b = (1, 2, 3) - (-1, 2, 7) = (2, 0, -4)$

$n = \vec{AP} \times \vec{BP}$ is NORMAL TO PLANE

$n = (0, 4, 0) \times (2, 0, -4) = (16, 0, -8)$

Summary

PLANE THROUGH GIVEN POINT CONTAINING TWO PRESCRIBED DIRECTIONS



$u = (2, -1, 3) = 2i - j + 3k$

$v = (-1, 3, 2) = -i + 3j + 2k$

$p : (1, 2, 3)$ GIVEN POINT.

- u AND v BOTH LIE IN THE PLANE

$n = v \times u$

$n = (2i - j + 3k) \times (-i + 3j + 2k)$

$= -11i - 7j + 5k$

$n = (-11, -7, 5)$

$p = (1, 2, 3)$

$p \cdot n = -10$

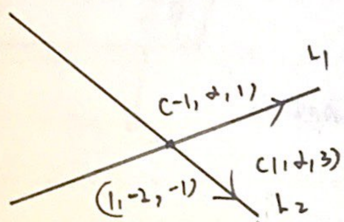
$\therefore \boxed{r \cdot n = p \cdot n}$

$(x, y, z) \cdot (-11, -7, 5) = -10$

$11x + 7y - 5z = 10.$

Summary

PLANES CONTAINING TWO STRAIGHT LINES



$\frac{x-1}{-1} = \frac{y+2}{2} = \frac{z+1}{1} = L_1$

$r = (1,-2,-1) + t(1,2,3) = L_2$

L_1: $x=1-s$, $y=-2+2s$, $z=-1+s$

OR $r = (1,-2,-1) + s(-1,2,1)$

PLANE CONTAINS THE VECTORS

$u=(-1,2,1)$ $v=(1,2,3)$ direction vector of line.

$n = u\times v = (4,4,-4)$

POINT ON PLANE P.

$p=(1,-2,-1)$

$r\cdot n = p\cdot n$.

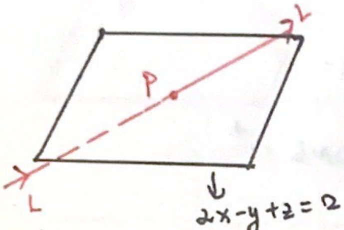
$(x,y,z)\cdot(4,4,-4) = (1,-2,-1)\cdot(4,4,-4)$

$4x+4y-4z=0$

$x+y-z=0$.

Summary

INTERSECTION OF A STRAIGHT LINE WITH A PLANE AT P.



$\frac{x+2}{1} = \frac{y-4}{-1} = \frac{z+5}{2}$

$\Rightarrow \begin{cases} x = -2+t \\ y = 4-t \\ z = -5+2t \end{cases}$

$2\times(-2+t) - (4-t) + (-5+2t) = 2$

$-13 + 5t = 2$

$\Rightarrow t = 3$

$\therefore\ x = -2+3 = 1 \quad y = 4-3 = 1 \quad z = -5+6 = 1$

$\therefore$ POINT OF INTERSECTION IS $P = (1, 1, 1)$

{VECTOR APPOACH}

$2x - y + z = 2 \Rightarrow (x, y, z)\cdot(2, -1, 1) = 2$

i.e. $\boxed{r\cdot(2, -1, 1) = 2}$. ①

$\frac{x+2}{1} = \frac{y-4}{-1} = \frac{z+5}{2} = t$

$\Rightarrow x = -2+t \quad y = 4-t \quad z = -5+2t$

$\Rightarrow \boxed{r = (x, y, z) = (-2+t, 4-t, -5+2t) = (-2, 4, -5) + t(1, -1, 2)}$ ②

$[(-2, 4, -5) + t(1, -1, 2)]\cdot(2, -1, 1) = 2$

$\Rightarrow (-2, 4, -5)\cdot(2, -1, 1) + t(1, -1, 2)\cdot(2, -1, 1) = 2$

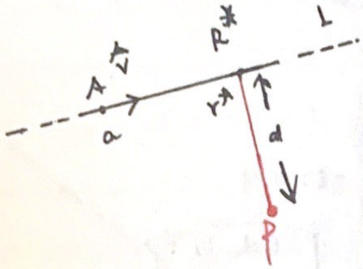
$5t = 15, \ t = 3.$

Summary

5 . .

SHORTEST DISTANCES

• POINT FROM A LINE



vector equation of L

$$r = a + t\hat{v}$$

R^* on L $\Rightarrow$ $r^* = a + t^*\hat{v}$ FOR SOME t^*

ALSO

$$r^* + \overrightarrow{R^*P} = p$$

i.e. $\overrightarrow{OR^*} + \overrightarrow{R^*P} = \overrightarrow{OP}$

$\overrightarrow{R^*P}$ PERPENDICULAR TO L $\Rightarrow$

$$\overrightarrow{R^*P} \cdot \hat{v} = 0$$

$\therefore$ $\overrightarrow{R^*P} = p - r$ $\Rightarrow$ $(p - r^*) \cdot \hat{v} = 0$

$\therefore$ $r^* \cdot \hat{v} = p \cdot \hat{v}$

$\Rightarrow$ $(a + t^*\hat{v}) \cdot \hat{v} = p \cdot \hat{v}$

$t^* = p \cdot \hat{v} - a\hat{v} = (p - a) \cdot \hat{v}$

$r^* = a + [(p-a) \cdot \hat{v}]\hat{v}$

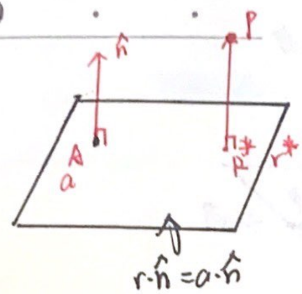
$\overrightarrow{R^*P} = p - r^*$

$= p - a - [(p-a) \cdot \hat{v}]\hat{v}$

$d = \| \overrightarrow{R^*P} \|$

$d = \| p - a - [(p-a) \cdot \hat{v}]\hat{v} \|$

Summary



POINT FROM A PLANE

- $\overrightarrow{R^*P} = \mu\hat{n}$
- R^* LIES ON PLANE, SO

$$r^*\cdot\hat{n} = a\cdot\hat{n}$$

$$\overrightarrow{OP} = \overrightarrow{OR^*} + \overrightarrow{R^*P} \Rightarrow p = r^* + \mu\hat{n}$$

$$\therefore \quad p\cdot\hat{n} = r^*\cdot\hat{n} + \mu\hat{n}\cdot\hat{n}$$

$$= a\cdot\hat{n} + \mu$$

$$\Rightarrow \mu = (p-a)\cdot\hat{n}$$

SO $p = r^* + \mu\hat{n} \Rightarrow$

$$r^* = p - [(p-a)\cdot\hat{n}]\hat{n}$$

Distance: $d = \|\overrightarrow{R^*P}\| = \|p - r^*\|$

$$= \|[(p-a)\cdot\hat{n}]\hat{n}\|$$

$$= |(p-a)\cdot\hat{n}|\,\|\hat{n}\|$$

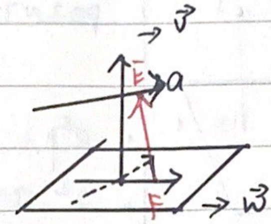
$$= |(p-a)\cdot\hat{n}|$$

$$d = |(p-a)\cdot\hat{n}|$$

Summary

George Willsons

空间向量距离问题



① 两条异面直线距离

先求两条异面直线的

公共法向量（$\vec{v}\times\vec{w}$）叉乘

然后取两点 E、F、

距离即为 EF 在公共法向量上的投影

$$d = \frac{|\vec{EF}\cdot\vec{n}|}{|\vec{n}|} = \frac{\|(f-e)\cdot(v\times w)\|}{\|v\times w\|}$$

APP

⑱ A particle moves in three dimensions at constant (nonzero) speed and nonzero acceleration. Show that its velocity and acceleration vectors must be perpendicular to one another.

Constant speed. $S = |v| =$ constant.

$$\underline{v}\cdot\underline{v} = \text{constant}$$

$$\Rightarrow \frac{d}{dt}(\underline{v}\cdot\underline{v}) = 0$$

$$\Rightarrow \frac{dv}{dt}\cdot\underline{v} + v\cdot\frac{dv}{dt} = 0$$ (product rules.)

$$\Rightarrow \underline{a}\cdot\underline{v} + \underline{v}\cdot\underline{a} = 0$$

$$\Rightarrow 2\,\underline{a}\,\underline{v} = 0$$

$$\Rightarrow \underline{a}\cdot\underline{v} = 0$$

$$\therefore\ \underline{a} \perp \underline{v}.$$

Summary

Calculus

(79) Suppose that f and its first $n+1$ derivatives exist on some interval $(a-\varepsilon, a+\varepsilon)$ where $\varepsilon>0$. $f^{(n+1)}$ is continuous on this interval, $f^{(k)}(a)=0$ when $0<k<n$, and $f^{(n)}(a)\neq 0$. Use Taylor's Theorem to show

(a) if n is even and $f^{(n)}(a)>0$ then f has a local minimum at a

(b) if n is even and $f^{(n)}(a)<0$ then f has a local maximum at a.

(c) if n is odd then f has neither a local minimum nor a local maximum at a.

Calculus.

(80) Use integration by parts to evaluate

$$\int_1^y \ln(a^2+x^2)/x^2\,dx$$

$$= \ln(a^2+x^2)\cdot(-x^{-1})\Big|_1^y + \int \frac{2x}{a^2+x^2}\cdot\frac{1}{x}\,dx$$

$$= -\ln(a^2+x^2)(x^{-1})\Big|_1^y + \boxed{\int \frac{1}{a^2+x^2}\,dx}$$

$\cancel{t=a\tan t}$ $x=a\tan t$

$dx = a\sec^2 t\,dt$

$$\int_1^y \frac{2a\sec^2 t}{a^2(1+\tan^2 t)}\,dt \qquad y=a\tan t. \quad t=\tan^{-1}(\tfrac{y}{a})$$

$$= \int_1^y \frac{2a}{a^2}\,dt = \int_{\tan^{-1}(\frac{1}{a})}^{\tan^{-1}(\frac{y}{a})} \frac{2}{a}\,dt = \frac{2}{a}\tan^{-1}(\tfrac{y}{a}) - \frac{2}{a}\tan^{-1}(\tfrac{1}{a})$$

Summary

(81) A factory produces nuts and bolts on two independent machines. The external diameter of the bolts is normally distributed with mean 0.5 cm. and the internal diameter of the nuts is normally distributed with mean 0.52 cm. The two machines have the same variance ~~which~~ is determined by the rate of production.

The nuts and bolts are produced at rate which corresponds to a standard deviation $\sigma = 0.01$ cm and a third machine fits each nut on to the corresponding bolt as they are produced, provided the diameter of the nut is strictly greater than that of the bolt, otherwise it rejects both.

(i) Find the probability that a typical pair of nut and bolt is rejected

(ii) If successive pair of nut and bolt are produced independently, find the probability that in 20 pairs of nuts and bolt ~~are produced independently~~. ~~find the probability that~~ at least 1 pair is rejected

(iii) The management wishes to reduce the probaby that a typical pair of nut and bolt is rejected to 0.01. What is the largest value of σ to acheive this.

probability

Application of Standard Normal distribution and transformation

Summary

5 . . .

(i) $X \to$ internal diameter $\quad Y \to$ external diameter

$X \sim N(0.52, 0.01^2) \qquad Y \sim N(0.5, 0.01^2)$

if reject. $W = X - Y \leq 0$. since. X, Y. indept.

indept $\Rightarrow$ $var(X-Y) = var(X) + var(-Y)$. $\to (-1)^2 \cdot \sigma^2$.

$W \sim N(0.52-0.5, \ 0.01^2 + 0.01^2)$

using transformation $\boxed{W \sim N(0.02, \ 0.0002)}$

$P(W \leq 0) \xrightarrow{\text{transformation}} P\left[\frac{W-0.02}{\sqrt{0.0002}} \leq \frac{0-0.02}{\sqrt{0.0002}}\right]$

$= P(Z < -\sqrt{2})$ where $\boxed{Z = \frac{W-0.02}{\sqrt{0.0002}} \sim N(0,1)}$

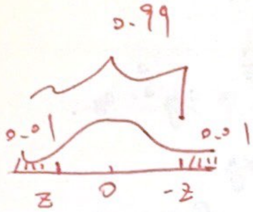
$\Phi(-\sqrt{2}) = \Phi(-1.414) = 0.0787$

(ii). no pair reject $\to (1-0.0787)^{20}$.

$P(\geq 1) = 1 - (1-0.0787)^{20} = 0.8059$

(iii) a pair is rejected

$\Phi = \left(\frac{-0.02}{\sqrt{\cancel{0.0002}}}\right) = 0.01$ ~~from~~
$\sqrt{2\sigma^2}$



$z = \Phi^{-1}(0.01)$

$-z = \Phi^{-1}(0.99)$

$= 2.3263$

$z = -2.3263$

$\therefore \frac{-0.02}{\sqrt{2\sigma^2}} = -2.3263$.

$\frac{-0.02}{\sqrt{2} \cdot \sigma} \leq -2.3263$ from table.

$\sigma \leq 0.02 / (2.3263\sqrt{2}) = 0.0061$

$\therefore \sigma_{max} = 0.0061$

Summary

Probability

82 The continuous random variable X has probability density function.

$$f_X(x) = \begin{cases} 6x(1-x) & 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density function of $Y = X^3$.

$0 < y < 1$

$$F_Y(y) = P(Y \le y) = P(X^3 \le y)$$
$$= P(X \le y^{\frac{1}{3}})$$
$$= \int_0^{y^{\frac{1}{3}}} 6x(1-x)\,dx$$
$$= \left[3x^2 - 2x^3\right]_0^{y^{\frac{1}{3}}} = 3y^{\frac{2}{3}} - 2y.$$

$f_Y(y) = F_Y'(y) = 2(y^{-\frac{1}{3}} - 1)$ for $0<y<1$.

if $y \le 0$. $F_Y(y) = P(Y \le y) = 0$ so $f_Y(y) = 0$.

if $y \ge 1$ $F_Y(y) = P(Y \le y) = 1$ $f_Y(y) = 0$.

$$f_Y(y) = \begin{cases} 2(y^{-\frac{1}{3}} - 1) & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

这种一定要分类讨论.
分布必须将范围覆盖完全.

$$F_Y(y) = \begin{cases} 0 & Y \le 0 \\ 3y^{\frac{2}{3}} - 2y & 0<Y<1 \\ 1 & Y \ge 1 \end{cases}$$

Summary

Probability.

(83) Let X be a continuous random variable with distribution function F. Show that the random variable Y, defined by $Y = F(X)$, has a Uniform $(0,1)$ distribution.

$$P(Y \le y) = P(F(x) \le y) = P(X \le F^{-1}(y))$$
$$= F(F^{-1}(y))$$
$$= y \quad , \underline{0 < y < 1}.$$

$\therefore Y \sim U(0,1)$.

Probability.

(84) Let $X \geq 0$ be continuous random variable

$$\int_0^\infty P(X > x)\,dx = E(X).$$

(Hint: Using integration by parts. $P(X>x) = 1 - F_X(x)$)

$$\int_0^\infty P(X>x)\,dx = \int_0^\infty 1 - F_X(x)\,dx$$

$f(x) = F_X'(x)$

$\therefore f(x) = F_X'(x)$

$\therefore$ 我们把 1 设为 u.

$(1-F(x))$ 设为 $\frac{du}{dx}$.

$$= \int_0^\infty 1 \times (1 - F_X(x))\,dx$$
$$= [x(1-F_X(x))]_0^\infty + \int_0^\infty \underline{x f(x)}\,dx$$
$$= 0 + E(X)$$
$$= E(X).$$

Summary

probability

to get mode.
use. 1^{st} derivative.

(85) Let X have a Weibull distribution with PDF.

$$f_X(x) = \begin{cases} \beta x^{\beta-1} e^{-x^\beta} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

where $\beta > 0$ is a parameter. Find the mode of X.

$\therefore$ For $x > 0$. $f_X'(x) = \beta\left((\beta-1)x^{\beta-2}e^{-x^\beta} - x^{\beta-1}\beta x^{\beta-1}e^{-x^\beta}\right)$.

$$= \beta x^{\beta-2} e^{-x^\beta}(\beta - 1 - \beta x^\beta)$$

if $\beta \leq 1$. $f_X'(x) \leq 0$ for $x > 0$. so the mode $X = 0$

let $f_X'(x) = 0$.

$$x = \left(\frac{\beta-1}{\beta}\right)^{\frac{1}{\beta}}$$

Summary

Notice: when get the pdf. we should check the domain of x. $\begin{cases} x > 0 \\ x \leq 0. \end{cases}$ respectively.

Probability

对比 (108)

(86) A field contains n animals and traps are set each day. Each animal has the same probability p of being trapped on a given day and trapped animals are not released. Animals are trapped independently and for each animal, its fates on different days are mutually independent. Show that the expected number of animals trapped on the rth day is $np(1-p)^{r-1}$ $(r=1,2,\dots)$

这个题太经典了. 关键就是label这些小动物 $\begin{cases}1 & 抓到就计数1.\\ 0 & 未抓到就计数0.\end{cases}$ 因为题要求我们求抓到的个数

然后. 总个数要从每一个动物算起, 即每个动物被抓期望. 然后全部加起来.

Label the animal $1,2,\dots n$. For $i=1,2,\dots n$ let

$$X_i = \begin{cases} 1 & \text{if animal } i \text{ trapped on day } r \\ 0 & \text{otherwise.} \end{cases}$$

and let N = number of animal trapped on day r.

Then $N = X_1 + X_2 + \dots + X_n$. so

$$E[N] = E[X_1 + X_2 \dots X_n] = E(X_1) + \dots E(X_n)$$

$$E[N] = nE(X_1)$$

$$\begin{aligned} E[X_1] &= P(X_1 = 1) \\ &= P(\text{animal 1 trapped on day } r) \\ &= \underbrace{(1-p)^{r-1}}_{\text{animal 1 not trapped on day } 1,2,\dots r-1}p \end{aligned}$$

$$E[N] = n(1-p)^{r-1}p.$$

Summary

Probability

⑧ Let X be continuous random variable with pdf:

$$f(x) = \begin{cases} kx^3, & 0<x<1 \\ ke^{1-x}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Evaluate K and find the distribution function of X

(b) Calculate $P(0.5<X<2)$ and $P(X>2 \mid X>1)$

$F_X(x)=\int_{-\infty}^{x} f(x)\,dx.$

(a). $\int_0^1 kx^3\,dx + \int_1^{\infty} ke^{1-x} = 1 \qquad k=\frac{4}{5}.$

$$\boxed{F_X(x) = \int_{-\infty}^{x} f(x)\,dx}$$

这个题求分布不同于以往的分布. 这题有3个段. 而且. 在到正无穷的地方竟然不是单纯的1. 段数一多了. 最容易错的地方也显现出来了 那就是很容易忘掉累加之前的部分.

① $F_X(x)=0.$ if $x\leq 0.$

② $F_X(x)=\int_0^1 \frac{4}{5}x^3\,dx = \frac{1}{5}x^4$ if $0<x\leq 1$

③ $F_X(x) = \cancel{\quad}$

$= \boxed{\int_0^1 \frac{4}{5}x^3\,dx + \int_1^x \frac{4}{5}e^{1-x}\,dx}$ don't forget to add the previous one

$= 1-\frac{4}{5}e^{1-x}$ if $x>1$

$$\therefore F_X(x) = \begin{cases} 0 & \text{if } x\leq 0 \\ \frac{x^4}{5} & \text{if } 0<x\leq 1 \\ 1-\frac{4}{5}e^{1-x} & \text{if } x>1 \end{cases}$$

Summary

(b) $P(\frac{1}{2}<x<2)$

$= \int_{\frac{1}{2}}^{1} \frac{1}{5}x^{3}\,dx + \int_{1}^{2} \frac{4}{5}e^{1-x}\,dx$

$= \left[\frac{x^4}{5}\right]_{\frac{1}{2}}^{1} + \left[-\frac{4}{5}e^{1-x}\right]_{1}^{2}$

$= \frac{3}{16} + \frac{4}{5}(1-e^{-1}).$

或. $P(X<2) - P(X<\frac{1}{2})$

$= F_X(2) - F_X(\frac{1}{2})$

$= (1-\frac{4}{5}e^{1-2}) - (\frac{1}{5}\times 0.5^4)$

$= \frac{79}{80} - \frac{4}{5}e^{-1}$

$P(X>2) = 1 - P(X<2) = 1-(1-\frac{4}{5}e^{1-2}) = \frac{4}{5}e^{-1}$

$P(X>1) = 1 - P(X<1) = 1-\frac{4}{5} = \frac{4}{5}$

$\therefore P(X>2 \mid X>1) = \dfrac{P(X>2)}{P(X>1)} = e^{-1}$

Probability

88 If X is a negative exponential random variable with parameter $\lambda=1$, i.e.

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x>0, \\ 0 & \text{if } x\le 0, \end{cases}$$

find the pdf of the random variable $Y=\log X$.

$F_Y(x) = P(Y\le x) = P(\log X \le x)$

自变量得以统一. $= P(X\le e^x)$

$= \int_0^{e^x} e^{-y}\,dy$

$= 1-\exp(-e^x)$

$f_Y(x) = \frac{d}{dx}F_Y(x) = \exp(x-e^x)$ $(x\in\mathbb{R})$.

我想说的是最后这个函数理应是用 y 来表达的吧.

Summary

Calculus

(89) Trigonom Substitution.

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Let $x = a\tan t$ $\quad \frac{dx}{dt} = a\sec^2 t.$ $\quad dx = a\cdot\sec^2 t\cdot dt$

$$\int \frac{1}{a^2\tan^2 t + a^2}\cdot a\cdot\sec^2 t\cdot dt$$

$$= \int \frac{a\sec^2 t}{a^2(1+\tan^2 t)}\,dt$$

$$= \int \frac{1}{a}\cdot dt.$$

$$= \frac{1}{a}\cdot t$$

$$= \frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Summary

Trigonometric Substitutions.

$\sqrt{a^2-x^2}$ $\sqrt{a^2+x^2}$ $\sqrt{x^2-a^2}$

Basic Substitutions

$x = a\tan\theta$ $x = a\sin\theta$ $x = a\sec\theta$

With $x = a\tan\theta$

$$a^2+x^2 = a^2+a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

With $x = a\sin\theta$

$$a^2-x^2 = a^2-a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

With $x = a\sec\theta$

$$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta-1) = a^2\tan^2\theta$$

Summary