5

L.M.

① Find all the 6^{th} roots of -64, that is $(-\frac{1}{64})^{1/6}$

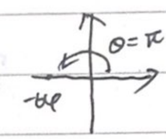
i) in exponential form

ii) in the form of $x+iy$.

Solution:

i) we want to solve z, where $z^6 = -64$

Change the form firstly ⇐ $-64 = 64 \cdot e^{i \cdot \pi}$



ii) $z_k = 64^{1/6} \cdot e^{i(2k\pi+\pi)/6}$ $\quad k = 0, 1, 2, 3, 4, 5$

$k=0 \quad z_0 = 64^{1/6} \cdot e^{i\frac{\pi}{6}} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{3} + i$

$k=1 \quad z_1 = 64^{\frac{1}{6}} \cdot e^{i \cdot \frac{\pi}{2}} = 2i$

$k=2 \quad z_2 = 64^{\frac{1}{6}} \cdot e^{i(5\pi/6)} = 2 \cdot (-\frac{\sqrt{3}}{2} + i\frac{1}{2}) = -\sqrt{3} + i$

$k=3 \quad z_3 = -\sqrt{3} - i$

$k=4 \quad z_4 = -2i$

$k=5 \quad z_5 = \sqrt{3} - i$

L.M

② Consider $z = 1 + \frac{w}{i}$, where $w \in \mathbb{C}$ statisfies $w^2 + w + \frac{1-i}{4} = 0$. Express z in the form $z = a+ib$, where the values $a, b \in \mathbb{R}$ are to be determined.

key step ⇐ $\boxed{w = \frac{-b \pm z}{2a}}$ $\quad z^2 = b^2 - 4ac$ ---- $z^2 = i$, $i = 1 \cdot e^{i\frac{\pi}{2}}$

$z_k = 1^{\frac{1}{2}} e^{i(2k\pi + \frac{\pi}{2})/2}$. $\quad z_1 = e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2}(1+i)$ $\quad z_2 = e^{\frac{9\pi}{4}i} = \frac{\sqrt{2}}{2}(1+i)$

$\therefore w_1 = \frac{-1 + \frac{\sqrt{2}}{2}(1+i)}{2}$ $\quad w_2 = \frac{-1 - \frac{\sqrt{2}}{2}(1+i)}{2}$

Summary

$\boxed{\text{Theorem 1.5}}$ (Solution of complex polynomial equation)

Let. a be NON-zero complex number, $n > 0, n \in \mathbb{Z}$.

So i.e. n different solutions to the equation $z^n = a$, if $a = Re^{i\delta}$

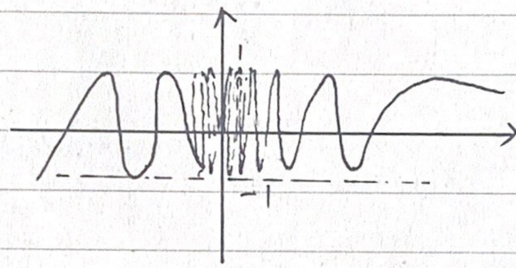
$z_k = R^{\frac{1}{n}} e^{i(2k\pi + \delta)/n}$, for $k = 0, 1, \cdots n-1$. 注意.

Calculus

③ Investigate $\lim_{x \to 0} \sin \frac{1}{x}$.

ANS: $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

the value of $\sin(\frac{1}{x})$ oscillate between 1 and -1 infinitely often as $x \to 0$.



Calculus

④ Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

we cannot use $\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$ because

$\lim_{x \to 0} \sin(1/x)$ does not exist. However since $-1 \leq \sin \frac{1}{x} \leq 1$

$$\boxed{-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2}$$

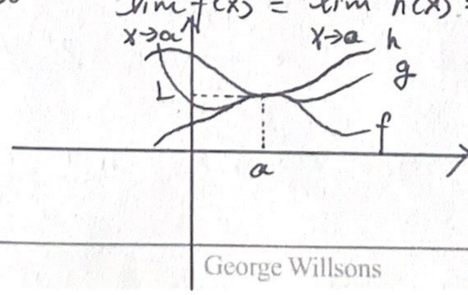
we know that $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} (-x^2) = 0$

Taking $f(x) = -x^2$ $g(x) = x^2 \sin(\frac{1}{x})$ $h(x) = x^2$ in Squeeze Theorem we obtain, $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

Summary

$\boxed{\text{The SQUEEZE THEOREM}}$. If $f(x) \leq g(x) \leq h(x)$, when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then

$\lim_{x \to a} g(x) = L$



本学期钻研的最深的一门课就是 probability，历经期中复习，期末系统复习以及专题复习，所有的错题全部都在这本改错本里了，高中也用改错本，不过大学由于科目较少而且学术资料渠道齐全让我正经地享受了一番自主做数学的乐趣。考试还有三天，现在我准备从头开始温习这些题，并希望能在考前有多次完成这个浩大工程的机会。

Probability

⑤ Toss 3 coins, All outcomes (eg. HTH) are equally likely and have probability 1/8.

Let E = 'Tail on 1st toss, and at least one head on'

F = 'Exactly one pair of consecutive heads'

list all outcomes ⇐

Then $E = \{THH, \cancel{HH} TTH, THT\}$

$F = \{HHT, THH\}$

$$p(E|F) = \frac{p(E\cap F)}{p(F)} = \frac{p(THH)}{p(F)} = \frac{1/8}{1/4} = \frac{1}{2}$$

$$p(F|E) = \frac{p(E\cap F)}{p(E)} = \frac{p(THH)}{p(E)} = \frac{1/8}{3/8} = \frac{1}{3}$$

Probability

⑥ 10% of men and 8% of women carry a certain gene. An individual is randomly chosen from a group of 25 men and 35 women.

i) what the probability ~~of that~~ they carry the gene?

ii) if they do have the gene, what is the probability they are a) male b) female

iii) What is the probability that they are female and do not have the gene?

Summary

$\boxed{\text{Theorem of Total Probability}}$ Let $E_1, E_2 \ldots$ be a partition of Ω and Let $F \subseteq \Omega$ be any event. Then $p(F) = \sum_i p(F|E_i)\cdot p(E_i)$

Let G = 'carries gene', M = 'individual male'

F = 'individual female'.

This is the premise condition. ⇐ $\{M, F\}$ is a partition $M \cup F = \Omega$ $M \cap F = \emptyset$.

Finding the partition firstly

i) $p(G) = p(G|M)p(M) + p(G|F) \cdot p(F)$

$= 0.1 \times \frac{25}{60} + 0.08 \times \frac{35}{60}$

$= 53/600$

ii) (a) $p(M|G) = \frac{p(M \cap G)}{p(G)} = \frac{p(G|M) \cdot p(M)}{p(G)} = \frac{0.1 \times (25/60)}{53/600}$

$= 25/53$

Applying the result $p(A|F) = 1 - p(A^c|F)$ ⇐ (b) $p(F|G) = p(M^c|G)$

$= 1 - p(M|G)$

$= 28/53$

iii) $p(F \cap G^c) = p(G^c|F)\, p(F)$

$= [1 - p(G|F)]\, p(F)$

$= (1 - 0.08) \cdot (35/60)$

$= 161/300$

Probability

⑦ 3 cards: One is red on both sides. One is black on both side. One is red one side, black other side.

The cards are shuffled and one is chosen at random.

If the upper side of the chosen card is red, what is the probability that the other side is red.

Summary

$\boxed{\text{Bayes' Theorem}}$

Simple form: for events E and F $p(E|F) = \frac{p(F|E)p(E)}{p(F)}$

General form: If $E_1, E_2, \cdots$ forms a partition of Ω, then

$$p(E_i|F) = \frac{p(F|E_i)\, p(E_i)}{\sum_i p(F|E_i)\, p(E_i)}$$

The key step is to find the partition

命名的过程十分重要 然后分别考虑内部的概率情况

Let F = 'Upper side red'. E_1 = 'RR card' E_2 = 'BB card' and E_3 = 'RB card'.

We want $p(E_1|F)$.

$p(E_1) = p(E_2) = p(E_3) = 1/3$

$p(F|E_1) = 1 \quad p(F|E_2) = 0 \quad p(F|E_3) = \frac{1}{2}$

Bayes' Theorem $\Leftarrow$

$$p(E_1|F) = \frac{p(F|E_1)\,p(E_1)}{p(F|E_1)\,p(E_1) + p(F|E_2)\,p(E_2) + p(F|E_3)\,p(E_3)}$$

$$= \frac{1 \times \frac{1}{3}}{1\times\frac{1}{3} + 0\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3}}$$

$$= \frac{2}{3}$$

Probability

⑧ The Birthday Problem: How large a group do you need to have at least a 50% chance that at least two people share the same birthday?

$p(k) := P($k people have no shared birthday$)$

$$p_k = \frac{365 \times 364 \times \cdots (365-k+1)}{365^k}$$

$1 - p_k \geq 0.5$ i.e. $p_k \leq 0.5$

$p_1 = 1$, $p_2 = 0.9973$, $p_3 = 0.9918 \cdots$ $p_{23} = 0.4961$

$\therefore$ need 23 people.

所以说，一个班里有同生日的人是非常大概率的.

Summary

Probability

9 Show that De Morgan's Law is true

i.e. for $E, F \subseteq \Omega$

i) $(E \cup F)^c = E^c \cap F^c$

ii) $(E \cap F)^c = \cancel{E^c \cap F^c}\ E^c \cup F^c$

De Morgan's Law

$(E \cup F)^c = E^c \cap F^c$

$(E \cap F)^c = E^c \cup F^c$

proof: i)

if $x \in (E \cup F)^c \Rightarrow x \notin E \cup F \Rightarrow x \notin E$ and $x \notin F$

$\Rightarrow x \in E^c \cap F^c$

proof ii)

if $x \in (E \cap F)^c \Rightarrow x \notin E \cap F \Rightarrow x \notin E$ or $x \notin F$

$\Rightarrow x \in E^c \cup F^c$

Probability

10 Suppose that Ω is a sample space on which are defined a probability measure P and events. E, F, and G. Use the axioms of probability prove.

i) $P(E \cap F) \geq P(E) + P(F) - 1$

ii) if $E \subseteq F$ then $P(E) \leq P(F)$

iii) $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$

(Hint. Set $F \cup G = H$, and write down an expression for $P(E \cup H)$

iv) $P(E \cup F \cup G) \leq P(E) + P(F) + P(G)$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\boxed{\text{But } P(E \cap F) > 0 \quad \therefore -P(E \cap F) < 0}$

$\therefore P(E \cup F) \leq P(E) + P(F)$

iv). $P(E \cup F \cup G) = P(E \cup H) \leq P(E) + P(H)$

$\because P(H) = P(F \cup G) \leq P(F) + P(G)$

$\therefore P(E \cup F \cup G) \leq P(E) + P(F) + P(G)$

$P(\bigcup_{j=1}^{n} E_j) = P(E_1 \cup E_2 \cup \cdots E_n) \leq P(E_1) + P(E_2) + \cdots + P(E_n) = \sum_{j=1}^{n} P(E_j)$

Summary

Boole's inequality: $P(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} P(E_j)$.

$\boxed{\text{Three axioms}}$

A_1 : $P(E) \geq 0$ for any even E

A_2 : $P(\Omega) = 1$

A_3 : If $E_1, E_2 \cdots$ are such that $E_i \cap E_j = \phi$ for $i \neq j$, then

$P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots$

proof

i) $p(E\cup F) = p(E)+p(F)-p(E\cap F)$

$p(E\cup F) \le 1$ Axiom 1 $\quad 1-p(E\cup F)^c \ge 0$

$p(E)+p(F)-p(E\cap F) \le 1$

$p(E\cap F) \ge p(E)+p(F)-1$

ii) common trick $F=(F\cap A)\cup(F\cap A^c)$

common trick ⇐ 用常用的一个小技巧 将F展开进而用交集将其化简.

$F=\underbrace{(F\cap E)}_{E}\cup(F\cap E^c)$

$p(F) = p(E) + p(F\cap E^c)$

since $p(F)\ge 0$, $p(E)\ge 0$

$\therefore p(E)\le p(F)$

iii) proof:

~~p(E∩F~~ $p(E\cup \overbrace{F\cup G}^{H})$ set $F\cup G = H$

$= p(E)+p(H)-p(E\cap H)$

$= p(E)+p(H)-p(E\cap(F\cup G))$

$\Downarrow$

$p((E\cap F)\cup(E\cap G))$

distributive law ⇐

$= p(E)+\overbrace{p(F)+p(G)-p(F\cap G)}-p((E\cap F)\cup(E\cap G))$

$= p(E)+p(F)+p(G)-p(F\cap G)-p(E\cap F)-p(E\cap G)$

$+\, \underset{\Downarrow}{p(E\cap F\cap E\cap G)}$

$p(E\cap F\cap G)$

$= p(E)+p(F)+p(G)-p(E\cap F)-p(E\cap G)-p(F\cap G)+p(E\cap F\cap G)$

Summary

The Addition RULE

$p(A\cup B)=p(A)+p(B)-p(A\cap B)$

When A, B are mutually exclusive or disjoint simplified to

$p(A\cup B)=p(A)+p(B)$

CHECKING FOR INDEPENDENCE

$p(A\cap B)=p(A)\cdot p(B)$ or

$p(B|A)=p(B)$ or

$p(A|B)=p(A)$

probability

(1) Show , by an intuitive argument , that for $n \geq 1$

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

(Hint : How many possible subsets are there of n objects)

Probability

(12) Understanding the Bayes' Theorem deeply

colorblind → men, women → S (sample space).

B : the person selected is a men

B^c : the person selected is a women

event A → A and B, A and B^c

$A = (A \cap B) \cup (A \cap B^c)$

mutually exclusive

⇓ to General

Sample space canbe partitioned into K subpopulations. $S_1\ S_2\ S_3 \cdots S_K$. ⇒ are mutually exclusive and exhaustive.

The total Probability ⇐

$A = (A \cap S_1) \cup (A \cap S_2) \cup (A \cap S_3) \cup \cdots \cup (A \cap S_K)$

Summary

$\boxed{\text{Binomial Theorem}}$ for any integer n.

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Probability

POKER:

四种花色: heart 红桃

spade 黑桃 剑

club 梅花 三叶草

diamond 方块

Joker 大小王

Jack J

Queen Q

King K

Ace A.

face card/court card 花牌 (JQK)

straight 顺子

royal flush 同花大顺

straight flush 同花顺

full house 三张相同和两张相同的牌

four of kind 四张相同的牌

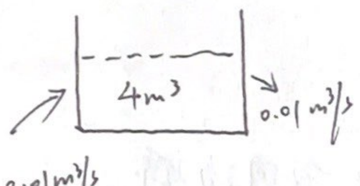
Summary

App.

(13) A tank contains 2kg of salt dissolved in $4m^3$ of water. Starting at time $t=0$. a salt solution of concentration $2\,kg/m^3$ flows into the tank at an inflow rate of $0.01\,m^3/s$. Assume that this incoming salt solution is rapidly mixed with the solution in the tank. The well mixed solution leaves the tank at an outflow rate $0.01\,m^3/s$.

a) Derive an ODE for $m(t)$. the mass of salt in the tank and an initial condition for $m(t)$.

b) Solve the ODE in (a) to detemine $m(t)$. Sketch and describe its behaviour in words.



$m(0) = 2$. Let $m =$ ~~salt of~~ mass of salt.

Flow in − Flow out $= 0.01 - 0.01 = 0$. volume is constant.

$\frac{dm}{dt} =$ salt flow in − salt flow out → salt in tank

$= 2kg \times 0.01\,m^3/s - 0.01\,m^3/s \times \boxed{\frac{m}{4m^3}}$ fraction.

separable 1st order ODE ⇐

$= 0.02 - 0.0025\,m.$

$\int \frac{dm}{0.02 - 0.0025m} = \int \cdot\, dt$

$\frac{\ln(0.02 - 0.0025m)}{-0.0025} = t + k$

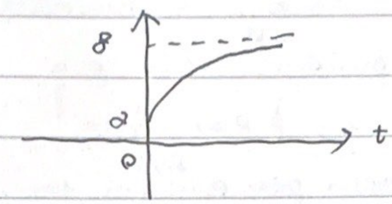
put the constant forward ⇐

$0.02 - 0.0025m = C \cdot e^{-0.0025t}$

$m(0) = 2 \quad C = 0.015$

Summary

$m = 8 - 6e^{-0.0025t}$

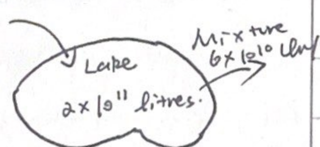


As $t \to \infty$ $m \to 8$

describe the behavior. ⇐ Mass of salt increases monotonically to 8 kg.

App

(14) A lake of volume 2×10^{11} litres has a supply of fresh water entering at a rate of 6×10^{10} litres/year. The fresh water and the lake water mix rapidly. 6×10^{10} litres of (well mixed) lake water leave the lake per year. Some pesticide is dropped into the lake, so that the concentration in the lake is five times the safe level for use by the stock that drink from the lake water. How long will it be until the pesticide concentration is back to a safe level.



Flow in − Flow out $= 6\times10^{10} - 6\times10^{10} = 0$ implies the volume of the lake is constant.

Let. p = Mass of pesticide.

$$\frac{dp}{dt} = \text{pesticide flow in} - \text{pesticide flow out}$$

$$= 0 - \left(\frac{p}{2\times10^{11}}\right)\times 6\times10^{10}.$$

flow in is zero because ⇐ pesticide has been already in the lake no more pesticide flows in.

$$\frac{dp}{dt} = -0.3p.$$

Summary

5 . .

$$\int \frac{dp}{p} = \int -0.3\,dt.$$

$$p(t) = p(0)\, e^{-0.3t}$$

Given. $p = \frac{1}{5} p(0)$, so.

we can remove the $p(0)$.

$$\frac{1}{5} p(0) = p(0)\, e^{-0.3t}.$$

$$t = \frac{\ln 5}{0.3} = 5 \text{ years.}$$

ACF

(15) Using mathematical induction to prove that $n < 2^n$. for all positive integers.

give $p(n)$ definition

Let $\boxed{p(n)}$ be the statement that $n < 2^n$, ~~for all positive integers~~.

Anchor step

$\boxed{\text{Anchor step}}$: $p(1): 1 < 2^1$

$p(1)$ is true

Inductive step.

$\boxed{\text{Inductive step}}$: $k \in N$. $p(k) \Rightarrow p(k+1)$

$\boxed{\text{Assume}}$ $p(k)$ holds: $k < 2^k$ for some n.

process.

it follows that:

$k < 2^k$.

$k+1 < 2^k + 1 \leq 2^k + 2^k = 2^k \cdot 2^k = 2^{k+1}$

Hence.

$p(k) \Rightarrow p(k+1)$

$\boxed{p(k) \Rightarrow p(k+1)}$

Conclusion.

$\boxed{\text{By mathematical induction}}$ $n < 2^n$. for all positive integers.

Summary

ACF. (16) Proposition: $2^n < n!$ for every integer $n \geq 4$.

$\begin{cases} n! = n(n-1)\cdots 1 \\ 0! = 1 \end{cases}$

Let $p(n)$ be the statement that $2^n < n!$

Anchor step: $p(4)$: $2^4 = 16 < 4! = 24$

$p(4)$ is true

Inductive step: Assume $p(k)$ holds. i.e. $2^k < k!$

for an arbitrary integers $k \geq 4$.

To show the $p(k+1)$ holds

$2^{k+1} = \boxed{2 \cdot 2^k} < 2 \cdot k!$

$< (k+1) k!$

$= (k+1)!$

So $p(k) \Rightarrow p(k+1)$

By mathematical Induction

$p(n)$ is true for $n \geq 4$.

Summary

5

ACF. ☆ (17) Proposition: $n^5 < 5^n$ for all integers $n \geq 6$.

Let $p(n)$ be the statement that $n^5 < 5^n$ for all integers $n \geq 6$.

Anchor step: $p(6)$: $6^5 = 7776 < 5^6 = 15625$

$p(6)$ is true

$(n+1)^5 = n^5 (\frac{n+1}{n})^5$

$= n^5 (1 + \frac{1}{n})^5$

$n \geq 6, \Rightarrow \frac{1}{n} \leq \frac{1}{6}$

$(1+\frac{1}{n})^5 \leq (1+\frac{1}{6})^5$

Inductive step: Assume $p(k)$ holds. $k^5 < 6^k$. $k \geq 6$.

$$LHS = (n+1)^5 = n^5 (1+\frac{1}{n})^5 \leq n^5 (1+\frac{1}{6})^5$$

$$= n^5 \cdot (2.\cdots) < n^5 \cdot 5 < 5^n \cdot 5 = 5^{n+1}$$

Hence

$p(k) \Rightarrow p(k+1)$

By mathematical induction $n^5 < 5^n$ for all integers $n \geq 6$.

LM. (18) Using only de Moivre's Theorem and Euler's formula and no other methods, prove that.

a) $\sin(5t) = 16 \sin^5 t - 20 \sin^3 t + 5 \sin t$;

b) $\cos^5 t = \frac{1}{16} \cos(5t) + \frac{5}{16} \cos(3t) + \frac{5}{8} \cos t$;

key step $\Leftarrow$ $\begin{cases} e^{it} = \cos t + i \sin t \\ e^{-it} = \cos t - i \sin t. \end{cases}$

$\Leftarrow$ $\begin{cases} \cos t = \frac{1}{2} (e^{it} + e^{-it}) \\ \sin t = \frac{1}{2i} (e^{it} - e^{-it}) \end{cases}$

Summary

$\boxed{\text{De Moivre's Theorem}}$

For any complex number x and any integer n.

$$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$$

a). $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$

$\sin^5 t = (\frac{1}{2i})^5 (e^{it} - e^{-it})^5$

$= \frac{1}{32i^5} \times (\binom{5}{0}(e^{-it})^0 (e^{it})^5 + \binom{5}{1}(-e^{-it})^1 (e^{it})^4 + \binom{5}{2}(-e^{-it})^2 (e^{it})^3$

$+ \binom{5}{3}(-e^{-it})^3 (e^{it})^2 + \binom{5}{4}(-e^{-it})^4 (e^{it})^1 + \binom{5}{5}(-e^{-it})^5 (e^{it})^0)$

$= \frac{1}{32i^5} \times (e^{i \cdot 5t} - 5 \cdot e^{i \cdot 3t} + 10 e^{i \cdot t} - 10 e^{-it} + 5 e^{-3it} - e^{-5it})$

$= \frac{1}{32i^5} \times [(e^{i \cdot 5t} - e^{-i \cdot 5t}) + 5 (e^{-i \cdot 3t} + e^{i \cdot 3t}) + 10 (e^{i \cdot t} - e^{-it})]$

$= \frac{1}{32i^5} \times [2 \cdot i \cdot \sin 5t - 10 \cdot i \cdot \sin 3t + 20 \cdot i \cdot \sin t]$

? $= \frac{1}{16i^5} i \cdot \sin 5t - \frac{5}{16i^5} \cdot i \cdot \sin 3t + \frac{5}{8i^5} i \cdot \sin t.$

b). 同理.

$\cos(t) = \frac{1}{2}(e^{it} - e^{-it})$

$\cos^5(t) = (\frac{1}{2})^5 (e^{it} - e^{-it})^5$

$= \frac{1}{32} (\binom{5}{0} (e^{-it})^0 (e^{it})^5 \cdots$

$= \frac{1}{32} \times ((e^{i5t} + e^{-i5t}) + 5 (e^{-i3t} + e^{i3t}) + 10 (e^{it} + e^{-it}))$

$= \frac{1}{32} \times (2 \cos 5t + 10 \cos 3t + 20 \cos t)$

$= \frac{1}{16} \cos 5t + \frac{5}{16} \cos(3t) + \frac{5}{8} \cos t.$

Summary

L.M. (1) Using the standard properties of the scalar (or dot) product. prove the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2+\|y\|^2).$$

$x = (x_1, x_2, x_3 \cdots x_n)$ $y = (y_1, y_2 \cdots y_n)$

$$\|x+y\|^2 = \sum_{i=1}^{n} (x_i + y_i)^2$$

$$= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} 2x_i y_i$$

$$\|x-y\|^2 = \sum_{i=1}^{n} (x_i - y_i)^2$$

$$= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} 2 x_i y_i$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} y_i^2$$

$$= 2\|x\|^2 + 2\|y\|^2$$

$$= RHS.$$

Summary

$\boxed{\text{Parallelogram Law.}}$

Let x, y be vectors. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + 2\|y\|^2).$$

Hyperbolic Trigonometric Function

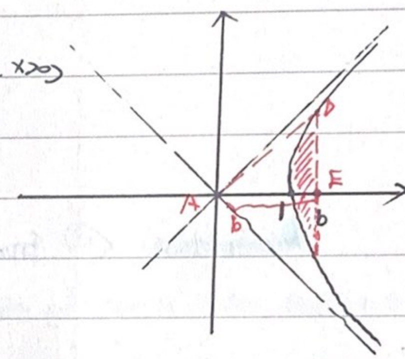
$$\begin{cases} x = \cosh a = \dfrac{e^a + e^{-a}}{2} \\ y = \sinh a = \dfrac{e^a - e^{-a}}{2} \end{cases}$$

Derivation:

$e^{\pm i\theta} = \cos\theta \pm \sin\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ $\quad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cosh a = \dfrac{e^a + e^{-a}}{2}$ $\quad \sinh a = \dfrac{e^a - e^{-a}}{2}$



proof: $x^2 - y^2 = 1$ $\quad y = \sqrt{x^2 - 1}$ $\quad (x > 0)$

Integrate from 1 to b. $\sqrt{x^2-1}$

red area $= \int_1^b \sqrt{x^2 - 1}$

$= \dfrac{b\sqrt{b^2-1} - \ln(b + \sqrt{b^2-1})}{2}$

Substract red area from $\triangle AED$

b is the distance from $A \to E$ $\quad \cancel{k = \int_1^b \sqrt{x^2-1}\, dx}$

$k = \dfrac{b\sqrt{b^2-1}}{2} - \dfrac{b\sqrt{b^2-1} - \ln(b+\sqrt{b^2-1})}{2} = \dfrac{\ln(b + \sqrt{b^2-1})}{2}$

area below x. $\quad k = \dfrac{a}{2}$.

$a = \ln(b + \sqrt{b^2 - 1})$

$x = b = \cosh a = \dfrac{e^a + e^{-a}}{2}$

$\cosh^2 a - \sinh^2 a = 1$, which is from $x^2 - y^2 = 1$

$y = \sin a = \dfrac{e^a - e^{-a}}{2}$

Summary

Lists of HTF:

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\quad \coth x = \dfrac{1}{\tanh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\quad \operatorname{sech} x = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\quad \operatorname{csch} x = \dfrac{1}{\sinh x}$

$$\boxed{e^{\pm a} = \cosh a \pm \sinh a}$$

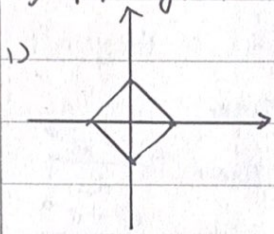
Calculus. (20) Sketch the following curves

1) $|x| + |y| = 1$

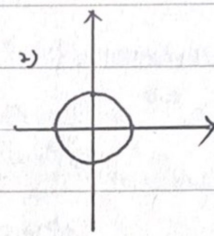
2) $|x|^2 + |y|^2 = 1$

3) $|x|^3 + |y|^3 = 1$

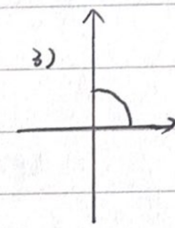
1)

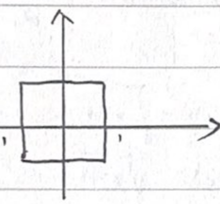


2)



3)





$|x|^n + |y|^n = 1 \quad n \to \infty.$

Calculus (21) From the definition of the hyperbolic function, verify:

1) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$e^{ia} = \cos a \pm \sin a$ ⇐ 2) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

sinx cos · same . sign. 3) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$

3) $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)} = \dfrac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$ $/\cosh y \cosh x$

$$= \frac{\frac{\sinh x}{\cosh x} + \frac{\sinh y}{\cosh y}}{1 + \tanh x \cdot \tanh y}.$$

$$= \frac{\tanh x + \tanh y}{1 + \tanh x \cdot \tanh y}.$$

Summary

Probability ㉒ Suppose that of the N people interviewed, R answer 'Yes' to the question that they received. Let Y be the event that a person gives a 'Yes' answer, E the event that they received a card asking the embarrassing question. Assuming that half the population have birthdays between January and June inclusive.

i) $P(Y)$ ii) $P(E)$ iii) $P(Y|E^c)$.

Y = Yes answer, E = Embarrassing question.

i) $P(Y) = \frac{R}{N}$

ii) $P(E) = \frac{1}{2}$

iii) $P(Y|E^c) = \frac{1}{2}$

Hence, calculate the proportion of people who answered 'Yes' to the embarrasing question.

$p(Y|E) = \frac{R - N(\frac{1}{4})}{\frac{N}{2}}$ → for non-embarrassing group.

$p(Y) = \boxed{p(Y|E)}\,p(E) + \underbrace{p(Y|E^c)\,p(E^c)}_{\frac{1}{2}\times\frac{1}{2}}$

$p(Y|E) = \frac{2R}{N} - \frac{1}{2}$

Summary

Probability (23) Write down the sample space of the eight possible boy-girl compositions of a family with three children. Assume each of the eight composition is equally likely. Let E be the event 'all the children are the same sex', and F the event 'at least two children are boys'.

i) Can you decide intuitively whether or not E and F are independent.?

ii) Now check formally (i.e. using the definition of independence)

The is a order inside.

whether or not E and F are independent.

iii) What happens for children families with two children.

Sample space: $\Omega = \{BBB, GBB, BGB, BBG, GGB, GBG, BGG, GGG\}$

ii) $E = \{BBB, GGG\}$ $F = \{BBB, BBG, BGB, GBB\}$ thus

$P(E)P(F) = \frac{2}{8} \times \frac{4}{8} = \frac{1}{8}$ However $P(E \cap F) = P(\{BBB\}) = \frac{1}{8}$

Thus $P(E \cap F) = P(E) \cdot P(F)$ so, E and F → independent.

iii) For two children $\Omega = \{BB, BG, GB, GG\}$

$E = \{BB, GG\}$ $F = \{BB\}$. $P(E)P(F) = (2/4)(1/4) = \frac{1}{8}$

$P(E \cap F) = P(\{BB\}) = \frac{1}{4}$ not independent.

Shows that independence can be hard to think about intuitively.

Summary

Probability

(20) A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5% of the population are suffers. Suppose that the test is applied to a person randomly chosen at random from the population.

i) the test will be positive

ii) the person is a sufferer, given a positive result.

iii) the person is a non-sufferer, given a negative result.

iv) the person is misclassified.

实际上让我们求事件的交集.

Let. T: test positive. S: sufferer.

$\cancel{P(T)} = P(T|S) = 0.95 \quad P(S) = 0.005$

$P(T|S^c) = 0.1$

$T = (T \cap S) \cup (T \cap S^c)$

i) $P(T) = P(T|S)\cdot P(S) + P(T|S^c)\cdot P(S^c)$

$= 0.95 \times 0.005 + 0.1 \times (1 - 0.005)$

$= 0.10425.$

贝叶斯定理中哪些是条件概率. $P(X|X)$ 哪些是积概率 $P(X \cap X)$ 要从题目中揣测.

ii) $P(S|T) = \dfrac{P(S \cap T)}{P(T)} = \dfrac{P(T|S)\cdot P(S)}{P(T)} = \dfrac{0.95 \times 0.005}{0.10425} = 0.04556$

iii) $P(S^c|T^c) = \dfrac{P(T^c|S^c)\,P(S^c)}{P(T^c)} = \dfrac{\boxed{[1 - P(T|S^c)]}\,P(S^c)}{P(T^c)} = \dfrac{0.9 \times 0.995}{0.89575} = 0.9997.$

iv) Misclassied: sufferer → negative or non-sufferer, positive.

$P(S \cap T^c) + P(S^c \cap T) = P(T^c|S)\cdot P(S) + P(S^c|T)\,P(T)$

$= [1 - P(T|S)]\,P(S) + P(T|S^c)\cdot P(S^c)$

$= (1 - 0.95) \times 0.005 + 0.1 \times (1 - 0.005)$

$= 0.09775.$

probability

* (2b) complete process.

let $J \Rightarrow$ be the person gets the job

$N \Rightarrow$ be the ~~same~~ name mentioned by director

The sample space:

$J = \{A \ B \ C\}$ $N = \{B \ C\}$ → the judge won't mention A

$$P(J=A) = P(J=B) = P(J=C) = \frac{1}{3}$$

$$P(J=A) = \underbrace{P(J=A, N=B)}_{\frac{1}{6}} + P(J=A, N=C) = \frac{1}{3}$$ → once A got job either B or C

$$P(J=B) = P(J=B, N=C) = \frac{1}{3}$$

$$P(J=C) = P(J=C, N=B) = \frac{1}{3}$$

$$\therefore P(J=A \mid N=B) = \frac{P(J=A, N=B)}{\underbrace{P(N=B)}_{\frac{1}{6}+\frac{1}{3}}} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{3}$$

$$P(J=C \mid N=B) = 1 - P(J=A \mid N=B) = \frac{2}{3}$$

Summary

George Willsons

5

Probability

(25) Suppose that E and F are two arbitrary events and that $p(E)>0$, $p(F)>0$, for each of the following statements, either prove that it is true, or give a counterexampler to show that it is not always true.

i) if $p(E)=p(F)$ then $p(E|F)=p(F|E)$

☆ ii) If $p(E|F)=p(F|E)$ then $p(E)=p(F)$

i) true: $p(E|F)=p(E\cap F)/p(F)=p(E\cap F)/p(E)$
$=p(F|E)$

ii) Statement is not necessarily true.

eg. if a fair die is rolled, E is the event that a six is rolled. and F is the event an odd number is rolled

numerator is decisive ⟸ $\boxed{p(E|F)=p(F|E)=0.}$ (since $E\cap F=0$)

but $p(E)=1/6$. $p(F)=1/2$

举个特殊例子 分子都为零. 即 $p(E\cap F)=p(E\cap F)=0$

Summary