5

$$\phi = -\frac{\sqrt{2}g}{a}\sin(\theta) + \frac{U^2}{a^2}e^{-\sqrt{2}\theta}$$

$$\dot{\theta}^2 = \frac{U^2}{a^2}\left\{e^{-\sqrt{2}\theta} - \frac{\sqrt{2}ga}{U^2}\sin(\theta)\right\}$$

e) $\phi = \dot{\theta}^2 \geq 0 \Rightarrow$

$$e^{-\sqrt{2}\theta} - \frac{\sqrt{2}ga}{U^2}\sin\theta \geq 0 \quad \text{if } \theta = \pi/2$$

$$e^{-\sqrt{2}\theta} - \frac{\sqrt{2}ga}{U^2} \geq 0$$

$$U^2 \geq \sqrt{2}ga\, e^{\frac{\pi}{\sqrt{2}}}.$$

Calculus

(122) Using an approximate Taylor Polynomial compute $\sqrt{(1.02)^3 + (1.97)^3}$

Define $f(x,y) = \sqrt{x^3+y^3}$

Note that $f(1,2) = \sqrt{1^3+2^3} = 3$, The linear approximation is $f(x,y) \approx f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$$= 3 + \left.\frac{3x^2}{2\sqrt{x^3+y^3}}\right|_{(1,2)}(x-1) + \left.\frac{3y^2}{2\sqrt{x^3+y^3}}\right|_{(1,2)}(y-2)$$

$$= 3 + \frac{3}{2\times3}(x-1) + \frac{12}{2\times3}(y-2)$$

$$= -\frac{3}{2} + \frac{1}{2}x + 2y$$

先在(1,2)处展开

后直接代值

Therefore $\sqrt{(1.02)^3+(1.97)^3}$

$$= f(1.02, 1.97)$$

$$= -\frac{3}{2} + \frac{1}{2}(1.02) + 2\times(1.97)$$

$$= 2.95$$

Compare with $\sqrt{(1.02)^3+(1.97)^3} = 2.9507$.

Summary

Taylor Theorem: i) $f(a+h) = \sum_{r=0}^{n} \frac{1}{r!} h^r f^{(r)}(a) + R_n$

$$R_n = \frac{1}{(n+1)!} h^{n+1} f^{(n+1)}(a+th) \quad 0 \leq t \leq 1$$

ii) $f(a+h, b+k) = \sum_{r=0}^{n} \frac{1}{r!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(x,y)\Big|_{a,b} + R_n$

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x,y)\Big|_{a+th,\, b+tk} \quad 0 \leq t \leq 1$$

5

Calculus

(123) Expand $f(x,y) = \sin xy$ in powers of $(x-1)$ and $(y-\frac{\pi}{2})$ to quadratic order.

Recall $f(x,y) \approx f(1,\frac{\pi}{2}) + f_x(1,\pi/2)(x-1) + f_y(1,\pi/2)(y-\frac{\pi}{2})$
$+ \frac{1}{2}[(x-1)^2 f_{xx}(1,\pi/2) + 2(x-1)(y-\pi/2) f_{xy} + f_{yy}(1,\pi/2) \cdot (y-\pi/2)^2]$

$= 1 + y\cos xy\Big|_{(1,\pi/2)}(x-1) + x\cos xy\Big|_{(1,\pi/2)}(y-\pi/2)$

$+ \frac{1}{2}(-y^2\sin xy\Big|_{(1,\pi/2)}(x-1)^2 + 2y\sin xy\Big|_{(1,\pi/2)}(x-1)(y-\pi/2) - x^2\sin xy\Big|_{(1,\pi/2)} \cdot (y-\pi/2)^2)$

$= 1 - \frac{1}{2}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2$

Summary

Linear approximation: $f(x,y) \approx f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) = L(x,y)$

Quadratic approximation:

$f(x,y) \approx f(a,b) + (x-a)f_x + (y-b)f_y + \frac{1}{2}[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}] = Q(x,y)$

5

(124) Investigate $x_{n+1} = \dfrac{x_n^2}{1+x_n^2}$.

sol

$\boxed{\text{Step I}}$: Determine which limits are possible.

Suppose $x_n \to L$, so $x_{n+1} \to L$ as $n \to \infty$

$\boxed{\text{By Algebra of Limits}}$

$$x_{n+1} = \frac{x_n^2}{1+x_n^2} \to \frac{L^2}{1+L^2}$$

$$L = \frac{L^2}{1+L^2} = f(L)$$

In other words, L is a fixpoint of f

$$L = 0 \quad \text{or} \quad 1 = \frac{L}{1+L^2} \quad \text{no root}$$

the only possible limit (x_n) is $-\infty, 0, +\infty$

$\boxed{\text{Step II}}$: Determine actual limit behaviour

if $x_0 = 0$, then $x_n = 0$.

if $x_0 = 2$. $x_0 = 2$. $x_1 = \frac{4}{5}$ $x_2 = \frac{16}{41}$ ...

so we suspect that $x_n \to 0$.

Note that $0 \le f(x) = \dfrac{x^2}{1+x^2} \le 1$. Bounded

since $0 \le x_n \le 1$ $\quad x_{n+1} = \dfrac{x_n^2}{1+x_n^2} \le x_n^2 \le x_n$ Decreasing

$\boxed{\text{By Montone Sequence Theorem}}$

x_n converges

By Step I. the only possible limit is 0

$\therefore x_n \to 0$

Summary

$\boxed{\text{Rule}}$ In general, if a sequence is given by $x_{n+1} = f(x_n)$ and if x_0 is a fixpoint $(x_0 = f(x_0))$, then the sequence (x_n) is constant : $x_n = x_0$ for all n.

5

ACF

(125) Let the sequence (x_n) be given by the rule $x_{n+1} = \frac{4x_n}{x_n+1}$, with a starting value $0 < x_0 < 3$

(i) For which non-negative real numbers L might it be possible that $x_n \to L$ as $n \to \infty$?

(ii) Prove by induction that $0 < x_n < 3$ for all $n \geq 0$. (Hint: $x_{n+1} - 3$)

(iii) By considering $x_{n+1} - x_n$, show that $x_{n+1} > x_n$ for all $n \geq 0$ (The result of (ii) should be useful)

(iv) Use (ii), (iii) and a theorem from Chapter I to deduce that (x_n) must converge to some limit. What must this limit be, and why?

Things need to be considered:

- Limit L by AL.
- Increase or decrease? } MST.
- Bounded? }

(i) Suppose $x_n \to L$. $x_{n+1} \to L$ as well. as $n \to \infty$

By Algebra of Limit: $x_{n+1} = \frac{4x_n}{x_n+1} \to \frac{4L}{L+1}$

$L = f(L) = \frac{4L}{L+1}$, $L(L+1) = 4L$, $L = 0$ or $L = 3$.

(ii) Let $p(n)$ be the statement that $\boxed{0 < x_n < 3}$

Since $0 < x_0 < 3$. $p(0)$ is true.

Assume $\boxed{p(n)}$ is true. for some $n \geq 0$.

$x_{n+1} - 3 = \frac{4x_n}{x_n+1} - 3 = \frac{x_n - 3}{x_n+1}$

Since $x_n - 3 < 0$ (Assumption) & $x_n + 1 > 0$.

$x_{n+1} < 3$

Summary

Now prove that $0 < x_{n+1} < 3$, $p(n+1)$ is true.

Hence, $p(n)$ is true for all $n \geq 0$ by induction.

iii) $x_{n+1} - x_n = \dfrac{4x_n}{x_n+1} - x_n$

$$= \frac{4x_n - x_n^2 - x_n}{x_n + 1}$$

$$= \frac{x_n(3-x_n)}{x_n+1}$$

By (ii). $3 - x_n > 0$.

$x_{n+1} - x_n > 0$.

$\therefore x_{n+1} > x_n$ $\uparrow$

(iv). $0 < x_n < 3$ bounded above & (x_n) is non-decreasing.

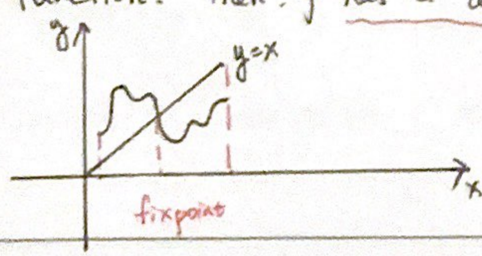
$x_n \to L$ by MST.

the only possible limit is $L = 3$.

$\therefore x_n \to 3$.

Summary

Theorem 7.5: Let $I = [a, b]$ be an interval in $\mathbb{R}$. Let $f: I \to I$ be a continuous function. Then, f has a least one fixpoint $c = f(c)$ in I.



$\boxed{\text{Proposition 7.7}}$ Let $f: I \to I$ be a function

(i) if f is a contraction, then f has exactly one fixpoint L in I.

(ii) If f is a contraction, then for any choice of $x_0 \in I$, the sequence given by $x_{n+1} = f(x_n)$. converges to the unique fixpoint $L \in I$ of f.

Proof:

(i) If L_1, L_2 are both fixpoints. then

$$|L_1 - L_2| = |f(L_1) - f(L_2)| \le K|f_1 - L_2| \quad K \in (0,1)$$

This can only happen if $L_1 = L_2$.

Our goal: $|x_n - L| \le K|x_{n-1} - L|$

(ii) 回归到 contraction 上.

$$|\boxed{x_n - L}| = |f(x_{n-1}) - f(L)| \le K|x_{n-1} - L|$$

$$|x_1 - L| \le K|x_0 - L|$$

$$|x_2 - L| \le K|x_1 - L| \le K^2|x_0 - L| \text{ etc.}$$

$$|x_n - L| \le K^n|x_0 - L| \xrightarrow{n \to \infty} 0$$

$$x_n \to L$$

$\boxed{\text{Useful Fact}}$: If $f: I \to \mathbb{R}$ is differentiable and we can find $K < 1$ s.t. that $|f'(t)| \le K$ for all $t \in I$, then f is a contraction.

非常重要

Proof: take $x < y \in I$. By. Mean Value theorem. there is $t \in (x, y)$ with $\frac{f(y) - f(x)}{y - x} = f'(t)$

By Assumption $|f'(t)| \le K$

$$\Rightarrow \left|\frac{f(y) - f(x)}{y - x}\right| \le K$$

$$|f(x) - f(y)| \le K|x - y|$$

Summary

$\boxed{\text{Contraction}}$ Let $I = [a, b]$. Let $f: I \to \mathbb{R}$ be a function. Suppose that there is a number $\boxed{0 < K < 1}$ s.t. $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in I$, then f is called a contraction on I.

f is a contraction if $\exists K \in (0,1)$ s.t. $|f(x) - f(y)| \le K|x - y|$, $\forall x, y \in I$

5

Act

(1.2.6) $f(x) = \frac{2x}{1+x^2}$ $\quad x_{n+1} = f(x_n)$

Fixpoint $= -1, 0, 1$

$f'(x) = \frac{2-2x^2}{(1+x^2)^2}$

$|f'(-1)| = 0 < 1$ attracting point

$|f'(0)| = 2 > 1$ repelling point

$|f'(1)| = 0 < 1$ attracting point

小结:

Fixpoint: $x_0 = f(x_0)$ → possible limit

Attracting point $|f'(x_0)| < 1$

Repelling point $|f'(x_0)| > 1$

Neutral point $|f'(x_0)| = 1$

Summary

Attracting Points

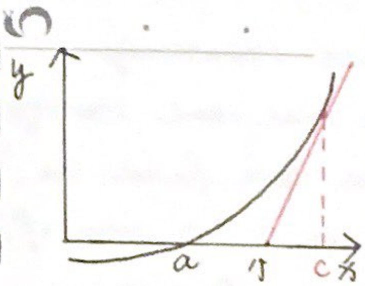
Proposition 7.7

Suppose $f(L) = L$ and the derivative $f'(L)$ exists and $|f'(L)| < 1$.

L is called attracting point.

proof: let $|f'(L)| < s < 1$. Then $|x_{n+1} - L| = |f(x_n) - f(L)| \approx (x_n - L) f'(L) \leq s|x_n - L|$

$|x_n - L| \leq s^n |x_0 - L| \to 0 \quad x_n \to L$



Suppose $f(a)=0$. we want to determine a.

Suppose c is near a

$\frac{f(x)-f(c)}{x-c} \approx f'(c)$ ↳ line.

Let $x=a$, $f(x)=0$.

$a \approx c - \frac{f(c)}{f'(c)}$

Newton-Raphson Method

Newton Function of f:

$$F(x) = x - \frac{f(x)}{f'(x)}$$

claim: if f' f'' exists.

- $f(a)=0$
- $f'(a) \neq 0$
- x_0 "close" to a

the sequence given by $x_{n+1} = F(x_n)$ converges to a

proof: $F'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$.

since $f(a)=0$ $f'(a) \neq 0$.

$\Rightarrow F(a)=0$ $F'(a)=0$

$\therefore F'(a) = 0 < 1$ so a is an attracting point

Summary

5

ACF

(12) Investigate the sequence given iteratively by $x_{n+1} = (10 - 2x_n)/3$, for all real starting value x_0. Hint: once you have found the possible limits α, write $x_{n+1} - \alpha$ in terms of $x_n - \alpha$ then write it in terms of $x_0 - \alpha$ and then consider the limit $n \to \infty$.

Do the same for the sequence given by $y_{n+1} = 6 - 2y_n$.

$x_{n+1} = (10 - 2x_n)/3$, $x_n \to \alpha$, $x_{n+1} \to \alpha$ as $n \to \infty$

By A.L. $x_{n+1} = \dfrac{10 - 2x_n}{3} \xrightarrow{n \to +\infty} \dfrac{10 - 2\alpha}{3}$

$\alpha = \dfrac{10 - 2\alpha}{3} \Rightarrow \alpha = \cancel{3}\ 2$

following the question. write it in terms of $x_n - \alpha$

possible limit.

$$\boxed{x_{n+1} - \alpha} = \frac{10 - 2x_n}{3} - 2 = \frac{10 - 2x_n - 6}{3} = \frac{-2(x_n - 2)}{3}$$

write it in terms of $x_0 - \alpha$

$x_{n+1} - \alpha = -\frac{2}{3}(x_n - 2)$ *.

$x_1 - \alpha = -\frac{2}{3}(x_0 - 2)$

$x_2 - 2 = -\frac{2}{3}(x_1 - 2) = (-\frac{2}{3})^2 (x_0 - 2)$

$x_3 - 2 = -\frac{2}{3}(x_2 - 2) = (-\frac{2}{3})^3 (x_0 - 2)$

⋮

联系 proposition 1.2 证明 发现这个 $|x_n - L|$ 的过程. 很重要.

$x_n - 2 = \boxed{(-\frac{2}{3})^n} (x_0 - 2)$

$n \to +\infty$ $(-\frac{2}{3})^n \to 0$

$x_n \to 2$ as $n \to +\infty$ regardless of the value x_0.

Summary

- 对于找到这种 n 次迭代的范围. $|x_n - L| \to 0$.
  $x_n \to L$.
- 注意 n 次方下面的数是什么范围
  $(-\frac{2}{3})^n \xrightarrow{n \to +\infty} 0$
  $(-2)^n \xrightarrow{n \to +\infty}$ 离0 越来越远.

5

Similarly : for $y_{n+1} = 6-2y_n$

Suppose $y_n \to \beta$, so $y_{n+1} \to \beta$ as $n \to +\infty$.

$\therefore \quad y_{n+1} = 6-2y_n \longrightarrow 6-2\beta$

$\beta = 6-2\beta \Rightarrow \beta = 2$

$\therefore$ Consider $y_{n+1} - 2 = 6-2y_n - 2$

$= 4-2y_n$

$= -2(y_n - 2)$

Since $y_{n+1} - 2 = -2(y_n - 2)$

$y_1 - 2 = -2(y_0 - 2)$

$y_2 - 2 = -2(y_1 - 2) = (-2)^2(y_0 - 2)$

$y_3 - 2 = -2(y_2 - 2) = (-2)^3(y_0 - 2)$

$y_n - 2 = \boxed{(-2)^n}(y_0 - 2)$

一定要说明初始值

Thus. if $y_0 = 2$. the sequence stays at 2 $y_n = 2$

if $y_0 \neq 2$ y_n can't converge. because the distance from 2 keeps doubling.

Summary

5 . .

ACF

(1.8) A sequence (x_n) is defined iteratively by $x_0 = 2$ and $x_{n+1} = \frac{x_n}{2+\sin(x_n)}$ for $n = 0, 1, 2$

Show that $x_0 \geq x_1 \geq x_2 \geq x_3 \cdots > 0$ and (x_n) converges. What is the limit?

$$\frac{x_{n+1}}{x_n} = \frac{1}{2+\underbrace{\sin x_n}_{\in[-1,1]}} \leq 1 \Rightarrow x_{n+1} \leq x_n$$

x_n is non-increasing

Additionally, $x_{n+1} = \frac{x_n}{2+\sin x_n} \boxed{>0}$ since $x_0 = 2$

$$x_1 = \frac{x_0}{2+\sin x_0} = \frac{2}{2+\sin 2} > 0$$

$$x_2 = \frac{x_1}{2+\sin x_1} > 0$$

Bounded below & non-increasing

By MST. $(0 < x_{n+1} \leq x_n)$ $x_n \to L$

Note that $\boxed{0 \leq L < 2}$

As $x_n \to L$ $x_{n+1} \to L$

By A. L.

$L = \frac{L}{2+\sin L}$ $L(1+\sin L) = 0$

$L = 0$ or $L = \sin^{-1}(-1) = \frac{3\pi}{2}$ (not satisfied)

$\therefore (x_n) \to L = 0$.

Summary

ALF

(129) A sequence (x_n) is defined iteratively by

$x_{n+1} = (x_n^2 + 2)/3$.

(i) Determine real number L_1, L_2 with $L_1 < L_2$

s.t. if (x_n) converges then $x_n \to L_1$ or $x_n \to L_2$

(ii) Show that if $x_n > L_2$, then $x_{n+1} > x_n$

(consider $x_{n+1} - x_n$), and deduce what happens

to the sequence if $x_0 > L_2$.

(iii) For $L_1 < x_n < L_2$ show that $L_1 < x_{n+1} < x_n$.

(iv) Hence determine the behaviour of the sequence

for all different positive values of x_0.

(i) $x_n \to L$, $x_{n+1} \to L$ By A.L. $L = \frac{L^2+2}{3} \Rightarrow L^2 - 3L + 2 = 0$

$L_1 = 1$ or $L_2 = 2$

(ii) $x_{n+1} - x_n = \frac{(x_n - 1)(x_n - 2)}{3}$ since $x_n > 2 \Rightarrow x_{n+1} > x_n$

if $x_n > 2$. $x_{n+1} > x_n > 2$

可多加这一步 用具体例子 检验 $\Rightarrow$ $x_1 = \frac{x_0^2 + 2}{3} > 2$ $x_2 = \frac{x_1^2 + 2}{3} > 2$

By MST. $x_n \to +\infty$

(iii) $1 < x_n < 2 \Rightarrow 1 < x_{n+1} < x_n$. $x_n \to 1$

(iv) if $0 < x_0 < 1$, $x_n < x_{n+1} \boxed{< 1}$ ⇒ 代值检验即可

$x_n \to 1$

if $1 < x_0 < 2 \Rightarrow 1 < x_{n+1} < x_n$ $x_n \to 1$

if $x_0 > 2 \Rightarrow$ $+\infty$

小于0的情况不用考虑

Summary

5

AcF

(130) Show that $f(x) = x^2/32 + \cos(\pi x/4)$ maps $[0, 2]$ into itself. Is f a contraction on $[0, 2]$?

For $x \in [0, 2]$

$x^2 \in [0, 4]$

$\cos(\pi x/4) \in [0, 1]$

$0 \le f(x) \le 4/32 + 1 < 2$

make sure $f(x)$ is on $[0, 2]$.

so f does map $[0, 2]$ into itself.

since $f'(c) = c/16 - (\pi/4)\sin(\pi c/4)$

$|f'(c)| \le \frac{1}{8} + \frac{\pi}{4}$ Mean Value Theorem.

Hence $|f(x) - f(y)| \le (\frac{1}{8} + \frac{\pi}{4})|x-y|$

so f is contraction on $[0, 2]$ because $\frac{1}{8} + \frac{\pi}{4} < 1$.

Summary

5

ACF

(31) A real sequence (x_n) is defined, from a positive starting value x_0, by $x_{n+1} = \frac{2}{1+x_n}$

(i) Show that $x_n > 0$ for every $n \in N$

(ii) If the sequence converges, what are the possible values of the limit?

(iii) Show that $|x_{n+1} - 1| \le |x_n - 1|$ and deduce that the sequence $(|x_n - 1|)$ converges to b say.

(iv) Show that $b = 0$ (so that $\lim_{n \to +\infty} x_n = 1$)

(i) $x_0 > 0$. $1 + x_0 > 0 \Rightarrow x_1 > 0$ Repeating this $\Rightarrow x_n > 0$ for all $n \in N$

一定要看好 x_n 的约束条件

(ii) $L = 1$ or $L = -2$ $\boxed{\text{Since } x_n > 0}$

$\therefore L = 1$

(iii) Since $x_n > 0$, $x_{n+1} - 1 = (1 - x_n)/(1 + x_n)$

$|x_{n+1} - 1| = \frac{|1 - x_n|}{1 + x_n}$

把绝对值看成整体

$\Rightarrow |x_{n+1} - 1| \le |x_n - 1|$

so $(|x_n - 1|)$ is non-increasing and bounded by 0.

$\Rightarrow$ must converge to b say

(iv) If $b \ne 0$, then from (iii)

$(1 + x_n) = |1 - x_n| / |x_{n+1} - 1| \to b/b = 1$

so $x_n \to 0$, which is impossible by (ii).

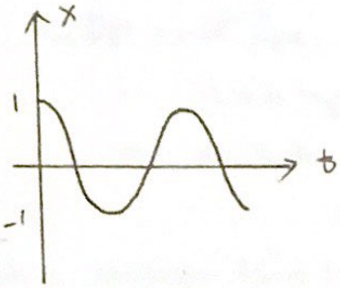
so $b = 0$

x_n 只能有 lim 为 1 的情况.

所以 提前算出的可能极限要引起重视

Summary

APP Spring semester   Modeling Revision



$x(t) = \cos t$

$f = m\ddot{x} = -\cos t\, m = -mx$

Restoring force.

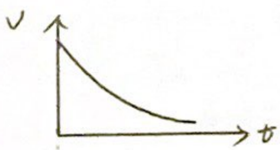
Similarly: $x(t) = A\cos\omega t$

$f = m\ddot{x} = -mA\omega^2 \cos\omega t$

$f = -mk\dot{x}(t)$

Drag force

Considering $x(0) = 0$ $\dot{x}(0) = V$

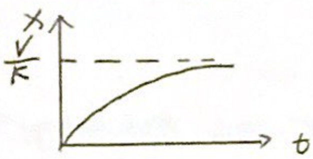


$f = -mk\dot{x}(t) = m\ddot{x}$

$\ddot{x} + k\dot{x} = 0$ let $v = \dot{x}$

$\dot{v} + kv = 0$

$v(t) = Ve^{-kt}$ $\frac{dx}{dt} = Ve^{-kt}$



$x(t) = \frac{V}{k}(1 - e^{-kt})$

Common forces in 1D.

1) $f = 0$: equilibrium

2) $f = mg$: uniform gravitational acceleration

3) $f = -mkv = -mk\dot{x}$ $(k>0)$ drag force

4) $f = -m\omega^2 x$ restoring force.

Summary
Addition: Newton's Law of Cooling

$$\frac{dT}{dt} = k(T_s - T) \Rightarrow T(t) = T_s + (T_0 - T_s)e^{-kt}$$

T_s: surrounding temperature.

Projectiles

$f = m\ddot{x}$ $\quad \dot{x}(0) = V \quad x(0) = 0$

$x(t) = \frac{1}{2}gt^2 + Vt + B \quad$ since $x(0) = 0$

$x(t) = \frac{1}{2}gt^2 + Vt$

Linear combination of $\underline{g}$ and $\underline{V}$

$\underline{V} = (V\cos\alpha, V\sin\alpha)$

$\underline{g} = (0, -g)$

2D

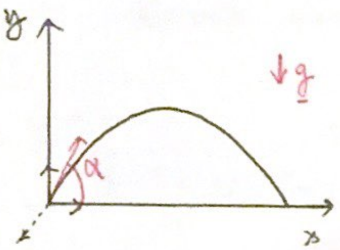
$\underline{x} = (x(t), y(t)) = \frac{1}{2}(0, -g)t^2 + (V\cos\alpha, V\sin\alpha)t$

where $\begin{cases} x(t) = V\cos(\alpha)\cdot t \\ y(t) = -\frac{1}{2}gt^2 + V\sin(\alpha)t. \end{cases}$

3D

$\underline{x} = (x_1(t), x_2(t), x_3(t)) = \frac{1}{2}(0, -g, 0)t^2 + (V\cos\alpha, V\sin\alpha, 0)t$

where $\begin{cases} x_1(t) = Vt\cos\alpha \\ x_2(t) = Vt\sin\alpha - \frac{1}{2}gt^2 \\ x_3(t) = 0 \end{cases}$



Solving:

$t = 0$ or $t_f = \frac{2V\sin\alpha}{g}$ (time of flight)

Range: $x_1 = Vt_f\cos\alpha = \frac{2V^2}{g}\sin\alpha\cos\alpha = \frac{V^2}{g}\sin 2\alpha$

$\alpha = \pi/2$. rises vertically.

$\alpha = \pi/4$ Maximum range.

eliminating t, get a parabola.

$y = V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 = x\tan\alpha - \left(\frac{g}{2V^2}\right)x^2\sec^2\alpha$

since: $\sec^2 x = \tan^2\alpha + 1$ if $(X, Y, 0)$ pass through.

$\cancel{Y = X\tan\alpha - \left(\frac{g}{2V^2}\right)X^2\sec^2\alpha}$

$Y = -\left(\frac{g}{2V^2}\right)X^2(\tan^2\alpha + 1) + X\tan\alpha.$ $\Delta = 0$ 重根

联立 Enveloping 和 原方程(无x)，配平方解t

$\tan^2\alpha - \left(\frac{2V^2}{gX}\right)\tan\alpha + \left(1 + \frac{2YV^2}{gX^2}\right) = 0 \Rightarrow$ parabola

$V_0 = V$

Summary ① ①

提 $\frac{1}{2}g$

$V\sin(\alpha)t - \frac{1}{2}gt^2 = \frac{V^2}{2g} - \frac{g(V\cos(\alpha)t)^2}{2V^2}$

$\Rightarrow t = \frac{V}{g\sin\alpha}$ (repeated root on the boundary)

verify it:

$V(t) = \underline{V} + \underline{g}t$

$V\cdot V_0 = V^2 + \underline{g}\cdot\underline{V}\cdot\frac{V}{g\sin\alpha}$

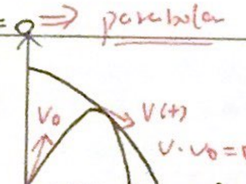
[illegible]

$= V^2 + g\cdot V\cdot\cos\left(\frac{\pi}{2}+\alpha\right)\cdot\frac{V}{g\sin\alpha}$

$= V^2 - V^2$

$= 0.$

$V \perp V_0.$



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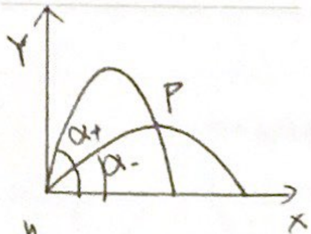
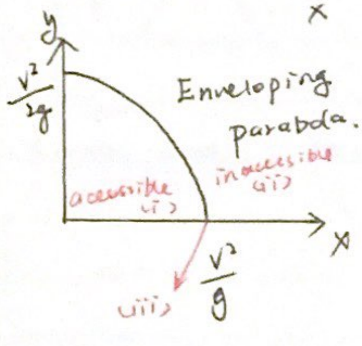




(i) two real roots $Y < \frac{V^2}{2g} - \frac{gX^2}{2V^2}$

(ii) two complex conjugate root

(iii) equal root $\left(\frac{2V^2}{gX}\right)^2 - 4\left(1 + \frac{2YV^2}{gX^2}\right) = 0$

$\Rightarrow$ Boundary between (i) and (ii)

$$Y = \frac{V^2}{2g} - \frac{gX^2}{2V^2}$$ 把Y想成常数即可

Air resistance gravity, resistance.

$$m\frac{d^2x}{dt^2} = mg - mk\frac{dx}{dt} \qquad \text{let } v = \dot{x}$$

$$m\frac{dv}{dt} = mg - mkv$$

$$\frac{dv}{dt} + kv = g \quad \text{solve by I.F}$$

$$v(t) = \frac{1}{k}g + Ae^{-kt} \qquad x(0) = 0. \quad \dot{x}(0) = V_0$$

$$v(t) = \frac{1}{k}(1 - e^{-kt})g + V_0 e^{-kt}$$

since $v = \dot{x}$

$$x(t) = \frac{1}{k^2}(-1 + kt + e^{-kt})\underset{\uparrow\, g=(0,g)}{g} + \frac{1}{k}(1 - e^{-kt})\underset{\uparrow\, V=(V_0\cos\alpha,\, V_0\sin\alpha)}{V_0}$$ 分部来.

Note that the acceleration is not a constant

$$\ddot{x}(t) = (g - kV_0)e^{-kt}$$

Projectiles with Air resistance

$$x(t) = \frac{V\cos\alpha}{k}(1 - e^{-kt}) \quad \text{since } g = -ge_2$$

$$y(t) = \frac{g}{k^2}(1 - kt - e^{-kt}) + \frac{V\sin\alpha}{k}(1 - e^{-kt})$$

Find y_{max}:

$$\dot{y} = 0.$$

$$-\frac{g}{k}(1 - e^{-kt}) + V\sin(\alpha)e^{-kt} = 0$$

$$t_{max} = \frac{1}{k}\ln\left(1 + \frac{Vk}{g}\sin\alpha\right)$$

$$y_{max} = \frac{V\sin\alpha}{k} - \frac{g}{k^2}\ln\left(1 + \frac{Vk}{g}\sin\alpha\right)$$

先把t代入再最后用taylor

Summary

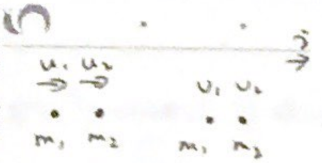
Taylor series: $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ when $k \to 0$ (k is very small)

$$\ln\left(1 + \frac{Vk\sin\alpha}{g}\right) = \frac{kV\sin(\alpha)}{g} - \frac{1}{2}\frac{k^2V^2\sin^2(\alpha)}{g^2}$$

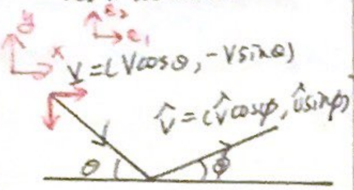
$$y_{max} \approx -\frac{g}{k^2}\left(\frac{kV\sin\alpha}{g} - \frac{1}{2}\cdot\frac{k^2V^2\sin^2(\alpha)}{g^2}\right) + \frac{V\sin\alpha}{k}$$

$$= \boxed{\frac{V^2\sin^2(\alpha)}{2g}}$$ (same as the case projectile without Air resistance.)

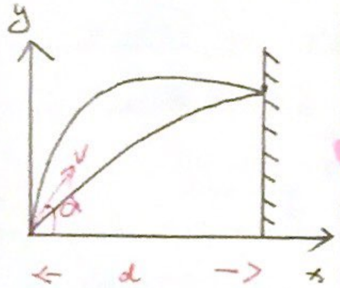
5



e: coefficient of restitution.



水平方向速度不变
竖直方向 collision.



关键在于 x方向 y方向
时间相同

Collision:

Conservation of Momentum: $m_1u_1+m_2u_2=m_1v_1+m_2v_2$

collision law: $\boxed{v_2-v_1=e(u_1-u_2)}$

$0\le e\le 1$

$e=1$: Perfect elastic collision

$e=0$: Perfect plastic collision (stick together)

Oblique Collisions

since Earth >> m.

$V_{before} = V\cos\theta\, e_1 - V\sin\theta\, e_2$ $\quad V_{after} = V\cos\theta\, e_1 + eV\sin\theta\, e_2$

$V_{after} = \hat{V}\cos\phi\, e_1 + \hat{V}\sin\phi\, e_2$

$\hat{V}\cos\phi = V\cos\theta \quad \hat{V}\sin\phi = eV\sin\theta$

$\hat{V} = V\sqrt{\cos^2\theta + e^2\sin^2\theta}$ since $0\le e\le 1$

$\boxed{\hat{V} \le V\sqrt{\cos^2\theta+\sin^2\theta} = V}$

$\tan\phi = \dfrac{eV\sin\theta}{V\cos\theta} = e\tan\theta$ since $0\le e\le 1$

$\tan\phi \le \tan\theta$

$\boxed{\phi\le\theta}$

Throwing a ball to a wall [t 相同]

$x(t) = V\cos\alpha\, t$, $y(t) = -\frac{1}{2}gt^2 + V\sin\alpha\, t$

x-direction: $t_1 = \dfrac{d}{V\cos(\alpha)}$ (to the wall t_1)

after collision, the velocity becomes $-eV\cos(\alpha)$

$t_2 = \dfrac{d}{eV\cos\alpha}$ (return to $x=0$)

$T = t_1+t_2 = \dfrac{d(1+e)}{eV\cos\alpha}$.

Y-direction: $y(T)=0$. $T = \dfrac{2V\sin\alpha}{g} = \dfrac{d(1+e)}{eV\cos\alpha}$

$d = \dfrac{eV^2\sin(2\alpha)}{g(1+e)}$

Summary

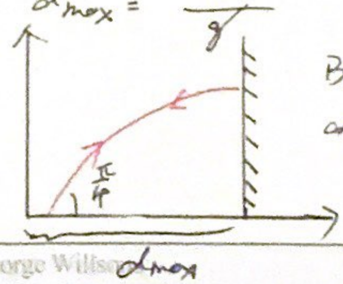
$\boxed{\text{Maximise } d}$: use $V=V_{max}$.

$d = \dfrac{eV_{max}^2\sin(2\alpha)}{g(1+e)}$, $\alpha = \dfrac{\pi}{4}$

$d_{max} = \dfrac{eV_{max}^2}{g(1+e)}$

if perfect elastic collision $\boxed{e=1}$

$d_{max} = \dfrac{V_{max}^2}{g}$



Bounds back from wall at same speed.

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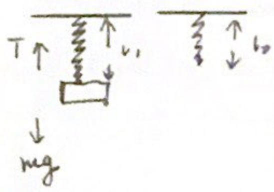
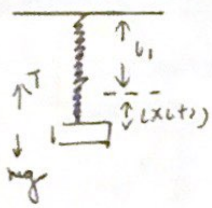
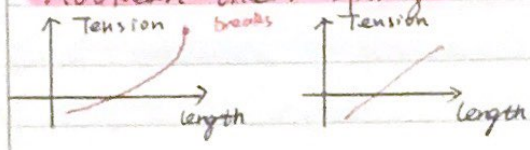




Hookean Linear Spring



Tension $\propto$ Extension of spring

Hooke's law:

$T = \frac{\lambda(l-l_0)}{l_0}$ (λ = ~~the~~ Elastic Modulus)

$[T] = [\frac{\lambda(l-l_0)}{l_0}] = \frac{[\lambda]L}{L}$ $[\lambda] = [T]$ = force $kg\,m/s^2$

Equilibrium: $mg\,\hat{i} - \frac{\lambda(l_1-l_0)}{l_0}\hat{i} = 0$

$mg = \frac{\lambda(l_1-l_0)}{l_0}$

$l_1 = (1 + mg/\lambda)\, l_0$.

Motion: $m\ddot{x} = mg - \lambda\frac{x+l_1-l_0}{l_0} = mg - \frac{\lambda x}{l_0} - \lambda\frac{l_1-l_0}{l_0} = -\frac{\lambda x}{l_0}$

$\ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{\lambda}{m l_0}}$

Simple Harmonic Motion (SHM)

$\ddot{x} + \omega^2 x = 0$

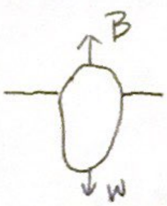
General solution: $x = A\cos\omega t + B\sin\omega t$

$x = a\cos(\omega t - \phi)$

Amplitude: a $\qquad [\omega] = \frac{1}{T} \qquad [f] = \frac{1}{T}$

phase: ϕ

$$\begin{cases} f = \frac{\omega}{2\pi} \\ T = \frac{2\pi}{\omega} = \frac{1}{f} \end{cases}$$



Buoyancy (SHM)

Archimedes' principle: the buoyancy force is equal in magnitude to the weight of water displaced.

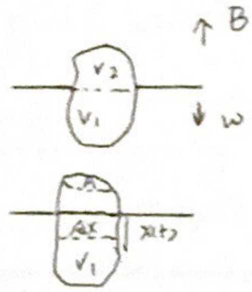
Summary

Equilibrium: $B = W$

$$\rho_i(V_1+V_2)g = \rho_w V_1 g \qquad \boxed{\frac{V_1}{V_1+V_2} = \frac{\rho_i}{\rho_w}} \approx 92\%$$

在解ODE时不用到

消去ρ



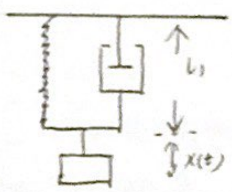
Motion: $W - B = ma$

$$\rho_i(V_1+V_2)g - \rho_w(V_1+A x)g = \rho_i(V_1+V_2)\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{\rho_w A g}{\rho_i(V_1+V_2)}x = 0 \qquad \text{Note that } \frac{\rho_w}{\rho_i(V_1+V_2)} = \frac{V_1+V_2}{V_1(V_1+V_2)} = \frac{1}{V_1}$$

$$\frac{d^2x}{dt^2} + \frac{Ag}{V_1}x = 0 \qquad \omega = \sqrt{\frac{Ag}{V_1}}$$

Damping (Damped Simple Harmonic Motion)



$$F_{damp} = -k\dot{x}$$

$$mg - \lambda\frac{x+l_1-l_0}{l_0} - k\dot{x} = -\frac{\lambda x}{l_0} - k\dot{x}$$

$$\boxed{\ddot{x} + p\dot{x} + \omega^2 x = 0} \qquad \omega = \sqrt{\frac{\lambda}{m l_0}}, \quad p = \frac{k}{m}$$

Auxiliary equation

$$m_+ = \frac{-p+\sqrt{p^2-4\omega^2}}{2} < 0$$

$$m_- = \frac{-p-\sqrt{p^2-4\omega^2}}{2} < 0$$

$\boxed{\text{Case 1}}$: $x(t) = Ae^{m_+ t} + Be^{m_- t}$ (Heavy damping)

$x(t) \to 0$ as $t \to +\infty$.

$$t = \frac{1}{\sqrt{p^2-4\omega^2}}\ln\left(-\frac{B}{A}\right) \qquad \left(-\frac{B}{A} > 1 \text{ needed}\right)$$

$\boxed{\text{Case 2}}$: $x(t) = e^{-\frac{1}{2}t}\{A\cos(\frac{1}{2}\sqrt{4\omega^2-p^2}\,t) + B\sin(\frac{1}{2}\sqrt{4\omega^2-p^2}\,t)\}$

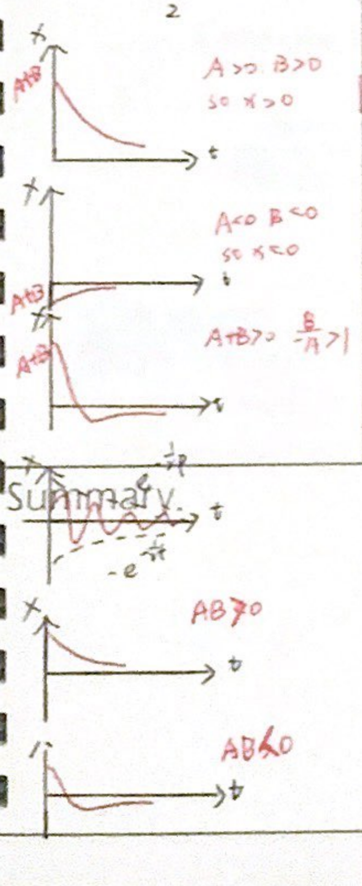
$x(t) \to 0$ as $t \to +\infty$ (light damping)

$$m_+ = \frac{-p+i\sqrt{4\omega^2-p^2}}{2}$$

$$m_- = \frac{-p-i\sqrt{4\omega^2-p^2}}{2}$$

$\boxed{\text{Case 3}}$: $x(t) = e^{-\frac{1}{2}p}(A+Bt)$ (critical damping)

$t = -\frac{A}{B}$



Summary

Summary:

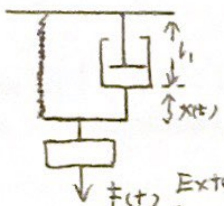
$\ddot{x} + p\dot{x} + \omega^2 x = 0 \qquad m_\pm = \frac{-p \pm \sqrt{p^2 - 4\omega^2}}{2}$

① $p^2 - 4\omega^2 > 0$ Heavy damping.

② $p^2 - 4\omega^2 < 0$ light damping

③ $p^2 - 4\omega^2 = 0$ Critical damping

Forcing:



External force

$\ddot{x} + p\dot{x} + \omega^2 x = f(t)$ where $f(t) = \frac{F(t)}{m}$

$f(t) = b\cos\Omega t$.

$x(t)$ = Particular integral + complementary function

For P.I.: $x_{PI}(t) = \lambda\cos\Omega t + \nu\sin\Omega t$ Substituting

$(-\Omega^2 + \omega^2)\lambda + p\Omega\nu = b$ (coefficient of $\cos\Omega t$)

$-p\Omega\lambda + (-\Omega^2 + \omega^2)\nu = 0$ (coefficient of $\sin\Omega t$)

$\lambda = \frac{(\omega^2 - \Omega^2)b}{(\omega^2 - \Omega^2)^2 + p^2\Omega^2} \qquad \nu = \frac{p\Omega b}{(\omega^2 - \Omega^2)^2 + p^2\Omega^2}$

$x_{PI} = \frac{b}{(\omega^2 - \Omega^2)^2 + p^2\Omega^2}\left[(\omega^2 - \Omega^2)\cos\Omega t + p\Omega\sin\Omega t\right]$

Alternatively: $x_{PI}(t) = \frac{b\cos(\Omega t - \phi)}{((\omega^2 - \Omega^2)^2 + p^2\Omega^2)^{1/2}}$

$\tan\phi = \frac{p\Omega}{\omega^2 - \Omega^2}$

for light/heavy damping ←

$x(t) = \underbrace{Ae^{(-\frac{p}{2} + \frac{1}{2}\sqrt{p^2 - 4\omega^2})t} + Be^{(-\frac{p}{2} - \frac{1}{2}\sqrt{p^2 - 4\omega^2})t}}_{\text{transient } \to 0 \text{ as } t \to \infty} + \underbrace{\frac{b\cos(\Omega t - \phi)}{((\omega^2 - \Omega^2)^2 + p^2\Omega^2)^{1/2}}}_{\text{persistent (Stead-state)}}$

Summary

Amplitude: $\dfrac{b}{((\omega^2-\Omega^2)^2+p^2\Omega^2)^{1/2}}$

Maximise it: $\frac{d}{d\Omega} \Rightarrow \Omega^2 = \omega^2 - p^2/2$

when $\Omega = \sqrt{\omega^2 - p^2/2} \approx \omega$.

forcing a system near its natural frequency can generate a very strong response.

Work and Energy

$F = m\cdot\frac{dv}{dt}$ take scalar v.

$m\frac{dv}{dt}\cdot v = F\cdot v$

$m\frac{dv}{dt}\cdot v = \frac{d}{dt}(\frac{1}{2}m v\cdot v) = \frac{d}{dt}(\frac{1}{2}m|v|^2) = \frac{dK}{dt}$

Kinetic energy: $\boxed{K = \frac{1}{2}m|v|^2} = \frac{1}{2}m\ \text{speed}^2$

$\frac{dK}{dt} = F\cdot v$

$K(t_2) - K(t_1) = \boxed{\int_{t_1}^{t_2} F\cdot v\,dt}$ work done by force F.

Energy principle: The increase in kinetic energy of a particle in a given time interval is equal to the work done by the applied forces during that time interval.

A projectile is launched when $t=t_1$, with initial velocity u and angle of elevation α. Let $t=t_2$ be the time of at maximum height.

verify Energy principle.

Summary

$x(t) = (u\cos(\alpha)(t-t_1),\ u\sin\alpha(t-t_1) - \frac{1}{2}g(t-t_1)^2)$

$v(t) = \frac{dx}{dt} = (u\cos\alpha,\ u\sin\alpha - g(t-t_1))$

External force: $F = (0, -mg)$

Work done by F: $= \int_{t_1}^{t_2} F \cdot v \, dt$

$= \int_{t_1}^{t_2} (0, -mg)(u\cos\alpha,\ u\sin\alpha - g(t-t_1))\,dt$

$= \int_{t_1}^{t_2} \{-mg\,u\sin\alpha + mg^2(t-t_1)\}\,dt$

$= -mg\left[u\sin\alpha\, t - \frac{1}{2}g(t-t_1)^2\right]_{t=t_1}^{t=t_2}$

$= -mg\{u\sin(\alpha)(t_2-t_1) - \frac{1}{2}g(t_2-t_1)^2\}$

$= \frac{1}{2}mg^2(t_2-t_1)^2 - mgu\sin(\alpha)(t_2-t_1)$

since. highest point. $V_y = 0$ $\quad t_2 - t_1 = \frac{1}{g}u\sin(\alpha)$

$= -\frac{1}{2}mu^2\sin^2(\alpha)$.

Change in Kinetic Energy: $= \frac{1}{2}mu^2\cos^2(\alpha) - \frac{1}{2}mu^2$

$= -\frac{1}{2}mu^2\sin^2(\alpha)$

δK = Work done by force.

Potential Energy

动能和势能都用到同样技巧

$x \quad \dot{x} \quad \dot{u}$

1D: $f = ma = m\ddot{x}$

$\dot{x}f = m\ddot{x}\dot{x} = \frac{d}{dt}(\frac{1}{2}m\dot{x}^2)$

E.P. $\Leftarrow$ $\int_{t_1}^{t_2} \dot{x}f\,dt = \frac{1}{2}m\dot{x}^2\Big|_{t=t_2} - \frac{1}{2}m\dot{x}^2\Big|_{t=t_1} = K(t_2) - K(t_1)$

force is a $\boxed{\text{function of } x}$

$\int_{t_1}^{t_2} f(x)\frac{dx}{dt} \cdot dt = \int_{x_1}^{x_2} f(x)\,dx$

eg: gravity. Hook's law for a spring $\Rightarrow$ conservative force

Define: $f(x) = -\frac{dV(x)}{dx}$ $\quad V(x) = -\int f(x)\,dx$ (potential Energy)

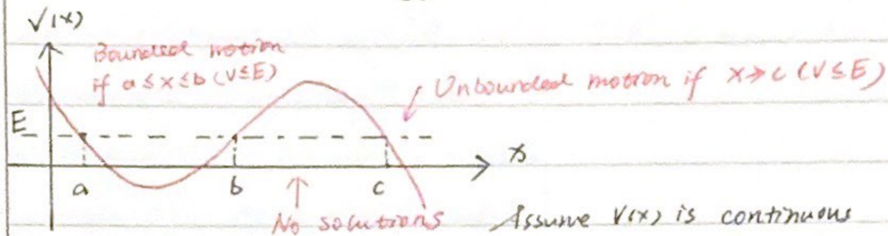
$K(t_2) - K(t_1) = \int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{x_2} -\frac{dV}{dx}\,dx = [-V]_{x_1}^{x_2} = -V(x_2) + V(x_1)$

Summary

$\Rightarrow K(t_2) + V(x_2) = K(t_1) + V(x_1)$

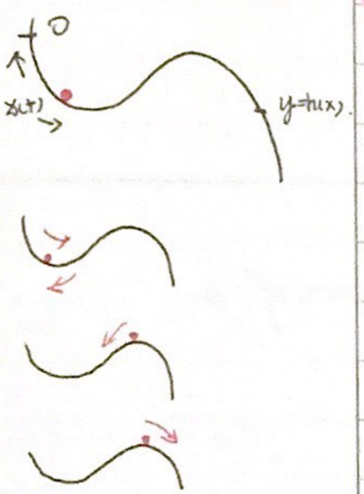
$E = K(t) + V(x(t)) = \text{constant}.$

If the applied force ONLY depends on position, then the total energy is constant.



Assume V(x) is continuous and smooth.

Potential Energy (Continued)

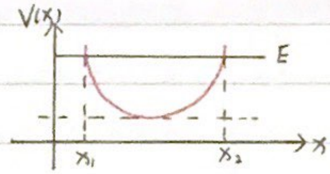


$E = \frac{1}{2} m \dot{x}^2 + mgh(x)$

Local minimum : stable equilibrium point (potential well)

Local maximum : unstable

Period:



$\frac{1}{2} m \dot{x}^2 + V(x) = E$

$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - V(x))}$

$\int_0^{\frac{1}{2}P} dt = \int_{x_1}^{x_2} \sqrt{\frac{m}{2(E-V(x))}}\, dx$

$$\boxed{P = \int_{x_1}^{x_2} \sqrt{\frac{2m}{E-V(x)}}\, dx}$$

Summary