Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

Newton - Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n=0,\ 1,\ 2,\ 3\cdots$$

Critical Point

- Stationary point $f'(x)=0$
  - Turning point
    $f'(x)=0$
    $f''(x)\neq 0$
    - point of Maxima $f'(x)=0$ $f''(x)<0$
    - point of Minima $f'(x)=0$ $f''(x)>0$.
  - Inflection Point
    $f'(x)=0$ $f''(x)=0$
    but $f'''(x)\neq 0$ k is odd.
- point where derivative is not defined.

Summary

① Method of Substitution.

$$F(x) = \int f(x)\,dx \Rightarrow \frac{dF}{dx} = f(x)$$

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

Let $g(x) = t \Rightarrow g'(x) = \frac{dt}{dx} \Rightarrow g'(x) \cdot dx = dt$

$$\int f(t)\,dt = F(t) + C.$$

② Integrating Functions with leaner term.

$\int f(x)\,dx = F(x) + C$ then $F' = f$

$$\int f(ax+b) = \frac{1}{a} F(ax+b) + C.$$

③ Integral of the form $\int \sin^m x \cos^n x \cdot dx$. $m.n \in N$.

1. $m.n$ are odd. $\begin{cases} m > n, & \sin x = t \\ m < n, & \cos x = t \end{cases}$ 把大化小为t.

2. one of $m.n.$ is even. even term $\to t$.

3. both are even. $\begin{cases} \cos^2 x = \dfrac{1+\cos 2x}{2} \\ \sin^2 x = \dfrac{1-\cos 2x}{2} \end{cases}$

Summary

④ Integrating algebraic fractions.

separate denominator and numerator.

⑤ t - substitution.

$$\begin{cases} \tan\left(\frac{x}{2}\right) = t \\ \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2}{1+t^2}\,dt. \end{cases}$$

product rule.

Result: $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C.$

⑥ The method of Partial fraction.

1. Non-repeated linear factor.

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

2. Repeated linear factor

$$\frac{1}{(x+a)^2(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}$$

3. Non-repeated Qudratic factors

$$\frac{1}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$$

Summary

⑦ Integrating by parts.

$$\int u \cdot \frac{dv}{dx} \cdot dx = \int \frac{d}{dx}[u \cdot v] \cdot dx - \int v \cdot \frac{du}{dx} \cdot dx,$$

$$= u \cdot v - \int v \cdot \frac{du}{dx} \, dx$$

$\frac{dv}{dx}$ must be the easiest one to integrate.

Choose $\cancel{\frac{dv}{dx}}$ u.

LIATE

Properties of Definite Integration.

① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)\,dx$

④ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

⑤ $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

⑥ $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ [f is even]

⑦ $\int_{-a}^a f(x)\,dx = 0$ [f is odd]

Summary

Differential equation

- Order : the order of the highest derivative
- degree : highest power.

Solution of an ODE

- Analytical method
- Numerical method.

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)\,dy = f(x).dx.$$

App.

$\boxed{\text{Dimension.}}$

eg: length, time. weight.

$\downarrow L \quad \downarrow T \quad \downarrow$

unit:

eg $\left[\frac{dh}{dt}\right] = \frac{L}{T}$. both sides need be the same.

so, we add constant k.

(b1) Determine whether each of the following function leads to sensible model for the growth of a tree:

(a) $f(h) = k(h-h_1)(h-2h_1)$

(b) $f(h) = k\cos(\pi h/2h_1)$

(c) $f(h) = k(\cosh(1-\frac{h}{h_1})-1)$.

Summary

5

(a): $\left[\frac{dh}{dt}\right] = \frac{L}{T}$.

$K \cdot L \cdot L = \frac{L}{T}$. $K = L^{-1} \cdot T^{-1}$

The dimension of angle. is 1.

(b) $\left[\frac{dh}{dt}\right] = \left[K \cos(\underbrace{\frac{\pi h}{2h_1}}_{1})\right]$

$\frac{L}{T} = K$

$K = L \cdot T^{-1}$

(c). $\left[\frac{dh}{dt}\right] = \left[K \underbrace{\cosh(1 - \frac{h}{h_1})}_{\text{dimensionless}} - 1\right]$

dimensionless.

$\frac{L}{T} = K$.

$K = L \cdot T^{-1}$.

(Q2) The increasing surface area $A(t)$ of a puddle is governed by $\frac{dA}{dt} = \alpha A^{1.5}$ where $\alpha > 0$. Given that L represents length and T represents time, the dimensions of α are:

(a) $L^{-1}T^{-1}$ (b) LT^{-1} (c) $L^{-1}T$ (d) $L^{2}T^{-1}$ (e) None of the above.

Firstly. using the bracket to rewrite the function. , Then. repsent them with dimension. Capital letter.

$\left[\frac{dA}{dt}\right] = [\alpha][A^{1.5}]$

$\frac{L^2}{T} = [\alpha][L^2]^{1.5}$

$\frac{L^2}{T} = [\alpha] \; L^3$

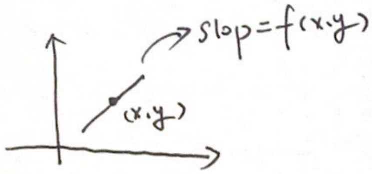
$[\alpha] = \frac{1}{LT} = L^{-1} \cdot T^{-1}$ choose (a).

Summary

ODE.

Analytic

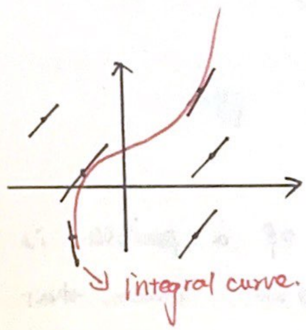
$y' = f(x, y)$



Geometric view of ODE

- Direction field
- Integral curve.

every point is tagent to line.



- $y_1(x)$ is a solution to $y' = f(x,y)$ $\Leftrightarrow$ graph of $y_1(x)$ is an intcure.

$\boxed{\text{First-order ODE linear}}$

$a(x)y' + b(x)y = c(x)$

(linear $ay_1 + by_2 = c$)

$c = 0$. homogeneous

$\boxed{\text{STANDARD LINEAR FORM.}}$

$y' + p(x)y = q(x)$

Summary

5

Autonomous

$\frac{dy}{dt} = f(y)$ ← no "t" on RHS.

no independent variable on RHS.

① critical point (Equilibrium point)

② graph $f(y) > 0$ or $f(y) < 0$?

eg: BANK ACCOUNT.

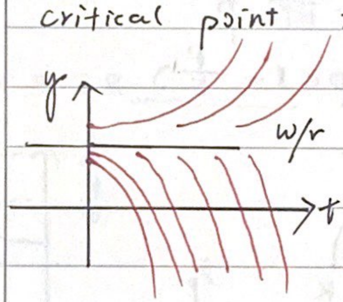
y = money. r = continuous interest rate.

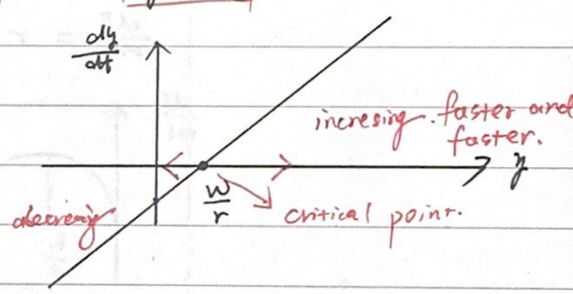
w = someone stole your money (embezzlement rate)

$$\boxed{\frac{dy}{dt} = ry - w}$$

Analytic

critical point $ry - w = 0$. $\boxed{y = \frac{w}{r}}$





Summary

5 . .

logistic 模型的建立

Original Model.

$\frac{dp}{dt} = r \cdot p$. Actually, due to the limitation of sources and environment spaces, r will decrease as p increases

The simpliest one is linear function (To dipict this kind of effect),

$r(p) = a + bp$.

$\begin{cases} r(0) = r \\ r(K) = 0 \end{cases}$

① $\boxed{r(0) = r}$ so, $\boxed{a = r}$

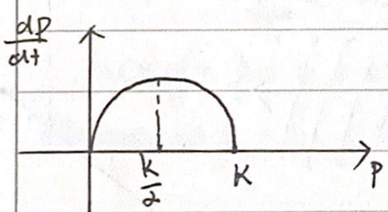
② when $\boxed{p = K}$. $r(p) = 0$. so. $Kb + r = 0$. $b = -\frac{r}{K}$

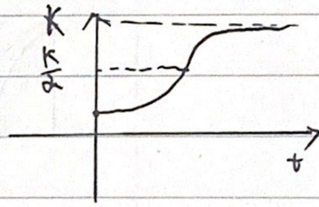
$\therefore$ $r(p) = r - \frac{r}{K} \cdot p$

so

$$\frac{dp}{dt} = (r - \frac{r}{K}p) \cdot p$$

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}) \to$$ source and environment, impede.





By evaluation

$$p(t) = \frac{p(0) K e^{rt}}{K + p(0)(e^{rt} - 1)}.$$

Summary

logistic Model

$$f'(x) = r(1 - \frac{f(x)}{K}) f(x).$$

Model for harvesting

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}) - h$$

$$h(p) = \begin{cases} Ep & 0 \le p \le Y_0/E \quad \text{constant effort} \\ Y_0 & Y_0/E \le p \quad \text{constant yield.} \end{cases}$$

5 . .

Calculus

(63) If $a_0, \cdots a_n$ are real numbers s.t

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0$$

Show using Rolle's theorem that the equation

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n = 0$$

has at least one real root between 0 and 1

try to $\boxed{\text{construct}}$ a function.

Let set $f(x) = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n} x^{n+1}$

$f'(x) = a_0 + a_1 x + \cdots a_{n-1} x^{n-1} + a_n x^n = 0$

$f(x)$ is ✓ continuous on $[0,1]$

$f(x)$ is ✓ differentiable on $(0,1)$

$f(0) = 0$, $f(1) = 0$

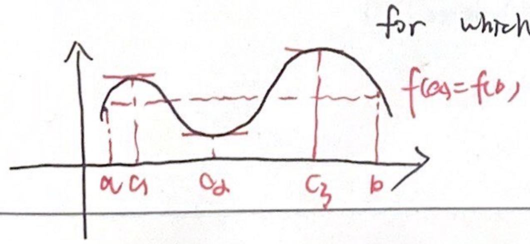
there exist $c \in (0,1)$ $f'(c) = 0$.

→ root.

Summary

$\boxed{\text{Roller's theorem}}$

Suppose f is $\boxed{\text{continuous}}$ on $[a, b]$ and $\boxed{\text{differentiable}}$ on (a, b). If $f(a) = f(b)$. there exists a point $c \in (a, b)$ for which $f'(c) = 0$



5 . .

Probability.

(64) The number of cases of a certain blood diseases, in Nottingham per year is well-described by Poisson distribution with mean λ. Find. for a given year.

(i) The probability of no cases.

(ii) the probability of no more than two cases

(iii) the probability of exactly one cases given that there is at least one case.

(iv) Each case has, independently of all other cases, a probability p of being a certain rare form of the disease. Let Y denote the number of cases of this rare form of the diseases in a given year. Find the probality that $Y=0$.

(i): $p(X=0) = e^{-\lambda}$

(ii): $p(X\le 2) = e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})$

(iii): $p(X\geq 1) = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$ ← $p(X=1|X\geq 1) = \frac{p(X=1)}{p(X\geq 1)} = \frac{p(X=1)}{1-p(0)}$

(iv). $p(Y=0|X=?)$ → we don't know the X. which means that we don't know how many partitions.

$Y \begin{cases} x_1 \\ x_2 \\ x \\ \vdots \\ x_n \end{cases}$?

因为Y是取决于x的个数的. 就像后面那个摇奖中不中奖, 取决于他的对手.

so. if we know x eg: $p(Y=0|X=x) = (1-p)^x$.

Summary

$$\therefore P(Y=0) = \sum_{x=0}^{\infty} P(Y=0 \mid X=x)\, P(X=x)$$

$$= \sum_{x=0}^{\infty} (1-p)^x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{[(1-p)\lambda]^x}{x!} e^{-\lambda}$$

$$= e^{-\lambda} \cdot e^{[(1-p)\lambda]} \sum_{x=0}^{\infty} \frac{[(1-p)\lambda]^x}{x!} e^{-[(1-p)\lambda]}$$

$$= e^{-\lambda p}.$$

Probability

(65) A Professor of Statistics has just solved a difficult brain teaser (set in a Sunday newspaper). and is wondering whether to pay the £2 entry fee and compete for this week's prize of £50.

The prize is awarded by making a random choice from among all this week's correct entries and the Professor believes that the number of other correct entries (in addition to this) is well modelled as a Poisson random variable with expectation λ.

Let W be the event that the Professor wins and X represent the number of other correct entries. By noting that.

win: gain 48.

lose: lose 2.

理论上这个概率是非常容易算出来的.

$$P(W) = \sum_{x=0}^{\infty} P(W \mid X=x)\, P(X=x)$$

show that the probability that the Professor wins is $(1-e^{-\lambda})/\lambda$. Hence deduce that the Professor should not enter if λ is great than 25.

Summary

(Consider the Professor's gain. if positive. he should enter.):

if k people enter.

$$p(w \mid X=k) = \frac{1}{k+1}$$

$$p(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

We are not sure how many partitions.

$$p(w) = \sum_{x=0}^{\infty} p(w \mid X=x)\, p(X=x)$$

$$= \sum_{x=0}^{\infty} \frac{1}{x+1} \cdot \frac{\lambda^x}{x!} \cdot e^{-\lambda}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x \cdot e^{-\lambda}}{(x+1)!}$$

$$\lambda^1 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$

发现少了一项

利用作差.

$$= \frac{e^{-\lambda}}{\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{(x+1)!} \qquad e^{\lambda} = 1 + \lambda^1 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$

$$= \frac{e^{-\lambda}}{\lambda} \cdot (e^{\lambda} - 1)$$

像这种题不仅考了题目处理技巧. 还有泊松分布的处理技巧.

$$= \frac{1 - e^{-\lambda}}{\lambda}.$$

~~E(X) =~~ -

Let D be the Professor net gain

$D = 48$ if he win. $D = -2$ otherwise.

$$E(D) = 48\, p(w) + (-2)(1 - p(w))$$

$$= 50\, p(w) - 2 > 0$$

$$p(w) > 1/25 \qquad (1 - e^{-\lambda})/\lambda > 1/25$$

Summary

Two special cases

① $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$

② $e^a = \lim_{x \to \infty} \left(1 + \frac{a}{n}\right)^n$

$\Leftrightarrow f(x) = 25(1-e^{-x}) - x > 0$

$x < 25.$

Calculus.

(66) Writings each of the following functions. defined on R. as a sum of an even function and an odd function..

(i), $f(x) = e^x \cos x.$

(ii) $f(x) = \begin{cases} 2 & \text{for } x > 0 \\ 1 & \text{for } x = 0 \\ 0 & \text{for } x < 0. \end{cases}$

$$\boxed{f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]}$$

$\cos(-x) = \cos x,$

so. (i). $f(x) = \frac{1}{2}[e^x + e^{-x}]\cos x + \frac{1}{2}[e^x - e^{-x}]\cos x$

$= \underbrace{\cosh \cos x}_{\text{even.}} + \underbrace{\sinh \cos x.}_{\text{odd}}$

(ii). $f(x) = \underbrace{1}_{\text{even}} + \underbrace{g(x)}_{\text{odd.}}.$

$g(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$

BTW.

$e^{2x}\sin x = \frac{1}{2}(e^{2x} - e^{-2x})\sin x + \frac{1}{2}(e^{2x} + e^{-2x})\sin x$

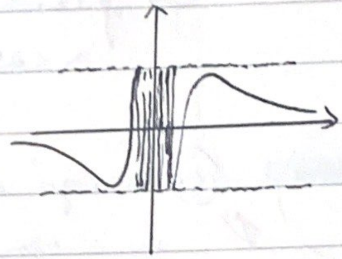
$= \sinh 2x \sin x + \cosh 2x \sin x,$

Summary

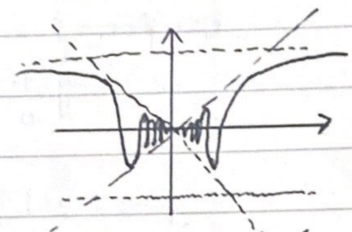
$\boxed{\text{Each function}}$ can be decomposed into the sum of an even function and odd function.

$f(x) = \underbrace{\frac{1}{2}[e^x + e^{-x}]}_{\text{even.}} + \underbrace{\frac{1}{2}[e^x - e^{-x}]}_{\text{odd.}}$

$f(x) = \sin\frac{1}{x}$

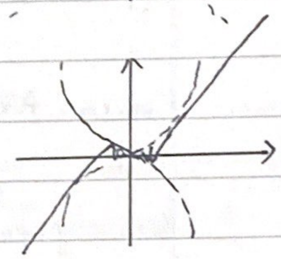


$f(x) = x\sin(\frac{1}{x})$



偶

$f(x) = x^2\sin(\frac{1}{x})$



奇

Summary

$\boxed{\text{立方和公式:}}$ $\cancel{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ 按第一个字母降幂. +, -, +, -

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Calculus

(67) Testing definition of limit.

Show that $\lim_{x \to 1} (5x-3) = 2$.

$x_0 = 1$. $L = 2$. $f(x) = 5x-3$.

For any given ϵ. we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$ that is, whenever

$$0 < |x - x_0| < \delta$$

$$|5x-3-2| = |5x-5| < \epsilon$$

$$5|x-1| < \epsilon$$

$$|x-1| < \frac{\epsilon}{5}$$

Thus, we can take $\delta = \frac{1}{5}\epsilon$. $0 < |x-1| < \delta = \epsilon/5$

then $|f(x) - L| = |5x-3-2| = 5|x-1| < 5 \times \epsilon/5 = \epsilon$

which proves that $\lim_{x \to 1} (5x-3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x-1| < \delta \Rightarrow |5x-5| < \epsilon$. Any smaller positive δ will do as well.

Summary

Limit of a Function. Let $f(x)$ be defined on an open interval about x_0. except at x_0 itself. We say limit of $f(x)$ as x approaches x_0 is number L. $\lim_{x \to x_0} f(x) = L$.

if. for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ s.t. for all x. $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

5 . .

How to find Algebraically a δ for a Given f, L, x_0 and $\epsilon > 0$.

The process of find a $\delta > 0$ s.t. for all x.

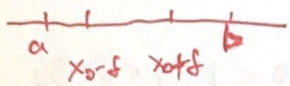
$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

can be accomplished in two steps.

Solve $|f(x) - L| < \epsilon$

① Solve inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) containg x_0 on which the inequality holds for all $x \neq x_0$

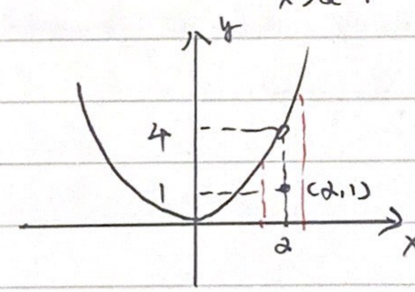
find suitable δ.



② Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality ~~will~~ ~~hold~~ $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in the this δ-interval.

Calculus

(68) Prove that $\lim_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$



$x_0 = 2$. $L = 4$.

Our task. to show $0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \epsilon$

Summary

5 . .

① $\boxed{\text{Solve. inequality.}}$

$$|f(x)-4|<\epsilon$$
$$|x^2-4|<\epsilon$$
$$-\epsilon<x^2-4<\epsilon$$
$$4-\epsilon<x^2<\epsilon+4$$
$$\sqrt{4-\epsilon}<|x|<\sqrt{4+\epsilon}$$
$$\sqrt{4-\epsilon}<x<\sqrt{4+\epsilon}.$$

so. $|f(x)-4|<\epsilon$ holds for all $x\neq 2$. in $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$.

② $\boxed{\text{Find } \delta>0}$

$$0<|x-2|<\delta.$$
$$2-\delta<x<\delta+2 \qquad (2-\delta, 2+\delta).$$

so. place. $(2-\delta, 2+\delta)$ in $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$.

$0<|x-a|<\delta \Rightarrow |f(x)-4|<\epsilon$

↳

$(a-\delta, a+\delta) \subseteq (\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$

$$\begin{cases} 2-\delta \geq \sqrt{4-\epsilon} \\ 2+\delta \leq \sqrt{4+\epsilon} \end{cases} \Rightarrow \begin{cases} \delta \leq 2-\sqrt{4-\epsilon} \\ \delta \leq \sqrt{4+\epsilon}-2 \end{cases}$$

so. take $\delta = \min\{2-\sqrt{4-\epsilon}, \sqrt{4+\epsilon}-2\}$

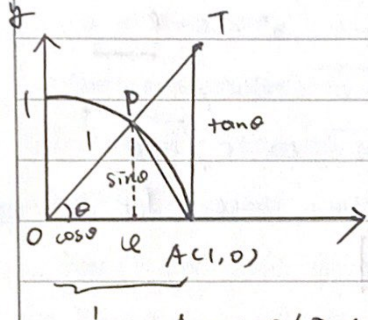
Summary

5 . .

Limit involving $\frac{\sin\theta}{\theta}$.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Proof.



Area $\triangle OAP$ < area sector OAP < area $\triangle OAT$

$$\frac{1}{2}\sin\theta \times 1 < \frac{1}{2}\theta \cdot r^2 < \frac{1}{2}\tan\theta$$

$\theta = \frac{l}{r}$. $l = r\theta$

$A = \frac{1}{2} r \cdot l = \frac{1}{2}\theta r^2$.

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$\lim_{x \to 0^+} \cos\theta = 1$ By Sandwich Theorem

$$\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$$

$\Rightarrow$. $\cos h = 1 - 2\sin^2(h/2)$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \sin\theta = (-1)(0) = 0$$

Summary

Limit as x approches ∞ or $-\infty$

1. $f(x)$ has the limit L as x approach infinity and

$$\lim_{x\to\infty} f(x) = L.$$

if for every number $\epsilon > 0$, there exists a corresponding number M s.t. for all x.

$$x > M \Rightarrow |f(x) - L| < \epsilon.$$

2. $f(x)$ has the limit L as x approaches minus infinity

$$\lim_{x\to-\infty} f(x) = L$$

if for every number $\epsilon > 0$, there exists a correspond number N s.t. for all x

$$x < N \Rightarrow |f(x) - L| < \epsilon.$$

Horizontal Asymptote

Definition. A line $y = b$ is a horizontal asmypotoe of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

Vertical Asymptote

A line $x = a$ is a vertical asmyptore of the graph if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

Summary

5

Oblique Asymptotes

deg(Numerator) > deg(Denominator)

⇒ has an oblique (slanted) asymptote

eg: Find an Oblique Asmptote.

$$f(x) = \frac{2x^2-3}{7x+4}$$ → degree 2. → degree 1.

$$f(x) = \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear}} + \underbrace{\frac{-115}{49(7x+4)}}_{\text{remainder}}$$

As $x \to \pm\infty$.

$$g(x) = \frac{2}{7}x - \frac{8}{49}$$ → an asymptote.

Summary

$y = \sqrt{x}$ is not differentiable at $x=0$

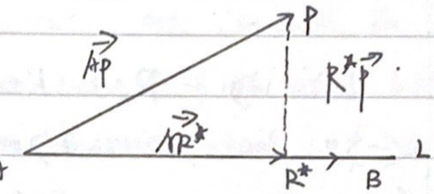
$$\lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

right limit is not finite.

there is no derivative at $x=0$.

LM

Three ways to calculate the distance between one point and a line.



向量相减 指向外点 $\Leftarrow$ ① 方法一：$d(p, L) = \|\vec{AP} - \vec{AR^*}\|$

写成两个单位向量

投影向量的另一种写法

$$= \| p - a - [(p-a)\cdot\hat{v}]\,\hat{v} \|. \qquad \hat{v} = \frac{v}{\|v\|}.$$

转换为两点之间距离 ② 方法二：specify t^*.

$$t^* = \frac{(p-a)\cdot v}{\|v\|^2}. \qquad \frac{(p-a)\hat{v}}{\|v\|}$$ 单位向量 单位向量除以模长

$r^* = a + t^* v. \Rightarrow$ 转换为求 p 和 r^* 两点距离

等面积法. ③ 方法三：等面积法

叉乘几何意义

$$\text{Area} = \|\vec{AB}\| \times d(p, L) = \|\vec{AB} \times \vec{AP}\|$$

$$\Rightarrow d(p, L) = \frac{\|\vec{AB} \times \vec{AP}\|}{\|\vec{AB}\|}$$

eg: Compute the distance between $p = (2, 0, 0)$ and line L passing through point $A = (0, -1, 0)$ and parallel to vector $v = (1, 2, 0)$.

L: $r = a + tv = (t, -1+2t, 0)$

$$t^* = \frac{(p-a)\cdot v}{\|v\|^2} = \frac{(2,1,0)\cdot(1,2,0)}{1+4} = \frac{4}{5}$$

Summary

$\therefore r^* = (\frac{4}{5}, \frac{3}{5}, 0)$.

$$\therefore d = d(p, R^*) = \|r^* - p\| = \|(\tfrac{4}{5}, \tfrac{3}{5}, 0) - (2, 0, 0)\|$$

$$= \sqrt{(\tfrac{6}{5})^2 + (\tfrac{3}{5})^2 + 0}$$

$$= \sqrt{\frac{45}{25}}$$

$$= \frac{3}{\sqrt{5}} \approx 1.34.$$

LM.

(69) for which pairs of complex numbers z and w is it true that $Re(zw) = Re(z) \cdot R(w)$.

$Re(zw) = Re(z) \cdot R(w) - Im(z)\, Im(w). = Re(z) \cdot Re(w)$

$\iff$. $Im(z) \cdot Im(w) = 0$.

$\iff$ at least one of $Im(z)$. $Im(w)$ is 0.

$\iff$. at least. one of the numbers z, w is a real number.

pf: let $\begin{cases} z = a+bi \\ w = x+yi \end{cases}$

$zw = (ax-by) + (ay+bx)i$.

use this to deduce Re. Im.

LM.

(70) Let L be the line through the origin and the point $(1,2,-2)$ and let a be the point $(-1,2,5)$

(a) Find a unit vector e so that L has parametric form $x = te$

(b) Find the components of a (i) along e (ii) perpendicular to e

(c) Find the closest point p of L to a

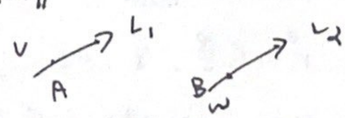
(d) Calculate the shortest distance from a to the line L.

(a). $\cancel{e = \frac{(1,2,-2)}{\|(1,2,-2)\|}}$

$e = \frac{(1,2,-2)}{\|(1,2,-2)\|} =$

Summary

The distance between two line



$d = \frac{|(a-b) \cdot (v \times w)|}{\|v \times w\|}$

2M. (a) Let a and b vector in $\mathbb{R}^3$. Suppose that $a \times b = b \times a$. Show that, as claimed in the notes, we must have $a \times b = 0$ (and also $b \times a = 0$). Can this happen without either of a, b being the zero vector?

① $a \times b = b \times a$ given.

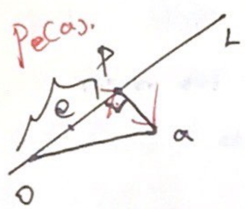
② $a \times b = -b \times a$ ⇒ facts.

① + ②

$2(a \times b) = 0 \iff a \times b = 0$

It happens whenever a and b are parallel.

eg: $a = b = (1, 0, 0)$



(b) (i) $P_e(a) = \dfrac{a \cdot e}{\|e\|^2} e = \left(-\frac{7}{9}, -\frac{14}{9}, \frac{14}{9}\right)$

(ii) component of a perpendicular to e.

$= a - P_e(a)$

$= (-1, 2, 5) - \left(-\frac{7}{9}, -\frac{14}{9}, \frac{14}{9}\right)$

$= \left(-\frac{2}{9}, \frac{32}{9}, \frac{31}{9}\right)$

(c). $p = \left(-\frac{7}{9}, -\frac{14}{9}, \frac{14}{9}\right)$. Closest point.

公式加减.

(d) $d = \|a - p\| = \left\|\left(-\frac{2}{9}, \frac{32}{9}, \frac{31}{9}\right)\right\| = \frac{\sqrt{221}}{3}$

Summary