$$\hat{\sigma}^2 = \frac{1}{n-2}\sum(y_i - \hat{y}_i)^2 = \frac{1}{n-2}\sum r_i^2$$

换成 $\hat{\beta}_1$，因为 $\hat{\beta}_1$ 有方差形式!

$$\sum r_i^2 = \sum(y_i - \hat{y}_i)^2 = \sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

$$= \sum(y_i - (\bar{y} - \hat{\beta}_1\bar{x}) - \hat{\beta}_1 x_i)^2$$

$$= \sum(y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x}))^2$$ 合并 $\hat{\beta}_1$

$$= \sum(y_i - \bar{y})^2 + \hat{\beta}_1^2 \sum(x_i - \bar{x})^2 - 2\hat{\beta}_1 \sum(y_i - \bar{y})(x_i - \bar{x})$$

$$= (n-1)S_y^2 + \hat{\beta}_1^2 (n-1) S_x^2 - 2\hat{\beta}_1 S_{xy}(n-1)$$ $S_{xy} = \hat{\beta}_1 S_x^2$

$$= (n-1)S_y^2 + \hat{\beta}_1^2 (n-1) S_x^2 - \cancel{2S_{xy}}\, 2\hat{\beta}_1^2 S_x^2 (n-1)$$

$$= (n-1)[S_y^2 + \hat{\beta}_1^2 S_x^2 - 2\hat{\beta}_1^2 S_x^2]$$

$$= (n-1)[S_y^2 - \hat{\beta}_1^2 S_x^2]$$

$$\hat{\sigma} = \frac{n-1}{n-2}[S_y^2 - \hat{\beta}_1^2 S_x^2]$$

Inferene in Regression Assumption

1. $E(\varepsilon_i) = 0$
2. $\mathrm{cov}(\varepsilon_i, \varepsilon_j) = 0 \quad i \neq j$
3. $\mathrm{var}(\varepsilon_i) = \sigma^2$
4. $Y_i | x_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2) \quad i = 1, \dots, n$

→ check whether or not the residuals have a normal distribution.

Summary

$TSS = RSS + ESS$

$\Sigma(y_i - \bar{y})^2 = \Sigma(y_i - \hat{y}_i)^2 + \underbrace{\Sigma(\hat{y}_i - \bar{y})^2}_{\text{explain}}$

$LHS = \Sigma(y_i - \bar{y})^2$

$= \Sigma((y_i - \hat{y}_i) + (\hat{y}_i - \bar{y}))^2$

$= \Sigma(y_i - \hat{y}_i)^2 + \Sigma(\hat{y}_i - \bar{y})^2 + \underbrace{2\Sigma(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})}_{\text{required} = 0}$

$\Sigma(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$

$= \Sigma(y_i - \hat{y}_i)(\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})$

$= \hat{\beta}_0 \underbrace{\Sigma(y_i - \hat{y}_i)}_{=0} + \hat{\beta}_1 \underbrace{\Sigma(y_i - \hat{y}_i)x_i}_{0} - \bar{y} \underbrace{\Sigma(y_i - \hat{y}_i)}_{0}$

不等过.

$= \Sigma(y_i - \hat{y})\hat{y}_i - \Sigma(y_i - \hat{y}_i)$
$\overset{\shortparallel}{0} \quad \overset{\shortparallel}{0}$

why R^2 (coefficient of determination) $= r^2$

$R^2 = \frac{ESS}{TSS}$

$ESS = \Sigma(\hat{y}_i - \bar{y})^2$

$= \Sigma(\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})^2$

$\hat{\beta}_0$ 替换

$= \Sigma(\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i - \bar{y})^2$

$= \hat{\beta}_1^2 \Sigma(x_i - \bar{x})^2$

$= \hat{\beta}_1^2 \cdot (n-1) S_x^2$

$= \frac{S_{xy}^2}{S_x^4} \cdot (n-1) S_x^2$

$= \frac{S_{xy}^2 (n-1)}{S_x^2}$

Total Sum of Square $= (n-1) S_y^2$

Summary

$R^2 = \frac{S_{xy}^2 (n-1)}{S_x^2} \Big/ (n-1) S_y^2 = \frac{S_{xy}^2}{S_x^2 S_y^2} = r^2$

$$t_1 = r\sqrt{\frac{n-2}{1-r^2}} \qquad t_2 = \frac{\hat{\beta}_1}{\hat{\sigma}/\sqrt{(n-1)S_x^2}}$$

$$\frac{\hat{\beta}_1}{\hat{\sigma}/\sqrt{(n-1)S_x^2}} = \frac{S_{xy}}{S_x^2}\sqrt{\frac{(n-1)S_x^2}{\frac{(n-1)}{(n-2)}[S_y^2-\hat{\beta}_1^2S_x^2]}}$$

$$= \frac{S_{xy}}{S_x^2}\sqrt{\frac{(n-2)}{[S_y^2-\frac{S_{xy}^2}{S_x^2}]}}$$

$$= \frac{S_{xy}}{S_xS_y}\sqrt{\frac{n-2}{1-\frac{S_{xy}^2}{S_x^2S_y^2}}}$$

$$= r\sqrt{\frac{n-2}{1-r^2}}$$

Spearman's rank order correlation coefficient

$$r_s = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}$$

There is no equal ranks

$$\bar{r}_x = \frac{1}{n}\sum r_{x_i} = \frac{(n+1)n}{2}\cdot\frac{1}{n} = \frac{n+1}{2}$$

$$\bar{r}_y = \frac{n+1}{2}$$

$$r_s = \frac{\sum(r_{x_i}-\bar{r}_x)(r_{y_i}-\bar{r}_y)}{\sqrt{\sum(r_{x_i}-\bar{r}_x)^2\sum(r_{y_i}-\bar{r}_y)^2}} \Rightarrow \frac{\sum(r_{x_i}-\bar{r}_x)(r_{y_i}-\bar{r}_y)}{\sum(r_{x_i}-\bar{r})^2}$$

相等的

$$\sum d_i^2 = \sum(r_{x_i}-r_{y_i})^2 = \sum((r_{x_i}-\bar{r}_x)-(r_{y_i}-\bar{r}_{y}))^2$$

$$= \sum[(r_{x_i}-\bar{r}_x)^2 - 2(r_x-\bar{r}_x)(r_{y_i}-\bar{r}_y) + (r_{y_i}-\bar{r}_y)^2]$$

$$= 2\sum(r_{x_i}-\bar{r}_x)^2 - 2(r_x-\bar{r}_x)(r_{y_i}-\bar{r}_y)$$

$$2\sum(r_{x_i}-\bar{r}_x)^2 = 2\sum(r_x^2 - 2r_{x_i}\bar{r}_x + \bar{r}_x^2)$$

$$= 2(\sum r_x^2 - 2\bar{r}_x\sum \bar{r}_{x_i} + n\bar{r}_x^2)$$

$$= 2[\sum r_{x_i}^2 - n(\bar{r}_x)^2]$$

$$= 2\cdot\left[\frac{n(n+1)(2n+1)}{6} - n\left(\frac{n+1}{2}\right)^2\right] = \frac{n^3-n}{6}$$

Summary

$$r_s = \frac{\frac{2\sum(r_{x_i}-\bar{r}_x)^2 - \sum d_i^2}{2}}{\frac{n^3-n}{6}} = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

Unbiased estimators and estimates

let $X_1 \cdots X_n$ iid. random variables

θ be unknow population parameters

$T = T(X_1 \cdots X_n)$ as a functin of X called esitimetor of θ

Unbiased estimator: $E(T) = \theta$

MVUE: Among all unbiased estimetes of θ has smallest variance, call 𝕋 T a minimum variance unbiased estimator

$$S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 \quad S^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$$

Consider: $E[\sum (x_i - \bar{x})^2] = E[\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)]$

$= E[\sum x_i^2 - 2\bar{x}\sum x_i + \sum \bar{x}^2]$

$= E[\sum x_i^2 - n\bar{x}^2]$

$= \sum E[x_i^2] - nE(\bar{x}^2)$

$var(x_i) = \sigma^2 = E(x_i^2) - E(x_i)^2 \Rightarrow E(x_i^2) = \sigma^2 + \mu^2$

$var(\bar{x}) = \frac{\sigma^2}{n} = E(\bar{x}^2) - E(\bar{x})^2 \Rightarrow E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2$

$E[\sum (x_i - \bar{x})^2] = \sum (\sigma^2 + \mu^2) - n(\frac{\sigma^2}{n} + \mu^2)$

$= n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2$

$= (n-1)\sigma^2$

$E(S^2) = \frac{1}{n-1} E[\sum (x_i - \bar{x})^2] = \sigma^2$ ✓

Summary

5

ALF 2011-2012 试卷总结报告

1. 命题: "and" "or"

2. 命题的否定: such that 的位置.

For evey ε there exist δ such that δ<ε and δ²>ε²

⇒ There exist ε such that for every δ we have δ≥ε or δ²≤ε²

3. PMI: 设定命题: 不能加 for all n∈N

4. 不等式: Algebraic 分情况讨论。题目中关键词 carefully algebraic

5. 确界: 通过算 limit 说明是确界是不会分的.

直接写式子

写出来的式子, 不要忘了考虑 $\frac{1}{n+2}$ 这种代数项

是恒大于0的 (always positive)

6. 数列极限: ratio test. 求出比值的极限 exits and

equal $L<1$. → zero.

$\cos(\pi n)+3^{-n}$. ⇒ 说明 Algebra of limit.

7. Chaos: 极限是 fixpoint, 用 algebra of limit.

单调有界理论 (MST)

指出可能极限 (最开始的问题)

$\boxed{\text{初始值不要忽略}}$

"Diverges to ∞"

8. Contraction: 定义.

Summary

5

2012-2013试卷总结报告

1. 命题的否定：

for all $y\in\mathbb{R}$, either $y=0$ or there exist $\varepsilon>0$ such that $\frac{1}{y^2}>\varepsilon$.

$\Rightarrow$ There exist $y\in\mathbb{R}$ such that $y\neq 0$ and for all $\varepsilon>0$ we have $\frac{1}{y^2}\leq\varepsilon$.

2. PMI: $p(n)$ 写成文字形式而非函数

$p(n)$ 设定时不可加 for all $n\in\mathbb{N}$

字母区分

3. 确界 不要用求极限来说明确界

(不要忽略附加项的正负情况)

4. 极限: ratio 求出的并不是数列的极限

说明数列收敛于哪一个具体的值.

5. chaos 不能跳过 Algebra of Limit 直接写

$f(L)=L$.

If $\lim\limits_{n\to\infty} x_n = L\in\mathbb{R}$

then also $\lim\limits_{n\to\infty} x_{n+1} = L$

$\therefore L=\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} f(x_n)$

$= f(\lim\limits_{n\to\infty} x_n)$

$= f(L)$

cite 单调有界原理

说清楚 fixpoint

$>$ 和 $\geq$

(一定要回归到这个数列收敛到哪一点(因为之前的结论)

6. 反常积分: 如果是字母, 分情况)

7. ratio test 不要忘了求 lim.

8. 最后绝对值打开的情况

Summary

ACP 易错.

Integral test 三个前提 ① non-negative

② continuous

③ non-increasing

前提需要首先 <u>check</u>. 这三个条件

limit Comparison test 可以理解成用一个已经收敛的极数去构造一个数列的极限.

Alternating Series Test 三个条件 ① $a_k \geq 0$ non-negative

② $a_{k+1} \leq a_k$ non-increasing

③ $a_k \to 0$ $a_k \to 0$.

eg: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{k^3+1}$ $a_k = \frac{k}{k^3+1}$

① $a_k \geq 0$

② $f'(x) = \frac{1-2x^3}{(x+1)^2(x^2-x+1)^2} < 0$. $x \geq \frac{1}{2}$. $a_{k+1} \leq a_k$ for $k \geq 1$

③ $\lim_{k \to \infty} \frac{k}{k^3+1} = 0$ (分母次数大于分子次数)

↳ converges

数列收敛必有界

算极限最简单方法，上下同除.

$$\int_{2}^{\infty} \frac{x^2-1}{x^p+4x^2+2}\,dx$$

① $p \leq 2$ $\Rightarrow$ $\frac{1}{4}$ $\int_2^{\infty} \frac{1}{4}$ $\Rightarrow$ diverges.

② $p > 2$ $\Rightarrow$ $\frac{x^2}{x^p}$ $\Rightarrow$ $\frac{1}{x^{p-2}}$ $p-2>1$ $\boxed{\{p>3\}}$ ✓

Summary

5

2013~2014 试卷总结报告

1. 命题：仔细判断 true or False

2. 命题的否定：

For every $\varepsilon>0$ there exist $n\in N$ such that $1/n<\varepsilon$ and $n>\varepsilon$

$\Rightarrow$ There exist $\varepsilon>0$ such that for every $n\in N$ we have $1/n \geq \varepsilon$ or $n\leq\varepsilon$

3. PMI: principle of Mathematical Induction

注意强调 $n\in N$. ($0\notin N$)

从1开始试 $P(1)$.

4. 确界：定义非常重要

证明最小上界或最小下界，根据定义写出的只是分析过程，最好写明取值以及取值是如何联系到定义上去的

4. 极限：强调有加减的说一句 Algebra of Limit

如果有用到 e^x，说一句

By the continuity of exponential function.

用洛必达，说明分子分母的情况

5. chaos：用MST之前说一句 ~~两~~

① Bounded ② non-increasing

6. 级数：该分奇偶就分奇偶.

重要收获：级数和反常积分相似的东西.

我之前以为只能偷偷判断，结果两者最完美的联系其实就是 $\boxed{\text{integral test.}}$

如果积分算简单就用积分.

Summary

累积求和公式$p(n+1)$只用改变 Σ 上的 n 即可.
原始公式里面不变

$$\frac{\ln k}{k^2} \le \frac{\ln k}{k^{3/2}} \to con.$$

K的取值在求和符号下面

PMI: $p(1) \wedge \forall n [p(n) \Rightarrow p(n+1)] \Rightarrow p(n) \quad \forall n \in N$

$p(N) \wedge \forall n \geq N [p(n) \Rightarrow p(n+1)] \Rightarrow p(n) \quad \forall n \geq N$

确界前提 A is non-empty subset
attracting point 求出具体值.

$p(x) = \frac{x^2}{10} + 1$ is a contraction on $[-1, 4]$ two ways

① Based on definition: factorising

$p(x) - p(y)$

For all $x, y \in [-1, 4]$

$|p(x) - p(y)| = (\frac{x^2}{10} - 1) - (\frac{y^2}{10} + 1) = \underbrace{\frac{1}{10}}|x+y||x-y|$

$x+y$ is at most $|4+4| = 8$

$\therefore |p(x) - p(y)| = \frac{|x+y|}{10}|x-y| \le \frac{8}{10}|x-y| = \frac{4}{5}|x-y| \quad \frac{4}{5} < 1$

② p is differentiable. Mean Value theorem:

if $-1 \le x < y \le 4$ there exist $t \in (x, y)$ s.t

$$p'(t) = \frac{p(x) - p(y)}{x - y}$$

$p'(t) = \frac{2}{10}t \le \frac{4}{5}$ at most

$\therefore |p(x) - p(y)| = |p'(t)||x-y| \le \frac{4}{5}|x-y| \quad \frac{4}{5} < 1$

Summary

$\therefore p$ is a contraction.

5

Show that $\sum_{k=1}^{\infty} ke^{-k} = \frac{e}{(e-1)^2}$

引用. 级数中求导

$f(x) = \sum_{k=1}^{\infty} C_k (x-a)^k$

$f(x) = \sum x^k = \frac{1}{1-x} \quad |x|<1$

$f(x)' = \sum k x^{k-1} = \frac{1}{(1-x)^2}$

let $e^{-1} = x$.

$\sum k \left(\frac{1}{e}\right)^{k-1} = \frac{1}{(1-\frac{1}{e})^2}$

$\therefore \sum k \cdot \left(\frac{1}{e}\right)^k = \sum k \left(\frac{1}{e}\right)^{k-1} \cdot \left(\frac{1}{e}\right)$

$= \frac{\frac{1}{e}}{(1-\frac{1}{e})^2} = \frac{e}{(e-1)^2}$

limit Comparison test 就像自己构造极限

$\sum_{k=1}^{\infty} \frac{4k^3+7}{k^6+9k^5}$ >0 choose $b_k = \frac{1}{k^3} > 0$.

$a_k = \frac{4k^3+7}{k^6+9k^5} > 0$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{(4k^3+7)k^3}{k^6+9k^5} = \lim_{k\to\infty} \frac{4+7k^{-3}}{1+9k^{-1}} = 4 \in (0, +\infty)$

since $\sum \frac{1}{k^3}$ conveges. $\therefore \sum \frac{4k^3+7}{k^6+9k^5}$ conveges

Summary

命题的否定:

For every positive $\varepsilon \in \mathbb{R}$, there exists $n \in \mathbb{N}$, such that $\frac{\ln n}{n} < \varepsilon$.

$\Rightarrow$ There exist a positive $\varepsilon \in \mathbb{R}$, such that for all $n \in \mathbb{N}$, we have $\frac{\ln n}{n} \geq \varepsilon$

$\forall x, A$

$\exists x, \neg A$

Only if $A \Rightarrow B$. A only if B

if $A \Leftarrow B$ A if B

命题的否定:

For all $x>0$ there exist $N \in \mathbb{N}$ such that $N > x$

$\Rightarrow$ There exists $x>0$ such that for all $N \in \mathbb{N}$, we have $N \leq x$.

命题的否定:

For all $\varepsilon>0$ there exist $N \in \mathbb{N}$ such that $1/N < \varepsilon$

$\Rightarrow$ There exist $\varepsilon>0$ such that for all $N \in \mathbb{N}$ we have $1/N \geq \varepsilon$

数学归纳法可能用到之前结论.

确界的商式子注意符号

数列的 Ratio test $l<1$ 即可说明 $x_n \to 0$

用 Sandwich Theo 不一定两边对称. $0 \leq |\cos x| \leq 1$ 也可以.

Summary

再强调一下. chaos 里的 x_0 非常重要.

相关 behavior 先代临界初始值观察.

$0 \le x_0 \le 1$. $\underline{x_0=0}$ $\underline{x_0=1}$ 可能之间有联系.

attracting point 判断记得加 $|f(x)|$ 绝对值

limit comparison test: $a_k, b_k > 0$

$L \in \underline{(0, \infty)}$

ratio test. a_k/分母不能为 0

反常积分 能积分就积分

Summary

linear 复习

Principle Argument $-\pi < \text{Arg}\, z \le \pi$

$\arg z_1 z_2 = \arg z_1 + \arg z_2$

$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$

De Moivre's formula:

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z = w^n \qquad w = \sqrt[n]{z}$

$z = r(\cos\theta + i\sin\theta) \qquad w = R(\cos\phi + i\sin\phi)$

$R^n(\cos\phi + i\sin\phi) = z = r(\cos\theta + i\sin\theta)$

$R = r^{\frac{1}{n}}$

$n\phi = \theta + 2k\pi$

$w = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$ $\quad k = 0, 1, 2, \cdots n-1$

$w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2k\pi}{n}\right)}$

异面直线距离

$d = \frac{|\vec{EF}\cdot\vec{n}|}{|\vec{n}|} = \frac{\|(f-e)\cdot(u\times w)\|}{\|u\times w\|}$

E, F 为 l_1 l_2 上的点

(距离 = EF 在公共法向量上投影)

Summary

5

矩阵乘法的多项式形式

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Gaussian-Jordan

$(A \mid I) \to (I \mid A^{-1})$

Vector Space 的八个 condition:

A_1: $v+w=w+v$ 交换律

A_2: $u+(v+w)=(u+v)+w$ 结合律

A_3: $v+0=v$

A_4: $v+(-v)=0$

M_1: $k(v+w)=k(v)+kw$ 分配律

M_2: $(k+k')v=kv+k'v$ 分配律

M_3: $(kk')v=k(k'v)$ 结合律

M_4: $1 \cdot v=v$

~~Subspace~~

Subspace 三个 condition:

(i) $0 \in W$

(ii) $v \in W$, $w \in W$. $v+w \in W$

(iii) $v \in W$. $kv \in W$

Summary

kernel 和 Range 定义

$Ker(T) = \{ \underline{v} \in V \mid T(\underline{v}) = 0 \}$

$Range(T) = \{ \underline{w} \in W \mid \underline{w} = T(\underline{v})$ for some $\underline{v} \in W \}$

↳ 可具体化: eg: 4×4 Matrix

$Range(T) = \{ v \in R^4 \mid \exists x \in R^4 \ \ Tx = v \}$

第一题要画图

向量叉乘

$\bar{a} = \langle a_1, a_2, a_3 \rangle \quad \bar{b} = \langle b_1, b_2, b_3 \rangle$

$\begin{matrix} a & b \\ c & d \end{matrix}$ 主: ad 副: bc

$c = \bar{a} \times \bar{b} = ($主-副, $\boxed{副-主}$, 主-副$)$

慢一点 看清楚

$u_1 \ u_2 \ u_3$

↓ Gram-Schmidt process

$\boxed{orthogonal}$ basis $(v_1 \ v_2 \ v_3)$

↓ Normalise

$(q_1 \ q_2 \ q_3)$

$\boxed{orthonormal}$ basis

Summary

George Willsons

Linear Mathematic chapter.

chapter 1:

$z = x + yi \quad w = u + vi$

Modulus:

$\sqrt{(x-u)^2 + (y-v)^2}$

$|z+w| \leq |z| + |w|$

$|z-w| \geq ||z| - |w||$

$|z| = |\bar{z}|$

$z\bar{z} = \bar{z}z = |z|^2$

Argument:

$Re(z) = |z| \cos(\arg z) \quad Im(z) = |z| \sin(\arg z)$

Principle argument $\quad Arg z \quad -\pi < \theta \leq \pi$

Exponential form

$e^{it} = \cos t + i \sin t$

$z = |z| e^{i \arg(z)}$

Solution of complex polynomial equation

$z^n = a \quad$ if $\quad a = Re^{is}$

solutions: $R^{1/n} e^{i(2k\pi + s)/n} \quad k = 0, 1 \cdots n-1$

Roots of unity:

$z^n = 1 \Rightarrow$ solution: $e^{i\frac{2k\pi}{n}} \quad k = 0, \cdots n-1$

set $\cancel{w = e^{i\frac{2k\pi}{n}}}$ $w = e^{i\frac{2\pi}{n}}$

n^{th} roots: $1, w, w^2, \cdots w^{n-1}$

Summary

Chapter 2.

vector a. b.

$$\cos\theta = \frac{a \cdot b}{\|a\|\|b\|}$$

$a_b = \|a\| \cos\theta = \frac{a \cdot b}{\|b\|}$ length of projection

$\bar{a}_b = a_b \hat{b} = \frac{a \cdot b}{\|b\|} \hat{b} = \frac{a \cdot b}{\|b\|^2} b$ component $P_b(a)$

Cross product:

Magnitude: $|a \times b| = \|a\|\|b\| \sin\theta$

Lagrange identity: $\|a \times b\|^2 = \|a\|^2\|b\|^2 - (a \cdot b)^2$

Triple product: $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

$(a \times b) \times c = (c \cdot a) b - (c \cdot b) a$

Vector Geometry:

Lines: $\mathcal{L}(A; v)$: through point A with direction v

$R = A + tv.$

Cartesian equation of a line.

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$ (v is direction vector)

Planes: parametric form of a plane $\Pi(A; v; w)$

$P = A + tv + sw$ (v is not parallel to w)

$n = v \times w$

Cartesian equation of a plane

$ax + by + cz = d$

Summary

Angle: two lines: $\cos\theta = \hat{u}\cdot\hat{v}$ $\quad 0 \le \theta \le \pi/2$

$\mathcal{L}$ and Π: $\theta = \pi/2 - \phi$

$\sin\theta = \frac{n\cdot v}{\|n\|\|v\|}$ $\quad 0 \le \phi \le \frac{\pi}{2}$

Nearest points and distances

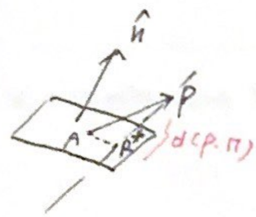
$\mathcal{L}(A; v)$

1. $R^* = A + (\vec{AP}\cdot\hat{v})\hat{v}$

2. Specify R^* $\quad R^* = A + t^* v$

$t^* = (\vec{AP}\cdot\hat{v})\hat{v}$

3. Area: $d(P, \mathcal{L}) = \frac{\|\vec{AB}\times\vec{AP}\|}{\|\vec{AB}\|}$



Π & A.

$R^* = P - (\vec{AP}\cdot\hat{n})\hat{n}$ $\quad (\vec{RP} = (\vec{AP}\cdot\hat{n})\hat{n}$

Matrix Algebra

$[AB]_{ij} = R_i \cdot C_j$

$a_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}$

$A^T = (a^T_{ij}) = a_{ji}$

Summary

数字在上或下

Upper triangle: $a_{ij}=0 \quad \forall i>j$

lower triangle: $a_{ij}=0 \quad \forall j>i$

diagonal triangle: $a_{ij}=0 \quad \forall i\neq j$

Identity matrix: $a_{ii}=1 \quad \forall i=1,\cdots n$

$$[I_n]_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

A, B are lower tri. Prove A·B is low tri.

$[A]\ a_{ij} \quad [B]=b_{ij} \quad \forall i<j,\ a_{ij}=b_{ij}=0$

$C=c_{ij}=A\cdot B$

$c_{ij}=\sum_{k=1}^{n} a_{ik}a_{kj}$

$i<j$

$$c_{ij}=\underbrace{\sum_{k=1}^{j-1} a_{ik}\cdot b_{kj}}_{k<j} + \underbrace{\sum_{k=j}^{n} a_{ik}b_{kj}}_{i<k} = 0$$

$AI=IA=A$

$$[AI]_{ij}=\sum_{k=1}^{n}[A]_{ik}[I]_{kj} = [A]_{ij}[I]_{jj} + \sum_{k\neq j}[A]_{kj}[I]_{kj}$$

0

$$= [A]_{ij}[I]_{jj}$$

$$= [A]_{ij}$$

$$[IA]_{ij}=\sum_{k=1}^{n}[I]_{ik}[A]_{kj} = [I]_{ii}[A]_{ij} + \sum_{k\neq i}^{n}[I]_{ik}[A]_{kj}$$

$$= [A]_{ij}\cdot 1 + 0$$

$$= [A]_{ij}$$

Summary

Conjugate $\bar{A}$

Hermitian conjugate $A^* = (\bar{A})^T = \overline{A^T}$

symmetric $A^T = A$

orthognal $A^T = A^{-1}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Adjugate $\text{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^{-1} = \frac{1}{\det A} \text{Adj}(A)$$

$$\det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det(M_{1k})$$

$$\det(A) = \sum_{k=1}^{n} \left[(-1)^{i+k} a_{ik} \det(M_{ik}) \right] = \sum_{k=1}^{n} \left[(-1)^{k+j} a_{kj} \det(M_{kj}) \right]$$

$$1 \le i \le n, \quad 1 \le j \le n$$

co-factor: $C_{ij} = (-1)^{i+j} \det(M_{ij})$

Determent of triangular matrixes:

$$\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11} a_{22} \cdots a_{nn}$$

proof: A $(n+1) \times (n+1)$

$$\det(A) = a_{11} \det(M_{11}) + \sum_{k=2}^{n+1} (-1)^{k+1} a_{k1} \det(M_{k1})$$ $k>1$ Upper $=0$

$$= a_{11} \det(M_{11})$$

$[M_{11}]_{ij} = [A]_{(i+1)(j+1)} = 0$

$$\det(M_{11}) = \prod_{k=1}^{n} [M_{11}]_{kk} = \prod_{k=1}^{n} [A]_{(k+1)(k+1)} = \prod_{k=2}^{n+1} [A]_{(k+1)(k+1)}$$

set $i = k+1$

$$\det(A) = a_{11} \det(M_{11}) = a_{11} \prod_{i=2}^{n+1} [A]_{(k+1)(k+1)} = \prod_{i=1}^{n+1} [A]_{ii}$$

since A is lower triangular, then A^T is upper.

Summary

$$\det(A) = \det(A^T)$$

$$\det(A^T) = [A^T]_{11} [A^T]_{22} \cdots [A^T]_{nn}$$

$$= [A]_{11} [A]_{22} \cdots [A]_{nn}$$

Chapter 4 Vector Space.

$\mathbb{R}^1 = \mathbb{R}$

$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

$v = (x_1, \cdots x_n) = x_1 e_1 + \cdots + x_n e_n = \sum_{i=1}^{n} x_i e_i$

$\mathbb{R}^n$: n-dimension.

$M_{m,n}(\mathbb{R})$ m, n - dimension

$A = (a_{ij}) = a_{11} E_{11} + a_{12} E_{12} + \cdots + a_{mn} E_{mn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij}$

E_{ij} (i, j) entry equal to 1 and all other 0.

Vector space: $u, v, w \in V$ $\quad k, k' \in F$ Field F of scalars (such as $\mathbb{R}$ or $\mathbb{C}$)

A_1: $v + w = w + v$

A_2: $u + (v + w) = (u + v) + w$

A_3: $v + 0 = v$

A_4: $v + (-v) = 0$

M_1: $k(v + w) = k(v) + k(w)$

M_2: $(k + k')v = kv + k'v$

M_3: $(kk')v = k(k'v)$

M_4: $1v = v$

题中如果是函数
就写成 f g 形式

Subspace

(S_1) $0 \in W$

(S_2) $v \in W$ $\quad w \in W$ $\quad v + w \in W$ 封闭加法

(S_3) $v \in W$ $\quad kv \in W$ $\quad k \in F$ 封闭乘法

$W \leq V$ (W is a subspace of V)

Summary

5

该向量空间对象是函数

Every vector space is a subspace of itself

Other call proper subspaces

eg: $W = \{ f \in V \mid f(x_0) = 0 \} \leq V$

$0(x_0) = 0 \quad 0 \in W$ (zero function is in W)

$f, g \in W \Rightarrow (f+g)(x_0) = f(x_0) + g(x_0) = 0 + 0 = 0 \quad f+g \in W$

$k \in \mathbb{R},\ f \in W \Rightarrow (kf)(x_0) = k(f(x_0)) = k \cdot 0 = 0 \quad kf \in W$

$W = \{ f \in V \mid f'' = f \} \leq V$

$0'' = 0 \Rightarrow 0 \in W$

$f, g \in W \Rightarrow (f+g)'' = f'' + g'' = f + g \in W$

$k \in \mathbb{R},\ f \in W \Rightarrow (kf)'' = kf'' = kf \Rightarrow kf \in W$

$W_1 \leq V \quad W_2 \leq V \qquad W_1 \cap W_2 \leq V$

proof: $0 \in W_1,\ 0 \in W_2 \qquad 0 \in W_1 \cap W_2$

$a, b \in W_1,\ a, b \in W_2 \qquad a + b \in W_1 \cap W_2$

$v_1 \in W_1, W_2\ \lambda v_1 \in W_1,\ \lambda v_1 \in W_2 \qquad \lambda v \in W_1 \cap W_2$

$W_1 \cup W_2$ not $\leq V$

$W_1 = \{ (x, 0) \mid x \in \mathbb{R} \} \qquad W_2 = \{ (0, y) \mid y \in \mathbb{R} \}$

$e_1 \in W_1 \quad e_2 \in W_2$

but $e_1 + e_2 = (1,1) \notin W_1 \cup W_2$

Summary

Chapter 5 Linear System

$Ax=b \quad x=A^{-1}b$

Gauss Jordon

$(A \mid I) \to (I \mid A^{-1})$

Chapter 6 ~

Coordinate transformation

$$\begin{cases} x' = ax+by \\ y' = cx+dy \end{cases} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Eigenvector

$Av = \lambda v \quad (v \neq 0)$

$A^k v_1 = \lambda^k v_1$

The Cayley - Hamilton Theorem

Matrix A satisfied its own characteristic polynomial

$p(A) = 0$ where $p(\lambda)$ is the characteristic polynomial

Diagonalisable of Matrix

$A_{n\times n}$ is said to be diagonalisable if there is an invertible $n\times n$ matrix P such that $P^{-1}AP$ is a diagonal matrix

Two matrix A and B are similar if there exists an invertible matrix C such that $A = C^{-1}BC$

Summary

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Constnet $n \times n$ matrix P

$P = [v_1 | v_2 | \cdots | v_n]$

P is invertible since ccolum are linearly independent

$$AP = A[v_1 | v_2 | \cdots | v_n]$$
$$= [Av_1 | Av_2 | \cdots | Av_n]$$
$$= [\lambda v_1 | \lambda v_2 | \cdots | \lambda v_n]$$
$$= [v_1 | v_2 | \cdots | v_n] \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda \end{bmatrix}$$

$$AP = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

$$P^{-1}AP = D$$

Definition Basis

The set $\{v_1, \cdots v_n\}$ of vectors forms a vector space V form basis if $\{v_1, \cdots v_n\}$ are linearly indeperelent and if the space they span is V.

$T: V \to W$

$Ker(T) = \{v \in V \mid T(v) = 0\} \leq V$ nullity(T)

$R(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\} \leq W$ rank(T)

rank(T) + nullity(T) = n

rank is the number of linearly independent rows of A

Summary