NA ① Newton-Raphson Method 推导.

Solving the nonlinear algebraic equation

$$y_{n+s} = \sigma_s h f(t_{n+s}, y_{n+s}) + v$$

$F(y_{n+s}) = y_{n+s} - \sigma_s h f(t_{n+s}, y_{n+s}) - v = 0$. 解这个方程.

在 $(t_{n+s}, y_{n+s}^{[j]})$ 处线性化. 或者将 $F(y_{n+s})$ 在 $(t_{n+s}, y_{n+s}^{[j]})$ 处展开.

$$F(y_{n+s}^{[j]}) + \frac{\partial F}{\partial y}(y_{n+s}^{[j]})\left(y_{n+s} - y_{n+s}^{[j]}\right) \approx 0$$

$$y_{n+s}^{[j]} - \sigma_s h f(t_{n+s}, y_{n+s}^{[j]}) - v + \left[1 - \sigma_s h \frac{\partial f(t_{n+s}, y_{n+s}^{[j]})}{\partial y}\right]\left(\boxed{y_{n+s}} - y_{n+s}^{[j]}\right) = 0$$

$$\boxed{y_{n+s}} = y_{n+s}^{[j]} - \left[1 - \sigma_s h \frac{\partial f(t_{n+s}, y_{n+s}^{[j]})}{\partial y}\right]^{-1}\left[y_{n+s}^{[j]} - \sigma_s h f(t_{n+s}, y_{n+s}^{[j]}) - v\right]$$

$\Downarrow$

$$y_{n+s}^{[j+1]} = y_{n+s}^{[j]} - \left[1 - \sigma_s h \frac{\partial f(t_{n+s}, y_{n+s}^{[j]})}{\partial y}\right]^{-1}\left[y_{n+s}^{[j]} - \sigma_s h f(t_{n+s}, y_{n+s}^{[j]}) - v\right]$$

Modified Newton-Raphson Method

$$y_{n+s}^{[j+1]} = y_{n+s}^{[j]} - \left[1 - \sigma_s h \frac{\partial f(t_{n+s}, y_{n+s}^{[0]})}{\partial y}\right]^{-1}\left[y_{n+s}^{[j]} - \sigma_s h f(t_{n+s}, y_{n+s}^{[j]}) - v\right]$$

NA ② Derive the coefficients of the BDF method for $m = 2, 3, 4$.

Are these method A-stable?

只演示 $m=2$.

BDF: $\rho(w)$. $\sigma(w) = \sigma_m w^m$, order m

$$\rho(w) - \log(w)\,\sigma(w) = O(|w-1|^{m+1})$$

$$\rho(w) = \sigma_m \log(1+(w-1))\,[1+(w-1)]^m$$

let $v = w-1$

$$= \sigma_m \log(1+v)(1+v)^m + \text{h.o.t.}$$

$m=2$: $\sigma_m\left(v - \frac{1}{2}v^2 + \cdots\right)(1 + 2v + v^2) + O(v^3)$

$$= \sigma_m\left[v + \frac{3}{2}v^2 + O(v^3)\right]$$

$$= \sigma_m\left[(w-1) + \frac{3}{2}(w^2 - 2w + 1)\right]$$

$= \sigma_m\left[\frac{1}{2} - 2w + \frac{3}{2}w^2\right]$ choose σ_m to make $\rho(w)$ monic polynomial.

$\sigma_m = \frac{2}{3}$. $= \frac{1}{3} - \frac{4}{3}w + w^2$

$m=2$: $\frac{1}{3}\underline{y}_n - \frac{4}{3}\underline{y}_{n+1} + \underline{y}_{n+2} = \frac{2}{3}h\underline{f}(\underline{y}_{n+2})$

Stability Analysis: $y' = \lambda y$

$$\frac{1}{3}\underline{y}_n - \frac{4}{3}y_{n+1} + \underline{y}_{n+2} = \frac{2}{3}h\lambda y_{n+2}$$

$$(1-\frac{2}{3}h\lambda)\underline{y}_{n+2} - \frac{4}{3}\underline{y}_{n+1} + \frac{1}{3}\underline{y}_n = 0, \quad z = h\lambda$$

$$(1-\frac{2}{3}z)\,w^2 - \frac{4}{3}w + \frac{1}{3} = 0$$

plug. it

$$\underbrace{(1-\frac{2}{3}it)}_{A}w^2 \underbrace{- \frac{4}{3}}_{B}w + \underbrace{\frac{1}{3}}_{C} = 0$$

Cohn - Schur criterion:

The quadratic aw^2+bw+c, $a,b,c \in \mathbb{C}$, $a \neq 0$ obeys the root condition iff (a) $|a| \geq |c|$ (b) $(|a|^2-|c|^2)^2 \geq |a\bar{b}-b\bar{c}|^2$

if (b) is obeyed as an equality then $|b| \leq 2|a|$.

$$A = 1-\frac{2}{3}it, \quad B = -\frac{4}{3}, \quad C = \frac{1}{3}$$

$$|A|^2 - |C|^2 = 1+\frac{4}{9}t^2 - \frac{1}{9} = \frac{4}{9}t^2 + \frac{8}{9} = \frac{4}{9}(2+t^2) > 0$$

$$|A\bar{B} - B\bar{C}|^2 = \left|-\frac{4}{3}(1-\frac{2}{3}it) + \frac{4}{9}\right|$$

$$= \left|-\frac{8}{9} + \frac{8}{9}it\right|$$

$$= \frac{64}{81}(1+t^2) \leq \frac{16}{81}(4+4t^2+t^4)$$

$\Leftrightarrow$ $t^4 \geq 0$. A-stable

NA ⑱ Determine conditions on b, c and A s.t. the method

$$\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

is of order $p \geq 3$.

关键：
1. 预判展开到多少项（根据 order）
2. 内部的 k 逐步递减次数代入
3. k_1 算出后没必要算 k_2，根据对称性换个字母即可。

$k_1 = f(y + h(a_{11}k_1 + a_{12}k_2))$

$= f + hf_y(a_{11}k_1 + a_{12}k_2) + \frac{h^2}{2}f_{yy}(a_{11}k_1 + a_{12}k_2)^2 + O(h^3)$

同理：$k_2 = f + hf_y(a_{21}k_1 + a_{22}k_2) + \frac{h^2}{2}f_{yy}(a_{21}k_1 + a_{22}k_2)^2 + O(h^3)$

停在这里，因为 k 在最后还要乘上 h. $O(h^4)$ $\Rightarrow p=3$.

关键的步骤就是代入多项式到 k 中去，这里的项数要有数。

可直接换成 f

$k_1 = f + hf_y(a_{11}(f + hf_y(a_{11}k_1 + a_{12}k_2)) + \cdots$
$a_{12}(f + hf_y(a_{21}k_1 + a_{22}k_2)) + \cdots)$
$+ \frac{1}{2}h^2 f_{yy} f^2 c_1^2 + O(h^3)$

直接换成 f.

$= f + hc_1 f_y f + h^2(a_{11}c_1 + a_{12}c_2)f_y^2 f + \frac{1}{2}h^2 c_1^2 f_{yy} f^2 + O(h^3)$

同理，根据对称性，换字母即可得 k_2.

$k_2 = f + hc_2 f_y f + h^2(a_{21}c_1 + a_{22}c_2) f_y^2 f + \frac{1}{2}h^2 c_2^2 f_{yy} f^2 + O(h^3)$

$y(t+h) = y + hf + \frac{1}{2}h^2 f_y f + \frac{1}{6}h^3(f_{yy}f^2 + f_y^2 f) + O(h^4)$

$y_{n+1} = y + h(b_1 + b_2)f + h^2(b_1c_1 + b_2c_2)f_y f$
$+ h^3\left[(b_1(a_{11}c_1 + a_{12}c_2) + b_2(a_{21}c_1 + a_{22}c_2))f_y^2 f + \right.$
$\left.\frac{1}{2}(b_1c_1^2 + b_2c_2^2)f_{yy}f^2\right] + O(h^4)$

对比即可得 order condition.

□

NA (19) Prove that the implicit Runge-Kutta scheme

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

can be expressed as a collocation method with a cubic collocation polynomial. Determine the order of this scheme.

这题理论依据是

Corollary 2: Quadrature $\int_a^b g(\tau)w(\tau)d\tau \approx \sum_{l=1}^{s} b_l g(c_l)$ is of order $s+r$ for $r\in\{0,1,\cdots s\}$ iff $b_1 \cdots b_s$ are chosen in the theorem, whereas

$$\int_0^1 \tau^j \boxed{w(\tau)}\, \omega(\tau)\, d\tau = 0 \quad j=0,\cdots r-1$$

where $\boxed{w(t)} = \prod_{k=1}^{s}(t-c_k)$

Corollary 3 Letting $(a,b)=(0,1)$ and $\omega\equiv 1$. the highest order of quadrature is obtained when $c_1 \cdots c_s$ are the zeros of a Legendre polynomial P_s shifted from $[-1,1]$ to $[0,1]$.

Theorem 14. Provided that $w(t)=\prod_{k=1}^{s}(t-c_k)$ with $(a,b)=(0,1)$ $\omega\equiv 1$ and $r\in\{0,1\cdots s\}$, the integral method is of order $s+r$.

collocation poly: $w(t)=(t-0)(t-\frac{1}{2})(t-1)=t^3-\frac{3}{2}t^2+\frac{1}{2}t$

$$\int_0^1 t^{l-1} w(t)\, dt = 0, \quad l=1\cdots m, \quad p=\underline{m}+\underset{\uparrow}{3} \text{ 3-stages.}$$

$l=1$: $\int_0^1 t^0 (t^3-\frac{3}{2}t^2+\frac{1}{2}t)\, dt = 0$ ✓

这里用来找m 判断order用的.

$l=2$: $\int_0^1 t^1 (t^3-\frac{3}{2}t^2+\frac{1}{2}t)\, dt \neq 0$

$\therefore$ $m=1$, $p=1+3=4$ order.

就到这里了.

2022.11.13.

NA

(22) The diffusion equation $u_t = u_{xx}$ is being approximated by the FD method

$$\left(\frac{1}{12}-\frac{1}{2}\mu\right)u_{m-1}^{n+1}+\left(\frac{5}{6}+\mu\right)u_m^{n+1}+\left(\frac{1}{12}-\frac{1}{2}\mu\right)u_{m+1}^{n+1}$$
$$=\left(\frac{1}{12}+\frac{1}{2}\mu\right)u_{m-1}^{n}+\left(\frac{5}{6}-\mu\right)u_m^{n}+\left(\frac{1}{12}+\frac{1}{2}\mu\right)u_{m+1}^{n}$$

μ is courant number.

1. Prove that the method is of order 4 in (Δx)

2. Prove that the method is stable for the Cauchy problem

$u_t = u_{xx}$

$$H(z,\mu)=\frac{\left(\frac{1}{12}+\frac{\mu}{2}\right)\left(z+\frac{1}{z}\right)+\left(\frac{5}{6}-\mu\right)}{\left(\frac{1}{12}-\frac{\mu}{2}\right)\left(z+\frac{1}{z}\right)+\left(\frac{5}{6}+\mu\right)}$$

这个表达式可以告诉我们关于 order stability 的各种信息.

$$=e^{\mu(\log z)^2}+O\left(|z-1|^{p+2}\right)$$

$z=e^{i\theta}$

~~time~~ space derivative u_{xx}

$$H(e^{i\theta},\mu)=e^{-\mu\theta^2}+O\left(\theta^{p+2}\right)$$

这个方法也很重要. 预判之后相乘, 进行计算.

$$\left(\frac{1}{6}+\mu\right)\cos\theta+\frac{5}{6}-\mu-e^{-\mu\theta^2}\left[\left(\frac{1}{6}-\mu\right)\cos\theta+\frac{5}{6}+\mu\right]$$

$= \cdots$

$= \cdots + 0\cdot\theta^4 + \underline{\theta^6}$

$p+2=6 \quad p=4.$

□

NA ㉓ Prove the Gerschgorin theorem.

$$\lambda_1 \cdots \lambda_n \in \bigcup_{k=1}^{n} S_k, \quad \text{where } S_k = \{z \in \mathbb{C} : |z - a_{k,k}| \le \sum_{\substack{l=1 \\ l \ne k}}^{n} |a_{k,l}|\}$$

证明: λ_k, $\underline{v}_k \ne 0$

取出 v_k 中最大的值. $|v_r| \ge |v_i|$, $i=1 \cdots n$.

$A\underline{v} = \lambda \underline{v}$ 写成求和形式

$$\sum_{j=1}^{n} a_{r,j} v_j = \lambda v_r \quad \text{最大值}$$

$$\Rightarrow \sum_{\substack{j=1 \\ j \ne r}}^{n} a_{r,j} v_j = (\lambda - a_{r,r}) v_r$$

$$|\lambda - a_{r,r}|\,|v_r| \le \sum_{j \ne r}^{n} |a_{r,j}|\,|v_j| \quad \text{divided by } |v_r|$$

$$|\lambda - a_{r,r}| \le \sum_{j \ne r}^{n} |a_{r,j}| \underbrace{\left|\frac{v_j}{v_r}\right|}_{\le 1} \le \sum_{j \ne r}^{n} |a_{r,j}|$$

因为 v_r 是最大的值

$\lambda \in S_r$.

NA ㉔ $u_t = G u_x$, $G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 得到 wave equation

$u_{tt} = u_{xx}$

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$u_t = \begin{pmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{pmatrix} \quad u_x = \begin{pmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \end{pmatrix}$$

$$u_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_x$$

$$\Rightarrow \begin{cases} \frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial x} \\ \frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial x} \end{cases}$$

$$\therefore u_t = \begin{pmatrix} \frac{\partial u_2}{\partial x} \\ \frac{\partial u_1}{\partial x} \end{pmatrix} \quad u_x = \begin{pmatrix} \frac{\partial u_2}{\partial t} \\ \frac{\partial u_1}{\partial t} \end{pmatrix}$$

$$u_{tt} = \begin{pmatrix} \frac{\partial}{\partial t} \frac{\partial u_2}{\partial x} \\ \frac{\partial}{\partial t} \frac{\partial u_1}{\partial x} \end{pmatrix} \quad u_{xx} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{\partial u_2}{\partial t} \\ \frac{\partial}{\partial x} \frac{\partial u_1}{\partial t} \end{pmatrix} \Rightarrow u_{tt} = u_{xx}$$