5 . .

Probability. (26) Three applicants for a job at a company. A, B and C are interviewed, and then told that the job will be offered to one of them by telephone the following day. On his way home, A happens to meet the company director, who was on the interview panel. A is desperate to find out who has been chosen, and asks the director if he can tell him. The director, of course, refuses to tell A, but does tell him that B will not be offered the job.

A is delighted, believing that his chances of being offered have now increasised from $1/3$ to $1/2$.

The director (who, if A had been chosen, would have been equally likely to given C's name rather B's) is mystified by this response, since he feels that he ~~was~~ has essentially not told A anything new. Who is correct?

Suppose C overhears the conversation. Have his chance of being offered changed or not?

let. J: be the person offered the job (so J is either A, B, or C). and N be the person named by the director. (so $N=B$, or $N=C$)

Summary

Thus the sample space $\Omega = \{(J=A, N=B)\ (J=A, N=C), (J=B, N=C), (J=C, N=B)\}$.

$p(N=A) = p(N=B) = p(N=C) = \frac{1}{2}$

$p(A) = p(B) = p(C) = \frac{1}{3}$, so the direct is equally likely to name B or C, if $J=A$

$$\left.\begin{aligned} p(J=A, N=B) &= \tfrac{1}{6} \\ p(J=A, N=C) &= \tfrac{1}{6} \\ p(J=B, N=C) &= \tfrac{1}{3} \\ p(J=C, N=B) &= \tfrac{1}{3} \end{aligned}\right\} = 1$$

Now. $p(J=A \mid N=B)$ $\quad p(N=B) = p(J=A, N=B) + p(J=C, N=B) = \frac{1}{2}$

$$p(J=A \mid N=B) = \frac{p(J=A, N=B)}{p(N=B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

so the diretor is correct.

However, $p(J=C \mid N=B) = \dfrac{p(J=C, N=B)}{p(N=B)} = \dfrac{\frac{1}{3}}{\frac{1}{2}} = \dfrac{2}{3}$

$\therefore$ C $\uparrow$.

'Monty Hall.'

Summary

(Monty Hall problem)

goat car goat

do not swapp : car 1/3 goat 2/3

swap : car 2/3 goat 1/3 if you chose goat, the host has no choice, he should chose goat, so if swap, car is coming. so the key step is what you chose in the first chance.

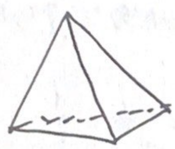
Probability

(27) Five identically-sized regular tetrahedra are available. denoted by $i=1,2,3,4,5$. Tetra hehdra i is colured red on $5-i$ sides and black on the other $i-1$ sides. $(i=1,2,3,4,5)$

ii) All tetrahedra are thrown at random on the ground and a tetrahedron is chosen at random from those. whose uppermost faces are all red. what is ~~the~~ its fourth face is red.

i) A tetrahedron is chosen at random and rolled on the ground. The three uppermost faces are red. What is the probability its fourth face is also red?

这种题直接用RRRB这种字母形式标记事件要容易得多.



① $\begin{cases} 4 \text{ red} \\ 0 \text{ black} \end{cases}$

② $\begin{cases} 3 \text{ red} \\ 1 \text{ black} \end{cases}$

$\begin{cases} 2 \text{ red} \\ 2 \text{ black} \end{cases}$

$\begin{cases} 1 \text{ red} \\ 3 \text{ black} \end{cases}$

$\begin{cases} 0 \text{ side} \\ 4 \text{ black.} \end{cases}$

i). Let E_i be the event that tetrahedron i is chosen $(i=1,2,3,4,5)$ and F be the event that all the upperside faces are red. F作了限定，其他状况自然P为0，只剩下 RRRB or RRRR.

$p(E_i)=\frac{1}{5}$. $(i=1,2,3,4,5)$

$p(F|E_1)=1$ $p(F|E_2)=\frac{1}{4}$ $p(F|E_3)=0$ $p(F|E_4)=0$

$p(F|E_5)=0$.

Then: $p(E_i|F)=\dfrac{p(F|E_i)\,p(E_i)}{\sum_{i=1}^{5} p(F|E_i)\,p(E_i)} = \dfrac{\frac{1}{5}\times 1}{1\times\frac{1}{5}+\frac{1}{4}\times\frac{1}{5}} = \frac{4}{5}$

E RRRR $\begin{cases} ①.② \text{ 都是三面红} \\ ② \text{ 是投出了黑在上.} \end{cases}$

我们不会选择②.

这时只选择①. 100%

ii) ~~E be the tetrahedron ②, H: tetrahed ① is chosen.~~

~~$p(E)=\frac{1}{4}$, so $p(E^c)=3/4$, $p(H|E)=1/2$ and $p(H|E^c)=1$~~

~~Thus $p(H)=p(H|E)p(E)+p(H|E^c)p(E^c)$~~

~~$=\frac{1}{2}\times\frac{1}{4}+1\times\frac{3}{4}=\frac{7}{8}$~~

~~②③④都无 red 在上.~~

Summary

E: chose RRRR.

H: 选 RRRB 的概率(即② uppermost red)

$p(H)=\frac{3}{4}$ $p(H^c)=\frac{1}{4}$

$p(E|H)=\frac{1}{2}$

$p(E|H^c)=1$

$p(E)=p(E|H)\times p(H)+p(E|H^c)\,p(H^c)$

$=\frac{1}{2}\times\frac{3}{4}+1\times\frac{1}{2}=\frac{7}{8}$

相当于我们已经锁定了 RRRR，RRRB 两种情况，但是不总得到RRRR，还得看RRRB

ACF

(28) Suppose that f is a real-valued function with the property that, for every pair of real numbers x and y, we have $f(x+y)=f(x)+f(y)$.

(a) let $x \in R$, Prove by induction that $f(nx)=nf(x)$. for all positive integers n;

(b) show that $f(0)=0$ (hint $0+0=\cdots$)

(c) show that $f(-x)=-f(x)$

(d) deduce that, for every rational number $\frac{p}{q}$, where p and q are integers with $q>0$, we have $f(\frac{p}{q})=\frac{p}{q}f(1)$

(a): Anchor step: $p(1)$ is the statement that $f(1\cdot x)=1\cdot f(x)$, is true.

hypothesis. Induction step: $\boxed{\text{Assume } p(n) \text{ is true}}$, prove $p(n+1)$

$$f((n+1)x)=f(xn+x)$$
$$=f(xn)+f(x)$$
$$=n\cdot f(x)+f(x)=(n+1)f(x)$$

$\therefore p(n) \Rightarrow p(n+1)$ for $n \geq 1$.

b) $f(0)=f(0+0)=f(0)+f(0)=2f(0)$. $f(0)=0$

c) $f(-x)=f(-1\cdot x)=-1\cdot f(x)=-f(x)$

d) $f(\frac{p}{q})=\frac{p}{q}f(1)$.

$pf(1)=f(p)=f(\frac{qp}{q})=f(\frac{p}{q}\cdot q)=qf(\frac{p}{q})$

$\therefore f(\frac{p}{q})=\frac{p}{q}f(1)$

Summary

AcF ⑳ The n^{th} positive odd integer can be written as $2n-1$, let.

$$A_n = \sum_{k=1}^{n} (2k-1) = 1+3+\dots+2n-1$$

i.e. A_n is the sum of the first n positive odd integers.

Determine A_2, A_3. determine (or guess) a simple formula for A_n, and prove it by induction.

$A_1=1$ $A_2=4$. $A_3=9$. $A_4=16$ --- $A_n=n^2$.

Anchor step: let $p(1)$ be the statement that

$$A_1 = \sum_{k=1}^{1} (2\times1-1) = 1$$, which is true

Induction step: Assuming $p(n)$ is true

so we should prove $\sum_{k=1}^{n+1} (2k-1) = (n+1)^2$

$$\begin{aligned} LHS = \sum_{k=1}^{n+1} (2k-1) &= \boxed{\sum_{k=1}^{n} (2k-1)} + 2(n+1)-1 \\ &= \boxed{n^2} + 2(n+1)-1 \\ &= n^2+2n+2-1 \\ &= n^2+2n+1 \\ &= (n+1)^2 = RHS. \end{aligned}$$

$p(n) \Rightarrow p(n+1)$.

Summary

Proof by mathematical induction PMI

PMI (version 1) : $p(1) \wedge \forall n\,[p(n) \Rightarrow p(n+1)] \Rightarrow p(n)$, $\forall n \in N$

PMI (version 2) : $p(N) \wedge \forall n \geq N\,[p(n) \Rightarrow p(n+1)] \Rightarrow p(n)$, $\forall n \geq N$.

App.

(30) The population of squirrels on an island at time t is given by $p(t)$. When the population of squirrels is too high, there is not enough food and the population tends to fall as a result. When the population is low, there is plenty of food and the population tends to rise. It is intended to model the squirrel population by an ODE of the form.

$$\frac{dp}{dt} = f(p)$$

Which of the following would be suitable form for $f(p)$?

(a) $f(p) = \frac{ap}{(b+p)^2}$ (b) $f(p) = \frac{a-p}{b+p}$ (c) $f(p) = \frac{(a-p)p}{b+p}$ (d) $f(p) = -ap^2 + \tanh bp$

a. b are non-negative constant.

find the key info. make main criteria.

I) when $p=0 \Rightarrow f=0$

II) when $0<p<1$ $f>0$

III) when $p \gg 1$ $f<0$

I means infinite. big enough.

(a) I) $p=0$. $f(0)=0$ V III) $p \to \infty$ $f(p)>0$. X

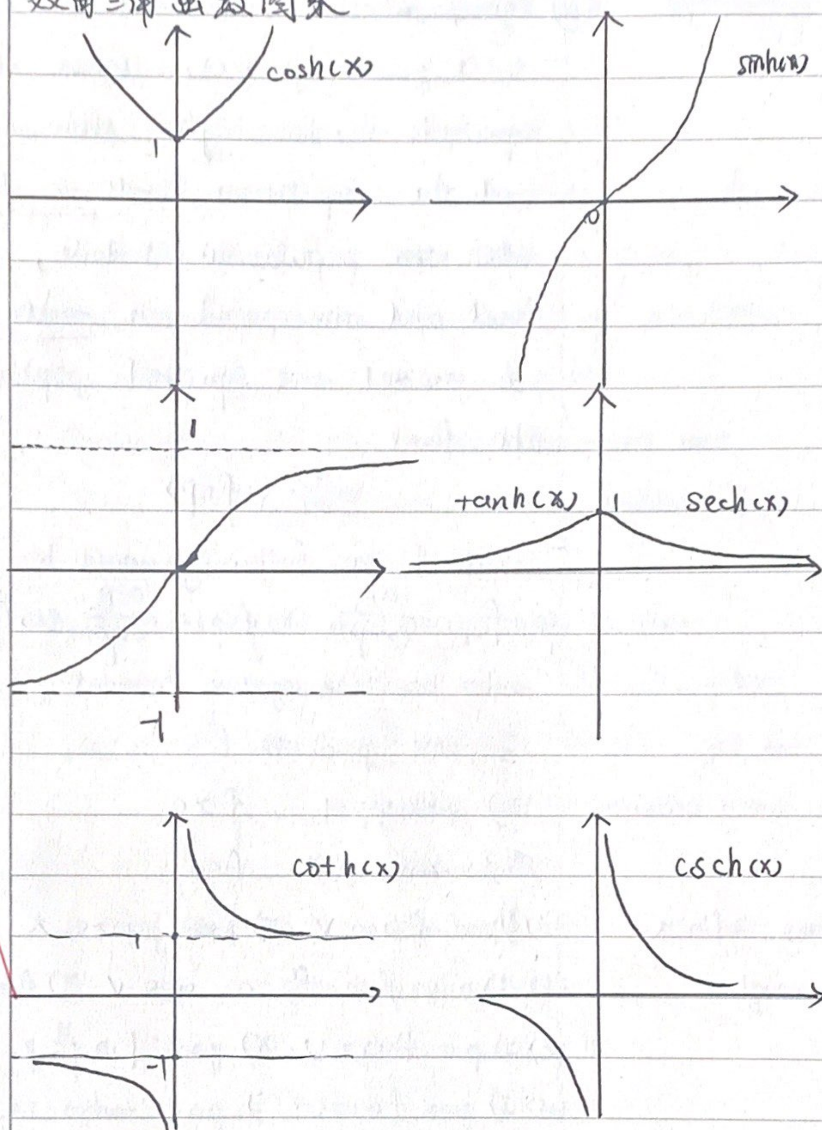
(b) I) $p=0$ $f(0)=\frac{a}{b}=0$. $a=0$. V II) $f(p) = -\frac{p}{b+p} < 0$. X

(c) I) $p=0$. $f(0)=0$ V II) $p \to 0$. $f \to \frac{a}{b}p$. >0 V III) $p \to \infty$. $f \to \frac{-p^2}{p} \approx -p < 0$

(d) I) $p=0$ $f(0)=0$ ✓ II) $p \to 0$ $\tanh \to 0$ >0. V III) $p \to \infty$ $f(p) \approx$ <0. V.

Summary

双曲三角函数图象



都有渐近线!

Summary

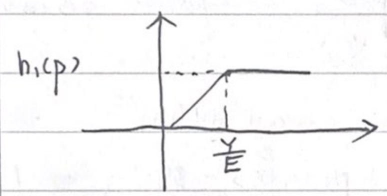
App. (31) Sketch the two functions

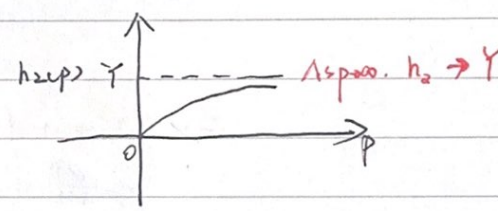
$$h_1(p) = \begin{cases} Ep & \text{for } 0 \le p < Y/E \\ Y & \text{for } Y/E \le p \end{cases} \quad \text{and} \quad h_2(p) = \frac{Ep}{1+Ep/Y}$$

for $p \geq 0$. where E and Y are positive-constants

Discuss in detail ways in which $h_1(p)$ and $h_2(p)$ approximate one another.

[Hint: What is the behaviour of $h_1(p)$ and $h_2(p)$ for small p? For large p?]





h_1 and h_2 have same behavior. when $p \to 0$ and $p \to \infty$

h_1 and h_2 are both increasing functions of p.

However, h_1 is not diffientiable. but. h_2 is.

Summary

App.

③ Models for harvesting.

p the population of fish.

b the net growth rate.

K the maximum capacity.

Due to the logistic model, we can get

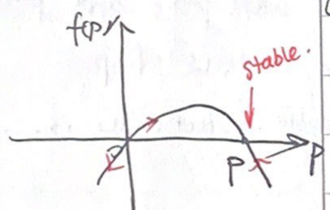
$$\frac{dP}{dt} = f(p) = bp(1-\frac{P}{K}) - h(p).$$

$$h(p) = \begin{cases} E \cdot p, & 0 \le p < Y_0/E \\ Y_0, & Y_0/E \le p \end{cases}$$ → constant effort. → constant yeild

⇒ Harvest for constant effort.

$f(p) = \frac{dP}{dt} = bp(1-\frac{P}{K}) - Ep$, to find the Equilibrium point.

Let $f(p) = 0$. $p = 0$ or $p = K(1-\frac{E}{b})$

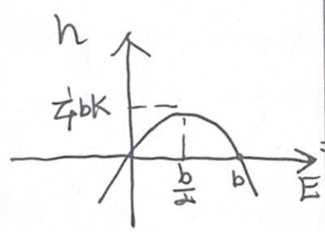


① if $E < b$ ~~appropriate fish~~ $\boxed{\text{sustainable fishing.}}$

$t \to \infty$ $p \to K(1-\frac{E}{b})$.

And if we want to maximize the yeild.

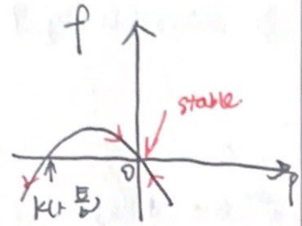
$$\begin{cases} p = K(1-\frac{E}{b}) \\ h = Ep. \end{cases}$$



$\Rightarrow h = EK(1-\frac{E}{b})$ h vs E.

$\therefore h_{max} = \frac{1}{4}bK$ when $E = \frac{1}{2}b$.

Summary



② If $E > b$ $\boxed{\text{overfishing}}$

so $p \to 0$ as $~~p~~$ $t \to \infty$.

Extinction!!!!

$\Rightarrow$ Harvest with constant ~~effort~~ yeild.

$f(p) = \frac{dP}{dt} = bp(1-\frac{P}{K}) - Y_0$ find the equilibrium points.

$f(p) = 0$.

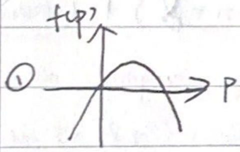
$bp - \frac{bp^2}{K} - Y_0 = 0$

$p^2 - Kp + \frac{Y_0 K}{b} = 0$ $\quad \Delta = K^2 - 4 \times 1 \times \frac{Y_0 K}{b}$

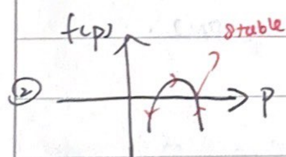
$\Rightarrow p = \frac{K \pm \sqrt{K^2 - \frac{4Y_0 K}{b}}}{2} = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4Y_0}{bK}}\right)$

Real solution $1 - \frac{4Y_0}{bK} > 0$

$\Leftrightarrow$ $Y_0 < \frac{1}{4} bK$

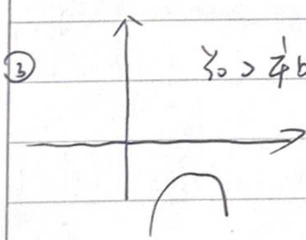
①


$\Rightarrow$ logistic model

②


p will approach equilibrium point.
as $t \to \infty$

③ $Y_0 > \frac{1}{4} bK$



Extinction!!!!!

Summary

Two classifications

- Harvest for constant effort
  - $E < b$ (小于生长率) sustainable fishing $\Rightarrow$ max yeild.
  - $E > b$ overfishing.
- Harvest for constant yeild.

5 . .

ACF

(33) Prove by induction that $5^{2n} - 6n + 8$ is divisible by 9 for every integer $n \geq 0$.

Let $p(n)$ be the statement that $5^{2n} - 6n + 8$ is divisible by 9. ~~for~~ $n \geq 0$.

Anchor step: $p(0)$ is the statement that $5^0 - 6 \cdot 0 + 8 = 9$, so clearly $p(0)$ is true

Induction step: Assume that $n \geq 0$ and $p(n)$ is true. Then

$$5^{2(n+1)} - 6(n+1) + 8$$

$$= 5^{2n+2} - 6n + 2$$

$$= \underbrace{25(5^{2n} - 6n + 8)}_{\text{divisible by assumption}} + \underbrace{144n - 198}_{\text{divisible by } 9}$$

Hence $5^{2(n+1)} - 6(n+1) + 8$ is divisible by 9.

$p(n) \Rightarrow p(n+1)$ by PMI.

$p(n)$ is true.

Summary

George Willsons

5

ACF

(34) The Fibonacci sequence is defined by the rules:

$f_1 = f_2 = 1$ for $n \geq 3$. $f_n = f_{n-1} + f_{n-2}$.

Use induction to prove that if $n \in N$. is divisible by 5 then f_n is also divisible by 5.

$f_1 = f_2 = 1$ $f_3 = 2$ $f_4 = 3$ $f_5 = 5$ $f_6 = 8$ $f_7 = 13$ $f_8 = 21$ $f_9 = 42$

$f_{10} = 63$ $f_{11} = 105$

let $p(n)$ be the statement that f_n is divisible by 5 if n is divisible by 5

Anchor step: $p(1)$ $f_5 = 5$ so $p(1)$ is true

Induction step: let $n \geq 1$ and assume that $p(n)$ is true

$$
\begin{aligned}
p(n+1):\ f_{5n+5} &= f_{5n+4} + f_{5n+3} \\
&= 2f_{5n+3} + f_{5n+2} \\
&= 2f_{5n+3} + f_{5n+1} + f_{5n} \\
&= 2(f_{5n+2} + f_{5n+1}) + f_{5n+1} + f_{5n} \\
&= 2(2f_{5n+1} + f_{5n}) + f_{5n+1} + f_{5n} \\
&= \underbrace{5f_{5n+1}}_{\text{divisible by 5.}} + \underbrace{3f_{5n}}_{p(n)}
\end{aligned}
$$

Hence. $p(n) \Rightarrow p(n+1)$.

$p(n)$ is true by PMI.

Summary

ACF

(35) Find all possible retangles, having sides of positive integers length, which have the property that the perimeter is equal (numerically) to the area.
Hint: you can label the side-lengths as x and y in such a way that x is lesser. i.e. $x \le y$.
Obtain a relation between x and y and think about $1/x$ and $1/y$. How large can x be?

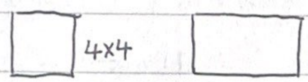
Assum $x \le y$.

$2x + 2y = xy$

$\frac{1}{y} + \frac{1}{x} = \frac{1}{2}$ so x at most 4.

key step. $\Leftarrow$ $\therefore$ $x = 4$. $y = 4$. $x = 3$. $y = 6$. and $x < 2$ it's impossible.

Since x, y are integers.



4x4 6x3

rewrite the equation

$2x + 2y + 4 = xy + 4$

$\boxed{xy - 2x - 2y} + 4 = 4$.

$(x-2)(y-2) = 4$

Since $4 = 4 \cdot 1 = 2 \cdot 2$.

so the only solution

$x - 2 = 2 = y - 2$

$x - 2 = 1, y - 2 = 4$

Summary

5

ACF

36 Is the argument valid?

Here, x, y, z are real number.

(a) P_1: All cats have wings

P_2: All wings creatures have four legs.

Q: All cats have four legs.

Valid: P_1: cat $\Rightarrow$ wings P_2: wings $\Rightarrow$ 4 legs.

so we get cat $\Rightarrow$ 4 legs. (So they can imply Q)

(b) P_1: If $x^2+y^2 \leq 1$, then $-1 \leq x \leq 1$.

P_2: $4x=1$

Q: $x^2+y^2 \leq 1$.

False. If we take $x=\frac{1}{4}$, P_1, P_2 both satisfied.

But Q is not satisfied.

(c) P_1: $y \leq 4$ only if $x \geq 3$

P_2: If $y \leq 4$, then ~~$y \Rightarrow$~~ $z^2 > x$

Q: If $z=0$, then ~~$y \geq 5$~~ $y > 0$

Valid. $y \leq 4 \Rightarrow x \geq 3$ and $y \leq 4 \Rightarrow z^2 > x$. combine them

$y \leq 4 \Rightarrow z^2 > x \geq 3$ $\therefore y \leq 4 \Rightarrow z^2 > 3$

Then take the contrapositive.

$z^2 \leq 3 \Rightarrow y > 4$.

$\therefore z=0 \Rightarrow y > 4 > 0$.

Summary

ACF

(3) Mathematical reasoning

which conclusion can you legitimately draw about citizens of the faraway Republic of Freedonia. ?

(1) No Freedonian that eats fish cannot swim

(2) No Freedonian without a television plays golf

(3) Freedonians who have a car all eat fish

(4) No freedonians who can swim likes basketball

(5) No Freedonions have televisions unless they have a car.

F= fish G: golf S=swim C: have a car. B. basketball. T=TV.

(1). $F \Rightarrow S$

(2) $\neg T \Rightarrow \neg G \quad G \Rightarrow T$

(3) $C \Rightarrow F$

(4) $S \Rightarrow \neg B$

(5) $\neg C \Rightarrow \neg T \quad T \Rightarrow C$

$\therefore G \Rightarrow T \Rightarrow C \Rightarrow F \Rightarrow S \Rightarrow \neg B.$

$\therefore$ Freedonians who play golf never like basketball

Summary

"Unless". we will play cricket unless it rains: "not rain $\Rightarrow$ play"

A unless B : $\neg B \Rightarrow A$, By contrapositive rule.

$\neg A \Rightarrow B$

so, mathematically at least. "A unless B" is same as "B unless A".

George Willsons

ACF (38) A collection of four cards each have a number on one side and a circle or a triangle drawn on the other.

$\boxed{2}$	$\boxed{\bigcirc}$	$\boxed{5}$	$\boxed{\triangle}$
a	b	c	d

Which card(s) must be turn over to determine whether the following statement is true or false?

" An even number on one side implies a circle on the other".

A = number is even B = figure is circle.

$A \Rightarrow B$.

everything is based on the "True table".

$\boxed{2}$ a i) A is True, B need to turn.

ii) B is True, no need to turn

iii) A is false, no need to turn

iv) B is false. need to turn.

Summary

$\boxed{\text{Compound Statement}}$

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

$A \Rightarrow B$ is false only in the case A is true. B is false.

5 . .

ACF

(39) Suppose that $p(x)$ is a polynomial, all the coefficients of which are integers, and that this polynomial takes the value 0 at four distinct integer values a, b, c and d, Show that there is no integer value n for which $p(n)=29$

Hint: What kind of number is 29?

Suppose $p(x)=0$. when a, b, c and d (all in $\mathbb{Z}$)

Then we can write

$p(x) = (x-a)Q(x)$. so.

$p(x) = (x-a)(x-b)(x-c)(x-d)\,q(x)$

Now if for some $n \in \mathbb{Z}$ we have $p(n)=29$ then clearly

$29 = p(n) = (n-a)(n-b)(n-c)(n-d)\,q(n)$

In particular, each of them must divide 29.

But 29 is prime number so its only factors are

1, -1, 29, -29, the last two cannot both occur on the RHS.

Hence, at least 3 of $n-a$, $n-b$, $n-c$, $n-d$ must be ± 1

so at least 2 of them must be the same.

Since a, b, c, d are all distinct, it is impossible.

Thus there is no integer n with $p(n)=29$.

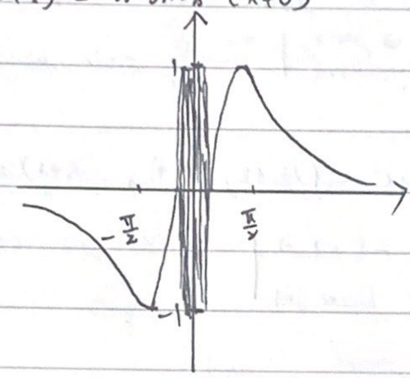
Summary

Calculus ④ Sketch the following function

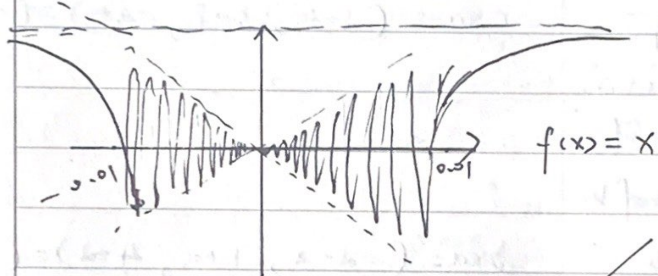
$f(x) = \sin\frac{1}{x} \quad (x \neq 0)$

$f(x) = x\sin\frac{1}{x} \quad (x \neq 0)$

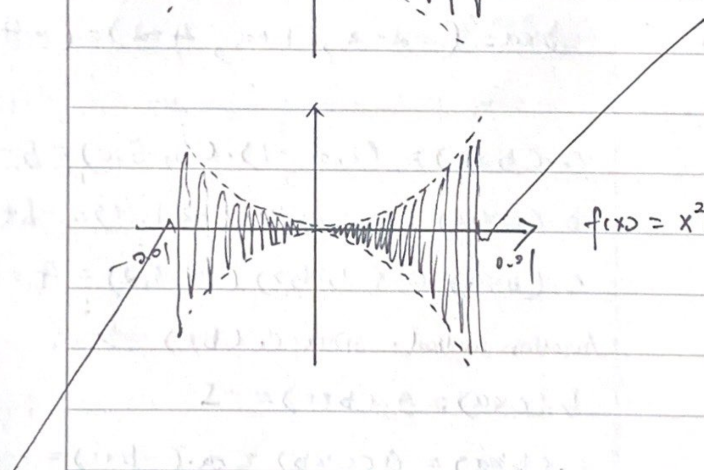
$f(x) = x^2\sin\frac{1}{x} \quad (x \neq 0)$



$f(x) = \sin\frac{1}{x} \quad (x \neq 0)$



$f(x) = x\sin\frac{1}{x} \quad (x \neq 0)$



$f(x) = x^2\sin\frac{1}{x}$

Summary

LM.

(41) Consider the vector $a = (1, 2, -1)$, $b = (2, 2, 1)$ and $c = (-1, -1, 2)$. Calculate the scalar triple product $a \cdot (b \times c)$, $b \cdot (c \times a)$ and $c \cdot (b \times a)$ (two of these should be equal) Does these 3 vectors lie on same plane through the origin. No.

$$\begin{vmatrix} 2 & 2 & 1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$b \times c = (4+1, -1-4, -2+2) = (5, -5, 0)$$

$$\begin{vmatrix} -1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$c \times a = (1-4, 2-1, -2+1) = (-3, 1, -1)$$

$$\begin{vmatrix} 2 & 2 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$b \times a = (-2-2, 1+2, 4-2) = (-4, 3, 2)$$

$$a \cdot (b \times c) = (1, 2, -1) \cdot (5, -5, 0) = 5 - 10 + 0 = -5$$

$$b \cdot (c \times a) = (2, 2, 1) \cdot (-3, 1, -1) = -6 + 2 - 1 = -5$$

$$c \cdot (b \times a) = (-1, -1, 2) \cdot (-4, 3, 2) = 4 - 3 + 4 = 5$$

Another method. since $a \cdot (b \times c) = -5$

$$b \cdot (c \times a) = a \cdot (b \times c) = -5$$

$$c \cdot (b \times a) = a (c \times b) = a \cdot (-b \times c) = -a (b \times c) = 5$$

Some concepts

a_b: the projection of a in the direction of b. → length

$$a_b = \|a\| \cos\theta = \frac{a \cdot b}{\|b\|}$$

$P_b(a)$: the component of a in the direction of b. → component (vector).

$$P_b(a) = a_b \cdot \hat{b} = \frac{a \cdot b}{\|b\|^2} \cdot b$$

Summary

$\boxed{\text{Triple product}}$

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$$

$$(a \times b) \times c = (c \cdot a) \cdot b - (c \cdot b) \cdot a$$

$\boxed{\text{Scalar Triple product}}$

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) = -a \cdot (c \times b)$$
$$= -b \cdot (a \times c)$$
$$= -c (b \times a)$$

ACF

Principle of mathematical induction (PMI)

classification

Summation / Inequalities / Divisibility

Summation

Prove that $\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$ for all positive integers.

Let $p(n)$ be the statement that $\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$ for all positive integers.

Anchor step: $p(1)$ is the statement that $2^1 = 2^{1+1} - 2 = 2$

$p(1)$ is true.

Induction step: Assume that $p(n)$ is true.

$$p(n+1) = \underbrace{2^1 + 2^2 + \dots + 2^n}_{p(n)} + 2^{n+1}$$
$$= 2^{n+1} - 2 + 2^{n+1}$$
$$= 2 \cdot 2^{n+1} - 2$$
$$= 2^{(n+1)+1} - 2$$

remarks:
Thus if the statement holds when $n=k$, it also holds for $n=k+1$

Hence. $p(n) \Rightarrow p(n+1)$

$p(n)$ is true.

Summary

Inequalities

ACF

① Prove that $2^n > n$ for all positive integer n.

let $p(n)$ be the statement that $2^n > n$ for all positive integer n.

Anchor step: $p(1)$ is the statement that $2^1 > 1$, $p(1)$ is true.

Induction step: Assume $p(n)$ is true, for $n > 1$

$$p(n+1) = 2^{n+1}$$
$$= 2 \cdot 2^n > 2n > n+1$$

Since we assume $n > 1$, $2n > n+1$ is always true.

$\therefore p(n) \Rightarrow p(n+1)$

$p(n)$ is true.

② Prove that ~~for~~ when $h > 0$, the inequality $(1+h)^n > 1+nh$ holds for all positive integers $n \geq 2$

let $p(n)$ be the statement $(1+h)^n > 1+nh$. for all positive integers $n \geq 2$.

Anchor step: $p(2)$ is the statement

$$(1+h)^2 = 1+2h+h^2 > 1+2h$$, it is true

Induction step: $(1+h)^{n+1} = (1+h)\cdot(1+h)^n > (1+h)(1+nh)$

$$= 1+nh+h+nh^2$$
$$= nh^2 + (n+1)h + 1$$
$$> (n+1)h + 1$$

Summary

$p(n) \Rightarrow p(n+1)$

$p(n)$ is true.

注意n的取值

③ Prove that $n^2 > 2n+1$ for $n \geq 4$.

let $p(n)$ be the statement that $n^2 > 2n+1$ for $\cancel{n \geq 4}$.

Anchor step: $p(4) = 4^2 = 16 > 2\times 4+1 = 9$

$p(4)$ is true.

Induction step: Assume $p(n)$ is true.

$$
\begin{aligned}
p(n+1) &= (n+1)^2 \\
&= n^2 + 2n+1 \\
&> (2n+1) + 2n+1 \\
&= 4n+2 = 2n+2n+2 \\
&\geq 2n+10 \quad \text{since we define } n \geq 4 \\
&> 2n+3 \\
&= 2n+2+1 \\
&= 2(n+1)+1
\end{aligned}
$$

$p(n) \Rightarrow p(n+1)$

$p(n)$ is true

Summary

5 . .

Probability

(4.2) Two cards are dealt from a standard pack of 52, what is the probability that the second card is a heart given that the first card is red?

H_2

$R_1 < \begin{matrix} D_1 \\ H_1 \end{matrix}$

Let $D_1 = \{$ first card is diamond $\}$
$H_1 = \{$ first card is a heart $\}$ > red.
$H_2 = \{$ second card is a heart $\}$
$R_1 = \{$ first card is red $\}$

If it is red. there are two possible outcomes < diamond / heart.

Require $p(H_2|R_1) = \dfrac{p(R_1 \cap H_2)}{p(R_1)}$

D_1 and H_1 are disjoint.

$p(R_1) = p(D_1 \cup H_1) = p(D_1) + p(H_1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$p(R_1 \cap H_2) = p((D_1 \cup H_1) \cap H_2)$

$= p((D_1 \cap H_2) \cup (H_1 \cap H_2))$ distributive law.

$= p(D_1 \cap H_2) + p(H_1 \cap H_2)$ disjoint. events.

So the key step is to find the sample space.

$= \frac{13}{52} \cdot \frac{13}{51} + \frac{13}{52} \cdot \frac{12}{51}$

$= \frac{1}{4} \cdot \frac{25}{51}$

Hence. $p(H_2|R_1) = \dfrac{\frac{1}{4} \cdot \frac{25}{51}}{\frac{1}{2}} = \frac{25}{102}$

Summary

Probability

(P3) Prove or disprove the following statement

If E_1 and E_2 are disjoint events then, for any event F,

$$P(F \mid E_1 \cup E_2) = P(F \mid E_1) + P(F \mid E_2).$$

LHS $P(F \mid E_1 \cup E_2) = \dfrac{P(F \cap (E_1 \cup E_2))}{P(E_1 \cup E_2)}$

Since E_1, E_2 disjoint

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$ and

$$P(F \cap (E_1 \cup E_2)) = P((F \cap E_1) \cup (F \cap E_2)) \quad \text{distributive law}$$

$$= P(F \cap E_1) + P(F \cap E_2) \quad \text{disjoint events.}$$

Thus

$$P(F \mid E_1 \cup E_2) = \frac{P(F \cap (E_1 \cup E_2))}{P(E_1 \cup E_2)} = \frac{P(F \cap E_1) + P(F \cap E_2)}{P(E_1) + P(E_2)}$$

and RHS:

$$P(F \mid E_1) + P(F \mid E_2) = \boxed{\frac{P(F \cap E_1)}{P(E_1)} + \frac{P(F \cap E_2)}{P(E_2)}}$$

放缩.

master the method.
how to deal with
the denominator.
to build an inequality

Now $P(E_1) > 0$, $P(E_2) > 0$.

Thus

$$\cancel{\frac{\quad}{P(E_1)}} \quad P(F \mid E_1) + P(F \mid E_2) > \frac{P(F \cap E_1)}{P(E_1) + P(E_2)} + \frac{P(F \cap E_2)}{P(E_1) + P(E_2)}$$

$$= P(F \mid E_1 \cup E_2).$$

So the result is false.

Summary

这个题的关键就在于分子的 distributive law 还有分母的 disjoint.