Probability

(44) A pack of cards is shuffled and then divided into two piles, A and B, each consisting of 26 cards. A card is drawn at random from A and added to B. The 27 cards in B are the shuffled and a card is draw from B randomly. What's the probability ~~that~~ the card drawn from B is an ace? Suppose that you notice the card draw from A is ace. What now the probability that the card ~~from~~ draw from B is an ace?

just an elaborate way of selecting a card randomly. ⇐ 1. $p(\text{card drawn from B is an ace}) = \frac{4}{52} = \frac{1}{13}$

2. Let $E = \{\text{card draw from B is an ace}\}$,
$F = \{\text{card draw from B is the interchanged card}\}$.

$$p(E) = p((E\cap F)\cup(E\cap F^c))$$
$$= p(F)\cdot p(E|F) + p(F^c)\cdot p(E|F^c)$$
$$= \frac{1}{27}\cdot 1 + \frac{26}{27}\cdot \frac{3}{51} = \frac{43}{459}$$

I see it is ace.

其他张洗牌打乱.

不用考虑洗牌的A, B分配问题
因为generally、本身就是随机分配的.

Summary

拿不拿得到张有那两种情况 { 拿到change的牌. / 拿到其他牌.

Probability.

(45) A friend of yours goes to Zermatt to climb the Matterhorn by one of its three principle ridges. In talking to him before he leaves, you estimate that the probabilities that he will attempt the Hornli, Italian and Zmutt ridges are $\frac{3}{5}$, $\frac{3}{10}$ and $\frac{1}{10}$ respectively. From past experience you also estimate that the probability for the Italian and Zmutt ridges are $\frac{1}{10}$ and $\frac{1}{4}$. You learn that your friend has made a successful ascent. What is the probality that he climed the Hornli ridge?

having accident given he attempts Hornli is $\frac{1}{20}$

Suppose now the weather condition can be classified as either good or doubtful. with the probability of good conditions being $\frac{2}{3}$. Suppose that if conditions are good the probability your friend chooses Hornli is $\frac{9}{20}$. and the probability he has an ~~attempt~~ accident give ~~an~~ he attempts the Hornli is $\frac{1}{100}$. The corresponding probability for ~~H0~~ doubtful conditions are $\frac{9}{10}$ and $\frac{9}{100}$. Show that these probabilities are consistent with your earlier model. Given that you friend made an accident-free ascent of the Hornli ridge, what is the probability that the weather conditions were good.

Summary

1. Let $H = \{\text{climbs Hornli}\}$

$I = \{\text{climbs Italian}\}$

$Z = \{\text{climbs Zmutt}\}$

$S = \{\text{successful ascent}\}$

So $P(H) = \frac{3}{5}$ $P(I) = \frac{3}{10}$ $P(Z) = \frac{1}{10}$ $P(S|H) = \frac{19}{20}$

$P(S|I) = \frac{9}{10}$ $P(S|Z) = \frac{3}{4}$.

Thus by Bayes' Theorem.

$$P(H|S) = \frac{P(S|H)\,P(H)}{P(S|H)\,P(H) + P(S|I)\,P(I) + P(S|Z)\,P(Z)}$$

$$= \frac{\frac{19}{20}\cdot\frac{3}{5}}{\frac{19}{20}\cdot\frac{3}{5} + \frac{9}{10}\cdot\frac{3}{10} + \frac{3}{4}\cdot\frac{1}{10}} = \frac{38}{61}$$

2. Let $G = \{\text{Good weather}\}$ $D = \{\text{weather doubtful}\}$

$P(G) = \frac{2}{3}$ $P(D) = \frac{1}{3}$

$\boxed{P(H|G) = \frac{9}{20}}$ $P(S|H\cap G) = \frac{99}{100}$

$P(S|H\cap D) = \frac{91}{100}$

$P(H) = P(G)P(H|G) + P(D)P(H|D) = \frac{2}{3}\cdot\frac{9}{20} + \frac{1}{3}\cdot\frac{9}{10} = \frac{3}{5}$

Also.

$P(S\cap H\cap G)$.

How to deal with $S\cap H\cap G$.

$P(S\cap H) = P(S\cap H|G)\,P(G) + P(S\cap H|D)\,P(D)$

$= P(S|H\cap G)\,P(H\cap G) + P(S|H\cap D)\cdot P(H\cap D)$

用三个连续的贝叶斯

$= P(S|H\cap G)\,P(H|G)\cdot P(G) + P(S|H\cap D)\cdot P(H|D)\cdot P(D)$

$= \frac{99}{100}\cdot\frac{9}{20}\cdot\frac{2}{3} + \frac{91}{100}\cdot\frac{9}{10}\cdot\frac{1}{3} = \frac{57}{100}$

Summary

$$P(G|H\cap S) = \frac{P(G\cap H\cap S)}{P(H\cap S)}$$

$$= \frac{P(S|H\cap G)\,P(H|G)\,P(G)}{P(H\cap S)}$$

$\frac{1}{100}$ $\frac{9}{20}$ $\frac{2}{3}$ 分别求

分别求!

$P(H|S)\,P(S)$

$= \frac{38}{61} \times \frac{183}{200}$ (from Q1)

So.

$$p(S|H) = \frac{p(S\cap H)}{p(H)} = \frac{57}{100} \Big/ \frac{3}{5} = \frac{19}{20}.$$

$$p(G|S\cap H) = \frac{p(G\cap S\cap H)}{p(S\cap H)}$$

关键步骤.

$$= \frac{p(G)\,p(H|G)\,p(S|H\cap G)}{p(S\cap H)}$$

$$= \frac{\frac{2}{3} \times \frac{9}{10} \times \frac{99}{100}}{\frac{57}{100}} = \frac{99}{190}$$

Probability

(40) A crime has been committed and a suspect is being held by police. He is either guilty, G, or not G^c, and $p(G) = p_0$ say. Forensic evidence is now produced which shows that the criminal must have a property A, which occurs in a proportion, π, of the general population. Suppose that if the suspect is innocent he can be treated as a member of the general population so that $p(A|G^c) = \pi$. The suspect is now ~~interrogate~~ interrogated and found to have property A. Prove that the odds on his guilt have now risen from $\lambda_0 = p_0/1-p_0$ to $\lambda_1 = \lambda_0/\pi$. What is the connection between this result and the way in which we are influenced by "rare coincidence"?

N.B. The odds on E are defined to be the ratio $p(E)/p(E^c)$. The odds against E $p(E^c)/p(E)$.

Summary

5

If he is guilty he must have A.

By Bayes' Theorem

$P(A|G)=1$.

$$P(G|A) = \frac{P(A|G).P(G)}{P(A|G).P(G) + P(A|G^c)P(G^c)} = \frac{p_0}{p_0+\pi(1-p_0)}$$

Hence

$$\frac{P(G|A)}{P(G^c|A)} = \frac{p_0}{\pi(1-p_0)} = \frac{p_0}{\pi}$$

Rare coincidents $\Rightarrow$ π small $\Rightarrow$ odd on guilt will increase dramatically.

Probability

(e) Suppose that in answering a question in a multiple-choice test, an examee either knows the answer (and suppose we assess the chance of this to be π) or guesses — i.e. is equally likely to pick any of the list of possible answers. Suppose there are m possible answer to a given question. If an examinee gives the correct answer, show that the probability he wasn't guessing is equal to $m\pi/\{1+(m-1)\pi\}$

Let $C=\{\text{correct}\}$ $G=\{\text{Guess}\}$ $G^c=\{\text{Knows}\}$

You can not say that $P(C)=m^{-1}$ directly. You should put it into some special condition.

If he know, it's correct. if he guess.

$P(G^c)=\pi$ $P(G)=1-\pi$ $P(C|G^c)=1$ $P(C|G)=m^{-1}$

$$P(G^c|C) = \frac{P(C|G^c).P(G^c)}{P(C|G^c).P(G^c)+P(C|G)P(G)}$$

$$= \frac{1.\pi}{1.\pi + m^{-1}(1-\pi)}$$

$$= \frac{m\pi}{1+(m-1)\pi}$$

Summary

其实最后发现也两题一模一样.

考虑分完类后. $\left\{\begin{array}{l}\text{guilt.}\\ \text{not guilt}\end{array}\right.$ $\left\{\begin{array}{l}\text{know}\\ \text{not know.}\end{array}\right.$

关键是发现 $P(A|G)=1$ $P(C|K)=1$

即 guilty 一定有A物质；知道答案，一定会正确。

Probability

(P8) In considering the definition provided for the mutual independent of event $E_1, E_2 \cdots E_n$ you might have wondered wheth the condition

$$P(E_1 \cap E_2 \cap E_3 \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdots P(E_n)$$

would suffice. (in the sense that it might imply other required conditions?). Find a counter example to show that this is not the case.

find E_1, E_2, E_3, which have intersections each other.

Let $\Omega = \{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\}$

$$P(\{i\}) = \frac{1}{8} \quad i = 1, 2, \cdots 8$$

$E_1 = \{1, 2, 3, 4\}$ $E_2 = \{1, 5, 6, 7\}$ $E_3 = \{1, 5, 6, 8\}$

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{2}$ and $P(E_1 \cap E_2 \cap E_3) = P(\{1\}) = \frac{1}{8}$

So $P(E_1 \cap E_2 \cap E_3) = P(E_1) P(E_2) P(E_3)$.

· ~~[illegible]~~举了4个数字当例子，只有一个数是共同的.

But

$$P(E_1 \cap E_2) = P(\{1\}) = \frac{1}{8} \neq P(E_1) \cdot P(E_2).$$

$$P(E_1 \cap E_3) = P(\{1\}) = \frac{1}{8} \neq P(E_1) P(E_3)$$

$$P(E_2 \cap E_3) = P(\{1, 5, 6\}) = \frac{3}{8} \neq P(E_2) P(E_3)$$

This is a very good example since at the beginning, it holds but when it comes to certain even E_1 or E_2 or E_3, it's not corret. it's a wise choice to select 8 as sample space since it could be written as $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, which is the form of the question.

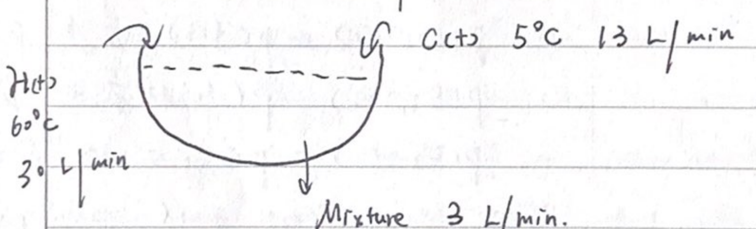
Summary

App.

Variable volume mixings.

A leaky bath of capacity 600 L is to be run from a a hot tap that produces water at 60°C and a cold tap that produces water at 5°C The bath leaks at a rice of 3 L/min and the hot and cold taps run at 30 L/min and 13 L/min respectively. There is no less of heat from the bath water to the surroundings. For safety reasons, the bath is first filled with 50 L of cold water, then both taps are used to fill the bath, each at full raw.

Q: How does the temperature of the bath water vary with time.? How hot is the water wt the bath is complete fill?



Let $C(t)$ = litres of cold water

$H(t)$ = litres of hot water

Temperature

$$T = \frac{V_1 T_1 + V_2 T_2}{V_1 + V_2}$$

Summary

Total volume: $V(t) = C(t) + H(t)$ we don't minus 3.

$V \sim t$.

$\frac{dV}{dt}$ = flow in − flow out

$= (30 + 13)$ L/min − 3 L/min

$= 40$ L/min

$\int \frac{dV}{dt} = 40$.

$H \sim t$.

$V(t) = 40t + K$. $V(0) = 50$ L. $K = 50$

$\therefore V(t) = 40t + 50$.

When it is full. $600 = 40t + 50$ $t = \frac{55}{4}$ min.

Now we want to calculate $\frac{dH}{dt}$.

$T \sim t$.

$\frac{dH}{dt}$ = Hot water in − Hot water out

$= 30 - (\frac{H}{V}) \cdot 3$

$= 30 - \frac{3H}{50 + 40t}$

by using integrating factor to solve the equation

$\frac{dH}{dt} + \frac{3H}{50+40t} = 30$ inhomogenous linear 1st order

$I \cdot \frac{dH}{dt} + I \cdot \frac{3}{50+40t} \cdot H = 30I$.

$I \cdot \frac{3H}{50+40t} = I' H$

$I \frac{3}{50+40t} = I'$

$\int \frac{I'}{I} = \int \frac{3}{50+40t} \cdot dt$

$\ln I = \int \frac{3}{50+40t} \cdot dt$

$I = e^{\int \frac{3}{50+40t} \cdot dt} = (50 + 40t)^{\frac{3}{40}}$

$\therefore (I \cdot H)' = 30 I$

$\therefore \int (I \cdot H)' = \int 30 \cdot I$

$I \cdot H = 30 \int I$

Summary

$$(50+40t)^{\frac{3}{40}} \cdot H = \frac{30}{43}(50+40t)^{\frac{43}{40}} + K$$

$H(0) = 0.$ $\quad K = -\frac{30}{43} \times 50^{\frac{43}{40}}$

$$H(t) = \frac{30}{43}\left\{\underbrace{50+40t}_{\checkmark} - \underbrace{50\left(1+\frac{4}{5}t\right)^{-\frac{3}{40}}}_{t\to\infty \;\to 0}\right\}$$

so if bath is infinitely large.

$t \to \infty$. $\quad H = \frac{30}{43} V$

What 's the temperature?

$T = \frac{60H+5C}{H+C}$ $\quad V = H+C$

$$\Rightarrow T = \frac{60H+5(V-H)}{V} = 5 + \frac{55H}{V}$$

$$T = 5 + 55 \times \frac{30}{43}\left\{1 - \frac{50}{50+40t}\left(1+\frac{4}{5}t\right)^{-\frac{3}{40}}\right\}$$

$$= 5 + \frac{1650}{43}\left\{1 - \left(1+\frac{4}{5}t\right)^{-\frac{3}{40}}\right\}$$

if bath infinitely large $t \to \infty$

$T \to 5 + \frac{1650}{43} \approx 43.4°C$

Bath is full. at $t = \frac{55}{4}$

$\Rightarrow T\left(\frac{55}{4}\right) \approx 40.7°C.$

Summary

ACF

For any real number x and y.

$|x+y| \leq |x| + |y|$

Proof:

Case 1: $x+y \geq 0$ Then $|x+y| = x+y \leq |x| + |y|$ since $a \leq |a|$

Case 2: $x+y < 0$ Then $|x+y| = -x-y \leq |x| + |y|$ since $-a \leq |a|$

Therefore $|x+y| \leq |x| + |y|$ QED.

Corollary:

For any real number x and y.

$|x-y| \geq |x| - |y|$

Proof:

$|x| = |(x-y) + (y)| \leq |x-y| + |y|$

$|x| - |y| \leq |x-y|$ QED.

Summary

ACF

(49) Use Cauchy-Schwarz to estimate

$$S = \sum_{k=1}^{99} \sqrt{k(100-k)} = \sqrt{99} + \sqrt{2\cdot 98} + \cdots + \sqrt{98\cdot 2} + \sqrt{99}$$

$$S^2 \leq \sum_{k=1}^{99} (\sqrt{k})^2 \cdot \sum_{k=1}^{99} (\sqrt{100-k})^2$$

$1+\cdots+99$ $\quad$ $99+\cdots+1$

$$S^2 \leq \underbrace{\left(\sum_{k=1}^{99} k\right)\left(\sum_{k=1}^{99} 100-k\right)}_{\text{they are same thing}} = \left(\sum_{k=1}^{99} k\right)^2$$

so.

$$S \leq \sum_{k=1}^{99} k = \frac{99\cdot 100}{2} = 4950.$$

Summary

$\boxed{\text{Cauchy-Schwarz Inequality}}$

Let $n \in \mathbb{N}$ and let $a_1, \cdots a_n$ and $b_1 \cdots b_n$ be real number.

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

ACF.

(30) Let x be a real number with $x \neq 1, -1$. Prove by induction that for $n \in \mathbb{N}$.

$$\sum_{k=1}^{n} \frac{x^{2^{k-1}}}{1-x^{2^k}} = \frac{1}{1-x} - \frac{1}{1-x^{2^n}}$$

Let $p(n)$ be the statement that $\sum_{k=1}^{n} \frac{x^{2^{k-1}}}{1-x^{2^k}} = \frac{1}{1-x} - \frac{1}{1-x^{2^n}}$

x be a real number with $x \neq 1, -1$

Anchor step: $p(1)$: $\cancel{\sum_{k=1}^{1}}$ $\frac{x^{0}}{1-x} - \frac{1}{1-x^2} = \frac{x}{(1+x)(1-x)} = \frac{x}{1-x^2}$

$$= \sum_{k=1}^{1} \frac{x^{2^{k-1}}}{1-x^{2^k}}$$

$p(1)$ is true

Induction step: Assume that $n \geq 1$ and $p(n)$ is true.

累积求和的公式

$n+1$ 项 变 $\sum^{n}$

$$\sum_{k=1}^{n+1} \frac{x^{2^{k-1}}}{1-x^{2^k}} = \sum_{k=1}^{n} \frac{x^{2^{k-1}}}{1-x^{2^k}} + \frac{x^{2^n}}{1-x^{2^{n+1}}}$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{2^n}} + \frac{x^{2^n}}{1-x^{2^{n+1}}}$$

$$= \frac{1}{1-x} + \frac{x^{2^n}}{1-x^{2^{n+1}}} - \frac{1}{1-x^{2^n}}$$

$$= \frac{1}{1-x} + \frac{x^{2^n}}{(1+x^{2^n})(1-x^{2^n})} - \frac{1}{1-x^{2^n}}$$

$\alpha \cdot \alpha^{n} = \alpha^{n+1}$

$\alpha^{n} + \alpha^{n}$

$$= \frac{1}{1-x} + \frac{x^{2^n} - (1+x^{2^n})}{(1+x^{2^n})(1-x^{2^n})}$$

$$= \frac{1}{1-x} + \frac{-1}{(1+x^{2^n})(1-x^{2^n})}$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{2^{n+1}}}$$

Summary

Hence $p(n) \Rightarrow p(n+1)$. Thus $p(n)$ holds for all $n \in \mathbb{N}$.

ACF

(51) A sequence (i.e. non-terminating list) of positive real number is defined by the rule.

$x_1 = \frac{4}{3}$. $x_{n+1} = \sqrt{6+x_n}$ $(n=1,2\cdots)$

Prove by induction that $x_n < 3$ for ~~all~~ every $n \in \mathbb{N}$.

Let $p(n)$ be the statement that. $x_n < 3$.

Then.

Anchor step: $p(1)$: $x_1 = \frac{4}{3} < 3$ is true.

Induction step: Assume that $p(n)$ is true.,

$p(n+1)$: $x_{n+1} = \sqrt{6+x_n} < \sqrt{6+3} = \sqrt{9} = 3$

$\therefore p(n) \Rightarrow p(n+1)$, Thus

$p(n)$ holds for all $n \geq 0$ by induction.

(52). $\frac{d^n}{dx^n} f = (x+n)e^x$.

$\downarrow$

$\frac{d^{n+1}}{dx^{n+1}} f = \frac{d}{dx}((x+n)e^x) = (x+n)e^x + e^x = (x+n+1)e^x$

Summary

5

ACF

(52) Suppose that the sequence (u_n) is defined by

$u_1 = 4,\ u_2 = 26$; for $n \geq 3$ $u_n = 4u_{n-1} + 5u_{n-2}$.

Prove that $u_n = 5^n + (-1)^n$ for every $n \in \mathbb{N}$.

Hint. ① Let $n \in \mathbb{N}$ be the least positive integer for which it is not true that $u_n = 5^n + (-1)^n$. and seek a contradiction

② Let $p(n)$ be the statement that $u_m = 5^m + (-1)^m$ for all integers m with $1 \leq m \leq n$. check $p(1)$ and $p(2)$, and prove by induction that $p(n)$ holds for all $n \geq 2$.

Anchor step: $Q(1)$: $u_1 = 4 = 5^1 + (-1)^1$ $Q(2)$ $u_2 = 26 = 5^2 + (-1)^2$

$Q(1)$ & $Q(2)$ are true.

Method ①. Assume $Q(n)$ fails for some $n \in \mathbb{N}$, $n \geq 3$.

$Q(m)$ must be true for all integers $1 \leq m < n$.

Since $n \geq 3$, $Q(n-1)$ $Q(n-2)$ must be true.

$$
\begin{aligned}
u_n &= 4u_{n-1} + 5u_{n-2} \\
&= 4(5^{n-1} + (-1)^{n-1}) + 5(5^{n-2} + (-1)^{n-2}) \quad \text{since } Q(n-1),\ Q(n-2) \text{ true} \\
&= 4 \cdot 5^{n-1} + 4(-1)^{n-1} + 5^{n-1} + 5(-1)^{n-2} \\
&= 5 \cdot 5^{n-1} + (5(-1)^n + 4(-1)^{n-1}) \\
&= 5^n + (5(-1)^n - 4(-1)^n) \\
&= 5^n + (-1)^n
\end{aligned}
$$

$(-1)^n \cdot (-1)^{-2}$

as $(-1)^{n-2} = (-1)^n$

as $(-1)^{n-1} = -(-1)^n$

common trick.

But this show that $Q(n)$ is true. contrary to our assumption. ~~so~~ cannot be any $n \in \mathbb{N}$ for $Q(n)$ fails.

Method ②: Let $p(n)$ be $u_m = 5^m + (-1)^m$ for $1 \leq m < n$ that $Q(m)$ holds for $1 \leq m < n$. Since $Q(1)$ and $Q(2)$ are true, so is $p(2)$. Let $n \geq 2$, and assume $p(n)$ holds. Then $n-1 \geq 1$. so $Q(n-1)$ $Q(n)$ are true.

Summary

George Willsons

$u_{n+1} = 4u_n + 5u_{n-1} = 4(5^n + (-1)^n) + 5(5^{n-1} + (-1)^{n-1})$

$= 5 \cdot 5^n - 4(-1)^{n+1} + 5(-1)^{n+1}$

$= 5^{n+1} + (-1)^{n+1}$,

which tells us $Q(n+1)$. Thus ~~p(m)~~ $Q(m)$ is true.

for $1 \le m \le n$. (by $(p(n))$. and for $m = n+1$.

so we have deduced $p(n+1)$. Thus $p(n)$ is true

for $n \geq 2$.

probability ⑬ Let X be a Geometric random variable with parameter p. Prove that, for intergers $n, m \geq 0$

$$p(X > n+m \mid X > m) = p(X > n)$$

General solution:

$$p(X > n+m \mid X > m) = \frac{p(X > n+m,\ X > m)}{p(X > m)} = \frac{p(X > m+n)}{p(X > m)}$$

But. $X > K$ if and only if the first K of an independent sequence of Bernoulli(p) trials are failure.

That means we don't care about what exactly happen afterwards. ⇐

So. $p(X > K) = (1-p)^K$.

$$\therefore p(X > n+m \mid X > m) = \frac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n = p(X > n)$$

as required.

Summary

$\boxed{\text{Common trick.}}$ How to make the $K = 1$?

eg: $\sum_{i=5}^{\infty} p^i = p^5 + p^6 + p^7 + \dots + p^{\infty}$

$= p^4 (p^1 + p^2 + \dots + p^{\infty})$

$= p^4 \sum_{i=1}^{\infty} p^i$

5 . .

Actually what if we want to care about what exactly happen afterwards? it's fine. well.

$X \sim Geo(p)$

$P(X=k) = (1-p)^{k-1} p. \quad k=1, 2, \cdots$

$$\sum_{k=1}^{\infty} (1-p)^{k-1} p = 1$$

$$\therefore P(X>m) = P(X \geq m+1) = \sum_{k=m+1}^{\infty} (1-p)^{m} p$$

$$= (1-p)^m \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= (1-p)^m.$$

L.M.

(ex) Find the distance between $P(2,4,5)$ and the line, which go through the point A, B. , $A(1,-2,3)$ $B(3,-1,1)$

$\vec{AB} = B - A = (2,1,-2)$

$\vec{AP} = P - A = (1,6,2)$

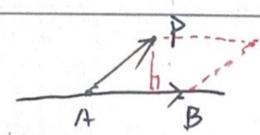
$$\|\vec{AB} \times \vec{AP}\|^2 = \|\vec{AB}\|^2 \|\vec{AP}\|^2 - (\vec{AB} \cdot \vec{AP})^2$$

$= 363$

$$d = \frac{\|\vec{AB} \times \vec{AP}\|}{\|\vec{AB}\|} = \frac{\sqrt{363}}{3}$$

Summary

$\boxed{\text{The distance between point and line.}}$



$$d(P, L). = \frac{\|AB \times \vec{AP}\|}{\|\vec{AB}\|}$$

$\boxed{\text{Lagrange's identity}}$

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2$$

ACF. (55) let $n \in N$. and let $x_1, x \cdots x_n$ be positive real number. Use the Cauchy-Schwarz inequality to prove the arithmetic-harmonic mean inequality.

$$\frac{x_1+x_2+\cdots+x_n}{n} \geq \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}$$

$$n^2 \leq (x_1+x_2+\cdots+x_n)(\frac{1}{x_1}+\cdots+\frac{1}{x_n})$$

$$n^2 = (\sum_{k=1}^{n} 1)^2 = (\sum_{k=1}^{n} \sqrt{x_k} \cdot \frac{1}{\sqrt{x_k}})^2 \leq (x_1+\cdots+x_n)(\frac{1}{x_1}+\cdots+\frac{1}{x_n})$$

(56) If x and y are positive number the $0 \leq (\sqrt{y}-\sqrt{x})^2 = x+y-2\sqrt{xy}$ and so $x+y/2 \geq \sqrt{xy}$.

Suppose $a+b+c=1$

Prove the minimum value of

$$(\frac{1-b}{a})(\frac{1-a}{b})(\frac{1-c}{c}) \text{ is } 8.$$

$$1-a = b+c \geq 2\sqrt{bc}$$

$$(\frac{b+c}{a})(\frac{a+c}{b})(\frac{a+b}{c}) \geq \frac{2\sqrt{bc}}{a} \cdot \frac{2\sqrt{ac}}{b} \cdot \frac{2\sqrt{ab}}{c} = 8$$

$\therefore$ if $a=b=c=\frac{1}{3}$ so 8 is the minimum.

Summary

Probability. (5) A fair die is rolled three times. Let X denote the number of sixes obtained.

(i) Find the probability mass function of X

(ii) Find the distribution function of X.

我发现我当时做的时候很难发现这是一个二项分布.

(i) $P(X=x)=p_X(x)=\binom{3}{x}(\frac{1}{6})^x(\frac{5}{6})^{3-x}$, $x=0,1,2,3$

(ii) $P(X\le x)=F_X(x)=\begin{cases}0 & \text{if } x<0,\\ 125/216 & \text{if } 0\le x<1,\\ 200/216 & \text{if } 1\le x<2\\ 215/216 & \text{if } 2\le x<3\\ 1 & \text{if } x\ge 3.\end{cases}$

cumulative

累加.

算分布时，$F_X(x)$ 要把范围给考虑完整，$x<0$ 和 $x\ge3$ 不要漏掉 而且还要累加

Summary

Cumulative distribution function

$F_X(x)=P(X\le x)=\sum_{y\le x}p_X(y)$, $x\in\mathbb{R}$.

(58) Prove. $0.\dot{9} = 1$

$$0.\dot{9} = 0.9 + 0.09 + 0.009 + \dots$$
$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$$
$$= \lim_{n\to\infty} \left(\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots + \frac{9}{10^n}\right)$$
$$= \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

Calculus.

(59) $f(x) = \begin{cases} x & \text{if } x \le \frac{4}{3} \\ \frac{1}{4}x+1 & \text{if } x > \frac{4}{3}. \end{cases}$

f is not differentiable at $\frac{4}{3}$.

$\lim_{x\to\frac{4}{3}^-} f(x) \ne \lim_{x\to\frac{4}{3}^+} f(x)$

Proof:

$\lim_{x\to\frac{4}{3}^-} f(x) = \frac{4}{3}$ $\quad \lim_{x\to\frac{4}{3}^+} f(x) = \frac{4}{3}$

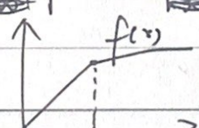
$\therefore \lim_{x\to\frac{4}{3}^+} f(x) = \lim_{x\to\frac{4}{3}^-} f(x) = \frac{4}{3}$. f is $\boxed{\text{continuous}}$

Continuity is to check the limit of f.
But differentiability is to check the limit of the ~~derivative~~ of f.

$\cancel{\lim_{h\to0}} f'(x) = \lim_{h\to0^+} \frac{f(x+h)-f(x)}{h} = \frac{f(\frac{4}{3}+h)-f(\frac{4}{3})}{h} = \frac{1}{4}$

$\cancel{\lim} f'(x) = \lim_{h\to0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0} \frac{\frac{4}{3}+h-\frac{4}{3}}{h} = 1$

$\cancel{\lim} f'(x) \ne \cancel{\lim} f'(x)$ not differentiable.



$\therefore f(x)$ is continuous but not differentiable.

Summary

等比数列

$a_1 + a_1q + a_1q^2 + \dots + a_1q^{n-1} = \frac{a_1(1-q^n)}{1-q}$ ⇐ 求极限

$a_1 + a_2 + \dots + a_n = \frac{a_1 - a_nq}{1-q}$ ⇐ 求数值，代入a_n(末项)，不需要考虑多少项。

Calculus

(60) prove $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $x>0$

$$\begin{cases} \lim_{n\to\infty} (1+\frac{1}{n})^{n} = e^{1} \\ \lim_{n\to\infty} (1+\frac{x}{n})^{n} = e^{x} \\ \lim_{h\to 0} (1+\frac{h}{x})^{\frac{1}{h}} = e^{\frac{1}{x}} \end{cases}$$

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \lim_{h\to 0} \frac{\ln(\frac{x+h}{x})}{h}$$

$$= \lim_{h\to 0} \ln(\frac{x+h}{x})^{\frac{1}{h}}$$

$$= \lim_{h\to 0} \ln(1+\frac{h}{x})^{\frac{1}{h}}$$

$$= \lim_{h\to 0} \ln e^{\frac{1}{x}}$$

$$= \lim_{h\to 0} \frac{1}{x}$$

$$= \frac{1}{x} \quad \text{q.e.d.}$$

Summary

important fact

$$\begin{cases} \lim_{n\to\infty} (1+\frac{1}{n})^{n} = e^{1} \\ \lim_{n\to\infty} (1+\frac{x}{n})^{n} = e^{x} \\ \lim_{h\to\infty} (1+\frac{h}{x})^{\frac{1}{h}} = e^{\frac{1}{x}} \end{cases}$$

5

柯西不等式:

形状:

① 小于方向:

$$|a_1b_1+a_2b_2+\cdots+a_nb_n| \le \sqrt{a_1^2+a_2^2+\cdots+a_n^2}\cdot\sqrt{b_1^2+b_2^2+\cdots+b_n^2}.$$

$$\sqrt{a_1b_1}+\sqrt{a_2b_2}+\cdots+\sqrt{a_nb_n} \le \sqrt{a_1+a_2+\cdots+a_n}\cdot\sqrt{b_1+b_2+\cdots+b_n}.$$

$$|A\sin x+B\cos x| \le \sqrt{A^2+B^2}\cdot\underbrace{1}_{\sqrt{\sin^2x+\cos^2x}}$$

② 大于方向:

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \ge (a_1b_1+a_2b_2+\cdots+a_nb_n)^2$$

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \ge n^2$$

$$\left(\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\right)(b_1+b_2+\cdots+b_n) \ge (a_1+a_2+\cdots+a_n)^2$$

$$\boxed{\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n} \ge \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}.}$$

Summary

eg: 设 $a+b+c=1$，$a,b,c>0$. 求下面最大值

① $\sqrt{15a+1}+\sqrt{15b+1}+\sqrt{15c+1}$

$=1\cdot\sqrt{15a+1}+1\cdot\sqrt{15b+1}+1\cdot\sqrt{15c+1}$

$\leq\sqrt{1+1+1}\cdot\sqrt{15a+15b+15c+3}$

$=\sqrt{3}\times3\sqrt{2}$

$=3\sqrt{6}$.

当且仅当 $a=b=c=\frac{1}{3}$ 取等.

② $\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$

$=1\cdot\sqrt{a+b}+1\cdot\sqrt{b+c}+1\cdot\sqrt{c+a}$

$\leq\sqrt{1+1+1}\cdot\sqrt{2(a+b+c)}=\sqrt{3}\cdot\sqrt{2}=\sqrt{6}$.

当且仅当 $a=b=c=\frac{1}{3}$ 取等.

eg: 求最小值.

① $a^2+b^2+c^2$

$=\frac{a^2}{1}+\frac{b^2}{1}+\frac{c^2}{1}$

$\geq\frac{(a+b+c)^2}{1+1+1}$

$=\frac{1}{3}$

② $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

$\geq\frac{(1+1+1)^2}{a+b+c}$

$=9$

③ $\frac{1}{a}+\frac{4}{b}+\frac{9}{c}$

$\geq\frac{(1+2+3)^2}{a+b+c}=36$.

Summary

5

Foundation Calculus Review.

Result: $\lim_{h \to 0} \frac{e^h - 1}{h} = \log_e e = 1$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Trigonometric function

$\frac{d}{dx}(\sin x) = \cos x$ $\quad \frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

sps → sc $\quad \sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

sms - cs $\quad \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

cpc → cc $\quad \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

cmc → -ss $\quad \cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

Summary

$$\begin{cases} 1 + \tan^2 x = \sec^2 x \\ 1 + \cot^2 x = \operatorname{cosec}^2 x \end{cases}$$

5

SPS → SC

$$\sin\alpha\cos\beta \quad \cancel{\cos\alpha\sin\beta} = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

SMS → CS

$$\sin\beta\cos\alpha = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

CPC → CC

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

CMC → -SS.

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

① Logarithmic differentiation ⇒ $(\sin x)^{\tan x}$

② Implicit function ⇒ collect $\frac{dy}{dx}$.

③ Derivatives of Inverse function.

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}.$$

④ Parametric differentiation

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\boxed{\frac{dx}{dt}}}$$

Summary