Mathematics of the USSR-Izvestiya

TOPOLOGICAL INVARIANTS OF ELLIPTIC OPERATORS. I: *K*-HOMOLOGY

To cite this article: G G Kasparov 1975 *Math. USSR Izv.* **9** 751

Izv. Akad. Nauk SSSR
Ser. Mat. Tom 39 (1975), No. 4

Math. USSR Izvestija
Vol. 9 (1975), No. 4

TOPOLOGICAL INVARIANTS OF ELLIPTIC OPERATORS. I: K-HOMOLOGY

UDC 513.8

G. G. KASPAROV

Abstract. In this paper the homological K-functor is defined on the category of involutory Banach algebras, and Bott periodicity is proved, along with a series of theorems corresponding to the Eilenberg-Steenrod axioms. As an application, a generalization of the Atiyah-Singer index theorem is obtained, and some problems connected with representation rings of discrete groups and higher signatures of smooth manifolds are discussed.

Bibliography: 16 items.

Introduction

With the appearance of the Atiyah-Singer theorem it became clear that in the theory of the index of elliptic operators the K-functor plays a leading role. This circumstance has profound causes. Atiyah [2] first noticed that an elliptic operator on a smooth manifold M^n can in some sense be considered as an element of a homological K-functor (the dual of the ordinary K-functor). By means of axiomatization of two fundamental properties of elliptic operators (being Fredholm and commuting modulo compact operators with multiplication by functions) Atiyah defined for any compact X a class of objects $\mathrm{Ell}(X)$ (with an epimorphism $\mathrm{Ell}(X) \to K_0(X)$ for CW-complexes) and proposed to consider the elements of $\mathrm{Ell}(X)$ as "representative cycles" for $K_0(X)$ (not introducing here any equivalence relation on $\mathrm{Ell}(X)$).

It should be noted that the homological K-functor was introduced by Whitehead purely homotopically and has no "geometric" definition, such as, for example, ordinary homology theory and bordism theory. Hence its specific character as a generalized homology theory, and, in particular, the connection with the theory of elliptic operators was not observed until recently.

The present paper is devoted to the construction of a homological K-functor on the category of involutory Banach algebras and to the study of its interrelations with elliptic operators. The basic results were announced in the note [10](1) and in the report [9].

AMS (MOS) subject classifications (1970). Primary 58G10; Secondary 55F50, 55B20.

(1) In the note [10] there are some errors. In points 2° and 3° of Theorem 1, the condition that (X, A) be a Borsuk pair was omitted. In addition, Definition 5 is suitable only for discrete groups. In this connection, cf. §8 of the present paper.

The contents of the paper are as follows. The definitions are given in §1, the necessary examples are examined in §2, and §3 contains technical results. In §4 the external product is constructed; in §5 the homotopy and excision axioms are proved. §6 contains the construction of the intersection index of the homology and cohomology K-functors, on the basis of which the exactness axiom and Bott periodicity are proved and a functorial isomorphism of Whitehead's K-functor (which we shall denote by $K_*^{(t)}(X)$) onto the one constructed by us $K_*(X)$ (t: $K_*^{(t)}(X) \to K_*(X)$) is constructed. In §7 *a generalization of the Atiyah-Singer theorem is presented: the agreement of the* analytic and topological indices of an elliptic operator as elements of $K_*^G(M^n)$ is proved. The possibility of generalizing the Atiyah-Singer theorem was first noticed by Atiyah [2]. He defined the analytic and topological indices as elements of $K_0^{(t)}(M^n)$. Here the definition of the analytic index is obtained unnaturally: one has to consider a family of elliptic operators. The new formulation seems to us more natural. We note that it contains two special cases in the complex situation (K_0 and K_1) and eight cases in the real situation. §8 contains the definition of the representation ring for infinite discrete groups. In addition, questions connected with the problem of homotopy invariance of higher signatures of smooth manifolds are discussed.

All Banach algebras and homomorphisms of algebras considered are assumed to be involutory. Beginning with §4, the basic theorems about $K_*(B)$ are proved under the assumption that B is topologically finitely generated.

The author thanks Professor S. P. Novikov for his attention to this work.

Notation. 1. H is a separable Hilbert space (real or complex), and $H \otimes H$ is the tensor product of Hilbert spaces (completion of the algebraic tensor product).

$L(H)$ is the Banach algebra of continuous linear operators in H.

$K(H)$ is the ideal of compact operators, and $A(H) = L(H)/K(H)$.

$L^U(H)$ is the group of unitary operators. If $F \in L(H)$, then F^* is the adjoint operator. Equality of operators modulo $K(H)$ is denoted by $F_1 \sim F_2$.

2. B is an involutory Banach algebra with unit (real or complex).

3. C is the algebra of real $(\mathbf{R})$ or complex $(\mathbf{C})$ numbers. The involution in $\mathbf{R}$ is the identity; that in $\mathbf{C}$ is complex conjugation.

4. X is a compactum, and $C(X, B)$ is the Banach algebra of continuous functions on X with values in B. The norm is

$$\|f\| = \sup_{x \in X} \|f(x)\|_B.$$

The involution induces an involution in B; also $C(X) = C(X, C)$.

5. $C_{p,q}$ is the Clifford algebra of the quadratic form $\Sigma_1^p(-x_i^2) + \Sigma_1^q y_j^2$ in the euclidean space of dimension $p + q$ (real or complex), $e_1, \ldots, e_p$, $\epsilon_1, \ldots, \epsilon_q$ are the canonical generators, and the involution is $e_i^* = -e_i$, $\epsilon_j^* = \epsilon_j$.

6. G is a compact topological group; $RU(G)$, $RO(G)$ and $RR(G)$ are the rings of complex, real and "real" representations (the latter is defined in the case when there is defined on G a "real" involution $g \to \bar{g}$). The action of G on the space Y is written on the left ($G \times Y \to Y$) or the right ($Y \times G \to Y$), while $g \cdot y = y \cdot g^{-1}$.

7. If X is a smooth manifold, then $C^\infty(X)$ is the space of smooth functions and

$L^2(X)$ is the Hilbert space of square summable functions. For a smooth bundle ξ, $C^\infty(\xi)$ and $L^2(\xi)$ are the spaces of smooth and L^2-sections respectively.

§1. Basic definitions

In this section the definition of the homology K-functor will be given. The construction of the real $KO_*(B)$ and complex $KU_*(B)$ for real and complex Banach algebras B is carried out simultaneously. The notation $K_*(B)$ refers both to the real and the complex case. Then one defines the "real" $KR_*(B)$ for a complex algebra. We begin at once with the definition of the equivariant $K_*^G(B)$ (Definitions 1 and 2), and then we shall indicate the necessary changes for the construction of $KR_*^G(B)$.

Definition 1. Let B be an involutory Banach algebra with unit, and let G be a compact topological group, acting as a continuous group of automorphisms of B. We denote by $\mathcal{E}_{p,q}^G(B)$ the set of quadruples (χ, ϕ, ψ, F), where χ: $G \to L^U(H)$ is a unitary representation, continuous in the strong operator (i.e. compact-open) topology, ϕ: $B \to L(H)$ is an involutory homomorphism of algebras with unit, compatible with χ, i.e.

$$\forall g \in G,\ \forall b \in B\ \ \varphi(g(b)) = \chi(g)\varphi(b)\chi(g^{-1});$$

ψ: $C_{p,q+1} \to L(H)$ is an involutory homomorphism which commutes with ϕ and χ, and F is a Fredholm operator in H satisfying the following conditions:

1) $F^* \sim -F$.
2) $F^2 \sim -1$.
3) F commutes with $\chi(G)$.
4) F anticommutes with the elements $\psi(e_1), \dots, \psi(e_p)$ and $\psi(\epsilon_1), \dots, \psi(\epsilon_{q+1})$.
5) $\forall b \in B \phi(b) \cdot F \sim F \cdot \phi(b)$.

In addition, we include in $\mathcal{E}_{p,q}^G(B)$ the empty quadruple. By $\mathfrak{D}_{p,q}^G(B)$ we denote the set of degenerate quadruples, i.e. those for which in points 1), 2) and 5) one has exact equations: $F^* = -F$, $F^2 = -1$, and $\forall b \in B\ \phi(b) \cdot F = F \cdot \phi(b)$. The empty quadruple is considered degenerate.

Definition 2. We introduce in the set $\mathcal{E}_{p,q}^G(B)$ the equivalence relations of homotopy and conjugacy. Quadruples (χ, ϕ, ψ, F_0) and (χ, ϕ, ψ, F_1) are *homotopic* if there exists a continuous map $[0, 1] \to L(H)$: $t \to F_t$ such that, for all t, $(\chi, \phi, \psi, F_t) \in \mathcal{E}_{p,q}^G(B)$. If u is an isometric automorphism of H, then the quadruples (χ, ϕ, ψ, F) and $(u\chi u^{-1}, u\phi u^{-1}, u\psi u^{-1}, uFu^{-1})$ are *conjugate*. The set of equivalence classes $\bar{\mathcal{E}}_{p,q}^G(B)$ is a commutative semigroup relative to the operation of direct sum:

$$(\chi_0, \varphi_0, \psi_0, F_0) \oplus (\chi_1, \varphi_1, \psi_1, F_1) = (\chi_0 \oplus \chi_1, \varphi_0 \oplus \varphi_1, \psi_0 \oplus \psi_1, F_0 \oplus F_1)$$

(the space $H \oplus H$ is identified with H by means of some isometry). The image of $\mathfrak{D}_{p,q}^G(B)$ in $\bar{\mathcal{E}}_{p,q}^G(B)$ we denote by $\bar{\mathfrak{D}}_{p,q}^G(B)$. The factor semigroup(1) $\bar{\mathcal{E}}_{p,q}^G(B)/\bar{\mathfrak{D}}_{p,q}^G(B)$ is denoted by $K_{p,q}^G(B)$. If X is a compactum on which G acts, then we set

$$K_{p,q}^G(X) = K_{p,q}^G(C(X)), \quad K_{p,q}^G(X;\ B) = K_{p,q}^G(C(X, B)).$$

(1) The factor semigroup M/N is the quotient set of the set M modulo the equivalence relation $x \sim y \Leftrightarrow \exists z, z' \in N$: $x + z = y + z'$. The addition of equivalence classes is by representatives.

Proposition 1. *$K^G_{p,q}(B)$ is a group.*

Proof. We denote by $(-\psi)$ the representation $C_{p,q+1} \to L(H)$ which is uniquely determined on the generators by the formulas

$$(-\psi)(e_i) = -\psi(e_i), \quad (-\psi)(\varepsilon_j) = -\psi(\varepsilon_j).$$

The element $(\chi, \phi, -\psi, -F)$ will be inverse to (χ, ϕ, ψ, F). In fact, the homotopy

$$\left(\begin{pmatrix}\chi & 0\\ 0 & \chi\end{pmatrix}, \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix}, \begin{pmatrix}\psi & 0\\ 0 & -\psi\end{pmatrix}, \begin{pmatrix}F\cos t & -\sin t\\ \sin t & -F\cos t\end{pmatrix}\right), \quad 0 \leqslant t \leqslant \frac{\pi}{2},$$

carries the sum $(\chi, \phi, \psi, F) \oplus (\chi, \phi, -\psi, -F)$ into the element

$$\left(\begin{pmatrix}\chi & 0\\ 0 & \chi\end{pmatrix}, \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix}, \begin{pmatrix}\psi & 0\\ 0 & -\psi\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) \in \mathcal{D}^G_{p,q}(B).$$

The proposition is proved.

Remark 1. The element (χ, ϕ, ψ', F') which is inverse to (χ, ϕ, ψ, F) can be constructed also in the following two ways:

1. Change ψ to $-\psi$ on only one of the generators e_i (or ϵ_j), leaving it unchanged on the others; $F' = F$.

2. Interchange the places of two generators e_i and e_k (or ϵ_j and ϵ_l), i.e. set $\psi'(e_i) = \psi(e_k)$, $\psi'(e_k) = \psi(e_i)$, and $\psi'(e_m) = \psi(e_m)$ for $m \neq i, k$; $F' = F$.

The quadruple (χ, ϕ, ψ', F') is conjugate with $(\chi, \phi, -\psi, -F)$: in the first case by means of the automorphism $u = \psi(e_i)$, and in the second by means of the automorphism $u = (1/\sqrt{2})(\psi(e_i) - \psi(e_k))$. As a consequence we note that the quadruple $(\chi, \phi, \psi, -F)$ differs from (χ, ϕ, ψ, F) by the sign $(-1)^{p+q}$. In fact, the automorphism $\psi(\epsilon_{q+1})$ in composition with the change of sign on the $(p+q)$ Clifford generators carries (χ, ϕ, ψ, F) into $(\chi, \phi, \psi, -F)$.

Definition 3. We construct $KR^G_{p,q}(B)$. Let there be defined on the complex algebra B a "real" structure, i.e. there is given an involution (action of $\mathbf{Z}_2$) $b \to \bar{b}$ with the properties that $\overline{b_1 b_2} = \overline{b_1} \cdot \overline{b_2}$, $\overline{\lambda b} = \bar{\lambda} \cdot \bar{b}$ $(\lambda \in \mathbf{C})$, and G is a "real" compact group (i.e. a group with involution $g \to \bar{g}$) acting on B so that $\overline{g(b)} = \bar{g}(\bar{b})$. We denote by $H_{\mathbf{C}}$ the complexification of the real Hilbert space $H_{\mathbf{R}}$, and by τ the complex conjugation; and we define a "real" structure on $L(H)$ and $L^U(H)$ by the formula $\bar{P} = \tau P \tau$. The necessary modifications in the definition of $\mathfrak{E}R^G_{p,q}(B)$ in comparison with $\mathfrak{E}^G_{p,q}(B)$ are included in the following: the homomorphisms χ and ϕ preserve the "real" structure; $C_{p,q+1}$ is a real Clifford algebra, while $\psi\colon C_{p,q+1} \to L(H_{\mathbf{R}}) \hookrightarrow L(H_{\mathbf{C}})$, $F \in L(H_{\mathbf{R}})$ (in other words, all $\psi(e_i)$, $\psi(\epsilon_j)$ and F commute with τ). Further one constructs $\mathfrak{D}R^G_{p,q}(B)$ and $KR^G_{p,q}(B)$ in accordance with Definitions 1 and 2. For compacta, $KR^G_{p,q}(X) = KR^G_{p,q}(\mathbf{C}(X))$. (We note that antilinear involutions on $\mathbf{C}(X)$ are always induced by some involution $f\colon X \to X$, i.e. $b \to \bar{b}$ is the composition of f^* and complex conjugation.)

Remark 2. All constructions and assertions of this section formulated for K^G_* carry over word for word to the case of KR^G_*.

Proposition 2. *Definition* 1 *admits various modifications, under which $K^G_{p,q}(B)$ is unchanged*:

a) *Point* 1 *can be strengthened*: $F^* = -F$.

b) *Point* 1 *can be omitted.*

c) *Point* 2 *can be omitted if instead one requires in the definition of* $\mathfrak{D}^G_{p,q}(B)$ *that* F *be invertible.*

Proof. a) The operator F is replaced by the homotopic operator $F_1 = \frac{1}{2}(F - F^*)$.

b) If F does not satisfy point 1, we replace it by

$$F_1 = (1 + F^*F)^{\frac{1}{2}}F(1 + F^*F)^{-\frac{1}{2}}.$$

A homotopy of F to F_1 is $(1 + tF^*F)^{\frac{1}{2}} F(1 + tF^*F)^{-\frac{1}{2}}, 0 \leq t \leq 1$.

c) Replacing F by $F_1 = \frac{1}{2}(F - F^*)$, one can then select a real continuous odd function $f(x)$ such that in the algebra $L(H) \otimes \mathbf{C}$ the element $F_2 = if(-iF_1)$ is real and satisfies point 2. The homotopy $tx + (1 - t)f(x)$, $0 \leq t \leq 1$, carries F_1 into F_2.

Definition 4. If $\omega: B_2 \to B_1$ is a continuous involutory homomorphism preserving all structures (the action of G, the "real" structure), then the homomorphism ω^*: $K^G_{p,q}(B_1) \to K^G_{p,q}(B_2)$ is defined by the formula $\omega^*(\chi, \phi, \psi, F) = (\chi, \phi \cdot \omega, \psi, F)$. (In case $B_i = C(X_i)$ the asterisk will be put below sometimes: ω_*.) Analogously one defines the restriction $K^{G_1}_{p,q}(B) \to K^{G_2}_{p,q}(B)$ for a homomorphism $G_2 \to G_1$.

Now suppose fixed an epimorphism $\omega: B_2 \to B_1$. We shall construct relative groups $K^G_{p,q}(\omega)$ (analogously one constructs $KR^G_{p,q}(\omega)$).

Definition 5. An element of $\mathfrak{E}^G_{p,q}(\omega)$ is a homotopy $(\chi, \phi, \psi, F_t) \in \mathfrak{E}^G_{p-1,q}(B_2)$, $0 \leq t \leq 1$, satisfying the following conditions:

a) $(\chi, \phi, \psi, F_1) \in \mathfrak{D}^G_{p-1,q}(B_2)$.

b) $H = H' \oplus H''$, and for $t = 0$ we have

$$(\chi, \varphi, \psi, F_0) = (\chi', \varphi', \psi', F_0') \oplus (\chi'', \varphi'', \psi', F_0''),$$

where $(\chi', \phi', \psi', F_0') \in \omega^*(\mathfrak{E}^G_{p-1,q}(B_1))$ and $(\chi'', \phi'', \psi'', F_0'') \in \mathfrak{D}^G_{p-1,q}(B_2)$. The element $(\chi, \phi, \psi, F_t) \in \mathfrak{E}^G_{p,q}(\omega)$ is called degenerate if condition b) holds for all $t \in [0, 1]$ (for a fixed decomposition $H = H' \oplus H''$). We denote the set of degenerate elements by $\mathfrak{D}^G_{p,q}(\omega)$.

Definition 6. We introduce in $\mathfrak{E}^G_{p,q}(\omega)$ the equivalence relations of *conjugacy* (just as in Definition 1) and homotopy. Elements $(\chi, \phi, \psi, F_{t,0})$ and $(\chi, \phi, \psi, F_{t,1})$ of $\mathfrak{E}^G_{p,q}(\omega)$ are *homotopic* if there exists a continuous map $[0, 1]^2 \to L(H)$: $(t, s) \to F_{t,s}$ such that $\forall s \in [0, 1]$ $(\chi, \phi, \psi, F_{t,s}) \in \mathfrak{E}^G_{p,q}(\omega)$ (for a fixed decomposition $H = H' \oplus H''$). The set of equivalence classes will be denoted by $\overline{\mathfrak{E}}^G_{p,q}(\omega)$, and the image of $\mathfrak{D}^G_{p,q}(\omega)$ in $\overline{\mathfrak{E}}^G_{p,q}(\omega)$ by $\overline{\mathfrak{D}}^G_{p,q}(\omega)$. We set

$$K^G_{p,q}(\omega) = \overline{\mathfrak{E}}^G_{p,q}(\omega)/\overline{\mathfrak{D}}^G_{p,q}(\omega).$$

This is a group (cf. Proposition 1). If X is a compactum, Y closed in X and ω: $C(X, B) \to C(Y, B)$ is the restriction, then $K^G_{p,q}(\omega)$ will be denoted by $K^G_{p,q}(X, Y; B)$.

Definition 7. We shall construct a map $\partial: K^G_{p,q}(\omega) \to K^G_{p-1,q}(B_1)$. Let $(\chi, \phi, \psi, F_t) \in \mathfrak{E}^G_{p,q}(\omega)$. Then (cf. Definition 5)

$$(\chi', \varphi', \psi', F_0') = \omega^*(\chi', \widetilde{\varphi}', \psi', F_0').$$

We set $\partial(\chi, \phi, \psi, F_t) = (\chi', \widetilde{\phi}', \psi', F_0')$ (here $\widetilde{\phi}'$: $B_1 \to L(H)$ is uniquely defined by ϕ': $B_2 \to L(H)$, since $\omega: B_2 \to B_1$ is an epimorphism. It is clear that $\partial(\mathfrak{D}^G_{p,q}(\omega)) \subset \overline{\mathfrak{D}}^G_{p-1,q}(B_1)$, so ∂ is well-defined).

Definition 8. We construct the group $K^G_{p,q}(B, \varnothing)$. For this in Definitions 5 and 6 it is necessary to set formally $B_1 = \varnothing$ and $B_2 = B$. This means that an element of $\mathcal{E}^G_{p,q}(B, \varnothing)$ is a homotopy degenerate on both ends, an element of $\mathfrak{D}^G_{p,q}(B, \varnothing)$ is a homotopy degenerate for all t, etc.; $K^G_{p,q}(B, \varnothing) = \bar{\mathcal{E}}^G_{p,q}(B, \varnothing)/\bar{\mathfrak{D}}^G_{p,q}(B, \varnothing)$.

Theorem 1. *Let $\omega: B_2 \to B_1$ be an epimorphism. Then the sequence of groups*

$$K^G_{p,q}(B_1, \varnothing) \xrightarrow{\omega^*} K^G_{p,q}(B_2, \varnothing) \to K^G_{p,q}(\omega) \xrightarrow{\partial} K^G_{p-1,q}(B_1) \xrightarrow{\omega^*} K^G_{p-1,q}(B_2)$$

is exact.

The proof is a simple exercise based on the definitions.

Definition 9. We construct a homomorphism $i_*: K^G_{p,q}(B) \to K^G_{p,q}(B, \varnothing)$. We set

$$i_*(\chi, \varphi, \psi, F) = (\chi, \varphi, \psi|_{C_{p-1,q+1}}, (-1)^q \psi(e_p) \cos \pi t + F \sin \pi t).$$

(In §6 it will be proved that i_* is an isomorphism.)

Theorem 2. *The groups $K^G_{p,q}$ (absolute and relative) depend only on the difference $(p - q)$. The isomorphisms $K^G_{p,q} \simeq K^G_{p+1,q+1}$ are functorial and commute with ∂ and i_*.*

Proof. The map $\mathcal{E}^G_{p,q}(B) \to \mathcal{E}^G_{p+1,q+1}(B)$ is defined by the formula

$$(\chi, \varphi, \psi, F) \to \left(\chi \oplus \chi,\ \varphi \oplus \varphi,\ \tilde{\psi}, \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}\right),$$

where

$$\tilde{\psi}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \psi(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tilde{\psi}(e_{i+1}) = -\begin{pmatrix} \psi(e_i) & 0 \\ 0 & -\psi(e_i) \end{pmatrix}, \quad 1 \leqslant i \leqslant p,$$

$$\tilde{\psi}(\varepsilon_{j+1}) = \begin{pmatrix} \psi(\varepsilon_j) & 0 \\ 0 & -\psi(\varepsilon_j) \end{pmatrix}, \quad 1 \leqslant j \leqslant q+1.$$

Conversely, let $(\tilde{\chi}, \tilde{\phi}, \tilde{\psi}, \tilde{F}) \in \mathcal{E}^G_{p+1,q+1}(B)$. The element $\xi = \psi(\epsilon_1)\psi(e_1)$ commutes with $\chi(G)$, $\phi(B)$, $\psi(C_{p,q+1})$ and F. In addition, $\xi^2 = 1$ and $\xi^* = \xi$. We set $H_1 = \operatorname{Im} \frac{1}{2}(1 + \xi)$ and $H_2 = \operatorname{Im} \frac{1}{2}(1 - \xi)$. Then $H = H_1 \oplus H_2$ and $H_1 \perp H_2$. Identifying H_1 with H_2 by means of $\psi(\epsilon_1)$, we get

$$\xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \psi(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It remains to set

$$\chi = \tilde{\chi}|_{H_1}, \quad \varphi = \tilde{\varphi}|_{H_1}, \quad \psi(e_i) = -\tilde{\psi}(e_{i+1})|_{H_1}, \quad \psi(\varepsilon_j) = \tilde{\psi}(\varepsilon_{j+1})|_{H_1}, \quad F = \tilde{F}|_{H_1}.$$

The theorem is proved.

Definition 10. Let n be an integer. We set

$$K^G_n = \begin{cases} K^G_{n,0}, & n \geqslant 0, \\ K^G_{0,n}, & n \leqslant 0. \end{cases}$$

Theorem 3. *The groups* K_n^G *are periodic with period* 2 *in the complex case and with period* 8 *in the real case. The groups* KR_n^G *are periodic with period* 8. *The periodicity is functorial and compatible with* ∂ *and* i_*.

Proof. In the complex case one has an isomorphism f: $C_{p+1,q} \rightleftarrows C_{p,q+1}$: $f(e_k) = e_k$, $k \le p$; $f(e_{p+1}) = i\epsilon_1$; $f(\epsilon_j) = \epsilon_{j+1}$. In the real case $C_{p+4,q} \overset{g}{\rightleftarrows} C_{q,p+4}$: $g(e_i) = e_{i-4}$, $i > 4$; $g(e_i) = \epsilon_1 \cdots \hat{\epsilon}_i \cdots \epsilon_4$, $i \le 4$; $g(\epsilon_j) = \epsilon_{j+4}$. Applying Theorem 2, we get what is needed. The compatibility with i_* is verified using Remark 1. The theorem is proved.

The terminology adopted in this section is also useful for defining the ordinary (cohomology) K-functor (cf. [6], [8] and [15]). In what follows we shall make use of such definitions.

Definition 11. Let X be compact with action of G, and let Y be an invariant closed subset. We denote by $\mathcal{E}_G^{p,q}(X)$ the set of triples (χ, ψ, f), where χ: $G \to L^U(H)$ is a strongly continuous representation, ψ: $C_{p,q+1} \to L(H)$ is an involutory representation commuting with χ, and f: $X \to L(H)$ is a map which is continuous in the norm, satisfying $\forall x \in X$ conditions 1, 2 and 4 of Definition 1 and also $f(gx) = \chi(g)f(x)\chi(g^{-1})$. The sets $\mathfrak{D}_G^{p,q}(X)$, $\overline{\mathcal{E}}_G^{p,q}(X)$ and $\overline{\mathfrak{D}}_G^{p,q}(X)$ are constructed by analogy with Definitions 1 and 2. We set

$$K_G^{q-p}(X) = K_G^{p,q}(X) = \overline{\mathcal{E}}_G^{p,q}(X)/\overline{\mathfrak{D}}_G^{p,q}(X).$$

Further, let $\mathcal{E}_G^{p,q}(X, Y)$ be the subset of $\mathcal{E}_G^{p,q}(X)$ consisting of triples whose restriction to Y is degenerate, and define

$$\mathfrak{D}_G^{p,q}(X, Y) = \mathfrak{D}_G^{p,q}(X), \qquad K_G^{q-p}(X, Y) = K_G^{p,q}(X, Y) = \overline{\mathcal{E}}_G^{p,q}(X, Y)/\overline{\mathfrak{D}}_G^{p,q}(X, Y).$$

Remark 3. Let ξ be a vector G-bundle over X with fixed invariant riemannian metric, and let the algebra $C_{p,q+1}$ act involutorily on ξ. If a homomorphism f: $\xi \to \xi$ anticommutes with the generators of $C_{p,q+1}$, and $f^*(y) = -f(y)$ and $f^2(y) = -1$ for $y \in Y$, then the pair (ξ, f) determines an element of the group $K_G^{q-p}(X, Y)$. In fact, adding to (ξ, f) a trivial direct summand (an element of $\mathfrak{D}_G^{p,q}(X)$), we get an element of $\mathcal{E}_G^{p,q}(X, Y)$.

§2. Examples

The basic examples of this section are obtained from the consideration of pseudodifferential elliptic operators on smooth manifolds, so we immediately describe the general construction. Let X be a smooth closed manifold, TX the tangent bundle (which we identify with the cotangent bundle by means of a riemannian metric), p: $TX \to X$ the projection, and BX and SX the bundles of unit tangent balls and spheres, respectively. We introduce analogous notation in the case of a compact manifold Y with boundary ∂Y. By $\breve{Y}$ we denote the interior of Y. If ξ is a smooth complex hermitian bundle and F: $C^\infty(\xi) \to C^\infty(\xi)$ is an elliptic operator of order 0 (of class $\overline{\mathcal{P}}^0$; cf. [3]), then there exists an extension of F to a Fredholm operator F: $L^2(\xi) \to L^2(\xi)$. This operator, as is easy to verify, commutes modulo compact operators with multiplication by continuous functions from $C(X)$. We suppose that the operator F is skew-hermitian and anticommutes with the generators of the Clifford algebra $C_{p,q+1}$, acting on ξ involutorily, and the compact group G acts smoothly on X and on ξ, commuting with $C_{p,q+1}$ and F. We denote the action of $C(X)$ on $L^2(\xi)$ by ϕ, the action of G by χ, and the action of $C_{p,q+1}$ by ψ. The quadruple (χ, ϕ, ψ, F) in correspondence with Proposition 2c) determines an element of $KU^G_{p-q}(X)$. (Requiring in addition that $F^2 \sim -1$, we would get a real element of $\mathfrak{S}U^G_{p-q}(C(X))$.) Introducing into consideration a "real" structure on X, ξ, G, we get an element of $KR^G_{p-q}(X)$. If here the involution on X and G is trivial, one gets an element of $KO^G_{p-q}(X)$.

Definition 1. The element $(\chi, \phi, \psi, F) \in KU^G_{p-q}(X)$ (or in $KR^G_{p-q}(X)$) is called the *analytic index* of the elliptic operator F; notation: $\operatorname{ind}^a_{p,q}(F) = \operatorname{ind}^a_{p-q}(F)$.

Remark 1. An arbitrary complex or real elliptic operator $\mathfrak{D}$: $C^\infty(\eta) \to C^\infty(\eta)$ is included in our scheme for $p = q = 0$:

$$\xi = \eta + \zeta, \quad F = \begin{pmatrix} 0 & -\mathscr{D}^* \\ \mathscr{D} & 0 \end{pmatrix}, \quad \psi(\varepsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In what follows, this fact will be used frequently. A skew-hermitian operator $\mathfrak{D}$: $C^\infty(\eta) \to C^\infty(\eta)$ corresponds to the case $p = 1$, $q = 0$:

$$\xi = \eta \oplus \eta, \quad F = \begin{pmatrix} 0 & \mathscr{D} \\ \mathscr{D} & 0 \end{pmatrix}, \quad \psi(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \psi(\varepsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a real hermitian elliptic operator the index is defined as an element of $KO_{7,0}(X)$, for a quaternionic one as an element of $KO_{4,0}(X)$, etc. (for a complete list of the 8 cases cf. Karoubi [8]).

The symbol of a skew-hermitian elliptic operator F is a skew-hermitian homomorphism σ: $p^*(\xi) \to p^*(\xi)$, continuous outside the zero section of TX. After multiplying it by a smooth function equal to 0 in a small neighborhood of the zero section and to 1 outside BX, and after a homotopy (so that $\sigma^2 = -1$ on SX) we get according to Remark 3 of §1 an element $(p^*(\xi), \sigma) \in KU^{q-p}_G(BX, SX)$. In the "real" case we introduce in TX a "real" structure as the composition of the "real" structure on X and the automorphism multiplying tangent vectors to X by -1. We get an element $(p^*(\xi), \sigma) \in KR^{q-p}_G(BX, SX)$.

Definition 2. The element thus constructed of the group $KU^{q-p}_G(BX, SX)$ (or of $KR^{q-p}_G(BX, SX)$) is called the *symbol* of the operator F; notation: $\sigma_{p,q}(F) = \sigma_{q-p}(F)$.

For a manifold with boundary we consider two cases.

1. F is degenerate (algebraically) on the boundary, i.e. in a neighborhood of the boundary it induces a continuous invertible homomorphism $\xi \to \xi$. In this case F is Fredholm and there is defined the absolute index

$$\operatorname{ind}^a_{p,q}(F) \in KU^G_{p-q}(Y)$$

and the symbol

$$\sigma_{p,q}(F) \in KU^{q-p}_G(T\widetilde{Y}) = KU^{q-p}_G(BY, SY \cup BY|_{\partial Y}).$$

2. F extends to an elliptic operator $\bar{F}$ on some closed manifold X of greater volume. In this case there is defined the relative index

$$\operatorname{ind}^a_{p,q}(F) \in KU^G_{p-q}(Y, \partial Y)$$

and symbol

$$\sigma_{p,q}(F) \in KU_G^{q-p}(TY) = KU_G^{q-p}(BY, SY).$$

The index is constructed as follows. We glue to Y along the boundary the manifold $\partial Y \times [0, 1]$ (identifying $\partial Y \subset Y$ and $\partial Y \times \{0\}$). We get Y_1. It will be assumed that $Y \subset Y_1 \subset X$. Let ξ_1 be the extension of ξ to Y_1. The obvious retraction $r\colon Y_1 \to Y$ allows one to define an action ϕ of the algebra of functions $C(Y)$ on the Hilbert space $L^2(\xi_1) = L^2(\xi) + L^2(\xi_1|_{\partial Y\times[0,1]})$: $\forall f \in C(Y)$ $\phi(f)$ is multiplication by $r^*(f)$. Let the function $a(y) \in C(Y_1)$ be equal to 1 on Y, and to $\cos \pi s$ at points of the form $(y, s) \in \partial Y \times [0, 1]$. For $0 \le t \le 1$ we denote by a_t the operator of multiplication by $(-t + (1-t)a(y))$ in the space $L^2(\xi_1)$. The quadruple

$$(\chi, \varphi, \psi|_{C_{p-1,q+1}}, F_t = (-1)^q \psi(e_p)\, a_t + (1-a_t^2)^{1/4} \overline{F} (1-a_t^2)^{1/4})$$

determines the desired element of $K^G_{p-q}(Y, \partial Y)$.

The verification of Definitions 1 and 5 of §1 is sufficiently elementary; it is necessary to make use of the fact that the function $(1-a_t^2)^{1/4}$ can be extended by zero to X. In Definition 5 we let

$$H' = L^2(\xi_1|_{\partial Y\times[0,1]}), \quad H'' = L^2(\xi).$$

The operator $(1-a_0^2)^{1/4}$ is equal to zero on H''.

We note that for operators which are degenerate on the boundary, the relative index and symbol are obtained from the absolute ones by the homomorphisms

$$KU^G_{p-q}(Y) \to KU^G_{p-q}(Y, \partial Y)$$

(cf. Definition 9 of §1) and

$$KU_G^{q-p}(T\widetilde{Y}) \to KU_G^{q-p}(TY).$$

Example 1. $K^G_{0,0}(\text{point})$ is isomorphic to the ring of representations of the group G. The isomorphism is constructed as follows: Let $F^* = -F$. On the kernel of F there is defined a representation of G. The operator $\psi(\epsilon_1)$ splits the kernel of F into two parts: that where $\psi(\epsilon_1) = 1$, and that where $\psi(\epsilon_1) = -1$. The difference of the representations of G on these parts is an element of $R(G)$. We note that the usual G-index of the elliptic operator F is obtained from $\mathrm{ind}^a_{0,0}(F)$ by the augmentation $K^G_0(X) \to K^G_0(\text{point}) = R(G)$.

Example 2. $X = G = S^1$. We define an operator $d\colon L^2(S^1) \to L^2(S^1)$ on the orthogonal basis $(1), \ldots, (\cos n\theta), \ldots, (\sin n\theta), \ldots$ by the formulas $d(1) = 0$, $d(\cos n\theta) = -(\sin n\theta)$ and $d(\sin n\theta) = (\cos n\theta)$ (in the complex case $d(e^{in\theta}) = i\,\mathrm{sgn}(n)(e^{in\theta})$). The operator d is skew-hermitian, $d^2 \sim -1$, d commutes exactly with the action of G and modulo compact operators with multiplication by continuous functions. (The last is verified first for the function $e^{i\theta}$, and then is extended by multiplicativity and continuity.) Setting

$$H = L^2(S^1) \oplus L^2(S^1), \quad \psi(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\psi(\varepsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -d \\ -d & 0 \end{pmatrix},$$

we get an element of the group $K^{S^1}_{1,0}(S^1)$. Here we proceeded without verifying that d is a pseudodifferential operator, but this is easy to verify. The symbol $\sigma(\theta, \zeta)$ is equal to $+i$ for $\zeta > 0$, and to $-i$ for $\zeta < 0$ (ζ is a covector).

Example 3. $Y = [0, 2\pi]$. We consider the operator

$$\mathcal{D} = \cos\frac{\theta}{2} - \left(\sin\frac{\theta}{2}\right)d\colon\ L^2[0, 2\pi] \to L^2[0, 2\pi].$$

It is asserted that this is a Fredholm operator of index 1, commuting modulo compact operators with multiplication by functions from $C[0, 2\pi]$. In fact, $\sin(\theta/2) = 0$ for $\theta = 0$ and 2π, so $\forall b \in C[0, 2\pi]$

$$\left(\sin\frac{\theta}{2}\, d\right) b \sim d\left(\sin\frac{\theta}{2}\, b\right) \sim \left(\sin\frac{\theta}{2}\, b\right) d = b\left(\sin\frac{\theta}{2}\, d\right).$$

It is easy to verify that $\mathfrak{D}\mathfrak{D}^* \sim \mathfrak{D}^*\mathfrak{D} \sim 1$. The kernel and cokernel of the operator $e^{i\theta/2}\mathfrak{D} = \frac{1}{2}(1 + e^{i\theta}) + (1/2i)(1 - e^{i\theta})d$ are calculated from the basis $\{e^{in\theta}\}$. The cokernel is equal to 0, and the kernel is spanned by the function $1 - \cos\theta$. By analogy with Example 2, it is easy to construct the corresponding element of $K_{0,0}(Y)$ (cf. Remark 1).

The operator $\mathfrak{D}$ in composition with the exterior product by the form $d\theta$ can be considered as an operator from $L^2(\Lambda^0([0, 2\pi]))$ into $L^2(\Lambda^1([0, 2\pi]))$. It is easy to verify that this new operator commutes with the group $O(1) = \mathbf{Z}_2$, which acts on $[0, 2\pi]$ as group of symmetries relative to the point π. Its $\mathbf{Z}_2$-index as an element of $R(O(1))$ is equal to 1 (since the function $1 - \cos\theta$ is invariant relative to $\mathbf{Z}_2$). The symbol, as an element of $K^0_{O(1)}(T\widetilde{Y})$, coincides with the generator of $K^0_{O(1)}(T\mathbf{R}^1)$, defined by the complex exterior algebra $\Lambda^0(\mathbf{C}^1) \overset{z}{\to} \Lambda^1(\mathbf{C}^1)$, where $z = \theta + i\zeta$.

Example 4. $Y = [0, 2\pi]$. Let the real function $b \in C[0, 2\pi]$ satisfy the conditions $-1 \leq b(x) \leq 1$, $b(0) = 1$ and $b(2\pi) = -1$. For what follows we fix the following notation:

$$T_1(b) = b - \sqrt{1-b^2}\,d : L^2[0, 2\pi] \to L^2[0, 2\pi].$$

By analogy with Example 3 this is a Fredholm operator which commutes modulo compact operators with multiplication by continuous functions. Its index is equal to 1, since it is homotopic to $\mathfrak{D}$.

Example 5. Suppose defined on $Y = [0, 2\pi]$ the involution $\theta \to (2\pi - \theta)$. The corresponding "real" structure on $\mathbf{C}(Y)$ extends uniquely to some complex conjugation on $H_{\mathbf{C}} = L^2(Y)$. Here $H_{\mathbf{R}}$ is generated by the basis $\{e^{in\theta}\}$. The operator $i\mathfrak{D}$ (Example 3) in composition with exterior multiplication by $d\theta$ represents a "real" operator $L^2(\Lambda^0(Y)) \to L^2(\Lambda^1(Y))$. Its integral index (i.e. the image under the homomorphism $KR_0(Y) \to KR_0(\text{point}) = \mathbf{Z}$) is equal to 1. The symbol, as an element of $KR^0(T\widetilde{Y})$, coincides with the generator of $KP^0(T\mathbf{R}^1)$, defined by the complex exterior algebra $\Lambda^0(C^1) \overset{\sigma}{\to} \Lambda^1(C^1)$: $\sigma(\theta, \zeta) = -i\theta + \zeta$.

Example 6. $Y = D^n$ and $G = O(n)$. The complex exterior algebra $\Lambda^*(C^n)$ under the standard decomposition $\Lambda^{ev} \overset{\sigma}{\to} \Lambda^{od}$ defines the symbol of some real elliptic operator (this is the canonical generator of $KR^0_{O(n)}(T\mathbf{R}^n)$). More precisely,

$$\sigma(x, \zeta) = (x^e + x^i) + i(\zeta^e + \zeta^i) f(\|x\|),$$

where $x \in D^n \subset \mathbf{R}^n$, ζ is a covector, x^e and $\zeta^e(x^i, \zeta^i)$ are the exterior (interior) multiplications, f: $[0, 1] \to [0, 1]$, $f(0) = 1$, and $f(t) = 0$ in a neighborhood of 1. The corresponding operator of order 0 determines (after averaging over $O(n)$) an element of $K_0^{O(n)}(D^n)$. In Example 3 it was shown that for $n = 1$ the $O(n)$-index of this operator is equal to 1. We verify that for $n = 2$ the $SO(2)$-index is equal to 1 (cf.[3]).

We note that the symbols

$$\sigma_0(x, \zeta) = i(x^e + x^i) + (\zeta^e + \zeta^i) f(\|x\|), \quad \sigma_\infty(x, \zeta) = -i(x^e + x^i) + (\zeta^e + \zeta^i) f(\|x\|)$$

are homotopic to $\sigma(x, \zeta)$: σ_∞ differs from σ by multiplication by $-i$, and the homotopy of σ_∞ to σ_0 is given by rotation in x from 0 to π and is constant in ζ (this is an $SO(2)$-homotopy). We glue two disks D_0 and D_∞ along the boundary. We get a sphere S^2 whose tangent bundle and exterior algebra are glued from the respective bundles on the disks. The complexes σ_0 (over D_0) and σ_∞ (over D_∞) are glued on the equators of the sphere, since the exterior normal to the boundary in D_0 is identified with the interior normal in D_∞. We denote the complex obtained by τ. The index of the operator F with symbol τ is equal to the sum of the indices of the operators $\sigma_0(x, \mathfrak{D})$ and $\sigma_\infty(x, \mathfrak{D})$, since F is degenerate on the equator. Hence it suffices to show that the $SO(2)$-index of F is equal to 2.

We construct a homotopy $\tau^{(s)}$ of the symbol τ: for $0 \leq s \leq 1$, we glue $\tau^{(s)}$ from $\sigma_0^{(s)}$ and $\sigma_\infty^{(s)}$, where

$$\sigma_0^{(s)} = is(x^e + x^i) + (\zeta^e + \zeta^i) f(s\|x\|), \quad \sigma_\infty^{(s)} = -is(x^e + x^i) + (\zeta^e + \zeta^i) f(s\|x\|).$$

For $s = 0$ we get the symbol of the Euler characteristic operator on the sphere S^2, whose index is equal to 2, since $SO(2)$ acts trivially on the cohomologies of the sphere $H^0(S^2)$ and $H^2(S^2)$. The assertion is proved.

Example 7. $Y = D^{2n} \subset \mathbf{C}^n$, the "real" involution is induced by complex conjugation in $\mathbf{C}^n$; and $G = U(n)$; "real" structure on G is the complex conjugation of matrices. We consider the elliptic operator $\bar\partial + \bar\partial^*$: $\Omega^{ev} \to \Omega^{od}$ on $\mathbf{C}P^n$, where Ω^{ev} is forms of type $(0, 2k)$ and Ω^{od} is forms of type $(0, 2k + 1)$. Passing to an operator of order 0, in correspondence with point 2 of the general construction of the present section we get an element of $KR_0^{U(n)}(D^{2n}, S^{2n-1})$. Its symbol as an element of $KR^0_{U(n)}(TD^{2n})$, under the identification by means of a hermitian metric of $TD^{2n} = D^{2n} \times \mathbf{C}^n$ with $\bar T^* D^{2n} = D^{2n} \times \bar{\mathbf{C}}^n$ (here $\bar{\mathbf{C}}^n = \mathrm{Hom}_{\mathbf{C}\text{-anti}}(\mathbf{C}^n, \mathbf{C})$) is determined by the complex exterior algebra on $\bar{\mathbf{C}}^n$, i.e. coincides with the canonical generator of the group $KR^0_{U(n)}(TD^{2n})$.

§3. Integration of operator functions

This section has a purely technical character. We consider some ways of integrating operator functions on compacta (which are essential for the definition of the intersection index), we prove some facts concerning averaging operator functions over groups, and we make some remarks concerning spectral measures.

We consider the following situation. Let X be a compactum, A a C^*-algebra, ϕ: $C(X) \to A$ an involutory homomorphism of algebras with unit, and F: $X \to A$ a continuous map, while $\forall x \in X\, F(x)$ commutes with the image of ϕ. In this case one can construct the integral $\int_X F(x) d\phi \in A$.

Definition 1. Let $X=\bigcup_1^n U_i$ be an open covering, and $\Sigma_1^n \alpha_i(x)=1$ a partition of unity subordinate to it. We choose points $\xi_i \in U_i$ and make up the integral sum

$$\Sigma(F, \{U_i\}, \{\alpha_i\}, \{\xi_i\}) = \sum_{i=1}^{n} F(\xi_i)\varphi(\alpha_i).$$

The limit of these sums upon refinement of the covering (if it exists) is called the *integral* of $F(x)$ with respect to ϕ.

Theorem 1. 1) *The integral of a continuous function exists.*

2) *If* $\|F(x)\| < \alpha$ *for all* $x \in X$, *then* $\|\int_X F(x)d\phi\| < \alpha$.

3) *The integral is additive and multiplicative.*

4) $(\int_X F(x)d\phi)^* = \int_X F^*(x)d\phi$.

5) *If* $F(x)$ *is a numerical function, i.e.* $F(x)=f(x)\cdot 1$, *where* $f(x)\in C(X)$, *then* $\int_X F(x)d\phi = \phi(f)$.

6) *If* $A=L(H)$ *and* $F(x)\in K(H)$ *for all* $x\in X$, *then* $\int_X F(x)d\phi \in K(H)$.

7) *If the homomorphism* $\phi\colon C(X)\to A$ *decomposes into a composition* $C(X)\xrightarrow{\omega^*} C(Y) \xrightarrow{\phi'} A$, *where* $\omega\colon Y\to X$ *is a continuous map, then*

$$\int_X F(x)\,d\varphi = \int_Y \omega F(y)\,d\varphi'.$$

8) *If* $X = Y\times Z$, *then*

$$\int_X F(x)\,d\varphi = \int_Z \left(\int_Y F(y,z)\,d(\varphi|_{C(Y)})\right) d(\varphi|_{C(Z)}).$$

The following lemma is used in the proof.

Lemma 1. *Let the elements* $\alpha_1,\dots,\alpha_n$, $F_1,\dots,F_n \in A$ *satisfy the conditions* $\Sigma_1^n\alpha_i=1$ *and* $\forall i$, $\alpha_i\geq 0$, $\|F_i\|\leq C$, $F_i\alpha_i=\alpha_iF_i$. *Then* $\|\Sigma_1^n F_i\alpha_i\|\leq C$.

Proof. Since $\alpha_i\geq 0$, $\exists\beta_i=\sqrt{\alpha_i}\in A$. Obviously, $\Sigma_i\beta_i^2=1$. We imbed A in the ring of operators in some Hilbert space T. Then $\forall f,\ g\in T$

$$\left|\left(\sum_i F_i\alpha_i f, g\right)\right| = \left|\left(\sum_i F_i\beta_i f, \beta_i g\right)\right|$$

$$\leqslant \sum_i \|F_i\|\cdot\|\beta_i f\|\cdot\|\beta_i g\| \leqslant C\left(\sum_i\|\beta_i f\|^2\right)^{\frac12}\left(\sum_i\|\beta_i g\|^2\right)^{\frac12} = C\cdot\|f\|\cdot\|g\|,$$

since $\Sigma_i\|\beta_i f\|^2=\Sigma_i(\beta_i^2 f, f)=\|f\|^2$. The lemma is proved.

Proof of Theorem 1. 1) We shall call $\{U_i\}$ an ϵ-covering if $\forall i$, $\forall x,\ y\in U_i$, $\|F(x)-F(y)\|\leq\epsilon$. In view of the continuity of F, every sufficiently fine covering will be an ϵ-covering. We shall verify that the integral sums $\Sigma_i F(\xi_i)\phi(\alpha_i)$ and $\Sigma_j F(\eta_j)\phi(\beta_j)$ of two ϵ-coverings differ (in norm) by not more than 2ϵ. As usual, we let $W_{ij}=U_i\cap V_j$ and $\gamma_{ij}=\alpha_i\cdot\beta_j$, and we apply Lemma 1. We get

$$\left\|\sum_i F(\xi_i)\varphi(\alpha_i) - \sum_{i,j} F(\zeta_{ij})\varphi(\alpha_i\beta_j)\right\| = \left\|\sum_{i,j}(F(\xi_i)-F(\zeta_{ij}))\varphi(\alpha_i\beta_j)\right\| \leqslant \varepsilon.$$

Assertion 1) is proved.

2) This is verified at the level of integral sums with the use of Lemma 1.

3) Additivity is obvious; we verify multiplicativity. Let $\{U_i\}$ be an ϵ-covering for F_1 and F_2 simultaneously. The product of the integral sums is equal to

$$\sum_{i,j} F_1(\xi_i) F_2(\xi_j) \varphi(\alpha_i \alpha_j).$$

In this sum those summands automatically vanish for which $U_i \cap U_j = \emptyset$; hence they can be thrown out. For $U_i \cap U_j \neq \emptyset$ we have $\|F_2(\xi_i) - F_2(\xi_j)\| \leq 2\epsilon$, which means that

$$\| F_1(\xi_i)(F_2(\xi_i) - F_2(\xi_j)) \| \leqslant 2\varepsilon \sup_{x \in X} \| F_1(x) \|.$$

Applying Lemma 1, we get

$$\left\| \sum_{i,j} F_1(\xi_i) F_2(\xi_j) \varphi(\alpha_i \alpha_j) - \sum_{i,j} F_1(\xi_i) F_2(\xi_i) \varphi(\alpha_i \alpha_j) \right\| \leqslant 2\varepsilon \sup_{x \in X} \| F_1(x) \|.$$

Here

$$\sum_{i,j} F_1(\xi_i) F_2(\xi_i) \varphi(\alpha_i \alpha_j) = \sum (F_1 F_2, \{U_i\}, \{\alpha_i\}, \{\xi_i\}).$$

4), 6) and 8) are obvious; 5) is a simple exercise.

7) The covering $\{U_i\}$ and partition of unity $\{\alpha_i\}$ can be restricted to Y. If here the function $\alpha_i(x)$ does not vanish, then $\omega^{-1}(U_i) \neq \emptyset$ and one can take $\xi_i \in \omega^{-1}(U_i)$. The theorem is proved.

Corollary 1. *Let $X \supset U \supset Y$, where U is open and Y is closed. If $F(x)$ commutes with $\phi(C(X))$ only for $x \in U$, and, for $x \in X \setminus Y$, $F(X)$ commutes with the subring $C(X/Y) \subset C(X)$, then all assertions of Theorem 1 for $F(x)$ remain valid.*

Proof. Let $\alpha(x) \in C(X)$, $0 \leq \alpha(x) \leq 1$, $\alpha(x) = 1$ on Y and $\alpha(x) = 0$ on $X \setminus U$. We set $F_1(x) = F(x)\phi(1 - \alpha) + F(x)\alpha(x)$. The function $F_1(x)$ commutes with $C(X)$ for all x, and it is easy to verify that the integral sums of $F(x)$ and $F_1(x)$ differ only slightly from one another if the covering is sufficiently fine. The assertion is proved.

Corollary 2. *Suppose given continuous homomorphisms ϕ_1: $B \to L(H)$ and ϕ_2: $C(X) \to L(H)$, where ϕ_2 is involutory. Then there is a uniquely defined continuous homomorphism $\phi_1 \otimes \phi_2$: $S(X, B) \to L(H \otimes H)$ which coincides with $\phi_1 \otimes 1$ on $B \subset C(X, B)$ and with $1 \otimes \phi_2$ on $C(X) \subset C(X, B)$.*

Proof. The uniqueness follows from the fact that any element $b(x) \in C(X, B)$ is approximated by elements of the form $\Sigma_1^n b(\xi_i)\alpha_i(x)$, $\xi_i \in X$, $\alpha_i(x) \in C(X)$. The existence follows from the relation

$$(\varphi_1 \otimes \varphi_2)(b(x)) = \int_X (\varphi_1(b(x)) \otimes 1) \, d(1 \otimes \varphi_2).$$

The assertion is proved.

Remark 1. In what follows, Theorem 1 will be used in the following situation ϕ: $C(X) \to L(H)$, F: $X \to L(H)$ and $F(x)$ commutes with $\phi(C(X))$ modulo $K(H)$. To apply the theorem one must pass to $A(H) = L(H)/K(H)$. The element $\int_X F(x) d\phi$ can be lifted to $L(H)$. It is defined modulo $K(H)$. If F depends continuously on a parameter $z \in Z$, where Z is compact, then the lifting of $\int_X F_z(x) d\phi$ in $L(H)$ can be chosen continuously 810 in z. This follows from the existence of continuous sections $r: A(H) \to L(H)$ (cf. [11]). Here if the lifting is already fixed over $Z_0 \subset Z$, then there exists a continuous extension of it to Z (cf. [11]). Such a situation arises when $F_z(x)$ commutes with $\phi(C(X))$ exactly for $z \in Z_0$ and hence $\int_X F_z(x)d\phi \in L(H)$ is defined.

We proceed to questions connected with averaging over a compact group G: Let $Q: G \to L(H)$ be a function continuous in the strong operator topology. The Haar measure on G allows one to integrate Q. By definition

$$\left(\left(\int_G Q(g)\,dg\right)x,\ y\right) = \int_G (Q(g)x,\ y)\,dg,$$

where $x,\ y \in H$ and $(\ ,\ )$ is the scalar product in H. This integral is linear, and $\forall R \in L(H)$

$$\int_G RQ(g)\,dg = R\int_G Q(g)\,dg, \quad \int_G Q(g)R\,dg = \left(\int_G Q(g)\,dg\right)R.$$

If the function Q is continuous in the norm, then $\int_G Q(g)dg$ can be constructed in complete analogy with Definition 1 (it suffices to replace $\phi: C(X) \to A$ by the Haar measure $\phi: C(G) \to C$):

$$\int_G Q(g)\,dg = \lim \sum_i Q(\xi_i) \int_G \alpha_i(g)\,dg.$$

In particular, one has the following assertions.

Proposition 1. *If $Q: G \to K(H) \subset L(H)$ is continuous in the norm, then $\int_G Q(g)dg \in K(H)$.*

Proposition 2. *Let $\chi: G \to L^U(H)$ be a representation that is continuous in the strong topology.*

1) *If $k \in K(H)$, then the function $Q(g) = \chi(g)k\chi(g^{-1})$ is continuous on G in the norm.*

2) *If $S,\ T \in L(H)$, where $\forall g \in G$ $\chi(g)S\chi(g^{-1})T \sim T\chi(g)S\chi(g^{-1})$ and the function $\chi(g)S\chi(g^{-1})$ is continuous in norm, then*

$$S\left(\int_G \chi(g)T\chi(g^{-1})\,dg\right) \sim \left(\int_G \chi(g)T\chi(g^{-1})\,dg\right)S.$$

Proof. 1) We note that $\chi(g)k$ is continuous in norm, since from the pointwise con-
811 tinuity of $\chi(g)$ on H follows its uniform continuity on compact subsets of H. The
function $k\chi(g^{-1}) = (\chi(g)k^*)^*$ is also continuous in norm. If $k \geq 0$, then

$$\chi(g)k\chi(g^{-1}) = (\chi(g)\sqrt{k}) \cdot (\sqrt{k}\chi(g^{-1})).$$

It remains to complexify H and represent k as a linear combination of four positive compact operators.

2) The function

$$S\chi(g)T\chi(g^{-1}) - \chi(g)T\chi(g^{-1})S$$
$$= \chi(g)[\chi(g^{-1})S\chi(g^{-1})^{-1}T - T\chi(g^{-1})S\chi(g^{-1})^{-1}]\chi(g^{-1})$$

is continuous in norm in correspondence with point 1. Applying Proposition 1, we get what is required. The assertion is proved.

Proposition 3. *Let the group G act continuously on X, let $\chi\colon G \to L^U(H)$ be a strongly continuous representation, and let the pair $\phi\colon C(X) \to L(H)$, $F\colon X \to L(H)$ satisfy the conditions of Remark 1, while*

$$\varphi(g(f)) = \chi(g)\varphi(f)\chi(g^{-1}), \qquad F(gx) = \chi(g)F(x)\cdot\chi(g^{-1}).$$

Then $\chi(g)(\int_X F(x)d\phi)\chi(g^{-1})$ is a function on G which is continuous in norm.

Proof. It suffices to verify continuity at the identity $e \in G$. For any $\epsilon > 0$ there exists an integral sum $\sigma = \sum_i F(\xi_i)\phi(\alpha_i)$ and a compact operator k such that

$$\left\| \int_X F(x)\,d\varphi - \sigma - k \right\| < \varepsilon \quad (\int_X F(x)\,d\varphi$$

is some lifting of the integral from $A(H)$ into $L(H)$; the norm is taken in $L(H)$). Hence it follows immediately that $\forall g \in G$

$$\left\| \chi(g)\left[\int_X F(x)\,d\varphi - \sigma - k\right]\chi(g^{-1}) \right\| < \varepsilon.$$

We find a neighborhood of the identity $U \subset G$ such that for $g \in U$ the following conditions are satisfied:

1°. $\|\chi(g)\sigma\chi(g^{-1}) - \sigma\| = \left\| \sum_i F(g\xi_i)\varphi(g\alpha_i) - \sum_i F(\xi_i)\varphi(\alpha_i) \right\| \leqslant \varepsilon;$

2°. $\|\chi(g)k\chi(g^{-1}) - k\| < \epsilon$ (cf. Proposition 2).

Then

$$\left\| \chi(g)\left(\int_X F(x)\,d\varphi\right)\chi(g^{-1}) - \int_X F(x)\,d\varphi \right\| < 3\varepsilon.$$

The proposition is proved.

Remark 2. An involutory homomorphism $\phi\colon C(X) \to L(H)$ generates a spectral measure P_ϕ on X: for any Borel set $\Delta \subset X$, $P_\phi(\Delta)$ is a hermitian projector. For open Δ we have $P_\phi(\Delta) = \sup \phi(f)$, where the supremum is taken over all $f \in C(X)$: $0 \leq f(x) \leq 1$, $f = 0$ on $X \setminus \Delta$ (sup is understood in the sense of inequality in the class of hermitian operators). The spectral measure is additive and multiplicative, and $P_\phi(X) = 1$.

The following assertion clarifies the meaning of condition 5 of Definition 1 of §1 (cf. the quasi-locality of pseudodifferential operators).

Proposition 4. *If $\phi\colon C(X) \to C(H)$ is an involutory homomorphism, then the following conditions are equivalent*:

1) $\forall f \in C(X)$ $\phi(f)F \sim F\phi(f)$.

2) $\forall f_1, f_2 \in C(X)$ $f_1 f_2 = 0 \Rightarrow \phi(f_1)F\phi(f_2) \sim 0$.

3) *If $X_1, X_2 \subset X$ are closed and disjoint, then* $P_\phi(X_1)FP_\phi(X_2) \sim 0$.

We shall not give the proof, since this fact will not be used in what follows. We note only that the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious.

Remark 3. Let Y be a closed G-invariant subset of X, and let $(\chi, \phi, \psi, F) \in \mathcal{E}^G_{p,q}(C(X))$. If F commutes with $P_\phi(Y)$, then our quadruple splits into the direct sum of two quadruples corresponding to the decomposition

$$H = \operatorname{Im} P_\varphi(Y) \oplus \operatorname{Im} P_\varphi(X \setminus Y).$$

(If one of these spaces is finite dimensional, then one should add to (χ, ϕ, ψ, F) a degenerate quadruple.) The first of the quadruples obtained lies in the image $\operatorname{Im}[\mathcal{E}^G_{p,q}(C(Y)) \to \mathcal{E}^G_{p,q}(C(X))]$. We shall call (χ, ϕ, ψ, F) *degenerate outside* Y if the second of the quadruples obtained is degenerate. In the definition of $K^G_{p,q}(X, Y)$ [cf. Definition 5 of §1], point b) can be replaced by the requirement of degeneracy of (χ, ϕ, ψ, F_0) outside Y.

§4. Construction of the product

In this section a basic technical construction is introduced: the external product

$$K^{G_1}_{p,q}(B_1) \otimes K^{G_2}_{p',q'}(B_2) \to K^{G_1 \times G_2}_{p+p',q+q'}(B_1 \otimes B_2).$$

The topological tensor product $B_1 \otimes B_2$ is the completion of the algebraic tensor product with respect to the norm:

$$\|z\| = \inf\left\{\sum_i \|x_i\| \cdot \|y_i\| \,\middle|\, z = \sum_i x_i \otimes y_i\right\}$$

(projective topology). The operations of multiplication, involution, action of $G_1 \times G_2$ and the "real" structure extend from the algebraic tensor product by continuity.

For C^*-algebras it would be more suitable to define the tensor product as the enveloping C^*-algebra: $C^*(B_1 \otimes B_2)$. Then $C(X) \otimes C(Y) = C(X \times Y)$. For our goals any definition is suitable which satisfies the following conditions:

1. Homomorphisms $\phi_1\colon B_1 \to L(H)$ and $\phi_2\colon B_2 \to L(H')$ uniquely determine a homomorphism $\phi_1 \otimes \phi_2\colon B_1 \otimes B_2 \to L(H \otimes H')$, whose image is contained in the closure of the subalgebra generated by the images of $\phi_1 \otimes 1\colon B_1 \to L(H \otimes H')$ and $1 \otimes \phi_2\colon B_2 \to L(H \otimes H')$.

2. The products $B_1 \otimes B_2$ and $\phi_1 \otimes \phi_2$ are associative and functorial (i.e. if $\omega_1\colon B_1' \to B_1$ is a homomorphism, then $(\phi_1 \circ \omega_1) \otimes \phi_2 = (\phi_1 \otimes \phi_2) \circ (\omega_1 \otimes 1)$).

3. If $g\colon B_1 \to B_1$ and $g\colon H \to H$ are (linear or antilinear) automorphisms which are compatible with one another (i.e. $\phi_1(g(b)) = g\phi_1(b)g^{-1}$), then $g \otimes 1\colon B_1 \otimes B_2 \to B_1 \otimes B_2$ and $g \otimes 1\colon H \otimes H' \to H \otimes H'$ are also compatible.

Unfortunately, our construction of the external product is dependent on some restrictions on the admissible class of algebras: further we shall assume that the Banach algebras considered are topologically finitely generated (i.e. there exists an everywhere dense finitely generated subalgebra). In the case of $C(X)$ this corresponds to finite dimensionality of X.

Let $(\chi_1, \phi_1, \psi_1, F_1) \in K^{G_1}_{p,q}(B_1)$ and $(\chi_2, \phi_2, \psi_2, F_2) \in K^{G_2}_{p',q'}(B_2)$. In the Hilbert space $H \otimes H'$ there are defined compatible (in view of point 3) representations $\chi_1 \otimes \chi_2$ and $\phi_1 \otimes \phi_2$. (In the "real" case the complex conjugation on $H \otimes H'$ is defined as $r \otimes r'$. The representation $\phi_1 \times \phi_2$ preserves "real" structure.) We construct $\psi_1 \otimes \psi_2$.

Definition 1. We shall call H a *graded Hilbert space* if there is fixed an orthogonal decomposition $H = H_0 \oplus H_1$. The elements of $L(H)$ which preserve (interchange)

H_0 and H_1 will be called operators of *degree* 0 and of *degree* 1, respectively. A representation $\psi: C_{p,q} \to L(H)$ will be called *graded* if all the operators $\psi(e_i)$ and $\psi(\epsilon_j)$ have degree 1. The tensor product of graded Hilbert spaces is graded:

$$(H \otimes H')_0 = H_0 \otimes H_0' + H_1 \otimes H_1', \quad (H \otimes H')_1 = H_1 \otimes H_0' + H_0 \otimes H_1'.$$

If the elements $D_1 \in L(H)$ and $D_2 \in L(H')$ have degrees, then their graded tensor product is defined by the formula

$$(D_1 \hat{\otimes} D_2)\left(\sum_i x_i \otimes y_i\right) = \sum_i (-1)^{\deg D_2 \deg x_i} D_1(x_i) \otimes D_2(y_i).$$

Lemma 1. *There exists a one-to-one correspondence between representations* ψ: $C_{p,q+1} \to L(H)$ *and graded representations* $\tilde{\psi}: C_{p,q} \to L(H)$.

Proof. We set

$$H_0 = \operatorname{Im} \frac{1}{2}(1 + \psi(\varepsilon_{q+1})), \qquad H_1 = \operatorname{Im} \frac{1}{2}(1 - \psi(\varepsilon_{q+1})).$$

Conversely,

$$\psi(\varepsilon_{q+1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The lemma is proved.

Definition 2. We identify the algebra $C_{p+p',q+q'}$ with the graded tensor product $C_{p,q} \otimes C_{p',q'}$ by the formulas

$$e_i \to e_i \hat{\otimes} 1, \quad i \leqslant p; \qquad \varepsilon_j \to \varepsilon_j \hat{\otimes} 1, \quad j \leqslant q;$$
$$e_{i+p} \to (-1)^q (1 \hat{\otimes} e_i), \quad i \leqslant p'; \qquad \varepsilon_{j+q} \to 1 \hat{\otimes} \varepsilon_j, \quad j \leqslant q'.$$

If there are given graded representations $\psi_1: C_{p,q} \to L(H)$ and $\psi_2: C_{p',q'} \to L(H')$, then a graded representation $\psi_1 \otimes \psi_2: C_{p,q} \otimes C_{p',q'} \to L(H \otimes H')$ is determined by the formula

$$[(\psi_1 \otimes \psi_2)(a \otimes b)](x \otimes y) = (-1)^{\deg b \cdot \deg x} [\psi_1(a)](x) \otimes [\psi_2(b)](y).$$

It remains for us to construct the tensor product $F_1 \# F_2$ of Fredholm operators. These are operators of degree 1 in H and H' respectively which anticommute with the generators of the Clifford algebras $C_{p,q}$ and $C_{p',q'}$. It is easy to verify that the operators $F_1 \hat{\otimes} 1$ and $1 \hat{\otimes} F_2$ anticommute with each other and with the generators of the Clifford algebra $C_{p+p',q+q'}$. The basic difficulty consists of the fact that it is impossible, in analogy with [6], to set $F_1 \# F_2 = F_1 \hat{\otimes} 1 + 1 \hat{\otimes} F_2$, since conditions 1, 2 and 5 of Definition 1 of §1 are violated.

Definition 3. Let $x_1 = (\chi_1, \phi_1, \psi_1, F_1) \in \mathcal{E}^{G_1}_{p,q}(B_1)$ and $x_2 = (\chi_2, \phi_2, \psi_2, F_2) \in \mathcal{E}^{G_2}_{p',q'}(B_2)$. We denote by $S(x_1, x_2)$ the set of pairs $M, N \subset L(H \otimes H')$ satisfying the following conditions:

1) M and N are hermitian and positive.

2) $M^2 + N^2 \sim 1$.

3) M and N commute with $G_1 \times G_2$.

4) M and N commute with $C_{p+p',q+q'}$ and have degree 0.

5) M and N commute modulo $K(H \otimes H')$ with $\operatorname{Im}\phi_1 \otimes 1$, $1 \otimes \operatorname{Im}\phi_2$, $F_1 \hat{\otimes} 1$ and $1 \hat{\otimes} F_2$.

6) $M(F_1 \hat{\otimes} 1)$ and $N(1 \hat{\otimes} F_2)$ commute modulo $K(H \otimes H')$ with $\operatorname{Im}\phi_1 \otimes 1$ and $1 \otimes \operatorname{Im}\phi_2$.

7) $M^2(F_1 \hat{\otimes} 1)^2 \sim -M^2$, $N^2(1 \hat{\otimes} F_2)^2 \sim -N^2$.

8) $[(F_1 + F_1^*) \hat{\otimes} 1]M \sim 0$, $[1 \hat{\otimes} (F_2 + F_2^*)]N \sim 0$.

9) In the "real" case M and N are real.

If for a pair M, N all the above conditions are satisfied exactly, and not modulo $K(H \otimes H')$, it is called a *degenerate pair.*

Lemma 2. *If $M, N \in S(x_1, x_2)$, then the quadruple*

$$(\chi_1 \otimes \chi_2, \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2, M(F_1 \hat{\otimes} 1) + N(1 \hat{\otimes} F_2))$$

is an element of $\mathfrak{E}^{G_1 \otimes G_2}_{p+p',q+q'}(B_1 \otimes B_2)$. Up to homotopy it is independent of the choice of $(M, N) \in S(x_1, x_2)$.

Proof. Definition 1 of §1 is easily verified. Let (M_1, N_0) be another pair from $S(x_1, x_2)$. It is asserted that for $0 \le t \le \pi/2$ the pair

$$M_t = (M^2 \cos^2 t + M_0^2 \sin^2 t)^{1/2}, \quad N_t = (N^2 \cos^2 t + N_0^2 \sin^2 t)^{1/2} \in S(x_1, x_2).$$

For the verification of points 5)–8) of Definition 3, we note that any positive hermitian operator M is uniformly approximated by polynomials with zero free terms from M^2. In particular, point 5) need only be verified for the operators M_t^2 and N_t^2, for which it is obvious. Computing the commutator $[\phi_1(b_1) \otimes \phi_2(b_2), M_t(F_1 \hat{\otimes} 1)]$ using point 5), we get modulo $K(H \otimes H')$

$$M_t((\varphi_1(b_1)F_1 - F_1\varphi_1(b_1)) \hat{\otimes} \varphi_2(b_2)).$$

Since M_t is approximated by polynomials of the form $P(M_t^2) \cdot M_t^2$, the above operator is compact if and only if

$$M_t^2((\varphi_1(b_1)F_1 - F_1\varphi_1(b_1)) \hat{\otimes} \varphi_2(b_2))$$

is compact. But this operator is obviously compact. Points 7) and 8) are verified analogously. The lemma is proved.

Definition 4. We denote by S^+ the set of pairs $(M, N) \in L(H \otimes H)$ satisfying the following conditions:

1°. M and N are hermitian and positive.

2°. $M^2 + N^2 \sim 1$.

3°. If $k \in K(H)$, then

$$M(k \otimes 1),\ (k \otimes 1)M,\ N(1 \otimes k),\ (1 \otimes k)N \in K(H \otimes H).$$

If $\Omega_1, \Omega_2 \subset L(H)$, then by $S^+(\Omega_1; \Omega_2)$ we denote the subset of S^+ consisting of pairs satisfying the condition

4°. M and N commute modulo $K(H \otimes H)$ with operators of the form $A \otimes 1$ for $A \in \Omega_1$, and with operators of the form $1 \otimes A$ for $A \in \Omega_2$.

Lemma 3. *Let T be a topological space, let the pair $(M, N) \in S^+$, and let $F: T \to L(H)$ be a continuous map such that $F(t)^2 \sim -1$ and $F(t)^* \sim -F(t)$. Denote $(1 + F(t)F(t)^*)^{1/2}$ by $U(t)$. Then the pair*

$$M(t) = \{(U(t)^{-1} \otimes 1)[M^2 + (F(t) \otimes 1) M^2 (F(t) \otimes 1)^*](U(t)^{-1} \otimes 1)\}^{1/2},$$
$$N(t) = \{(U(t)^{-1} \otimes 1)[N^2 + (F(t) \otimes 1) N^2 (F(t) \otimes 1)^*](U(t)^{-1} \otimes 1)\}^{1/2}$$

for each fixed $t \in T$ belongs to $S^+(F(t);\ )$ and depends continuously on t.

Proof. Points 1° and 2° of Definition 4 are obvious, point 3° is verified by analogy with the proof of Lemma 2, and point 4° is verified first for $M(t)$, and for $N(t)$ is obtained from the equation $N(t)^2 \sim 1 - M(t)^2$. The lemma is proved.

Theorem 1. *If Ω_1 and Ω_2 are finite sets, then $S^+(\Omega_1; \Omega_2) \neq \varnothing$.*

The proof is given in the Supplement to this section.

Corollary 1. *If $x_1 \in \mathcal{E}^{G_1}_{p,q}(B_1)$ and $x_2 \in \mathcal{E}^{G_2}_{p',q'}(B_2)$, then $S(x_1, x_2) \neq \varnothing$.*

Proof. Let $\sigma_1 \subset B_1$ and $\sigma_2 \subset B_2$ be finite collections of topological generators. We choose $(M, N) \in S^+(\phi_1(\sigma_1), \phi_2(\sigma_2))$. Then automatically $(M, N) \in S^+(\phi_1(B_1); \phi_2(B_2))$. We average the pair (M, N) by the actions of the groups G_1, G_2 and the two finite groups generated by the generators of the Clifford algebras $C_{p,q+1}$ and $C_{p',q'+1}$ (and in the "real" case also by τ), i.e. we set

$$\overline{M} = \left(\int gM^2g^{-1}\right)^{1/2}, \quad \overline{N} = \left(\int gN^2g^{-1}\right)^{1/2}.$$

Applying Propositions 1 and 2 of §3, we get that the new pair $(\overline{M}, \overline{N})$ belongs to $S^+(\phi_1(B_1); \phi_2(B_2))$ and satisfies points 3) and 4) of Definition 3 (for example, the fact that it commutes with $\phi_1(B_1) \otimes 1$ follows from point 2) of Proposition 2 of §3: $S = \phi_1(b_1) \otimes 1$ and $T = M^2$ or N^2). Further, we average the pair M, N in correspondence with Lemma 3 first with respect to $F_1 \hat{\otimes} 1$ and then $1 \hat{\otimes} F_2$. This pair already satisfies Definition 3 (points 6), 7) and 8) are obtained from 3° of Definition 4). The corollary is proved.

Definition 5. We set $F_1 \# F_2 = M(F_1 \hat{\otimes} 1) + N(1 \hat{\otimes} F_2)$, where $M, N \in S(x_1, x_2)$. The quadruple

$$(\chi_1 \otimes \chi_2, \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2, F_1 \# F_2)$$

is called the *product* of x_1 and x_2.

Theorem 2. *The product of quadruples determines a functorial, distributive, skew-commutative, associative pairing:*

$$K^{G_1}_{p,q}(B_1) \otimes K^{G_2}_{p',q'}(B_2) \to K^{G_1 \times G_2}_{p+q',q+q'}(B_1 \otimes B_2).$$

This pairing commutes with the periodicity isomorphisms (Theorems 2 and 3 of §1).

Proof. That it is well-defined with respect to replacement of x_1 or x_2 by something equivalent is verified using Lemma 3. If x_1 is a degenerate quadruple, then one can take $M = 1$ and $N = 0$, and $x_1 \cdot x_2$ will be degenerate. That the pairing is functorial and distributive is obvious. We verify skew-commutativity. Let u be an isometric automorphism of the graded Hilbert space $H \otimes H$ determined on homogeneous elements by the formula

$$u(x \otimes y) = (-1)^{\deg x \cdot \deg y}(y \otimes x).$$

Then

$$u(F_1 \hat{\otimes} 1)u^{-1} = 1 \hat{\otimes} F_1, \quad u(1 \hat{\otimes} F_2)u^{-1} = F_2 \hat{\otimes} 1,$$

and the representation $\psi_2 \otimes \psi_1$ differs from $u(\psi_1 \otimes \psi_2)u^{-1}$ by the automorphism of Clifford algebras

$$C_{p+p',q+q'} \simeq C_{p,q} \otimes C_{p',q'} \xrightarrow{T} C_{p',q'} \otimes C_{p,q} \simeq C_{p+p',q+q'}:$$

$$T(a \otimes b) = (-1)^{\deg a \cdot \deg b}(b \otimes a).$$

On the generators, this automorphism is the composition of $pp' + qq'$ transpositions and $pq' + qp'$ sign changes. Applying Remark 1 of §1, we find that $x_2 \cdot x_1$ differs from $x_1 \cdot x_2$ by the sign $(-1)^{(p+q)(p'+q')}$.

We shall prove associativity. Let $x_1 \in \mathcal{E}^{G_1}_{p,q}(B_1)$, $x_2 \in \mathcal{E}^{G_2}_{p',q'}(B_2)$ and $x_3 \in \mathcal{E}^{G_3}_{p'',q''}(B_3)$. The representations $\chi_1 \otimes \chi_2 \otimes \chi_3$, $\phi_1 \otimes \phi_2 \otimes \phi_3$ and $\psi_1 \otimes \psi_2 \otimes \psi_3$ are independent of the arrangement of parentheses. Let $(M, N) \in S(x_1, x_2)$ and $(P, Q) \in S(x_1 \cdot x_2, x_3)$. By Theorem 1 (using the scheme of proof of Corollary 1) one can choose (P, Q) so that the additional condition

$$(P, Q) \in S^+(M, N, \operatorname{Im}\varphi_1 \otimes 1, 1 \otimes \operatorname{Im}\varphi_2; \operatorname{Im}\varphi_3)$$

is satisfied. We set $R = \sqrt{P}(M \otimes 1)\sqrt{P}$ and $T = \sqrt{P}(N \otimes 1)\sqrt{P}$. The triple $(R, T, Q) \in L(H \otimes H \otimes H)$ satisfies in relation to the operators $F_1 \hat{\otimes} 1 \hat{\otimes} 1$, $1 \hat{\otimes} F_2 \hat{\otimes} 1$ and $1 \hat{\otimes} 1 \hat{\otimes} F_3$ a system of conditions analogous to points 1)–9) of Definition 3. The operator

$$F_1 \# F_2 \# F_3 = R(F_1 \hat{\otimes} 1 \hat{\otimes} 1) + T(1 \hat{\otimes} F_2 \hat{\otimes} 1) + Q(1 \hat{\otimes} 1 \hat{\otimes} F_3)$$

is independent of the choice of R, T and Q (cf. Lemma 2), so it is independent of the arrangement of parentheses.

The verification that our pairing commutes with the periodicity isomorphisms is analogous to the verification of skew-commutativity. The theorem is proved.

Definition 6. Let $\omega_1: B_1' \to B_1$ be an epimorphism. $B_3 = B_1 \otimes B_2$ and $B_3' = B_1' \otimes B_2$, so that $\omega_3 = \omega_1 \otimes 1$ is an epimorphism. We construct pairings of the absolute and relative K-functors:

$$K_{p,q}(\omega_1) \otimes K_{p',q'}(B_2) \to K_{p+p',q+q'}(\omega_3), \tag{1}$$

$$K_{p'q'}(B_2) \otimes K_{p,q}(\omega_1) \to K_{p+p',q+q'}(\omega_3). \tag{2}$$

We identify $C_{p+p'-1,q+q'}$ with $C_{p-1,q} \otimes C_{p',q'}$ by the formulas

$$e_i \to e_i \hat{\otimes} 1, \quad i \leqslant p-1; \quad \varepsilon_j \to \varepsilon_j \hat{\otimes} 1, \quad j \leqslant q;$$
$$e_{i+p-1} \to (-1)^{q+1}(1 \hat{\otimes} e_i), \quad i \leqslant p'; \quad \varepsilon_{j+q} \to -1 \hat{\otimes} \varepsilon_j, \quad j \leqslant q'.$$

Moreover, we identify $C_{p+p'-1,q+q'}$ with $C_{p',q'} \otimes C_{p-1,q}$ by the formulas

$$e_i \to e_i \hat{\otimes} 1, \quad i \leqslant p'; \quad \varepsilon_j \to \varepsilon_j \hat{\otimes} 1, \quad j \leqslant q';$$
$$e_{i+p'} \to (-1)^{q'} (1 \hat{\otimes} e_i), \quad i \leqslant p-1; \quad \varepsilon_{j+q} \to 1 \hat{\otimes} \varepsilon_j, \quad j \leqslant q.$$

Let

$$x_t = (\chi_1, \varphi_1, \psi_1, F_t) \in \mathscr{E}^{G_1}_{p,q}(\omega_1), \quad y = (\chi_2, \varphi_2, \psi_2, F_2) \in \mathscr{E}^{G_2}_{p',q'}(B_2).$$

The representations $\chi_1 \otimes \chi_2$, $\chi_2 \otimes \chi_1$, and $\phi_1 \otimes \phi_2$, $\phi_2 \otimes \phi_1$ are defined; $\psi_1 \otimes \psi_2$ and $\psi_2 \otimes \psi_1$ are obtained from the identifications written out above, so it remains to construct the homotopies $F_t \# F_2$ and $F_2 \# F_t$. According to Definition 5 of §1,

$$x_0 = (\chi_1, \varphi_1, \psi_1, F_0) = (\chi', \varphi', \psi', F_0') \oplus (\chi'', \varphi'', \psi'', F_0'') = x_0' \oplus x_0''.$$

We denote by $S(\{x_t\}, y)$ the set consisting of continuous homotopies of pairs $(M_t, N_t) \in S(x_t, y)$, where for $t = 0$

$$(M_0, N_0) = (M_0', N_0') \oplus (M_0'', N_0''),$$

where $(M_0', N_0') \in S(x_0', y)$ and $(M_0'', N_0'') \in S(x_0'', y)$. It is required that (M_0'', N_0'') and (M_1, N_1) be degenerate (cf. Definition 3). If $(M_t, N_t) \in S(\{x_t\}, y)$, then we set $F_t \# F_2 = M_t(F_t \hat{\otimes} 1) + N_t(1 \hat{\otimes} F_2)$. Analogously one defines $S(y, \{x_t\})$ and $F_2 \# F_t$.

Theorem 3. *The pairings* (1) *and* (2) *are well-defined and differ from one another only by the sign* $(-1)^{(p+q)(p'+q')}$. *The homomorphism* $i_*: K_{p,q}(B) \to K_{p,q}(B, \varnothing)$ *(Definition 9 of* §1) *commutes with these pairings. The boundary homomorphism* $\partial: K_{p,q}(\omega_1) \to K_{p-1,q}(B_1)$ *commutes with the pairing* (2) *(i.e. with left multiplication).*

Proof. Any element $\{x_t\} \in \mathcal{E}^{G_1}_{p,q}(\omega_1)$ can be changed by a homotopy so that $\forall y \in \mathcal{E}^{G_2}_{p',q'}(B_2)$ $S(\{x_t\}, y) \neq \varnothing$. It suffices to make the homotopy F_t constant in an ϵ-neighborhood of the points $t = 0$ and $t = 1$. Then one can define (M_t, N_t) for $\epsilon \leq t \leq 1 - \epsilon$ with the help of Lemma 3; for $t = 0$, $(M_0', N_0') \in S(x_0', y)$ is chosen arbitrarily, and $M_0'' = 1$, $N_0'' = 0$; for $t = 1$, $(M_1, N_1) = (1, 0)$. For $0 \leq t \leq \epsilon$ we set

$$M_t = \left(M_0^2 \cos^2 \frac{\pi t}{2\varepsilon} + M_\varepsilon^2 \sin^2 \frac{\pi t}{2\varepsilon}\right)^{1/2}, \quad N_t = \left(N_0^2 \cos^2 \frac{\pi t}{2\varepsilon} + N_\varepsilon^2 \sin^2 \frac{\pi t}{2\varepsilon}\right)^{1/2}.$$

Analogously we define (M_t, N_t) on the segment $[1 - \epsilon, 1]$.

The independence of $F_t \# F_2$ from the choice of (M_t, N_t) is verified just as in Lemma 2. If $\{x_t\}$ or y is degenerate, it is not hard to construct the corresponding (M_t, N_t). Now let $\{x_{t,s}\}$ be a homotopy of $\{x_t\}$. One can assume $F_{t,s}$ is constant in t in an ϵ-neighborhood of the points $t = 0$ and $t = 1$. We construct $(M_{t,s}, N_{t,s})$ in the rectangle $[\epsilon, 1 - \epsilon] \times [0, 1]$ by Lemma 3, and for $0 \leq t \leq \epsilon$ and for $1 - \epsilon \leq t \leq 1$ we define it as above (here one first constructs $(M_{0,s}', N_{0,s}')$ by Lemma 3). Thus we get a homotopy of $\{x_{t,0}\} \cdot y$ to $\{x_{t,1}\} \cdot y$. Analogously one constructs the homotopy $\{x_t\} \cdot y_s$. The well-definedness is proved.

The fact that the pairings (1) and (2) differ by $(-1)^{(p+q)(p'+q')}$ is verified just as was skew-commutativity in Theorem 2. The assertion about ∂ is obvious. The assertion about i_* need be verified only for the pairing (2). Let

$$y = (\chi_2, \varphi_2, \psi_2, F_2) \in \mathcal{E}^{G_2}_{p',q'}(B_2), x = (\chi_1, \varphi_1, \psi_1, F) \in \mathcal{E}^{G_1}_{p,q}(B_1).$$

Then

$$i_*(y \cdot x) = (\chi_2 \otimes \chi_1, \varphi_2 \otimes \varphi_1, (\psi_2 \otimes \psi_1)|_{C_{p+p'-1,q+q'}}, (-1)^{q+q'} (\psi_2 \otimes \psi_1)$$
$$\times (e_{p+p'}) \cdot \cos \pi t + (F_2 \# F) \sin \pi t) = (\chi_2 \otimes \chi_1, \varphi_2 \otimes \varphi_1, \psi_2 \otimes (\psi_1|_{C_{p-1,q}}),$$
$$(-1)^q (1 \hat{\otimes} \psi_1(e_p)) \cos \pi t + F_2 \# F \sin \pi t).$$

It is necessary to verify that the last operator is homotopic to $F_2 \# F_t$, where

$$F_t = (-1)^q \psi_1(e_p) \cos \pi t + F \sin \pi t, \quad 0 \leqslant t \leqslant 1.$$

This is a straightforward exercise. The theorem is proved.

Theorem 4. *The element*

$$a_1 = \left(1 : C \to L(H), \psi(\varepsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, T = \begin{pmatrix} 0 & -T_1^* \\ T_1 & 0 \end{pmatrix}\right) \in K_{00} \text{ (point)}$$

where T_1 is a Fredholm operator of index 1, *is a multiplicative unit.*

We shall give the proof in the absolute case (in the relative one everything is just the same). One can assume that $T_1 T_1^* = 1$ and $T_1^* T_1 = 1 - p_1$, where p_1 is a one-dimensional projector. We denote $\operatorname{Ker} p_1$ by H_1 and $\operatorname{Im} p_1$ by E_1. The product of the element $x = (\chi, \phi, \psi, F) \in \mathcal{E}^G_{p,q}(B)$ by a_1, using Remark 1 of §1, can be written as

$$\left(\begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix} \otimes 1, \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \otimes 1, \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix} \otimes 1, \right.$$
$$\left. \begin{pmatrix} M_1(F \otimes 1) & -N_1(1 \otimes T_1^*) \\ N_2(1 \otimes T_1) & -M_2(F \otimes 1) \end{pmatrix}\right),$$

where

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \in S(x, a_1).$$

According to Definition 3, $N_2(1 \otimes T_1) \sim (1 \otimes T_1) N_1$. If with the help of Theorem 1 one assumes in addition that $\begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$ commutes modulo $K(H \otimes H)$ with $\begin{pmatrix} 1 \otimes T_1 & 0 \\ 0 & 1 \otimes T_1 \end{pmatrix}$, then $N_1 \sim N_2$. Consequently $M_1 \sim M_2$. We set $M = M_1$ and $N = N_1$. It is easy to verify that $N(1 \otimes p_1) \sim 0$ and $M(1 \otimes p_1) \sim 1 \otimes p_1$. We replace M by

$$(1 \otimes p_1) + (1 \otimes (1 - p_1)) M (1 \otimes (1 - p_1)),$$

and N by

$$(1 \otimes (1 - p_1)) N (1 \otimes (1 - p_1)).$$

The new pair (M, N) modulo $K(H \otimes H)$ coincides with the old, and $M(1 \otimes p_1) = 1 \otimes p_1$ and $N(1 \otimes p_1) = 0$.

We consider the decomposition $H \otimes H = (H \otimes E_1) \oplus (H \otimes H_1 + H \otimes H)$. The operator

$$\begin{pmatrix} M(F\otimes 1) & -N(1\otimes T_1^*) \\ N(1\otimes T_1) & -M(F\otimes 1) \end{pmatrix}$$

leaves both summands invariant. On the first summand this operator coincides with $F \otimes 1 = F$, and on the second with the help of the identification of H_1 with H by means of T_1 it reduces (after adding a compact operator) to the form

$$\begin{pmatrix} M(F\otimes 1) & -N \\ N & -M(F\otimes 1) \end{pmatrix},$$

i.e. it coincides with $F \# \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus $x \cdot a_1 = x \oplus$ an element of $\overline{\mathfrak{D}}^G_{p,q}(B)$. The theorem is proved.

Remark 1. That the above-constructed pairings commute with periodicity allows us to define

$$K_m^{G_1}(B_1) \otimes K_n^{G_2}(B_2) \to K_{m+n}^{G_1\times G_2}(B_1 \otimes B_2)$$

(analogously in the relative case). If $G_1 = G_2 = G$, then composition with restriction to the diagonal gives

$$K_m^G(B_1) \otimes K_n^G(B_2) \to K_{m+n}^G(B_1 \otimes B_2).$$

Remark 2. The product in $K_G^*(X)$ can also be defined by means of the present section. The product of the triples (χ_1, ψ_1, f_1) and (χ_2, ψ_2, f_2) is

$$(\chi_1\otimes\chi_2,\ \psi_1\otimes\psi_2,\ M(x)\,(f_1(x)\hat{\otimes}1) + N(x)\,(1\hat{\otimes}f_2(x))),$$

where $M(x)$ and $N(x)$ are constructed by Lemma 3. This definition coincides with the definition given in [6] in view of the fact that one has a homotopy $M_t(F_1 \hat{\otimes} 1) + N_t(1 \hat{\otimes} F_2)$ of the operator $M(F_1 \hat{\otimes} 1) + N(1 \hat{\otimes} F_2)$ to the operator $F_1 \hat{\otimes} 1 + 1 \hat{\otimes} F_2$: $M_t = t + (1-t)M$ and $N_t = t + (1-t)N$. For finite-dimensional elements of $K_G^*(X)$ (cf. Remark 3 of §1) the product has the form

$$\begin{aligned}(\xi, f) \otimes (\eta, g) &= (\xi \otimes \eta, f \hat{\otimes} \sqrt{1+g^2} + 1 \hat{\otimes} g) \\ &= (\xi \otimes \eta, f \hat{\otimes} 1 + \sqrt{1+f^2} \hat{\otimes} g).\end{aligned}$$

(We assume that $f^* = -f$, $\|f\| \le 1$, $g^* = -g$ and $\|g\| \le 1$. This is achieved by a homotopy.)

Remark 3. The method described for constructing the product is borrowed from the theory of elliptic operators. We note that the analytic index with values in K_*^G is also defined for operators D of order 1; it suffices to choose a positive elliptic operator $\Lambda\colon H^0(\xi) \to H^{-1}(\xi)$ (here H^k is a Sobolev space) which is invariant relative to G and $C_{p,q+1}$, and set

$$\operatorname{ind}^a_{p,q}(D) = \operatorname{ind}^a_{p,q}(\Lambda^{-1}D).$$

Up to homotopy, nothing depends on the choice of Λ. Now let D_1 and D_2 be elliptic operators of order 1 in the bundles ξ_1 and ξ_2 over manifolds X_1 and X_2. Then

$$\mathrm{ind}^{a}_{p+p',q+q'}(D_1 \# D_2) = \mathrm{ind}^{a}_{p,q}(D_1) \# \mathrm{ind}^{a}_{p',q'}(D_2) \in K^{G_1\times G_2}_{p+p',q+q'}(X_1 \times X_2).$$

In fact, we set

$$F_1 = \Lambda_1^{-1} D_1, \quad F_2 = \Lambda_2^{-1} D_2;$$
$$\Lambda = (\Lambda_1^2 \otimes 1 + 1 \otimes \Lambda_2^2)^{1/2}: H^0(\xi_1 \otimes \xi_2) \to H^{-1}(\xi_1 \otimes \xi_2).$$

By definition, $D_1 \# D_2 = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$. Reducing this operator to an operator of order 0 with the help of left multiplication by Λ^{-1}, we get

$$\Lambda^{-1}(D_1 \# D_2) = M(F_1 \hat{\otimes} 1) + N(1 \hat{\otimes} F_2),$$

where $M = (\Lambda_1 \otimes 1)\Lambda^{-1}$ and $N = (1 \otimes \Lambda_2)\Lambda^{-1}$.

Theorem 5 (cf. [3]) *The index of the elliptic operator of Example 6 of §2 with values in the ring of representations $RU(O(n))$ or $RO(O(n))$ is equal to 1. If the "real" involution and "real" group G act on D^k linearly, then the "real" operator whose symbol in $KR^0_G(T\widetilde{D}^k)$ is given by the complex exterior algebra has index $1 \in RR(G)$.*

Proof. Using the multiplicative property of the index, we conclude that the $G_1 \times G_2$-index of the operator $F_1 \hat{\otimes} 1 + 1 \hat{\otimes} F_2$ is equal to the product (in the ring $R(G_1 \times G_2)$) of the indices of F_1 and F_2. In view of Remarks 2 and 3 this is also true for the product of elliptic operators. An element of the ring $R(O(n))$ is uniquely determined by its restriction to subgroups of the form $\Pi_i G_i$, where $G_i = O(1)$ or $SO(2)$.

In view of the multiplicativity of the complex exterior algebra the symbol of our operator splits into a product $\Pi_i \sigma_i$, where $\sigma_i \in K^0_{G_i}(TR^{n_i})$. The assertion relative to $RU(O(n))$ and $RO(O(n))$ now follows from Examples 3 and 6 of §2.

In the "real" case it is necessary to multiply by operators constructed in Examples 3 and 5 of §2 in order for everything to be reduced to the case $k = 2n$, $G = U(n)$. In this case the assertion follows from what has been proved, in view of the existence of the isomorphism $RR(U(n)) \simeq RU(U(n))$. The theorem is proved.

Supplement to §4

It suffices to prove Theorem 1 in the complex case, since in the real one the desired M and N are obtained by "averaging" with respect to complex conjugation. Theorem 1 follows immediately from the two theorems proved below (Theorems 6 and 7).

Definition 7. We denote by K^+ the set of selfadjoint positive compact operators with zero kernel in H. If $\Omega \subset L(H)$, then by $K^+(\Omega)$ we denote the set of those $k \in K^+$ for which

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} \|\alpha(k+\alpha)^{-1}F - F\alpha(k+\alpha)^{-1}\| = 0$$

for all $F \in \Omega$ (the limit is taken over positive numbers α).

Lemma 4. *If $k \in K^+$, then as $\alpha \to 0$ the operator $\alpha(K + \alpha)^{-1}$ tends strongly to 0. For any $l \in K(H)$*

$$\lim_{\alpha\to 0} \left\| \frac{\alpha}{k+\alpha} l \right\| = \lim_{\alpha\to 0} \left\| l \frac{\alpha}{k+\alpha} \right\| = 0.$$

Proof. The first assertion is verified easily if one uses a basis $\{e_i\} \subset H$ in which $k(e_i) = \lambda_i e_i$ ($\lambda_i > 0$, $\lim_{i\to\infty}\lambda_i = 0$). The second follows from the first. The lemma is proved.

Theorem 6. *Let* $h \in K^+(\Omega_1)$ *and* $k \in K^+(\Omega_2)$. *Set*

$$M = (1 \otimes k)(1 \otimes k + h \otimes 1)^{-1}, \quad N = (h \otimes 1)(1 \otimes k + h \otimes 1)^{-1}.$$

Then the pair $M_0 = M(M^2 + N^2)^{-1/2}$, $N_0 = N(M^2 + N^2)^{-1/2}$ *is an element of* $S^+(\Omega_1; \Omega_2)$.

Proof. On the subspace $H \otimes e_i \subset H \otimes H$ the operator $M = \lambda_i(\lambda_i + h)^{-1}$ is bounded, and its norm is ≤ 1. This means it can be extended from $\bigoplus_{i=1}^{\infty}(H \otimes e_i)$ to $H \otimes H$. A similar result holds for N, and $M + N = 1$. Points 3° and 4° of Definition 4 of §4 are first verified for operators M and N (for example,

$$\forall\, l \in K(H) \ \ \|(l \otimes 1) \cdot M|_{H\otimes e_i}\| = \left\| l\, \frac{\lambda_i}{\lambda_i + h} \right\| \to 0 \text{ as } i \to \infty$$

by Lemma 4). For M_0 and N_0 points 1° and 2° of Definition 4 are obvious; points 3° and 4° are obtained from what was proved for M and N. The theorem is proved.

Theorem 7. *If* Ω *is a finite set, then* $K^+(\Omega) \neq \varnothing$.

Proof. In the complex case each element of $L(H)$ can be represented as a linear combination of four unitary ones; hence one can assume $\Omega \subset L^U(H)$. Let $\Omega = \{u_1, \ldots, u_n\}$. We choose an orthonormal basis $\{e_j\}_{j\in Z}$ and denote by u_0 the operator determined by $u_0(e_j) = e_{j+1}$. We set

$$V_{-1} = 0, \ V_0 = \mathbf{C} \cdot e_0, \quad V_{k+1} = V_k + \sum_{i=0}^{n} u_i(V_k) + \sum_{i=0}^{n} u_i^{-1}(V_k).$$

Obviously, $V_k \subset V_{k+1}$, $V_k \subset u_i(V_{k+1})$ and $u_i(V_k) \subset V_{k+1}$.

We denote by $V_k \ominus V_l$ the orthogonal complement of V_l in V_k. If $x_k \in V_k \ominus V_{k-1}$, then $u_i(x_k) \in u_i(V_k)$ and $u_i(x_k) \perp u_i(V_{k-1})$, so $u_i(x_k) \in V_{k+1}$ and $u_i(x_k) \perp V_{k-2}$, i.e. $u_i(x_k) \in V_{k+1} \ominus V_{k-2}$. This means that $u_i(x_k) = y_k + z_k + t_k$, where $y_k \in V_{k+1} \ominus V_k$, $z_k \in V_k \ominus V_{k-1}$ and $t_k \in V_{k-1} \ominus V_{k-2}$.

We set $h(x_k) = x_k/k$ for $x_k \in V_k \ominus V_{k-1}$. Since $\bigoplus_0^\infty V_k$ is dense in H (the basis $\{e_j\} \subset \bigoplus_k V_k$), h extends to an element of K^+. We shall verify that $h \in K^+(u_0, \ldots, u_n)$. If $x_k \in V_k \ominus V_{k-1}$, then

$$\left(\frac{\alpha}{h+\alpha} u_i - u_i \frac{\alpha}{h+\alpha}\right)(x_k) = \lambda_k y_k + \mu_k t_k,$$

where

$$\lambda_k = \alpha\left(\frac{1}{k+1} + \alpha\right)^{-1} - \alpha\left(\frac{1}{k} + \alpha\right)^{-1}, \quad \mu_k = \alpha\left(\frac{1}{k-1} + \alpha\right)^{-1} - \alpha\left(\frac{1}{k} + \alpha\right)^{-1}.$$

Obviously $|\lambda_k| \leq \alpha$ and $|\mu_k| \leq \alpha$. An arbitrary element $x \in H$ can be written in the form $\sum_0^\infty x_k$, $x_k \in V_k \ominus V_{k-1}$. Here

$$\|x\|^2 = \|u_i(x)\|^2 = \sum_{k=0}^{\infty} \|u_i(x_k)\|^2 = \sum_{k=0}^{\infty} \|y_k\|^2 + \sum_{k=0}^{\infty} \|z_k\|^2 + \sum_{k=0}^{\infty} \|t_k\|^2.$$

This means that

$$\left\| \left(\frac{a}{h+a} u_i - u_i \frac{a}{h+a}\right)(x) \right\| \leqslant \left\| \sum_{k=0}^{\infty} \lambda_k y_k \right\| + \left\| \sum_{k=0}^{\infty} \mu_k t_k \right\| \leqslant 2\alpha \| x \|,$$

since for $k \neq l$ we have $y_k \perp y_l$ and $t_k \perp t_l$. The theorem is proved.

§5. Homotopy and excision axioms

The homology K-functor we have constructed satisfies the Eilenberg-Steenrod axioms. The homotopy and excision axioms are verified in this section (Theorems 1 and 2), the exactness axiom in §6.

Definition 1. By a *homotopy of homomorphisms* $\omega_t \colon B' \to B$ $(0 \le t \le 2\pi)$ is meant a homomorphism $[\omega_t]\colon B' \to C([0, 2\pi], B)$ whose composition with the restriction $C([0, 2\pi], B) \to C(t, B) = B$ coincides with ω_t for all t. By the *integral* of a homotopy $\phi_t\colon B' \to L(H)$ is meant the homomorphism $\int_0^{2\pi} \phi_t dt\colon B' \to L(H \otimes L^2[0, 2\pi])$, which associates with the element $b \in B'$ the element

$$\int_{[0,2\pi]} (\varphi_t(b) \otimes 1)\, d(1 \otimes \mathrm{id}), \text{ where } \mathrm{id}: C[0, 2\pi] \to L(L^2[0, 2\pi]).$$

Theorem 1. *If the homomorphisms $\omega_0, \omega_{2\pi}\colon B' \to B$ are homotopic, the mappings $\omega_0^*, \omega_{2\pi}^*\colon K^G_{p,q}(B) \to K^G_{p,q}(B')$ coincide. The analogous assertion is valid in the relative case.*

Proof. It will be assumed that the homotopy $\omega_t\colon B' \to B$ is constant for $0 \le t \le 2\pi/3$ and $4\pi/3 \le t \le 2\pi$. Let $x = (\chi, \phi, \psi, F) \in \mathcal{E}^G_{p,q}(B)$ and

$$a_1 = \left(1\colon C \to L(H),\ \psi(\epsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ F' = \begin{pmatrix} 0 & -T^*(b) \\ T_1(b) & 0 \end{pmatrix}\right) \in \mathcal{E}_{0,0}(C)$$

(for the definition of $T_1(b)$ cf. Example 4 of §2; we shall choose the function b below). By Theorem 4 of §4 the element a_1 is a multiplicative unit. We set

$$\Phi(b) = F \# F' = \begin{pmatrix} M(F \otimes 1) & -N(1 \otimes T_1^*(b)) \\ N(1 \otimes T_1(b)) & -M(F \otimes 1) \end{pmatrix}.$$

Theorem 1 of §4 allows one to assume that M and N commute modulo compact operators with $1 \otimes d$ and $1 \otimes C[0, 2\pi]$. We shall show that the element

$$\left(\chi \otimes 1, \int_0^{2\pi} (\varphi \cdot \omega_t)\, dt, \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix} \otimes 1, \Phi(b)\right) \in K_{p,q}(B')$$

for some choice of b coincides with $(\chi, \phi \cdot \omega_0, \psi, F)$, and for some other choice with $(\chi, \phi \cdot \omega_{2\pi}, \psi, F)$.

First let

$$b = \begin{cases} \cos 3t, & 0 \leqslant t \leqslant \dfrac{\pi}{3}, \\ -1, & \dfrac{\pi}{3} \leqslant t \leqslant 1. \end{cases}$$

Then $T_1(b)$ commutes with the projection onto $L^2[0, 2\pi/3]$, while on $L^2[0, 2\pi/3]$ it is equal to an operator of index 1, and on $L^2[2\pi/3, 2\pi]$ it is multiplication by -1. We choose $\alpha(t) \in C[0, 2\pi]$ so that $0 \leq \alpha(t) \leq 1$; $\alpha(t) = 0$ for $t \leq \pi/3$ and $\alpha(t) = 1$ for $t \geq 2\pi/3$. The operator $\Phi(b)$ is homotopic to

$$\begin{pmatrix} M(F \otimes \sqrt{1-\alpha}) & -\sqrt{1 \otimes \alpha + N^2(1 \otimes (1-\alpha))}(1 \otimes T_1^*(b)) \\ \sqrt{1 \otimes \alpha + N^2(1 \otimes (1-\alpha))}(1 \otimes T_1(b)) & -M(F \otimes \sqrt{1-\alpha}) \end{pmatrix}.$$

The latter commutes with projection onto $L^2[0, 2\pi/3]$, while on $L^2[0, 2\pi/3]$ it represents the product

$$F \# \begin{pmatrix} 0 & -T_1^* \\ T_1 & 0 \end{pmatrix},$$

where T_1 is an operator of index 1, and on $L^2[2\pi/3, 2\pi]$ it is equal to $\begin{pmatrix} 0 & 1 \\ -0 & 0 \end{pmatrix}$. This proves the first part of our assertion.

The second part is obtained if one sets

$$b = \begin{cases} 1, & 0 \leqslant t \leqslant \frac{5\pi}{3}, \\ -\cos 3t, & \frac{5\pi}{3} \leqslant t \leqslant 2\pi. \end{cases}$$

The theorem is proved.

Theorem 2. *The excision axiom holds: if U is open in X, and $\bar{U} \subset \operatorname{Int} Y$ (the interior of Y), then*

$$K^G_{p,q}(X \setminus U, Y \setminus U; B) \to K^G_{p,q}(X, Y; B)$$

is an isomorphism.

Proof. We shall use Remark 3 of §3. To prove that it is an epimorphism it suffices to change the quadruple $(\chi, \phi, \psi, F_s) \in \mathfrak{E}^G_{p,q}(C(X, B) \to C(Y, B))$ so that it becomes degenerate outside $X \setminus U$. To prove that it is a monomorphism it is necessary to change a homotopy between quadruples of this form to a homotopy which is degenerate outside $X \setminus U$. The idea of the proof is that it is necessary to "crush" the homotopy $\{F_s\}$ on the set U to the operator F_1, which is degenerate. For convenience of notation it will be assumed that s varies from 0 to 2π (dilatation of the segment).

First we prove that it is an epimorphism. Let the function $\beta(x) \in C(X)$ be G-invariant, $0 \leq \beta(x) \leq 1$, $\beta(x) = 0$ for $x \in X \setminus Y$ and $\beta(x) = 1$ for $x \in U$. For $0 \leq r \leq 2\pi$ we set

$$f_r(x, t) = \begin{cases} \cos\left[3t - \frac{5}{2}(r + (2\pi - r)\beta(x))\right], & \frac{5}{6}(r + (2\pi - r)\beta(x)) \leqslant t \\ & \leqslant \frac{5}{6}(r + (2\pi - r)\beta(x)) + \frac{\pi}{3}, \\ 1, & t \leqslant \frac{5}{6}(r + (2\pi - r)\beta(x)), \\ -1, & t \geqslant \frac{5}{6}(r + (2\pi - r)\beta(x)) + \frac{\pi}{3}. \end{cases}$$

Let f_r be the operator

$$\int_{[0,2\pi]} [\varphi(f_r(x,t))\otimes 1]\, d(1\otimes \mathrm{id})$$

in the space $H \otimes L^2[0, 2\pi]$, and set

$$T_1(f_r) = f_r - \sqrt{1-f_r^2}\,(1\otimes d).$$

We construct in $L(H \otimes L^2[0, 2\pi])$ a family of pairs $(M_{s,r}, N_{s,r})$, satisfying conditions 1–4 of Definition 3 of §4 ($G_1 \times G_2$ is replaced by $G \times 1$, and $C_{p+p',q+q'+1}$ by $C_{p-1,q+1}$), and also the following conditions:

5′. $(M_{s,r}, N_{s,r})$ commutes modulo compact operators with $\operatorname{Im}\phi \otimes 1$, $1 \otimes C[0, 2\pi]$, $F_s \otimes 1$ and $1 \otimes d$.

6′. $M_{s,r}(F_s \otimes 1)$ and $N_{s,r} \cdot T_1(f_r)$ commute modulo compact operators with $\operatorname{Im}\phi \otimes 1$ and $1 \otimes C[0, 2\pi]$.

7′. $M_{s,r}^2(F_s\otimes 1)^2 \sim -M_{s,r}^2;\qquad N_{s,r}^2T_1(f_r)T_1^*(f_r) \sim N_{s,r}^2T_1^*(f_r)T_1(f_r) \sim N_{s,r}^2;$

$$M_{s,r}[(F_s+F_s^*)\otimes 1] \sim 0.$$

For $(M_{s,r}, N_{s,r}) \in S^+$ points 6′ and 7′ follow from 5′, since $f_r(x, t)$ is approximated by a sum of functions of the form $a(x) \cdot b(t)$.

8′. For $s = 0$

$$(M_{0,r}, N_{0,r}) = (M'_{0,r}, N'_{0,r}) \oplus (M''_{0,r}, N''_{0,r})$$

(cf. Definition 6 of §4). The pairs $(M''_{0,r}, N''_{0,r})$ and $(M_{2\pi,r}, N_{2\pi,r})$ are degenerate.

Let $\alpha(t) \in C(-\infty, +\infty)$: $0 \le \alpha(t) \le 1$; $\alpha(t) = 0$ for $t \le -\pi/6$; $\alpha(t) = 1$ for $0 \le t \le \pi/3$; and $\alpha(t) = 0$ for $t \ge \pi/2$. We define for $0 \le r \le 2\pi$ a function $\gamma_r(x, t) \in C(X \times [0, 2\pi])$ by the formula

$$\gamma_r(x,t) = \alpha\left(t - \frac{5}{6}(r + (2\pi - r)\beta(x))\right).$$

By $\gamma_r \in L(H \otimes L^2[0, 2\pi])$ we denote the operator

$$\int_{[0,2\pi]} (\varphi(\gamma_r(x,t))\otimes 1)\cdot d(1\otimes \mathrm{id}).$$

We replace $(M_{s,r}, N_{s,r})$ by

$$\left(M_{s,r}\sqrt{\gamma_r},\ \sqrt{(1-\gamma_r) + N_{s,r}^2\cdot\gamma_r}\right).$$

We denote the new pair again by $(M_{s,r}, N_{s,r})$. In addition to properties 1–4 and 5′–8′, it has the following:

9′. For $r = 0$ it commutes with $P_\phi(X\setminus Y) \otimes P[\pi/2, 2\pi]$, where $P(a, b)$ is the projector onto $L^2[a, b]$, and is equal to $(0, 1)$ on $\operatorname{Im} P_\phi(X\setminus Y) \otimes P[\pi/2, 2\pi]$.

10′. For $r = 2\pi$ it commutes with $1 \otimes P[0, 3\pi/2]$ and is equal to $(0, 1)$ on $H \otimes L^2[0, 3\pi/2]$.

11′. For all r it commutes with $P_\phi(U) \otimes P[0, 3\pi/2]$ and is equal to $(0, 1)$ on $\operatorname{Im} P_\phi(U) \otimes P[0, 3\pi/2]$.

We denote the operator

$$\begin{pmatrix} M_{s,r}(F_s \otimes 1) & -N_{s,r}T_1^*(f_r) \\ N_{s,r}T_1(f_r) & -M_{s,r}(F_s \otimes 1) \end{pmatrix}$$

by $\Phi_{s,r}$. For fixed r, $\Phi_{s,r}$ is homotopic to

$$F_s \# \begin{pmatrix} 0 & -T_1^* \\ T_1 & 0 \end{pmatrix},$$

where T_1 is an operator of index 1, so the quadruple $x = (\chi, \phi, \psi, F_s)$ can be replaced by

$$y_r = \left(\begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix} \otimes 1, \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \otimes 1, \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix} \otimes 1, \Phi_{s,r} \right)$$

for $r = \text{const}$. It will be assumed that the homotopies F_s and $(M_{s,r}, N_{s,r})$ are constant in s for $s \leq 2\pi/3$ and $s \geq 4\pi/3$. Then the operator(2) $\int_{[0,2\pi]} \Phi_{s,r} ds$ for $r = 0$ is degenerate outside Y, for $r = 2\pi$ is simply degenerate, and for all r is degenerate outside $X \setminus U$. It is asserted that the quadruple y_r is homotopic to the quadruple

$$\left(\begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix} \otimes 1, \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \otimes 1, \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix} \otimes 1, \int_{[0,2\pi]} \Phi_{s,r}\, ds \,|\, 0 \leqslant r \leqslant 2\pi \right).$$

To prove this we replace $\{\Phi_{s,r}, 0 \leq s \leq 2\pi, r = \text{const}\}$ first by $\{\Phi_{r,r}, 0 \leq r \leq 2\pi\}$, and then this latter by $\{\int_{[0,2\pi]} \Phi_{s,r} ds, 0 \leq r \leq 2\pi\}$. For fixed r a homotopy of the operator $\Phi_{r,r}$ to $\int_{[0,2\pi]} \Phi_{s,r} ds$ is defined by the formula

$$\int_{[0,2\pi]} \Phi_{ps+(1-p)r,r}\, ds, \quad 0 \leqslant p \leqslant 1.$$

That it is an epimorphism is thus proved. To prove that it is a monomorphism one applies the same argument. It is only necessary to note that if the initial quadruple is degenerate outside $X \setminus U$, then all the transformations can be performed so as to preserve this property. The theorem is proved.

§6. The intersection index

The duality of the homology and cohomology K-functors is expressed by the existence of the intersection index. In this section intersection indices are constructed:

$$K_i^G(B_1 \otimes B_2) \cap K_G^0(B_2) \to K_i^G(B_1) \ [B_2 \text{ is a } C^*\text{-algebra}], \tag{1}$$

$$K_i^G(X; B) \cap K_G^j(X) \to K_{i-j}^G(X; B), \tag{2}$$

$$K_G^i(Y \times X) \cap K_j^G(X) \to K_G^{i-j}(Y), \tag{3}$$

$$K_i^G(X, Y \cup Z; B) \cap K_G^j(X, Y) \to K_{i-j}^G(X, Z; B). \tag{4}$$

With the help of these pairings the axiom of exactness and Bott periodicity are proved, and also the agreement on the category of cell complexes of our K-functor $K_*(X)$ with the Whitehead K-functor, which we denote by $K_*^{(t)}(X)$.

(2) Instead of $d(1 \otimes \text{id})$ we write here ds in order to indicate that the integral is with respect to s.

Definition 1. We shall construct (1). Let $x = (\chi, \phi, \psi, F) \in \mathcal{E}^G_{p,q}(B_1 \otimes B_2)$, and let the element $y \in K^0_G(B_2)$ be realized by the hermitian G-invariant idempotent p in the algebra of matrices $M_n(B_2)$. The restriction of ϕ to B_2 induces a homomorphism $\phi_n: M_n(B_2) \to L(H \otimes C^n)$. We denote $\operatorname{Im}\phi_n(p)$ by H'. We set

$$x \cap y = (\chi|_{H'}, \varphi|_{H'}, \psi|_{H'}, \varphi_n(p)F|_{H'}) \in \mathcal{E}^G_{p,q}(B_1).$$

Definition 2. We shall construct (2). Let

$$a = (\chi, \varphi, \psi, F) \in \mathcal{E}^G_{p,q}(C(X, B)), b = (\chi', \psi', f') \in \mathcal{E}^{p',q'}_G(X).$$

Using Lemma 3 and the method of the proof of Corollary 1 of §4, we construct in the Hilbert space $H \otimes H$ a continuous family of operators $F \# f'(x)$. We integrate it in $A(H \otimes H)$: $\int_X (F \# f'(x)) d(\phi \otimes 1)$; then we lift it to $L(H \otimes H)$ (Remark 1 of §3) and average over G (using Propositions 2 and 3 of §3). We set

$$a \cap b = (\chi \otimes \chi', \varphi \otimes 1, \psi \otimes \psi', \int_G (\chi \otimes \chi')(g) \left[\int_X (F \# f'(x)) d(\varphi \otimes 1)\right] \times (\chi \otimes \chi')(g^{-1}) dg) \in \mathcal{E}^G_{p+p',q+q'}(C(X, B)).$$

Definition 3. We construct (3). Let

$$a = (\chi, \psi, f) \in \mathcal{E}^{p,q}_G(Y \times X), \qquad b = (\chi', \varphi', \psi', F') \in \mathcal{E}^G_{p',q'}(C(X)).$$

We set

$$a \cap b = \left(\chi \otimes \chi', \psi \otimes \psi', \int_G \left[\int_X (f(y, gx) \# F') d(1 \otimes \varphi'_g)\right] dg\right) \in \mathcal{E}^{p+p',q+q'}_G(Y),$$

where $\phi'_g = \chi'(g) \cdot \phi' \cdot \chi'(g^{-1})$.

Definition 4. We construct (4). Let

$$a = (\chi, \varphi, \psi, F_t) \in \mathcal{E}^G_{p,q}(C(X, B) \to C(Y \cup Z, B)), \qquad b = (\chi', \psi', f') \in \mathcal{E}^{p',q'}_G(X, Y).$$

We construct (using the method of Theorem 3 of §4) a continuous family of operators $F_t \# f'(x)$, which for $t = 1$ is equal to $F_1 \hat{\otimes} 1$ and for $t = 0$ is equal to $(F'_0 \# f'(x)) \oplus (F''_0 \hat{\otimes} 1)$, where since f' is degenerate over Y one can assume that $F'_0 \# f'(x) = 1 \hat{\otimes} f'(x)$ for $x \in Y$. We set

$$a \cap b = (\chi \otimes \chi', \varphi \otimes 1', \psi \otimes \psi', \int_G (\chi \otimes \chi')(g) \cdot \left[\int_X (F_t \# f'(x)) d(\varphi \otimes 1)\right] \times (\chi \otimes \chi')(g^{-1}) dg) \in \mathcal{E}^G_{p+p',q+q'}(C(X, B) \to C(Y, B)).$$

(The degeneracy outside Z for $t = 0$ follows from point 7) of Theorem 1 of §3. The identification of $C_{p-1,q} \otimes C_{p',q'}$ with $C_{p+p'-1,q+q'}$ is carried out in correspondence with Definition 6 of §4.)

Proposition 1. *The pairings* (1)–(4) *are functorial, bilinear, and have two types of associative properties, which can be written briefly as follows (the multiplication in K_* or K^* is denoted by $\otimes$):*

1°. $(a \cap b) \cap c = a \cap (b \otimes c)$.

2°. $(a \otimes b) \cap c = a \otimes (b \cap c)$.

Proof. That they are functorial and bilinear is obvious; associativity is verified by analogy with Theorem 2 of §4. We omit the details.

Theorem 1. *$i_*: K^G_{p,q}(B) \to K^G_{p,q}(B, \varnothing)$ is an isomorphism.*

Proof. We define a homomorphism $j_*: K^G_{p,q}(B, \varnothing) \to K^G_{p,q}(B)$ as "intersection index" with the generator of the group $K_{1,0}(S^1)$, constructed in Example 2 of §2, i.e.

$$j_*(\chi, \varphi, \psi, F_t) = (\chi \otimes 1, \varphi \otimes 1, \psi \otimes \psi_1, \int_G (\chi \otimes 1)(g)$$
$$\times \left[\int_{[0,2\pi]} F_t \mathbin{\#} \begin{pmatrix} 0 & -d \\ -d & 0 \end{pmatrix} d(1 \otimes \varphi_1) \right] (\chi \otimes 1)(g^{-1})\,dg),$$

where $\phi_1: C(S^1) \to L(H')$, $\psi_1: C_{1,1} \to L(H')$ and $H' = L^2(S^1) \oplus L^2(S^1)$. (Throughout the proof we assume that all homotopies are defined for $0 \le t \le 2\pi$.) The existence of the integral in square brackets is obtained from Corollary 1 of §3, if one sets $X = [0, 2\pi]$ and $Y = \{0\} \cup \{2\pi\}$ and assumes that $F_t \mathbin{\#} \begin{pmatrix} 0 & -d \\ -d & 0 \end{pmatrix}$ is constant in t and degenerate in a neighborhood of the points 0 and 2π.

It suffices to verify two assertions.

1. $j_* \circ i_* = -1$. We have

$$j_* \circ i_*(\chi, \varphi, \psi, F) = \left(\begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix} \otimes 1, \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \otimes 1, \right.$$
$$\left. \tilde{\psi}|_{C_{p-1,q+1}} = \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix} \Big|_{C_{p-1,q+1}} \otimes 1, \tilde{\psi}(e_p) = (-1)^q \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tilde{F} \right),$$

where

$$\tilde{F} = M\alpha(F \otimes 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (-1)^q M\beta(\psi(e_p) \otimes 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$- N(1 \otimes d) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here we have written

$$M = \int_G (\chi \otimes 1)(g) \left[\int_{[0,2\pi]} M_t d(1 \otimes \varphi_1) \right] (\chi \otimes 1)(g^{-1})\,dg,$$
$$N = \int_G (\chi \otimes 1)(g) \left[\int_{[0,2\pi]} N_t d(1 \otimes \varphi_1) \right] (\chi \otimes 1)(g^{-1})\,dg,$$
$$\alpha = 1 \otimes \varphi_1\left(\sin \frac{t}{2}\right), \quad \beta = 1 \otimes \varphi_1\left(\cos \frac{t}{2}\right).$$

We define a homotopy of the operator $\tilde{F}$ to the operator

$$\tilde{F}' = M(F \otimes 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (-1)^q N\beta(\psi(e_p) \otimes 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - N\alpha(1 \otimes d) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

by means of homotopies of triples of "coefficients":

$$\{M\sqrt{(1-r)\alpha^2+r},\ (-1)^q\sqrt{N^2r+M^2(1-r)}\,\beta,\\ -N\sqrt{r\alpha^2+1-r}\},\quad 0\leqslant r\leqslant 1.$$

The isometry

$$u=\frac{1}{\sqrt{2}}\begin{pmatrix}(-1)^q & \psi(e_p)\\ \psi(e_p) & (-1)^q\end{pmatrix}\otimes 1$$

carries $\widetilde{\psi}(e_p)$ into

$$\widetilde{\psi}(e_p) \text{ в } \begin{pmatrix}\psi(e_p) & 0\\ 0 & -\psi(e_p)\end{pmatrix}\otimes 1,$$

and the operator $\widetilde{F}'$ into the operator

$$\widetilde{F}''=M(F\otimes 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+N\beta\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}-N\alpha(1\otimes d)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=F\,\#\begin{pmatrix}0 & T_1\\ -T_1^* & 0\end{pmatrix},$$

where T_1^* is an operator of index -1.

2. $i_*\circ j_*=-1$. We have

$$i_*\circ j_*(\chi,\varphi,\psi,F_s)=\left(\begin{pmatrix}\chi & 0\\ 0 & \chi\end{pmatrix}\right)\otimes 1,\begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix}\otimes 1,\begin{pmatrix}\psi & 0\\ 0 & -\psi\end{pmatrix}\otimes 1,$$

$$\widetilde{F}_t=\Big(\int_G(\chi\otimes 1)(g)\Big[\int_{[0,2\pi]}\Phi_s ds\Big](\chi\otimes 1)(g^{-1})\,dg\Big)\sin\frac{t}{2}+1\otimes\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cos\frac{t}{2},$$

where

$$\Phi_s=M_s(F_s\otimes 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-N_s(1\otimes d)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

We consider the family of operators

$$\Phi_{s,t}=\Phi_s\sin\frac{t}{2}+1\otimes\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cos\frac{t}{2},$$

parametrized by points of the square $[0,2\pi]^2$, and we shall show that $\int_{[0,2\pi]}\Phi_{s,t}ds$ is homotopic to $\int_{[0,2\pi]}\Phi_{s,2\pi-t}dt$. It suffices to map the square $[0,2\pi]^2$ onto the disk D^2 with the help of the projection from the center and to note that under rotation by the angle $\pi/2$ the system of curves $t=\text{const}$, over which the first integral is taken, is carried into the system of curves $s=\text{const}$, over which the second integral is taken. The desired homotopy is given by the collection of rotations by angles $0\leq\theta\leq\pi/2$. That integration over curves in the disk D^2 is well-defined is guaranteed by Corollary 1 of §3 (one easily sees that the operators $\Phi_{s,t}$ were degenerate in a neighborhood of the boundary of the square $[0,2\pi]^2$). Denoting $1\otimes\phi_1(\sin(t/2))$ by α and $1\otimes\phi_1(\cos(t/2))$ by β, we get

$$\int_{[0,2\pi]}\Phi_{s,1-t}\,dt=M_s\alpha(F_s\otimes 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-N_s\alpha(1\otimes d)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\beta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

The homotopy of triples of "coefficients"

$$\left\{M_s\sqrt{r+(1-r)\alpha^2},\ -N_s\alpha,\ -\sqrt{rN_s^2+1-r}\,\beta\right\},\quad 0\leqslant r\leqslant 1,$$

carries our operator into the operator

$$M_s(F_s\otimes 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-N_s\alpha\,(1\otimes d)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-N_s\beta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

which coincides with

$$M_s \# \begin{pmatrix}0 & T_1\\ -T_1^* & 0\end{pmatrix},$$

where T_1^* is an operator of index -1. The theorem is proved.

Remark 1. With the help of the methods of Karoubi [7], Theorem 1 can be proved without using the fact that the algebra B is finitely generated.

Theorem 2. *Let $\omega\colon B_2\to B_1$ be an epimorphism. There is an exact sequence*

$$\ldots\to K_{p,q}(B_1)\xrightarrow{\omega_1^*}K_{p,q}(B_2)\to K_{p,q}(\omega)\xrightarrow{\partial}K_{p-1,q}(B_1)\xrightarrow{\omega^*}K_{p-1,q}(B_2)\to\ldots.$$

The proof is obtained by the juxtaposition of Theorem 1 with Theorem 1 of §1.

Remark 2. In the pairings (4) and (2) an element of $K_G^j(X, Y)$ (respectively of the image of $K_G^j(X, Y)\to K_G^j(X)$) can have the form indicated in Remark 3 of §1. In this case the pairings (4) and (2) are constructed as follows. Let $a=(\chi, \phi, \psi, F)$ and $b=(\xi, f)$, and denote the action of G and $C_{p',q'+1}$ on ξ by χ' and ψ'. One can assume that $f^*=-f$ and $\|f\|\leq 1$. In the tensor product $H'=H\otimes_{C(X)}C(\xi)$, where $C(\xi)$ is the module of continuous sections of ξ over X, we introduce a metric with the help of a scalar product:

$$(h\otimes s, h'\otimes s')_{H'}=(h, \varphi(s, s')h')_H.$$

Here $(s, s')\in C(X)$ is the scalar product of s and s' in the sense of a riemannian metric on ξ. The completeness of H' follows from the projectivity and finite generation of $C(\xi)$ over $C(X)$. We set

$$a\cap b=(\chi\otimes\chi',\ \varphi\otimes 1,\ \psi\otimes\psi',$$

$$\int_G(\chi\otimes\chi')(g)\left[\int_X\left(F\hat{\otimes}\sqrt{1+f^2(x)}+1\hat{\otimes}f(x)\right)d(\varphi\otimes 1)\right](\chi\otimes\chi')(g^{-1})dg).$$

In this formula it is necessary to make more precise what is meant by $F\hat{\otimes}1$. We choose an open covering $X=\bigcup_i U_i$, trivializing ξ, and a partition of unity $\Sigma_i\alpha_i^2=1$. If $H_i=\operatorname{Im}P_\phi(U_i)$, where P_ϕ is the spectral measure of ϕ, then $\operatorname{Im}P_{\phi\otimes 1}(U_i)=H_i\otimes_{C(X)}C(\xi)=H_i\otimes C$, where $n=\dim\xi$. The operator $F\hat{\otimes}1=\Sigma_i\alpha_iF_{\alpha_i}\otimes 1$ modulo compact operators is independent of the arbitrariness in the construction.

Lemma 1. *Let F be an elliptic operator on the manifold Y (acting on the sections of bundle ξ), let $\operatorname{ind}_i^a(F)\in KU_i^G(Y, \partial Y)$ be its relative index, and let the element $(\eta, f)\in KU_G^j(Y, \partial Y)$. Then $\operatorname{ind}_i^a(F)\cap(\eta, f)\in KU_{i-j}^G(Y)$ is the absolute index of the elliptic operator with symbol $\sigma_{-i}(F)\otimes(\eta, f)\in KU_G^{j-i}(BY, SY\cup BY|_{\partial Y})$. An analogous assertion holds in the "real" case.*

Proof. The desired operator $F \# f$ (acting on sections of the bundle $\xi \otimes \eta$) is constructed with the help of a trivialization of the bundle η and a corresponding smooth partition of unity by the formula

$$F \# f = F \hat{\otimes} \sqrt{1+f^2(x)} + 1 \hat{\otimes} f(x).$$

If $f^2 = -1$ in a neighborhood of the boundary, then $F \# f$ is degenerate on the boundary. Remark 2 of this section and Remark 2 of §4 show that the index and the symbol of this operator are constructed just as asserted. The lemma is proved.

Lemma 2. *If $\alpha \in KR_0^{U(n)}(D^{2n}, S^{2n-1})$ is the element constructed in Example 7 of §2 and $\beta \in KR^0_{U(n)}(D^{2n}, S^{2n-1})$ is the canonical element defined by the complex exterior algebra, then $\alpha \cap \beta = 1 \in RR(U(n))$.*

Proof. Let

$$f: KR_0^{U(n)}(D^{2n}) \to KR_0^{U(n)}(D^{2n}, S^{2n-1}), \; g: KR^0_{U(n)}(D^{2n}, S^{2n-1}) \to KR^0_{U(n)}(D^{2n}).$$

In the group $KR_0^{U(n)}(D^{2n})$ one has a distinguished element (cf. Example 6 of §2 and Theorem 5 of §4), which we denote by γ. It is easy to verify (integrating over D^{2n} using Corollary 1 of §3), that $f(\gamma) \cap \beta = \gamma \cap g(\beta)$. The right side is equal to $g(\beta)$, if one considers $g(\beta)$ as an element of $RR(U(n))$.

The symbol σ of the elliptic operator γ in the group

$$KR^0_{U(n)}(D^{2n} \times D^{2n}, S^{2n-1} \times D^{2n} \cup D^{2n} \times S^{2n-1})$$

is equal to $\beta \otimes \beta$, and its image under the homomorphism into

$$KR^0_{U(n)}(D^{2n} \times D^{2n}, D^{2n} \times S^{2n-1}) = KR^0_{U(n)}(TD^{2n})$$

is equal to $g(\beta) \otimes \beta$. This means that the symbol of the elliptic operator $f(\gamma)$ is equal to $g(\beta) \otimes \beta$. Since the symbol of the operator α is equal to $1 \otimes \beta$, from the coincidence of the symbols (cf. Lemma 1) we get $g(\beta)(\alpha \cap \beta) f(\gamma) \cap \beta = g(\beta)$ (here $g(\beta)$ is again considered as an element of $RR(U(n))$). The element $g(\beta)$ is not equal to 0 in $RR(U(n))$ (under the restriction $RR(U(n)) \to RR(T^n)$ it goes into $\Pi_1^n(1-\alpha_i)$, where α_i are canonical one-dimensional representations), so $\alpha \cap \beta = 1$. The lemma is proved.

Lemma 3. *Let $a \in K_{1,0}(S^1)$ be the element constructed in Example 2 of §2, and let $b = (\xi, f) \in K^{0,1}(D^1, S^0)$, where ξ is a trivial two-dimensional bundle with action of $C_{0,2}$:*

$$C_{0,2}: \psi'(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \psi'(\varepsilon_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; f = \begin{pmatrix} 0 & \cos\frac{\theta}{2} \\ -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

(Here $D^1 = [0, 2\pi]$.) Then $a \cap b = 1$.

Proof. The element $a \cap b \in K_{1,1}(\text{point})$ is defined by the graded representation

$$\psi: C_{1,1} \to L(H \otimes C^2): \psi(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{\otimes} 1, \; \psi(\varepsilon_1) = 1 \hat{\otimes} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the Fredholm operator

$$F=\begin{pmatrix}0 & -d\\ -d & 0\end{pmatrix}\sin\frac{\theta}{2}\hat{\otimes}1+\cos\frac{\theta}{2}\hat{\otimes}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

The automorphism

$$u=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\hat{\otimes}1+1\hat{\otimes}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]$$

carries (ψ, F) into $(\widetilde{\psi}, \widetilde{F})$, where

$$\widetilde{\psi}(e_1)=1\hat{\otimes}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \widetilde{\psi}(\varepsilon_1)=-1\hat{\otimes}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\widetilde{F}=\begin{pmatrix}0 & d\\ d & 0\end{pmatrix}\sin\frac{\theta}{2}\hat{\otimes}1+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cos\frac{\theta}{2}\hat{\otimes}1=\begin{pmatrix}0 & T_1^*\\ -T_1 & 0\end{pmatrix}\hat{\otimes}1.$$

The pair $(\widetilde{\psi}, \widetilde{F})$ is obtained from $1\in K_{00}(\text{point})$ under the periodicity isomorphism (Theorem 2 of §1). The lemma is proved.

Theorem 3 (Bott periodicity). *If a representation $G\to U(n)$ is fixed, then multiplication by the element $\alpha\in KU_0^G(D^{2n}, S^{2n-1})$ defines an isomorphism α_*: $KU_i^G(B)\to KU_i^G(D^{2n}, S^{2n-1}; B)$.*

The analogous assertion is valid in the "real" case. In addition, multiplication by the element $i_*(a)\in K_1(D^1, S^0)$ defines an isomorphism $K_i^G(B)\to K_{i+1}^G(D^1, S^0; B)$.

Proof. We define the inverse homomorphism β_*: $KU_i^G(D^{2n}, S^{2n-1}; B)\to KU_i^G(B)$ as the intersection index with the element $\beta\in KU_G^0(D^{2n}, S^{2n-1})$. It follows from Proposition 1 and Lemma 2 that $\beta_*\alpha_*=1$. We denote by $\bar{\alpha}$ the element of the group

$$\operatorname{Ker}[KU_0^G(S^{2n})\to KU_0^G(\text{point})]=KU_0^G(S^{2n}, \text{point}),$$

corresponding to α under excision (here $S^{2n}=D^{2n}/S^{2n-1}$), and by α_*, multiplication by $\bar{\alpha}$. It is sufficient to verify that $\alpha_*\beta_*=\beta_*\bar{\alpha}_*(=1)$.

Writing out explicitly for $x\in KU_i^G(D^{2n}, S^{2n-1}; B)$ the definition of the elements $(x\cap\beta)\otimes\alpha$ and $(x\otimes\bar{\alpha})\cap\beta$, it is easy to see that they differ only in which of the two disks of the product $D^{2n}\times D^{2n}$ the integration goes over. Just as in the proof of Theorem 1, mapping $D^{2n}\times D^{2n}$ onto D^{4n}, it is not hard to construct a homotopy consisting of rotations. The last assertion of the theorem is verified analogously. The theorem is proved.

Corollary 1. $K_{n+1}(X\times S^1; B)\simeq K_n(X; B)\oplus K_{n+1}(X; B)$.

The proof is a simple exercise.

Remark 3. The periodicity theorem in K^*-theory has nowhere been used by us up to now and can be proved completely analogously to the proof of Theorem 3. The only difference of our proof from the proof of Atiyah [1] consists in the fact that references to Kodaira's theorems are replaced by the simple Lemma 2. There is also a spinor periodicity. The needed element of $KO_0^{\mathrm{Spin}(8n)}(D^{8n}, S^{8n-1})$ is constructed in [1].

Definition 5. We recall the (topological) definition of the homology K-functor given by Whitehead [16]. In [6] a classifying space was constructed for the functor K^n; we shall denote it by $\mathcal{F}_n$. It is a subspace of the space of Fredholm skew-hermitian operators in H which anticommute with the representation $C_{p,q+1} \to L(H)$, $q-p=n$. Let $\mathcal{F}_n \wedge S^1 \to \mathcal{F}_{n+1}$ be the homotopy equivalence defined in [6]. For a CW-pair (X, Y) we set

$$K_n^{(t)}(X, Y) = \varinjlim_k \pi_{n+k}(\mathcal{F}_k \wedge X/Y).$$

Theorem 4. *On the category of CW-complexes there is an isomorphism of homology theories* $t\colon K_*^{(t)} \to K_*$.

Proof. We fix in the groups $\tilde{K}_n(S^n) = K_n(D^n, S^{n-1})$ canonical generators α_n. Let the map $f\colon S^{n+i} \to \mathcal{F}_i \wedge X/Y$ define some element of $K_n^{(t)}(X, Y)$. We choose a finite subcomplex $Z \subset \mathcal{F}_i$ such that f passes through $Z \wedge X/Y \to \mathcal{F}_i \wedge X/Y$. The intersection index of the element

$$f_*(\alpha_{n+i}) \in \tilde{K}_{n+i}(Z \wedge X/Y) \subset \tilde{K}_{n+i}(Z \times X/Y)$$

with the element $[g\colon Z \to \mathcal{F}_i] \in K^i(Z)$ in composition with the projection

$$\tilde{K}_n(Z \times X/Y) \to \tilde{K}_n(X/Y) = \quad K_n(X, Y)$$

carries $f(\alpha_{n+i})$ into some element $h \in K_n(X, Y)$. We set $t(f) = h$.

The independence of the construction from the choice of Z is obvious. Increasing the number i by one does not change h in view of Lemma 3 (the map $\mathcal{F}_n \wedge S^1 \to \mathcal{F}_{n+1}$ actually coincides with exterior multiplication by an element $b \in K^{01}(D^1, S^0)$). It is not hard to verify that t is an isomorphism on homology points. The theorem is proved.

§7. The Atiyah-Singer index theorem

The Atiyah-Singer theorem establishes the coincidence of the analytic and topological indices of an elliptic operator as elements of $R(G)$. In this section we shall prove the coincidence of indices as elements of $K_*^G(X)$. The complex and "real" cases are completely analogous.

Definition 1. We construct the topological index. We (equivariantly) imbed the manifold X in $\mathbf{R}^k$, and we denote by N a closed tubular neighborhood, and by $\tilde{N}$ its interior. $T\tilde{N}$ is identified with $p^*(\tilde{N} \otimes_{\mathbf{R}} \mathbf{C})$. The Thom isomorphism $KR_G^{q-p}(TX) \to KR_G^{q-p}(T\tilde{N})$ carries the symbol $\sigma_{q-p}(F)$ into some element $\sigma \in KR_G^{q-p}(T\tilde{N})$. We denote by f the composition

$$S^{2k} \to BN/(SN \cup BN\,|_{\partial N}) \xrightarrow{\Delta} (BN/(SN \cup BN\,|_{\partial N})) \wedge BN^+,$$

where Δ is the diagonal map and $BN^+ = BN \cup$ (point). If $\alpha \in K\tilde{R}_0^G(S^{2k})$ is the canonical generator, then the projection of the element

$$f_*(\alpha) \cap \sigma \in KR_{p-q}^G((BN/(SN \cup BN\,|_{\partial N})) \times BN)$$

in the group $KR_{p-q}^G(BN) = KR_{p-q}^G(X)$ is called the *topological index* of the operator F, and denoted by $\operatorname{ind}_{p-q}^t(F)$.

Lemma 1. *Let* $\zeta\colon X \hookrightarrow Y$ *be a closed (equivariant, "real") imbedding in a smooth* *manifold* Y, *and let* F *be an elliptic operator of order* 0 *on* X. *Then the element* $\zeta_*(\mathrm{ind}^a_{p,q}(F))$ *is the analytic index of the elliptic operator on* Y *whose symbol is equal to* $\zeta_!(\sigma_{p,q}(F))$, *where* $\zeta_!\colon KR^{p,q}_G(X) \to KR^{p,q}_G(Y)$ *is the Gysin homomorphism.*

Proof. One can assume immediately that Y is the space of a bundle of disks D^k over X, i.e. $Y = Z \times_{G_1} D^k$, where Z is a principal G_1-bundle over X and the group G acts on Z and on D^k, commuting with G_1. In correspondence with Theorem 5 of §4 we consider a $G_1 \times G$-invariant elliptic operator I on D^k. We construct a "fiber bundle" F on I. We choose an open covering $X = \bigcup_i U_i$ trivializing the bundle Z, and a partition of unity $\Sigma_i \alpha_i^2 = 1$. The product $\alpha_i F \alpha_i \# I$ over U_i is defined as

$$M_i(F\hat{\otimes}1) + N_i\left(1\hat{\otimes}\begin{pmatrix}0 & -I^*\\ I & 0\end{pmatrix}\right),$$

where $M_i^2 + N_i^2 = \alpha_i^2 \otimes 1$ (the positive elliptic operators M_i and N_i are constructed by analogy with Remark 3 of §4). The transition functions commute with $\alpha_i F \alpha_i \# I$ modulo operators of lower order. Averaging the sum $\Sigma_i \alpha_i F \alpha_i \# I$ over G, we get an operator with the required symbol.

One can assume that $II^* = 1$ and $I^*I = 1 - p_1$, where p_1 is a one-dimensional $G_1 \times G$-invariant projector. In fact (cf. [4], §4), since the index of I is equal to 1, there exists an epimorphism $\gamma\colon \mathrm{Ker}\, I \to \mathrm{Ker}\, I^*$, invariant with respect to $G_1 \times G$, while $\mathrm{Ker}\,\gamma = 1 \in RR(G_1 \times G)$. We extend by zero to the orthogonal complement of $\mathrm{Ker}\,\gamma$, and we replace I by $I + \gamma$. The order of the operator γ is equal to $-\infty$. The operator II^* we now invert, since $\mathrm{Ker}\, II^* = \mathrm{Ker}\, I^* = 0$. The desired operator is equal to $(II^*)^{-1/2} I$.

It remains to apply to each of the products $\alpha_i F \alpha_i \# I$ the argument given in the proof of Theorem 4 of §4. The lemma is proved.

Theorem 1. $\mathrm{ind}^a_{p-q}(F) = \mathrm{ind}^t_{p-q}(F)$.

Proof. By Lemma 1, lifting F to BN does not change the analytic index. We denote this lift by $\widetilde{F}$. Since $\mathrm{ind}^t_{p-q}(F) = f_*(\alpha) \cap \sigma$, it suffices to show that $\mathrm{ind}^a_{p-q}(\widetilde{F}) = f_*(\alpha) \cap \sigma$. Lemma 1 of §6 allows one to restrict the verification to the fact that $\sigma_0(\alpha) \otimes \sigma = \sigma_{q-p}(\widetilde{F})$, where $\sigma_0(\alpha)$ is the symbol of the elliptic operator corresponding to α. This last equation is obvious. The theorem is proved.

Remark 1. The theorem proved contains two assertions in the complex case ($p - q \equiv 0$ or $1 \bmod 2$) and eight assertions in the real case ($p - q \equiv 0, \ldots, 7 \bmod 8$; cf. Remark 1 of §2). One of its corollaries is, for example, Theorem 2.1 of [5].

§8. Representation rings and their duals

Definition 1. Let π be a finitely generated discrete group. We set $R_i(\pi) = K_i(L^1(\pi)) = K_i(C^*(\pi))$ ($L^1(\pi)$ is the algebra of summable functions on π (the multiplication is convolution); $C^*(\pi)$ is the enveloping C^*-algebra) and

$$RU_*(\pi) = \bigoplus_{i=0}^{1} RU_i(\pi), \qquad RO_*(\pi) = \bigoplus_{i=0}^{7} RO_i(\pi).$$

$R_*(\pi)$ is the ring

$$K_i(L^1(\pi)) \otimes K_j(L^1(\pi)) \to K_{i+j}(L^1(\pi \times \pi)) \xrightarrow{\Delta^*} K_{i+j}(L^1(\pi))$$

($\Delta\colon \pi \to \pi \times \pi$ is the diagonal imbedding).

Proposition 1. $RU_{n+1}(\pi \times \mathbf{Z}) = RU_n(\pi) \oplus RU_{n+1}(\pi)$.

Proof. Cf. Corollary 1 of §6.

Corollary 1. *The ring* $Ru_*(\mathbf{Z}^n)$ *is isomorphic to* $KU_*(T^n)$ (*the ring structure on* $KU_*(T^n)$ *is induced by the group structure on* T^n).

Definition 2. Let $E_\pi \overset{\omega}{\to} B_\pi$ be the universal bundle. We construct a homomorphism $\alpha: R_i(\pi) \to K^{-i}(B_\pi)$. If $(\phi, \psi, F) \in R_{p,q}(\pi)$, then we define a Hilbert bundle $E_\pi \times_\pi H$ over B_π as the quotient of $E_\pi \times H$ by the action of π: $g(x, h) = (xg^{-1}, \phi(g)h)$. We choose a locally finite covering of $B_\pi = \bigcup_i U_i$ such that $\omega^{-1}(U_i) = U_i \times \pi$, and a partition of unity $\Sigma_i \alpha_i = 1$, and also we fix isometries $r_i: U_i \times H \to (E_\pi \times_\pi H)|_{U_i}$. A homomorphism $\Phi: E_\pi \times_\pi H \to E_\pi \times_\pi H$ is defined by the formula

$$\Phi = \sum_i \alpha_i r_i (1 \times F) r_i^{-1}.$$

The Fredholm complex thus obtained (cf. [15]) gives the desired element $\alpha(\phi, \psi, F) \in K^{p,q}(B_\pi)$.

Proposition 2. α *is a well-defined ring homomorphism.*

Proof. We have $r_j^{-1}\Phi r_j = \Sigma_i \alpha_i r_j^{-1} r_i (1 \times F) r_i^{-1} r_j \sim 1 \times F$, since $r_j^{-1} r_i$ over $U_i \cap U_j$ is simply $1 \times \phi(g_{ij})$, where $g_{ij} \in \pi$. Hence Φ is defined by the fact that its lifting to $E_\pi \times H$ coincides with $1 \times F$ modulo compact operators. That α is multiplicative is obvious. The proposition is proved.

Remark 1. If π is finite, then $R_0(\pi) = R(\pi)$, and the homomorphism α coincides with the usual homomorphism $R(\pi) \to K^0(B_\pi)$.

Theorem 1. $\alpha: RU_*(\mathbf{Z}^n) \to KU^*(T^n)$ *is an isomorphism.*

Proof. Since α is a ring homomorphism, it suffices (in view of Corollary 1) to prove that $\alpha: RU_1(\mathbf{Z}) \approx KU^1(S^1)$. It will be convenient for us to realize generators of $RU_1(\mathbf{Z})$ as elements

$$z = \left(\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}, \quad \psi(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi(\varepsilon_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \right.$$
$$\left. F = \begin{pmatrix} 0 & -\frac{d}{i} \\ \frac{d}{i} & 0 \end{pmatrix} \right) \in RU_{0,1}(\mathbf{Z}).$$

Here $\phi: \mathbf{Z} \to L^U(L^2(S^1))$ carries 1 into the operator of multiplication by $e^{i\theta}$, and d is the operator of Example 2 of §2. We define a trivialization of the bundle $\mathbf{R} \times_{\mathbf{Z}} L^2(S^1)$ over S^1 by the formula $T(x, h) = (x \bmod 2\pi, u_x(h))$ (u_x is a one-parameter subgroup in $L^U(L^2(S^1))$ passing through $e^{i\theta}$: $u_0 = 1$ and $u_{2\pi} = e^{i\theta}$). Using T, we find that $\alpha(z)$ is determined by the family of Fredholm operators over S^1:

$$\left(\psi(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi(\varepsilon_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\frac{d_x}{i} \\ \frac{d_x}{i} & 0 \end{pmatrix} \right),$$

where $d_x \sim u_x d u_x^{-1}$. The family $u_x d u_x^{-1}$ is discontinuous at the point $x = 0$; however, a continuous family is defined uniquely modulo compact operators.

We replace the operator d by $\widetilde{d}$:

$$\widetilde{d}\,(e^{in\theta}) = i \cdot \begin{cases} e^{in\theta}, & n \geqslant 0, \\ -e^{in\theta}, & n < 0, \end{cases}$$

and we replace the family d_x by the homotopic family $\widetilde{d}_x$ such that $\widetilde{d}_x = u_{2x}\widetilde{d}u_{2x}$ for $0 \leq x \leq \pi$. The operators $\widetilde{d}_0 = \widetilde{d}$ and $\widetilde{d}_\pi = e^{i\theta}\widetilde{d}e^{-i\theta}$ in the basis $\{e^{in\theta}\}$ will differ only on the vector (1): $\widetilde{d}_0(1) = (1)$ and $\widetilde{d}_\pi(1) = -(1)$. One can therefore assume that, for $\pi \leq x \leq 2\pi$, $\widetilde{d}_x$ behaves as follows:

$$\widetilde{d}_x\,(e^{in\theta}) = \begin{cases} e^{in\theta}, & n \leqslant 0. \\ i\cos x \cdot (1), & n = 0. \end{cases}$$

For $0 \leq x \leq \pi$ our family is degenerate; hence upon excision of the upper half of the circumference $K^{01}(S^1) \to K^{01}(D^1, S^0)$ the element $\alpha(z)$ goes into

$$\left(\psi(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \psi(\varepsilon_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \begin{pmatrix} 0 & -\cos x \\ \cos x & 0 \end{pmatrix}\right).$$

Identifying $[\pi, 2\pi]$ with $[0, 1]$ carries it into

$$\left(\psi(\varepsilon_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \psi(\varepsilon_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \begin{pmatrix} 0 & \cos \pi t \\ -\cos \pi t & 0 \end{pmatrix}\right) \in K^{0,1}(D^1, S^0).$$

The theorem is proved.

Conjecture. *The image of α is dense in the topology on $K^*(B_\pi)$ induced by the filtration of B_π by compact sets.*

Theorem 2. *The conjecture is true for discretely imbedded subgroups of the full linear group $GL(n, \mathbf{C})$, while for torsion-free groups α is an epimorphism.*

The proof is somewhat involved and will be published in a later paper of this series.

The object dual to the representation ring is the group $K^*(C^*(\pi))$. It turns out that there exists a homomorphism β: $K_i(B_\pi) \to K^{-i}(C^*(\pi))$ dual to α. For its construction a new definition of $K^*(B)$ is needed. We shall briefly describe the basic construction.

Let B be a C^*-algebra. We denote by H_B the set of sequences $\{b = (b_1, b_2, \ldots) \mid b_i \in B\}$ satisfying the Cauchy condition:

$$\forall\, \varepsilon > 0 \ \exists\, N, \forall\, m, n > N \ \left\| \sum_{i=m}^{n} b_i^* b_i \right\|_B < \varepsilon.$$

We define a scalar product with values in B: $(a, b) = \Sigma_1^\infty a_i^* b_i$, and a norm $\|b\| = \|\Sigma_1^\infty b_i^* b_i\|^{1/2}$. It is easy to verify that, as a right B-module, H_B is a complete normed space. We denote by p_n the projector onto the first n coordinates. A module homomorphism f: $H_B \to B$ will be a compact functional if $\lim_{n\to\infty}\|f \cdot p_n - f\| = 0$. The space $L(H_B)$ is defined as the set of continuous module endomorphisms F: $H_B \to H_B$ satisfying the condition $\forall y \in H_B$, $F_y(x) = (y, F(x))$ is a compact functional. The norm on $L(H_B)$ is induced from the space of all continuous endomorphisms of H_B. $L(H_B)$ is a C^*-algebra. The ideal of compact operators $K(H_B)$ by definition is

$$\{k\in L(H_B)\,|\,\lim_{n\to\infty}\|kp_n-k\|=0\}.$$

We define $K^{q-p}(B)=K^{p,q}(B)$ by analogy with §1, starting from the set $\mathfrak{S}^{p,q}(B)$ of pairs $(\psi\colon C_{p,q+1}\to L(H_B),\ F\in L(H_B))$, where $F^*\sim -F$, $F^2\sim -1$ and F anticommutes with the generators of $C_{p,q+1}$ ($\sim$ denotes equality modulo $K(H_B)$). For this K-functor the homotopy and excision axioms are satisfied, Bott periodicity holds (the proof is obtained from the existence of the intersection index with K_*) and also there is the exact sequence

$$\dots\to K^i(I)\to K^i(B_1)\xrightarrow{\omega_*}K^i(B_2)\to\dots$$

(here $\omega\colon B_1\to B_2$ is an epimorphism and $I=\operatorname{Ker}\omega$). In addition, $K^0(B)$ is isomorphic to the Grothendieck group $K^0(B)$; hence our $K^*(B)$ coincides with Karoubi's K-functor [7].

Let π be a discrete group. For the construction of the homomorphism $\beta\colon K_i(B_\pi)\to K^{-i}(C^*(\pi))$ we consider an arbitrary principal π-bundle $\widetilde{X}\to X$ over the compactum X. For an element $(\phi,\psi,F)\in K_{p,q}(X)$ one constructs the space $H\otimes_{C(X)}C_0(\widetilde{X})$ ($C_0(\widetilde{X})$ are the finitary functions on $\widetilde{X}$), whose completion with respect to the norm is isomorphic to $H_{C^*(\pi)}$. We set $\beta(\phi,\psi,F)=(\psi,1\otimes F)$ (the operator $1\otimes F$ is defined with the help of a partition of unity). It is easy to verify that α and β are dual.

The homomorphisms α and β are closely connected with the problem of homotopy invariance of higher signatures of smooth manifolds, which consists of the following (cf. [14]). Let M^n be a smooth manifold with fundamental group π, let $f\colon M^n\to B_\pi$, and let $L_*(M^n)$ be the L-genus of Hirzebruch. One asks: are the numbers

$$\langle L_*(M^n)\cdot f^*(x),\ [M^n]\rangle,$$

where $x\in H^*(B_\pi;\mathbf{Q})$, homotopy invariants? An equivalent question: Is $f_*(DL_*(M^n))\in H_*(B_\pi;\mathbf{Q})$, where D is Poincaré duality, a homotopy invariant? With the help of Theorem 1 of §7 it is easy to verify that the signature operator $[d+\delta\colon\Omega^+\to\Omega^-]\in K_0(M^n)$ under the projection into $H_*(B_\pi;\mathbf{Q})$ goes into the element $f_*(DL_*(M^n))$.

Theorem 3. *The element* $\beta[d+\delta]\in K^0(C^*(\pi))$ *is a homotopy invariant.*

The proof will be published in a separate paper.

Theorem 3 shows that if the homomorphism

$$\beta\otimes\mathbf{Q}:K_0(B_\pi)\otimes\mathbf{Q}\to K^0(C^*(\pi))\otimes\mathbf{Q}$$

is a monomorphism, then the higher signatures of manifolds with fundamental group are homotopy invariants. In its own right, that $B\otimes\mathbf{Q}$ is a monomorphism follows from the conjecture formulated above in view of the adjointness of α and β. In particular, for the class of groups described in Theorem 2, one has the theorem on the homotopy invariance of the higher signatures. In the general case the question remains open. The problem of homotopy invariance has a long history, described in detail in

[13]. The last advance (belonging to A. S. Miščenko [12], [13]) consisted of the proof of homotopy invariance for the class of groups π for which B_π is a compact riemannian manifold with metric of nonpositive curvature.

In conclusion we consider the generalization of our construction of $R_*(\pi)$ to nondiscrete groups. Unfortunately, the definition of $R_*(\pi)$ given above is not suitable, not only because of the absence of a unit in $L^1(\pi)$, but essentially. Here it is necessary to "strengthen" the basic definitions of §1. Let B and B_1 be involutory Banach algebras (possibly without unit). We consider the imbedding of $B \otimes B_1$ in the algebra of continuous linear operators on $B \otimes B_1$, and we denote by $\overline{B \otimes B_1}$ the closure of the image in the topology of strong operator convergence. The norm on $\overline{B \otimes B_1}$ is induced from $L(B \otimes B_1)$. Point 5 of Definition 1 of §1 must be strengthened in the following way. For any algebra B_1 there is uniquely defined a homomorphism

$$\overline{B \otimes B_1} \xrightarrow{\varphi \otimes 1} L(H \otimes B_1)$$

($L(H \otimes B_1)$ is the analogue of $L(H_{B_1})$ constructed above). It is required that the operator $F \otimes 1$ should commute modulo $K(H \otimes B_1)$ with the image of $\phi \otimes 1$. For this definition of K_* (setting $R_*(\pi) = K_*(L^{-1}(\pi))$ for locally compact π) one can construct the homomorphisms α and β. In addition, for compact groups π, $R^0(\pi)$ will coincide with $R(\pi)$.

Under the new definition the question of existence of a multiplication in K_* is still open. If the multiplication exists, then there is a Thom isomorphism (which there wasn't under the old definition).

We note, finally, that one can construct $R_*(\pi)$ also for arbitrary topological groups so as to preserve some of the basic properties (for example, the homomorphism α will exist). Here it is necessary to start from triples (ϕ, ψ, F), where $\phi: \pi \to L^U(H)$ is a representation which is continuous in the strong operator topology, and a family of operators $\widetilde{F}: \pi \times H \to \pi \times H$, where $\widetilde{F}(g, h) = (g, (\phi(g)F - F\phi(g))h)$, compact in the sense of [15].

Received 31/JULY/74

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Translated by N. STEIN