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THE OPERATOR K-FUNCTOR AND EXTENSIONS OF C^*-ALGEBRAS

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ABSTRACT. In this paper a general operator K-functor $K_*K^*(A, B)$ is constructed, depending on a pair A, B of C^*-algebras. Special cases of this functor are the ordinary cohomological K-functor $K^*(B)$ and the homological K-functor $K_*(A)$. The results (homotopy invariance, Bott periodicity, exact sequences, etc.) permit one to compute $K_*K^*(A, B)$ effectively in concrete examples. The main result, concerning extensions of C^*-algebras, consists in a description of a "stable type" of extensions of the most general form: $0 \to B \to D \to A \to 0$. It is shown that the sum of such an extension with a fixed decomposable extension of the form $0 \to \mathcal{K} \otimes B \to D_0 \to A \to 0$ is uniquely determined by an element of the group $KK^1(A, B)$.

Bibliography: 25 titles.

The study of operator algebras by methods borrowed from homotopy theory began comparatively recently. As in every field of algebra, in the theory of C^*-algebras there arises the problem of classification. As usual, a class of simple objects is chosen from which more complicated objects are obtained by means of given operations. One of these operations is extension. The study of the simplest extensions of the form $0 \to \mathcal{K} \to D \to C(X) \to 0$, where $\mathcal{K}$ is the algebra of compact operators and $C(X)$ is the algebra of continuous functions on a compact space X, carried out by Brown, Douglas, and Fillmore [9], led to an important result (which has been at the center of interest of specialists in C^*-algebras in the past years). It has turned out that the group constructed from extensions of the form indicated above is isomorphic to the homological K-functor $K_{-1}(X)$, well known in topology. Under certain additional assumptions, the initial extension (and consequently, the algebra D, as well) is defined by an element of this group uniquely up to isomorphism. This has made it possible in particular to describe completely C^*-algebras of some solvable Lie groups by indicating the spaces X and elements in the groups $K_{-1}(X)$ corresponding to them (see [23] and [25]).

In this paper the results of Brown, Douglas, and Fillmore [9] are extended to the case of more general extensions of the form

$$0 \to B \to D \to A \to 0. \qquad (1)$$

We construct a general K-functor $K_*K^*(A, B)$, special cases of which are the ordinary cohomological K-functor $K^*(B)$ and the homological K-functor $K_*(A)$ (defined in [15]). This new K-functor has several simple properties (homotopy invariance, Bott periodicity, exact sequences, etc.), which enable us to calculate it efficiently in concrete examples. All

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basic results concerning the K-functor (except exact sequences) are obtained under the assumption that the algebra A is separable and B has a countable approximate identity.

In order to describe the connection between the K-functor and extensions it is convenient to pass from extensions of the form (1) to extensions of the form

$$0 \to \mathcal{K} \otimes B \to D \to A \to 0. \tag{2}$$

To every extension of the form (1) there corresponds a unique extension of the form (2) (see Remark 2 in §7). Under the condition that A is nuclear, every extension of the form (2) determines an element of the group $K_{-1}K(A, B) = KK^1(A, B)$. To decomposable extensions (i.e., extensions admitting a section homomorphism $A \to D$) there corresponds the null element of the group indicated above. Conversely, to every element of $KK^{-1}(A, B)$ there corresponds a unique (up to unitary equivalence) so-called *absorbing* extension of the form (2). Consequently, classes of unitary equivalence of absorbing extensions are in a one-to-one correspondence with elements of $KK^1(A, B)$ (in analogy with [9]). Any extension (of the form (1) or (2)) becomes absorbing upon adding to it a (uniquely determined) decomposable absorbing extension. (For the reason for adding an extension, see §7.) By the same token, an element of the group $KK^1(A, B)$ determines the so-called "stable type" of an extension.

We note that besides the theory of extensions, the operator K-functor depending on a pair of arguments may be useful in other questions, as well. For example, the family of elliptic operators on a closed manifold X parametrized by points of a compact space Y defines an element of the group $KK^0(C(X), C(Y))$ and not only an element of $K^0(Y)$ as in [7]. The groups $K_*K^*(C(X), C(Y))$ also admit a good homotopy definition (see §6).

All our results relate at the same time to three categories of C^*-algebras: complex, real, and "real" algebras (see §1). Moreover, C^*-algebras are considered with an action of a fixed compact group G (equivariant K-theory and theory of extensions). It has also turned out that the K-functor $KK(A, B)$ is defined more naturally on the category of $\mathbf{Z}_2$-graded C^*-algebras (see §2). Ordinary C^*-algebras are then considered as trivially graded: $B^{(0)} = B$ and $B^{(1)} = 0$. Then, setting

$$K_{p-q}K(A, B) = KK^{q-p}(A, B) = KK\big(A \hat{\otimes} C_{p,q}, B\big) = KK\big(A, B \hat{\otimes} C_{q,p}\big),$$

where $C_{p,q}$ is the Clifford algebra, we obtain the graded K-functor $K_*K^*(A, B)$.

The basic results were announced in the note [16]. The content of the paper is as follows. In §1 we give the notation and basic facts which are used in the sequel. §2 contains information concerning graded algebras and Clifford algebras. The definition of the K-functor is given in §4. We note that this definition actually includes homotopy invariance. The basic part of §4 and the whole of §3 deals with the most complicated technical construction of the work, the intersection product, which combines (and essentially generalizes) the construction of the outer product for the homological or cohomological K-functor and the construction of the index of intersection between these K-functors. An immediate consequence of the existence of the intersection product is Theorem 6 of §4, which can be considered as the most general theorem concerning the periodicity of the K-functor. In §5 from this theorem we obtain the theorem on formal (Clifford) periodicity, the Bott periodicity theorem, and the Thom isomorphism theorem.

We must mention that in the definition of the graded K-functor (§5) there arises the question of the choice of signs. For example, the isomorphism $C_{p,q} \hat{\otimes} C_{1,1} \simeq C_{p+1,q+1}$

may be defined by various methods, which lead to different (in sign) identifications of $KK(A \hat{\otimes} C_{p,q}, B)$ with $KK(A \hat{\otimes} C_{p+1,q+1}, B)$. Apparently there is no most successful "canonical" choice of all usable isomorphisms of a Clifford algebra. Nevertheless, some compatible choice is possible, and it is made in §2. Then the *orientation* of a Clifford algebra introduced in §2 turned out useful.

In §6 special cases of the operator K-functor $K_*K^*(A, B)$ are considered, where one of the two algebras A and B coincides with the field C of scalars. In the case $B = C$ this gives the homological K-functor $K_*(A)$ (see [15]), and in the case $A = C$ it gives the ordinary (cohomological) K-functor $K^*(B)$. Moreover, $K^*(B)$ can be realized by means of Fredholm operators over the C^*-algebra B anticommuting with the representation of the Clifford algebra. (Such an interpretation of $K^*(B)$ was first given in [15], §8.) The remaining part of §6 is devoted to the homotopy description of $K_*K^*(C(X), C(Y))$ in the case where X and Y are finite cell complexes.

We must mention Theorem 1 of §6 particularly. Together with Lemma 1 of §6 it is used in §7 for identifying the semigroup $\operatorname{Ext}(A, B)$ constructed from an extension of the form (2) with $KK^1(A, B)$. In this sense Theorem 1 of §6 can be considered as a theorem on the homotopy invariance of $\operatorname{Ext}(A, B)$. Besides, this theorem implies the coincidence of the following two definitions of $K_*(A)$: the definition of [15] and the definition of [14]. The proof of Theorem 1 of §6 is completely based on the construction of the intersection product. (See Theorem 1 of §5 in [15], as well.)

Besides the isomorphism $\operatorname{Ext}(A, B) \simeq KK^1(A, B)$, which is established by usual methods (see [3] and [4]) by application of the generalized Stinespring theorem, §7 contains theorems on exact sequences for the K-functor (Theorems 2 and 3). We have succeeded in describing the boundary homomorphisms in these exact sequences as the indices of the intersection. At the end of §7 extensions of graded algebras are considered.

In §7 an additional restriction on the first argument of the K-functor $K_*K^*(A, B)$ appears, the condition of *nuclearity*. (With respect to the notion of nuclearity, see, for example, [4].) In applications the condition of nuclearity is apparently not too stringent a restriction. We recall that, for example, all C^*-algebras of type I, all C^*-algebras of connected locally compact groups, and the C^*-algebras of discrete amenable groups are nuclear. The class of nuclear algebras is closed with respect to both extensions and passage to an ideal, a quotient algebra, or an inductive limit (see [19], [21], and [22]). Moreover, we would like to note that in §7 the nuclearity of the algebra A is not used directly. In fact, we only need two technical results, in the formulation of which the condition of nuclearity occurs: the Choi-Effros theorem on the existence of completely positive sections [11] and the generalized theorem of Voiculescu (see [17] and §1.16). Instead of the nuclearity of A we could have used any other condition on A and B ensuring the validity of these two theorems.

In the second part of the paper a whole series of examples of extensions connected with C^*-algebras of Lie groups will be examined. The construction of intersection product allows us to calculate the K-functor (both the homological and cohomological) completely for a large class of group C^*-algebras. In particular, if Γ is a simply connected solvable Lie group, then

$$K_i(C^*(\Gamma)) \simeq K_{i+\dim \Gamma}(C), \qquad K^i(C^*(\Gamma)) \simeq K^{i+\dim \Gamma}(C).$$

With the aid of exact sequences and some simple geometric arguments this often gives complete information on the class of an extension of type

$$0 \to B \to C^*(\Gamma) \to C^*(\Gamma/\Gamma_1) = A \to 0$$

in the group $KK^1(A, B)$, and the decomposability of extensions of this type is usually obtained simply by considering exact sequences. (The simplest example, the Heisenberg group, was mentioned in [16].) Regretfully, extensions connected with C^*-algebras are not, as a rule, absorbing. Therefore, the corresponding group C^*-algebras cannot be calculated up to isomorphism by using only the K-functor.

In conclusion, the author expresses his gratitude to Professor A. A. Kirillov for posing the problem of "calculating" the C^*-algebras of the Heisenberg group. This problem stimulated the present study. The author is also grateful to L. G. Brown, R. G. Douglas, and P. A. Fillmore for a preprint of their paper [9].

§1. Notation and introductory remarks

1. In what follows, by an algebra we always mean a C^*-algebra, by a homomorphism a $*$-homomorphism, and by an ideal a two-sided ideal. All results relate simultaneously to the categories of real, complex, and "real" algebras. A "*real*" algebra is a complex algebra equipped with an antilinear involution $b \to \bar{b}$ with the properties that $\overline{b_1 b_2} = \bar{b}_1 \cdot \bar{b}_2$ and $\overline{(b^*)} = (\bar{b})^*$. The homomorphisms of such algebras have to preserve "real" involution. Besides, we shall assume that a fixed compact group G satisfying the second axiom of countability acts as an automorphism group on all algebras under consideration, and all homomorphisms are equivariant. The action of G on the algebra B is said to be continuous if the mapping $G \times B \to B$: $(g, b) \to g(b)$ is norm continuous. The group G acting on a "real" algebra has to be equipped with a ("real") involution $g \to \bar{g}$, and the condition $\overline{g(b)} = \bar{g}(\bar{b})$ has to be satisfied. An element $x \in B$ is said to be *invariant* if $\bar{x} = x$ and $g(x) = x$ for all $g \in G$. If B is an algebra with identity 1 and 1 is an invariant element, then B is said to be *unital*.

2. By C we denote the field of scalars, i.e. the algebra $\mathbf{R}$ or $\mathbf{C}$. The "real" involution is complex conjugation, and the action of G is trivial.

3. All tensor products of algebras are considered with the minimal C^*-norm. The "real" involution and the action of G on a tensor product of algebras are uniquely defined by the relations

$$\overline{b_1 \otimes b_2} = \bar{b}_1 \otimes \bar{b}_2, \quad g(b_1 \otimes b_2) = g(b_1) \otimes g(b_2) \quad \forall g \in G.$$

4. M_n is the algebra of $n \times n$ matrices over C, "real" involution is elementwise complex conjugation, and the action of G is defined by some unitary ("real") representation of G on the space C^n.

5. If B is an algebra and X is a locally compact space, then $B(X)$ is the algebra of continuous functions on X with values in B, vanishing at ∞. A *homotopy of homomorphisms* φ_t: $A \to B$, $\alpha \leqslant t \leqslant \beta$, is, by definition, a homomorphism $\{\varphi_t\}$: $A \to B[\alpha, \beta]$.

6. By $\tilde{B}$ we denote an algebra B with adjoined 1 (moreover, $\bar{1} = 1$ and $g(1) = 1$ for all $g \in G$). Every homomorphism φ: $A \to B$ can be extended uniquely to a unital homomorphism $\tilde{\varphi}$: $\tilde{A} \to \tilde{B}$. If the algebra B itself is unital, then $\tilde{B} \simeq B \oplus C$, and therefore any homomorphism $A \to B$ can be extended to a unital homomorphism $\tilde{A} \to \tilde{B} \xrightarrow{p} B$, where p is the projection.

7. We recall that any algebra with a countable approximate identity has a strictly positive element, and the presence of a strictly positive element k implies the presence of a countable increasing Abelian approximate identity, which can be defined, for example, by the formula $u_i = k^{1/i}$ or $u_i = k(k + 1/i)^{-1}$, $i = 1, 2, \ldots$ (see [1]). If the element k is invariant (which, for algebras with continuous action of G, can easily be achieved by averaging), then all the u_i are invariant.

CRITERION OF STRICT POSITIVITY. *An element $k \in B$ is strictly positive if and only if for every Hermitian $b \in B$ and every $\varepsilon > 0$ there exists a number $c > 0$ such that $b \leqslant ck + \varepsilon$ in the algebra $\tilde{B}$.*

PROOF. *Necessity*. Since for every $\varepsilon > 0$ there exists an i such that

$$\left[\left(\frac{1}{i}\right)\left(k+\frac{1}{i}\right)^{-1}\right]\cdot b\cdot\left[\left(\frac{1}{i}\right)\left(k+\frac{1}{i}\right)^{-1}\right]\leqslant\varepsilon,$$

we have $b \leqslant \varepsilon(ik + 1)^2 \leqslant ck + \varepsilon$.

Sufficiency. If f is a positive functional on B and $\tilde{f}$ is its canonical extension to $\tilde{B}$, then for every $b \in B$ and $\varepsilon > 0$ there exists $c > 0$ such that $f(b^*b) \leqslant \tilde{f}(ck + \varepsilon) = c \cdot f(k) + \varepsilon \cdot \|f\|$. Therefore from $f(k) = 0$ it follows that $f(b^*b) = 0$, and so $f(b) = 0$. ■

COROLLARY 1. *From the presence of a strictly positive element in an ideal $J \subset B$ and in the quotient algebra B/J it follows that there exists a strictly positive element in B.* ■

COROLLARY 2. *If the subalgebras A and B of D have strictly positive elements, then the minimal subalgebra containing A and B also has a strictly positive element. If A is unital and separable and B has a strictly positive element, then the minimal subalgebra containing AB has a strictly positive element.* ■

8. $\mathfrak{M}(B)$ is the algebra of multipliers (double centralizers) of the algebra B. We recall (see [10]) that the pair of mappings $T_{(1)}, T_{(2)}\colon B \to B$ is called a *multiplier* if for all $x, y \in B$ we have $x \cdot T_{(1)}(y) = T_{(2)}(x) \cdot y$.

The *strong topology on* $\mathfrak{M}(B)$ is induced by the family

$$\{\|T\|_b=\|T_{(1)}(b)\|+\|T_{(2)}(b)\|\mid b\in B\},$$

of seminorms, where $T = (T_{(1)}, T_{(2)}) \in \mathfrak{M}(B)$. The continuous action of G on B defines the following action of G on $\mathfrak{M}(B)$: $g(T_{(i)})(b) = g(T_{(i)}g^{-1}(b))$, $i = 1, 2$. In the general case this action of G is continuous in the strong topology (but not in norm). The "real" involution on $\mathfrak{M}(B)$ is defined by the formula $\overline{T}_{(i)}(b) = \overline{T_{(i)}(\bar{b})}$, $i = 1, 2$. The algebra B is imbedded in $\mathfrak{M}(B)$ as an ideal. The quotient algebra $\mathfrak{M}(B)/B$ is denoted by $\mathcal{O}(B)$. If J is an ideal in B, then the restriction homomorphism $\mathfrak{M}(B) \to \mathfrak{M}(J)$ is defined, and it is the identity on J.

9. We recall that the algebra $\mathfrak{M}(B(X))$ is isomorphic to the algebra of functions continuous in the strong topology, bounded in norm, and defined on X with values in $\mathfrak{M}(B)$ (see [2]).

10. Let B be an algebra with continuous action of G. We recall the definition of a Hilbert B-module (see [18] and [17]). We consider a linear space E over the field C, which is a right B-module at the same time, and $\forall\, x \in E$, $\forall\, b \in B$, $\forall\, \lambda \in C$ $\lambda(xb) = (\lambda x)b = x(\lambda b)$. The space E is called a *pre-Hilbert* B-module if an inner product

$E \times E \to B$ is defined which satisfies the following conditions for all $x, y, z \in E$, all $b \in B$ and all $\lambda \in C$:

1°. $(x, y + z) = (x, y) + (x, z)$; $(x, \lambda y) = \lambda(x, y)$;

2°. $(x, yb) = (x, y)b$;

3°. $(y, x) = (x, y)^*$;

4°. $(x, x) \geqslant 0$, and if $(x, x) = 0$, then $x = 0$;

Besides, on E a linear action of G which is norm continuous and (in the "real" case) an antilinear involution $x \to \bar{x}$ are defined satisfying the conditions

5°. $g(xb) = g(x)g(b)$; $\overline{xb} = \bar{x}\bar{b}$;

6°. $(g(x), g(y)) = g((x, y))$; $(\bar{x}, \bar{y}) = \overline{(x, y)}$;

for every $g \in G$, besides the usual condition $\overline{g(x)} = \bar{g}(\bar{x})$.

An element $x \in E$ is said to be *invariant* if $\bar{x} = x$ and $g(x) = x$ for all $g \in G$. Setting $\|x\| = \|(x, x)\|^{1/2}$, we obtain a norm on E. If the space E is complete in this norm, it is called a *Hilbert* B-module. The Hilbert direct sum $\bigoplus_{i \in I} E_i$ is, by definition, the completion of the algebraic direct sum in the norm defined by the inner product $(\bigoplus_i x_i, \bigoplus_i y_i) = \sum_{i \in I}(x_i, y_i)$.

11. The Hilbert space over B is defined as the Hilbert direct sum

$$\mathcal{H}_B = \bigoplus_{i=1}^{\infty} (V_i \otimes_C B),$$

where $\{V_i\}$ is a countable collection of finite-dimensional Euclidean spaces in which (up to isomorphism) all irreducible unitary representations of G are realized (in the "real" case we consider "real" representations). Each of the V_i has to be repeated infinitely many times in the collection under consideration. The inner product on V_i is assumed to be antilinear in the first argument. The inner product on $V_i \otimes B$ is defined by

$$(x_1 \otimes b_1, x_2 \otimes b_2) = (x_1, x_2)\, b_1^* b_2.$$

In the case $B = C$ an ordinary Hilbert space $\mathcal{H}$ is obtained. (Throughout the paper, inner products are assumed to be linear in the second argument and antilinear in the first argument.)

12. A system of generators of a Hilbert B-module E is, by definition, a set of elements $\{x_i\}_{i \in I} \subset E$, such that the finite sums $\{\sum_k x_{i_k} b_k | b_k \in B\}$ are dense in E. If B has a countable approximate identity, then $\mathcal{H}_B$ has a countable system of generators.

STABILIZATION THEOREM [17]. *Let B be an algebra with a continuous action of G, and assume that the Hilbert B-module E has a countable system of generators. Then $E \oplus \mathcal{H}_B \simeq \mathcal{H}_B$.*

13. If E is a Hilbert B-module, then by $\mathcal{L}(E)$ we denote the set of mappings T: $E \to E$ for which there exists T^*: $E \to E$ satisfying the condition $(T(x), y) = (x, T^*(y))$ $\forall\, x, y \in E$. The action of G and the "real" involution are defined on $\mathcal{L}(E)$ by the formulas $\bar{T}(x) = \overline{T(\bar{x})}$ and $g(T)(x) = g(Tg^{-1}(x))$, $g \in G$, $x \in E$. Every element $T \in \mathcal{L}(E)$ is bounded, linear, and B-modular (see [18] and [17]). $\mathcal{L}(E)$ is a C^*-algebra with the norm induced from the space of bounded linear operators on E.

14. For $x, y, z \in E$ set $\theta_{x,y}(z) = x \cdot (y, z)$. Then $\theta_{x,y} \in \mathcal{L}(E)$. The closure, in $\mathcal{L}(E)$, of the linear subspace generated by the operators $\theta_{x,y}$ is an ideal in $\mathcal{L}(E)$. We denote it by $\mathcal{K}(E)$. We set $\mathcal{K} = \mathcal{K}(\mathcal{H})$ and $\mathcal{K}_B = \mathcal{K}(\mathcal{H}_B)$. It is easy to verify that $\mathcal{K}(B) \simeq B$ and $\mathcal{K}_B \simeq \mathcal{K} \otimes B$.

THEOREM [17]. *For any Hilbert B-module E the correspondence $T \in \mathcal{L}(E) \to (T_{(1)}, T_{(2)}) \in \mathfrak{M}(\mathcal{K}(E))$, where $T_{(1)}(\theta_{x,y}) = \theta_{T(x),y}$ and $T_{(2)}(\theta_{x,y}) = \theta_{x,T^*(y)}$ $(x, y \in E)$, defines an isomorphism of $\mathcal{L}(E)$ onto $\mathfrak{M}(\mathcal{K}(E))$.*

COROLLARY. $\mathcal{L}(\mathcal{H}_B) \simeq \mathfrak{M}(\mathcal{K} \otimes B)$.

15. GENERALIZED THEOREM OF STINESPRING (see [17]). *Let A be a separable unital algebra, and assume that B has a countable approximate identity, the action of G on A and B is continuous, and φ: $A \to \mathfrak{M}(\mathcal{K} \otimes B)$ is unital, completely positive, equivariant and "real". There exists a unital homomorphism ρ: $A \to \mathfrak{M}(M_2 \otimes \mathcal{K} \otimes B) = \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$, such that $\varphi(a) \oplus 0 = q\rho(a)q$ for every $a \in A$, where $q \in \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$ is the projection onto the first direct summand.*

16. Let E_1 and E_2 be Hilbert B-modules. The mappings φ_1: $A \to \mathcal{L}(E_1)$ and φ_2: $A \to \mathcal{L}(E_2)$ are said to be *approximately equivalent* if there exists a sequence of G-equivariant ("real") isometries u_n: $E_2 \to E_1$ such that

$$u_n^*\varphi_1(a)\,u_n - \varphi_2(a) \in \mathcal{K}(E_2) \quad \text{and} \quad \lim_{k\to\infty} \| u_k^*\varphi_1(a)\,u_k - \varphi_2(a)\| = 0$$

for all $a \in A$ and all n.

We denote by A_{inv} and $\mathcal{H}_{\mathrm{inv}}$ the G-invariant elements in the algebra A and the Hilbert space $\mathcal{H}$, respectively. By restriction every homomorphism π: $A \to \mathcal{L}(\mathcal{H})$ defines a homomorphism π_{inv}: $A_{\mathrm{inv}} \to \mathcal{L}(\mathcal{H}_{\mathrm{inv}})$. An imbedding π: $A \hookrightarrow \mathcal{L}(\mathcal{H})$ is called a *G-imbedding* if $(1 \oplus \pi)_{\mathrm{inv}}$: $(\mathcal{L}(C^n) \otimes A)_{\mathrm{inv}} \to \mathcal{L}((C^n \otimes \mathcal{H})_{\mathrm{inv}})$ is an imbedding for every n and every unital action of G on C^n. If A is separable, then (unital) G-imbeddings always exist (see [17]). A G-imbedding $A \hookrightarrow \mathcal{L}(\mathcal{H})$ is said to be *regular* if its unital extension $\tilde{A} \to \mathcal{L}(\mathcal{H})$ is a G-imbedding and $1 \in \mathcal{L}(\mathcal{H})$ is not contained in $A + \mathcal{K}$. If π is a G-imbedding, then $\pi \oplus 0$: $A \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is a regular G-imbedding.

GENERALIZED THEOREM OF VOICULESCU. *Let the algebra A be separable and assume that B has a countable approximate identity, the action of G on A and B is continuous, and $A \hookrightarrow \mathcal{L}(\mathcal{H})$ is a regular G-imbedding. Assume, furthermore, that, for all n, every completely positive mapping $\tilde{A}/(A \cap \mathcal{K}) \to M_n \otimes B$ is nuclear (for example, it is sufficient that at least one of the algebras A and B be nuclear). Consider the algebra $\mathcal{L}(\mathcal{H}) = \mathfrak{M}(\mathcal{K}) \otimes C$ as a subalgebra of the scalar operators in $\mathcal{L}(\mathcal{H}_B) = \mathfrak{M}(\mathcal{K} \otimes B)$, and denote the composition $A \hookrightarrow \mathcal{L}(\mathcal{H}) \hookrightarrow \mathcal{L}(\mathcal{H}_B)$ by π. If φ: $A/(A \cap \mathcal{K}) \to \mathcal{L}(\mathcal{H}_B)$ is an arbitrary homomorphism, then $\varphi \oplus \pi$: $A \to \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$ is approximately equivalent to π.*

The proof can be obtained from the unital case ([17], Theorem 6) by replacing A, φ and π by $\tilde{A}$, $\tilde{\varphi}$ and $\tilde{\pi}$. ■

We note that if ψ: $A \hookrightarrow \mathcal{L}(\mathcal{H})$ is an arbitrary imbedding, then $\psi' = \bigoplus_1^\infty \psi$ has the property that $\psi'(A) \cap \mathcal{K} = 0$.

17. *If u is a unitary element in $\mathcal{L}(\mathcal{H}_B)$, then there exists a unitary homotopy $\{u_t\}_{t\in[0,1]} \in \mathcal{L}(\mathcal{H}_{B[0,1]})$ such that $u_0 = 1$ and $u_1 = u$. If u is invariant, so is $\{u_t\}$.*

PROOF. In the space $L^2[0, 1]$ for $t \in (0, 1]$ we define a unitary operator U_t by the formula $(U_t f)(x) = \sqrt{t}\, f(tx)$, $f \in L^2[0, 1]$. We realize $\mathcal{H}$ as $L^2[0, 1] \otimes \mathcal{H}$ and set $R_t = U_t \otimes 1$. The families $\{R_t\}_{t\in(0,1]}$ and $\{R_t^*\}_{t\in(0,1]}$ are continuous in the strong operator topology; we have $R_t \to 0$ strongly as $t \to 0$; $R_t R_t^* = 1$; $R_1 = 1$ (see [12], 10.8.1). Considering $\mathcal{L}(\mathcal{H})$ as the space of scalar operators in $\mathcal{L}(\mathcal{H}_B)$, we obtain a family

$\{R_t\}_{t\in(0,1]}$ in $\mathcal{L}(\mathcal{H}_B)$. For $t > 0$ let $u_t = R_t^* u R_t + (1 - R_t^* R_t)$; $u_0 = 1$. It is easy to verify that $\{u_t\}_{t\in[0,1]} \in \mathcal{L}(\mathcal{H}_{B[0,1]})$ is the desired element. ■

18. Let A, B and D be algebras, and let $\varphi\colon A \to D$ and $\psi\colon B \to D$ be homomorphisms. The subalgebra $A \oplus_D B = \{(a, b) \in A \oplus B \mid \varphi(a) = \psi(b)\} \subset A \oplus B$ is called the *fibered sum* of A and B over D with projections φ and ψ. In particular, for $\alpha = 0$ and $\alpha = 1$ we define the α-cylinder $Z_\alpha(A, D, \varphi)$ of the homomorphism $\varphi\colon A \to D$ as the fibered sum $A \oplus_D D[0, 1]$ with projections $\varphi\colon A \to D$ and $D[0, 1] \to D[\alpha] = D$. If φ is an imbedding or projection onto a quotient algebra, we denote the α-cylinder briefly by $Z_\alpha(A, D)$. The cone $S(A, D, \varphi)$ of a homomorphism $\varphi\colon A \to D$ is defined as the fibered sum $A \oplus_D D[0, 1)$ with projections $\varphi\colon A \to D$ and $D[0, 1) \to D[0]$. The abbreviated notation is $S(A, D)$. Using Corollary 1 of subsection 7, it is easy to verify that the presence of strictly positive elements in an algebra B and an ideal $J \subset B$ implies the presence of strictly positive elements in $Z_\alpha(J, B)$, $Z_\alpha(B, B/J)$, $S(J, B)$ and $S(B, B/J)$. For the separability or nuclearity of these algebras it is sufficient that the algebra B have these properties (see [21], Corollary 4).

19. Let B be an ideal in the algebra D and assume that B and D have countable approximate identities and the action of G on B and D is continuous. We fix an isomorphism $u\colon \mathcal{H}_D \oplus \mathcal{H}_D \simeq \mathcal{H}_B \oplus \mathcal{H}_D$ (see subsection 12). *There exists an isomorphism*

$$\{\varphi_t\}_{t\in[0,1]}\colon \mathcal{L}(\mathcal{H}_D \oplus \mathcal{H}_D) \to \mathcal{L}(\mathcal{H}_{D[0,1]} \oplus \mathcal{H}_{D[0,1]})$$

such that $\varphi_1 = 1$ and $u\varphi_0 u^{-1}\colon \mathcal{L}(\mathcal{H}_D \oplus \mathcal{H}_D) \to \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_D)$ coincides with the restriction homomorphism. Moreover,

$$\{\varphi_t\}(M_2 \otimes \mathcal{K}_D) \subset M_2 \otimes \mathcal{K}_{D[0,1]}, \quad \{\varphi_t\}(M_2 \otimes \mathcal{K}_B) \subset M_2 \otimes \mathcal{K}_{B[0,1]}$$

(here $\mathcal{H}_B$ is considered as an ideal in $\mathcal{H}_D$).

PROOF. By the stabilization theorem (subsection 12) there exists an isomorphism

$$\{\nu_t\}\colon \mathcal{H}_{D[0,1]} \oplus \mathcal{H}_{D[0,1]} \simeq \mathcal{H}_{Z_0(B,D)} \oplus \mathcal{H}_{D[0,1]}.$$

In view of subsection 17, we may assume that its restriction over the point $0 \in [0, 1]$ coincides with u, and its restriction over the point $1 \in [0, 1]$ with the identity mapping. We denote the restriction homomorphism

$$\mathcal{L}(\mathcal{H}_{D[0,1]} \oplus \mathcal{H}_{D[0,1]}) \to \mathcal{L}(\mathcal{H}_{Z_0(B,D)} \oplus \mathcal{H}_{D[0,1]})$$

by $\{\psi_t\}$. We set $\{\varphi_t\} = \{\nu_t\}^{-1} \cdot \{\psi_t\} \cdot \{\nu_t\}$. ■

20. Let A be an ideal in an algebra D and assume that A and D have countable approximate identities and the action of G on A and D is continuous. We identify $A \oplus \mathcal{H}_D$ with $D \oplus \mathcal{H}_D$ (see subsection 12) and denote the natural homomorphisms

$$D = D \oplus 0 \hookrightarrow \mathcal{L}(D \oplus \mathcal{H}_D), \qquad D = D \oplus 0 \to \mathcal{L}(A \oplus \mathcal{H}_D)$$

by φ and ψ, respectively. *There exists a homomorphism*

$$\{\varphi_t\}\colon D \to \mathcal{L}(D[0, 1] \oplus \mathcal{H}_{D[0,1]}),$$

such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$. Moreover,

$$\{\varphi_t\}(D) \subset \mathcal{K}(D[0, 1] \oplus \mathcal{H}_{D[0,1]}), \qquad \{\varphi_t\}(A) \subset \mathcal{K}(A[0, 1] \oplus \mathcal{H}_{A[0,1]}).$$

The proof can be obtained from the existence of an isomorphism

$$Z_1(A, D) \oplus \mathcal{H}_{D[0,1]} \simeq D[0, 1] \oplus \mathcal{H}_{D[0,1]}$$

by the same method as in the preceding subsection. ■

§2. Graded algebras and Clifford algebras

The operator K-functor is defined most naturally on the category of $\mathbf{Z}_2$-graded algebras. In this section we compile the basic facts on graded algebras necessary in what follows.

1. A C^*-algebra B is said to be $(\mathbf{Z}_2)$-*graded* if we have a decomposition $B = B^{(0)} \oplus B^{(1)}$ in which $B^{(0)}$ and $B^{(1)}$ are closed selfadjoint linear subspaces such that $B^{(i)}B^{(j)} \subset B^{(i+j)}$ for $i, j \in \mathbf{Z}_2$. The action of G and the "real" involution preserve this decomposition. A homomorphism $f\colon B_1 \to B_2$ is said to be graded (of degree 0) if $f(B_1^{(i)}) \subset B_2^{(i)}$ $(i \in \mathbf{Z}_2)$. The grading of B is said to be *trivial* if $B^{(1)} = 0$. Any algebra B can be considered trivially graded by putting $B^{(0)} = B$ and $B^{(1)} = 0$. In particular, the algebra C of scalars will be considered trivially graded. For a unital algebra B the condition $1 \in B^{(0)}$ is satisfied.

The equality $\deg x = i$ means that $x \in B^{(i)}$.

The *graded commutator* $[x, y]$ is defined on homogeneous elements $x, y \in B$ by the formula

$$[x, y] = xy - (-1)^{\deg x \cdot \deg y} yx.$$

This definition can be extended to all $x, y \in B$ by linearity.

2. A Hilbert module E is said to be *graded* if it admits a decomposition $E = E^{(0)} \oplus E^{(1)}$ into the direct sum of closed subspaces, where for every $i, j \in \mathbf{Z}_2$ we have $E^{(i)} \cdot B^{(j)} \subset E^{(i+j)}$ (action of B) and $(E^{(i)}, E^{(j)}) \subset B^{(i+j)}$ (inner product). The action of G and the "real" involution preserve this decomposition. The grading of E defines a grading of $\mathcal{L}(E)$ and $\mathcal{K}(E)$ in the following way: $\deg T = j$ if $T(E^{(i)}) \subset E^{(i+j)}$ $(i, j \in \mathbf{Z}_2)$. Since $\mathfrak{M}(B) \simeq \mathcal{L}(B)$ (see §1.14), the algebra $\mathfrak{M}(B)$ is also graded. It is easy to see that the isomorphism $\mathcal{L}(E) \simeq \mathfrak{M}(\mathcal{K}(E))$ is graded. The opposite grading of the Hilbert module E can be obtained by interchanging $E^{(0)}$ and $E^{(1)}$. It is obvious that the grading of $\mathcal{L}(E)$ is preserved under changing the grading of E to the opposite grading. Moreover, if it is known that the grading of $\mathcal{K}(E)$ is induced by the grading of E, then restoration of the grading of E by means of the grading of $\mathcal{K}(E)$ is possible in two ways: since for every $x, y \in E$ we have $\deg x - \deg y = \deg \theta_{x,y}$, it is sufficient to give $\deg x$ for a single element $x \in E$.

We note that if B has the trivial grading, then $E^{(0)}$ and $E^{(1)}$ are orthogonal B-modules and the grading operator $\varepsilon \in \mathcal{L}(E)$ is defined as follows: $\varepsilon = 1$ on $E^{(0)}$ and $\varepsilon = -1$ on $E^{(1)}$.

3. A *canonically graded* Hilbert space is defined as the direct sum $\mathcal{H}_B \oplus \mathcal{H}_B$, where the gradings of the two summands are opposite and the grading of the first one $\mathcal{H}_B = \bigoplus_1^\infty V_j \otimes B$ is defined in such a way that for all j, all $x \in V_j$, and all $b \in B$ we have $\deg(x \otimes b) = \deg b$. The stabilization theorem (§1.12) can be generalized verbatim to the graded case (in the proof a homogeneous system of generators has to be considered).

If $u \in \mathcal{L}(\mathcal{H}_B)$ is a unitary element of degree 0, then there exists a homotopy $\{u_t\} \in \mathcal{L}(\mathcal{H}_{B[0,1]})$ *of degree* 0 contracting u to 1 (see §1.17).

4. If in the algebra B there exists a positive element h, then there exists a strictly positive element k of degree 0, too. Indeed, let $h = h_0 + h_1$, $\deg h_i = i$. Then $h^2 \leqslant 2(h_0^2 + h_1^2) = k \in B^{(0)}$.

5. A graded imbedding $\pi\colon A \hookrightarrow \mathcal{L}(\mathcal{H})$ is called a *graded G-imbedding* if

$$(1 \otimes \pi)^{(0)}_{\text{inv}}\colon (\mathcal{L}(C^n) \otimes A)^{(0)}_{\text{inv}} \to \mathcal{L}((C^n \otimes \mathcal{H})^{(0)}_{\text{inv}})$$

is an imbedding for any n, any grading of C^n, and any unitary action of G on C^n (see §1.16). If A is separable, then (unitary) graded G-imbeddings always exist (the proof is the same).

6. We note that the tensor product $A \otimes B$ of the graded algebras A and B has a natural grading: $\deg(a \otimes b) = \deg a + \deg b$. Now we define the *skew-commutative* tensor product $A \hat{\otimes} B$. We denote by $A \hat{\odot} B$ the algebraic tensor product of A and B as linear spaces with the same grading as for $A \otimes B$. We define a product and an involution by the formulas

$$(a_1 \hat{\odot} b_1)(a_2 \hat{\odot} b_2) = (-1)^{\deg b_1 \cdot \deg a_2} (a_1 a_2 \hat{\odot} b_1 b_2),$$
$$(a \hat{\odot} b)^* = (-1)^{\deg a \cdot \deg b} (a^* \hat{\odot} b^*),$$

and a C^*-norm by the formula [19]:

$$\left\| \sum_{i=1}^{m} a_i \hat{\odot} b_i \right\|^2 = \sup \frac{(\rho \hat{\odot} \lambda)\left[\left(\sum_{j=1}^{n} x_j \hat{\odot} y_j\right)^* \left(\sum_{i=1}^{m} a_i \hat{\odot} b_i\right)^* \left(\sum_{i=1}^{m} a_i \hat{\odot} b_i\right)\left(\sum_{j=1}^{n} x_j \hat{\odot} y_j\right)\right]}{(\rho \hat{\odot} \lambda)\left[\left(\sum_{j=1}^{n} x_j \hat{\odot} y_j\right)^* \left(\sum_{j=1}^{n} x_j \hat{\odot} y_j\right)\right]}$$

where sup is taken for all nonzero elements $\sum_{j=1}^{n} x_j \hat{\odot} y_j \in A \hat{\odot} B$ and all *graded* states ρ and λ (i.e., for all states $\rho = 0$ on $A^{(1)}$ and all states $\lambda = 0$ on $B^{(1)}$). The completion of $A \hat{\odot} B$ in this norm will be denoted by $A \hat{\otimes} B$.

If in A (or $\mathfrak{M}(A)$) there is an invariant Hermitian element (*graded operator*) ε such that $\varepsilon^2 = 1$ and the grading of A is defined by the formula $a\varepsilon = (-1)^{\deg a}\varepsilon a$ (for homogeneous elements a), then $A \hat{\odot} B \simeq A \odot B$: $a \hat{\odot} b \leftrightarrow a\varepsilon^{\deg b} \odot b$. This graded isomorphism can be extended to an isomorphism $A \hat{\otimes} B \simeq A \otimes B$. (Analogously, if ε is a grading operator in B or $\mathfrak{M}(B)$, then the correspondence $a \hat{\otimes} b \leftrightarrow a \otimes \varepsilon^{\deg a} b$ also defines an isomorphism of $A \hat{\otimes} B$ onto $A \otimes B$.) In particular, $\mathcal{K} \hat{\otimes} B \simeq \mathcal{K} \otimes B$.

7. For $A \otimes B$ there exists a transposition isomorphism: $A \otimes B \simeq B \otimes A$. In the anticommutative case an analogous isomorphism is defined by the formula

$$a \hat{\otimes} b \to (-1)^{\deg a \cdot \deg b} b \hat{\otimes} a.$$

8. (cf. [24]). Let E_1 and E_2 be Hilbert modules over B_1 and B_2, respectively, and φ: $B_1 \to \mathcal{L}(E_2)$ a graded homomorphism. The algebraic tensor product $E_1 \odot E_2$ is a right B_2-module with the grading

$$\deg (x_1 \odot x_2) = \deg x_1 + \deg x_2$$

and inner product

$$(x_1 \odot x_2,\ y_1 \odot y_2) = (x_2,\ \varphi((x_1,\ y_1))y_2).$$

Factoring $E_1 \odot E_2$ over the B_2-module $N = \{z \in E_1 \odot E_2 | (z, z) = 0\}$, and then completing it in the norm $\|z\| = \|(z, z)\|^{1/2}$, we obtain a Hilbert B_2-module, which will be denoted by $E_1 \otimes_{B_1} E_2$ or $E_1 \hat{\otimes}_{B_1} E_2$. The correspondence $F \to F \otimes 1$ defines a homomorphism φ_*: $\mathcal{L}(E_1) \to \mathcal{L}(E_1 \otimes_{B_1} E_2)$. If φ is an imbedding, so is φ_*. In particular, every faithful representation $B_1 \to \mathcal{L}(H)$ induces a faithful representation $\mathcal{L}(E_1) \to \mathcal{L}(E_1 \otimes_{B_1} H)$.

In the case $E_2 = B_2$ and $\varphi: B_1 \to B_2 \hookrightarrow \mathcal{L}(B_2)$ we have $\varphi_*(\mathcal{K}(E_1)) \subset \mathcal{K}(E_1 \otimes_{B_1} B_2)$. Indeed,

$$\theta_{x,y} \otimes 1 = \lim_{\alpha \to +0} \theta_{x\otimes u_\alpha, y\otimes u_\alpha},$$

where $u_\alpha = \varphi(\langle y, y\rangle \cdot [\langle y, y\rangle + \alpha]^{-1})$.

For unital B_1 and φ we have the following isomorphisms:

$$E_1 \otimes_{B_1} E_2 \simeq E_2 \quad \text{for } E_1 = B_1,$$
$$E_1 \otimes_{B_1} E_2 \simeq \mathcal{H}_{B_2} \quad \text{for } E_1 = \mathcal{H}_{B_1}, \qquad E_2 = B_2 \quad \text{or } \mathcal{H}_{B_2}.$$

The above tensor product also has the following associative poperty:

$$(E_1 \otimes_{B_1} E_2) \otimes_{B_2} E_3 \simeq E_1 \otimes_{B_1} (E_2 \otimes_{B_2} E_3),$$

where $E_1 \otimes_{B_1} E_2$ and $E_2 \otimes_{B_2} E_3$ are constructed from the homomorphisms φ_1: $B_1 \to \mathcal{L}(E_2)$ and φ_2: $B_2 \to \mathcal{L}(E_3)$, respectively, and $E_1 \otimes_{B_1}(E_2 \otimes_{B_2} E_3)$ from the homomorphism $B_1 \to \mathcal{L}(E_2) \to \mathcal{L}(E_2 \otimes_{B_2} E_3)$. In particular, for φ_1: $B_1 \to B_2$ and a unital φ_2: $B_2 \to \mathcal{L}(E_3)$ the following operation of change of algebras is defined:

$$E_1 \otimes_{B_1} E_3 \simeq E_1 \otimes_{B_1} (B_2 \otimes_{B_2} E_3) \simeq (E_1 \otimes_{B_1} B_2) \otimes_{B_2} E_3.$$

9. If E_1 and E_2 are Hilbert modules over B_1 and B_2, respectively, then the algebraic tensor product $E_1 \hat{\odot} E_2$ is a $B_1 \hat{\odot} B_2$-module:

$$(x_1 \hat{\odot} x_2)(b_1 \hat{\odot} b_2) = (-1)^{\deg x_2 \cdot \deg b_1} (x_1 b_1 \hat{\odot} x_2 b_2),$$

with inner product

$$\langle x_1 \hat{\odot} x_2, \ y_1 \hat{\odot} y_2\rangle = (-1)^{\deg x_2 \cdot (\deg x_1 + \deg y_1)} \langle x_1, y_1\rangle \hat{\odot} \langle x_2, y_2\rangle$$

and grading $\deg(x_1 \hat{\odot} x_2) = \deg x_1 + \deg x_2$. (In verifying the positiveness of this inner product we may assume that E_1 and E_2 are *finitely* generated, and using the stabilization theorem we may imbed E_1 into $\mathcal{H}_{B_1}$ and E_2 into $\mathcal{H}_{B_2}$. After this all reduces to the case $E_1 = B_1$, $E_2 = B_2$.) The completion of $E_1 \hat{\odot} E_2$ in the norm $\|z\| = \|\langle z, z\rangle\|^{1/2}$ is the Hilbert $B_1 \hat{\otimes} B_2$-module $E_1 \hat{\otimes} E_2$. It is easy to verify that

$$B_1 \hat{\otimes} \mathcal{H}_{B_2} \simeq \mathcal{H}_{B_1} \hat{\otimes} B_2 \simeq \mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2} \simeq \mathcal{H}_{B_1 \hat{\otimes} B_2}.$$

If φ_i: $B_i \to \mathcal{L}(H_i)$ $(i = 1, 2)$ are exact representations, then, according to subsection 8, a faithful representation

$$\mathcal{L}(E_1 \hat{\otimes} E_2) \to \mathcal{L}\big((E_1 \hat{\otimes} E_2) \hat{\otimes}_{B_1 \hat{\otimes} B_2} (H_1 \hat{\otimes} H_2)\big) \simeq \mathcal{L}\big((E_1 \hat{\otimes}_{B_1} H_1) \hat{\otimes} (E_2 \hat{\otimes}_{B_2} H_2)\big)$$

is defined. Using this, it is easy to verify that the natural homomorphism

$$\mathcal{L}(E_1) \hat{\otimes} \mathcal{L}(E_2) \to \mathcal{L}(E_1 \hat{\otimes} E_2): (F_1 \hat{\otimes} F_2)(x_1 \hat{\otimes} x_2)$$
$$= (-1)^{\deg F_2 \cdot \deg x_1}(F_1(x_1) \hat{\otimes} F_2(x_2))$$

is an imbedding. Its restriction $\mathcal{K}(E_1) \hat{\otimes} \mathcal{K}(E_2) \to \mathcal{K}(E_1 \hat{\otimes} E_2)$ is an isomorphism.

We also need the following general relation for Hilbert B_i-modules $E_1, \ldots, E_4$ and homomorphisms φ: $B_1 \to \mathcal{L}(E_3)$ and ψ: $B_2 \to \mathcal{L}(E_4)$:

$$(E_1 \hat{\otimes} E_2) \hat{\otimes}_{B_1 \hat{\otimes} B_2} (E_3 \hat{\otimes} E_4) \simeq (E_1 \hat{\otimes}_{B_1} E_3) \hat{\otimes} (E_2 \hat{\otimes}_{B_2} E_4),$$

where

$$(e_1 \hat{\otimes} e_2) \hat{\otimes} (e_3 \hat{\otimes} e_4) \to (-1)^{\deg e_2 \cdot \deg e_3} (e_1 \hat{\otimes} e_3) \hat{\otimes} (e_2 \hat{\otimes} e_4).$$

10. The generalized theorems of Stinespring and Voiculescu (see §§1.15 and 1.16) can be carried over to the graded case in the following way. In Stinespring's theorem the mapping φ and the homomorphism ρ have to be considered graded; the proof can be preserved verbatim. In the definition of approximate equivalence all u_n have to have degree 0. In Voiculescu's theorem the imbedding $A \hookrightarrow \mathcal{L}(\mathcal{H})$ has to be a graded regular G-imbedding, and the homomorphism φ graded. In the proof we have to make a number of obvious modifications (for example, all mappings under consideration have to be graded; in Glimm's lemma a sequence of vectors of degree 0 is considered; the members of the approximate identity have degree 0; and so on). Only the following analogue of Lemma 7 of [17] deserves mention:

LEMMA. *If φ: $A \to B$ is a nuclear graded mapping, then φ belongs to the closure, in the topology of pointwise convergence in norm, of the set of mappings of the form $A \xrightarrow{\sigma} M_n \otimes M_2 \xrightarrow{\tau} B$, where σ and τ are completely positive and graded (the grading of $M_n \otimes M_2$ is defined by the grading operator $\varepsilon_0 = 1 \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$).*

PROOF. Let σ_0: $A \to M_n$ and τ_0: $M_n \to B$ be chosen so that $\|\varphi(a) - \tau_0\sigma_0(a)\| \leqslant \delta$ for all a from some compact subset $X \subset A$. We define σ: $A \to M_n \otimes M_2$ and τ_1: $M_n \otimes M_2 \to B$ by the formulas

$$\sigma(a) = \sigma_0(a) \otimes \varepsilon_1^{\deg a}, \qquad \tau_1\left(\sum_{i,j=1}^{2} a_{ij} \otimes e_{ij}\right) = \frac{1}{2} \sum_{i,j=1}^{2} \tau_0(a_{ij}),$$

where $\{e_{ij}\}$ is the standard system of matrix units in M_2 and $\varepsilon_1 = e_{12} + e_{21}$. The mappings σ and τ_1 are completely positive, σ is graded, and $\tau_1\sigma = \tau_0\sigma_0$. We denote by $p^{(i)}$ the projection onto $B^{(i)}$ in B, and we set $\tau(a) = p^{(\deg a)}(\tau_1(a))$. If the grading of B is defined by a grading operator ε, then

$$\tau(a) = \frac{1}{2}(\tau_1(a) + (-1)^{\deg a} \varepsilon \tau_1(a) \varepsilon),$$

from which we obtain easily that τ is completely positive. In the general case the complete positivity of τ follows from the existence of the imbedding $B \hookrightarrow \mathcal{L}(H)$. Since φ was graded, we have $\|\varphi(a) - \tau\sigma(a)\| \leqslant \delta$ for $a \in X$. ■

11. A finite-dimensional linear space V over C equipped with a positive definite inner product (linear in the second argument) and an isometric antilinear involution $x \to x^*$ is called a *linear *-space*. By $(-V)$ we denote the same space with involution $x \to -x^*$. We define a quadratic form Q by the formula $Q(x) = (x^*, x)$. The Clifford algebra Cliff(V, Q) (see [5]) will be denoted by C_V. Setting $(x_1 \cdots x_n)^* = x_n^* \cdots x_1^*$ for $x_1, \ldots, x_n \in V$, we obtain an antilinear involution on C_V. We introduce the C^*-norm in the following way. We denote by λ_x the operator of exterior multiplication by $x \in V$ in the exterior algebra $\bigwedge^*(V)$ (the word "algebra" does not mean C^*-algebra here). The mapping μ: $V \oplus (-V) \to \mathcal{L}(\bigwedge^*(V))$ is defined by the formula $\mu(x \oplus y) = \lambda_{x+y} + \lambda^*_{x^*-y^*}$, where λ_x^* is the adjoint of λ_x with respect to the inner product induced from V. Since $(\mu(x \oplus y))^2 = (x^* - y^*, x + y) = Q(x) - Q(y)$, the mapping μ can be extended uniquely to a *-homomorphism $C_{V \oplus (-V)} \to \mathcal{L}(\bigwedge^*(V))$. This is an isomorphism. In particular, $C_V \hookrightarrow \mathcal{L}(\bigwedge^*(V))$, which gives a C^*-norm on C_V.

The isometric action of G on V commuting with the involution $*$ can be extended uniquely to a continuous action of G on C_V: $g(x_1 \cdots x_n) = g(x_1) \cdots g(x_n)$. If $x \to \bar{x}$ is another isometric antilinear involution on V ("real" involution), commuting with the involution $*$, then, setting $\overline{(x_1 \cdots x_n)} = \bar{x}_1 \cdots \bar{x}_n$, we obtain a "real" structure on C_V (the action of G on V has to be "real": $\overline{g(x)} = \bar{g}(\bar{x})$).

We shall identify $C_V \hat{\otimes} C_W$ with $C_{V \oplus W}$ by means of the isomorphism mapping $v \hat{\otimes} 1$ onto $v \oplus 0$ and $1 \hat{\otimes} w$ onto $0 \oplus w$, where $v \in V$ and $w \in W$. The isomorphism μ: $C_{V \oplus (-V)} \xrightarrow{\sim} \mathfrak{L}(\bigwedge^*(V))$ fixed above allows us to identify $C_V \hat{\otimes} C_{-V}$ with $\mathfrak{L}(\bigwedge^*(V))$.

12. Let ξ be a vector bundle over a locally compact space X equipped with a Hermitian metric and an isometric involution $x \to x^*$ which is antilinear on the fibers. Such a bundle will be called a *vector $*$-bundle*. By $(-\xi)$ we denote the same bundle with involution $x \to -x^*$. We consider the Clifford bundle $\mathrm{Cliff}(\xi, Q)$ associated with the quadratic form $Q(x) = (x^*, x)$ on ξ. The space of continuous sections of this bundle converging to 0 at ∞ will be denoted by $C_\xi(X)$. The same construction as in the preceding subsection allows us to introduce a C^*-norm on $C_\xi(X)$ and obtain the isomorphism $C_{\xi \oplus (-\xi)}(X) \simeq \mathcal{K}(\Gamma(\bigwedge^*(\xi)))$, where $\Gamma(\bigwedge^*(\xi))$ is the Hilbert $C(X)$-module of continuous sections of the bundle $\bigwedge^*(\xi)$ converging to 0 at ∞. The action of G and the antilinear involution $x \to \bar{x}$ on ξ induce, as above, an action of G and a "real" involution on $C_\xi(X)$.

13. *The algebra $C_{p,q}$.* Let $V = V_{p,q} = C^p \oplus C^q$. Denote by τ coordinatewise complex conjugation on V. We set $x^* = \tau(x)$ for $x \in C^p$ and $x^* = -\tau(x)$ for $x \in C^q$; $\bar{x} = \tau(x)$; the action of G on V is trivial. The Clifford algebra $C_{V_{p,q}}$ will be denoted by $C_{p,q}$. Its generators are $\varepsilon_1, \ldots, \varepsilon_p, e_1, \ldots, e_q$ (coordinate basis in $V_{p,q}$), $\varepsilon_i^2 = 1$, $\varepsilon_i^* = \varepsilon_i$, $\bar{\varepsilon}_i = \varepsilon_i$ ($i \leqslant p$); $e_j^2 = -1$, $e_j^* = -e_j$, $\bar{e}_j = e_j$ ($j \leqslant q$). (We note that this notation differs from the notation $C_{p,q}$ in [15] in that p and q are interchanged.)

14. An *orientation* of the Clifford algebra C_V is, by definition, a homogeneous element $\omega \in C_V$ such that

1) $\omega^* = \pm\omega$; $\omega^*\omega = 1$; $\bar{\omega} = \omega$ (in the "real" case), and

2) $\forall\, x \in C_V$ $x\omega = (-1)^{\deg x(\deg \omega + 1)}\omega x$.

LEMMA. *The orientation exists and is defined uniquely up to multiplication by ± 1 in the real and "real" cases and up to multiplication by ± 1 or $\pm i$ in the complex case.*

PROOF. If $f_1, \ldots, f_n$ is an orthonormal basis in V such that $f_k^* = \pm f_k$ and $\bar{f}_k = f_k$, then $\omega = f_1 \cdots f_n$ is an orientation. On the other hand, an arbitrary element $\omega \in C_V$ can be written uniquely in the form of a linear combination of monomials of the form $f_{i_1} \cdots f_{i_k}$, where $i_1 < \cdots < i_k$. Let $\omega = a + f_1 b$, where the polynomials a and b belong to C_V and do not contain f_1. Then

$$f_1\omega = f_1 a + f_1^2 b, \quad \omega f_1 = (-1)^{\deg\omega} f_1 a + (-1)^{\deg\omega - 1} f_1^2 b.$$

A comparison with 2) shows that $f_1 a = 0$, from which we get $a = \pm f_1(f_1 a) = 0$, i.e., $\omega = f_1 b$. An analogous reasoning with different f_j leads to the relation $\omega = \alpha f_1 \cdots f_n$, where $\alpha \in C$. From the first two relations of section 1) it follows that $\bar{\alpha} = \pm\alpha$ and $\bar{\alpha} \cdot \alpha = 1$, from which we get $\alpha = \pm 1$ or $\pm i$. In the real and "real" cases we have $\bar{\alpha} = \alpha$, i.e. $\alpha = \pm 1$. ∎

15. An algebra C_V with fixed orientation ω_V is called an *oriented* Clifford algebra. If φ is a graded automorphism of C_V, then $\varphi(\omega_V) = \pm\omega_V$. Indeed, ω_V and $\varphi(\omega_V)$ are orientations of C_V, and either both of those elements are Hermitian, or both of them are skew-Hermitian, and therefore the assertion follows from the preceding lemma. We say that an isomorphism φ: $C_V \to C_W$ *preserves orientation* if $\varphi(\omega_V) = \omega_W$.

If C_V and C_W are oriented, then the orientation of $C_V \hat{\otimes} C_W \simeq C_{V\oplus W}$ will be the element $\omega_V \hat{\otimes} \omega_W$. The orientation of $\mathfrak{L}(\bigwedge^*(V))$ will be the element ε_V which is equal to $(-1)^k$ on $\bigwedge^k(V)$. If an orientation $\omega_V \in C_V$ is fixed, we define $\omega_{-V} \in C_{-V}$ as the (unique) orientation such that $\omega_V \hat{\otimes} \omega_{-V} = \varepsilon_V$ for the isomorphism of subsection 11. It is easy to see that if $\omega_V = f_1 \cdots f_n$ as in subsection 14, and among the elements $f_1, \ldots, f_n$ exactly q are skew-Hermitian, then $\omega_{-V} = (-1)^q f_n \cdots f_1$. (Generally speaking, $\omega_{-(-V)} \neq \omega_V$.)

16. The *standard* orientation on $C_{p,q}$ is defined in the following way: $\omega_{p,q} = (-i)^q \varepsilon_1 \cdots \varepsilon_p e_1 \cdots e_q$ in the complex case, and $\omega_{p,q} = \varepsilon_1 \cdots \varepsilon_p e_1 \cdots e_q$ in the real or "real" case. It is easy to verify that for the trivial action of G on V the algebra C_V is isomorphic with preservation of orientation to one of the algebras $C_{p,q}$. In particular, $C_{p,q} \hat{\otimes} C_{p',q'} \simeq C_{p+p',q+q'}$:

$$\begin{aligned} \varepsilon_i \hat{\otimes} 1 &\to \varepsilon_i, \quad i \leqslant p, \\ e_j \hat{\otimes} 1 &\to e_j, \quad j \leqslant q, \\ 1 \hat{\otimes} \varepsilon_i &\to (-1)^q \varepsilon_{i+p}, \quad i \leqslant p', \\ 1 \hat{\otimes} e_j &\to e_{j+q}, \quad j \leqslant q'. \end{aligned}$$

Moreover, $C_{-(p,q)} = C_{-V_{p,q}} \simeq C_{q,p}$. We define an orientation-preserving isomorphism by the formulas

$$\begin{aligned} \varepsilon_i &\to (-1)^{i-1} e_i, \quad i \leqslant p, \\ e_j &\to (-1)^j \varepsilon_j, \quad j \leqslant q. \end{aligned}$$

17. For what follows we fix the orientation-preserving graded isomorphisms f: $C_{p,q+1} \to C_{p+1,q}$ in the complex case and g: $C_{p,q+4} \to C_{p+4,q}$ in the general case. We set

$$f(\varepsilon_k) = \varepsilon_k \quad (k \leqslant p); \quad f(e_1) = i\varepsilon_{p+1}; \quad f(e_j) = e_{j-1} \quad (2 \leqslant j \leqslant q+1).$$

We define a homomorphism by the formulas

$$\begin{gathered} g(\varepsilon_k) = \varepsilon_k \quad (k \leqslant p); \quad g(e_j) = \varepsilon_{p+1} \cdots \hat{\varepsilon}_{p+j} \cdots \varepsilon_{p+4} \quad (j \leqslant 4); \\ g(e_j) = e_{j-4} \quad (j \geqslant 5). \end{gathered}$$

18. Let $V = C^n = C^{p+q}$, and let $\mathbf{R}^n \subset C^n$, the real coordinate subspace. We denote coordinatewise complex conjugation in V by τ. We set $x^* = \tau(x)$ and $\bar{x} = \tau(x)$ for $x \in C^p$; $\bar{x} = -\tau(x)$ for $x \in C^q$. The group Spin V is defined as the subgroup of the group of invertible elements $\{g\}$ of degree 0 in C_V satisfying the following conditions:

1) $g^*g = 1$, and
2) $gxg^{-1} \in \mathbf{R}^n$ for $x \in \mathbf{R}^n$.

The "real" structure on Spin V is induced from C_V. If we change the involution * to the opposite involution $x^* = -\tau(x)$, we obtain the group Spin$(-V)$ isomorphic to

$\mathrm{Spin}(V)$. Indeed, setting $\mu_V(x) = \lambda_x + \lambda_x^*$ and $\mu_{-V}(x) = \lambda_x - \lambda_x^*$ for $x \in V$ (see subsection 11), and $\varepsilon_V = (-1)^k$ and $u = (-1)^{k(k-1)/2}$ on $\bigwedge^k(V)$, we obtain $u\mu_V(x)u^{-1} = \mu_{-V}(x) \cdot \varepsilon_V$ for $x \in V$. Extending μ_V and μ_{-V} to C_V and C_{-V}, respectively, we obtain

$$u\mu_V(\mathrm{Spin}\ (V))u^{-1}=\mu_{-V}(\mathrm{Spin}\ (-V)).$$

The groups $\mathrm{Spin}(V)$ and $\mathrm{Spin}(-V)$ act on $\mathbf{R}^n$ according to the formula $g(x) = gxg^{-1}$, $x \in \mathbf{R}^n$. It is easy to verify that in our identification, the actions of $\mathrm{Spin}(V)$ and $\mathrm{Spin}(-V)$ coincide on $\mathbf{R}^n$. Besides, $\mathrm{Spin}(V)$ and $\mathrm{Spin}(-V)$ act on $\mathfrak{L}(\bigwedge^*(V))$ according to the formula

$$g(a) = \mu_{\pm V}(g) \cdot a \cdot \mu_{\pm V}(g^{-1}), \qquad a \in \mathfrak{L}(\textstyle\bigwedge^*(V)).$$

Since the operators $\mu_V(\mathrm{Spin}(V))$ commute with $\mu_{-V}(\mathrm{Spin}(-V))$ (and with all $\mu_{-V}(x)$ for $x \in V$), the following diagonal action of $\mathrm{Spin}(V)$ on $\mathfrak{L}(\bigwedge^*(V))$ is defined:

$$\delta(g)(a) = \mu_V(g)\mu_{-V}(g) \cdot a \cdot \mu_V(g^{-1})\mu_{-V}(g^{-1}) \quad \text{for } a \in \mathfrak{L}(\textstyle\bigwedge^*(V)).$$

This action of $\mathrm{Spin}(V)$ on $\mathfrak{L}(\bigwedge^*(V))$ is induced by the action of $\mathrm{Spin}(V)$ on V. For the proof it is sufficient to consider elements a of the form $\lambda_x \pm \lambda_x^*$, $x \in V$.

§3. Intersection product (technical part)

Theorems 4 and 5 of this section provide the technical basis for our construction of the intersection product in the K-functor which shall be considered in §4. We shall work in the category of graded algebras. We recall that $[a, b]$ means the graded commutator $ab - (-1)^{\deg a \cdot \deg b}\, ba$ (see §2.1).

LEMMA 1. *Let A be an algebra, and let $a, b, d \in A$. Then*

$$a^*db + b^*d^*a \leqslant \|d\| \cdot (a^*a + b^*b).$$

PROOF. We imbed A in the algebra of operators in some Hilbert space H. Then for all $\xi \in H$

$$((a^*db + b^*d^*a)\xi, \xi) = (db\xi, a\xi) + (d^*a\xi, b\xi)$$
$$\leqslant 2\|d\| \cdot \|a\xi\| \cdot \|b\xi\| \leqslant \|d\| \cdot (\|a\xi\|^2 + \|b\xi\|^2) = \|d\| \cdot ((a^*a + b^*b)\xi, \xi). \ \blacksquare$$

THEOREM 1. *Let A be a unital algebra, B a subalgebra with continuous action of G, having a strictly positive element. Assume that the family $\{F_x\}_{x \in X} \subset A$ of elements of A is bounded in norm, for every $x \in X$ and $b \in B$ we have $[F_x, b] \in B$, and the closure of the set $\{[F_x, b] | x \in X\}$ is compact in B for every $b \in B$. Then for every number $m \geqslant 0$ in the algebra B there exist a strictly positive invariant element h of degree 0 and a number $c > 0$ such that*

$$[h, F_x]\cdot[h, F_x]^* + [h, F_x]^*\cdot[h, F_x] \leqslant ch^m \tag{1}$$

for every $x \in X$.

PROOF. We may assume that $\|F_x\| \leqslant 1$ for every $x \in X$. We denote by $\{u_i\}$ a countable increasing Abelian approximate identity in B consisting of invariant elements of degree 0. We construct an increasing sequence of integers $\{k_i\}$ in the following way.

Let $k_1 \geqslant 1$. If $k_1, \dots, k_r$ are already constructed, then k_{r+1} is chosen so that

$$\|(1-u_{k_{r+1}})^{1/4}\cdot u_{k_r}^{1/4}\| \leqslant 2^{-(2r+2)}, \tag{2}$$

$$\|(1-u_{k_{r+1}})^{1/4}\cdot F_x\cdot u_{k_r}^{1/4}\| \leqslant 2^{-(2r+2)}, \tag{3}$$

$$\|(1-u_{k_{r+1}})^{1/4}\cdot F_x^*\cdot u_{k_r}^{1/4}\| \leqslant 2^{-(2r+2)} \tag{4}$$

for every $x \in X$. This is possible thanks to the compactness of the sets $\{F_x u_{k_r}^{1/4} - u_{k_r}^{1/4}F_x\}$ and $\{F_x^* u_{k_r}^{1/4} - u_{k_r}^{1/4}F_x^*\}$. We introduce the following notation: $b_1 = u_{k_1}$, $b_i = u_{k_i} - u_{k_{i-1}}$ for $i \geqslant 2$, $\varepsilon_{p,q} = 2^{-p-q}$ for $|p-q| \geqslant 2$, and $\varepsilon_{p,q} = 1$ for $|p-q| \leqslant 1$. Then

$$\forall b \in B \quad \sum_{i=1}^{\infty} b_i b = \sum_{i=1}^{\infty} b b_i = b, \tag{5}$$

$$\forall p, q \quad \|b_p^{1/4} b_q^{1/4}\| \leqslant \varepsilon_{p,q}, \tag{6}$$

$$\forall x \in X,\ \forall p, q \quad \|b_p^{1/4} F_x b_q^{1/4}\| \leqslant \varepsilon_{p,q}. \tag{7}$$

Indeed, if, for example, $p \geqslant q+2$, then for $p = i+2$ we obtain

$$q \leqslant i, \quad b_p^{1/2} \leqslant (1-u_{k_{i+1}})^{1/2}, \quad b_q^{1/2} \leqslant u_{k_i}^{1/2},$$

$$\|b_p^{1/4}F_x b_q^{1/4}\|^2 = \|b_p^{1/4}F_x b_q^{1/2} F_x^* b_p^{1/4}\| \leqslant \|b_p^{1/4} F_x u_{k_i}^{1/2} F_x^* b_p^{1/4}\| = \|b_p^{1/4}F_x u_{k_i}^{1/4}\|^2$$
$$= \|u_{k_i}^{1/4}F_x^* b_p^{1/2} F_x u_{k_i}^{1/4}\| \leqslant \|u_{k_i}^{1/4}F_x^*(1-u_{k_{i+1}})^{1/2}F_x u_{k_i}^{1/4}\| = \|(1-u_{k_{i+1}})^{1/4}F_x u_{k_i}^{1/4}\|^2 \leqslant 2^{-(2i+2)\cdot 2}.$$

Let n be an integer $\geqslant \max(1, (m-2)/2)$. Set $\lambda_i = (1/i)^{1/n}$ and $\mu_i = \lambda_i - \lambda_{i+1}$ $(i \geqslant 1)$. We define the desired strictly positive element h by the formula $h = \sum_1^\infty \mu_i u_{k_i} = \sum_1^\infty \lambda_i b_i$. We prove the required estimate for the first summand on the left side of (1). The estimate of the second summand can be obtained in exactly the same way. Using (5), we have

$$(hF_x - F_x h)(hF_x - F_x h)^* = \frac{1}{2} \sum_{\substack{i_1,\dots,i_{2n+3}\\ j_1,\dots,j_{2n+3}}} (\lambda_{i_2} - \lambda_{i_1})(\lambda_{j_2} - \lambda_{j_1})(R + R^*),$$

where

$$R = b_{i_{2n+3}} b_{i_{2n+2}} \cdots b_{i_2} F_x b_{i_1} b_{j_1} F_x^* b_{j_2} \cdots b_{j_{2n+2}} b_{j_{2n+3}}$$
$$= b_{i_{2n+3}}^{1/2} \cdots b_{i_2}^{1/2} (b_{i_{2n+3}}^{1/2} b_{i_{2n+2}}^{1/4})(b_{i_{2n+2}}^{1/4} b_{i_{2n+1}}^{1/4}) \cdots (b_{i_3}^{1/4} b_{i_2}^{1/4})(b_{i_2}^{1/4} F_x b_{i_1}^{1/2})$$
$$\times (b_{i_1}^{1/2} b_{j_1}^{1/2})(b_{j_1}^{1/2} F_x^* b_{j_2}^{1/4})(b_{j_2}^{1/4} b_{j_3}^{1/4}) \cdots (b_{j_{2n+1}}^{1/4} b_{j_{2n+2}}^{1/4})(b_{j_{2n+2}}^{1/4} b_{j_{2n+3}}^{1/2})\, b_{j_2}^{1/2} \cdots b_{j_{2n+3}}^{1/2}.$$

From (6), (7) and Lemma 1 we obtain

$$R + R^* \leqslant \varepsilon_{i_{2n+3}, i_{2n+2}} \cdots \varepsilon_{i_2, i_1} \varepsilon_{i_1, j_1} \varepsilon_{j_1, j_2} \cdots \varepsilon_{j_{2n+2}, j_{2n+3}} [b_{i_{2n+3}} \cdots b_{i_2} + b_{j_2} \cdots b_{j_{2n+3}}].$$

The expression in square brackets does not exceed

$$\frac{1}{2n+2}[b_{i_2}^{2n+2} + \cdots + b_{i_{2n+3}}^{2n+2} + b_{j_2}^{2n+2} + \cdots + b_{j_{2n+3}}^{2n+2}],$$

and therefore

$$(hF_x - F_x h)(hF_x - F_x h)^* \leqslant \frac{1}{4n+4} \sum_{\substack{i_1,\dots,i_{2n+3}\\ j_1,\dots,j_{2n+3}}} |\lambda_{i_2} - \lambda_{i_1}| \cdot |\lambda_{j_2} - \lambda_{j_1}|$$
$$\times \varepsilon_{i_{2n+3}, i_{2n+2}} \cdots \varepsilon_{i_1, j_1} \cdots \varepsilon_{j_{2n+2}, j_{2n+3}} [b_{i_2}^{2n+2} + \cdots + b_{i_{2n+3}}^{2n+2} + b_{j_2}^{2n+2} + \cdots + b_{j_{2n+3}}^{2n+2}].$$

We introduce the following new notation: $\varepsilon_{p,q}^{(k)} = 2^{-p-q}(\lambda_p\lambda_q)^{-k}$ for $|p-q| \geq 2$, and $\varepsilon_{p,q}^{(k)} = 1$ for $|p-q| \leq 1$. It is easy to verify that

$$\forall p, q \quad |\lambda_p - \lambda_q| \cdot \varepsilon_{p,q} \leqslant 2\lambda_p^{n+1} \cdot \varepsilon_{p,q}^{(n+1)},$$
$$\forall p, q, k, l \quad \lambda_p^k \cdot \varepsilon_{p,q}^{(l)} \leqslant 2\lambda_q^k \cdot \varepsilon_{p,q}^{(k+l)}.$$

Using these inequalities, we rewrite our estimate in the form

$$(hF_x - F_xh)(hF_x - F_xh)^* \leqslant \text{const} \cdot \sum_{\substack{i_1,\dots,i_{2n+3} \\ j_1,\dots,j_{2n+3}}} \varepsilon_{i_{2n+3},i_{2n+2}}^{(3n+3)} \cdots \varepsilon_{j_{2n+2},j_{2n+3}}^{(3n+3)}$$
$$\times [(\lambda_{i_2}b_{i_2})^{2n+2} + \cdots + (\lambda_{i_{2n+3}}b_{i_{2n+3}})^{2n+2} + (\lambda_{j_2}b_{j_2})^{2n+2} + \cdots + (\lambda_{j_{2n+3}}b_{j_{2n+3}})^{2n+2}].$$

The right side does not exceed $c \cdot h^{2n+2} \leqslant c \cdot h^m$. ■

LEMMA 2. *Let A be an algebra, and let B_1 and B_2 be subalgebras such that $B_1 \cdot B_2 \subset B_1$. Then $B_1 + B_2$ is a subalgebra of A and B_1 is an ideal in $B_1 + B_2$. (In particular, if the subalgebras B_1 and B_2 satisfy the condition $B_1 \cdot B_2 = B_1 \cap B_2 = B$, then $B_1 + B_2$ is a subalgebra of A, and B, B_1, and B_2 are ideals in $B_1 + B_2$.)*

If the elements h and k are strictly positive in B_1 and B_2, respectively, then $h + k$ is strictly positive in $B_1 + B_2$.

PROOF. The closure $E = \overline{B_1 + B_2}$ is a subalgebra of A, and B_1 is an ideal in E: $B_1 \cdot E \subset \overline{B_1 \cdot (B_1 + B_2)} \subset \overline{B}_1 \subset E$. Since the sum of an ideal and a subalgebra in E is closed (see [12], 1.8.4), we have $B_1 + B_2 = E$. The strong positivity of $h + k$ follows from the criterion in §1.7. ■

We recall (see [8]) that the functions $f(x) = x/(x + \alpha)$ for $\alpha \geqslant 0$ and $f(x) = x^\alpha$ for $0 \leqslant \alpha \leqslant 1$ are operator monotone functions in the interval $0 \leqslant x < \infty$, i.e. in any algebra, for any Hermitian elements y and z whose spectrum is in the indicated interval, $y \leqslant z$ implies that $f(y) \leqslant f(z)$. The monotonicity of $x/(x + \alpha)$ can be verified immediately, and the monotonicity of x^α is very simply obtained, for example, from the integral representation

$$x^\alpha = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{x}{x+t} \cdot \frac{dt}{t^{1-\alpha}}, \qquad 0 < \alpha < 1,$$

taking into account that all $x/(x + t)$ are monotone.

THEOREM 2. *Let A be a unital algebra and let B_1 and B_2 be subalgebras with a continuous action of G such that $B_1 \cdot B_2 = B_1 \cap B_2$; and assume that B_1 and B_2 have strictly positive elements. Assume, furthermore, that the family $\{F_x\}_{x \in X} \subset A$ is bounded in norm, $[F_x, b] \in B_i$ $(i = 1, 2)$ for every $b \in B_i$ and $x \in X$, and the closure of the set $\{[F_x, b] | x \in X\}$ is compact in $B_1 + B_2$ for every $b \in B$. There exist invariant elements of degree 0: a strictly positive element $h \in B_1$, a positive element $k \in B_2$, a strictly positive element $l \in B_1 + B_2$, and a constant $c > 0$ such that for every $x \in X$*

1) $h^2 + k^2 = l^2$,

2) $h^4 \leqslant l^4$, $k^4 \leqslant 4l^4$,

3) $[h, F_x] \cdot [h, F_x]^* + [h, F_x]^* \cdot [h, F_x] \leqslant ch^4$,

4) $[l, F_x] \cdot [l, F_x]^* + [l, F_x]^* \cdot [l, F_x] \leqslant cl^4$.

PROOF. Let the elements h_0 and k_0 of degree 0 be invariant and strictly positive in B_1 and B_2, respectively. Then $\{u_i = h_0^{1/i}\}$ and $\{v_i = (h_0 + k_0)^{1/i}\}$ are invariant Abelian increasing approximate identities in B_1 and $B_1 + B_2$, respectively. We choose an increasing sequence of integers $\{n_i\}$ so that conditions (2)–(4) are satisfied for $\{u_{n_i}\}$ and analogous conditions are satisfied for $\{v_{n_i}\}$ at the same time. We may assume that all $n_i \geqslant 4$. We set $h = \sum_1^\infty \mu_i u_{n_i}$ and $l = \sum_1^\infty \mu_i v_{n_i}$, where $\mu_i = 1/i - 1/(i+1)$. As is shown in the proof of Theorem 1, conditions 3) and 4) will be satisfied. Moreover, $v_{n_i} - u_{n_i} \in B_2$ for every i, and so $l^2 - h^2 \in B_2$. Since all $n_i \geqslant 4$, from the operator monotonicity of x^α for $0 \leqslant \alpha \leqslant 1$ it follows that $h^2 \leqslant l^2$ and $h^4 \leqslant l^4$. We set $k = (l^2 - h^2)^{1/2}$. Then $k^4 \leqslant 2l^4 + 2h^4 \leqslant 4l^4$. ■

LEMMA 3. *Let A be an algebra with* 1, *and let $x, y \in A$, $x \geqslant 0$. The existence of the limit* $\lim_{\alpha \to +0}(x+\alpha)^{-1}y$ *is equivalent to the condition*

$$\forall \varepsilon > 0 \quad \exists c > 0 \quad yy^* \leqslant cx^4 + \varepsilon x^2. \tag{8}$$

PROOF. By the Cauchy criterion the existence of the limit is equivalent to the condition that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$[(x+\alpha)^{-1} - (x+\beta)^{-1}]yy^*[(x+\alpha)^{-1} - (x+\beta)^{-1}] \leqslant \varepsilon$$

for $0 < \alpha$, $\beta \leqslant \delta$. This condition can be obtained easily from (8). Conversely, this condition implies that

$$yy^* \leqslant \varepsilon \left(\frac{(x+\alpha)(x+\beta)}{\beta - \alpha}\right)^2 \quad \text{for } \alpha, \beta \leqslant \delta.$$

Passing to the limit as $\beta \to 0$, we arrive at (8). ■

THEOREM 3. *Let B_1 and B_2 be subalgebras in $\mathfrak{M}(D)$ with a continuous action of G such that $B_1 \cdot B_2 = B_1 \cap B_2 = B$, $D \subset B_1 + B_2$ and the algebras B_1 and B_2 have strictly positive elements. Assume that X is compact and $X \to \mathfrak{M}(D)$: $x \mapsto F_x$ is a mapping such that the norms of $\{\|F_x\|\}_{x \in X}$ constitute a bounded set, $[F_x, b] \in B_i$ for every $b \in B_i$ and $x \in X$, and the mapping $x \to [F_x, b]$ is norm continuous on X ($i = 1, 2$) for every $b \in B_i$. Then in the algebra $\mathfrak{M}(D)$ there exists a pair of invariant elements M, N of degree* 0 *satisfying the following conditions*:

1) $M + N = 1$, $M \geqslant 0$, $N \geqslant 0$.

2) $M \cdot B_1 \subset B$, $N \cdot B_2 \subset B$; $M \cdot B_2 \subset B_2$, $N \cdot B_1 \subset B_1$.

3) $[M, F_x] \in B$, $[N, F_x] \in B$, *for every* $x \in X$, *and these commutators are norm continuous as functions of* $x \in X$.

PROOF. We apply Theorem 2 by setting $A = \mathfrak{M}(D)$. For $\alpha > 0$ let $M_\alpha = (l+\alpha)^{-1}k^2(l+\alpha)^{-1}$ and $N_\alpha = (l+\alpha)^{-1}h^2(l+\alpha)^{-1}$. We show that for every $b \in B_1 + B_2$ the limits of $M_\alpha b$, bM_α, $N_\alpha b$ and bN_α exist as $\alpha \to +0$. According to Lemma 3 and part 2) of Theorem 2 the limits $\lim_{\alpha \to +0}(l+\alpha)^{-1}k^2$ and $\lim_{\alpha \to +0}(l+\alpha)^{-1}l^2$ exist, and therefore so does the limit

$$\lim_{\alpha \to +0} (l+\alpha)^{-1} k^2 (l+\alpha)^{-1} l^2 = \lim_{\alpha \to +0} M_\alpha l^2.$$

According to the criterion of §1.7, there exists $c > 0$ such that $bb^* \leqslant cl^4 + \varepsilon$. Hence

$$\|(M_\alpha - M_\beta) bb^* (M_\alpha - M_\beta)\| \leqslant c\|(M_\alpha - M_\beta) l^4 (M_\alpha - M_\beta)\| + 4\varepsilon$$

for every $\alpha, \beta > 0$. Now the existence of $\lim_{\alpha\to+0} M_\alpha b$ can be obtained from the Cauchy criterion. The existence of the remaining limits can be proved analogously.

We put $M = \lim_{\alpha\to+0} M_\alpha$ and $N = \lim_{\alpha\to+0} N_\alpha$, where the limits are understood in the sense of strong convergence. Part 1) is obviously satisfied. 2) follows from the fact that $M_\alpha \in B_2$ and $N_\alpha \in B_1$. We now prove that the expressions

$$(M_\alpha+N_\alpha)F_x-F_x(M_\alpha+N_\alpha) \quad \text{and} \quad N_\alpha F_x-F_xN_\alpha$$

converge in norm uniformly in $x \in X$ as $\alpha \to +0$. This implies that $M_\alpha F_x - F_x M_\alpha$ also converges, and since $N_\alpha F_x - F_x N_\alpha \in B_1$ and $M_\alpha F_x - F_x M_\alpha \in B_2$, we have

$$NF_x-F_xN= -(MF_x-F_xM)\in B_1\cap B_2=B.$$

According to Lemma 3 and part 4) of Theorem 2, the limit

$$\lim_{\alpha\to+0} (l+\alpha)^{-1}(lF_x-F_xl)$$

exists (uniformly in $x \in X$), and therefore

$$\lim_{\alpha\to+0}\left\|\frac{l}{l+\alpha}F_x-F_x\frac{l}{l+\alpha}\right\|=\lim_{\alpha\to+0}\left\|(l+\alpha)^{-1}(lF_x-F_xl)\frac{\alpha}{l+\alpha}\right\|=0,$$

from which we get

$$\lim_{\alpha\to+0}\|(M_\alpha+N_\alpha)F_x-F_x(M_\alpha+N_\alpha)\|=0.$$

For the proof of the convergence of the expression $F_xN_\alpha - N_\alpha F_x$ we rewrite it in the form

$$\begin{aligned}&[(l+\alpha)^{-1}(lF_x-F_xl)]\cdot N_\alpha+N_\alpha\cdot[(lF_x-F_xl)(l+\alpha)^{-1}]\\ &+[(l+\alpha)^{-1}(F_xh-hF_x)]\,h\,(l+\alpha)^{-1}+(l+\alpha)^{-1}h\,[(F_xh-hF_x)(l+\alpha)^{-1}].\end{aligned}$$

On the basis of Theorem 2 and Lemma 3 all expressions in square brackets converge as $\alpha \to +0$ (uniformly in x). Since bN_α and $N_\alpha b$ converge for $b \in B_1 + B_2$ and N_α is bounded in α, the first two summands converge. We set $a = \lim_{\alpha\to+0}(l+\alpha)^{-1} \times (F_xh - hF_x)$. It is obvious that $a \in B_1$. According to the criterion of §1.7, for every $\varepsilon > 0$ there is a $c > 0$ such that $a^*a \leqslant ch^2 + \varepsilon$; hence $ha^*ah \leqslant ch^4 + \varepsilon h^2 \leqslant cl^4 + \varepsilon l^2$. According to Lemma 3 the limit $\lim_{\alpha\to+0} ah(l+\alpha)^{-1}$ exists. Since $h(l+\alpha)^{-1}$ is bounded in α, the third summand is convergent. The convergence of the fourth summand can be obtained analogously. ■

THEOREM 4. *Let E_1, E_2, and E be subalgebras of $\mathfrak{M}(D)$ with a continuous action of G, and assume that E is an ideal in E_1, $E_1 \cdot E_2 \subset E$, $D \subset E_1 + E_2$, E_1 and E have strictly positive elements, and E_2 is separable. Assume that $\mathfrak{F}$ is a graded (i.e. $\mathfrak{F} = \mathfrak{F}^{(0)} + \mathfrak{F}^{(1)}$) separable linear subspace of $\mathfrak{M}(D)$ invariant with respect to the action of G and the "real" involution, and G acts continuously on $\mathfrak{F}$ and $[f, b] \in E$, $[f, b_1] \in E_1$ for every $f \in \mathfrak{F}$, $b \in E$, and $b_1 \in E_1$. Then in the algebra $\mathfrak{M}(D)$ there exists a pair of invariant elements M and N of degree 0 satisfying the following conditions*:

1) $M + N = 1$, $M \geqslant 0$, $N \geqslant 0$.

2) $M \cdot E_1 \subset E$, $N \cdot E_2 \subset E$, $N \cdot E_1 \subset E_1$.

3) $\forall f \in \mathfrak{F}\ [f, M] \in E$, $[f, N] \in E$.

PROOF. Replacing $\mathfrak{F}$ by $\mathfrak{F} + \mathfrak{F}^*$, we may assume that $\mathfrak{F}$ is a selfadjoint linear subspace. We denote by $E_{2,i}$ the set consisting of the elements of the form $[f_1, [f_2, \ldots, [f_i, b] \ldots]$, where $b \in E_2$ and $f_1, \ldots, f_i \in \mathfrak{F}$. Let E_2' be the minimal subalgebra in $\mathfrak{M}(D)$ containing E_2 and all $E_{2,i}$, for $i \geqslant 1$. It is clear that E_2' is separable and $[\mathfrak{F}, E_2'] \subset E_2'$. It is easy to prove by induction that $E_1 \cdot E_{2,i} \subset E$, and therefore $E_1 \cdot E_2' \subset E$. According to Lemma 2, $E_2'' = E_2' + E$ is a subalgebra in $\mathfrak{M}(D)$. Let $\{f_i\}$ be a countable everywhere dense subset in $\mathfrak{F}$. We denote by X the compact set $\{0\} \cup \{f_i/(i \cdot \|f_i\|)\}$. The subalgebras E_1, E_2'' and the identical imbedding $X \to \mathfrak{M}(D)$ satisfy the hypotheses of Theorem 3. The pair M, N constructed there is the desired one. ■

DEFINITION 1. The subalgebras $\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_n \supset D$ in the algebra $\mathfrak{M}(D)$ form an *n-chain* if all the $\mathfrak{B}_i$ have strictly positive elements, G acts on every $\mathfrak{B}_i$ continuously and B_j is an ideal in $\mathfrak{B}_i$ for $i \leqslant j$. Besides, it is required that there exist a graded separable linear subspace $L \subset \mathfrak{B}_0$ invariant with respect to the action of G and the "real" involution and such that $\mathfrak{B}_n + L = \mathfrak{B}_0$.

THEOREM 5. *Let $\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_n$ be an n-chain in $\mathfrak{M}(D)$, and let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be separable subalgebras of $\mathfrak{M}(D)$ with a continuous action of G. Let $F_1, \ldots, F_n \in \mathfrak{M}(D)$ be invariant elements of degree* 1, *and $Q_1, \ldots, Q_n \subset \mathfrak{M}(D)$ finite or countable sets consisting of invariant elements of degree* 0. *Assume that the following conditions are satisfied*:

1°. $\mathfrak{A}_1 \subset \mathfrak{B}_1$, $\mathfrak{B}_i \cdot \mathfrak{A}_{i+1} \subset \mathfrak{B}_{i+1}$ *for* $1 \leqslant i \leqslant n-1$.

2°. $F_i \cdot \mathfrak{B}_j \subset \mathfrak{B}_j$, $\mathfrak{B}_j \cdot F_i \subset \mathfrak{B}_j$ *for* $1 \leqslant i \leqslant j \leqslant n$.

3°. $\forall\, b_i \in \mathfrak{B}_i\ [b_i, F_{i+1}] \in \mathfrak{B}_{i+1}$ *for* $0 \leqslant i \leqslant n-1$.

4°. $\forall\, b_j \in \mathfrak{B}_j$, $\forall\, q_i \in Q_i\ [b_j, q_i] \in \mathfrak{B}_j$ *for* $0 \leqslant j \leqslant n$, $1 \leqslant i \leqslant n$.

5°. $\forall\, b_j \in \mathfrak{B}_j$, $\forall\, q_i \in Q_i\ [b_j, q_i F_i] \in \mathfrak{B}_j$ *for* $0 \leqslant j \leqslant n$, $1 \leqslant i \leqslant n$.

6°. $\forall\, q_i \in Q_i\ \mathfrak{B}_j \cdot [q_i, F_{j+1}] \subset \mathfrak{B}_{j+1}$, $[q_i, F_{j+1}] \cdot \mathfrak{B}_j \subset \mathfrak{B}_{j+1}$ *for* $1 \leqslant i \leqslant j \leqslant n-1$.

In the algebra $\mathfrak{M}(D)$ there exist invariant elements $M_1, \ldots, M_n$ of degree 0 *satisfying the following conditions*:

1) $M_i \geqslant 0$ *for* $1 \leqslant i \leqslant n$; $\sum_{i=1}^n M_i = 1$.

2) $M_i \cdot \mathfrak{A}_i \subset \mathfrak{B}_n$, $M_i \cdot \mathfrak{B}_n \subset \mathfrak{B}_n$ *for* $1 \leqslant i \leqslant n$.

3) $[M_j, F_i] \in \mathfrak{B}_n$ *for* $1 \leqslant i \leqslant j \leqslant n$; $M_i[M_j, F_i] \in \mathfrak{B}_n$ *for* $1 \leqslant j \leqslant i \leqslant n$; $[M_i, M_j] \in \mathfrak{B}_n$ *for* $1 \leqslant i, j \leqslant n$.

4) $\forall\, q_j \in Q_j$ *for* $1 \leqslant i, j \leqslant n$: $[M_i, q_j] \in \mathfrak{B}_n$, $[M_i, q_j F_j] \in \mathfrak{B}_n$; $M_i[q_j, F_i] \in \mathfrak{B}_n$ *for* $i > j$.

5) $\forall\, b \in \mathfrak{B}_0\ [b, M_i] \in \mathfrak{B}_n$ *for* $1 \leqslant i \leqslant n$.

6) $\forall\, b \in \mathfrak{B}_0\ M_i[b, F_i] \in \mathfrak{B}_n$ *for* $1 \leqslant i \leqslant n$.

7) $\mathfrak{B}_{j-1} \cdot [M_i, F_j] \subset \mathfrak{B}_n$, $[M_i, F_j] \cdot B_{j-1} \subset B_n$ *for* $1 \leqslant i \leqslant j \leqslant n$.

Such a choice of operators is determined uniquely up to homotopy, i.e. any two choices of operators $(M_1', \ldots, M_n')$ and $(M_1'', \ldots, M_n'')$ satisfying conditions 1)–7) *are connected by a norm continuous homotopy $(M_1(t), \ldots, M_n(t))$, $0 \leqslant t \leqslant 1$, satisfying conditions* 1)–7) *for all t*.

PROOF. We construct the desired operators $M_1, \ldots, M_n$ by induction. The induction hypothesis of index k is the following: there exist operators $M_1, \ldots, M_k$ satisfying conditions 1)–3) and 5)–7) with n replaced by k and also condition 4) with $\mathfrak{B}_n$ replaced by $\mathfrak{B}_k$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$ (in place of $1 \leqslant i, j \leqslant n$). For $k = 1$ we may take $M_1 = 1$. We assume that the operators $M_1', \ldots, M_k'$ satisfy the induction hypothesis of index k. Let $\mathfrak{A}'_{k+1}$ be the minimal subalgebra in $\mathfrak{M}(D)$ containing $\mathfrak{A}_{k+1}$, $[M'_j, F_{k+1}]$ for all $j \leqslant k$, $[q_j, F_{k+1}]$ for all $q_j \in Q_j$ and $j \leqslant k$, and $[b, F_{k+1}]$ for all $b \in L$ (see Definition 1). From the conditions 1°–3°, 6°, and $M'_i \cdot \mathfrak{B}_k \subset \mathfrak{B}_k$ it is easy to obtain that $\mathfrak{B}_k \cdot \mathfrak{A}'_{k+1} \subset \mathfrak{B}_{k+1}$.

Let $\mathfrak{F}$ be the linear subspace in $\mathfrak{M}(D)$ spanned by $M'_1, \dots, M'_k$; $F_1, \dots, F_{k+1}$; $Q_1, \dots, Q_n$; $Q_1F_1, \dots, Q_nF_n$; L. We put $E_1 = \mathfrak{B}_k$, $E_2 = \mathfrak{A}'_{k+1}$ and $E = \mathfrak{B}_{k+1}$. Theorem 4 gives a pair of elements $M, N \in \mathfrak{M}(D)$. The operators $M_i = \sqrt{M}\, M'_i \sqrt{M}$ for $i \leqslant k$ and $M_{k+1} = N$ satisfy the induction hypothesis of index $(k+1)$. The verification does not represent any difficulty.

The assertion concerning the existence of a homotopy can be proved in the following way. Let $Q'_i = Q_i \cup \{M'_i\} \cup \{M''_i\}$. Applying the part of the theorem already proved to the n-chain $\mathfrak{B}_0 \supset \cdots \supset \mathfrak{B}_n$, the subalgebras $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, the operators $F_1, \dots, F_n$ and the sets $Q'_1, \dots, Q'_n$, we obtain the collection $M_1, \dots, M_n$. The desired homotopy is given by the formula $M_i(t) = 2tM_i + (1-2t)M'_i$ for $0 \leqslant t \leqslant \frac{1}{2}$, and by the formula $M_i(t) = (2-2t)M_i + (2t-1)M''_i$ for $\frac{1}{2} \leqslant t \leqslant 1$. The satisfaction of conditions 1)–7) is obvious except for the second and third relations of 3). These two relations follow from 3) for the collections $(M'_1, \dots, M'_n)$, $(M''_1, \dots, M''_n)$ and $(M_1, \dots, M_n)$ and from 4) for the collection $(M_1, \dots, M_n)$. ■

§4. Operator K-functor. Intersection product

In this section we define the operator homological-cohomological K-functor $KK(A, B)$. All algebras and subalgebras under consideration are assumed to be graded. (We recall that $[x, y]$ means the graded commutator $xy - (-1)^{\deg x \cdot \deg y} yx$.) The notation $KU^GK(A, B)$, $KO^GK(A, B)$ and $KR^GK(A, B)$ refers to the K-functor on the category of complex, real, and "real" algebras, respectively. We usually omit the letters U, O, R and G. The actions of G on all algebras considered as arguments of the K-functor are assumed to be continuous.

DEFINITION 1. Let A and B be algebras with continuous action of G. We denote by $\mathfrak{E}(A, B)$ the set of triples $(\varepsilon, \varphi, F)$, where ε is a grading of the space $\mathcal{H}_B$ (see §2.3), φ: $A \to \mathfrak{L}(\mathcal{H}_B)$ is a homomorphism, $F \in \mathfrak{L}(\mathcal{H}_B)$ is an invariant operator of degree 1 and the elements

$$[\varphi(a), F], \quad (F^2-1)\varphi(a), \quad (F-F^*)\varphi(a) \tag{1}$$

belong to $\mathcal{K}_B = \mathcal{K}(\mathcal{H}_B)$ for every $a \in A$. By $\mathfrak{D}(A, B)$ we denote the set of degenerate triples, i.e. those for which all elements (1) are equal to 0.

DEFINITION 2. 1°. The triples $(\varepsilon_1, \varphi_1, F_1), (\varepsilon_2, \varphi_2, F_2) \in \mathfrak{E}(A, B)$ are said to be *unitarily equivalent* if there exists an invariant unitary element $u \in \mathfrak{L}(\mathcal{H}_B)$ which converts $(\varepsilon_1, \varphi_1, F_1)$ into $(\varepsilon_2, \varphi_2, F_2)$, i.e. $\deg_2(u(z)) = \deg_1(z)$ for $z \in \mathcal{H}_B$, $\varphi_2(a) = u\varphi_1(a)u^{-1}$ for $a \in A$, and $F_2 = uF_1u^{-1}$.

2°. A *homotopy connecting the triples* $x_\alpha = (\varepsilon_\alpha, \varphi_\alpha, F_\alpha)$ and $x_\beta = (\varepsilon_\beta, \varphi_\beta, F_\beta) \in \mathfrak{E}(A, B)$, is, by definition, a triple

$$\{x_t\} = (\{\varepsilon_t\}, \{\varphi_t\}, \{F_t\})_{t \in [\alpha,\beta]} \in \mathfrak{E}(A, B[\alpha, \beta]),$$

the restrictions of which to the endpoints of the interval $[\alpha, \beta]$ (i.e. the images under the restriction $\mathcal{H}_{B[\alpha,\beta]} \to \mathcal{H}_{B[t]} \simeq \mathcal{H}_{B[\alpha,\beta]} \otimes_{B[\alpha,\beta]} B[t_0]$ and $\mathfrak{L}(\mathcal{H}_{B[\alpha,\beta]}) \to \mathfrak{L}(\mathcal{H}_{B[t]})$, where $t_0 = \alpha$ or β) coincide with the given triples x_α and x_β.

3°. An *operator homotopy* is, by definition, a homotopy such that $\varepsilon_t = \varepsilon_\alpha = \varepsilon_\beta$ and $\varphi_t = \varphi_\alpha = \varphi_\beta$ for every $t \in [\alpha, \beta]$ and the function $t \to F_t$ is continuous in norm.

DEFINITION 3. Let $\overline{\mathcal{E}}(A, B)$ be the set of homotopy classes of triples and let $\overline{\mathcal{D}}(A, B)$ be the image of $\mathcal{D}(A, B)$ in $\overline{\mathcal{E}}(A, B)$. We identify $\mathcal{H}_B \oplus \mathcal{H}_B$ with $\mathcal{H}_B$ by means of an invariant isometry of degree 0 (any two different isometries are homotopic–see §§1.17 and 2.3) and introduce the following addition on $\overline{\mathcal{E}}(A, B)$:

$$(\varepsilon_1, \varphi_1, F_1) \oplus (\varepsilon_2, \varphi_2, F_2) = (\varepsilon_1 \oplus \varepsilon_2, \varphi_1 \oplus \varphi_2, F_1 \oplus F_2).$$

We denote by $KK(A, B)$ the factor group $\overline{\mathcal{E}}(A, B)/\overline{\mathcal{D}}(A, B)$.

REMARK 1. If the Hilbert B-module E has a countable system of generators, then any triple (ε is a grading of E, $\varphi: A \to \mathcal{L}(E)$, and $F \in \mathcal{L}(E)$) satisfying the conditions of Definition 1 (with $\mathcal{H}_B$ replaced by $\mathcal{K}(E)$) can be considered as an element of $KK(A, B)$. The stabilization

$$E \Rightarrow E \oplus \mathcal{H}_B, \qquad (\varepsilon, \varphi, F) \to (\varepsilon, \varphi, F) \oplus (\varepsilon_0, \varphi_0, F_0),$$

where $(\varepsilon_0, \varphi_0, F_0) \in \mathcal{D}(A, B)$, converts $(\varepsilon, \varphi, F)$ into an element of $\mathcal{E}(A, B)$.

As follows, for example, from the same stabilization theorem (see §§2.3 and 1.12), any two canonical gradings of $\mathcal{H}_B$ differ by an isometry. Therefore, it would be possible to fix one grading ε for all triples $(\varepsilon, \varphi, F)$. However, it is more convenient not to fix ε.

THEOREM 1. *$KK(A, B)$ is a group.*

PROOF. Let $(\varepsilon, \varphi, F) \in \mathcal{E}(A, B)$. We denote by $(-\varepsilon)$ the grading of $\mathcal{H}_B$ opposite to ε and by $(-\varphi)$: $A \to \mathcal{L}(\mathcal{H}_B)$ the homomorphism defined by $(-\varphi)(a) = (-1)^{\deg a} \varphi(a)$. Then the element $(-\varepsilon, -\varphi, -F)$ is the inverse of $(\varepsilon, \varphi, F)$. Indeed, the operator homotopy

$$\left(\varepsilon \oplus (-\varepsilon), \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}, \begin{pmatrix} F\cos t & \sin t \\ \sin t & -F\cos t \end{pmatrix}\right)$$

for $0 \leqslant t \leqslant \pi/2$ reduces $(\varepsilon, \varphi, F) \oplus (-\varepsilon, -\varphi, -F)$ to

$$\left(\varepsilon \oplus (-\varepsilon), \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \in \mathcal{D}(A, B). \blacksquare$$

REMARK 2. If the algebra A is unital and B has a countable approximate identity, then in Definition 1 it can be required additionally that the homomorphism φ be unital. This does not change the group $KK(A, B)$. Indeed, firstly, the set of triples $(\varepsilon, \varphi, F)$ in which φ is unital is nonempty: if $\psi: A \hat{\otimes} C_{1,0} \to \mathcal{L}(\mathcal{H}) \hookrightarrow \mathcal{L}(\mathcal{H}_B)$ is a unital representation (see §2.5), then setting $\varphi_0 = \psi|_{A\hat{\otimes}C}$ and $F_0 = \psi(1 \hat{\otimes} \varepsilon_1)$, we obtain an element $(\varepsilon_0, \varphi_0, F_0) \in \mathcal{D}(A, B)$. Moreover, if $(\varepsilon, \varphi, F) \in \mathcal{E}(A, B)$, then $P = \varphi(1)$ is a Hermitian projection of degree 0 in $\mathcal{L}(\mathcal{H}_B)$. With respect to the decomposition $\mathcal{H}_B = \operatorname{Im} P \oplus \operatorname{Im}(1 - P)$, the triple $(\varepsilon, \varphi, F)$ is operator homotopic to $(\varepsilon', \varphi, PFP) \oplus (\varepsilon'', 0, 0)$. Applying Remark 1 with $E = \operatorname{Im} P$, we obtain the required assertion.

We note that for triples $(\varepsilon, \varphi, F) \in \mathcal{E}(A, B)$ in which φ is unital we have $F^2 - 1 \in \mathcal{K}_B$ and $F - F^* \in \mathcal{K}_B$. In the case $A = C$ we shall usually assume that φ is unital and omit φ in the notation of the element $(\varepsilon, \varphi, F) \in \mathcal{E}(C, B)$.

THEOREM 2. *Let A be an ideal in A_1 and B an ideal in B_1, and assume that A, B, A_1 and B_1 are algebras with countable approximate identities, and the action of G on A, B, A_1 and B_1 is continuous.*

1°. *The group $KK(A, B)$ does not change if in Definitions 1–3 $\mathcal{L}(\mathcal{H}_B)$ is replaced by $\mathcal{L}(\mathcal{H}_{B_1})$ and in Definition 1 the condition that the elements (1) belong to the ideal $\mathcal{K}_B$ is preserved (note that $\mathcal{K}_B \subset \mathcal{K}_{B_1} \subset \mathcal{L}(\mathcal{H}_{B_1})$).*

2°. *The group $KK(A, B)$ does not change if in Definitions 1–3 it is required that the homomorphism φ be extended to A_1 (i.e. instead of $\mathcal{E}(A, B)$ the set of triples $(\varepsilon, \varphi: A_1 \to \mathcal{L}(\mathcal{H}_B), F)$, is considered; in (1) the same elements occur as before for $a \in A$, and, in 1° of Definition 2, $a \in A_1$).*

Both of these changes are permissible at the same time.

PROOF. 1°. We denote the new group by $KK'(A, B)$. The homomorphism $r: KK'(A, B) \to KK(A, B)$ can be obtained by the restriction $\mathcal{L}(\mathcal{H}_{B_1}) \to \mathcal{L}(\mathcal{H}_B)$. The homomorphism $s: KK(A, B) \to KK'(A, B)$ can be obtained by means of stabilization (see Remark 1): considering $\mathcal{H}_B$ as a Hilbert B_1-module and using the isomorphism $\mathcal{H}_B \oplus \mathcal{H}_{B_1} \simeq \mathcal{H}_{B_1}$, we append a degenerate triple $(\varepsilon_0, \varphi_0, F_0)$ to $(\varepsilon, \varphi, F)$ and obtain an element of $KK'(A, B)$. The composite homomorphism $r \cdot s$ is the identity for obvious reasons, and $s \cdot r$ is the identity in view of the fact that the restriction homomorphism $\mathcal{L}(\mathcal{H}_{B_1} \oplus \mathcal{H}_{B_1}) \to \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_{B_1})$ is homotopic to the identity automorphism (see §1.19).

2°. We denote the new group by $K'K(A, B)$. The homomorphism $K'K(A, B) \to KK(A, B)$ is obtained in an obvious way. For the construction of the inverse homomorphism we consider a triple $(\varepsilon, \varphi, F) \in \mathcal{E}(A, B)$. We extend φ to $\tilde{\varphi}: \tilde{A} \to \mathcal{L}(\mathcal{H}_B)$ and denote by ψ the homomorphism $\mathcal{L}(\tilde{A} \oplus \mathcal{H}_{\tilde{A}}) \to \mathcal{L}(\mathcal{H}_B \oplus (\mathcal{H} \otimes \mathcal{H}_B))$ induced by $\tilde{\varphi}$, by Φ the operator $F \oplus (1 \otimes F) \in \mathcal{L}(\mathcal{H}_B \oplus (\mathcal{H} \otimes \mathcal{H}_B))$, and by $\bar{\varepsilon}$ the grading $\varepsilon \oplus (\varepsilon' \otimes \varepsilon)$ of the space $\mathcal{H}_B \oplus (\mathcal{H} \otimes \mathcal{H}_B)$.

Let $\xi: A = A \oplus 0 \hookrightarrow \mathcal{L}(\tilde{A} \oplus \mathcal{H}_{\tilde{A}})$ and $\eta: A = A \oplus 0 \hookrightarrow \mathcal{L}(A \oplus \mathcal{H}_{\tilde{A}})$ be the natural imbeddings. It is obvious that $(\bar{\varepsilon}, \psi, \Phi)$ can be considered as an element of $\mathcal{E}(\mathcal{K}(A \oplus \mathcal{H}_A), B)$, and $(\varepsilon, \varphi, F) = (\bar{\varepsilon}, \psi \cdot \xi, \Phi)$ in $KK(A, B)$. We identify $\mathcal{L}(A \oplus \mathcal{H}_{\tilde{A}})$ with $\mathcal{L}(\tilde{A} \oplus \mathcal{H}_{\tilde{A}})$ by means of the isomorphism $A \oplus \mathcal{H}_{\tilde{A}} \simeq \tilde{A} \oplus \mathcal{H}_{\tilde{A}}$. According to §1.20, the triples $(\bar{\varepsilon}, \psi \cdot \xi, \Phi)$ and $(\bar{\varepsilon}, \psi \cdot \eta, \Phi)$ are homotopic. The homomorphism $\psi \cdot \eta$ obviously extends to $\mathcal{L}(A) = \mathcal{M}(A)$, and consequently to A_1. Therefore, we may define a homomorphism $KK(A, B) \to K'K(A, B)$, by sending $(\varepsilon, \varphi, F)$ to $(\bar{\varepsilon}, \psi \cdot \eta, \Phi)$. The fact that the homomorphisms thus constructed are one-to-one follows from §1.20. ■

COROLLARY 1. 1) $KK(A, B_1 \oplus B_2) \simeq KK(A, B_1) \oplus KK(A, B_2)$.

2) *If the algebras A_1, A_2, and B have countable approximate identities, then*

$$KK(A_1 \oplus A_2, B) \simeq KK(A_1, B) \oplus KK(A_2, B).$$

PROOF. 1) is obvious, since $\mathcal{L}(\mathcal{H}_{B_1 \oplus B_2}) \simeq \mathcal{L}(\mathcal{H}_{B_1}) \oplus \mathcal{L}(\mathcal{H}_{B_2})$.

2) If $(\varepsilon_i, \varphi_i, F_i) \in \mathcal{E}(A_i, B)$ for $i = 1, 2$, then

$$(\varepsilon_1 \oplus \varepsilon_2, \varphi_1 \oplus \varphi_2, F_1 \oplus F_2) \in \mathcal{E}(A_1 \oplus A_2, B).$$

Conversely, let $(\varepsilon, \varphi, F) \in \mathcal{E}(A_1 \oplus A_2, B)$. Using Theorem 2, we may assume that φ can be extended to $\tilde{A}_1 \oplus \tilde{A}_2$. Let $P = \varphi(1 \oplus 0)$, $E_1 = \operatorname{Im} P$ and $E_2 = \operatorname{Im}(1 - P)$, and let ε_i be the grading of E_i. It is easy to see that the triples $(\varepsilon_1, \varphi|_{A_1 \oplus 0}, PFP)$ and $(\varepsilon_2, \varphi|_{0 \oplus A_2}, (1 - P)F(1 - P))$ satisfy the conditions of Remark 1, and consequently define elements of $KK(A_1, B)$ and $KK(A_2, B)$, respectively. ■

DEFINITION 4 (*functorial properties*). A homomorphism $f: A_2 \to A_1$ induces a group homomorphism $f^*: KK(A_1, B) \to KK(A_2, B)$ by the formula $f^*(\varepsilon, \varphi, F) = (\varepsilon, \varphi \cdot f, F)$. Let the algebras B_1 and B_2 have countable approximate identities. According to §2.8, a homomorphism $g\colon B_1 \to B_2$ generates $g_*\colon \mathcal{L}(\mathcal{H}_{\tilde{B}_1}) \to \mathcal{L}(\mathcal{H}_{\tilde{B}_1} \otimes_{\tilde{B}_1} \tilde{B}_2)$ and induces a group homomorphism $g_*\colon KK(A, B_1) \to KK(A, B_2)$ by the formula $g_*(\varepsilon, \varphi, F) = (\bar{\varepsilon}, g_* \cdot \varphi, g_*(F))$, where $\bar{\varepsilon}$ is the grading of $\mathcal{H}_{\tilde{B}_1} \otimes_{\tilde{B}_1} \tilde{B}_2$ corresponding to ε.

For any algebra D the homomorphism $\tau_D\colon KK(A, B) \to KK(A \hat{\otimes} D, B \hat{\otimes} D)$ is defined by the formula $\tau_D(\varepsilon, \varphi, F) = (\bar{\varepsilon}, \varphi \hat{\otimes} 1, F \hat{\otimes} 1)$, where $\bar{\varepsilon}$ is the grading of $\mathcal{H}_B \hat{\otimes} D$ corresponding to ε.

THEOREM 3 (homotopy invariance). *If the homomorphisms f_0 and $f_1\colon A_2 \to A_1$ are homotopic, then f_0^* and $f_1^*\colon KK(A_1, B) \to KK(A_2, B)$ coincide. If the homomorphisms g_0 and $g_1\colon B_1 \to B_2$ are homotopic, then $(g_0)_*$ and $(g_1)_*\colon KK(A, B_1) \to KK(A, B_2)$ coincide.* ■

THEOREM 4. *Let the algebras A_1 and A_2 be separable. Let D, B_1 and B_2 have strictly positive elements, and assume that the action of G on these algebras is continuous. The following bilinear (distributive) coupling (intersection product) is defined:*

$$KK(A_1, B_1 \hat{\otimes} D) \otimes_D KK(D \hat{\otimes} A_2, B_2) \to KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2). \tag{2}$$

This coupling is contravariant in A_1 and A_2 and covariant in B_1 and B_2; and, for any homomorphism $f\colon D \to D_1$,

$$f_*(x) \otimes_{D_1} y = x \otimes_D f^*(y).$$

Moreover, the coupling (2) is associative, i.e.

$$(x_1 \otimes_{D_1} x_2) \otimes_{D_2} x_3 = x_1 \otimes_{D_1} (x_2 \otimes_{D_2} x_3)$$

for

$$x_1 \in KK(A_1, B_1 \hat{\otimes} D_1), \qquad x_2 \in KK(D_1 \hat{\otimes} A_2, B_2 \hat{\otimes} D_2), \qquad x_3 \in KK(D_2 \hat{\otimes} A_3, B_3)$$

(here the algebras A_1, A_2, A_3 and D_1 have to be separable and the remaining ones have to have strictly positive elements). Moreover, this coupling commutes with the homomorphism τ from Definition 4, i.e.

1) $\tau_{D_2}(x_1) \otimes_{D_1 \hat{\otimes} D \hat{\otimes} D_2} \tau_{D_1}(x_2) = x_1 \otimes_D x_2$ *for* $x_1 \in KK(A_1, B_1 \hat{\otimes} D_1 \hat{\otimes} D)$, $x_2 \in KK(D \hat{\otimes} D_2 \hat{\otimes} A_2, B_2)$ *(here the algebras A_1, A_2 and D_2 have to be separable and the remaining ones have to have strictly positive elements); and*

2) $\tau_{D_1}(x_1 \otimes_D x_2) = \tau_{D_1}(x_1) \otimes_{D \hat{\otimes} D_1} \tau_{D_1}(x_2)$ *for* $x_1 \in KK(A_1, B_1 \hat{\otimes} D)$, $x_2 \in KK(D \hat{\otimes} A_2, B_2)$ *(here the algebras A_1, A_2 and D_1 have to be separable, and the remaining ones have to have positive elements).*

PROOF. We fix elements

$$x_1 = (\varepsilon_1, \varphi_1, F_1) \in \mathcal{E}(A_1, B_1 \hat{\otimes} D), \qquad x_2 = (\varepsilon_2, \varphi_2, F_2) \in \mathcal{E}(D \hat{\otimes} A_2, B_2).$$

Using Theorem 2, we may assume that

$$\varphi_1\colon \tilde{A}_1 \to \mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}}), \qquad F_1 \in \mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}}),$$
$$\varphi_2\colon \tilde{D} \hat{\otimes} \tilde{A}_2 \to \mathcal{L}(\mathcal{H}_{\tilde{B}_2}), \qquad F_2 \in \mathcal{L}(\mathcal{H}_{\tilde{B}_2}),$$

and the homomorphisms φ_1 and φ_2 are unital. Identifying $(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}} \hat{\otimes} \tilde{A}_2) \hat{\otimes}_{\tilde{B}_1 \hat{\otimes} \tilde{D} \hat{\otimes} \tilde{A}_2} (\tilde{B}_1 \hat{\otimes} \mathcal{H}_{\tilde{B}_2})$ with $\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2}$ (see §§2.8 and 2.9), we obtain a grading $\varepsilon_1 \otimes_D \varepsilon_2$ on $\mathcal{H}_{\tilde{B}_1 \cdot \tilde{B}_2}$ and a homomorphism

$$\mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}}) \hat{\otimes} \tilde{A}_2 \to \mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2})$$

(see §2.8). Composition with the restriction $\mathcal{L}(\mathcal{H}_{\tilde{B}_1\hat{\otimes}\tilde{B}_2}) \to \mathcal{L}(\mathcal{H}_{B_1\hat{\otimes}B_2})$ gives

$$\Phi_2\colon \mathcal{L}(\mathcal{H}_{\tilde{B}_1\hat{\otimes}\tilde{D}}) \hat{\otimes} \tilde{A}_2 \to \mathcal{L}(\mathcal{H}_{B_1\hat{\otimes}B_2}).$$

This homomorphism can be obtained as a result of the composition

$$\Phi_2\colon \mathfrak{M}(\mathcal{K} \hat{\otimes} \tilde{B}_1 \hat{\otimes} \tilde{D}) \hat{\otimes} \tilde{A}_2 \to \mathfrak{M}(\mathcal{K} \hat{\otimes} \tilde{B}_1 \hat{\otimes} \tilde{D} \hat{\otimes} \tilde{A}_2)$$

$$\xrightarrow{(1\hat{\otimes}\varphi_2)_*} \mathfrak{M}(\mathcal{K} \hat{\otimes} B_1 \hat{\otimes} \mathfrak{M}(\mathcal{K} \hat{\otimes} B_2)) \to \mathfrak{M}(\mathcal{K} \hat{\otimes} B_1 \hat{\otimes} \mathcal{K} \hat{\otimes} B_2) \underset{\sim}{\to} \mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2}).$$

We set

$$\varphi_1 \otimes_D \varphi_2 = \Phi_2 \cdot (\varphi_1 \hat{\otimes} 1)\colon A_1 \hat{\otimes} A_2 \to \mathcal{L}(\mathcal{H}_{B_1\hat{\otimes}B_2}) \simeq \mathfrak{M}(\mathcal{H}_{B_1\hat{\otimes}B_2}).$$

For the construction of the operator $F_1 \#_D F_2$ we introduce the following notation.

Definition 5. Let $\mathfrak{A}_1$ be the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2})$ containing the elements

$$\Phi_2((F_1 - F_1^*) \hat{\otimes} 1) \cdot (\varphi_1 \otimes_D \varphi_2)(a),$$
$$\Phi_2((F_1^2 - 1) \hat{\otimes} 1) \cdot (\varphi_1 \otimes_D \varphi_2)(a),$$
$$[\Phi_2(F_1 \hat{\otimes} 1),\ (\varphi_1 \otimes_D \varphi_2)(a)]$$

for all $a \in A_1 \hat{\otimes} A_2$. Let $\mathfrak{A}_2$ be the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2})$ containing the elements

$$1 \hat{\otimes} (F_2 - F_2^*);\quad 1 \hat{\otimes} (F_2^2 - 1);$$
$$[(1 \hat{\otimes} F_2),\ (\varphi_1 \otimes_D \varphi_2)(a)];$$
$$[\Phi_2(F_1 \hat{\otimes} 1),\ 1 \hat{\otimes} F_2]$$

for all $a \in A_1 \hat{\otimes} A_2$. We denote by $S(x_1, x_2)$ the set of pairs of invariant elements $M_1, M_2 \in \mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2})$ of degree 0 satisfying the following conditions:

1) $M_1 \geqslant 0$, $M_2 \geqslant 0$, $M_1 + M_2 = 1$.

2) $M_i \cdot \mathfrak{A}_i \subset \mathcal{K}_{B_1} \hat{\otimes} \mathcal{K}_{B_2}$.

3) M_1 and M_2 commute with $\Phi_2(F_1 \hat{\otimes} 1)$, $1 \hat{\otimes} F_2$ and $(\varphi_1 \otimes_D \varphi_2)(A_1 \hat{\otimes} A_2)$ modulo $\mathcal{K}_{B_1} \hat{\otimes} \mathcal{K}_{B_2}$.

For $(M_1, M_2) \in S(x_1, x_2)$ we set

$$F_1 \#_D F_2 = \sqrt{M_1} \cdot \Phi_2(F_1 \hat{\otimes} 1) + \sqrt{M_2} \cdot (1 \hat{\otimes} F_2).$$

Using the fact that $\sqrt{M_i}$ can be approximated in norm by polynomials in M_i with vanishing absolute term, it is easy to verify that

$$x_1 \otimes_D x_2 = (\varepsilon_1 \otimes_D \varepsilon_2, \varphi_1 \otimes_D \varphi_2, F_1 \#_D F_2) \in \mathcal{E}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

If (M_1', M_2') is another element from $S(x_1, x_2)$, then $(tM_1' + (1-t)M_1,\ tM_2' + (1-t)M_2) \in S(x_1, x_2)$ for every $t \in [0, 1]$. Therefore $(\varepsilon_1 \otimes_D \varepsilon_2, \varphi_1 \otimes_D \varphi_2, F_1 \#_D F_2)$ does not depend on the choice of $(M_1, M_2) \in S(x_1, x_2)$ up to operator homotopy.

We show that the set $S(x_1, x_2)$ is not empty. We denote by $\mathfrak{F}_1$ the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2})$ containing 1, $\Phi_2(F_1 \hat{\otimes} 1)$ and $(\varphi_1 \otimes_D \varphi_2)(A_1 \hat{\otimes} A_2)$. Let $\mathfrak{B}_1'$ be the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2})$ containing $\mathfrak{F}_1 \cdot \mathfrak{A}_1$. It is clear that $\mathfrak{B}_1' \subset \Phi_2(\mathcal{K}_{B_1\hat{\otimes}D} \hat{\otimes} A_2)$, from which it follows easily that $\mathfrak{B}_1' \cdot \mathfrak{A}_2 \subset \mathcal{K}_{B_1} \hat{\otimes} \mathcal{K}_B$. We set

$$\mathfrak{B}_2 = \mathcal{K}_{B_1} \hat{\otimes} \mathcal{K}_{B_2},\ \mathfrak{B}_1 = \mathfrak{B}_1' + \mathfrak{B}_2,\ \mathfrak{B}_0 = (\varphi_1 \otimes_D \varphi_2)(A_1 \hat{\otimes} A_2) + \mathfrak{B}_1.$$

Then $\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \mathfrak{B}_2$ is a 2-chain (see Definition 1 of §3 and Lemma 2 of §3). The subalgebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ and the elements $\Phi_2(F_1 \hat{\otimes} 1)$, $1 \hat{\otimes} F_2$ in $\mathfrak{M}(\mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2})$ satisfy the hypotheses of Theorem 5 in §3 (for $Q_1 = Q_2 = \{0\}$). This theorem provides the desired pair $(M_1, M_2) \in S(x_1, x_2)$.

If $x_1^{(t)} = (\varepsilon_1^{(t)}, \varphi_1^{(t)}, F_1^{(t)}) \in \mathcal{E}(A_1, B_1[0,1] \otimes D)$ is the homotopy of the triple x_1, then $x_1^{(t)} \otimes_D x_2$ is the homotopy of $x_1 \otimes_D x_2$. An analogous assertion holds for the homotopy of x_2. Finally, if x_1 is degenerate, then, setting $M_1 = 1$ and $M_2 = 0$, we obtain a degenerate $x_1 \otimes_D x_2$. If x_2 is degenerate, then we may take $M_1 = 0$ and $M_2 = 1$. By the same token, the coupling (2) is defined unambiguously.

The bilinearity of the coupling just constructed is obvious. The functoriality in A_1 is also obvious, and the functoriality in A_2 and D follows from relations of §§2.8 and 2.9. Analogously, for any homomorphisms $B_1 \to B_1'$, $B_2 \to B_2'$ we obtain (change of algebras)

$$\left[\left(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}} \hat{\otimes} \tilde{A}_2\right) \hat{\otimes}_{\tilde{B}_1 \hat{\otimes} \tilde{D} \hat{\otimes} \tilde{A}_2} \left(\tilde{B}_1 \hat{\otimes} \mathcal{H}_{\tilde{B}_2}\right)\right] \hat{\otimes}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2} \left(\tilde{B}_1' \hat{\otimes} \tilde{B}_2'\right)$$
$$\simeq \left(\mathcal{H}_{\tilde{B}_1' \hat{\otimes} \tilde{D}} \hat{\otimes} \tilde{A}_2\right) \hat{\otimes}_{\tilde{B}_1' \hat{\otimes} \tilde{D} \hat{\otimes} \tilde{A}_2} \left(\tilde{B}_1' \hat{\otimes} \mathcal{H}_{\tilde{B}_2'}\right).$$

If the operator $F_1 \#_D F_2$ were constructed in $\mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2})$ rather than $\mathcal{L}(\mathcal{H}_{B_1 \hat{\otimes} B_2})$, this would immediately imply functoriality in B_1 and B_2. In order to remove this obstacle, we use stabilization:

$$\mathcal{H}_{B_i} \Rightarrow \mathcal{H}_{B_i} \oplus \mathcal{H}_{\tilde{B}_i}, \quad x_i = (\varepsilon_i, \varphi_i, F_i) \Rightarrow y_i = (\varepsilon_i, \varphi_i, F_i) \oplus (\varepsilon_i', 0, 0), \qquad i = 1, 2.$$

The product of the operators $(F_1 \oplus 0) \#_D (F_2 \oplus 0)$ can be written in the following form:

$$\left[\sqrt{M_1} \cdot \Phi_2(F_1 \hat{\otimes} 1) + \sqrt{M_2} \cdot (1 \hat{\otimes} F_2)\right] \oplus \left[0 \cdot \Phi_2(F_1 \hat{\otimes} 1) + 1 \cdot 0\right]$$
$$\oplus \left[1 \cdot 0 + 0 \cdot (1 \hat{\otimes} F_2)\right] \oplus \left[1 \cdot 0 + 0 \cdot 0\right]$$
$$\in \mathcal{L}\left(\mathcal{H}_{B_1 \hat{\otimes} B_2} \oplus \mathcal{H}_{B_1 \hat{\otimes} \tilde{B}_2} \oplus \mathcal{H}_{\tilde{B}_1 \hat{\otimes} B_2} \oplus \mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2}\right) \simeq \mathcal{L}\left(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2}\right).$$

The operators $M_1' = M_1 \oplus 0 \oplus 1 \oplus 1$, $M_2' = M_2 \oplus 1 \oplus 0 \oplus 0 \in \mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2})$ satisfy conditions 1)–3) of Definition 5, with respect to $F_1' = F_1 \oplus 0$ and $F_2' = F_2 \oplus 0$. By the same token, the product $F_1' \#_D F_2'$ is defined in $\mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2})$, from which the functoriality in B_1 and B_2 follows.

For the verification of the commutation of the coupling (2) with τ we again have to use a change of algebras: $\widetilde{B_1 \hat{\otimes} D_1}$ to $\tilde{B}_1 \hat{\otimes} \tilde{D}_1$, $\widetilde{D_2 \hat{\otimes} A_2}$ to $\tilde{D}_2 \hat{\otimes} \tilde{A}_2$, ad $\widetilde{(D_1 \hat{\otimes} D \hat{\otimes} D_2)}$ to $\tilde{D}_1 \hat{\otimes} \tilde{D} \hat{\otimes} \tilde{D}_2$. The second relation of commutation with τ can be obtained analogously.

It remains to prove associativity. Since the coupling (2) commutes with τ, we may rewrite the desired formula in the form

$$(\tau_{A_2 \hat{\otimes} A_3}(x_1) \otimes_{D_1 \hat{\otimes} A_2 \hat{\otimes} A_3} \tau_{A_3}(x_2)) \otimes_{D_2 \hat{\otimes} A_3} x_3 = \tau_{A_2 \hat{\otimes} A_3}(x_1) \otimes_{D_1 \hat{\otimes} A_2 \hat{\otimes} A_3} (\tau_{A_3}(x_2) \otimes_{D_2 \hat{\otimes} A_3} x_3).$$

After setting $A = A_1 \hat{\otimes} A_2 \hat{\otimes} A_3$, $D_1 = D_1 \hat{\otimes} A_2 \hat{\otimes} A_3$, $D_2 = D_2 \hat{\otimes} A_3$, $x_1 = \tau_{A_2 \hat{\otimes} A_3}(x_1)$ and $x_2 = \tau_{A_3}(x_2)$, everything reduces to the proof of the relation

$$(x_1 \otimes_{D_1} x_2) \otimes_{D_2} x_3 = x_1 \otimes_{D_1} (x_2 \otimes_{D_2} x_3),$$

where $x_1 \in KK(A, B_1 \hat{\otimes} D_1)$, $x_2 \in KK(D_1, B_2 \hat{\otimes} D_2)$, $x_3 \in KK(D_2, B_3)$, and the algebras A and D_1 are separable and the remaining ones have strictly positive elements.

We assume that $x_i = (\varepsilon_i, \varphi_i, F_i)$, $i = 1, 2, 3$, where

$$\varphi_1\colon\ \tilde{A} \to \mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}_1}), \qquad F_1 \in \mathcal{L}(\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{D}_1});$$
$$\varphi_2\colon\ \tilde{D}_1 \to \mathcal{L}(\mathcal{H}_{\tilde{B}_2 \hat{\otimes} \tilde{D}_2}), \qquad F_2 \in \mathcal{L}(\mathcal{H}_{\tilde{B}_2 \hat{\otimes} \tilde{D}_2});$$
$$\varphi_3\colon\ \tilde{D}_2 \to \mathcal{L}(\mathcal{H}_{\tilde{B}_3}), \qquad F_3 \in \mathcal{L}(\mathcal{H}_{\tilde{B}_3}).$$

We note immediately that the grading $\varepsilon_1 \otimes_{D_1} \varepsilon_2 \otimes_{D_2} \varepsilon_3$ on $\mathcal{H}_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2 \hat{\otimes} \tilde{B}_3}$ does not depend on the arrangement of parentheses in our product. This follows from the relations of §§2.8 and 2.9. In analogy with the above we consider the homomorphisms

$$\Phi_2 = (1 \hat{\otimes} \varphi_2)_*: \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} \tilde{D}_1}) \to \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} \tilde{D}_2}),$$

$$\Phi_3' = (1 \hat{\otimes} \varphi_3)_*: \mathfrak{M}(\mathcal{H}_{B_2 \hat{\otimes} \tilde{D}_2}) \to \mathfrak{M}(\mathcal{H}_{B_2 \hat{\otimes} B_3}),$$

$$\Phi_3 = 1 \hat{\otimes} \Phi_3': \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} \tilde{D}_2}) \to \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3}).$$

It is obvious that

$$(\varphi_1 \otimes_{D_1} \varphi_2) \otimes_{D_2} \varphi_3 = \Phi_3 \cdot \Phi_2 \cdot \varphi_1 = \varphi_1 \otimes_{D_1} (\varphi_2 \otimes_{D_2} \varphi_3),$$

and therefore it remains to verify that the operators $(F_1 \#_{D_1} F_2) \#_{D_2} F_3$ and $F_1 \#_{D_1} (F_2 \#_{D_2} F_3)$ are homotopic.

We denote $\Phi_3 \cdot \Phi_2(F_1)$ by $\tilde{F}_1$, $\Phi_3(1 \hat{\otimes} F_2)$ by $\tilde{F}_2$, $1 \hat{\otimes} 1 \hat{\otimes} F_3$ by $\tilde{F}_3$, and $\Phi_3 \cdot \Phi_2 \cdot \varphi_1(A)$ by A'. The subalgebras $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \subset \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3})$ are defined in the following way. $\mathfrak{A}_1$ is the minimal subalgebra containing the elements

$$(\tilde{F}_1^2 - 1)a, \quad (\tilde{F}_1 - \tilde{F}_1^*)a, \quad [\tilde{F}_1, a]$$

for all $a \in A'$. For $i = 2, 3$ $\mathfrak{A}_i$ is the minimal subalgebra containing the elements

$$(\tilde{F}_i^2 - 1), \quad (\tilde{F}_i - \tilde{F}_i^*), \quad [\tilde{F}_i, a], \quad [\tilde{F}_i, \tilde{F}_j] \quad \text{for all } j < i$$

and for every $a \in A'$. Moreover, for $i = 1, 2$ we denote by $\mathfrak{F}_i$ the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3})$, containing 1, $\tilde{F}_1$, $\tilde{F}_i$, and A'. Now let $\mathfrak{B}_1'$ be the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3})$ containing $\mathfrak{F}_1 \cdot \mathfrak{A}_1$, $\mathfrak{B}_2''$ the minimal subalgebra containing $\mathfrak{B}_1 \cdot \mathfrak{A}_2$ and $[b, \tilde{F}_2]$ for all $b \in \mathfrak{B}_1'$, and $\mathfrak{B}_2'$ the minimal subalgebra containing $\mathfrak{F}_2 \cdot \mathfrak{B}_2''$. It is clear that $\mathfrak{A}_1 \subset \mathfrak{B}_1' \subset \Phi_3 \cdot \Phi_2(\mathcal{H}_{B_1 \hat{\otimes} D_1})$; $\mathfrak{B}_2' \subset \Phi_3(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} D_2})$. Finally, we set $\mathfrak{B}_3 = \mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3}$, $\mathfrak{B}_2 = \mathfrak{B}_2' + \mathfrak{B}_3$, $\mathfrak{B}_1 = \mathfrak{B}_1' + \mathfrak{B}_2$ and $\mathfrak{B}_0 = A' + \mathfrak{B}_1$. It is easy to see that $\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \mathfrak{B}_3$ is a 3-chain and the subalgebras $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ and elements $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ satisfy conditions 1°–3° of Theorem 5 of §3. Now we represent each of the two operators $(F_1 \#_{D_1} F_2) \#_{D_2} F_3$ and $F_1 \#_{D_1}(F_2 \#_{D_2} F_3)$ in the form $\sqrt{N_1}\,\tilde{F}_1 + \sqrt{N_2}\,\tilde{F}_2 + \sqrt{N_3}\,\tilde{F}_3$, where N_1, N_2, and N_3 satisfy conditions 1)–3) and 5)–7) of Theorem 5 of §3. On the basis of this theorem, our operators will be homotopic. (It is easy to verify that conditions 1)–3) and 5)–7) are sufficient in order that

$$\left(\varepsilon_1 \otimes_{D_1} \varepsilon_2 \otimes_{D_2} \varepsilon_3, \varphi_1 \otimes_{D_1} \varphi_2 \otimes_{D_2} \varphi_3, \sqrt{N_1}\,\tilde{F}_1 + \sqrt{N_2}\,\tilde{F}_2 + \sqrt{N_3}\,\tilde{F}_3\right) \in \mathcal{E}(A, B_1 \hat{\otimes} B_2 \hat{\otimes} B_3)).$$

We begin with the operator $(F_1 \#_{D_1} F_2) \#_{D_2} F_3$. Applying the same device as in the proof of functoriality, we may assume that

$$F_1 \#_{D_1} F_2 = \sqrt{M_1} \cdot \Phi_2(F_1) + \sqrt{M_2} \cdot (1 \hat{\otimes} F_2) \in \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} \tilde{D}_2}),$$

where $M_1, M_2 \in \mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} \tilde{D}_2})$ satisfy conditions 1)–3) of Definition 5 (with B_2 replaced by $B_2 \hat{\otimes} D_2$). Let $M_i' = \Phi_3(M_i)$, $i = 1, 2$. We denote by $\mathfrak{A}_3'$ the minimal subalgebra in $\mathfrak{M}(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3})$, containing $\mathfrak{A}_3$, $[M_i', \tilde{F}_3]$ for $i = 1, 2$, and $[b, \tilde{F}_3]$ for all $b \in A' + \mathfrak{B}_1' + \mathfrak{B}_2'$. Let $\mathfrak{F}$ be the linear space spanned by M_1', M_2', $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$ and A'. We set

$$E_1 = \Phi_3(\mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} D_2}) + \mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3}, \; E_2 = \mathfrak{A}_3', \; E = \mathcal{H}_{B_1 \hat{\otimes} B_2 \hat{\otimes} B_3}.$$

Theorem 4 of §3 gives a pair of elements $M, N \in \mathcal{M}(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}B_3})$. It is easy to verify that these elements satisfy conditions analogous to 1)–3) of Definition 5 with respect to the operators $F_1 \#_{D_1} F_2$ and F_3, and besides, the operators $N_1 = \sqrt{M}\, M_1' \sqrt{M}$, $N_2 = \sqrt{M}\, M_2' \sqrt{M}$ and $N_3 = N$ satisfy conditions 1)–3) and 5)–7) of Theorem 5 of §3. Therefore, $(F_1 \#_{D_1} F_2) \#_{D_2} F_3$ can be written in the desired form $\sqrt{N_1}\,\tilde{F}_1 + \sqrt{N_2}\,\tilde{F}_2 + \sqrt{N_3}\,\tilde{F}_3$.

Now we consider the operator $F_1 \#_{D_1} (F_2 \#_{D_2} F_3)$. Let

$$F_2 \#_{D_2} F_3 = \sqrt{M_2} \cdot \Phi_3'(F_2) + \sqrt{M_3}\,(1 \hat{\otimes} F_3),$$

where M_2 and $M_3 \in \mathcal{M}(\mathcal{K}_{B_2\hat{\otimes}B_3})$ satisfy conditions 1)–3) of Definition 5 with respect to the operators $\Phi_3'(F_2)$, $1 \hat{\otimes} F_3$, and the algebra D_1 (instead of $A_1 \hat{\otimes} A_2$). The additional condition we want to impose on M_2 and M_3 is that they have to transform the subalgebra $E_1' = \Phi_3'(\mathcal{K}_{B_2\hat{\otimes}D_2}) + \mathcal{K}_{B_2\hat{\otimes}B_3} \subset \mathcal{M}(\mathcal{K}_{B_2\hat{\otimes}B_3})$ into itself. For the construction of such M_2 and M_3 we denote by E_2' the minimal subalgebra in $\mathcal{M}(\mathcal{K}_{B_2\hat{\otimes}B_3})$ containing the elements $1 \hat{\otimes} (F_3^2 - 1)$; $1 \hat{\otimes} (F_3 - F_3^*)$; $[1 \hat{\otimes} F_3, \Phi_3' \cdot \varphi_2(d)]$ for all $d \in D_1$; and $[\Phi_3'(F_2), 1 \hat{\otimes} F_3]$. Applying Theorem 4 of §3 to the subalgebras E_1', E_2', $E' = \mathcal{K}_{B_2\hat{\otimes}B_3}$ and the linear subspace $\mathfrak{F}$ spanned by $\Phi_3'(F_2)$, $1 \hat{\otimes} F_3$, $\Phi_3 \cdot \varphi_2(D_1)$, we obtain the desired pair M_2, M_3.

In the algebra $\mathcal{M}(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}B_3})$ we consider the subalgebras $E_1 = \Phi_3 \cdot \Phi_2(\mathcal{K}_{B_1\hat{\otimes}D_1}) + \Phi_3(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}D_2}) + \mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}B_3}$ and $E = \mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}B_3}$ and also the elements $M_i' = 1 \hat{\otimes} M_i$, $i = 2, 3$. From the construction of M_2 and M_3 it follows that for all $x \in \Phi_3 \cdot \Phi_2(\mathcal{K}_{B_1\hat{\otimes}D_1})$ we have $[M_i', x] \in E$, $i = 2, 3$. Moreover, $M_2' \cdot \Phi_3(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}D_2}) \subset E$. Consequently (since $M_3' = 1 - M_2'$) the relation $[M_i', x] \in E$ is satisfied for $x \in \Phi_3(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}D_2})$, $i = 2, 3$, as well. Therefore, this relation holds for every $x \in E_1$. It can be proved analogously that $[M_i'\tilde{F}_i, x] \in E$, $i = 2, 3$, for every $x \in E_1$. (Here we additionally used the fact that $[\tilde{F}_3, x] \in E$ for $x \in \Phi_3(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}D_2})$.)

We denote by E_2 the minimal subalgebra in $\mathcal{M}(E)$ containing the elements

$$M_i'(\tilde{F}_i^2 - 1), \quad M_i'(\tilde{F}_i - \tilde{F}_i^*), \quad M_3'[\tilde{F}_2, \tilde{F}_3], \quad [M_i', \tilde{F}_j],$$
$$[M_i', a], \quad [M_i'\tilde{F}_i, a], \quad [M_i'\tilde{F}_i, \tilde{F}_1]$$

for $i = 2, 3$, $j = 1, 2, 3$, and for all $a \in A'$.

Taking account of what was said about M_2' and M_3', it is easy to verify that $E_1 \cdot E_2 \subset E$.

We apply Theorem 4 of §3 to the subalgebras E_1, E_2, E and the linear subspace $\mathfrak{F}$ spanned by $\tilde{F}_1$, $\tilde{F}_2$, M_2', M_3', $M_3'\tilde{F}_3$ and A'. We obtain a pair of elements $M, N \in \mathcal{M}(E)$. These elements satisfy conditions analogous to 1)–3) of Definition 5 with respect to the operators F_1 and $F_2 \#_{D_2} F_3$; and, moreover, the operators $N_1 = M$, $N_2 = \sqrt{N}\, M_2' \sqrt{N}$ and $N_3 = \sqrt{N}\, M_3' \sqrt{N}$ satisfy conditions 1)–3) and 5)–7) of Theorem 5 of §3. The verification of conditions 1)–3) of Theorem 5 of §3 does not cause any difficulty. Condition 5) follows from the fact that any element $b \in A' + \mathfrak{B}_1' + \mathfrak{B}_2'$ can be approximated in norm by sums of products of elements $a \in A'$, $\tilde{F}_1$, $\tilde{F}_1^*$, $\tilde{F}_2$, and $\tilde{F}_2^*$. Condition 6) can be obtained as follows. We have $[b, \tilde{F}_i] \in \mathfrak{A}_i$ for $b \in A'$, and therefore $N_i[b, \tilde{F}_i] \in E$. Since $\mathfrak{B}_1 \subset E_1$, for $b \in \mathfrak{B}_1$ we have $N_1 \cdot [b, \tilde{F}_1] \in E$. For $b \in \mathfrak{B}_2$ we have $[b, \tilde{F}_2] \in \Phi_3(K_{B_1\hat{\otimes}B_2\hat{\otimes}D_2}) + E$, and therefore $N_2 \cdot [b, \tilde{F}_2] \in E$. The relations $N_2[b, \tilde{F}_2] \in E$ for $b \in \mathfrak{B}_1'$ and $N_3[b, \tilde{F}_3] \in E$ for $b \in \mathfrak{B}_1' + \mathfrak{B}_2'$ can be obtained in the same way as 5).

For example, for $j < i$ we have

$$N_i\,[\widetilde{F}_j^*, \widetilde{F}_i] = ([\widetilde{F}_i^*, \widetilde{F}_j]\,N_i)^* \in ([\widetilde{F}_i, \widetilde{F}_j]\cdot N_i)^* + E \subset N_i \cdot \mathfrak{A}_i + E \subset E.$$

For $j = 3$ condition 7) follows from the fact that all the N_i transform the subalgebra $\Phi_3(\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}D_2}) + E$, which contains $\mathfrak{B}_2$, into itself. As a result, $F_1 \#_{D_1} (F_2 \#_{D_2} F_3)$ can be written in the form $\sqrt{N_1}\,\tilde{F}_1 + \sqrt{N_2}\,\tilde{F}_2 + \sqrt{N_3}\,\tilde{F}_3$ up to an element of $\mathcal{K}_{B_1\hat{\otimes}B_2\hat{\otimes}B_3}$. ■

REMARK 3. In the following two cases it is possible to define the coupling (2) without using the technique of §3:

a) $A_1 = C, A_2 = C$.

b) $A_1 = C, D = C$.

For this we first mention that, if $x_1 = (\varepsilon_1, F_1) \in \mathcal{E}(C, B_1 \hat{\otimes} D)$, then first by means of an operator homotopy we may replace F_1 by $F_1' = \frac{1}{2}(F_1 + F_1^*)$, and then F_1' by $\tilde{F}_1$ such that $\tilde{F}_1^* = \tilde{F}_1$ and $\tilde{F}_1^2 \leqslant 1$. We shall assume that the initial operator F_1 already has these properties. Let $x_2 = (\varepsilon_2, \varphi_2, F_2) \in \mathcal{E}(D \hat{\otimes} A_2, B_2)$. We define the operator $F_1 \#_D F_2$ by the formula

$$\Phi_2(F_1 \hat{\otimes} 1) + \Phi_2(\sqrt{1 - F_1^2}\,\hat{\otimes}\, 1)\cdot(1 \hat{\otimes} F_2).$$

This new definition is, of course, equivalent to the old one; namely, we have the homotopy

$$\sqrt{M_1(t)}\cdot\Phi_2(F_1 \hat{\otimes} 1) + \sqrt{M_2(t)}\cdot(1 \hat{\otimes} F_2), \quad 0 \leqslant t \leqslant 1,$$

where $M_1(t) = tM_1 + (1 - t)$, $M_2(t) = tM_2 + (1 - t)\cdot\Phi_2((1 - F_1^2)\hat{\otimes} 1)$, $(M_1, M_2) \in S(x_1, x_2)$.

THEOREM 5. *Let T_1: $\mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$ be an invariant Fredholm operator, and assume that $1 - T_1T_1^* \in \mathcal{K}(\mathcal{H}^{(1)})$, $1 - T_1^*T_1 \in \mathcal{K}(\mathcal{H}^{(0)})$ and the index of T_1 as an element of $R(G)$ (respectively, of $RU(G)$, $RO(G)$ or $RR(G)$) is equal to 1. In the graded Hilbert space $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$ consider the operator*

$$T = \begin{pmatrix} 0 & T_1^* \\ T_1 & 0 \end{pmatrix}$$

and denote by c_1 the element $(\varepsilon_1, T) \in KK(C, C)$, where ε_1 is the grading of $\mathcal{H}$. This element is the identity for the operation of taking intersection products, i.e. $x \otimes_C c_1 = x$ and $c_1 \otimes_C x = x$ for every $x \in KK(A, B)$.

PROOF. Adding a compact operator to T_1, we may assume that $T_1T_1^* = 1$ and $T_1^*T_1 = 1 - p_1$, where p_1 is the projection onto a one-dimensional subspace $E^0 \subset \mathcal{H}^{(0)}$ whose elements are all invariant. Indeed, since the index of T_1 is equal to 1, there exists an invariant epimorphism t: $\operatorname{Ker} T_1 \to \operatorname{Ker} T_1^*$. We extend t to zero on $(\operatorname{Ker} T_1)^\perp$ and replace T_1 by $T_1 + t$. The operator $T_1T_1^*$ becomes invertible. The desired operator is equal to $(T_1T_1^*)^{-1/2}\cdot T_1$.

Let $x = (\varepsilon, \varphi, F)$. According to Remark 3, $c_1 \otimes_C x$ can be represented in the form

$$(\varepsilon_1 \otimes \varepsilon,\ 1 \hat{\otimes} \varphi,\ R = T \hat{\otimes} 1 + \sqrt{1 - T^2}\,\hat{\otimes} F).$$

The decomposition

$$\mathcal{H} \hat{\otimes} \mathcal{H}_B = (E^0 \hat{\otimes} \mathcal{H}_B) \oplus ((E^0)^\perp \oplus \mathcal{H}^{(1)}) \hat{\otimes} \mathcal{H}_B$$

reduces the operator R. On the first direct summand it is equal to F, and on the second direct summand, after identifying $(E^0)^\perp$ with $\mathcal{H}^{(1)}$ by means of T_1, the operator R will be equal to $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\hat{\otimes}1$. By the same token, $c_1\otimes_C x = x\oplus$ (an element of $\mathfrak{D}(A, B)$). ■

THEOREM 6. *Let the algebra A be separable, and assume that B, D and E have strictly positive elements.*

1) *Assume that there exist elements $\alpha\in KK(D, E)$ and $\beta\in KK(E, D)$ such that*

$$\alpha\otimes_E\beta=\pm\tau_D(c_1)\in KK(D, D),\quad \beta\otimes_D\alpha=\pm\tau_E(c_1)\in KK(E, E).$$

Then the homomorphisms

$$\otimes_D\alpha: KK(A, B\hat{\otimes}D)\to KK(A, B\hat{\otimes}E),$$
$$\otimes_E\beta: KK(A, B\hat{\otimes}E)\to KK(A, B\hat{\otimes}D)$$

are isomorphisms. If D and E are separable, then

$$\beta\otimes_D: KK(A\hat{\otimes}D, B)\to KK(A\hat{\otimes}E, B),$$
$$\alpha\otimes_E: KK(A\hat{\otimes}E, B)\to KK(A\hat{\otimes}D, B)$$

are also isomorphisms.

2) *Assume that the algebras D and E are separable and there exist elements $\alpha\in KK(D\hat{\otimes}E, C)$ and $\beta\in KK(C, D\hat{\otimes}E)$, such that*

$$\beta\otimes_D\alpha=\pm\tau_E(c_1)\in KK(E, E),\quad \beta\otimes_E\alpha=\pm\tau_D(c_1)\in KK(D, D).$$

Then the homomorphisms

$$\beta\otimes_D: KK(A\hat{\otimes}D, B)\to KK(A, B\hat{\otimes}E),$$
$$\beta\otimes_E: KK(A\hat{\otimes}E, B)\to KK(A, B\hat{\otimes}D),$$
$$\otimes_D\alpha: KK(A, B\hat{\otimes}D)\to KK(A\hat{\otimes}E\ \ B),$$
$$\otimes_E\alpha: KK(A, B\hat{\otimes}E)\to KK(A\hat{\otimes}D, B)$$

are isomorphisms.

The proof immediately follows from Theorems 4 and 5. ■

§5. Periodicity. Thom isomorphism

In this section we define the groups $K_iK(A, B)$ and establish the formal (Clifford) periodicity of Bott. As a topological consequence, we obtain the Thom isomorphism. We continue working in the category of graded algebras. Moreover, it is assumed everywhere that A is a separable algebra, B has a countable approximate identity, and the action of G on A and B is continuous.

DEFINITION 1. We set

$$K_{p,q}K^{p',q'}(A, B)=KK(A\hat{\otimes}C_{p,q}, B\hat{\otimes}C_{p',q'}).$$

In general, if V and W are linear *-spaces, then

$$K_VK^W(A, B)=KK(A\hat{\otimes}C_V, B\hat{\otimes}C_W).$$

(In the case $V=\{0\}$ the notation $K_{\{0\}}K^W$ is abbreviated as K_0K^W or KK^W. We use an analogous abbreviation for $W=\{0\}$.) We define the homological and cohomological K-functor by the following formulas:

$$K_V(A)=K_VK(A, C),\qquad K^W(B)=KK^W(C, B).$$

The K-functor for a unital algebra A and arbitrary B is defined in the following way:

$$\widetilde{K}_V K^W(A, B) = \operatorname{Ker}[f^* : K_V K^W(A, B) \to K_V K^W(C, B)],$$
$$K_V \widetilde{K}^W(A, \widetilde{B}) = \operatorname{Ker}[g_* : K_V K^W(A, \widetilde{B}) \to K_V K^W(A, C)],$$
$$\widetilde{K}_V \widetilde{K}^W(A, \widetilde{B}) = \operatorname{Ker} f^* \cap \operatorname{Ker} g_*,$$

where $f\colon C \to A$, $f(1) = 1$, and $g\colon \tilde{B} \to C$, $g(1) = 1$.

THEOREM 1. *Let H be a separable Hilbert space (possibly finite-dimensional) with action of G and "real" involution. The algebra $\mathcal{K}(H)$ will be considered as graded by some decomposition $H = H^{(0)} \oplus H^{(1)}$ (it is allowed that $H^{(0)}$ or $H^{(1)} = \{0\}$). There exist canonical generators $\alpha \in K_0(\mathcal{K}(H))$ and $\beta \in K^0(\mathcal{K}(H))$, with which the intersection product induces the following isomorphisms*:

$$KK(A \hat{\otimes} \mathcal{K}(H), B) \simeq KK(A, B),$$
$$KK(A, B \hat{\otimes} \mathcal{K}(H)) \simeq KK(A, B).$$

PROOF. We denote the above-mentioned grading of $\mathcal{K}(H)$ by ε. Let $T \in \mathcal{L}(\mathcal{K})$ be the operator in Theorem 5 of §4. We set

$$\alpha = (\varepsilon_1 \otimes \varepsilon,\ 1 \hat{\otimes} \mathrm{id}\colon \mathcal{K}(H) \to \mathcal{L}(\mathcal{K} \hat{\otimes} H),\ T \hat{\otimes} 1).$$

We denote by $T_2 \in \mathcal{L}(\mathcal{K} \hat{\otimes} H) = \mathfrak{M}(\mathcal{K} \hat{\otimes} \mathcal{K}(H))$ the operator corresponding to T under an arbitrary isomorphism $\mathcal{K} \hat{\otimes} H \simeq \mathcal{K}$. We set $\beta = (\varepsilon_1 \otimes \varepsilon, T_2)$. The relation $\beta \otimes_{\mathcal{K}(H)} \alpha = c_1 \in KK(C, C)$ can be obtained immediately from the definitions. The relation $\beta \otimes_C \alpha = \tau_{\mathcal{K}(H)}(c_1)$ can be obtained by means of the obvious homotopy of the operator $T_2 \#_c (T \hat{\otimes} 1)$ to the form $T_3 \hat{\otimes} 1$, where $T_3 \in \mathcal{L}((\mathcal{K} \hat{\otimes} H) \hat{\otimes} \mathcal{K})$ is the operator corresponding to T under the isomorphism $\mathcal{K} \hat{\otimes} H \hat{\otimes} \mathcal{K} \simeq \mathcal{K}$. Applying Theorem 6 of §4, we obtain the desired isomorphism. ■

THEOREM 2.

$$K_{V \oplus (-V)} K(A, B) \simeq KK(A, B),$$
$$K_V K^V(A, B) \simeq KK(A, B),$$
$$K_V K(A, B) \simeq KK^{-V}(A, B).$$

PROOF. Since $C_V \hat{\otimes} C_{-V} \simeq \mathcal{K}(\wedge^*(V))$, the first relation follows from Theorem 1. The second relation can be obtained from the fact that the compositions

$$KK(A, B) \xrightarrow{\tau_{C_V}} K_V K^V(A, B) \xrightarrow{\tau_{C_{-V}}} K_{V \oplus (-V)} K^{V \oplus (-V)}(A, B) \simeq KK(A, B),$$
$$K_V K^V(A, B) \xrightarrow{\tau_{C_{-V}}} K_{V \oplus (-V)} K^{V \oplus (-V)}(A, B) \simeq KK(A, B) \xrightarrow{\tau_{C_V}} K_V K^V(A, B)$$

are identical. The third relation follows from the first two:

$$K_V K(A, B) \simeq K_{V \oplus (-V)} K^{-V}(A, B) \simeq KK^{-V}(A, B). \ ■$$

THEOREM 3. *Let the action of G on V be trivial, and let $\eta\colon C_V \to C_V$ be a graded automorphism. If η is orientation-preserving (see §2.15), then $(1 \hat{\otimes} \eta)^*(x) = x$ for every $x \in K_V K(A, B)$, and if η changes orientation, then $(1 \hat{\otimes} \eta)^*(x) = -x$. An analogous assertion holds for $KK^V(A, B)$.*

PROOF. First let η preserve orientation. Using Theorem 2, we pass from $K_V K(A, B)$ and the automorphism $\eta: C_V \to C_V$ to $K_{V \oplus [(-V) \oplus V]} K(A, B)$ and the automorphism $\eta \hat{\otimes} 1 \hat{\otimes} 1$. We denote by ζ the automorphism $\eta \hat{\otimes} 1$ of the algebra $C_V \hat{\otimes} C_{-V} \simeq \mathcal{L}(\bigwedge^*(V))$. One of the orientations of the algebra $\mathcal{L}(\bigwedge^*(V))$ is the grading operator $\varepsilon_V = (-1)^k$ on $\bigwedge^k(V)$. As is known, any automorphism ζ of $\mathcal{L}(\bigwedge^*(V))$ can be written in the form $\zeta(a) = uau^{-1}$, where $u \in \mathcal{L}(\bigwedge^*(V))$ is a unitary element ("real" in the "real" case). Since $\zeta(\varepsilon_V) = \varepsilon_V$, the element u commutes with ε_V, and so $\deg u = 0$. Let $x = (\varepsilon, \varphi, F) \in KK(A \hat{\otimes} \mathcal{L}(\bigwedge^*(V)) \hat{\otimes} C_V, B)$. Assuming φ can be extended to $\tilde{A} \hat{\otimes} \mathcal{L}(\bigwedge^*(V)) \hat{\otimes} C_V$ (Theorem 2 of §4), we denote $\varphi(1 \hat{\otimes} u \hat{\otimes} 1)$ by U. The triple $(\varepsilon, \varphi \cdot (1 \hat{\otimes} \zeta \hat{\otimes} 1), F) = \varepsilon,\ U\varphi U^{-1},\ F)$ is unitarily equivalent (and consequently, in view of §§2.3 and 1.17, homotopic) to the triple $(\varepsilon, \varphi, U^{-1}FU)$, which is in turn operator homotopic to $(\varepsilon, \varphi, F)$, since

$$(U^{-1}FU - F) \in \mathcal{K}_B \quad \forall\, a \in A \hat{\otimes} \mathcal{L}(\textstyle\bigwedge^*(V)) \hat{\otimes} C_V \varphi(a).$$

We now consider the case where η changes orientation. We denote by ξ the automorphism of C_V generated by the change of sign of one of the generators, say $f \in V$. Since ξ changes the orientation of C_V, $\xi \cdot \eta$ is orientation-preserving. The part of the theorem already proved reduces our assertion to verifying that $(\varepsilon, \varphi \circ (1 \hat{\otimes} \xi), F) = -(\varepsilon, \varphi, F)$ in the group $K_V K(A, B)$. This follows from the relation

$$(U\varepsilon U^{-1},\ U(\varphi\circ(1\hat{\otimes}\xi))U^{-1},\ UFU^{-1}) = (-\varepsilon,\ -\varphi,\ -F) \in K_V K(A, B),$$

where $U = \varphi(1 \hat{\otimes} f)$. ■

REMARK 1. If the action of G on V is trivial, then $C_V \simeq C_{p,q}$ for some p and q (see §2.16), and therefore $K_V K(A, B) \simeq K_{p,q} K(A, B)$. According to Theorem 3, in order to fix such an isomorphism of the K-functor, it is sufficient to prescribe the orientation of C_V.

THEOREM 4. *For a fixed difference $(p - q) - (p' - q')$ all groups $K_{p,q} K^{p',q'}(A, B)$ are canonically isomorphic.*

PROOF. By Theorem 2 we have

$$\begin{aligned} KK(A \hat{\otimes} C_{p,q},\ B \hat{\otimes} C_{p',q'}) &\simeq KK(A \hat{\otimes} C_{p,q} \hat{\otimes} C_{-(p',q')},\ B) \\ &\simeq KK(A \hat{\otimes} C_{p+q',\,q+p'},\ B). \end{aligned}$$

Our isomorphisms of Clifford algebras used here are the orientation-preserving isomorphisms of §§2.15 and 2.16. (Their concrete form is immaterial in view of Theorem 3.) Since for $k \geqslant l$

$$C_{k,l} \simeq C_{k-l,0} \hat{\otimes} (C_{l,0} \hat{\otimes} C_{-(l,0)}),$$

and for $k \leqslant l$

$$C_{k,l} \simeq C_{0,l-k} \hat{\otimes} (C_{k,0} \hat{\otimes} C_{-(k,0)}),$$

applying Theorem 2 again, we obtain the desired result. ■

DEFINITION 2. For any integers p and q we set

$$\begin{aligned} K_{p-q} K(A, B) &= K_p K^q(A, B) = KK^{q-p}(A, B) \\ &= \begin{cases} K_{p-q,0} K(A, B), & \text{if } p \geqslant q, \\ K_{0,q-p} K(A, B), & \text{if } p \leqslant q. \end{cases} \end{aligned}$$

THEOREM 5 (formal periodicity). *The groups $K_nK(A, B)$ are periodic in n with period* 8, *and with period* 2 *in the special case of $KU_nK(A, B)$.*

PROOF. The isomorphism of Clifford algebras fixed in §2.17 induces isomorphisms

$$KK^{p,q+4}(A, B) \to KK^{p+4,q}(A, B) \quad \text{and} \quad KUK^{p,q+1}(A, B) \to KUK^{p+1,q}(A, B).$$

As a consequence of Theorem 3, these isomorphisms are coordinated among themselves and with the isomorphisms of Theorem 4. ■

THEOREM 6. *The coupling* (2) *of* §4 *induces the coupling*

$$K_iK(A_1, B_1 \hat{\otimes} D) \otimes_D K_jK(D \hat{\otimes} A_2, B_2) \to K_{i+j}K(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2), \tag{1}$$

which commutes with the periodicity isomorphism and is skew-commutative for $D = C$ (the case of product):

$$x_2 \otimes_D x_1 = (-1)^{ij} x_1 \otimes_D x_2.$$

PROOF. The desired coupling can be obtained if in the coupling (2) of §4 we put $A_1 \hat{\otimes} C_{p,q}$ in place of A_1 and $A_2 \hat{\otimes} C_{p',q'}$ in place of A_2 for $p - q = i$, $p' - q' = j$, and then identify $C_{p,q} \hat{\otimes} C_{p',q'}$ with $C_{p+p',q+q'}$ (see §2.16). The commutation with the isomorphisms of Theorems 4 and 5 is obtained from Theorem 3. To verify skew-commutativity, we identify $(A_1 \hat{\otimes} C_{p,q}) \hat{\otimes} (A_2 \hat{\otimes} C_{p',q'})$ with $(A_2 \hat{\otimes} C_{p',q'}) \hat{\otimes} (A_1 \hat{\otimes} C_{p,q})$ and $\mathcal{K}_{B_1} \hat{\otimes} \mathcal{K}_{B_2}$ with $\mathcal{K}_{B_2} \hat{\otimes} \mathcal{K}_{B_1}$ by means of the isomorphism of §2.7. It is easy to verify that the automorphism

$$C_{p+p',q+q'} \simeq C_{p,q} \hat{\otimes} C_{p',q'} \simeq C_{p',q'} \hat{\otimes} C_{p,q} \simeq C_{p+p',q+q'}$$

multiplies orientation by exactly $(-1)^{(p+q)(p'+q')} = (-1)^{ij}$. ■

REMARK 2. If $D = C(X)$, where X is a locally compact space, then the coupling (1) can be defined with values in the group $K_{i+j}K(A_1 \hat{\otimes} D \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$. For this we have to use the existence of the diagonal mapping $\Delta\colon X \to X \times X$ inducing $(\Delta^*)^*$: $K_jK(D \hat{\otimes} A_2, B_2) \to K_jK(D \hat{\otimes} D \hat{\otimes} A_2, B_2)$, and then apply the coupling (1). Moreover, if I and J are ideals in $D = C(X)$, then the multiplication $\Delta^*\colon D \otimes D \to D$ converts $I \otimes J$ into $I \cap J$. Therefore the coupling

$$K_iK(A_1, B_1 \hat{\otimes} I) \otimes_I K_jK(I \hat{\otimes} J \hat{\otimes} A_2, B_2) \to K_{i+j}K(A_1 \hat{\otimes} J \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$$

implies the coupling

$$K_iK(A_1, B_1 \hat{\otimes} I) \otimes_I K_jK((I \cap J) \hat{\otimes} A_2, B_2) \to K_{i+j}K(A_1 \hat{\otimes} J \hat{\otimes} A_2, B_1 \hat{\otimes} B_2). \tag{2}$$

In particular, if Y and Z are closed in X, then the coupling

$$K^i(C(X \setminus Y)) \otimes_{C(X \setminus Y)} K_j(C(X \setminus (Y \cup Z))) \to K_{j-i}(C(X \setminus Z)) \tag{3}$$

is defined.

DEFINITION 3. Let X be a locally compact space. We set $K_V(X) = K_V(C(X))$ and $K^V(X) = K^V(C(X))$.

Canonical generators in $K^n(\mathbf{R}^n)$ and $K_n(\mathbf{R}^n)$.

I. We construct elements $\beta_n \in KU^n_{\mathrm{Spin}(V)}(\mathbf{R}^n)$, $\beta_n \in KO^n_{\mathrm{Spin}(V)}(\mathbf{R}^n)$ and $\beta_{p,q} \in KR^{p-q}_{\mathrm{Spin}(V)}(\mathbf{R}^{p,q})$. (The group $\mathrm{Spin}(V)$ is defined in §2.18, and $\mathbf{R}^{p,q}$ is the coordinate space $\mathbf{R}^n = \mathbf{R}^p \oplus \mathbf{R}^q$ with "real" involution: $\bar{x} = x$ for $x \in \mathbf{R}^p$ and $\bar{x} = -x$ for $x \in \mathbf{R}^q$.)

For definiteness we consider the "real" case. It is not difficult to see that the linear $*$-space V defined in §2.18 is isomorphic to $V_{p,q}$. Using Remark 1, we see that it is sufficient to fix an orientation $\omega_V = i^q\varepsilon_1 \cdots \varepsilon_{p+q}$ in C_V (here $\varepsilon_1, \ldots, \varepsilon_{p+q}$ is a coordinate basis in $V = \mathbf{R}^{p,q} \otimes \mathbf{C}$) and then construct the element $\beta_{p,q} \in KR^V_{\mathrm{Spin}(V)}(\mathbf{R}^{p,q})$.

LEMMA 1. *Let $G = \mathrm{Spin}(V)$ and $V_G = V$. Assume that G acts on V trivially and the action of G on V_G is induced by the action of G on $\mathbf{R}^{p,q}$. Then*

$$KK^V(A, B) \simeq KK^{V_G}(A, B), \quad K_VK(A, B) \simeq K_{V_G}K(A, B).$$

PROOF. By Theorem 2 we have

$$KK^{V_G}(A, B) \simeq K_{-V}K^{V_G \oplus (-V)}(A, B).$$

We identify $C_{V_G \oplus (-V)}$ with $\mathcal{L}(\bigwedge^*(V))$ by means of the isomorphism μ (see §2.11). The action of G on $C_{V_G \oplus (-V)}$ corresponds to a unitary representation μ_V of G in the space $\bigwedge^*(V)$ (see §2.18). By Theorems 1 and 2 we have

$$K_{-V}K^{V_G \oplus (-V)}(A, B) \simeq K_{-V}K(A, B) \simeq K_{-V}K^{V \oplus (-V)}(A, B) \simeq KK^V(A, B). \ \blacksquare$$

Lemma 1 reduces the construction of the element $\beta_{p,q}$ to the construction of $\beta_V \in KR^{V_G}_G(\mathbf{R}^{p,q})$. In view of the existence of the homomorphism $\mathrm{Spin}(V) \to O(p + q)$, where $O(p + q)$ is the orthogonal group of $\mathbf{R}^{p,q}$ (with "real" structure induced from $\mathbf{R}^{p,q}$), in what follows we can assume that $G = O(p + q)$. We denote by D the algebra $C(\mathbf{R}^{p,q}) \otimes C_V$. It is obvious that $\mathfrak{M}(D) = \mathcal{L}(D)$ can be considered as the algebra of continuous bounded functions on $\mathbf{R}^{p,q}$ with values in C_V. We define the invariant element $F \in \mathfrak{M}(D)$ of degree 1 to be the function $F(x) = x(1 + \|x\|^2)^{-1/2}$. Since $x^* = x$ and $x^2 = \|x\|^2$ for every $x \in \mathbf{R}^{p,q}$, we have $F^* = F$ and $1 - F^2 = (1 + \|x\|^2)^{-1/2} \in \mathcal{K}(D) = D$. If ε is the ordinary grading of D, then $\beta_V = (\varepsilon, F)$ can be considered as an element of $KR^{V_G}_G(\mathbf{R}^{p,q})$ according to Remark 1 in §4.

The family of generators thus constructed is multiplicative:

$$\beta_{V_1} \otimes_C \beta_{V_2} = \beta_{V_1 \oplus V_2} \in KR^{V_{G_1} \oplus V_{G_2}}_{G_1 \times G_2}(R^{p_1,q_1} \times R^{p_2,q_2})$$

(the group $O((p_1 + p_2) + (q_1 + q_2))$ is restricted to $G_1 \times G_2 = O(p_1 + q_1) \times O(p_2 + q_2)$). The elements $\beta_{p,q}$ have an analogous property.

In the case $p = q$ the space $\mathbf{R}^{p,q}$ coincides with $\mathbf{C}^p$, and there is another canonical generator in $KR^0_{U(p)}(\mathbf{C}^p)$:

$$\tilde{F}(x) = (\lambda_x + \lambda_x^*)(1 + \|x\|^2)^{-1/2} \in \mathfrak{M}(C(\mathbf{C}^p) \otimes \mathcal{L}(\textstyle\bigwedge^*(\mathbf{C}^p))).$$

Identifying C_V with $\mathcal{L}(\bigwedge^*(\mathbf{C}^p))$ by the mapping $x \to \lambda_x + \lambda_x^*$ and using the well-known imbedding $U(p) \subset \mathrm{Spin}(V)$, it is easy to verify that both of these generators of $KR^0_{U(p)}(\mathbf{C}^p)$ coincide up to the sign $(-1)^{p(p-1)/2}$. The sign is determined by our choice of the orientations in the Clifford algebras.

II. Now we construct the elements $\alpha_n \in KU_n^{\mathrm{Spin}(V)}(\mathbf{R}^n)$, $\alpha_n \in KO_n^{\mathrm{Spin}(V)}(\mathbf{R}^n)$ and $\alpha_{p,q} \in KR_{p-q}^{\mathrm{Spin}(V)}(\mathbf{R}^{p,q})$. Again, we restrict ourselves to the "real" case. Let $\Omega^*(\mathbf{R}^{p,q}) = C^\infty(\bigwedge^*_{\mathbf{C}}(\mathbf{R}^{p,q}))$ be the space of complex-valued C^∞-forms on $\mathbf{R}^{p,q}$, let d and δ be the operator of exterior differentiation and its adjoint in the sense of the Riemannian metric, and let $\Delta = d\delta + \delta d$ be the Laplace operator. We denote by H the Hilbert space $L^2(\bigwedge^*_{\mathbf{C}}(\mathbf{R}^{p,q}))$ of L^2-forms graded by the ordinary decomposition $\bigwedge^* = \bigwedge^{\mathrm{ev}} \oplus \bigwedge^{\mathrm{od}}$ (the forms of even and odd dimensions).

We denote the grading by ε'. By means of the Fourier transformation it is easy to define the (pseudodifferential) operator $Q = (d + \delta)(1 + \Delta)^{-1/2} \in \mathfrak{L}(H)$. This is an invariant Hermitian element of degree 1. Multiplication by a function from $C(\mathbf{R}^{p,q})$ and the action of C_V by means of the representation μ_V (see §§2.11 and 2.18) give the representation φ: $C(\mathbf{R}^{p,q}) \otimes C_V \to \mathfrak{L}(H)$. It is easy to verify that the triple $\alpha_V = (\varepsilon', \varphi, Q)$ is an element of the group $KR^G_{V_G}(\mathbf{R}^{p,q})$, where $G = O(p + q)$. By Lemma 1 this gives the required element $\alpha_{p,q} \in KR_{p-q}^{\mathrm{Spin}(V)}(\mathbf{R}^{p,q})$.

The family of generators thus constructed is multiplicative:

$$\alpha_{V_1} \otimes_C \alpha_{V_2} = \alpha_{V_1 \oplus V_2} \in KR^{G_1 \times G_2}_{V_{G_1} \oplus V_{G_2}} (\mathbf{R}^{p_1,q_1} \times \mathbf{R}^{p_2,q_2})$$

(the group $O((p_1 + p_2) + (q_1 + q_2))$ is restricted to $G_1 \times G_2 = O(p_1 + q_1) \times O(p_2 + q_2)$). An analogous property is enjoyed by the elements $\alpha_{p,q}$.

For $p = q$ in the group $KR_0^{U(p)}(\mathbf{C}^p)$ there exists still another canonical generator (see, for example, [15], §2.7). Let $\bar\partial$: $\Omega^{0,*}(\mathbf{C}^p) \to \Omega^{0,*}(\mathbf{C}^p)$ be the operator of antihomological differentiation, and let $\tilde\Delta = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $H = L^2(\bigwedge^{0,*}(\mathbf{C}^p))$. If we denote by ε' the grading of H by means of the decomposition $\bigwedge^{0,*} = \bigwedge^{0,\mathrm{ev}} \oplus \bigwedge^{0,\mathrm{od}}$, by φ: $C(\mathbf{C}^p) \to \mathfrak{L}(H)$ multiplication by a function and by $\tilde Q$ the operator $(\bar\partial + \bar\partial^*)(1 + \tilde\Delta)^{-1/2} \in \mathfrak{L}(H)$, then $(\varepsilon', \varphi, \tilde Q) \in KR_0^{U(p)}(\mathbf{C}^p)$. The isomorphism of Theorem 1 identifies these generators up to sign.

THEOREM 7 (Bott periodicity). *If the group G acts on $\mathbf{R}^n$ (on $\mathbf{R}^{p,q}$) by means of the spinor representation, then the intersection product with the elements α_n and β_n ($\alpha_{p,q}$ and $\beta_{p,q}$) constructed above defines the isomorphisms*

$$K_{i+n}K(A(\mathbf{R}^n), B) \simeq K_iK(A, B) \simeq K_{i-n}K(A, B(\mathbf{R}^n))$$

in the complex and real case, and the isomorphisms

$$K_{i+p-q}K(A(\mathbf{R}^{p,q}), B) \simeq K_iK(A, B) \simeq K_{i-p+q}K(A, B(\mathbf{R}^{p,q}))$$

in the "real" case.

PROOF. As a consequence of Theorem 6 of §4 it is sufficient to verify the relations

$$\text{a) } \beta_V \otimes_{C(\mathbf{R}^{p,q}) \otimes C_V} \alpha_V = c_1; \quad \text{b) } \beta_V \otimes_C \alpha_V = \tau_{C(\mathbf{R}^{p,q}) \otimes C_V}(c_1). \tag{4}$$

a) Using Remark 3 of §4, we represent the operator $F \#_{C(\mathbf{R}^{p,q}) \otimes C_V} Q$ in the form

$$(\lambda_x + \lambda_x^*)(1 + \|x\|^2)^{-1/2} + (1 + \|x\|^2)^{-1/2} \cdot (d + \delta)(1 + \Delta)^{-1/2}.$$

It is well known that its "real" $O(p + q)$-index as an operator $L^2(\bigwedge^{\mathrm{ev}}_{\mathbf{C}}(\mathbf{R}^{p,q})) \to L^2(\bigwedge^{\mathrm{od}}_{\mathbf{C}}(\mathbf{R}^{p,q}))$ is equal to 1 (see [6] or [15], §4, Theorem 5). This proves the first relation.

b) Passing to the proof of the second relation, we denote the algebra $C(\mathbf{R}^{p,q}) \otimes C_V$ by D and identify $D \hat\otimes D$ with $C(\mathbf{R}^{p,q} \oplus \mathbf{R}^{p,q}) \otimes C_{V \oplus V}$. The rotation

$$\mathbf{R}^{p,q} \oplus \mathbf{R}^{p,q} \to \mathbf{R}^{p,q} \oplus \mathbf{R}^{p,q}: (x, y) \to (\cos t \cdot x - \sin t \cdot y,\ \sin t \cdot x + \cos t \cdot y),$$

$0 \leqslant t \leqslant \pi/2$, induces a continuous family of automorphisms of the algebras

$C(\mathbf{R}^{p,q} \oplus \mathbf{R}^{p,q})$ and $C_{V \oplus V}$ (we recall that $V \oplus V = (\mathbf{R}^{p,q} \oplus \mathbf{R}^{p,q}) \otimes \mathbf{C}$). As a result, we obtain a homotopy of the automorphisms of the algebras $D \hat{\otimes} D$ and $\mathfrak{M}(D \hat{\otimes} D)$, which we denote by $\{\gamma_t\}$: $\mathfrak{M}(D \hat{\otimes} D) \to \mathfrak{M}(D \hat{\otimes} D \hat{\otimes} C[0, \pi/2])$. We denote temporarily by $\mathcal{K}$ the algebra $\mathcal{K}(L^2(\wedge^*_{\mathbf{C}}(\mathbf{R}^{p,q})))$. The natural homomorphism $1 \hat{\otimes} \varphi$: $D \to \mathfrak{M}(D \hat{\otimes} \mathcal{K})$ can be extended to $\Phi = (\mathrm{id} \hat{\otimes} \varphi)_*$: $\mathfrak{M}(D \hat{\otimes} D) \to \mathfrak{M}(D \hat{\otimes} \mathcal{K})$. Let $\Phi_t = \Phi \cdot \gamma_t$ and $\varphi_t = \Phi_t \cdot (1 \hat{\otimes} \mathrm{id})$. We denote by ω the orientation of $C_{V\oplus V}$, by u the element corresponding to it in $\mathfrak{M}(D \hat{\otimes} D)$, and by U the element $\Phi(u)$.

We construct a pair $\{M_t\}, \{N_t\} \in \mathfrak{M}(D \hat{\otimes} \mathcal{K} \hat{\otimes} C[0, \pi/2])$ such that the triple $\{(\beta_V \otimes \alpha_V)_t\} = (\varepsilon \otimes \varepsilon', \{\varphi_t\}, \{\sqrt{M_t} \cdot \Phi_t(F \hat{\otimes} 1) + \sqrt{N_t}\,(1 \hat{\otimes} Q)\})$ will be an element of $\mathcal{E}(D, D[0, \pi/2])$, and $(\beta_V \otimes \alpha_V)_t = \beta_V \otimes_C \alpha_V$ for $t = 0$ and $(\beta_V \otimes \alpha_V)_t = U\tau_D(\beta_V \otimes_D \alpha_V)U^{-1}$ for $t = \pi/2$. This completes the proof.

In the algebra $\mathfrak{M}(D \hat{\otimes} \mathcal{K} \otimes C[0, \pi/2])$ we consider the subalgebras $E = D \hat{\otimes} \mathcal{K} \hat{\otimes} C[0, \pi/2]$ and $E_1 = (\Phi(D \hat{\otimes} D) \hat{\otimes} 1) + E$. Let E_2 be the minimal subalgebra containing the elements

$$1\hat{\otimes}(Q^2-1)\hat{\otimes}1; \qquad [\{\Phi_t(F\hat{\otimes}1)\},\ 1\hat{\otimes}Q\hat{\otimes}1]; \qquad [\{\varphi_t\}(d),\ 1\hat{\otimes}Q\hat{\otimes}1]$$

for all $d \in D$. We denote by $\mathfrak{F}$ the linear space spanned by $\{\Phi_t(F \hat{\otimes} 1)\}$, $1 \hat{\otimes} Q \hat{\otimes} 1$ and $\{\varphi_t\}(D)$. Applying Theorem 4 of §3, we obtain the required pair $\{M_t\}$, $\{N_t\}$. ∎

Now we pass to the questions connected with the Thom isomorphism.

Definition 4. Let ξ and η be vector *-bundles (see §2.12) over the locally compact spaces X and Y, respectively. The group $KK(C_\xi(X), B)$ will be denoted by $K_\xi K(X, B)$, the group $KK(A, C_\eta(Y))$ by $KK^\eta(A, Y)$ and $KK(C_\xi(X), C_\eta(Y))$ by $K_\xi K^\eta(X, Y)$.

We consider the following situation. Let π: $P \to X$ be a principal G_1-bundle, Z a space with action of G_1 and the spaces X and Z locally compact, the group G_1 compact, and assume that X, Z and G_1 satisfy the second axiom of countability. Besides, let ξ be a vector *-bundle over X and η a vector *-bundle with action of G_1 over Z. We denote by Y the space $Z \times_{G_1} P$, by ρ the projection $Y \to X$, and by $\tilde{\xi}$ and $\tilde{\eta}$ the vector *-bundles $\rho^*(\xi)$ and $\eta \times_{G_1} P$ over Y, respectively. We shall assume that on P, X, Y, Z, ξ and η an action of G is defined which commutes with the action of G_1 on P, Z and η and in the "real" case a "real" structure is also defined (and the groups G and G_1 are "real").

Lemma 2. *We have the following natural homomorphisms*:

$$j_Z: K_\eta^{G\times G_1}(Z) \to K^G_{\tilde{\eta}\oplus\tilde{\xi}}K^{\xi}(Y, X),$$
$$j^Z: K^\eta_{G\times G_1}(Z) \to K^G_\xi K^{\widetilde{\eta}\oplus\widetilde{\xi}}(X, Y),$$
$$j^Z_Z: K_\eta^{G\times G_1}K^\eta(Z, Z) \to K^G_{\tilde{\eta}\oplus\tilde{\xi}}K^{\widetilde{\eta}\oplus\widetilde{\xi}}(Y, Y),$$

satisfying the conditions

1) $j_Z(a) \otimes_{C_\xi(X)} j^Z(b) = j^Z_Z(a \otimes_C b)$,

2) $j^Z(b) \otimes_{C_{\tilde{\eta}\oplus\tilde{\xi}}(Y)} j_Z(a) = \tau_{C_\xi(X)}(b \otimes_{C_\eta(Z)} a)$,

3) $j^Z_Z(\tau_{C_\eta(Z)}(c_1)) = \tau_{C_{\tilde{\eta}\oplus\tilde{\xi}}(Y)}(c_1)$.

Proof. We introduce the notation $\mathrm{Sect}_T(\zeta)$ for the space of continuous sections of the vector bundle ζ over the locally compact space T, converging to 0 at ∞. Our basic

Hilbert space $\mathcal{H}$ will be assumed to be endowed with an action of the group $G \times G_1$ in the sense defined in §1.11. It is obvious that $\mathrm{Sect}_X(C_\eta(Z) \times_{G_1} P) \simeq C_{\tilde{\eta}}(Y)$. The space $E = \mathrm{Sect}_X(\mathcal{H} \times_{G_1} P)$ is a Hilbert $C(X)$-module, and $E_Z = \mathrm{Sect}_X(\mathcal{H}_{C_\eta(Z)} \times_{G_1} P)$ is a Hilbert $C_{\tilde{\eta}}(Y)$-module. The algebra $C_{\tilde{\eta}\oplus\tilde{\xi}}(Y)$ can be considered as a skew-commutative algebraic tensor product over $C(X)$: $C_{\tilde{\eta}}(Y) \hat{\otimes}_{C(X)} C_\xi(X)$. On the algebraic tensor products $E \hat{\otimes}_{C(X)} C_\xi(X)$ and $E_Z \hat{\otimes}_{C(X)} C_\xi(X)$ we introduce the inner product

$$(u_1 \hat{\otimes} v_1,\ u_2 \hat{\otimes} v_2) = (-1)^{\deg v_1\cdot(\deg u_1+\deg u_2)}(u_1, u_2) \hat{\otimes} v_1^* v_2$$

with values in

$$C(X) \hat{\otimes}_{C(X)} C_\xi(X) = C_\xi(X) \quad \text{and} \quad C_{\tilde{\eta}}(Y) \hat{\otimes}_{C(X)} C_\xi(X) = C_{\tilde{\eta}\oplus\tilde{\xi}}(Y)$$

respectively. Moreover, on $E_Z \hat{\otimes}_{C(X)} C_\xi(X)$ we may define the following right action of the algebra:

$$C_{\tilde{\eta}\oplus\tilde{\xi}}(Y) = C_{\tilde{\eta}}(Y) \hat{\otimes}_{C(X)} C_\xi(X)\colon (u \hat{\otimes} v)(a \hat{\otimes} b) = (-1)^{\deg v\cdot\deg a}(ua \hat{\otimes} vb).$$

It is easy to verify that we obtain the Hilbert modules $E \hat{\otimes}_{C(X)} C_\xi(X)$ over $C_\xi(X)$ and $E_Z \hat{\otimes}_{C(X)} C_\xi(X)$ over $C_{\tilde{\eta}\oplus\tilde{\xi}}(Y)$.

Now let $(\varepsilon, \varphi, F) \in \mathcal{E}(C_\eta(Z), C)$. The homomorphism φ can be extended naturally to

$$\varphi \hat{\otimes} \mathrm{id}\colon C_{\tilde{\eta}\oplus\tilde{\xi}}(Y) = \mathrm{Sect}_X\big(C_{\tilde{\eta}}(Z) \times_{G_1} P\big) \hat{\otimes}_{C(X)} C_\xi(X) \to \mathcal{L}\big(E \hat{\otimes}_{C(X)} C_\xi(X)\big).$$

Since the operator F commutes with the action of G_1 on $\mathcal{H}$, there exists $F \hat{\otimes} 1 \in \mathcal{L}(E \hat{\otimes}_{C(X)} C_\xi(X))$. We set

$$j_Z(\varepsilon, \varphi, F) = (\varepsilon \otimes \varepsilon',\ \varphi \hat{\otimes} \mathrm{id},\ F \hat{\otimes} 1),$$

where ε' is the grading of $C_\xi(X)$. If $(\varepsilon, F) \in \mathcal{E}(C, C_\eta(Z))$, then there exists $F \hat{\otimes} 1 \in \mathcal{L}(E_Z \hat{\otimes}_{C(X)} C_\xi(X))$. We set

$$j^Z(\varepsilon, F) = (\varepsilon \otimes \varepsilon',\ 1 \hat{\otimes} \mathrm{id},\ F \hat{\otimes} 1).$$

j_Z^Z is defined analogously:

$$j_Z^Z(\varepsilon, \varphi, F) = (\varepsilon \otimes \varepsilon'\ \ \varphi \hat{\otimes} \mathrm{id},\ F \hat{\otimes} 1).$$

Relations 1)–3) can be verified easily. ∎

Every n-dimensional real bundle η over X can be considered as a bundle $\mathbf{R}^n \times_{G_1} P \to X$, where $G_1 = O(n)$ and $P \to X$ is the bundle of n-frames associated with η. (The action of G on η obviously defines an action of G on P commuting with the action of G_1.) In the "real" case we shall assume that on η a "real" structure is defined, i.e. a linear involution coordinated with the "real" involution on X. Let p and q be the dimensions of the positive and negative subspaces of this involution, respectively. We have $\eta = \mathbf{R}^{p,q} \times_{O(p+q)} P$. Denote by Y_η the bundle space of η, and by $\tilde{\eta}_0$ the lifting of η to Y_η. We set $\tilde{\eta} = \tilde{\eta}_0$ in the real case and $\tilde{\eta} = \tilde{\eta}_0 \times \mathbf{C}$ in the complex and "real" cases. On $\tilde{\eta}$ we define an involution * and a real structure as antilinear extensions of the identity mapping and the "real" involution on η, respectively.

THEOREM 8 (Thom isomorphism). *Let ξ be an arbitrary *-bundle on X and $\tilde{\xi}$ its lifting to Y_η. Then*

$$K_\xi(X) \simeq K_{\tilde{\eta}\oplus\tilde{\xi}}(Y_\eta), \qquad K^\xi(X) \simeq K^{\tilde{\eta}\oplus\tilde{\xi}}(Y_\eta).$$

If η is a spinor bundle, then

$$K_i(X) \simeq K_{i+n}(Y_\eta), \quad K^i(X) \simeq K^{i+n}(Y_\eta)$$

in the complex and real cases, and

$$K_i(X) \simeq K_{i+p-q}(Y_\eta), \quad K^i(X) \simeq K^{i+p-q}(Y_\eta)$$

in the "real" case.

PROOF. We restrict ourselves to the "real" case. Let $Z = \mathbf{R}^{p,q}$, let the action of G on Z be trivial, and let $G_1 = O(p+q)$ and $V = \mathbf{R}^{p,q} \otimes \mathbf{C}$. It is obvious that $\tilde{\eta} = V \times_{G_1} P$ and $Y_\eta = \mathbf{R}^{p,q} \times_{G_1} P$. We set $\alpha = j_Z(\alpha_V)$ and $\beta = j^Z(\beta_V)$, where $\alpha_V \in K_{V_{G_1}}^{G_1}(Z)$ and $\beta_V \in K_{G_1}^{V_{G_1}}(Z)$ are the generators constructed above. From Lemma 2 and (4) it follows that

$$\alpha \otimes_{C_\xi(X)} \beta = \tau_{C_{\widetilde{\eta} \oplus \widetilde{\xi}}(Y)}(c_1), \quad \beta \otimes_{C_{\widetilde{\eta} \oplus \widetilde{\xi}}(Y)} \alpha = \tau_{C_\xi(X)}(c_1).$$

On the basis of Theorem 6 of §4 the homomorphisms

$$K_\xi(X) \to K_{\widetilde{\eta} \oplus \widetilde{\xi}}(Y_\eta) : a \to \alpha \otimes_{C_\xi(X)} a,$$
$$K^\xi(X) \to K^{\widetilde{\eta} \oplus \widetilde{\xi}}(Y_\eta) : b \to b \otimes_{C_\xi(X)} \beta$$

are isomorphisms.

In the spinor case we replace G_1 by $\mathrm{Spin}(V)$ and the elements α_V and β_V by $\alpha_{p,q} \in K_V^{\mathrm{Spin}(V)}(Z)$ and $\beta_{p,q} \in K_{\mathrm{Spin}(V)}^V(Z)$, where the action of $\mathrm{Spin}(V)$ on V is trivial. The above reasoning can be repeated verbatim with $\tilde{\eta}$ replaced by $\tilde{\zeta} = V \times_{\mathrm{Spin}(V)} P$, which is trivial in view of the triviality of the action of $\mathrm{Spin}(V)$ on V.

§6. Other definitions of the K-functor

In this section we establish that for trivially graded unital algebras A and B the definition of the homological K-functor $K_i(A)$ is equivalent to the definition given in [15], and the group $K^0(B)$ is isomorphic to the ordinary Grothendieck group of projective modules over B. In conclusion, for finite cell complexes X and Y we give two homotopical definitions of the group $K_iK(X, Y)$. In this section we assume again that the algebra A is separable and B has a countable approximate identity. The action of G on A and B is continuous.

THEOREM 1. *Let A and B be graded algebras. The group $KK(A, B)$ does not change if in Definition 3 of §4 homotopy equivalence is replaced by the combination of unitary equivalence and operator homotopy. Moreover, if the triples $x_0 = (\varepsilon_0, \varphi_0, F_0)$ and $x_1 = (\varepsilon_1, \varphi_1, F_1)$ are connected by the homotopy $\{x_t\} = (\{\varepsilon_t\}, \{\varphi_t\}, \{F_t\})$, then there exist $x_0', x_1' \in \mathfrak{D}(A, B)$ such that $x_0 \oplus x_0'$ is operator homotopic to $x_1 \oplus x_1'$. (If $\{\varphi_t\}$ can be extended to the algebra $A_1 \supset A$ in which A is an ideal (see Theorem 2 of §4), then this is so for x_0' and x_1'. If $\{\varphi_t\}$ is unital, then the same is true for x_0' and x_1'.) Conversely, unitarily equivalent triples are homotopic.*

PROOF. It will be convenient to assume that the homotopy $\{x_t\}$ is parametrized by points of the interval $[0, 2\pi]$ and not $[0, 1]$, i.e. $\{x_t\} \in \mathfrak{E}(A, B[0, 2\pi])$. Also we assume that for $0 \leqslant t \leqslant 2\pi/3$ and $4\pi/3 \leqslant t \leqslant 2\pi$ the homotopy is constant in t, i.e. $x_t = x_0$ for $t \leqslant 2\pi/3$ and $x_t = x_{2\pi}$ for $t \geqslant 4\pi/3$. We recall some notation from §2 of [15]. The operator d: $L^2[0, 2\pi] \to L^2[0, 2\pi]$ is defined, using the basis $(1), \ldots, (\cos nx), \ldots, (\sin nx), \ldots$, by the formulas

$$d(1)=0, \quad d(\cos nx) = -(\sin nx), \quad d(\sin nx) = (\cos nx).$$

The operator d is skew-Hermitian, $d^2 + 1 \in \mathcal{K}(L^2[0, 2\pi])$, and d commutes with multiplication by functions from $C(0, 2\pi)$ modulo the compact operators. Let $f \in C[0, 2\pi]$ be a continuous real function satisfying the conditions $|f(x)| \leq 1$, $f(0) = 1$ and $f(2\pi) = -1$. We set $T_1(f) = f - \sqrt{1 - f^2} \cdot d \in \mathcal{L}(L^2[0, 2\pi])$, where by f we denote the operator of multiplication by f. The operator $T_1(f)$ is Fredholm of index 1, and $1 - T_1(f) \cdot T_1(f)^*$ and $1 - T_1(f)^* \cdot T_1(f)$ are compact. Moreover, $T_1(f)$ commutes, modulo the compact operators, with multiplication by functions from $C[0, 2\pi]$. It is clear that any two operators of the form $T_1(f)$ (for different f) are connected by a norm continuous homotopy consisting of operators of the same form.

Let $H^{(0)} = H^{(1)} = L^2[0, 2\pi]$, and let $H = H^{(0)} \oplus H^{(1)}$ be a graded Hilbert space (the action of G is trivial). The triple $c_1(f) = (\varepsilon, \mathrm{id}, T(f))$, where ε is the grading of H, id: $C[0, 2\pi] \to \mathcal{L}(H)$ is the natural representation, and

$$T(f) = \begin{pmatrix} 0 & T_1(f)^* \\ T_1(f) & 0 \end{pmatrix},$$

defines an element of $KK(C[0, 2\pi], C)$. The intersection $\{x_t\} \otimes_{C[0,2\pi]} c_1(f) \in KK(A, B)$ can be written in the form

$$\Big(\varepsilon_t \otimes \varepsilon, \mathrm{id}_* \circ \{\varphi_t\}: A \to \mathfrak{M}(\mathcal{K} \hat{\otimes} B \hat{\otimes} C[0, 2\pi])$$

$$\xrightarrow{\mathrm{id}_*} \mathfrak{M}(\mathcal{K} \hat{\otimes} B \hat{\otimes} \mathcal{K}(H)), \quad F(f) = \sqrt{M} \cdot \mathrm{id}_*(\{F_t\}) + \sqrt{N}\,(1 \hat{\otimes} T(f))\Big).$$

We may assume that the pair $(M, N) \in S(\{x_t\}, c_1(f))$ does not depend on f and commutes with $1 \hat{\otimes} C[0, 2\pi]$ modulo $\mathcal{K} \hat{\otimes} B \hat{\otimes} \mathcal{K}(H)$. For the construction of such M and N it is sufficient to use Theorem 4 of §3 in the case where

$$E_1 = \mathrm{id}_*(\mathcal{K} \hat{\otimes} B \hat{\otimes} C[0, 2\pi]) + \mathcal{K} \hat{\otimes} B \hat{\otimes} \mathcal{K}(H); \quad E = \mathcal{K} \hat{\otimes} B \hat{\otimes} \mathcal{K}(H);$$

E_2 is the minimal subalgebra in $\mathfrak{M}(\mathcal{K} \hat{\otimes} B \hat{\otimes} \mathcal{K}(H))$ containing $1 \hat{\otimes} \mathcal{K}(H)$;

$$[\mathrm{id}_* \circ \{\varphi_t\}(a),\ 1 \hat{\otimes} T(f)] \quad \forall a \in A,\ \forall f \in C[0, 2\pi];$$

$$[\mathrm{id}_*(\{F_t\}),\ 1 \hat{\otimes} T(f)] \quad \forall f \in C[0, 2\pi];$$

$\mathfrak{F}$ is the linear space spanned by

$$\mathrm{id}_*(\{F_t\});\ 1 \hat{\otimes} f \quad \text{and} \quad 1 \hat{\otimes} T(f)\ \forall f \in C[0, 2\pi];\ (\mathrm{id}_* \circ \{\varphi_t\})(A).$$

Now we set

$$f = \begin{cases} \cos 3t, & 0 \leq t \leq \dfrac{\pi}{3}, \\[2mm] -1, & \dfrac{\pi}{3} \leq t \leq 2\pi. \end{cases}$$

The operator $T_1(f)$ commutes with the projection P onto $L^2[0, 2\pi/3]$, and $P \cdot T_1(f)$ is an operator of index 1 on $L^2[0, 2\pi/3]$, and $(1 - P)T_1(f) = -1$ on $L^2[2\pi/3, 2\pi]$. We choose $\alpha(t) \in C[0, 2\pi]$ so that $0 \leq \alpha(t) \leq 1$, $\alpha(t) = 0$ for $t \leq \pi/3$, and $\alpha(t) = 1$ for $t \geq 2\pi/3$.

By means of a norm continuous homotopy, M and N can be replaced by

$$\widetilde{M} = \sqrt{1 \hat{\otimes} (1-\alpha)} \cdot M \cdot \sqrt{1 \hat{\otimes} (1-\alpha)},$$
$$\widetilde{N} = 1 \hat{\otimes} \alpha + \sqrt{1 \hat{\otimes} (1-\alpha)} \cdot N \cdot \sqrt{1 \hat{\otimes} (1-\alpha)}.$$

The new operator $F(f)$ commutes with $1 \hat{\otimes} (P \oplus P)$, and for the decomposition $H = \operatorname{Im}(P \oplus P) \oplus \operatorname{Im}(1 - P \oplus P)$ we obtain

$$\{x_t\} \otimes_{C[0,2\pi]} c_1(f) = (x_0 \otimes_C c_1) \oplus (\text{element of } \mathfrak{D}(A, B)).$$

(Here c_1 is the multiplicative identity; see Theorem 5 of §4. We recall that $x_t = x_0$ for $t \leqslant 2\pi/3$.)

If we now set

$$f = \begin{cases} 1, & 0 \leqslant t \leqslant \dfrac{5\pi}{3}, \\ -\cos 3t, & \dfrac{5\pi}{3} \leqslant t \leqslant 2\pi, \end{cases}$$

then we may prove in an analogous way that $\{x_t\} \otimes_{C[0,2\pi]} c_1(f)$ is operator homotopic to $(x_{2\pi} \otimes_C c_1) \oplus$ (element of $\mathfrak{D}(A, B)$). It remains to apply Theorem 5 of §4. The converse assertion follows from §§1.17 and 2.3. ■

DEFINITION 1. Let the algebras A, B and $\mathcal{K}$ have trivial grading. We denote by $\mathcal{E}_{p,q}(A, B)$ the set of pairs (φ, F), where $\varphi\colon A \otimes C_{p+1,q} \to \mathfrak{M}(\mathcal{K} \otimes B)$ is a homomorphism (not graded), F is an invariant element in $\mathfrak{M}(\mathcal{K} \otimes B)$ and for every $a \in A$ and $b \in C_{p+1,q}$ the elements

$$\varphi(a \otimes b) \cdot F - (-1)^{\deg b} F \cdot \varphi(a \otimes b), \quad (F^2 - 1)\varphi(a \otimes b), \quad (F - F^*)\varphi(a \otimes b)$$

belong to $\mathcal{K} \otimes B$. Let $\mathfrak{D}_{p,q}(A, B)$ be the set of (degenerate) pairs for which the indicated elements are equal to 0, and denote by $\mathcal{E}'_{p,q}(A, B)$ and $\mathfrak{D}'_{p,q}(A, B)$ the set of pairs (φ: $\tilde{A} \otimes C_{p+1,q} \to \mathfrak{M}(\mathcal{K} \otimes B)$, F), satisfying the same conditions (for $a \in A$). Homotopy and operator homotopy are defined in the same way as in §4 (for $\mathcal{E}'_{p,q}(A, B)$ and $\mathfrak{D}'_{p,q}(A, B)$ homotopy is assumed to be defined on $\tilde{A} \otimes C_{p+1,q}$). Pairs (φ, F) and (φ_1, F_1) are unitarily equivalent if there exists an invariant unitary element $u \in \mathfrak{M}(\mathcal{K} \otimes B)$ such that

$$\forall a \in A \otimes C_{p+1,q} \quad \varphi_1(a) = u\varphi(a)u^{-1}, \quad F_1 = uFu^{-1}$$

(for $\mathcal{E}'_{p,q}(A, B)$ we assume that $a \in \tilde{A} \otimes C_{p+1,q}$).

We denote by $\overline{\mathcal{E}}_{p,q}(A, B)$ the set of equivalence classes of $\mathcal{E}_{p,q}(A, B)$ modulo homotopy, and by $\overline{\mathfrak{D}}_{p,q}(A, B)$ the image of $\mathfrak{D}_{p,q}(A, B)$ in $\overline{\mathcal{E}}_{p,q}(A, B)$. We define $\overline{\mathcal{E}}'_{p,q}(A, B)$ and $\overline{\mathfrak{D}}'_{p,q}(A, B)$ analogously. Moreover, let $\tilde{\mathcal{E}}'_{p,q}(A, B)$ be the set of equivalence classes of $\mathcal{E}'_{p,q}(A, B)$ modulo unitary equivalence and operator homotopy, and let $\tilde{\mathfrak{D}}'_{p,q}(A, B)$ be the image of $\mathfrak{D}'_{p,q}(A, B)$ in $\tilde{\mathcal{E}}'_{p,q}(A, B)$. The direct sum (see Definition 3 of §4) turns these sets of equivalence classes into semigroups.

THEOREM 2. *If A and B are trivially graded, then*

$$\overline{\mathcal{E}}_{p,q}(A, B)/\overline{\mathfrak{D}}_{p,q}(A, B) \simeq \overline{\mathcal{E}}'_{p,q}(A, B)/\overline{\mathfrak{D}}'_{p,q}(A, B)$$
$$\simeq K_{p,q}K(A, B) \simeq \tilde{\mathcal{E}}'_{p,q}(A, B)/\tilde{\mathfrak{D}}'_{p,q}(A, B).$$

PROOF. The first isomorphism can be obtained in the same way as in the proof of Theorem 2 of §4. The second and third isomorphisms are obtained in the following way. Let $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$, and let ε be the grading operator $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. It is obvious that $\mathcal{K}(\mathcal{H}) \hat{\otimes} B = \mathcal{K}(\mathcal{H}) \otimes B$ and $A \hat{\otimes} C_{p,q} = A \otimes C_{p,q}$. To the triple $(\varepsilon, \varphi, F) \in \mathcal{E}(A \otimes C_{p,q}, B)$, where φ is extended to $\tilde{A} \otimes C_{p,q}$, we assign the pair $(\varphi', F) \in \mathcal{E}'_{p,q}(A, B)$, where $\varphi'|_{\tilde{A}\otimes C_{p,q}} = \varphi$ and $\varphi'(a \otimes \varepsilon_{p+1}) = \varphi(a \otimes 1) \cdot \varepsilon$. The inverse mapping $(\varphi', F) \to (\varepsilon, \varphi, F)$ can be obtained if we set $\varepsilon = \varphi'(1 \otimes \varepsilon_{p+1})$. The decomposition

$$\mathcal{H}_B = \operatorname{Im}((1+\varepsilon)/2) \oplus \operatorname{Im}((1-\varepsilon)/2)$$

defines a grading of $\mathcal{H}_B$. Operator homotopy enables us to replace the operator F by $\frac{1}{2}(F - \varepsilon F \varepsilon)$, which has degree 1 with respect to the grading ε. (In the case of the third isomorphism we have to apply Theorem 1.) ■

COROLLARY 1. *For a trivially graded unital algebra A the definition of $K_{p,q}(A)$ is equivalent to Definition* 1 *of* [15], §1.

PROOF. We note that, for a unital algebra A, in the formulation of Theorem 2 it is not necessary to require that the homomorphism $\varphi\colon A \otimes C_{p+1,q} \to \mathcal{M}(\mathcal{K} \otimes B)$ be extendable to $\tilde{A} \otimes C_{p+1,q}$ (the proof remains valid). The isomorphism of the Clifford algebras

$$C_{q+1,p} \rightleftarrows C_{p+1,q} : \varepsilon_i \to e_i\varepsilon_{p+1}, \quad i \leqslant q;$$
$$e_j \to \varepsilon_j\varepsilon_{p+1}, \quad j \leqslant p; \quad \varepsilon_{q+1} \to \varepsilon_{p+1}$$

with the operator F replaced by $F_1 = F \cdot \varphi(1 \otimes \varepsilon_{p+1})$ converts the pair $(\varphi\colon A \otimes C_{p+1,q} \to \mathcal{M}(\mathcal{K} \otimes B), F)$ into the pair $(\varphi_1\colon A \otimes C_{q+1,p} \to \mathcal{M}(\mathcal{K} \otimes B), F_1)$, which satisfies the conditions

$$F_1 \cdot \varphi(a) - (-1)^{\deg a} \varphi(a) F_1 \in \mathcal{K}_B, \quad (F_1^2 + 1)\varphi(a) \in \mathcal{K}_B,$$
$$(F_1 + F_1^*)\varphi(a) \in \mathcal{K}_B$$

for every $a \in A \otimes C_{q+1,p}$. Since $1 \in A$, the last two conditions are replaced by $F_1^2 + 1 \in \mathcal{K}_B$ and $F_1 + F_1^* \in \mathcal{K}_B$. ($B = C$ in [15].) ■

LEMMA 1. *Let A and B be graded algebras, and let $(\varepsilon, \varphi, \{F_t\}) \in \mathcal{E}(A, B[0, 1])$ be an operator homotopy. There exists a norm continuous function $[0, 1] \to \mathcal{L}(\mathcal{H}_B)$: $t \to u_t$, satisfying the following conditions for every $t \in [0, 1]$:*

1) *u_t is an invariant unitary element of degree* 0, *and $u_0 = 1$.*
2) *$\forall a \in A$ $u_t\varphi(a) - \varphi(a)u_t \in \mathcal{K}_B$.*
3) *$\forall a \in A$ $(F_tu_t - u_tF_0)\varphi(a) \in \mathcal{K}_B$.*

PROOF. Let $\delta > 0$ be such that for $|s - t| \leqslant \delta$ the operator $(2 - F_s^2 + F_tF_s)$ is invertible. We cover the interval $[0, 1]$ by intervals $\{[t_i, t_{i+1}]\}$ of length $\leqslant \delta$. First we construct by induction an invertible (and not unitary) function $v\colon [0, 1] \to \mathcal{L}(\mathcal{H}_B)$ satisfying the same conditions. We set $v_0 = 1$. If v_t is already constructed for $t \in [0, t_i]$, then for $t \in [t_i, t_{i+1}]$ we put $v_t = \frac{1}{2}(2 - F_{t_i}^2 + F_tF_{t_i}) \cdot v_{t_i}$. Conditions 2) and 3) can be verified easily. It remains to replace the function v_t by $u_t = v_t(v_t^*v_t)^{-1/2}$, which is unitary. ■

DEFINITION 2. Let B be a trivially graded unital algebra. We consider the category of finitely generated projective right B-modules with a left action of G compatible with the action of G on B. (B-modules are complex, real, or "real", depending on what category of algebras B is considered.) We denote by $\mathfrak{G}^0(B)$ the Grothendieck group of this category with respect to the operation of taking direct sums of modules. For a nonunital algebra B we set

$$\mathfrak{G}^0(B) = \widetilde{\mathfrak{G}}^0(\tilde{B}) = \operatorname{Ker}[g_* : \mathfrak{G}^0(\tilde{B}) \to \mathfrak{G}^0(C)],$$

where $g\colon \tilde{B} \to C$, $g(1) = 1$.

THEOREM 3. $K^0(B) \simeq \mathfrak{G}^0(B)$.

The proof uses several lemmas.

DEFINITION 3. Let B be a trivially graded algebra. We denote by $\mathcal{E}^0(B)$ the set of invariant elements $F \in \mathcal{L}(\mathcal{H}_B)$ satisfying the condition $1 - FF^*$, $1 - F^*F \in \mathcal{H}_B$, and by $\mathcal{D}^0(B)$ the set of invariant unitary elements in $\mathcal{L}(\mathcal{H}_B)$. We say that the elements F_1 and F_2 of $\mathcal{E}^0(B)$ are *equivalent* if there exist invariant unitary elements $u, v \in \mathcal{L}(\mathcal{H}_B)$ such that $uF_1v - F_2 \in \mathcal{H}_B$. We denote by $\overline{\mathcal{E}}^0(B)$ the set of equivalence classes in $\mathcal{E}^0(B)$, and by $\overline{\mathcal{D}}^0(B)$ the image of $\mathcal{D}^0(B)$ in $\overline{\mathcal{E}}^0(B)$. In $\overline{\mathcal{E}}^0(B)$ we introduce an addition, the direct sum.

LEMMA 2. $\overline{\mathcal{E}}^0(B)/\overline{\mathcal{D}}^0(B) \simeq K^0(B)$. *The element* $F^* \in \mathcal{E}^0(B)$ *is the inverse of* $F \in \mathcal{E}^0(B)$ *in the group* $K^0(B)$. *The sum* $F_1 \oplus F_2$ *is equal to* F_1F_2.

PROOF. In the algebra $\mathcal{L}(\mathcal{H}_B^{(0)} \oplus \mathcal{H}_B^{(1)})$, every element F of degree 1 has the form $\left(\begin{smallmatrix} 0 & F_2 \\ F_1 & 0 \end{smallmatrix}\right)$. The conditions $F^2 - 1 \in \mathcal{H}_B$ and $F^* - F \in \mathcal{H}_B$ imply that $F_2 - F_1^*$, $1 - F_1F_1^*$, $1 - F_1^*F_1 \in \mathcal{H}_B$. The correspondence $F \to F_1$ defines a mapping of $\mathcal{E}(C, B)$ into $\mathcal{E}^0(B)$. The inverse mapping is defined by

$$F_1 \to F = \begin{pmatrix} 0 & F_1^* \\ F_1 & 0 \end{pmatrix}.$$

It is not difficult to see that the unitary equivalence of the elements F and T in $\mathcal{E}(C, B)$ corresponds to the existence of invariant unitary elements $u, v \in \mathcal{L}(\mathcal{H}_B)$ satisfying the condition $uF_1v = T_1$. If F and $T \in \mathcal{E}(C, B)$ are operator homotopic, then by Lemma 1 there exists an invariant unitary element $u \in \mathcal{L}(\mathcal{H}_B)$ of degree 0 such that $T - uFu^{-1} \in \mathcal{H}_B$. Conversely, if the elements $F_1, T_1 \in \mathcal{E}^0(B)$ are connected by the relation $F_1 - T_1 \in \mathcal{H}_B$, then the elements $F, T \in \mathcal{E}(C, B)$ corresponding to them are operator homotopic: $tF + (1 - t)T$, $0 \leqslant t \leqslant 1$.

By Theorem 1 of §4 the inverse of $F = \left(\begin{smallmatrix} 0 & F_1^* \\ F_1 & 0 \end{smallmatrix}\right)$ is

$$\begin{pmatrix} 0 & -F_1 \\ -F_1^* & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}_B^{(1)} \oplus \mathcal{H}_B^{(0)})$$

(the grading of $\mathcal{H}_B$ is changed to the opposite). Therefore the element $(-F_1^*) = (-1) \cdot F_1^*$, and also F_1^* will be the inverse of $F_1 \in \mathcal{E}^0(B)$. The elements $F_1 \oplus F_2$ and $F_1F_2 \oplus 1 \in \mathcal{E}^0(B)$ are connected by the homotopy

$$\begin{pmatrix} F_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad 0 \leqslant t \leqslant \frac{\pi}{2},$$

and therefore the elements of $\mathcal{E}(C, B)$ corresponding to them are homotopic. ■

LEMMA 3. *Let the algebra B be unital. In the definition of $\mathfrak{G}^0(B)$, instead of finitely generated projective B-modules, it is possible to consider finitely generated projective Hilbert B-modules.*

PROOF. If $V \oplus W \simeq B^n$ and $p\colon B^n \to B^n$ is an invariant projection onto V, then, replacing p by

$$p_1 = [(2p^* - 1)(2p - 1) + 1]^{1/2} \cdot p \cdot [(2p^* - 1)(2p - 1) + 1]^{-1/2},$$

we obtain a Hermitian projection in $\mathcal{L}(B^n)$ with image isomorphic to V. The isomorphism $\operatorname{Im} p_1$ on V defines a Hilbert B-module structure on V. If $u\colon V_1 \to V_2$ is a (not isometric) isomorphism of two Hilbert B-modules, then $u_0 = u(u^*u)^{-1/2}$ is an isometric (i.e. inner product preserving) isomorphism. ■

LEMMA 4. *If p and q are invariant Hermitian projections in the algebra D, and $\|p - q\| < 1$, then there exists an invariant unitary element $u \in D$ such that $upu^{-1} = q$.*

PROOF. Set $v = (2q - 1)(2p - 1) + 1$. Then

$$\|2 - v\| = \|(2p-1)^2 - (2q-1)(2p-1)\| \leqslant 2\|p - q\| < 2.$$

Consequently, v is invertible. It is easy to verify that $v(2p - 1)v^{-1} = 2q - 1$, from which we obtain $vpv^{-1} = q$. It remains to set $u = v(v^*v)^{-1/2}$. ■

LEMMA 5. *Let the algebra B be unital. In order that the image of a Hermitian projection $p \in \mathcal{L}(\mathcal{H}_B)$ be a finitely generated projective B-module it is necessary and sufficient that $p \in \mathcal{K}_B$.*

PROOF. If $\operatorname{Im} p$ is finitely generated and projective, then there exists a B-module W such that $\operatorname{Im} p \oplus W \simeq B^n$. As can be seen from the proof of Lemma 3, we may assume that W is also a Hilbert B-module and the isomorphism preserves inner product. By the stabilization theorem (§1.12) we have $W \oplus \mathcal{H}_B \simeq \mathcal{H}_B$, and so in $\mathcal{H}_B$ there exists a Hermitian projection q with image isometrically isomorphic to W. We consider the projection $p \oplus q$ in $\mathcal{H}_B \oplus \mathcal{H}_B$. Its image is isometrically isomorphic to B^n. If $x_1, \ldots, x_n$ is an orthonormal basis in $\operatorname{Im}(p \oplus q)$, then $p \oplus q = \sum_{i=1}^n \theta_{x_i, x_i}$. Therefore $p \oplus q \in \mathcal{K}(\mathcal{H}_B \oplus \mathcal{H}_B)$, from which it follows that $p \in \mathcal{K}(\mathcal{H}_B)$.

Conversely, let $p \in \mathcal{K}(\mathcal{H}_B)$. Denote by $p_n \in \mathcal{K}(\mathcal{H}_B)$ the projection onto the first n coordinates in $\mathcal{H}_B$. There exists an n such that p_n is invariant and $\|(1 - p_n)p\| < 1/8$ (the latter follows from the easily verifiable relation $\lim_{n\to\infty}\|(1 - p_n)k\| = 0$ for every $k \in \mathcal{K}_B$); hence $\|p - p_n p p_n\| < 1/4$. We set $q = p_n \cdot f(p_n p p_n) \cdot p_n$, where $f(x)$ is the following real function:

$$f(x) = \begin{cases} 0, & x \leqslant \frac{1}{4}, \\ 2x - \frac{1}{2}, & \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ 1, & x \geqslant \frac{3}{4}. \end{cases}$$

Then q is a Hermitian vector and $\|p - q\| < 1/2$. By Lemma 4, p and q are unitarily equivalent, and therefore $\operatorname{Im} p \simeq \operatorname{Im} q$. On the other hand, q has the property that $p_n q p_n = q$, and therefore $\operatorname{Im} q$ is finitely generated and projective. ■

PROOF OF THEOREM 3. First we assume that B is unital. The homomorphism $\mathfrak{G}^0(B) \to K^0(B)$ is defined in the following way. Let V be a finitely generated projective Hilbert B-module. Then $\mathcal{H}_B \oplus V \simeq \mathcal{H}_B$, and so there exists an operator $F_V \in \mathcal{L}(\mathcal{H}_B \oplus V) \simeq \mathcal{L}(\mathcal{H}_B)$, isometrically mapping $\mathcal{H}_B \oplus \{0\}$ onto $\mathcal{H}_B \oplus V$ and equal to 0 on $\{0\} \oplus V$. It is clear that $F_V F_V^* = 1$ and $F_V^* F_V = 1 - p$, where $p \in \mathcal{K}(\mathcal{H}_B \oplus V)$ is the projection onto

V. The correspondence $[V] \to [F_V]$ is generated by the homomorphism $\mathfrak{G}^0(B) \to K^0(B)$, since $K^0(B)$ is a group.

Conversely, let $F \in \mathcal{E}^0(B)$. Then $FF^* = 1 - k$, $k \in \mathcal{K}(\mathcal{H}_B)$. There is an n such that $\|(1 - p_n)k(1 - p_n)\| < 1$ (where p_n is the invariant projection onto the first n coordinates). We set

$$h = (1 - p_n)\,k\,(1 - p_n), \quad l = (1 - p_n) + \sum_{r=1}^{\infty} h^r, \quad F_1 = \sqrt{l}\,F.$$

It is clear that $F_1 - F \in \mathcal{H}_B$. Besides, $\sqrt{l}\,(1 - p_n) = \sqrt{l}$, and therefore

$$F_1F_1^* = \sqrt{l}\,FF^*\sqrt{l} = \sqrt{l}\,(1 - p_n)\,FF^*\,(1 - p_n)\sqrt{l} = \sqrt{l}\,(1 - p_n - h)\sqrt{l} = 1 - p_n.$$

By the same token, $F_1F_1^*$ (and thus $F_1^*F_1$) is a Hermitian projection. The correspondence

$$[F] \to [\operatorname{Ker} F_1^*F_1] - [\operatorname{Ker} F_1F_1^*] = [\operatorname{Ker} F_1] - [\operatorname{Ker} F_1^*]$$

defines the homomorphism $K^0(B) \to \mathfrak{G}^0(B)$.

Unambiguity. We have to prove that if $F_1 - F_2 \in \mathcal{H}_B$, and F_1F and $F_2F_2^*$ are projections, then

$$[\operatorname{Ker} F_1] - [\operatorname{Ker} F_1^*] = [\operatorname{Ker} F_2] - [\operatorname{Ker} F_2^*].$$

Replacing F_1 by $\begin{pmatrix} F_1 & 0 \\ 0 & F_2^* \end{pmatrix}$, and F_2 by

$$\begin{pmatrix} F_2 & 1 - F_2F_2^* \\ 1 - F_2^*F_2 & F_2^* \end{pmatrix},$$

we may assume that $F_2 \in \mathcal{D}^0(B)$. Now replacing F_1 by $F_1F_2^*$, we can reduce everything to the case $F_1 - 1 \in \mathcal{H}_B$.

Hence, let $F - 1 \in \mathcal{H}_B$ and $FF^* = 1 - p$. Using the construction of Lemmas 4 and 5, it is easy to construct a sequence of integers $n_i \to \infty$ and invariant unitary elements $\{u_i\}$ such that

$$p_{n_i}(u_ipu_i^{-1})\,p_{n_i} = u_ipu_i^{-1} \quad \text{and} \quad \lim_{i\to\infty} \|u_i - 1\| = 0,$$

and all the p_{n_i} are invariant. Since $F - 1 \in \mathcal{H}_B$, we have

$$\|F^*(1-p_n)F - (1-p_n)\| \to 0$$

as $n \to \infty$, and therefore

$$a_i = \|u_iF^*u_i^{-1}(1 - p_{n_i})\,u_iFu_i^{-1} - (1 - p_{n_i})\| \to 0$$

as $i \to \infty$. Consequently there is an i such that $a_i < 1$. We replace F by $u_iFu_i^{-1}$ and denote n_i by n. We have

$$(1-p_n)\,FF^*(1-p_n) = 1-p_n, \quad \|F^*(1-p_n)F - (1-p_n)\| < 1.$$

Applying Lemma 4, we see that the operator $\Phi = (1 - p_n)F$ has "index" zero, i.e.

$$\operatorname{Im}(1-\Phi\Phi^*) = \operatorname{Im} p_n, \quad \operatorname{Im}(1-\Phi^*\Phi) \simeq \operatorname{Im} p_n.$$

On the other hand,

$$\operatorname{Im}(1-\Phi\Phi^*) = \operatorname{Im}(1-FF^*) \oplus \operatorname{Im}(p_n-p),$$

$$\operatorname{Im}(1-\Phi^*\Phi) = \operatorname{Im}(1-F^*F+F^*(p_n-p)F) = \operatorname{Im}(1-F^*F)$$

$$\oplus \operatorname{Im}(F^*(p_n-p)F).$$

(The last equality can be obtained from the orthogonality of the projections $(1 - F^*F)$ and $F^*(p_n - p)F$.) Since $\mathrm{Im}(p_n - p) \simeq \mathrm{Im}(F^*(p_n - p)F)$ (the operator F^* defines this isomorphism), it follows that $\mathrm{Im}(1 - F^*F)$ is stably isomorphic to $\mathrm{Im}(1 - FF^*)$. The unambiguity is proved.

It is easy to verify that the homomorphisms constructed above are inverses of each other. This gives the required isomorphism in the case where B is unital. In the general case it is sufficient to refer to Corollary 1 in §7. Such a reference is allowed, since none of the assertions of the present section except Theorems 1 and 2 and Lemma 1 are used in §7. ∎

COROLLARY 2. *Let B be a trivially graded algebra, let the operator $\varepsilon \in \mathcal{L}(\tilde{B}^n)$ define a grading of the algebra $\mathcal{L}(B^n)$, and let $F \in \mathcal{L}(B^n)$ be an invariant Hermitian operator of degree* 1 *satisfying the conditions* $\|F\| \leqslant 1$ *and* $F^2 - 1 \in \mathcal{K}(B^n)$. *The image of the element $(\varepsilon, F) \in K^0(B)$ in the group $K^0(\tilde{B})$ is equal to the difference of the two projective modules* $\mathrm{Im}((1 + \Phi)/2)$ *and* $\mathrm{Im}((1 - \varepsilon)/2)$, *where* $\Phi = (1 - 2F)\varepsilon + 2F\sqrt{1 - F^2}$.

PROOF. The operator ε can be considered as an element of $\mathcal{L}(B^n)$. We denote by $\tilde{E}_1$ and $\tilde{E}_2$ the Hilbert submodules $\mathrm{Im}((1 + \varepsilon)/2)$ and $\mathrm{Im}((1 - \varepsilon)/2)$ in $\tilde{B}^n$, and by E_1 and E_2 the analogous submodules in B^n. With respect to the decomposition $B^n = E_1 \oplus E_2$, the operator F has the form $\left(\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right)$. An operator $T\colon E_1 \to E_2$ can be considered (after stabilization: $T \oplus 1\colon E_1 \oplus \mathcal{K}_{\tilde{B}} \to E_2 \oplus \mathcal{K}_{\tilde{B}}$) as an element of $\mathcal{E}^0(\tilde{B})$. We consider another operator

$$S: E_1 \oplus E_2 \to E_2 : S(x, y) = T(x) + \sqrt{1 - TT^*}(y).$$

Since $1 - TT^* \in \mathcal{K}(E_2)$, in the group $K^0(\tilde{B})$ the difference $[S] - [T]$ coincides with the projective module $\tilde{E}_2 = \mathrm{Im}((1 - \varepsilon)/2)$. We also note that $SS^* = 1$. Therefore our assertion follows from the equality $1 - S^*S = \frac{1}{2}(1 + \Phi)$, which can be verified directly by using the relation $T \cdot \sqrt{1 - T^*T} = \sqrt{1 - TT^*} \cdot T$. This relation can be obtained by passing to the limit in $T \cdot p(T^*T) = p(TT^*) \cdot T$, where p is a polynomial. ∎

We now consider the homotopy definitions of the groups $K_iK(X, Y)$ for locally compact spaces X and Y, where the one-point compactifications X^+ and Y^+ are finite cell complexes. We restrict ourselves to the cases of the complex and real K-functor with the trivial action of G, neglecting the possible generalizations. In our considerations we shall use the fact that $K_*K(X, Y)$ is a generalized homology theory for X and a generalized cohomology theory for Y. This follows from Theorems 3 of §4, 7 of §5, and 2 and 3 of §7. (The reference to §7 is allowed, since the homotopy definitions used here are not used in §7.)

The classifying space $\tilde{\mathfrak{F}}_{p,q}$ for the K-functor $\widetilde{KU}^{p-q}(Z)$ or $\widetilde{KO}^{p-q}(Z)$, where Z is a compact set, is realized in the following way (see [13]). We consider a graded separable Hilbert space H and a fixed graded representation $\psi_{q,p}\colon C_{q+1,p} \to \mathcal{L}(H)$ having infinite multiplicity. By $\mathfrak{F}_{p,q}(H)$ we denote the space of operators $F \in \mathcal{L}(H)$ satisfying the following conditions:

$$F^* = F; \quad F^2 - 1 \in \mathcal{K}(H); \quad \deg F = 1;$$
$$F\psi_{q,p}(a) = -\psi_{q,p}(a)F \text{ for } a = \varepsilon_1, \ldots, \varepsilon_q, e_1, \ldots, e_p.$$

The topology on $\mathfrak{F}_{p,q}(H)$ is induced by the operator norm. By $\tilde{\mathfrak{F}}_{p,q}(H)$ we denote the connected component of the point $\psi_{q,p}(\varepsilon_{q+1}) \in \mathfrak{F}_{p,q}(H)$. For fixed $p - q$ all spaces

$\tilde{\mathfrak{F}}_{p,q}(H)$ are homeomorphic. The mapping $S\colon \tilde{\mathfrak{F}}_{p,q+1} \to \Omega\tilde{\mathfrak{F}}_{p,q}$ (where Ω denotes the loop space) is defined by the formulas

$$\psi_{q,p}(\varepsilon_i) = \psi_{q+1,p}(\varepsilon_{i+1}), \quad 1 \leqslant i \leqslant q+1,$$
$$\psi_{q,p}(e_i) = \psi_{q+1,p}(e_i), \quad 1 \leqslant i \leqslant p,$$
$$S(F)(t) = \psi_{q+1,p}(\varepsilon_1)\cos t + F\sin t, \quad 0 \leqslant t \leqslant \pi,$$
$$S(F)(t) = \psi_{q+1,p}(\varepsilon_1)\cos t + \psi_{q+1,p}(\varepsilon_{q+2})\sin t, \quad \pi \leqslant t \leqslant 2\pi.$$

As a result, we obtain the spectrum of the spaces $\tilde{\mathfrak{F}}_{p-q} = \tilde{\mathfrak{F}}_{p,q}(H)$. It is easy to verify that for a compact space Z the suspension isomorphism $\tilde{K}^n(Z) \simeq \tilde{K}^{n+1}(SZ)$, induced by the above mapping $S^1 \wedge \tilde{\mathfrak{F}}_n \to \tilde{\mathfrak{F}}_{n+1}$, coincides with the Bott periodicity $\tilde{K}^1(S^1) \otimes \tilde{K}^n(Z) \underset{\sim}{\to} \tilde{K}^{n+1}(SZ)$.

If P and Q are spaces with distinguished points, then by $Q(P)$ we denote the space of continuous mappings $P \to Q$ preserving the distinguished point ($Q(P)$ is equipped with the compact-open topology). For any space R with a distinguished point the natural mapping $R \times Q(P) \to (R \times Q)(P)$, converting (r, f) into $f_r\colon p \to (r, f(p))$, can be factored to the mapping $R \wedge Q(P) \to (R \wedge Q)(P)$.

DEFINITION 4.

$$H_i K(X, Y) = \varinjlim_n \pi_{n+i}(X^+ \wedge \tilde{\mathfrak{F}}_n(Y^+)),$$

$$KH^{-i}(X, Y) = \varinjlim_n \pi_{n+i}((X^+ \wedge \tilde{\mathfrak{F}}_n)(Y^+)) = \varinjlim_n [S^{n+i}Y^+, X^+ \wedge \tilde{\mathfrak{F}}_n].$$

According to Theorems 5.2 and 5.10 of [20], on the category of finite cell complexes $H_iK(X, Y)$ is a generalized homology theory with respect to X (for fixed Y) and $KH^{-i}(X, Y)$ is a generalized cohomology theory with respect to Y (for fixed X).

THEOREM 4. *If X^+ and Y^+ are finite cell complexes, then*

$$H_iK(X, Y) \simeq KH^{-i}(X, Y) \simeq K_iK(X, Y).$$

PROOF. We denote by $\alpha_i(Y) \in K_iK(\mathbf{R}^i \times Y, Y) = \tilde{K}_i\tilde{K}(S^i \wedge Y^+, Y^+)$ the image of the canonical generator $\alpha_i \in K_i(\mathbf{R}^i)$ under the homomorphism $\tau_{C(Y)}$. The homomorphism $t\colon KH^{-i}(X, Y) \to KK^{-i}(X, Y)$ is defined in the following way. For any mapping $f\colon S^{n+i}Y^+ \to X^+ \wedge \tilde{\mathfrak{F}}_n$ there exist a compact set Z and a mapping $g\colon Z \to \tilde{\mathfrak{F}}_n$ such that f splits into the composition

$$S^{n+i}Y^+ \to X^+ \wedge Z \xrightarrow{\mathrm{id}\wedge g} X^+ \wedge \tilde{\mathfrak{F}}_n.$$

We identify $\tilde{K}_{n+i}\tilde{K}(X^+ \wedge Z, Y^+)$ with

$$\operatorname{Ker}[\tilde{K}_{n+i}\tilde{K}(X^+ \times Z, Y^+) \to \tilde{K}_{n+i}\tilde{K}(X^+ \vee Z, Y^+)].$$

The image of $\alpha_{n+i}(Y)$ under the mapping

$$\tilde{K}_{n+i}\tilde{K}(S^{n+i}Y^+, Y^+) \to \tilde{K}_{n+i}\tilde{K}(X^+ \wedge Z, Y^+)$$

will be denoted by $[f]$, and the element of $\tilde{K}^n(Z)$ corresponding to g by $[g]$. We set

$$t(f) = [g] \otimes_{C(Z)} [f] \in \tilde{K}_i\tilde{K}(X^+, Y^+) = K_iK(X, Y).$$

The unambiguity can be verified easily.

For compact X and Y, t is an isomorphism. Indeed, if X and Y are one-point spaces, then t is the identity isomorphism. If Y is a point, then $KH^{-i}(X, Y)$ is a generalized homology theory for X. Therefore, in this case t is an isomorphism of homology theories:

$$KH^{-i}(X, \text{point}) \to KK^{-i}(X, \text{point}).$$

Now, considering $KH^{-i}(X, Y)$ as a cohomology theory for Y (for fixed X), we obtain that t is an isomorphism of cohomology theories. In the case where X and Y are compact, it is sufficient to pass to the corresponding groups (from X^+ and Y^+).

Now we note that a mapping $X^+ \wedge \tilde{\mathfrak{F}}_n(Y^+) \to (X^+ \wedge \tilde{\mathfrak{F}}_n)(Y^+)$ induces a homomorphism w: $H_iK(X, Y) \to KH^{-i}(X, Y)$, which is an isomorphism if at least one of the spaces X or Y is a singleton. Since $H_iK(X, Y)$ is a generalized homology theory for X, the homomorphism $t \cdot w$, and consequently w, too, is an isomorphism. ■

§7. Extensions

In this section we consider the group $\mathrm{Ext}(A, B)$ constructed from extensions of the form

$$0 \to \mathcal{K} \otimes B \to D \to A \to 0, \tag{1}$$

where all the algebras A, B, D and $\mathcal{K}$ are *graded trivially*. We establish the isomorphism $\mathrm{Ext}(A, B) \simeq KK^1(A, B)$ and also obtain exact sequences for the K-functor (homological and cohomological). At the end of the section we shall consider extensions of graded algebras. In what follows we assume that the algebra A is separable and B has a countable approximate identity.

We start with the following remarks (see [10]). The extension (1) unambiguously defines a homomorphism $D \to \mathfrak{M}(\mathcal{K} \otimes B)$, which gives, upon factorization with respect to $\mathcal{K} \otimes B$,

$$\varphi: A \to \mathfrak{M}(\mathcal{K} \otimes B)/\mathcal{K} \otimes B = \mathcal{O}(\mathcal{K} \otimes B).$$

We assume that the group G acts continuously on A, B, and D. This imposes the following condition on φ:

$$\left\{\begin{array}{c} \text{If } a \in A,\, b \in \mathfrak{M}(\mathcal{K} \otimes B) \text{ and } \varphi(a) = b \bmod \mathcal{K} \otimes B, \\ \text{then the mapping } G \to \mathfrak{M}(\mathcal{K} \otimes B)\text{: } g \to g(b) \text{ is norm continuous.} \end{array}\right\} \tag{2}$$

Conversely, if φ: $A \to \mathcal{O}(\mathcal{K} \otimes B)$ satisfies (2), then, setting $D = \mathfrak{M}(\mathcal{K} \otimes B) \oplus_{\mathcal{O}(\mathcal{K} \otimes B)} A$ (see §1.18), we obtain the extension (1), and G acts on D continuously. Under this correspondence the subset of decomposable extensions (i.e. extensions admitting a section homomorphism $A \to D$) corresponds to the set of homomorphisms φ admitting a lifting $A \to \mathfrak{M}(\mathcal{K} \otimes B)$. We shall often identify an extension with the homomorphism φ corresponding to it.

DEFINITION 1. For fixed A and B we denote the set of extensions of type (1) by $\mathfrak{E}xt(A, B)$ and the subset of decomposable extensions by $\mathfrak{D}xt(A, B)$. Two elements φ_1 and φ_2 of $\mathfrak{E}xt(A, B)$ are said to be *unitarily equivalent* if there exists an invariant unitary operator $u \in \mathfrak{M}(\mathcal{K} \otimes B)$ such that $\varphi_2(a) = u\varphi_1(a)u^{-1}$ for every $a \in A$. Let $\overline{\mathfrak{E}xt}(A, B)$ be the set of classes of unitarily equivalent extensions and let $\overline{\mathfrak{D}xt}(A, B)$ be the image of $\mathfrak{D}xt(A, B)$ in $\overline{\mathfrak{E}xt}(A, B)$. The addition $\varphi_1 \oplus \varphi_2$ in $\overline{\mathfrak{E}xt}(A, B)$ is defined as the direct sum

($\mathcal{K}(\mathcal{H}_B \oplus \mathcal{H}_B)$ is identified with $\mathcal{K}(\mathcal{H}_B)$ by means of an invariant isometry of $\mathcal{H}_B \oplus \mathcal{H}_B$ onto $\mathcal{H}_B$). We set

$$\operatorname{Ext}(A, B) = \overline{\mathcal{E}xt}(A, B) / \overline{\mathcal{D}xt}(A, B).$$

We note that the direct sum of two extensions $0 \to \mathcal{K} \otimes B \to D_i \xrightarrow{p_i} A \to 0$ $(i = 1, 2)$ represents an extension

$$0 \to M_2 \otimes (\mathcal{K} \otimes B) \to D_\oplus \xrightarrow{p} A \to 0,$$

where

$$D_\oplus = \left\{ \begin{pmatrix} x_1 & b_1 \\ b_2 & x_2 \end{pmatrix} \middle| x_i \in D_i,\ b_i \in \mathcal{K} \otimes B;\ p_1(x_1) = p_2(x_2) \right\},$$

$$p \begin{pmatrix} x_1 & b_1 \\ b_2 & x_2 \end{pmatrix} = p_i(x_i).$$

Definition 2. We say that an extension $\varphi \in \mathcal{E}xt(A, B)$ is *absorbing* if for every $\psi \in \mathcal{D}xt(A, B)$ the elements $\varphi \oplus \psi$ and φ are unitarily equivalent. We denote by $\mathcal{E}xt_a(A, B)$ the set of absorbing extensions and by $\operatorname{Ext}_a(A, B)$ the set of classes of unitarily equivalent absorbing extensions. Addition on $\operatorname{Ext}_a(A, B)$ is defined as the direct sum. (We note that an absorbing extension φ cannot be unital, since φ and $\varphi \oplus \psi$ are not unitarily equivalent.)

Definition 3. Let A and B be trivially graded algebras, and assume that the algebra $\mathcal{K}$ is also graded trivially. We denote by $\mathcal{E}^1(A, B)$ the set of pairs (φ, P), where φ: $A \to \mathcal{M}(\mathcal{K} \otimes B)$ is a homomorphism, $P \in \mathcal{M}(\mathcal{K} \otimes B)$ is an invariant element, and for every $a \in A$ the elements

$$\varphi(a)P - P\varphi(a), \qquad (P^2 - P)\varphi(a), \qquad (P^* - P)\varphi(a) \tag{3}$$

belong to $\mathcal{K} \otimes B$. Let $\mathcal{D}^1(A, B)$ be the set of (degenerate) pairs for which all elements (3) are equal to 0. Two pairs (φ_1, P_1) and (φ_2, P_2) are *unitarily equivalent* if there exists an invariant unitary element $u \in \mathcal{M}(\mathcal{K} \otimes B)$ such that $\varphi_2(a) = u\varphi_1(a)u^{-1}$ and $P_2 = uP_1u^{-1}$ for every $a \in A$. Two pairs (φ_1, P_1) and (φ_2, P_2) are said to be *homological* if $P_1\varphi_1(a) - P_2\varphi_2(a) \in \mathcal{K} \otimes B$ for every $a \in A$. Let $\overline{\mathcal{E}}^1(A, B)$ be the set of equivalence classes of $\mathcal{E}^1(A, B)$ modulo unitary equivalence and homology, and let $\overline{\mathcal{D}}^1(A, B)$ be the image of $\mathcal{D}^1(A, B)$ in $\overline{\mathcal{E}}^1(A, B)$. The addition of equivalence classes is defined to be the direct sum. We set

$$E^1(A, B) = \overline{\mathcal{E}}^1(A, B) / \overline{\mathcal{D}}^1(A, B).$$

Lemma 1. 1) *If at least one of the two algebras A and B is nuclear, then* $\operatorname{Ext}_a(A, B) \simeq \operatorname{Ext}(A, B)$.

2) *If A is nuclear, then the mapping* $(\varphi, P) \to P \cdot \varphi \bmod \mathcal{K} \otimes B$ *defines an isomorphism* γ: $E^1(A, B) \simeq \operatorname{Ext}(A, B)$.

Proof. 1) We fix a regular G-imbedding $A \hookrightarrow \mathcal{M}(\mathcal{K})$ such that $A \cap \mathcal{K} = 0$ (see §1.16). We denote by π the composition $A \hookrightarrow \mathcal{M}(\mathcal{K}) \hookrightarrow \mathcal{M}(\mathcal{K} \otimes B) \to \mathcal{O}(\mathcal{K} \otimes B)$. According to the generalized theorem of Voiculescu (§1.16), the extension π is absorbing. If $\varphi \in \mathcal{E}xt(A, B)$, we obviously have $\varphi \oplus \pi \in \mathcal{E}xt_a(A, B)$. Therefore the imbedding $\mathcal{E}xt_a(A, B) \subset \mathcal{E}xt(A, B)$ defines the desired isomorphism of $\operatorname{Ext}_a(A, B)$ onto $\operatorname{Ext}(A, B)$.

2) *The epimorphism property of* γ. Let $\varphi \in \mathcal{E}xt(A, B)$. We extend φ to a unital $\tilde{\varphi}$: $\tilde{A} \to \mathcal{O}(\mathcal{H} \otimes B)$. Since the algebra $\tilde{A}$ is nuclear, there exists a completely positive unital lifting χ: $\tilde{A} \to \mathfrak{M}(\mathcal{H} \otimes B)$ (see [11], Theorem 3.10, or [4], Theorem 7). From condition (2) it follows that for every $a \in \tilde{A}$ the function $g \to g^{-1}\chi(ga)$ is norm continuous on G. Besides, for every $g \in G$

$$g^{-1}\chi(ga)-\chi(a)\in\mathcal{K}\otimes B,\ \overline{\chi(\bar{a})}-\chi(a)\in\mathcal{K}\otimes B.$$

We set

$$\chi_1(a)=\int_G g^{-1}\chi(ga)\,dg,\quad \xi(a)=\frac{1}{2}(\chi_1(a)+\overline{\chi_1(\bar{a})}).$$

The mapping ξ: $A \to \mathfrak{M}(\mathcal{H} \otimes B)$ is unital, equivariant, "real", and is a lifting of φ. Applying the generalized theorem of Stinespring (see §1.15), we obtain a homomorphism ψ: $\tilde{A} \to \mathfrak{M}(M_2 \otimes \mathcal{H} \otimes B)$ such that $\tilde{\varphi}(a) \oplus 0 = P\psi(a)P$, for every $a \in \tilde{A}$, where $P = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$.

It is easy to verify that $\psi(a)P - P\psi(a) \in M_2 \otimes \mathcal{H} \otimes B$ for every $a \in \tilde{A}$. Indeed,

$$(P\psi(a^*)P)(P\psi(a)P)=P\psi(a^*a)P \mod M_2\otimes\mathcal{K}\otimes B.$$

Therefore,

$$[(1-P)\psi(a)P]^*[(1-P)\psi(a)P]\in M_2\otimes\mathcal{K}\otimes B,$$

and so $\psi(a)P - P\psi(a)P \in M_2 \otimes \mathcal{H} \otimes B$. Substituting a^* for a, passing to adjoints, and subtracting, we obtain the required result. By the same token,

$$\varphi\oplus 0=P\cdot\psi \mod M_2\otimes\mathcal{K}\otimes B,$$

where $(\psi, P) \in \mathcal{E}^1(A, B)$.

That γ is a monomorphism is obvious. ■

LEMMA 2. $E^1(A, B) \simeq KK^1(A, B)$.

PROOF. First we reformulate the definition of $\mathcal{E}(A, B \hat{\otimes} C_{1,0})$. We temporarily denote the algebra $\mathcal{H}_B$ graded by the grading operator $\varepsilon \in \mathcal{L}(\mathcal{H}_B)$ by $\mathcal{H}_B^{gr}$. The notation $\mathcal{H}_B$ is preserved for the algebra $\mathcal{H}_B$ with the trivial grading. We identify $\mathcal{H}_B^{gr} \hat{\otimes} C_{1,0}$ with $\mathcal{H}_B \otimes C_{1,0}$ by the formula

$$x\hat{\otimes} y \to (x\varepsilon^{\deg y})\otimes(y\varepsilon_1^{\deg x}).$$

This is a graded isomorphism. The space $\mathcal{H}_B$ will be assumed to be graded by the operator ε. By η we denote the corresponding grading of the space $\mathcal{H}_{B\hat{\otimes}C_{1,0}} = \mathcal{H}_B \hat{\otimes} C_{1,0}$. We note that any element of degree 0 in $\mathfrak{M}(\mathcal{H}_B \otimes C_{1,0})$ has the form $x \otimes 1$, and any element of degree 1 has the form $y \otimes \varepsilon_1$, where $x, y \in \mathfrak{M}(\mathcal{H}_B)$. Therefore, if

$$\left(\eta, \varphi: A \to \mathfrak{M}(\mathcal{H}_B^{gr} \hat{\otimes} C_{1,0}) = \mathfrak{M}(\mathcal{H}_B \otimes C_{1,0}), F\right) \in \mathcal{E}(A, B \hat{\otimes} C_{1,0}),$$

then $\varphi = \varphi' \otimes 1$, $F = F' \otimes \varepsilon_1$. It is easy to see that

$$\left(\varphi', \frac{F'+1}{2}\right) \in \mathcal{E}^1(A, B).$$

The mapping $(\varphi, P) \to (\eta, \varphi \otimes 1, (2P - 1) \otimes \varepsilon_1)$ identifies $\mathcal{E}^1(A, B)$ with $\mathcal{E}(A, B \hat{\otimes} C_{1,0})$.

If the pairs (φ, P) and (ψ, Q) are homological, then the pairs $(\varphi, P) \oplus (\psi, 0)$ and $(\varphi, 0) \oplus (\psi, Q)$ are operator homotopic. For $0 \leqslant t \leqslant \infty$ homotopy is defined by the formula

$$\left(\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix} P & tPQ \\ tQP & t^2Q \end{pmatrix}\right).$$

Consequently, the corresponding elements of $\mathcal{E}(A, B \hat{\otimes} C_{1,0})$ are also operator homotopic. Conversely, if (η, φ, F_t), $0 \leqslant t \leqslant 1$, is an operator homotopy in $\mathcal{E}(A, B \hat{\otimes} C_{1,0})$, then Lemma 1 in §6 gives an invariant unitary element $u_1 \in \mathfrak{M}(\mathcal{K}_B \otimes C_{1,0})$ of degree 0 such that

$$u_1\varphi(a)u_1^{-1} - \varphi(a) \in \mathcal{K}_B \otimes C_{1,0}, \qquad (F_1 - u_1F_0u_1^{-1}) \cdot \varphi(a) \in \mathcal{K}_B \otimes C_{1,0}$$

for every $a \in A$. Writing u_1 in the form $u \otimes 1$, F_t in the form $F_t' \otimes \varepsilon_1$, and φ in the form $\varphi' \otimes 1$, we obtain that

$$u\varphi'(a)u^{-1} - \varphi'(a) \in \mathcal{K}_B, \qquad (F_1' - uF_0'u^{-1})\varphi'(a) \in \mathcal{K}_B$$

for every $a \in A$. Therefore, the pairs $(\varphi, (F_1' + 1)/2)$ and $(u\varphi'u^{-1}, u((F_0' + 1)/2)u^{-1})$ are homologous. An application of Theorem 1 of §6 shows that our mapping establishes the desired isomorphism. ∎

Combining Lemmas 1 and 2, we obtain the following theorem.

THEOREM 1. *If the algebra A is nuclear, then*

$$\mathrm{Ext}_a(A, B) \simeq \mathrm{Ext}(A, B) \simeq E^1(A, B) \simeq KK^1(A, B). \blacksquare$$

REMARK 1. In the case $B = C$, by Voiculescu's theorem to say that an extension is absorbing is to say that the homomorphism $\tilde{\varphi}\colon \tilde{A} \to \mathcal{O}(K) = \mathfrak{L}(\mathcal{K})/\mathcal{K}$ is a monomorphism. Our equivalence relation on $\mathcal{E}xt(A, B)$ corresponds to the strong equivalence of [9]. However, since even for a unital algebra A we consider all extensions, and not only unital ones, we obtain the same group $\mathrm{Ext}(A, B)$ as in the case of weak equivalence (see [9], 3.1). We note that in the case $B \neq C$ the absorbing property of an extension is stronger than the monomorphic property. In particular, for $B = C(Y)$, where Y is locally compact, it is easy to verify that for absorbing extensions the following strong monomorphism condition must be satisfied: $\omega_y(\tilde{\varphi}(\tilde{A})) \simeq \tilde{A}$ for every $y \in Y$, where $\omega_y\colon \mathcal{O}(\mathcal{K}_{C(Y)}) \to \mathcal{O}(\mathcal{K})$ is the restriction over the point $y \in Y$. It would be useful to find an intrinsic description of absorbing extensions.

REMARK 2. Since there exists an imbedding $\mathcal{O}(B_1) \hookrightarrow \mathcal{O}(\mathcal{K} \otimes B_1)$: $x \to p_1 \otimes x$ (where p_1 is an orthogonal projection in $\mathfrak{L}(\mathcal{K})$ onto a homogeneous subspace all elements of which are invariant), every extension $0 \to B_1 \to D \to A \to 0$ defines an element

$$(\varphi_1\colon A \to \mathcal{O}(B_1) \hookrightarrow \mathcal{O}(\mathcal{K} \otimes B_1)) \in \mathcal{E}xt(A, B_1).$$

If we apply this construction to the extension (1) for $B_1 = \mathcal{K} \otimes B$, we obtain the element of $KK^1(A, \mathcal{K} \otimes B)$ which corresponds to the element of $KK^1(A, B)$ determined earlier under the isomorphism of Theorem 1 of §5.

LEMMA 3. *Let the algebra A be nuclear and the ideal $J \subset B$ have a countable approximate identity. The sequence of groups*

$$KK^1(A, J) \xrightarrow{i_*} KK^1(A, B) \xrightarrow{q_*} KK^1(A, B/J)$$

is exact. (Here $i\colon J \to B$ is an imbedding, and $q\colon B \to B/J$ is the projection.)

PROOF. Clearly $q_* i_* = 0$. We show that $\operatorname{Ker} q_* \subset \operatorname{Im} i_*$. We use Theorem 1. Let $\varphi \in \mathcal{E}xt(A, B)$ and $q_*(\varphi) = 0$ in the group $\operatorname{Ext}(A, B/J)$, i.e., after adding some $\psi \in \mathcal{D}xt(A, B/J)$ the composition $q_* \cdot \varphi \colon A \to \mathcal{O}(\mathcal{K} \otimes B) \to \mathcal{O}(\mathcal{K} \otimes B/J)$ becomes an element of $\mathcal{D}xt(A, B/J)$. We fix a regular G-imbedding $A \hookrightarrow \mathfrak{M}(\mathcal{K})$ such that $A \cap \mathcal{K} = 0$ (see §1.16). We denote by π the composition $A \hookrightarrow \mathfrak{M}(\mathcal{K}) \hookrightarrow \mathfrak{M}(\mathcal{K} \otimes B) \to \mathcal{O}(\mathcal{K} \otimes B)$. According to the generalized theorem of Voiculescu (§1.16), $\psi \oplus q_*(\pi)$ is unitarily equivalent to $q_*(\pi)$, and therefore the element $q_*(\varphi) \oplus \psi \oplus q_*(\pi) \in \mathcal{D}xt(A, B/J)$ is unitarily equivalent to $q_*(\varphi \oplus \pi)$. We denote by χ a lifting of $q_*(\varphi \oplus \pi)$ to a homomorphism $A \to \mathfrak{M}(\mathcal{K} \otimes B/J)$ and for convenience we change the notation of the element $\varphi \oplus \pi$ itself to φ. (Clearly $[\varphi] = [\varphi \oplus \pi]$ in $\operatorname{Ext}(A, B)$.)

The pair (φ, χ) of homomorphisms defines the homomorphism

$$\eta \colon A \to E = \mathcal{O}(\mathcal{K} \otimes B) \oplus_{\mathcal{O}(\mathcal{K} \otimes B/J)} \mathfrak{M}(\mathcal{K} \otimes B/J).$$

It is easy to verify that $E \simeq \mathfrak{M}(\mathcal{K} \otimes B)/\mathcal{K} \otimes J$ (the homomorphism $\mathfrak{M}(\mathcal{K} \otimes B) \to E \colon x \to (x \bmod \mathcal{K} \otimes B, q_*(x))$ is an epimorphism, and its kernel is equal to $\mathcal{K} \otimes J$). The composition of η with the restriction $E \simeq \mathfrak{M}(\mathcal{K} \otimes B)/\mathcal{K} \otimes J \to \mathcal{O}(\mathcal{K} \otimes J)$ gives an element $[\tilde{\eta}]$ of the group $\operatorname{Ext}(A, J) \simeq KK^1(A, J)$, and it is well known that the composition $A \xrightarrow{\eta} \mathfrak{M}(\mathcal{K} \otimes B)/\mathcal{K} \otimes J \to \mathcal{O}(\mathcal{K} \otimes B)$ coincides with φ. It remains to verify that in this case $i_*[\tilde{\eta}] = [\varphi]$ in the group $KK^1(A, B)$.

The construction given in the proof of Lemma 1 enables us to construct a homomorphism $\xi \colon A \to \mathfrak{M}(M_2 \otimes \mathcal{K} \otimes B)$ and an invariant Hermitian projection $P \in \mathfrak{M}(M_2 \otimes \mathcal{K} \otimes B)$, such that

$$\xi(a)P - P\xi(a) \in M_2 \otimes \mathcal{K} \otimes J \quad \text{and} \quad \eta(a) \oplus 0 = P\xi(a) \mod M_2 \otimes \mathcal{K} \otimes J$$

for every $a \in A$. If $(\tilde{\xi}, \tilde{P}) \in \mathcal{E}^1(A, J)$ is the element obtained from (ξ, P) by the restriction $\mathfrak{M}(M_2 \otimes \mathcal{K} \otimes B) \to \mathfrak{M}(M_2 \otimes \mathcal{K} \otimes J)$, then in the group $E^1(A, B)$ we have $i_*(\tilde{\xi}, \tilde{P}) = (\xi, P)$ in view of part 1 of Theorem 2 of §4. (Here the isomorphism of Lemma 2 is used.) ■

REMARK 3. If, according to Remark 2, to the extension $0 \to B \to D \to A \to 0$ there corresponds the element $\alpha \in KK^1(A, B)$, then to the extension $0 \to B \oplus 0 \to D \oplus_A A_1 \to A_1 \to 0$ induced by the homomorphism $f \colon A_1 \to A$ there obviously corresponds the element $f^*(\alpha) \in KK^1(A_1, B)$. Moreover, if the diagram of extensions

$$\begin{array}{ccccccccc} 0 & \to & B_1 & \to & D_1 & \to & A & \to & 0 \\ & & {\scriptstyle g}\downarrow & & \downarrow & & \| & & \\ 0 & \to & B_2 & \to & D_2 & \to & A & \to & 0 \end{array}$$

is commutative, and either g is an epimorphism or B_1 is an ideal in B_2 and g is an imbedding, then under the homomorphism $g_* \colon KK^1(A, B_1) \to KK^1(A, B_2)$ the element corresponding to the first extension turns into the element corresponding to the second extension. For an epimorphism the assertion is obvious, and for an imbedding it is contained in the proof of Lemma 3.

We now establish some known facts concerning homotopic functors on the category of C^*-algebras.

DEFINITION 4. We say that the (covariant or contravariant) functor L from the category of C^*-algebras (or from a subcategory of them) into the category of Abelian groups is a *homotopic functor* if $L(f_0) = L(f_1)$ for homotopic homomorphisms f_0 and f_1. (In what follows the homomorphism $L(f)$ will be denoted by f_* or f^*.) We say that L

satisfies the *axiom of exactness* if for any algebra D and ideal J there exists a short exact sequence $L(J) \xrightarrow{i_*} L(D) \xrightarrow{q_*} L(D/J)$ in the case of a covariant L and $L(D/J) \xrightarrow{q^*} L(D) \xrightarrow{i^*} L(J)$ in the case of a contravariant L, where i: $L \hookrightarrow D$ is an imbedding and q: $D \to D/J$ is the projection. The algebra D is said to be *contractible* if the identity homomorphism is homotopic to zero.

LEMMA 4. *Consider the homotopy functor L, satisfying the axiom of exactness defined on either the category of nuclear separable C^*-algebras or the category of C^*-algebras with countable approximate identity. (In fact, any category of C^*-algebras containing all algebras occurring in the proof could be considered.) Let J be an ideal in the algebra D.*

1) *If D/J is contractible, then an imbedding i: $J \hookrightarrow D$ induces an isomorphism $L(J) \simeq L(D)$.*

2) *An imbedding*

$$J = J[0] \hookrightarrow S(D, D/J)$$

induces an isomorphism $L(J) \simeq L(S(D, D/J))$. (For the definition of $S(D, D/J)$, see §1.18.)

PROOF. Since the quotient algebra $S(D, D/J)/J[0] \simeq (D/J)[0, 1)$ is contractible, 2) follows from 1). In the proof of 1), for definiteness we shall assume that L is a covariant functor. It is obvious that $L(E) = 0$ for a contractible algebra E, and therefore

$$L(D/J) = L((D/J)(0, 1)) = L(J(0, 1]) = 0.$$

From the exact sequences

$$L(J) \to L(D) \to L(D/J) = 0,$$
$$0 = L((D/J)(0, 1)) \to L(S(D, D/J)) \to L(D),$$
$$0 = L(J(0, 1]) \to L(Z_1(J, D)) \to L(S(D, D/J))$$

it follows that i_*: $L(J) \to L(D)$ is an epimorphism and $L(Z_1(J, D)) \to L(S(D, D/J)) \to L(D)$ is a monomorphism. However, the algebra $Z_1(J, D)$ is homotopically equivalent to J, and the composition $J \hookrightarrow J[0, 1] \hookrightarrow Z_1(J, D) \to S(D, D/J) \to D$ coincides with i. Therefore i_* is also a monomorphism. ∎

LEMMA 5. *Consider a covariant functor L satisfying the conditions of Lemma 4. Let J be an ideal in the algebra D, i: $J \hookrightarrow D$ an imbedding, and q: $D \to D/J$ the projection. We have the following exact sequence infinite to the left:*

$$\ldots \xrightarrow{i_*} L(D(0, 1)) \xrightarrow{q_*} L((D/J)(0, 1)) \xrightarrow{\delta} L(J) \xrightarrow{i_*} L(D) \xrightarrow{q_*} L(D/J),$$

where δ is the composition $L((D/J)(0, 1)) \to L(S(D, D/J)) \simeq L(J)$. For a contravariant functor L an analogous (dual) assertion holds.

PROOF. It is sufficient to verify the exactness at the first three terms. The exactness at $L(D)$ holds by assumption, and the exactness at $L(J)$ can be obtained from that of the sequence

$$L((D/J)(0, 1)) \to L(S(D, D/J)) \to L(D).$$

Now we consider the algebra

$$T(D, D/J) = D(-1, 0] \oplus_{D/J} (D/J)[0, 1).$$

We note that $L(T(D, D/J)) \simeq L((D/J)(0, 1))$ since the quotient algebra $T(D, D/J)/(D/J)(0, 1) \simeq D(-1, 0]$ is contractible. Besides, the imbedding $D(-1, 0) \hookrightarrow T(D, D/J)$ is homotopic to the quotient mapping $D(0, 1) \to (D/J)(0, 1) \hookrightarrow T(D, D/J)$. Therefore the exactness at $L((D/J)(0, 1))$ follows from the exactness of the sequence

$$L(D(-1, 0)) \to L(T(D, D/J)) \to L(S(D, D/J)). \blacksquare$$

LEMMA 6. *Let the algebra A be nuclear, and let $\alpha_D \in KK^1(A, B)$ be the element corresponding to the extension $0 \to B \to D \to A \to 0$ according to Remark 2. The homomorphism*

$$\begin{aligned}\delta: KK^0(A(0, 1), A(0, 1)) &\to KK^0(A(0, 1), S(D, A)) \\ &\simeq KK^0(A(0, 1), B) \simeq KK^1(A, B)\end{aligned}$$

converts the generator $\tau_{A(0,1)}(c_1)$ into α_D.

PROOF. First we prove our assertion for the extension $0 \to C(0, 1) \to C(0, 1] \to C \to 0$. The element from $\operatorname{Ext}(C, C(0, 1))$ corresponding to it is defined by the homomorphism $\varphi\colon C \to \mathcal{Q}(C(0, 1))$, under which $1 \in C$ turns into the element $t \bmod C(0, 1) \in C_b(0, 1)/C(0, 1)$, where $C_b(0, 1)$ is the algebra of bounded continuous functions on $(0, 1)$ and t is the coordinate function on $(0, 1)$. We set $P = t \in C_b(0, 1)$ and $F = 2P - 1$. Under an appropriate homeomorphism of $(0, 1)$ onto $\mathbf{R}^1$ the function $2t - 1$ turns into $x \cdot (1 + \|x\|^2)^{-1/2} \in C_b(\mathbf{R}^1)$. Therefore the isomorphism

$$E^1(C, C(0, 1)) \simeq KK^1(C, C(0, 1))$$

converts the element (id: $C \to C_b(0, 1), P$) into the generator $K^1(\mathbf{R}^1)$ (see §5).

On the other hand, δ coincides with the Bott periodicity, which proves the assertion in the special case under consideration.

Using now the homomorphism τ_A, we obtain our assertion for the extension $0 \to A(0, 1) \to A(0, 1] \to A \to 0$.

Applying the second part of Remark 3 to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A(0,1) & \to & A(0,1] & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & S(D,A) & \to & Z_0(D,A) & \to & A & \to & 0 \\ & & \uparrow & & \uparrow f & & \| & & \\ 0 & \to & B & \to & D & \to & A & \to & 0 \end{array}$$

where f is the composition $D \hookrightarrow D[0, 1] \to Z_0(D, A)$, we obtain the assertion in the general case. $\blacksquare$

THEOREM 2. *Let the algebras A and B be trivially graded, let A be nuclear, and let J be an ideal with a strictly positive element in B. Denote the imbedding $J \subset B$ by i and the projection $B \to B/J$ by q. We have the following doubly infinite (cohomological) exact sequence*

$$\dots \xrightarrow{i_*} KK^n(A, B) \xrightarrow{q_*} KK^n(A, B/J) \xrightarrow{\delta} KK^{n+1}(A, J) \xrightarrow{i_*} KK^{n+1}(A, B) \xrightarrow{q_*} \dots,$$

where the coboundary homomorphism δ is defined as the composition

$$KK^n(A, B/J) \simeq KK^{n+1}(A, (B/J)(0, 1)) \to KK^{n+1}(A, S(B, B/J)) \simeq KK^{n+1}(A, J).$$

If the algebra B/J is also nuclear and separable, then $\delta(x) = x \otimes_{B/J} [\varphi]$, where $[\varphi] \in KK^1(B/J, J)$ is the element corresponding to the extension $0 \to J \to B \to B/J \to 0$.

PROOF. In view of Theorem 3 of §4 and Lemma 3, the functor $L(B) = KK^1(A, B)$ satisfies the conditions of Lemma 4, which together with the Bott periodicity (Theorem 7 of §5) gives the desired exact sequence. For the proof of the second assertion it is sufficient to consider the commutative diagram

$$\begin{array}{ccc} KK^n(A, B/J) \otimes_{B/J} KK^0(B/J, B/J) & \to & KK^n(A, B/J) \\ \| \qquad\qquad \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta} \\ KK^n(A, B/J) \otimes_{B/J} KK^1(B/J, J) & \longrightarrow & KK^{n+1}(A, J) \end{array}$$

where δ_1 is defined like δ, and apply Lemma 6. ■

LEMMA 7. *Let the algebra A be nuclear, I an ideal in A, $i\colon I \hookrightarrow A$ an imbedding, and $q\colon A \to A/I$ the projection. Then the sequence*

$$KK^1(A/I, B) \xrightarrow{q^*} KK^1(A, B) \xrightarrow{i^*} KK^1(I, B)$$

of groups is exact.

PROOF. Clearly $i^*q^* = 0$. We verify that $\operatorname{Ker} i^* \subset \operatorname{Im} q^*$. Let $\varphi \in \mathcal{E}xt(A, B)$ and $i^*(\varphi) = 0$ in the group $\operatorname{Ext}(I, A)$, i.e. upon adding some $\psi \in \mathcal{D}xt(I, B)$ the composition $\varphi \cdot i\colon I \to \mathcal{Q}(\mathcal{K} \otimes B)$ becomes an element of $\mathcal{D}xt(I, B)$. We fix a regular G-imbedding $A \hookrightarrow \mathcal{M}(\mathcal{K})$ such that $A \cap \mathcal{K} = 0$. We denote the composition $A \hookrightarrow \mathcal{M}(\mathcal{K}) \hookrightarrow \mathcal{M}(\mathcal{K} \otimes B) \to \mathcal{Q}(\mathcal{K} \otimes B)$ by π. According to the generalized theorem of Voiculescu (§1.16), $\psi \oplus i^*(\pi)$ is unitarily equivalent to $i^*(\pi)$, and therefore the element $i^*(\varphi) \oplus \psi \oplus i^*(\pi) \in \mathcal{D}xt(I, B)$ is unitarily equivalent to $i^*(\varphi \oplus \pi)$. For convenience we change the notation of $\varphi \oplus \pi$ to φ.

Let the extension $0 \to \mathcal{K} \otimes B \xrightarrow{e} E \xrightarrow{p} A \to 0$ correspond to the element $\varphi \in \mathcal{E}xt(A, B)$. According to Remark 3, to the element $i^*(\varphi) \in \mathcal{E}xt(I, B)$ there corresponds the extension

$$0 \to \mathcal{K} \otimes B \xrightarrow{e} D = p^{-1}(I) \xrightarrow{p} I \to 0.$$

Since $i^*(\varphi) \in \mathcal{D}xt(I, B)$, the last extension is decomposable: there exists a homomorphism $s\colon I \to D$ such that $ps = 1$. Using the decomposition of s, it is not difficult to obtain from the exact sequence of Theorem 2 that

$$KK^1(A/I, D) \simeq KK^1(A/I, \mathcal{K} \otimes B) \oplus KK^1(A/I, I).$$

We denote by ω the projection onto the first direct summand. Let $\beta \in KK^1(A/I, D)$ be the element corresponding to the extension $0 \to D \to E \to E/D \simeq A/I \to 0$. We show that $q^*(\omega(\beta)) = [\varphi]$. (Here we identify the group $KK^1(A, \mathcal{K} \otimes B)$ with $KK^1(A, B)$ in the same way as in Remark 2.) It is sufficient to verify the coincidence of the elements $q^*(\beta)$ and $e_*([\varphi])$ in the group $KK^1(A, D)$, since their projections in $KK^1(A, \mathcal{K} \otimes B)$ under the analogous decomposition

$$KK^1(A, D) \simeq KK^1(A, \mathcal{K} \otimes B) \oplus KK^1(A, I)$$

are equal to $q^*(\omega(\beta))$ and $[\varphi]$, respectively. The coincidence of $q^*(\beta)$ and $e_*([\varphi])$ follows from Remark 3 and the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{K} \otimes B & \xrightarrow{e} & E & \xrightarrow{\quad p \quad} & A \to 0 \\ & {\scriptstyle e\oplus 0}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ 0 \to & (D \oplus 0) & \to & (E \oplus_{A/I} A) & \xrightarrow{g} & A \to 0 \end{array}$$

where

$$f(x) = (x, p(x));\ g(x, y) = y. \ ■$$

LEMMA 8. *Let both algebras A and B be nuclear and separable. Denote by $\alpha_D \in KK^1(A, B)$ the element corresponding to the extension $0 \to B \to D \to A \to 0$. Then under the homomorphism*

$$\partial : KK^0(B, B) \simeq KK^0(S(D, A), B) \to KK^0(A(0, 1), B) \simeq KK^1(A, B)$$

the canonical generator $\tau_B(c_1)$ turns into α_D.

PROOF. In the same way as in the proof of Lemma 6, the assertion can be verified easily for the extension $0 \to C(0, 1) \to C[0, 1) \to C \to 0$. (In this case the element α_D is equal to the canonical generator of $K^1(C(0, 1))$ taken with a minus sign.) Using the homomorphism τ_B, we obtain our assertion for the extension $0 \to B(0, 1) \to B[0, 1) \to B \to 0$. The general case follows from the commutative diagram

$$\begin{array}{ccccccc} 0 \to & B(0, 1) & \to & B[0, 1) & \to & B & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & B(0, 1) & \to & D[0, 1) & \to & S(D, A) & \to 0 \\ & \| & & \uparrow & & \uparrow & \\ 0 \to & B(0, 1) & \to & D(0, 1) & \to & A(0, 1) & \to 0 \end{array}$$

and Remark 3. ■

THEOREM 3. *Let the algebras A and B be trivially graded, and assume that A is nuclear and I is an ideal in A. Denote by i the imbedding $I \hookrightarrow A$ and by q the projection $A \to A/I$. We have the following two-sided infinite (homological) exact sequence:*

$$\ldots \xrightarrow{q^*} K_{n+1}K(A, B) \xrightarrow{i^*} K_{n+1}K(I, B) \xrightarrow{\partial} K_nK(A/I, B) \xrightarrow{q^*} K_nK(A, B) \xrightarrow{i^*} \ldots,$$

where the boundary homomorphism ∂ is defined as the composition

$$K_{n+1}K(I, B) \simeq K_{n+1}K(S(A, A/I), B) \to K_{n+1}K((A/I)(0, 1), B) \simeq K_nK(A/I, B).$$

The homomorphism ∂ also admits the following definition: $\partial(x) = [\varphi] \otimes_I x$, where $[\varphi] \in KK^1(A/I, I)$ is the element corresponding to the extension $0 \to I \to A \to A/I \to 0$.

The proof is analogous to that of Theorem 2. (The reference to Lemma 6 is replaced by a reference to Lemma 8.) ■

COROLLARY 1. *Let the algebras A and B be trivially graded, and assume that A is nuclear. Then*

$$K_iK(A, B) \simeq \widetilde{K}_iK(\widetilde{A}, B) \simeq K_i\widetilde{K}(A, \widetilde{B}) \simeq \widetilde{K}_i\widetilde{K}(\widetilde{A}, \widetilde{B}). \ ■$$

REMARK 4. If in the extension $0 \to B \to D \xrightarrow{p} A \to 0$ the algebras A and D and the homomorphism p are unital, then the corresponding element in $KK^1(A, B)$ belongs to the subgroup $\tilde{K}K^1(A, B)$. Indeed, upon restricting to $C \hookrightarrow A$, we obtain the extension $0 \to B \to \tilde{B} \to C \to 0$, which is splitting.

On the basis of Theorem 1 absorbing extensions can be classified by elements of $KK^1(A, B)$ up to unitary equivalence. We fix $\alpha \in KK^1(A, B)$. Let $0 \to \mathcal{K} \otimes B \to D_\alpha \to A \to 0$ be the absorbing extension corresponding to α. (A is assumed to be nuclear.) We also fix the natural imbedding $D_\alpha \subset \mathfrak{M}(\mathcal{K} \otimes B)$. How can one describe the class of all extensions $0 \to B \to D \to A \to 0$ (or $0 \to \mathcal{K} \otimes B \to D \to A \to 0$) which define the same element $\alpha \in KK^1(A, B)$? We have

COROLLARY 2. *If $0 \to B_1 \to D \to A \to 0$ (where $B_1 = B$ or $\mathcal{K} \otimes B$) is an arbitrary extension defining the element α, then there exist an imbedding $D \hookrightarrow D_\alpha$ and an invariant projection $P \in \mathfrak{M}(\mathcal{K} \otimes B)$ such that*

1) $D + (\mathcal{K} \otimes B) = D_\alpha$,

2) $P(\mathcal{K} \otimes B)P = B_1$,

3) $D \cap (\mathcal{K} \otimes B) = P(\mathcal{K} \otimes B)P$.

Conversely, if D is a subalgebra in D_α and P is an invariant projection in $\mathfrak{M}(\mathcal{K} \otimes B)$ such that $P(\mathcal{K} \otimes B)P \simeq B$ or $\mathcal{K} \otimes B$ and conditions 1) *and* 3) *are satisfied, then to the extension $0 \to P(\mathcal{K} \otimes B)P \to D \to A \to 0$ there corresponds the element $\alpha \in KK^1(A, B)$.*

PROOF. Adding the decomposable absorbing extension $0 \to \mathcal{K} \otimes B \to D_0 \to A \to 0$ to the extension $0 \to B_1 \to D \to A \to 0$, we obtain an absorbing extension which is unitarily equivalent to $0 \to \mathcal{K} \otimes B \to D_\alpha \to A \to 0$. Therefore, we may assume that $D_\alpha = D_\oplus$ is the sum of D and D_0 (see Definition 1). We define the imbedding $D \hookrightarrow D_\oplus$ by the formula

$$x \to \begin{pmatrix} x & 0 \\ 0 & sp(x) \end{pmatrix},$$

where $p\colon D \to A$ is the projection and $s\colon A \to D_0$ is a section homomorphism, and we also define the projection P by the formula $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then conditions 1)–3) are obviously satisfied. Conversely, for any pair (D, P), according to 3), $P(\mathcal{K} \otimes B)P = D \cap (\mathcal{K} \otimes B)$ is an ideal in D, and according to 1), we have $D/P(\mathcal{K} \otimes B)P \simeq D_\alpha/\mathcal{K} \otimes B \simeq A$. From the commutative diagram

$$\begin{array}{ccccccc} 0 \to & P(\mathcal{K} \otimes B)P & \to & D & \to & A & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & \mathcal{K} \otimes B & \longrightarrow & D_\alpha & \to & A & \to 0 \end{array}$$

it follows that under the imbedding $P(\mathcal{K} \otimes B)P \hookrightarrow \mathcal{K} \otimes B$ the element $\delta_D \tau_A(c_1) \in KK^1(A, P(\mathcal{K} \otimes B)P)$ turns into the element $\delta_{D_\alpha} \tau_A(c_1) \in KK^1(A, \mathcal{K} \otimes B)$. Therefore the extension $0 \to P(\mathcal{K} \otimes B)P \to D \to A \to 0$ defines the element $\alpha \in KK^1(A, B)$. ■

COROLLARY 3. *The K-functor $KK^i(E, D)$ for the algebra D, obtained as a result of the extension $0 \to B_1 \to D \to A \to 0$ (where $B_1 = B$ or $\mathcal{K} \otimes B$, and the algebras A and E are nuclear and separable) depends only on the class $\alpha \in KK^1(A, B)$ determined by this extension. An analogous assertion holds for $KK^i(D, E)$ in the case where D is nuclear and separable.*

PROOF. It is sufficient to use the uniqueness of the absorbing extension corresponding to α, the existence of the homomorphism of extensions $D \hookrightarrow D_\alpha$, the functoriality of the exact sequences of Theorems 2 and 3, and the lemma on five homomorphisms. ■

We now briefly consider extensions of graded algebras. We fix a canonical grading of the space $\mathcal{H}$ and the corresponding grading of the algebra $\mathcal{K}$. The definitions of the subgroups $\operatorname{Ext}(A, B)$, $\operatorname{Ext}_a(A, B)$ and $E^1(A, B)$ remain unchanged, except for the following: all homomorphisms have to be graded; in the definition of unitary equivalence and addition invariant isometries of degree 0 are used; and in Definition 3 the element P has degree 0. Lemma 1 is preserved verbatim. (In the proof of the second assertion of this lemma, in order to obtain a graded lifting χ we have to use the same device as in the lemma of §2.10: we have to replace χ by χ_0: $\chi_0(a) = p^{(\deg a)}(\chi(a))$, where $p^{(i)}$ is the projection onto $\mathfrak{M}(\mathcal{K} \otimes B)^{(i)}$ in $\mathfrak{M}(\mathcal{K} \otimes B)$.) As a result, for a nuclear algebra A we have

$$\operatorname{Ext}_a(A, B) \simeq \operatorname{Ext}(A, B) \simeq E^1(A, B).$$

Parallel with the case of trivially graded algebras, Lemma 2 ceases to be valid in the graded case. For example,

$$E^1(C_{1,0}, C) \simeq K^1(C) \not\simeq KK^1(C_{1,0}, C),$$
$$E^1(C, C_{1,0}) \simeq K^1(C) \not\simeq KK^1(C, C_{1,0}).$$

(The first isomorphism can be verified immediately, and the second one can be established most easily by means of the same arguments as in Lemma 2.) We also note that for trivially graded algebras the semigroup $E^1(A, B)$ is isomorphic to the direct sum of two copies of $KK^1(A, B)$, since the condition of commutation of φ and P with the grading operator $\varepsilon = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ of $\mathcal{H}_B$ implies that

$$\varphi = \begin{pmatrix}\varphi_1 & 0\\ 0 & \varphi_2\end{pmatrix}, \qquad P = \begin{pmatrix}P_1 & 0\\ 0 & P_2\end{pmatrix}.$$

The connection of $E^1(A, B)$ with the K-functor is not clear. Nevertheless, we may study $E^1(A, B)$ in complete analogy with the K-functor. We set $E^n(A, B) = E^1(A(\mathbf{R}^{n-1}), B)$ (the action of G and the "real" involution on $\mathbf{R}^{n-1}$ are trivial).

THEOREM 4. *For the graded algebras A and B the semigroup $E^n(A, B)$ is a group homotopically invariant with respect to A and B. If the algebras A_0, B_0, and D are graded trivially, and A_0 is separable and B_0 and D have strictly positive elements, then the bilinear pairings*

$$E^i(A, B \otimes D) \otimes_D KK^j(D \otimes A_0, B_0) \to E^{i+j}(A \otimes A_0, B \otimes B_0), \tag{4}$$
$$KK^j(A_0, B_0 \otimes D) \otimes_D E^i(D \otimes A, B) \to E^{i+j}(A_0 \otimes A, B_0 \otimes B) \tag{5}$$

are defined and have the ordinary properties of functoriality and associativity, and also commute with τ (see Theorem 4 of §4). Moreover, if H is a separable graded Hilbert space, then

$$E^i(A \otimes \mathcal{K}(H), B) \simeq E^i(A, B) \simeq E^i(A, B \otimes \mathcal{K}(H)).$$

If the group G acts on $\mathbf{R}^n$ by means of the spinor representation, then

$$E^i(A(\mathbf{R}^n), B) \simeq E^i(A, B) \simeq E^i(A, B(\mathbf{R}^n))$$

for $n \equiv 0 \pmod 2$ *in the complex case and for* $n \equiv 0 \pmod 8$ *in the real case*; *and*

$$E^i(A(\mathbf{R}^{p,q}), B) \simeq E^i(A, B) \simeq E^i(A, B(\mathbf{R}^{p,q}))$$

for $p - q \equiv 0 \pmod 8$ *in the "real" case. Moreover,*

$$E^i(A, B) \simeq E^i(A(\mathbf{R}^n), B(\mathbf{R}^n))$$

for all n. *If the algebra* A *is nuclear,* I *and* J *are graded ideals in* A *and* B, *respectively, and* J *has a strictly positive element, then we have the following infinite exact sequences*:

$$\dots \xrightarrow{p^*} E^n(A, B) \xrightarrow{i^*} E^n(I, B) \xrightarrow{\partial} E^{n+1}(A/I, B) \xrightarrow{p^*} E^{n+1}(A, B) \xrightarrow{i^*} \dots,$$

$$\dots \xrightarrow{j_*} E^n(A, B) \xrightarrow{q_*} E^n(A, B/J) \xrightarrow{\delta} E^{n+1}(A, J) \xrightarrow{j_*} E^{n+1}(A, B) \xrightarrow{q_*} \dots,$$

where $i\colon I \hookrightarrow A$ *and* $j\colon J \hookrightarrow B$ *are imbeddings and* $p\colon A \to A/I$ *and* $q\colon B \to B/J$ *are the projections.*

PROOF. Replacing homology by homotopy in the definition of $E^1(A, B)$, we obtain a new semigroup $E^1_{\mathrm{ht}}(A, B)$. It is a group: the inverse of (φ, P) is equal to $(\varphi, 1 - P)$. Indeed, the operator homotopy

$$\left(\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}, \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{1+t^2}\begin{pmatrix} (1-P)t^2 & (1-P)t \\ (1-P)t & (1-P) \end{pmatrix}\right), \quad 0 \leqslant t \leqslant \infty,$$

converts $(\varphi, P) \oplus (\varphi, 1 - P)$ into $(\varphi \oplus \varphi, 1 \oplus 0) \in \mathfrak{D}^1(A, B)$. Theorems 2 and 3 of §4 can be carried over to $E^1_{\mathrm{ht}}(A, B)$ verbatim.

We first define the pairings (4) and (5) for $i = 1, j = 0$ with E^1 replaced by E^1_{ht}. For convenience, instead of the pairs $(\varphi, P) \in E^1_{\mathrm{ht}}(A, B)$ we shall consider the pairs $(\varphi, F = 2P - 1)$, assuming that $\deg F = 0$ and the elements $\varphi(a)F - F\varphi(a)$, $(F^2 - 1)\varphi(a)$ and $(F - F^*)\varphi(a)$ belong to $\mathcal{K} \otimes B$ for $a \in A$. We realize the group $K_0K(D \otimes A_0, B_0)$ as indicated in Theorem 2 of §6. Let

$$(\varphi, F) \in \mathcal{E}^1(A, B \otimes D), \qquad (\varphi_0, F_0) \in \mathcal{E}_{0,0}(D \otimes A_0, B_0).$$

Carrying out the same constructions as in the proof of Theorem 4 of §4, we obtain the homomorphisms

$$\Phi_0\colon \mathfrak{L}(\mathcal{K}_{\tilde{B} \otimes \tilde{D}}) \otimes \tilde{A}_0 \otimes C_{1,0} \to \mathfrak{L}(\mathcal{K}_{B \otimes B_0})$$

and

$$\varphi \otimes_D \varphi_0\colon A \otimes A_0 \to \mathfrak{L}(\mathcal{K}_{B \otimes B_0}).$$

(Note that the space $\mathcal{K}_{B_0}$ is assumed to be trivially graded, so that the grading of $\mathcal{K}_{B \otimes B_0}$ is determined by the grading of $\mathcal{K}_B$.) We set

$$F \#_D F_0 = \sqrt{M} \cdot \Phi_0(F \otimes \varepsilon_1) + \sqrt{N} \cdot (1 \otimes F_0),$$

where ε_1 is the generator of $C_{1,0}$ and M and N satisfy an analogue of Definition 5 of §4. The second pairing can be obtained in exactly the same way. Here

$$F_0 \#_D F = \sqrt{M} \cdot \Phi(F_0 \otimes 1) + \sqrt{N} \cdot \Phi(\varphi_0(\varepsilon_1) \otimes 1) \cdot (1 \otimes F).$$

Theorem 1 of §6 and Lemma 1 of §6 can be carried over to $E^1_{\mathrm{ht}}(A, B)$ verbatim, which provides the isomorphism $E^1_{\mathrm{ht}}(A, B) \simeq E^1(A, B)$ (see the second part of the proof of Lemma 2). The periodicity can be established by taking intersection products with the elements $\alpha_n \in K_0(\mathbf{R}^n)$ and $\beta_n \in K^0(\mathbf{R}^n)$ as in §5 (here $n \equiv 0 \pmod 2$ or $\pmod 8$). From the periodicity it follows that the compositions

$$E^1(A, B) \xrightarrow{\tau_{C(\mathbf{R}^n)}} E^1(A(\mathbf{R}^n), B(\mathbf{R}^n)) \xrightarrow{\tau_{C(\mathbf{R}^{7n})}} E^1(A(\mathbf{R}^{8n}), B(\mathbf{R}^{8n})) \simeq E^1(A, B),$$

$$E^1(A(\mathbf{R}^n), B(\mathbf{R}^n)) \xrightarrow{\tau_{C(\mathbf{R}^{7n})}} E^1(A(\mathbf{R}^{8n}), B(\mathbf{R}^{8n})) \simeq E^1(A, B) \xrightarrow{\tau_{C(\mathbf{R}^n)}} E^1(A(\mathbf{R}^n), B(\mathbf{R}^n))$$

are identity mappings. Consequently, $E^1(A(\mathbf{R}^n), B(\mathbf{R}^n)) \simeq E^1(A, B)$. Now the pairings (4) and (5) can be defined for all i and j.

In the case of a trivially graded space H, Theorem 1 of §5 can be carried over to $E^i(A, B)$ verbatim. In the general case it is sufficient to prove the required isomorphisms for $H = \mathcal{H}$, since then

$$E^i(A \otimes \mathscr{K}(H), B) \simeq E^i(A \otimes \mathscr{K}(H) \otimes \mathscr{K}, B) \simeq E^i(A \otimes \mathscr{K}, B) \simeq E^i(A, B).$$

Temporarily we denote by $\mathcal{H}_0$ the algebra $\mathcal{H}$ with trivial grading. Clearly $\mathcal{H} \simeq \mathcal{H}_0 \otimes M_2$, where the grading of M_2 is determined by the grading operator $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (the action of G on M_2 is trivial). It remains to verify that

$$E^1(A, B) \simeq E^1(A \otimes M_2, B), \quad E^1(A, B) \simeq E^1(A, B \otimes M_2).$$

We denote by $\mathcal{H}_B'$ the space $\mathcal{H}_B$ with the opposite grading. The first isomorphism converts (φ, P) into

$$(\varphi \otimes 1 \colon A \otimes M_2 \to \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B'), P \oplus P).$$

The inverse mapping is obtained by restriction to $A \otimes e_{11}$ (where $e_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2$). The second isomorphism is induced by the imbedding $\mathcal{L}(\mathcal{H}_B) \hookrightarrow \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B')$, and its inverse by the identification of $\mathcal{H}_B \oplus \mathcal{H}_B'$ with $\mathcal{H}_B$.

The exact sequences for $E^i(A, B)$ are established in the same way as in Theorems 2 and 3. ∎

REMARK 5. The question of existence of exact sequences for the K-functor in the graded case remains open.

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