Coxeter 元素的性质

设 Φ 的基本根系是 $\{\alpha_i\}_i$,考虑最高元素 $\theta_l = \sum_{\alpha_i \in \Pi} n_i \alpha_i$ 以及基本权向量 ω_i 满足

$$egin{align} \omega_j(h_i) &= 2rac{\langle lpha_i, \omega_j
angle}{\langle lpha_i, lpha_i
angle} = \delta_{ij} \
ho &= \sum_i \omega_i = rac{1}{2} \sum_{i \in \mathcal{I}_+^+} lpha \ \end{gathered}$$

定义 $n(lpha,eta)=2rac{\langlelpha,eta
angle}{\langleeta,eta
angle}$,根据根系的关系

$$egin{aligned} n(lpha,eta) &= |\Phi \cap \{\mathbb{Z}lpha + eta\}| \ 2rac{\langle
ho,lpha_i
angle}{\langle lpha_i,lpha_i
angle} &= 1 \quad orall i \end{aligned}$$

(参考第八章第八节长短根的系数关系) 考虑伸缩后的根系 Φ^{\vee} 如下定义

$$lpha^ee = rac{2lpha}{\langle lpha, lpha
angle}$$

 Φ^\vee 的基本根是 $\{\alpha_i^\vee\}$ (参考第八章第八节长短根的系数关系)设全半根是如下的元素

$$ho^ee = \sum_{lpha \in \Phi^+} rac{1}{2} rac{lpha}{\langle lpha, lpha
angle} = rac{1}{2} \sum_{lpha^ee \in \Phi^ee} lpha^ee = \sum_i c_i \omega_i^ee$$

这里 ω_i^ee 是一组基满足 $2rac{\langle lpha_i^ee,lpha_i^ee
angle}{\langle lpha_i^ee,lpha_i^ee
angle}=\delta_{ij}$,考虑Weyl变换 $s_{lpha_i^ee}(a)=a-2rac{\langle lpha_i^ee,lpha_i^ee}{\langle lpha_i^ee,lpha_i^ee
angle}$ 满足

$$egin{aligned} s_{lpha_i^ee}(\omega_j^ee) &= \omega_j^ee - \delta_{ij}lpha_i^ee \ s_{lpha^ee}(
ho^ee) &=
ho^ee - c_ilpha_i^ee \quad orall i \end{aligned}$$

根据Weyl群的生成元 $s_{\alpha_i^\vee}$ 将 α_i^\vee 变为 $-\alpha_i^\vee$ 把其他 Φ^\vee 中的正根映射成正根,于是

$$s_{lpha_i^ee}(
ho^ee) =
ho^ee - lpha_i^ee \quad orall i$$

从而 $c_i = 1$ $\forall i$, 于是

$$ho^ee = rac{1}{2} \sum_{lpha^ee = \Phi^ee} lpha^ee = \sum_i \omega_i^ee$$

这样得到了

$$egin{aligned} lpha_i^{eeee} &= rac{2lpha_i^ee}{\langle lpha_i^eeee, lpha_i^ee
angle} = rac{2rac{2lpha_i}{\langle lpha_i, lpha_i
angle}}{\langle rac{2lpha_i}{\langle lpha_i, lpha_i
angle}, rac{2lpha_i}{\langle lpha_i, lpha_i
angle}
angle} = lpha_i \ & \omega_i^ee = 2rac{\omega_i}{\langle lpha_i, lpha_i
angle} \end{aligned}$$

其高度有如下公式

$$egin{aligned} \mathrm{ht} heta_l &= \sum_i n_i = \sum_{ij} n_i \delta_{ij} = \sum_i n_i \sum_j 2rac{\langle lpha_i, \omega_j
angle}{\langle lpha_j, lpha_j
angle} = \sum_i n_i \langle lpha_i, \sum_j \omega_j^ee
angle \ &= \sum_i n_i \langle lpha_i,
ho^ee
angle = \langle heta_l,
ho^ee
angle \end{aligned}$$

其实对任何根 α 都有

$$\operatorname{ht} \alpha = \langle \alpha, \rho^{\vee} \rangle$$

用带有正根的公式得到

$$\mathrm{ht} heta_l = rac{1}{2}\sum_{lpha^ee \in \Phi^ee} \langle heta_l, lpha^ee
angle = rac{1}{2}\sum_{lpha \in \Phi^+} n(heta_l, lpha) = 1 + rac{1}{2}\sum_{lpha \in \Phi^+ lpha
eq heta_l} n(heta_l, lpha) = 1 + \sum_{lpha \in \Phi^+ lpha
eq heta_l} rac{\langle heta_l, lpha
angle}{\langle lpha, lpha
angle}$$

由于 $\theta_l, -\theta_l + \alpha$ 才可能是根,那么就有 $n(\theta_l, \alpha) = 0, 1$,于是 $n(\theta_l, \alpha)^2 = n(\theta_l, \alpha)$,代入得到

$$1+\mathrm{ht}\theta_l=2+2\sum_{\alpha\in\Phi^+\alpha\neq\theta_l}\frac{\langle\theta_l,\alpha\rangle^2}{\langle\alpha,\alpha\rangle\langle\theta_l,\theta_l\rangle}=2\sum_{\alpha\in\Phi^+}\frac{\langle\theta_l,\alpha\rangle^2}{\langle\alpha,\alpha\rangle\langle\theta_l,\theta_l\rangle}=\sum_{\alpha\in\Phi}\frac{\langle\theta_l,\alpha\rangle^2}{\langle\alpha,\alpha\rangle\langle\theta_l,\theta_l\rangle}$$

考虑 Φ 上的二次函数

$$egin{aligned} f(x) &= B(x,x) = \sum_{lpha \in \Phi} \langle x, rac{lpha}{||lpha||}
angle^2 \ B(x,y) &= \sum_{lpha \in \Phi} \langle x, rac{lpha}{||lpha||}
angle \langle y, rac{lpha}{||lpha||}
angle \end{aligned}$$

注意到对于所有外尔群的元素 $w\in W$, B(wx,wy)=B(x,y) $\forall x,y$,作为二次型, $B(x,y)=\langle \phi(x),y\rangle$,这样根据内积非退化有这样的同态 ϕ 满足

$$\phi \cdot w = w \cdot \phi \quad \forall w \in W$$

注意到单李代数的根空间可作为其Weyl群的不可约表示,从而由Schur引理得到 $\phi=t\mathrm{Id}$,另一方面可以根据 Dykin 图选择正交的根反射 $s_{i_1},\ldots,s_{i_r};s_{j_1},\ldots,s_{j_{n-r}}$,可知 ϕ 在这两个空间上作用都是伸缩的,Dykin diagram的二步分解是唯一的,在 A_l,D_l,E_6,E_7,E_8 上考虑根的置换,在 B_l,C_l,G_2,F_4 上考虑图的对称置换可知,在单根系上 ϕ 是一致的伸缩,于是

$$f(x) = \sum_{\alpha \in \Phi} \langle x, \frac{\alpha}{||\alpha||} \rangle^2 = t||x||^2$$

进一步得到

$$t|\Pi| = \sum_{\omega \in \Pi} t\langle \omega_i, \omega_i \rangle = \sum_{\omega \in \Pi} f(\omega_i) = \sum_{\omega \in \Pi} \sum_{\alpha \in \Phi} \langle \omega_i, \frac{\alpha}{||\alpha||} \rangle^2 = \sum_{\alpha \in \Phi} 1 = |\Phi|$$

于是由于 $\mathrm{ord} c = rac{|\Phi|}{|\Pi|} = t$,计算 $f(heta_l) = \mathrm{ord} c || heta_l||^2$ 得到

$$1 + ht\theta_l = ordc$$

对于所有的根甚至 H^* 的元素 λ, μ ,则有 $e_{\lambda}e_{\mu}=e_{\lambda+\mu}$

$$s_i(lpha) = lpha - 2rac{\langle lpha, lpha_i
angle}{\langle lpha_i, lpha_i
angle} lpha_i = lpha - n(lpha, lpha_i) lpha_i$$

I 是 Weyl 群的一个交换子集, $s_is_j=s_js_i$, $s_ilpha_j=lpha_j-2\delta_{ij}lpha_i$

假设 $lpha=\sum_{i\in I}n_ilpha_i$ 是一个正根($n_1\neq 0$),那么有 $\omega=s_{i_1}\dots s_{i_k}$ 满足 $\omega\alpha=-\alpha$ 那么就有所有的 n_i 都相等,于是假设 $|I|>1,n_i>0$,考虑 $s_1\alpha$ 是根但是 $s_1\alpha=-n_1\alpha_1+\sum_{i\in I-\{1\}}n_1\alpha_i$ 不满足根向量条件,所以只有 $\alpha_i,i\in I$ 是根。

$$egin{aligned} wL:rac{2\pi}{h}\mathbb{Z}+rac{\pi}{h}\quad wM:rac{2\pi}{h}\mathbb{Z}\mod \mathbb{Z}\pi \ h=2s+1:\quad wL o wM\quad t o t-s \ h=2s:\quad wL-wM
e^{rac{1}{2}x}-e^{-rac{1}{2}x}=x+rac{2}{3!}x^3+\dots \end{aligned}$$

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例外李代数的基本模维数计算

 G_2 : 正根系为 $\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_1+2\alpha_2,\alpha_1+3\alpha_2,2\alpha_1+3\alpha_2$, 且 $w_1=3,w_2=1$, 利用 Weyl 特征公式计算得到

$$egin{aligned} \dim L(\omega_1) &= \prod_{lpha = \sum_i k_i lpha_i} (1 + rac{3k_1}{3k_1 + k_2}) = 2rac{7}{4}rac{8}{5}rac{3}{2}rac{15}{9} = 14 \ \dim L(\omega_2) &= \prod_{lpha = \sum_i k_i lpha_i} (1 + rac{k_2}{3k_1 + k_2}) = 2rac{5}{4}rac{7}{5}rac{3}{2}rac{4}{3} = 7 \end{aligned}$$

 F_4 : 正根系是

$$\beta_i:\alpha_1+\alpha_2+\alpha_3,\alpha_2+\alpha_3,\alpha_3,\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\\ \beta_i-\beta_j:\alpha_1,\alpha_1+\alpha_2,\alpha_2+2\alpha_3+2\alpha_4,\alpha_2,\alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4\\ \beta_i+\beta_j:\alpha_1+2\alpha_2+2\alpha_3,\alpha_1+\alpha_2+2\alpha_3,2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\\ \alpha_2+2\alpha_3,\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4\\ \frac{1}{2}(\pm\beta_1\pm\beta_2\pm\beta_3\pm\beta_4):\alpha_4,\alpha_3+\alpha_4,\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\alpha_1+\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4\\ \alpha_1+2\alpha_2+$$

计算基本模的维数

$$\dim L(\omega_1) = \prod_{\alpha = \sum_i k_i \alpha_i} (1 + \frac{2k_1}{2k_1 + 2k_2 + k_3 + k_4}) = \frac{7}{5} \frac{13}{11} \frac{9}{2} \frac{25}{9} \frac{11}{5} \frac{8}{7} = 13 \cdot 4 = 52$$

$$\dim L(\omega_2) = \prod_{\alpha = \sum_i k_i \alpha_i} (1 + \frac{2k_2}{2k_1 + 2k_2 + k_3 + k_4}) = 2\frac{5}{11} 7^3 \frac{11}{7} 5 \frac{2}{5^2} 13 = 13 \cdot 2 \cdot 49 = \binom{52}{2} - 52$$

$$\dim L(\omega_3) = \prod_{\alpha = \sum_i k_i \alpha_i} (1 + \frac{k_3}{2k_1 + 2k_2 + k_3 + k_4}) = \frac{2^5}{5} \frac{7}{11} 3 \frac{5^2}{7} \frac{7}{2^5} \frac{13}{5} 11 = 13 \cdot 3 \cdot 7 = \binom{26}{2} - 52$$

$$\dim L(\omega_4) = \prod_{\alpha = \sum_i k_i \alpha_i} (1 + \frac{k_4}{2k_1 + 2k_2 + k_3 + k_4}) = \frac{13}{11} 2 \frac{3}{2} \frac{11}{3} 2 = 13 \cdot 4 = 52$$

考虑 D_4 的正根有12个,如果考虑 $\sigma=(1,3,4)(2)$ 是在单根的置换作用,分别写成 $\sigma-$ 轨道是

$$egin{aligned} |\Phi_{\sigma}| &= 1: \{lpha_2\}, \{lpha_1 + lpha_2 + lpha_3 + lpha_4\}, \ &\{lpha_1 + 2lpha_2 + lpha_3 + lpha_4\} \ |\Phi_{\sigma}| &= 3: \{lpha_1, lpha_3, lpha_4\}, \{lpha_1 + lpha_2, lpha_2 + lpha_3, lpha_2 + lpha_3 + lpha_4, lpha_1 + lpha_2 + lpha_4\}, \ &\{lpha_1 + lpha_2 + lpha_3, lpha_2 + lpha_3 + lpha_4, lpha_1 + lpha_2 + lpha_4\} \end{aligned}$$

较难验证的是 E_2E_1 的关系注意到 $[e_i[e_ie_j]]=0$ 和正根的关系:

$$egin{aligned} [E_2E_1] &= [e_1e_2] + [e_3e_2] + [e_4e_2] \ [E_2[E_2E_1]] &= [([e_1e_3] + [e_1e_4] + [e_3e_4])e_2] \ [E_2[E_2[E_2E_1]]] &= [e_1[[e_3e_4]e_2]] + [e_3[[e_1e_4]e_2]] + [e_4[[e_1e_3]e_2]] \ [E_2[E_2[E_2[E_2E_1]]]] &= 0 \quad 1 - A_{21} = 4 \ &\pm eta_3 \pm eta_j \ &rac{1}{2}(eta_1 + eta_2 + \ldots + eta_8) \end{aligned}$$

通过 E_6 中的 $\pm \beta_k \pm \beta_j$ $3 \le k \ne j$ 对 $\Phi^+(E_7) - \Phi^+(E_6)$ 进行李代数乘法

$$egin{aligned} (eta_2 \pm eta_j) + (\pm eta_k \mp eta_j) &= eta_2 \pm eta_k \quad (k
eq j) \ &rac{1}{2} (-eta_1 - eta_2 + \ldots - eta_8) + \ldots = \end{aligned}$$

$$\alpha_{1} = \beta_{1} - \beta_{2} \quad \beta_{1} = \alpha_{1} + \alpha_{2} + \alpha_{3} + \alpha_{4} + \alpha_{5} + \frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{2} = \beta_{2} - \beta_{3} \quad \beta_{2} = \alpha_{2} + \alpha_{3} + \alpha_{4} + \alpha_{5} + \frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{3} = \beta_{3} - \beta_{4} \quad \beta_{3} = \alpha_{3} + \alpha_{4} + \alpha_{5} + \frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{4} = \beta_{4} - \beta_{5} \quad \beta_{4} = \alpha_{4} + \alpha_{5} + \frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{5} = \beta_{5} - \beta_{6} \quad \beta_{5} = \alpha_{5} + \frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{6} = \beta_{6} - \beta_{7} \quad \beta_{6} = \frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{7} = \beta_{6} + \beta_{7} \quad \beta_{7} = -\frac{1}{2}\alpha_{6} + \frac{1}{2}\alpha_{7}$$

$$\alpha_{8} = -\frac{1}{2}\sum_{i=1}^{i=8}\beta_{i} \quad -\beta_{8} = \alpha_{1} + 2\alpha_{2} + 3\alpha_{3} + 4\alpha_{4} + 5\alpha_{5} + \frac{5}{2}\alpha_{6} + \frac{7}{2}\alpha_{7} + 2\alpha_{8}$$

则 E_8 正根系(共120 = 56 + 64个):

64

$$\begin{aligned} \alpha_{i}+._{k}.+\alpha_{j} & 1 \leq i \leq k \leq j \leq 6 \rightarrow \frac{1}{j-i+1} \\ \alpha_{i}+._{k}.+2\alpha_{j}+..+2\alpha_{5}+\alpha_{6}+\alpha_{7} & 1 \leq i \leq k < j \leq 6 \rightarrow \frac{1}{14-j-i} \\ \alpha_{i}+..+2\alpha_{j}+._{k}.+2\alpha_{5}+\alpha_{6}+\alpha_{7} & 1 \leq i < j \leq k \leq 5 \rightarrow \frac{2}{14-j-i} \\ \alpha_{i_{1}}+._{k}.+2\alpha_{i_{2}}+..+3\alpha_{i_{3}}+..+4\alpha_{i_{4}}+..+2\alpha_{6}+2\alpha_{7} & i_{1} \leq k < i_{2} < i_{3} < i_{4} \\ \alpha_{i_{1}}+..+2\alpha_{i_{2}}+._{k}.+3\alpha_{i_{3}}+..+4\alpha_{i_{4}}+..+2\alpha_{6}+2\alpha_{7} & i_{1} < i_{2} \leq k < i_{3} < i_{4} \\ \alpha_{i_{1}}+..+2\alpha_{i_{2}}+..+3\alpha_{i_{3}}+..+4\alpha_{i_{4}}+..+2\alpha_{6}+2\alpha_{7} & i_{1} < i_{2} \leq k < i_{3} < i_{4} \\ \alpha_{i_{1}}+..+2\alpha_{i_{2}}+..+3\alpha_{i_{2}}+..+4\alpha_{i_{4}}+..+2\alpha_{6}+2\alpha_{7} & i_{1} < i_{2} \leq i_{3} < i_{4} \leq k \end{aligned}$$

维数可以分解成如下的形式

$$30380 = 2^25^17^231$$
 $2450240 = 2^65^113^119^131$
 $146325270 = 2^13^15^17^213^219^131^1$
 $6899079264 = 2^53^17^211^217^223^131$
 $6696000 = 2^63^35^331$
 $3875 = 5^331$
 $147250 = 2^15^319^131$

$$egin{aligned} heta_{\lambda}(e_j)v_{i_1}\cdots v_{i_r} &= \sum_{i_k} \delta_{i_k j}(\lambda + lpha_{i_k} - \sum_{i_k} lpha_{i_l})v_{i_1}\cdots v_{\hat{i_k}}\cdots v_{i_r} \ heta_{\lambda}(e_j)1 &= 0 \end{aligned}$$

这和第十四章的 θ_0 一致。(Cambridge的介绍李代数的书籍)

李代数 \mathfrak{gl}_n 的非平凡理想

在一般的文献中普遍将那个最常见的李代数记成如下的形式 $\mathfrak{gl}_n(k):=\mathrm{M}_{n\times n}(k), [a,b]=ab-ba$ 。 这里 k 通常指一个数域,李括号的定义就是类似于换位子的方式。那么在矩阵李代数 $\mathfrak{gl}_n(k)$ 中存在一个非常常见的子代数(这里说的"代数"的意思是其作为集合关于李括号是封闭的) $\mathfrak{sl}_n(k)$,也就是 $\mathfrak{gl}_n(k)$ 中对角线元素之和为零(这在抽象的数学中称为迹)的那些元素构成的集合。恒等式

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad \forall A, B \in M_{n \times n}(k)$$

和迹函数的线性立刻证实 $\mathfrak{sl}_n(k)$ 是一个李代数(甚至其还是一个李理想)。通常一个李代数(有限维)会对应一个李群,反过来也有某一个李群也对应一个李代数,前者是通过指数映射(需要处理几个例外情况 E_6,E_7,E_8,F_4,G_2),后者是典范的构造(考虑那个李群在单位元处的切向量场自然具有的李代数结构)。在这个层面上会说 $\mathfrak{gl}_n(k)$ 是一般线性群 $\mathrm{GL}_n(k)$ 的李代数,而 $\mathfrak{sl}_n(k)$ 是特殊线性群(行列式为一的可逆矩阵在矩阵乘法下构成的群)的李代数。有关有限维李代数的抽象理论诸如 Cartan 分解、Dynkin 图、Weyl 特征公式、Verma 模等都可以通过考虑上面那些具体的李代数被更好地理解。此处的注记是想说明两点: $\mathfrak{sl}_n(k)$ 是 $\mathfrak{gl}_n(k)$ 的唯一一个非平凡理想;此外 $\mathfrak{sl}_n(k)$ 还是一个单李代数("单"的意思是没有非平凡子理想。事实上在分类中其被称为 A-型李代数,通常记成 A_n)。

假设