On extremal 2-connected graphs avoiding (0 mod 4)-cycles

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Abstract

For two integers k and ℓ, a cycle of length m is an (ℓ mod k)-cycle if $m \equiv \ell \pmod{k}$. In 1977, Bollobás proved that for every two integers k and ℓ such that ℓ is even or k is odd, an n-vertex graph that contains no (ℓ mod k)-cycle has at most a linear number of edges in terms of n, which was a conjecture by Burr and Erdös. The problem of determining the maximum number of edges in a graph without (ℓ mod k)-cycle has been considered an interesting research topic, but it is known for only a few integers ℓ and k. Since the most of known extremal examples are not 2-connected, it is very natural to ask whether the maximum number of edges changes when the graph is restricted to be 2-connected. Recently, Györi, Li, Salia, Tompkins, Varga, and Zhu showed that for an n-vertex graph G if G has no (0 mod 4)-cycle, then $e(G) \leq \left\lfloor \frac{19}{12}(n-1) \right\rfloor$ and their extremal examples are not 2-connected. In this paper, we precisely determine the maximum number of edges in a 2-connected graph without (0 mod 4)-cycles. Precisely, we show that a 2-connected graph with n vertices has at most $\left\lfloor \frac{3n-1}{2} \right\rfloor$ edges when it does not have (0 mod 4)-cycles, and the bound is tight as we provide a method to construct infinitely many extremal examples.

Keywords: 2-connected graph;

1 Introduction

All graphs in the paper are simple. A graph G has no H if H is not a subgraph of G. We denote $e(G)$ by the number of edges in a graph G. For two integers k and ℓ, a cycle or a path of length m is called an (ℓ mod k)-cycle or an (ℓ mod k)-path if $m \equiv \ell \pmod{k}$.

Burr and Erdős [6] conjectured that if an n-vertex graph does not contain an (ℓ mod k)-cycle, where $k\mathbb{Z} + \ell$ includes an even number, then it can have at most a linear number of edges in terms of n. This was later confirmed by Bollobás [2]. Bollobás's result naturally led to the following question: What is the smallest constant $c_{\ell,k}$, where $k\mathbb{Z} + \ell$ contains an even number, such that every n-vertex graph with $c_{\ell,k}n$ edges contains an (ℓ mod k)-cycle? This problem has received significant attention, and some improvements to the general bounds on $c_{\ell,k}$ have been made including a result. In [13], Sudakov and Verstraëte,

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The exact value of $c_{\ell,k}$ is known only for a few specific values of ℓ and k. It is well known that $c_{0,2} = \frac{3}{2}$. Chen and Saito [3] proved that $c_{0,3} = 2$, and the extremal n-vertex graphs are $K_{2,n-2}$. Dean, Kaneko, Ota, and Toft [4], as well as Saito [12], showed that $c_{2,3} = 3$, with extremal graphs being $K_{3,n-3}$. Bai, Li, Pan, and Zhang [1] proved that $c_{1,3} = \frac{5}{3}$, and the extremal graphs are one vertex identifications of Peterson graphs.

About ($\ell \bmod 4$)-cycles, there has been meaningful recent progress, and several interesting results have been announced. In [8], Gao, Li, Ma, and Xie proved that an n-vertex graph G with at least $\frac{5(n-1)}{2}$ edges contains two consecutive even cycles unless $4 \mid (n-1)$ and every block is isomorphic to K_5. This result not only shows that $c_{2,4} = \frac{5}{2}$, but also resolves a special case of a conjecture by Verstraëte [14] on extremal graphs avoiding k cycles of consecutive lengths. The result was extended by Li, Pan and Shi in [10] most recently. Győri, Li, Salia, Tompkins, Varga, and Zhu in [9] determined that $c_{0,4} = \frac{19}{12}$, as stated below:

Theorem 1.1 ([9]). *Let G be an n-vertex graph. If $e(G) > \lfloor \frac{19}{12}(n-1) \rfloor$, then G contains a* $(0 \bmod 4)$-*cycle.*

In this paper, we study the same extremal problem for avoiding $(0 \bmod 4)$-cycles, under the assumption that the graph is 2-connected. Although previous works discuss extremal graphs attaining equality, the examples given in [9] are not 2-connected, as all known constructions contain cut-vertices.

Furthermore, it is known from the work of [5] that every 3-connected graph always contains a $(0 \bmod 4)$-cycle. This naturally leads to the question: what is the maximum number of edges in a 2-connected n-vertex graph that avoids $(0 \bmod 4)$-cycles? Our main result gives a complete answer to this question, and is stated as follows.

Theorem 1.2. *Let G be a 2-connected n-vertex graph. If $e(G) > \lfloor \frac{3n-1}{2} \rfloor$, then G contains a* $(0 \bmod 4)$-*cycle.*

We also construct infinitely many 2-connected graphs with exactly $\frac{3n-1}{2}$ edges that contain no $(0 \bmod 4)$-cycle (see Section 4).

The fact that known extremal graphs have cut-vertices naturally motivates the study of the extremal problem within the class of 2-connected graphs. In [7], Fan, Lv and Wang considered the extremal problem about 2-connected graphs avoiding a cycle of length at least c of some given integer c, since the extremal graphs on general graphs have cut-vertices. This line of research has already been proposed by other researchers and is considered an interesting and meaningful study. Similar questions can be asked for other values of ℓ and k when the known extremal exmaples have cut-vertices, and these remain interesting problems for further research.

2 Preliminaries

2.1 Basic definitions and notations

For a path $P : v_0v_1 \cdots v_m$, the subpath $v_iv_{i+1} \cdots v_j$ of P is denoted by $P[v_i, v_j]$. When $C : v_0v_1 \cdots v_mv_0$ is a cycle, the path $v_iv_{i+1} \cdots v_{i+j}$, where the subscripts are taken modulo $m+1$, is denoted by $C[v_i, v_{i+j}]$.

For a path or a cycle $W : v_0 v_1 \cdots v_m$, $\overleftarrow{W}$ means a path or a cycle $v_m \cdots v_1 v_0$ obtained by reversing W. The length of a path or a cycle W is the number of edges in W, denoted by $\ell(W)$. For two paths P and Q, if the terminal vertex of P and the initial vertex of Q are the same, then $P + Q$ denotes a walk along P and Q. For two vertex sets X and Y of a graph, a path P is called an *(X, Y)-path* if one end of P is contained in X, the other is contained in Y, and no interior vertex is contained in $X \cup Y$. When X or Y is a singleton, we drop the set notation from an (X, Y)-path for convenience. For example, if $X = \{x\}$ and $Y = \{y\}$, we call it an (x, y)-path. We say that a set S of vertices *separates* X and Y if every (X, Y)-path contains a vertex of S, and that S is an *(X, Y)-separating set*. The following is a well-known Menger's Theorem.

Theorem 2.1 ([11, Theorem 2.2])**.** *Let X and Y be sets of vertices in a graph G. For any positive integer k, there are k pairwise vertex-disjoint (X, Y)-paths in G if and only if every (X, Y)-separating set contains at least k vertices.*

The following is a folklore. For an edge e, $V(e)$ means the set of vertices of e.

Proposition 2.2. *Let G be a 2-connected graph, $X = \{x, y\}$ be a vertex cut of G, and S be a connected component of $G - X$. If every (x, y)-path in $G[V(S) \cup X]$ has the same parity, then $G[V(S) \cup X]$ is a bipartite.*

Proof. Let $H = G[V(S) \cup X]$. By Theorem 2.1, for each edge e in H, when $e \neq xy$, there are two vertex-disjoint $(X, V(e))$-paths P and Q in G since G is 2-connected. Since $X = \{x, y\}$ is a vertex cut of G, P and Q must be paths in H. Hence, for an edge e in H, if $e \neq xy$, then there is an (x, y)-path in H containing the edge e.

Suppose that H has an odd cycle C and every (x, y)-path in H has the same parity. Then $xy \notin E(C)$ even if xy is an edge of G. By the previous paragraph, there is an (x, y)-path P in H containing an edge of C. Let w_1 and w_2 be the first and the last vertices of P that are also in $V(P) \cap V(C)$. Then consider two (x, y)-paths $Q_1 : P[x, w_1] + C[w_1, w_2] + P[w_2, y]$ and $Q_2 : P[x, w_1] + \overleftarrow{C}[w_1, w_2] + P[w_2, y]$. Clearly $\ell(Q_1)$ and $\ell(Q_2)$ have different parity, which is a contradiction. Thus H has no odd cycle, and so H is bipartite. $\square$

2.2 Gadgets and small graphs

For two vertices u and v of G, for a nonempty set L of nonnegative integers, we say a pair (u, v) of vertices is an *L-type* in G if for every (u, v)-path P of G, $\ell(P) \equiv t \pmod 4$ for some $t \in L$. We define six graphs F_3, F_4, F_6, F_7, F_8, and F_9 in Figure 1 with two specified vertices a and b. It is clear that (a, b) is a $\{2\}$-type in F_3. One can also observe that (a, b) is a $\{0, 3\}$-type in F_6, a $\{0, 1\}$-type in F_7, a $\{1, 2\}$-type in F_8, and a $\{2, 3\}$-type in F_4 and F_9.

We denote by $(G; x, y)$ a graph G with two specified vertices x and y. Given two graphs $(G; x, y)$ and $(H; a, b)$, the *parallel sum* of $(G; x, y)$ and $(H; a, b)$ is the graph obtained from $G \cup H$ by identifying x and a, identifying y and b, and then deleting a loop or multiple edges. We denote it by $(G; x, y) \uplus (H; a, b)$. If $xy \in E(G)$ or $ab \in E(H)$, then the identified vertices are also adjacent in $(G; x, y) \uplus (H; a, b)$. If $x = y$ or

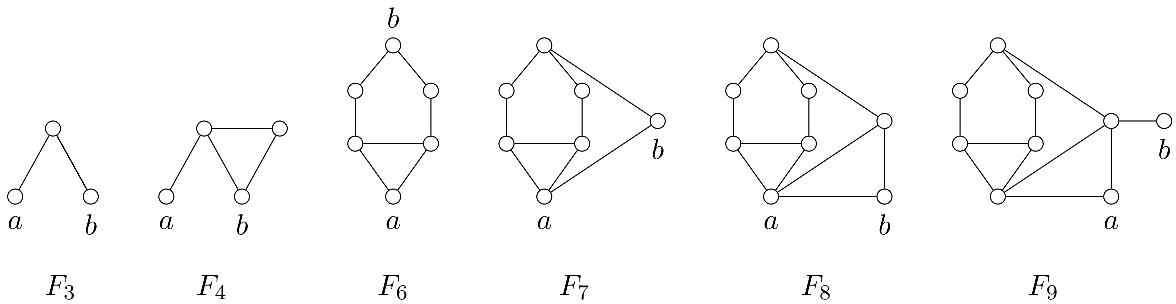


Figure 1: Graphs $(F_i; a, b)$

$a = b$, then $(G; x, y) \uplus (H; a, b)$ is the graph obtained from $G \cup H$ by identifying all the vertices x, y, a, b and deleting a loop or multiple edges. When P is a path and the ends of P are a and b, we simply denote $(G; x, y) \uplus (P; a, b)$ by $(G; x, y) \uplus P$.

For a vertex cut $\{x, y\}$ of a graph G, a connected component S of $G - \{x, y\}$, a *reverse* of G at $\{x, y\}$ with S means a graph $H = (H_1; x, y) \uplus (H_2; y, x)$, where $H_1 = G[V(S) \cup \{x, y\}]$ and $H_2 = G - V(S)$. We also say H is obtained by *reversing* G and the relation of reversing is an equivalence relation, since for a set X of size two, X is a vertex cut in G if and only if X is a vertex cut in H. If a graph H can be obtained from G by reversing a finite number of times, we say H is *reserving-equivalent* to G. The two graphs in Figure 2 and are reversing-equivalent to F_9 in Figure 1. From the definition, when G and H are reversing-equivalent, it is clear that $e(G) = e(H)$ and G has a cycle of length ℓ if and only if H has a cycle of length ℓ.

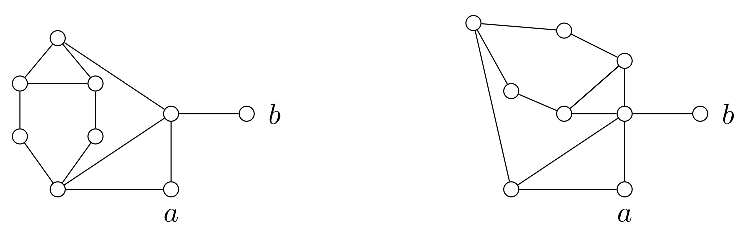


Figure 2: Graphs that are reversing-equivalent to F_9 in Figure 1

We finish the section with some observations on small graphs, whose proofs are given in the appendix.

Proposition 2.3. *Let $n \geq 3$ and let $(H; x, y)$ be an n-vertex graph without* (0 mod 4)*-cycle such that every edge in H is contained in an (x, y)-path of H. If (x, y) is an L-type in H, then*

$$e(H) \leq \begin{cases} \frac{3n-4}{2} & \text{if } L = \{0, 3\} \text{ and } n \leq 6; \\ \frac{3n-3}{2} & \text{if } L = \{0, 1\} \text{ and } n \leq 7; \\ \frac{3n-2}{2} & \text{if } L = \{1, 2\} \text{ and } n \leq 8; \\ \frac{3n-3}{2} & \text{if } L = \{2, 3\} \text{ and } n \leq 9. \end{cases}$$

Moreover, if the equality holds, then H is reversing-equivalent to F_n in Figure 1, where $\{x, y\} = \{a, b\}$, for some $n \in \{6, 7, 8, 9\}$

2.3 Odd cycles in a graph without $(0 \bmod 4)$-cycles

This subsection gathers some results of previous work in [5, 9].

Theorem 2.4 ([5])**.** *Every* 3*-connected graph has a* $(0 \bmod 4)$*-cycle.*

A *theta graph* Θ is a graph consisting of three internally vertex-disjoint paths Q_1, Q_2, Q_3 from a vertex x to a vertex y, denoted by $\Theta(Q_1, Q_2, Q_3)$. We sometimes call it an (x, y)-theta graph. In addition, if each path Q_i has even length, then we call it an *even* theta graph.

Theorem 2.5 ([9])**.** *The following holds.*

(i) (Lemma 1) *An even theta graph contains a* $(0 \bmod 4)$*-cycle.*

(ii) (Lemma 2) *Every non-planar graph contains a* $(0 \bmod 4)$*-cycle.*

(iii) (Lemma 9) *If G is a bipartite n-vertex graph containing no* $(0 \bmod 4)$*-cycle with $n \geq 4$, then* $e(G) \leq \frac{3(n-2)}{2}$.

Lemma 2.6 ([9])**.** *Let C_1 and C_2 be odd cycles of a graph G with* $\ell(C_1) \equiv \ell(C_2) \equiv \ell(C_3) \pmod 4$*.*

(i) (Lemma 6(2)) *If $V(C_1) \cap V(C_2) = \{x\}$, P is an even $(V(C_1), V(C_2))$-path with $x \notin V(P)$, then $C_1 \cup C_2 \cup P$ contains a* $(0 \bmod 4)$*-cycle.*

(ii) (Lemma 6(3)) *If C_1, C_2 are vertex-disjoint and P, Q, R are vertex-disjoint $(V(C_1), V(C_2))$-paths, then $C_1 \cup C_2 \cup P \cup Q \cup R$ contains a* $(0 \bmod 4)$*-cycle.*

(iii) (Lemma 7) *Let C_1, C_2, and C_3 pairwise intersect at a vertex x. Let Q_i be a path from C_i to C_{i+1} that is vertex-disjoint with C_{i+2} for $i = 1, 2, 3$ (the subscripts are taken modulo 3), such that Q_1, Q_2, and Q_3 are pairwise internally vertex-disjoint. Then $C_1 \cup C_2 \cup C_3 \cup Q_1 \cup Q_2 \cup Q_3$ contains a* $(0 \bmod 4)$*-cycle.*

The following properties are easily derived from the above lemma and will be frequently used.

Lemma 2.7. *Let G be a* 2*-connected graph without* $(0 \bmod 4)$*-cycles, and C_1, C_2, and C_3 be three edge-disjoint odd cycles of G.*

(i) *There is no cycle C such that $C \cap C_i$ induces a path in C_i for each $i \in \{1, 2, 3\}$ and the vertices in $V(C_1) - V(C)$, $V(C_2) - V(C)$, and $V(C_3) - V(C)$ are distinct.*

(ii) *There is no theta graph $\Theta := \Theta(Q_1, Q_2, Q_3)$ such that $\Theta \cap C_i$ induces a subpath of Q_i for each $i \in \{1, 2, 3\}$ and the vertices in $V(C_1) - V(\Theta)$, $V(C_2) - V(\Theta)$, and $V(C_3) - V(\Theta)$ are distinct.*

(iii) *If C_1 and C_2 are triangles, then there are no three vertex-disjoint $(V(C_1), V(C_2))$-paths and $\ell(P) + \ell(Q) \equiv 3$* $\pmod 4$ *for every two vertex-disjoint $(V(C_1), V(C_2))$-paths P and Q.*

(iv) *Suppose that* $|V(C_i) \cap V(C_j)| = 1$ *for every distinct* $i, j \in \{1, 2, 3\}$. *Then* $V(C_1) \cap V(C_2) \cap V(C_3) = \{v\}$ *for some vertex* v. *Moreover, if* $\ell(C_1) \equiv \ell(C_2) \equiv (C_3) \pmod 4$, *then there is no connected subgraph* H *of* $G - v$ *such that* $V(H) \cap V(C_i) \neq \emptyset$ *and* $E(H) \cap E(C_i) = \emptyset$ *for each* $i \in \{1, 2, 3\}$. *B:Check please. 5-face쪽도 다시 원래대로 돌렸어요. 결국 반복되는 argument만 모은 꼴이고 약간 분량이 준 정도.*

Figure 3: Illustrations for Lemma 2.7 (i) and (ii)

Proof. (i) Suppose to the contrary that there is a cycle C satisfying the condition. See the first figure of Figure 3. We may assume that $\ell(C_1) \equiv \ell(C_2) \pmod 4$. Note that there are vertex-disjoint $(V(C_1), V(C_2))$-paths P and Q in $C \cup C_3$ such that $\ell(P) + \ell(Q)$ is even, since C_3 is an odd cycle. Since $\ell(C_1) \equiv \ell(C_2) \pmod 4$, we can take subpaths R_1 and R_2 of C_1 and C_2 so that $\ell(R_1)+\ell(R_2) \equiv \ell(P)+\ell(Q) \pmod 4$. By using P, Q, R_1, R_2, we can find a (0 mod 4)-cycle, which is a contradiction.

(ii) Suppose to the contrary that there is an (a, b)-theta graph Θ satisfying the condition. Let $\Theta \cap C_i$ be an (x_i, y_i)-path and Q'_i be an (x_i, y_i)-path whose edge set is $E(C_i) - E(\Theta)$, for each $i \in \{1, 2, 3\}$. See the second figure of Figure 3. Then we can find an even (a, b)-theta graph by taking an even path among Q_i or $Q_i[a, x_i] + Q'_i + Q_i[y_i, b]$, which is a contradiction to Theorem 2.5 (i).

(iii) Suppose that C_1 and C_2 are triangles. If $V(C_1)$ and $V(C_2)$ are disjoint, then by Lemma 2.6 (ii), there are no three vertex-disjoint $(V(C_1), V(C_2))$-paths. In the following, if $V(C_1)$ and $V(C_2)$ are not disjoint, then we let z denote the vertex in $V(C_1) \cap V(C_2)$.

We take any two vertex-disjoint $(V(C_1), V(C_2))$-paths P and Q. Suppose that $V(C_1)$ and $V(C_2)$ are disjoint or one of P and Q is a trivial path z. By using paths of length one or two in C_1 and C_2, we find cycles of length $\ell(P) + \ell(Q) + 2$, $\ell(P) + \ell(Q) + 3$, and $\ell(P) + \ell(Q) + 4$ in G. Therefore, $\ell(P) + \ell(Q) \equiv 3 \pmod 4$ since G has no (0 mod 4)-cycle.

From the previous paragraphs, it is sufficient to show that when C_1 and C_2 share a vertex z, one of P and Q must be a trivial path. Suppose to the contrary that C_1 and C_2 share a vertex z and both P and Q are nontrivial paths. Let R be the trivial path with the vertex z. Then applying the previous paragraph to P and R, we have $\ell(P) \equiv 3 \pmod 4$. Similarly, $\ell(Q) \equiv 3 \pmod 4$. Then P, Q, an edge of C_1, and an edge of C_2 form a (0 mod 4)-cycle, which is a contradiction.

(iv) If $|V(C_i) \cap V(C_j)| = 1$ for every distinct $i, j \in \{1, 2, 3\}$ and $V(C_1) \cap V(C_2) \cap V(C_3) = \emptyset$, then one may easy to find a (0 mod 4)-cycle C such that $V(C) \subset V(C_1) \cup V(C_2) \cup V(C_3)$, which contradicts (i) of this lemma. Let v be the vertex in $V(C_1) \cup V(C_2) \cup V(C_3)$, and let $B_i = C_i - v$ for each $i \in \{1,2,3\}$. Suppose that $\ell(C_1) \equiv \ell(C_2) \equiv \ell(C_3) \pmod 4$. Suppose to the contrary that there is a connected subgraph H of $G - v$ such that $V(H) \cap V(C_i) \neq \emptyset$ and $E(H) \cap E(C_i) = \emptyset$ for each $i \in \{1,2,3\}$ We take such H as a smallest one. Then H is a tree. Take a shortest $(V(C_i), V(C_{i+1}))$-path $R_{i,i+1}$ in H for each $i \in \{1,2,3\}$, where subscripts are modulo 3, and say $R_{i,i+1}$ is an (x_i, y_i)-path. By Lemma 2.6 (i), $R_{i,j}$ has odd length. Without loss of generality, we assume that $\ell(R_{1,2}) \leq \ell(R_{2,3}) \leq \ell(R_{3,1})$.

If $R_{1,2}$ contains a vertex of B_3 or $R_{2,3}$ contains a vertex of B_1, then either $E(R_{1,2}) = E(H)$ or $E(R_{2,3}) = E(H)$, which is a contradiction to the assumption that $\ell(R_{1,2}) \leq \ell(R_{2,3}) \leq \ell(R_{3,1})$. Thus $R_{1,2}$ does not contain any vertex of B_3 and $R_{2,3}$ does not contain any vertex of B_1. Then y_2 is the only vertex in $V(B_3) \cap (V(R_{1,2}) \cup V(R_{2,3}))$ and x_1 is the only vertex in $V(B_1) \cap (V(R_{1,2}) \cup V(R_{2,3}))$.

Suppose that $y_1 = x_2$. Then the walk $R_{1,2} + R_{2,3}$ contains at least one vertex of each B_i, and therefore, $E(H) = E(R_{1,2}) \cup E(R_{2,3})$. The path $R_{3,1}$ is an (y_2, x_1)-path. Then $R_{1,2} + R_{2,3} + R_{3,1}$ is a closed walk of an odd length. Since a closed walk of an odd length contains an odd cycle, it is a contradiction to the fact that H is a tree. Thus $y_1 \neq x_2$.

Note that the existence of $z \in V(R_{1,2}) \cap V(R_{2,3})$ implies that $R_{1,2}[x_1, z] + R_{2,3}[z, y_2]$ is a $(V(C_1), V(C_3))$-path, and therefore $E(R_{1,2}) \cup E(R_{2,3}[z, y_2])$ induces a connected graph H' containing at least one vertex of each of C_1, C_2, C_3. Since H' is a subgraph of H and H' does not have an edge incident to x_2, H' has less edges than H which is a contradiction. Then $V(R_{1,2}) \cap V(R_{2,3}) = \emptyset$.

If $R_{3,1}$ does not intersect with any of $R_{1,2}$ and $R_{2,3}$, it is a contradiction to Lemma 2.6 (iii). Thus $R_{3,1}$ intersects with $R_{1,2}$ or $R_{2,3}$. If $R_{3,1}$ intersects with $R_{j,j+1}$ for some $j \in \{1,2\}$, then since $R_{3,1} \cup R_{j,j+1}$ does not have an edge incident to x_2 or y_1, it makes a proper connected subgraph H' of H such that $V(H') \cap V(C_i) \neq \emptyset$ and $V(H') \cap E(C_i) \neq \emptyset$ for each $i \in \{1,2,3\}$, which is a contradiction. $\square$

Proposition 2.8. *Let G be a 2-connected graph without $(0 \bmod 4)$-cycles, and T_1, T_2, T_3 be triangles in G. There is a permutation $\sigma : [3] \to [3]$ such that there are vertex-disjoint $(V(T_{\sigma(1)}), V(T_{\sigma(2)}))$-paths P and Q satisfying $ab \in E(P)$ and $c \in V(Q)$ where $V(T_{\sigma(3)}) = \{a, b, c\}$. (See Figure 4 for an illustration.)*

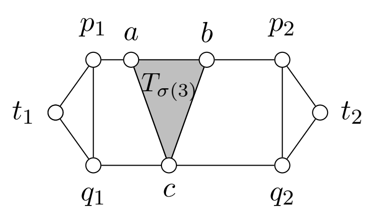


Figure 4: An illustration for Proposition 2.8

Proof. We note that since G has no C_4, $|V(T_i) \cap V(T_j)| \leq 1$ for every distinct $i, j \in \{1,2,3\}$. First, suppose that $|V(T_i) \cap V(T_j)| = 1$ for every distinct $i, j \in \{1,2,3\}$. By Lemma 2.7 (iv), there is a unique vertex v in $V(T_1) \cap V(T_2) \cap V(T_3)$. Since $G - v$ is connected, we can take a smallest connected subgraph

H of $G - v$ that $V(H) \cap V(T_i) \neq \emptyset$ for every $i \in \{1,2,3\}$. Since H contains an edge of some T_i by Lemma 2.7 (iv), it gives a desired conclusion from the minimality of H.

In the following, we may assume that $V(T_1) \cap V(T_2) = \emptyset$. Since G is 2-connected, there are two vertex-disjoint $(V(T_1), V(T_2))$-paths P and Q by Theorem 2.1. Let P and Q be (p_1,p_2)- and (q_1,q_2)-paths, respectively. We take such P as a shortest one so that $|V(P) \cap V(T_3)| \geq |V(Q) \cap V(T_3)|$. Let $\{t_i\} = V(T_i) - \{p_i, q_i\}$ for $i \in \{1,2\}$. For simplicity, we also let $U = V(P) \cup V(Q) \cup V(T_1) \cup V(T_2)$.

(Case 1) Suppose that $|V(T_3) \cap U| \leq 1$. Then by Theorem 2.1, there are vertex-disjoint $(V(T_3), U)$-paths R_1 and R_2. Let R_1 and R_2 be an (a, r_1)-path and a (b, r_2)-path, say R_1 is a shortest one. If $\{r_1, r_2\} = \{t_1, t_2\}$, then P, Q, and $\overleftarrow{R_1} + ab + R_2$ are three vertex-disjoint $(V(T_1), V(T_2))$-paths, which is a contradiction by Lemma 2.7 (iii). Without loss of generality, we may assume that $t_2 \notin \{r_1, r_2\}$. Then one of the following four cases must hold.

(1)-1 $r_1 = t_1$ and $r_2 \in \{p_1, q_1\}$;

(1)-2 $r_1 = t_1$ and $r_2 \in (V(P) \cup V(Q)) \setminus \{p_1, q_1\}$;

(1)-3 $\{r_1, r_2\} \subset V(P)$ or $\{r_1, r_2\} \subset V(Q)$;

(1)-4 $|V(P) \cap \{r_1, r_2\}| = |V(Q) \cap \{r_1, r_2\}| = 1$.

For the case (1)-1, we let $\sigma = (13)$ and then it satisfies the desired condition of the proposition. For each of the cases (1)-2, (1)-3, (1)-4, using R_1 and R_2, we reach a contradiction by finding C or Θ described in Lemma 2.7 (i) or (ii), where Figure 5 shows such subgraph with thick lines. Note that R_1 might be a trivial path and R_2 cannot be a trivial path by our case assumption that $|V(T_3) \cap U| \leq 1$.

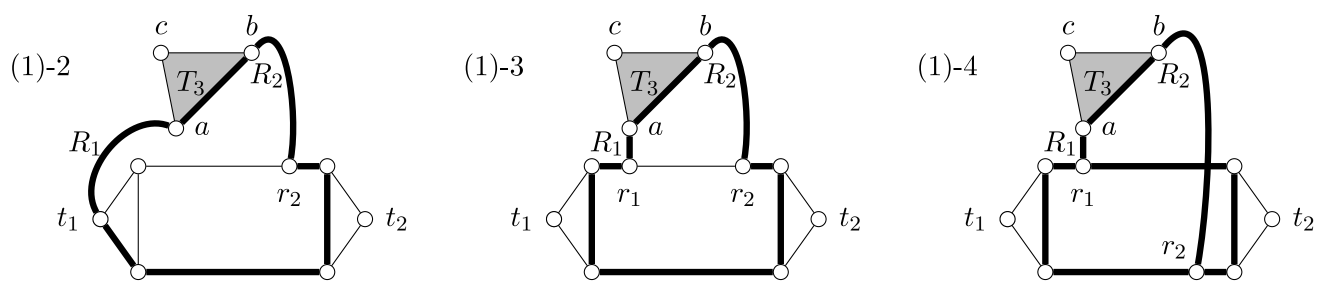


Figure 5: Illustrations for (Case 1): For the cases (1)-2 and (1)-3, the thick lines show a cycle satisfying the condition in Lemma 2.7 (i). For the case (1)-4, the think lines show an (r_1, r_2)-theta graph satisfying the condition in Lemma 2.7 (ii)

(Case 2) Now suppose that $|V(T_3) \cap U| \geq 2$. If $V(T_3) \cap U$ contains $\{t_1, t_2\}$, then P, Q, and $t_1 t_2$ are three vertex-disjoint $(V(T_1), V(T_2))$-paths, which is a contradiction by Lemma 2.7 (iii). Thus we may assume that $t_2 \notin V(T_3)$. Then $|V(T_3) \cap V(P)| \geq 1$ by the choice of P, and so let a be the vertex in $V(T_3) \cap V(P)$ that is closest to p_1. By the minimality of $\ell(P)$, $|V(T_3) \cap V(P)| \leq 2$. We suppose that it is not the case where $|V(T_3) \cap V(P)| = 2$ and $|V(T_3) \cap V(Q)| = 1$, since it is the desired condition of the proposition. Then one of the following four cases must hold.

(2)-1 $|V(T_3) \cap V(P)| = 2$, $|V(T_3) \cap V(Q)| = 0$;

(2)-2 $|V(T_3) \cap V(P)| = 1$, $t_1 \notin V(T_3)$ (and therefore $|V(T_3) \cap V(Q)| = 1$ by the case assumption);

(2)-3 $|V(T_3) \cap V(P)| = 1$, $t_1 \in V(T_3)$, $|V(T_3) \cap V(Q)| = 0$;

(2)-4 $|V(T_3) \cap V(P)| = 1$, $t_1 \in V(T_3)$, $|V(T_3) \cap V(Q)| = 1$.

For the cases (2)-1, (2)-2, and (2)-3, we reach a contradiction by finding C or Θ described in Lemma 2.7 (i) or (ii), where Figure 6 shows such subgraph with thick lines. For the case (2)-4, see the last figure of Figure 6 for an illustration, there are three vertex-disjoint $(V(T_1), V(T_3))$-paths, which is a contradiction to Lemma 2.7 (iii). $\square$

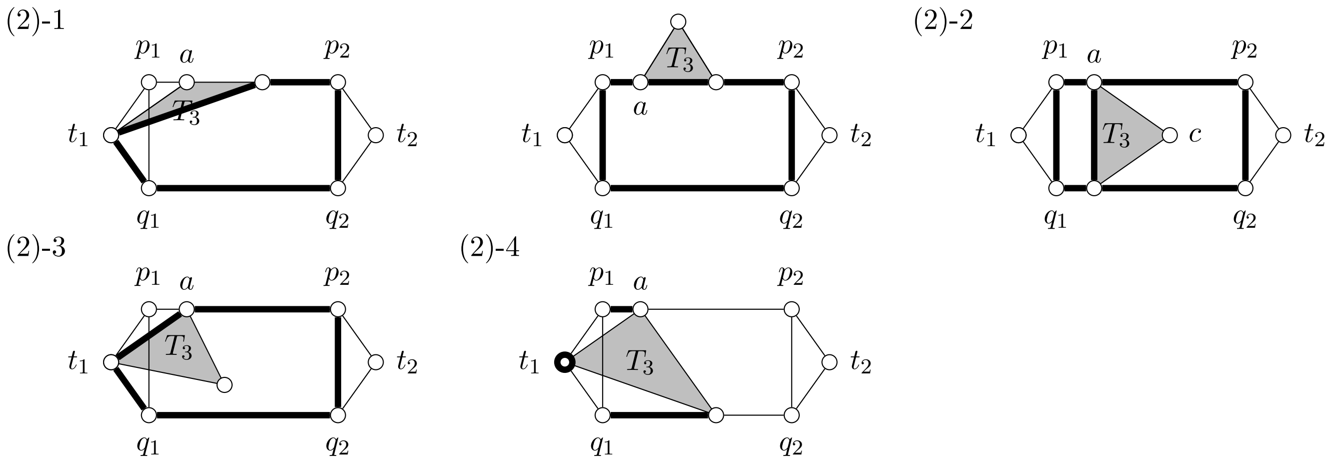


Figure 6: Illustrations for (Case 2): For the cases (2)-1, (2)-2, (2)-3, the thick lines shows a cycle or a theta graph. For the case (2)-4, two thick lines and the trivial path t_1 are three vertex-disjoint $(V(T_1), V(T_3))$-paths

3 Proof of Theorem 1.2

From now on, we assume that G is a minimal counterexample to Theorem 1.2 and let $n = |V(G)|$. That is, G is a 2-connected graph that contains no (0 mod 4)-cycle, $e(G) > \frac{3n-1}{2}$, and every 2-connected graph G' of order less than n has at most $\frac{3|V(G')|-1}{2}$ edges. It is easy to observe that if $n \leq 5$, then G must be a cycle, and so we assume that $n \geq 6$ in the following. Then by Theorem 2.5 (ii), G is planar. The following lemma holds, where its proof is given later.

Lemma 3.1. *In the graph G, there are at most two 3-faces and at most five 5-faces.*

Let $f(G)$ be the number of faces of G, and f_i be the number of i-faces of G. Then $f_3 \leq 2$ and $f_5 \leq 5$ by Lemma 3.1. Since G has no (0 mod 4)-cycles, $f_4 = 0$. Thus, by Euler's formula,

$$n + f(G) = 2 + e(G) = 2 + \frac{1}{2}\sum_{i \geq 3} i f_i \geq 2 + 3f(G) - \frac{3}{2}f_3 - f_4 - \frac{1}{2}f_5 \geq 3f(G) - \frac{7}{2}$$

and so $f(G) \leq \frac{2n+7}{4}$. Since $f(G)$ is an integer, $f(G) \leq \frac{2n+6}{4} = \frac{n+3}{2}$. Therefore, $e(G) = n+f(G)-2 \leq \frac{3n-1}{2}$, which is a contradiction.

To prove Lemma 3.1, we first examine useful structural properties of a vertex cut in our minimal counterexample G in Subsection 3.1, and then establish the proof of the lemma in Subsection 3.2.

3.1 The structure of vertex cuts of G

In this subsection, we aim to explore the structural properties of a vertex cut in the minimal counterexample G. By Theorem 2.4, G is not 3-connected. By Theorem 2.5 (iii), G is non-bipartite. The following lemma is the final goal of the subsection.

Lemma 3.2. *Let $X = \{x, y\}$ be a vertex cut of G. Then $G - X$ has exactly two components and there exists a connected component S of $G - X$ such that $G[V(S) \cup X]$ is one of K_3, $(F_3; a, b)$, $(F_4; a, b)$ in Figure 1, where $\{a, b\} = \{x, y\}$.*

We start with a simple observation.

Lemma 3.3. *Let $X = \{x, y\}$ be a vertex cut of G, and let Z_1 and Z_2 be a bipartition of $V(G) - X$ such that there are no edges between Z_1 and Z_2. Suppose that $H_i = G[Z_i \cup X]$ and $e(H_i) \leq \frac{3|V(H_i)|-1-\alpha_i}{2}$ for each $i \in \{1, 2\}$. Then $\alpha_1 + \alpha_2 \leq 4 - 2e(G[X])$.*

Proof. Suppose that $\alpha_1 + \alpha_2 \geq 5 - 2e(G[X])$. Note that $|V(H_1)| + |V(H_2)| = n + 2$. Then

$$e(G) \leq e(H_1) + e(H_2) - e(G[X]) \leq \frac{3(|V(H_1)| + |V(H_2)|) - 2 - (\alpha_1 + \alpha_2 + 2e(G[X]))}{2} \leq \frac{3n-1}{2},$$

which contradicts the choice of G. □

When S is a connected component of $G - X$ for some vertex cut $X = \{x, y\}$ of G, every edge in $G[V(S) \cup X]$ is contained in an (x, y)-path of $G[V(S) \cup X]$. Therefore, when $G[V(S) \cup X]$ has a small number of vertices, we often apply Proposition 2.3 to $G[V(S) \cup X]$.

We often recall the graphs $(F_i; a, b)$ in Figure 1, as they play a key role in our proofs. In addition, Table 1 shows the summations of the integers modulo 4, where each cell collects $\ell_1 + \ell_2 \pmod 4$ for every ℓ_1 and ℓ_2 from the sets indicated by its row and column, respectively.

$+$	$\{0,1\}$	$\{1,2\}$	$\{2,3\}$	$\{0,3\}$
$\{0,1\}$	$\{0,1,2\}$	$\{1,2,3\}$	$\{0,2,3\}$	$\{0,1,3\}$
$\{1,2\}$		$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,2\}$
$\{2,3\}$			$\{0,1,2\}$	$\{1,2,3\}$
$\{0,3\}$				$\{0,2,3\}$

Table 1: Summation table, where each cell represents the set of sums of the elements from the sets indicated by its row and column, taken modulo 4.

Lemma 3.4. *Let $X = \{x, y\}$ be a vertex cut of G. Then each of the following holds.*

(i) *$G-X$ has exactly two connected components.*

(ii) *For every connected component S of $G-X$, either $G[V(S)\cup X]$ is non-bipartite or $|S|=1$.*

Proof. Let $S_1,\dots,S_r$ be the connected components of $G-X$. Let $G_i=G[V(S_i)\cup X]$ and $n_i=|V(G_i)|$ for each $i\in\{1,2,\dots,r\}$.

(i) Suppose to the contrary that $r\geq 3$. If there exists an (x,y)-path in G_i of even length for each $i\in\{1,2,3\}$, then those three paths form an even theta graph, which is a contradiction to Lemma 2.5 (i). Thus we may assume that every (x,y)-path in G_1 has odd length. Then, by Proposition 2.2, G_1 is bipartite. Clearly $n_1\geq 4$. Thus, by Theorem 2.5 (iii), $e(G_1)\leq\frac{3|V(G_1)|-6}{2}$. Note that $H_2=G-V(S_1)$ is a 2-connected graph from our assumption that $r\geq 3$. By the minimality of G, $e(H_2)\leq\frac{3|V(H_2)|-1}{2}$. Therefore we reach a contradiction to Lemma 3.3. Hence, $r=2$.

(ii) Suppose that G_1 is bipartite. It suffices to show that $n_1\leq 3$. We choose a vertex cut X of G so that n_1 is maximized. Suppose that $n_1\geq 4$. Then $e(G_1)\leq\frac{3n_1-6}{2}$ by Theorem 2.5 (iii). If G_2 is 2-connected, then $e(G_2)\leq\frac{3n_2-1}{2}$ by the minimality of G, which is a contradiction to Lemma 3.3. Then G_2 has a cut-vertex z and it also holds that $xy\notin E(G)$. Note that $z\notin\{x,y\}$ and each of $\{x,z\}$ and $\{y,z\}$ is a vertex cut of G. Since z is not a cut-vertex of G, G_2-z has exactly two connected components S_1' and S_2' such that $x\in V(H_1)$ and $y\in V(H_2)$ where $H_i=G[V(S_i')\cup\{z\}]$ for each $i\in\{1,2\}$. Clearly, each H_i has no (0 mod 4)-cycle. We also $H_1':=G-(V(S_2')\setminus\{y\})$ and $H_2':=G-(V(S_1')\setminus\{x\})$. By the maximality of n_1, each of H_i' and H_i's is non-bipartite.

Claim 3.5. *G_1 is a path of length 3 and $e(H_i)\geq\frac{3|V(H_i)|-1}{2}$ for some $i\in\{1,2\}$. (See Figure 7.)*

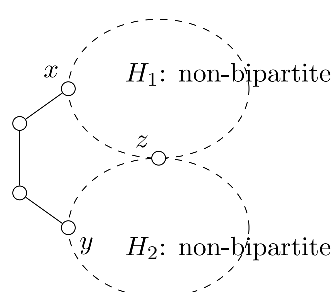


Figure 7: An Illustration for Claim 3.5

Proof. Take an (x,y)-path Q in G_1. Let $\ell(Q)\equiv t\pmod 4$ for some $t\in\{0,1,2,3\}$. Let $G_2^*=(G_2;x,y)\uplus P_{t+1}$. Then G_2^* is 2-connected and has no (0 mod 4)-cycle. Clearly, $|V(G_2^*)|\leq n$ and $xy\notin E(G)$.

Suppose that $|V(G_2^*)|<n$. Then $e(G_2^*)\leq\frac{3|V(G_2^*)|-1}{2}$ by the minimality of G. Note that $|V(G_2^*)|=n_2+t-1$, and so $e(G_2^*)\leq\frac{3|V(G_2^*)|-1}{2}=\frac{3n_2+3t-4}{2}$. If $t=0$, then $e(G_2)\leq e(G_2^*)+1$ which implies that $e(G_2)\leq\frac{3n_2-2}{2}$. For $t\in\{1,2,3\}$, $e(G_2)\leq e(G_2^*)-t\leq\frac{3n_2+t-4}{2}$. In any case, $e(G_2)\leq\frac{3n_2-1}{2}$, which is a contradiction to Lemma 3.3. Thus $|V(G_2^*)|=n$, which implies that $n_1=4$ and so $G_1=P_4$.

If $e(H_i)\leq\frac{3|V(H_i)|-2}{2}$ for each $i\in\{1,2\}$, then $e(G)\leq e(H_1)+e(H_2)+e(G_1)\leq\frac{3(n-1)-4}{2}+3=\frac{3n-1}{2}$, which is a contradiction. Thus $e(H_i)\geq\frac{3|V(H_i)|-1}{2}$ for some $i\in\{1,2\}$. $\square$

By Claim 3.5, we may assum that $e(H_1) \geq \frac{3|V(H_1)|-1}{2}$. Since $G_1 = P_4$ and H_2 is non-bipartite, it is clear that $|V(H_2')| \geq 6$.

Let L_i be the smallest subset of $\{0,1,2,3\}$ such that (x,z) is an L_1-type in H_1 and and (y,z) is an L_2-type in H_2. Since H_i is non-bipartite, Proposition 2.2 implies that L_i contains one of $\{0,1\}$, $\{1,2\}$, $\{2,3\}$, and $\{0,3\}$. Note that $G_1 = P_4$ and so there is no (x,y)-path P in G_2 such that $\ell(P) \equiv 1 \pmod 4$. By Table 1, one of the following (1)$\sim$(4) holds:

(1) $L_1 = L_2 = \{0,3\}$, (2) $L_1 = \{0,1\}, L_2 = \{2,3\}$ (3) $L_1 = \{2,3\}, L_2 = \{0,1\}$ (4) $L_1 = L_2 = \{1,2\}$.

In the following, we will define H_1^* such that $|V(H_1^*)| < n$ and H_1^* is a 2-connected graph that contains no (0 mod 4)-cycle. Then $e(H_1^*) \leq \frac{3|V(H_1^*)|-1}{2}$ by the minimality of G. If (1) or (2) holds, then let $H_1^* = (H_1; x, z) \uplus (F_3; a, b)$ and so

$$e(H_1) \leq e(H_1^*) - 2 \leq \frac{3(|V(H_1)|+1)-1}{2} - 2 \leq \frac{3|V(H_1)|-2}{2},$$

a contradiction. If (3) holds, then $|V(H_2')| \geq 8$, we let $H_1^* = (H_1; y, z) \uplus (F_6; a, b)$ and so

$$e(H_1) \leq e(H_1^*) - 7 \leq \frac{3(|V(H_1)|+4)-1}{2} - 7 = \frac{3|V(H_1)|-3}{2},$$

a contradiction. Suppose that (4) holds. If $|V(H_2')| \geq 8$, then let $H_1^* = (H_1; x, z) \uplus (F_7; a, b)$, and so

$$e(H_1) \leq e(H_1^*) - 8 \leq \frac{3(|V(H_1)|+5)-1}{2} - 8 \leq \frac{3|V(H_1)|-2}{2},$$

a contradiction. If $|V(H_2')| \leq 7$, then $H_2 = K_3$, let $H_1^* = H + xz$ and so $e(H_1) \leq e(H_1^*) \leq \frac{3|V(H_1)|-1}{2} = \frac{3(n-4)-1}{2}$ and $e(H_2') = 6$, which implies that $e(G) \leq \frac{3n-1}{2}$. $\square$

Let $X = \{x, y\}$ be any vertex cut of G. Lemma 3.4 (i), there are only two connected components of $G - X$. In the following of Lemmas 3.6 and 3.7, we always denote by S_1 and S_2 the connected components of $G - X$, and let $G_i = [V(S_i) \cup X]$ and $n_i = |V(G_i)|$ for each $i \in \{1,2\}$. Moreover, let L_i be the smallest subset of $\{0,1,2,3\}$ such that (x,y) is an L_i-type in G_i.

Lemma 3.6. *If $L_1 \subset \{0,1\}$ and $L_2 \subset \{1,2\}$ for some vertex cut $X = \{x,y\}$ of G, then $n_2 = 3$.*

Proof. Suppose to the contrary that $n_2 \geq 4$. Let $H_1 = (G_1; x, y) \uplus (K_3; a, b)$, where a, b are two vertices of K_3, and let $H_2 = G_2 + xy$, which is the graph obtained by adding an edge xy to G_2. Then both H_1 and H_2 are 2-connected and have no (0 mod 4)-cycle. Since $n_2 \geq 4$, it follows that $|V(H_1)| = n_1 + 1 < n$. Therefore, by the minimality of G,

$$e(G_1) = e(H_1) - 3 + e(G[X]) \leq \frac{3(n_1+1)-1}{2} - 3 + e(G[X]) = \frac{3n_1 - 4 + 2e(G[X])}{2}.$$

If $xy \notin E(G)$, then $e(G_1) \leq \frac{3n_1-4}{2}$ and $e(G_2) = e(H_2) - 1 \leq \frac{3n_2-1}{2} - 1 = \frac{3n_2-3}{2}$, which contradicts to Lemma 3.3. Therefore, $xy \in E(G)$, that is, $e(G[X]) = 1$. It also implies that $e(G_1) \leq \frac{3n_1-2}{2}$ and both G_1 and G_2 are 2-connected. By the minimality of G, $e(G_2) \leq \frac{3n_2-1}{2}$.

Suppose that $n_1 \leq 7$ or $n_2 \leq 8$. Then either $e(G_1) \leq \frac{3n_1-3}{2}$ or $e(G_2) \leq \frac{3n_2-2}{2}$ by Proposition 2.3. If G_1 is not reversing-equivalent to F_7 or G_2 is not reversing-equivalent to F_8, then by Proposition 2.3 again, either $e(G_1) \leq \frac{3n_1-4}{2}$ or $e(G_2) \leq \frac{3n_2-3}{2}$, which contradicts Lemma 3.3. If G_1 is reversing-equivalent to F_7 and G_2 is reversing-equivalent to F_8, then $|V(G)| = 13$ and $e(G) = 19$, a contradiction.

Suppose that $n_1 \geq 8$ and $n_2 \geq 9$. Let $H_1 = (G_1; x, y) \uplus (F_8; a, b)$ and $H_2 = (G_2; x, y) \uplus (F_7; a, b)$. Then for each $i \in \{1, 2\}$, $|V(H_i)| < n$ and H_i is a 2-connected graph without (0 mod 4)-cycles. By the minimality of G,

$$e(G_1) = e(H_1) - e(F_8) + 1 \leq \frac{3(n_1+6)-1}{2} - 11 + 1 = \frac{3n_1-3}{2}$$

$$e(G_2) = e(H_2) - e(F_7) + 1 \leq \frac{3(n_2+5)-1}{2} - 9 + 1 = \frac{3n_2-2}{2},$$

which contradicts Lemma 3.3. Therefore, $n_2 = 3$. $\square$

Lemma 3.7. *For a vertex cut $X = \{x, y\}$, there is $i \in \{1, 2\}$ such that $(G_i; x, y)$ is reversing-equivalent to one of K_3, $(F_3; a, b)$, $(F_4; a, b)$, and $(F_6; a, b)$ in Figure 1, where $\{a, b\} = \{x, y\}$. Moreover, if it is $(F_6; a, b)$, then x and y have a unique common neighbor in G_{3-i}.*

Proof. If $n_i = 3$ for some $i \in \{1, 2\}$, then $(G_i; x, y)$ is either K_3 or $(F_3; a, b)$. Suppose that $n_i \geq 4$ for each $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, G_i is non-bipartite by Lemma 3.4 (ii) and so L_i contains one of $\{0, 1\}$, $\{1, 2\}$, $\{2, 3\}$ and $\{0, 3\}$ by Proposition 2.2. By Table 1, we may assume that either $L_1 = \{0, 1\}$ and $L_2 = \{1, 2\}$, or $L_1 = \{0, 3\}$ and $L_2 = \{2, 3\}$. If $L_1 = \{0, 1\}$ and $L_2 = \{1, 2\}$, then by Lemma 3.6, $G_2 = K_3$, which is a desired conclusion.

Suppose that $L_1 = \{0, 3\}$ and $L_2 = \{2, 3\}$. If $n_2 \leq 4$, then $(F_4; a, b)$ is the only possible $(G; x, y)$ to have $L_2 = \{2, 3\}$, which is a desired conclusion. Suppose that $n_2 \geq 5$. Note that $0 \in L_1$ and so $n_1 \geq 5$.

Claim 3.8. *$n_2 \geq 10$, $n_1 \leq 6$, and $e(G_1) \leq \frac{3n_1-4}{2}$.*

Proof. Let $H = (G_1; x, y) \uplus (F_4; a, b)$. Then $|V(H)| = n_1 + 2 < n$ and H is a 2-connected graph without (0 mod 4)-cycles. By the minimality of G,

$$e(G_1) = e(H) - 4 \leq \frac{3(n_1+2)-1}{2} - 4 = \frac{3n_1-3}{2}.$$

Suppose to the contrary that $5 \leq n_2 \leq 9$. If G_2 is not reversing-equivalent to F_9, then $e(G_2) \leq \frac{3n_2-4}{2}$ by Proposition 2.3, which contradicts Lemma 3.3. Thus G_2 is reversing-equivalent to F_9. Let z be the common neighbor of a and b of F_9. Then $Y = \{x, z\}$ is a vertex cut of G and let $G - Y$ have two connected components Y_1 and Y_2, say $D_i = G[V(Y_i) \cup Y]$ for each $i \in \{1, 2\}$. Then (x, z) is a $\{1, 2\}$-type in D_i and (x, z) is a $\{0, 1\}$-type in D_j for $\{i, j\} = \{1, 2\}$. Since both D_1 and D_2 have more than three vertices, it contradicts Lemma 3.6. Thus, $n_2 \geq 10$.

Suppose to the contrary that $n_1 \geq 7$. Let $H_1 = (G_1; x, y) \uplus (F_9; a, b)$ and $H_2 = (G_2; x, y) \uplus (F_6; a, b)$. Then for each $i \in \{1, 2\}$, $|V(H_i)| < n$ and H_i is a 2-connected graph without (0 mod 4)-cycles. By the minimality of G,

$$e(G_1) = e(H_1) - 12 \leq \frac{3(n_1+7)-1}{2} - 12 = \frac{3n_1-4}{2}, \quad e(G_2) = e(H_2) - 7 \leq \frac{3(n_1+4)-1}{2} - 7 = \frac{3n_2-3}{2},$$

which contradicts Lemma 3.3. Therefore, $n_1 \leq 6$. By Proposition 2.3, $e(G_1) \leq \frac{3n_1-4}{2}$. $\square$

Claim 3.9. *x and y have a unique common neighbor in G_2, and $e(G_2) \leq \frac{3n_2-2}{2}$.*

Proof. Let G_2^* be the graph obtained from G_2 by identifying x and y, where v^* is the identified vertex of x and y in G_2^*. Then G_2^* has no (0 mod 4)-cycle, since (x, y) is a $\{2, 3\}$-type in G_2. If G_2^* is not 2-connected, then the vertex v^* is a cut-vertex of G_2^*, which implies that $G-X$ has at least three connected components, and it is a contradiction to Lemma 3.4 (i). Therefore G_2^* is 2-connected.

Note that $|V(G^*)| < n$ and G_2^* is a 2-connected graph without (0 mod 4)-cycles. By the minimality of G, $e(G_2^*) \leq \frac{3(n_2-1)-1}{2}$. If x and y have no common neighbor in G_2, then $e(G_2) = e(G_2^*) \leq \frac{3n_2-4}{2}$, which contradicts Lemma 3.3 since $e(G_1) \leq \frac{3n_1-4}{2}$ by Claim 3.8. Thus x and y have a common neighbor z in G_2. Then such z is unique since G_2 has no 4-cycle. Then $e(G_2) = e(G_2^*) + 1 \leq \frac{3n_2-2}{2}$. $\square$

By Claims 3.8 and 3.9, $e(G_1) \leq \frac{3n_1-4}{2}$ and $e(G_2) \leq \frac{3n_2-2}{2}$. By Lemma 3.3, $e(G_1) = \frac{3n_1-4}{2}$ and $e(G_2) = \frac{3n_2-2}{2}$. By Proposition 2.3, G_1 is isomorphic to F_6 where $\{x, y\} = \{a, b\}$. $\square$

Now we are ready to prove Lemma 3.2.

Proof of Lemma 3.2. By Lemma 3.4 (i), $G-X$ has exactly two connected components S_1 and S_2, and let $G_i = G[V(S_i) \cup X]$ and $n_i = |V(G_i)|$. By Lemma 3.7, suppose to the contrary that $(G_1; x, y) = (F_6; a, b)$ and x and y have a unique common neighbor z in G_2. Recall that (x, y) is a $\{0, 3\}$-type in G_1 and a $\{2, 3\}$-type in G_2.

Claim 3.10. *Both $G_2 - \{x, y\}$ and $G_2 - z$ are connected.*

Proof. Since $G_2 - \{x, y\}$ is equal to the connected component S_2, it is trivial that $G_2 - \{x, y\}$ is connected. Suppose to the contrary that $G_2 - z$ is not connected. Then each of $Z_1 = \{x, z\}$ and $Z_2 = \{y, z\}$ is a vertex cut of G. There is a connected component S_1' of $G - Z_1$ not containing y, and let $H_1 := G[V(S_1') \cup Z_1]$. Since (x, z) is an $\{1, 2\}$-type in H_1, K_3 is the only case where two vertices of a vertex cut are adjacent by Lemma 3.7, which implies that $H_1 = K_3$. By the same reason, there is a connected component S_2' of $G - Z_2$ not containing x, and let $H_2 := G[V(S_2') \cup Z_2]$. Since (y, z) is an $\{1, 2\}$-type in H_2 and so $H_2 = K_3$. Then we can find an (x, y)-path of length 4 in G_2, which is a contradiction to the fact that (x, y) is a $\{2, 3\}$-type in G_2. $\square$

In the following of the proof, we do not use the structure of F_6, but only the fact that (x, y) is a $\{0, 3\}$-type in F_6. By Claim 3.10, there is an (x, y)-path P in $G_2 - z$. For a $(V(P) - \{x, y\}, z)$-path Q in $G_2 - \{x, y\}$, we call this pair (P, Q) of paths a *feasible pair*.

Claim 3.11. *Let (P, Q) be a feasible pair, say Q is a (q, z)-path. Let R be a $(V(Q), V(P))$-path in $G_2 - \{z, q\}$, say R is an (r, r')-path. Then the following holds:*

(i) $\ell(P) \equiv 3 \pmod 4$.

(ii) *When $\{x', y'\} = \{x, y\}$, it holds that either $\ell(P[x', q]) \equiv 0$, $\ell(P[q, y']) \equiv 3$, $\ell(Q) \equiv 2$, or $\ell(P[x', q]) \equiv 2$, $\ell(P[q, y']) \equiv 1$, $\ell(Q) \equiv 0 \pmod 4$.*

(iii) *Both* $\ell(Q[q,r])$ *and* $\ell(Q[z,r])$ *are even;*

(iv) $r' \in \{x', y'\}$ *and the subpath of* P *from* q *to* r' *is an odd path;*

(v) *When* (P, Q') *is a also a feaxible pair, say* Q' *is a* (q', z)*-path with* $q \neq q'$*. Then* Q *and* Q' *intersect only at* z*.*

Proof. (i) If $\ell(P) \not\equiv 3 \pmod 4$, then it is easy to find a (0 mod 4)-cycle in $G[V(P) \cup \{z\} \cup V(G_1)]$.

(ii) Since $\ell(P) \equiv 3 \pmod 4$, without loss of generality, we may assume that $\ell(P[x,q])$ is even. Since (x,y) is a $\{2,3\}$-type in G_2, each of $P[x,q]+Q$ and $\overleftarrow{Q}+P[q,y]$ is a (1 mod 4)-path or a (2 mod 4)-path. Hence, if $\ell(P[x,q]) \equiv 0 \pmod 4$, then $\ell(P[q,y]) \equiv 3 \pmod 4$ and so $\ell(Q) \equiv 2 \pmod 4$. If $\ell(P[x,q]) \equiv 2 \pmod 4$, then $\ell(P[q,y]) \equiv 1 \pmod 4$ and $\ell(Q) \equiv 0 \pmod 4$.

By (ii), we may assume $P_e := P[x,q]$ is an even path and $P_o := P[q,y]$ is an odd path. Let $Q_1 = Q[q,r]$ and $Q_2 = Q[r,z]$. We define an (x,y)-path P^* in $G_2 - z$ as follows, see Figure 8 for an illustration: If $r' \in V(P_o)$, then let $P^* : P_e + Q_1 + R + P_o[r', y]$. If $r' \in V(P_e)$, let $P^* : P_e[x, r'] + \overleftarrow{R} + \overleftarrow{Q_1} + P_o$.

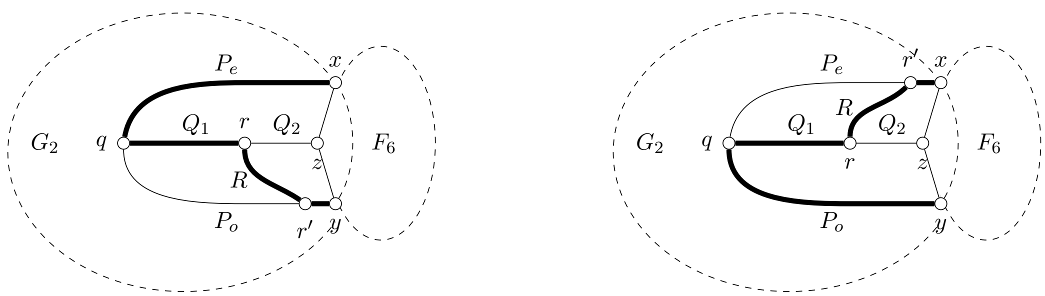


Figure 8: Illustrations for Claim 3.11, , where thick lines show the path P^*

(iii) Clearly, $\ell(Q_2)$ is even by applying (ii) to P^*. Note that $\ell(Q_1) + \ell(Q_2)$ is even by (ii), and therefore each of $\ell(Q_1)$ and $\ell(Q_2)$ is even.

(iv) Suppose to the contrary that $r' \neq y$. If $r' \neq x$, then for the path $Q^* = \overleftarrow{R} + Q_2$, (P, Q^*) is a feasible pair, which implies that $\ell(Q^*)$ is even by (ii) and so $\ell(R)$ is even. If $r' = x$, then in the fesaible pair (P^*, Q_2), since $P^*[r, y]$ is odd, $\ell(R)$ is even by (ii). Hence, in any case, $\ell(R)$ is even. By (i), $\ell(P^*)$ is odd and so the following hold from the definition of P^*:

- Suppose that $r' \in V(P_o)$. Then $\ell(P[r', y])$ is odd, and so $\ell(P[q, r'])$ is even. By comparing P and P^*, it follows that $\ell(\overleftarrow{Q_1} + R) \equiv \ell(P[q, r']) \pmod 4$ by (i). Since $\ell(P[q, r'])$ is even, $Q_1 + R + \overleftarrow{P}[r', q]$ is a (0 mod 4)-cycle, which is a contradiction.

- Suppose that $r' \in V(P_e)$. Then $\ell(P[x, r'])$ is even, and $\ell(P[r', q])$ is also even. By comparing P and P^*, it follows that $\ell(P[r', q]) \equiv \ell(R) + \ell(Q_1) \pmod 4$ by (i), and so $P[r', q] + Q_1 + R$ is a (0 mod 4)-cycle, which is a contradiction.

Thus, in any case, we reach a contradiction. Therefore $r' = y$.

(v) Without loss of generality, we assume that q is closer to x than q' along the path P. Suppose that Q and Q' intersect at some vertex other than z. Then there is a cycle $C : P[q, q'] + Q'[q', w] + \overleftarrow{Q}[w, q]$ for some vertex w, $w \neq z$. For simplicity, let $W : Q'[q'w] + \overleftarrow{Q}[w, q]$. Then $\ell(W) \equiv \ell(P[q, q']) \pmod 4$ by (i). Since C is not a (0 mod 4)-cycle, $\ell(W)$ must be odd. In addition, there is a $(V(W) - \{q, q'\}, z)$-path W', which is a subpath of $Q \cup Q'$. Say W' is a (w', z)-path. Let $Q_1 : Q[q, w'] + W'$ and $Q_2 : Q[q', w'] + W'$. Note that $\ell(Q_1) + \ell(Q_2) \equiv \ell(W) + 2\ell(W') \pmod 4$. Recall that $\ell(W)$ is odd, and each of $\ell(Q_1)$ and $\ell(Q_2)$ is even by (ii), which is a contradiction. $\square$

By Claim 3.11 (i), any (x, y)-path P in $G_2 - z$ is a (3 mod 4)-path, and so a feasible pair exists.

Claim 3.12. *There is a feasible pair (P, Q) such that $\ell(Q) = 2$ and $\deg_G(u) = 2$, where u is the middle vertex of Q. In this case, neither $\{x, q\}$ nor $\{y, q\}$ is a vertex cut of G.*

Proof. Suppose to the contrary that for every feasible pair (P, Q), the end vertices of Q do not form a vertex cut of G_2. We take such P and Q so that $\ell(Q)$ is small as possible. Say Q is a (q, z)-path. We may assume $P[x, q]$ is an even path and $P[q, y]$ is an odd path. Since $\{q, z\}$ is not a vertex cut of G_2, there is a $(V(Q), V(P))$-path R in $G_2 - \{q, z\}$. Say R is an (r, r')-path. We also denote $Q[q, r]$ and $Q[r, z]$ by Q_1 and Q_2 respectively. Recall that each $\ell(Q_1)$ of $\ell(Q_2)$ is even by Claim 3.11 (iii). In addition, by Claim 3.11 (iv), $r' = y$. Then $Q[r, z]$ has shorter than Q. We reach a contradiction to the choice of Q by considering a feasible pair $(P[x, q] + Q[q, r] + R, Q[r, z])$. Therefore, there is a feasible pair (P, Q) such that the ends q and z of Q form a vertex cut of G_2. By Lemma 3.7, Q must be P_3 and so $\ell(Q) = 2$, since Q always has even length (q, z)-path by Claim 3.11 (ii).

Let $P[x, q]$ and $P[q, y]$ have different parity by Claim 3.11 (ii). We denote an even path among $P[x, q]$ and $P[q, y]$ by P_e, and the other path by P_o. Suppose to the contrary that either $\{x, q\}$ or $\{q, y\}$ is a vertex cut of G, say we call this vertex cut $Y = \{y_1, y_2\}$. Note that the connected component of $G - Y$ containing the vertex z has more than 6 vertices. Thus Lemma 3.7 implies that there is a connected component D of $G - Y$ such that $H = G[V(D) \cup Y]$ contains P_e or P_o and H is one of K_3, F_3, F_4, and F_6. If H is one K_3, F_4, and F_6, then there are two (y_1, y_2)-paths of different parity and it contradicts Claim 3.11 (i) and (ii). Thus H is F_3 and so $\ell(P[y_1, y_2]) = 2$. Recall that $P[y_1, y_2]$ is P_e or P_o. Since $\ell(Q) = 2$, $\ell(P_e) \equiv 0 \pmod 4$ and $\ell(P_o) \equiv 3 \pmod 4$ by Claim 3.11 (ii), which is a contradiction. Hence neither $\{x, q\}$ nor $\{q, y\}$ is a vertex cut of G. $\square$

We take a feasible pair (P, Q) satisfying the conditions of Claim 3.12. Say Q is a (q, z)-path. Without loss of generality, assume that $P[x, q]$ is an even path denoted by P_e, and $P[q, y]$ is an odd path denoted by P_o. Note that since $\ell(Q) = 2$, we have $\ell(P_e) \equiv 0$, $\ell(P_o) \equiv 3 \pmod 4$ by Claim 3.11 (ii). Moreover, by Claim 3.12, both $G - \{x, q\}$ and $G - \{y, q\}$ are connected. Thus there is a $(V(P_e), V(P_o) \cup V(Q))$-path Q_1 in $G - \{x, q\}$, say Q_1 is a (q_1, t_1)-path, and there is a $(V(P_o), V(P_e) \cup V(Q))$-path Q_3 in $G - \{y, q\}$, say Q_3 is a (q_3, t_3)-path.

Claim 3.13. *$t_1 = t_3 = z$.*

Proof. Suppose to the contrary that $t_1 \notin \{z, y\}$. Since the ends of Q form a vertex cut of G, $t_1 \in V(P_o) \setminus \{y, q\}$. Let $P^* = P[x, q_1] + Q_1 + P[t_1, y]$, $Q_1^* : P[q_1, q] + Q$, and $Q_2^* : \overleftarrow{P}[t_1, q] + Q$. By applying Claim 3.11 (v) to a feasible pairs (P^*, Q_1^*) and (P^*, Q_2^*), we reach a contradiction since Q_1^* and Q_2^* share a vertex other than z. Thus $t_1 \in \{z, y\}$. Similarly, we can show that $t_3 \in \{z, x\}$.

Suppose to the contrary that $t_3 = x$. For a feasible pair $(\overleftarrow{Q_3} + P[q_3, y], \overleftarrow{P}[q_3, q] + Q)$, it follows that $\ell(P[q_3, q])$ is even by Claim 3.11 (ii), and so $P[x, q_3]$ is an even path. In addition, $\ell(P[x, q_3]) \equiv \ell(Q_3) \pmod 4$ by Claim 3.11 (i). Then $P[x, q_3] + Q_3$ becomes a (0 mod 4)-cycle, which is a contradiction. Therefore $t_3 = z$. Suppose to the contrary that that $t_1 = y$. Let $P^* : P[x, q_1] + Q_1$. For feasible pairs $(P^*, P[q_1, q] + Q)$ and $(P^*, P[q_1, q_3] + Q_3)$, it follows that $\ell(P[q_1, q] + Q) \equiv \ell(P[q_1, q_3] + Q_3) \pmod 4$ by Claim 3.11 (ii). Then $\ell(Q) \equiv \ell(P[q, q_3] + Q_3) \pmod 4$. Since $\ell(Q)$ is even, $P[q, q_3] + Q_3$ and Q form a (0 mod 4)-cycle, which is a contradiction. Thus $t_1 = z$. $\square$

By Claims 3.11 (v) and 3.13, Q_1 and Q_3 intersect only at z. We relabel Q and q by Q_2 and q_2, respectively. See Figure 9 for an illustration.

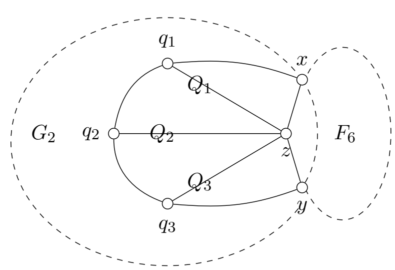


Figure 9: An illustration for Lemma 3.2

Note that Q_2 and $P[q_2, y] + yz$ are even paths, and so by Lemma 2.5 (i), $\overleftarrow{Q_1} + P[q_1, q_2]$ has odd length not to have an even theta graph. Then $P[q_1, q_2]$ has odd length by Claim 3.11 (ii), and so $P[x, q_1]$ also has odd length.

By considering Q_1, $\overleftarrow{P}[q_1, x] + xz$, it holds that $P[q_1, q_3] + Q_3$ has length odd by Lemma 2.5 (i). Then $P[q_1, q_3]$ has odd length and so $P[q_2, q_3]$ has even length, and $P[q_3, y]$ has odd length. Then Q_3, $P[q_3, y] + yz$, $\overleftarrow{P}[q_3, q_2] + Q_2$ are all even paths, they form an even theta graph, which is a contradiction to Lemma 2.5 (i). $\square$

3.2 Proof of Lemma 3.1

Lemma 3.14. *There are at most two triangles in G.*

Proof. Suppose, to the contrary, that there are three distinct triangles T_1, T_2, and T_3 in G. Let $V(T_i) = \{p_i, q_i, t_i\}$ for each $i \in \{1, 2, 3\}$. Then, by Proposition 2.8, we may assume that there are two vertex-disjoint (p_1, p_2)-path P and (q_1, q_2)-path Q such that $p_3t_3 \in E(P)$ and $q_3 \in V(Q)$. For simplicity, we define sets as follows (See Figure 10 for an illustration):

$$A_1 = V(P[p_1, p_3]) - \{p_3\}, \quad A_2 = V(Q[q_1, q_3]) - \{q_3\}, \quad A = \{t_1\} \cup A_1 \cup A_2,$$
$$B_1 = V(P[t_3, p_2]) - \{t_3\}, \quad B_2 = V(Q[q_3, q_2]) - \{q_3\}, \quad B = \{t_2, t_3\} \cup B_1 \cup B_2.$$

Note that A_i or B_i might be empty if a vertex of T_3 coincides with a vertex in T_1 or T_2. (For example, $A_1 = \emptyset$ if $p_1 = p_3$.)

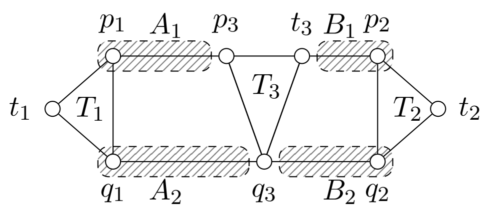


Figure 10: An illustration for Lemma 3.14, where the shaded parts represent A_1, A_2, B_1, and B_2.

Suppose that $\{p_3, q_3\}$ is a vertex cut of G. Then $G - \{p_3, q_3\}$ has exactly two components S_1 and S_2 by Lemma 3.2, one of which induces K_3 together with $\{p_3, q_3\}$, since K_3 is the only graph that two vertices of a vertex cut are adjacent. But we can observe that $G[V(S_i) \cup \{p_2, q_3\}]$ has at least four vertices, since two triangles share at most one vertex. Therefore $\{p_3, q_3\}$ is not a vertex cut of G. Hence, there is an (A, B)-path R, say R is an (a, b)-path. We divide the proofs into following 6 possible cases, and one can check the following cases cover all possibilities:

(1) $a \in A_1$ and $b \in B_1 \cup \{t_3\}$;

(2) $a \in A_2$ and $b \in B_2$;

(3) $a \in A_2$ and $b \in B_1$ (the case where $a \in A_1$ and $b \in B_2$ is similar.);

(4) $a = t_1$ and $b \in B_1 \cup B_2$ (the case where $a \in A_1 \cup A_2$ and $b = t_2$ is similar.);

(5) $a = t_1$, $b \in \{t_2, t_3\}$;

(6) $a \in A_2$, $b = t_3$.

In each of the cases (1)$\sim$(3), we reach a contradiction since it is not difficult to find a cycle C or a theta graph Θ satisfying the condition of Lemma 2.7 (i) or (ii), where Figure 11 shows such C or Θ with thick lines. Note that in each case, the vertices of T_1, T_2, T_3 not on C or Θ are distinct.

For the case (4), we can find three vertex-disjoint path between $V(T_1)$ and $V(T_3)$, two of them are $P[p_1, p_3]$ and $Q[q_1, q_3]$ and the other one is the path with thick lines in first two figures of Figure 12. For the case (5), if $b = t_2$, then P, Q, R are three vertex-disjoint paths between $V(T_1)$ and $V(T_2)$, and, if $b = t_3$, then $P[p_1, p_3]$, $Q[q_1, q_3]$, R are three vertex-disjoint paths between $V(T_1)$ and $V(T_3)$. We reach a contradiction to Lemma 2.7 (iii).

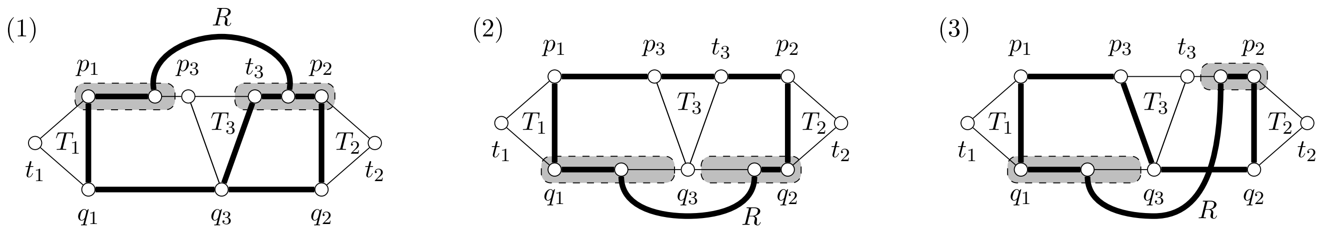


Figure 11: Illustrations for the cases (1)∼(3), where the shaded parts represent the parts containing the end vertices of R: For the case (1) the thick lines show a theta graph Θ satisfying the condition in Lemma 2.7 (ii). For the cases (2) and (3), the thick lines show a cycle C satisfying the condition in Lemma 2.7 (i)

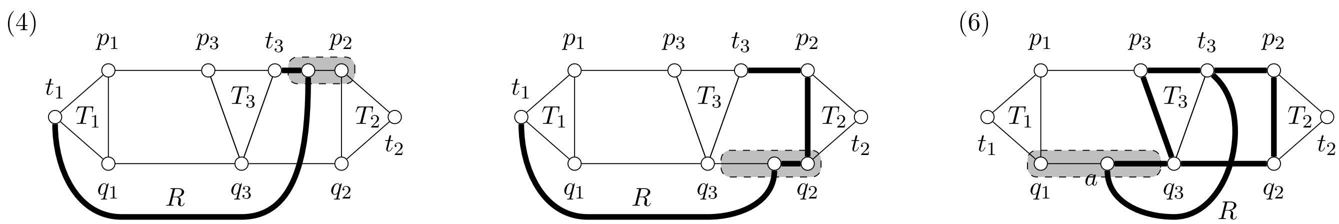


Figure 12: Illustrations for the cases (4) and (6), where the shaded part represent the part containing an end vertex of R: For the case (4), the thick lines show a $(V(T_1), V(T_3))$-path which is vertex-disjoint with $P[t_1, t_3]$ and $Q[q_1, t_3]$. For the case (6), the thick lines show an even theta graph.

For the case (6), that is, $a \in A_2$ and $t_3 = b$, we will reach a contradiction to Lemma 2.6 (i) by showing that there is an even (t_3, q_3)-theta graph. See the last figure of Figure 12. It is clear that $t_3p_3q_3$ is a path of length two. By Lemma 2.7 (iii), $P[t_3, p_2] + p_2q_2 + \overleftarrow{Q}[q_2, q_3]$ is a (0 mod 4)-path, and both $\overleftarrow{P}[p_3, p_1] + p_1q_1 + Q[q_1, a] + Q[a, q_3]$ and $\overleftarrow{P}[p_3, p_1] + p_1q_1 + Q[q_1, a] + R$ are (0 mod 4)-paths. Therefore $Q[a, q_3]$ and R have the same length modulo 4, and so $\overleftarrow{R} + Q[a, q_3]$ is an even (t_3, q_3)-path. It completes the proof. □

Lemma 3.15. *There are at most five* 5*-faces in* G.

Proof. Suppose to the contrary that there are two vertex-disjoint 5-cycles C_1 and C_2. Let X be a smallest $(V(C_1), V(C_2))$-separating set. Then $|X| \geq 2$ since G is 2-connected. If $|X| \geq 3$, then by Theorem 2.1, there are three vertex-disjoint $(V(C_1), V(C_2))$-paths, which is a contradiction to Lemma 2.6 (ii). Thus $|X| = 2$. By Lemma 3.4 (i), $G - X$ has exactly two connected components S_1 and S_2. Then each of $G[V(S_1) \cup X]$ and $G[V(S_2) \cup X]$ has either C_1 or C_2 as a subgraph, which is a contradiction to Lemma 3.2 since there is no 5-cycle in K_3, F_3, and F_4. Therefore G has no vertex-disjoint 5-cycles. Since G has no (0 mod 4)-cycle, each two 5-cycles intersect at a vertex or a path of length 2.

Claim 3.16. *There are no three* 5*-faces such that every two of them intersect at a vertex.*

Proof. Suppose that there are three 5-faces C_1, C_2, C_3 such that every two of them intersect at a vertex. By Lemma 2.7 (iv), $V(C_1) \cap V(C_2) \cap V(C_3) = \{v\}$ for some vertex v. Let $B_i = C_i - v$ for each $i \in \{1, 2, 3\}$.

Suppose to the contrary that $V(C_k)$ separates B_i and B_j for some i, j, k. We may say $i = 1$, $j = 2$ and $k = 3$. Since G is embedded on a plane and C_3 is a face, there is a vertex $u \in V(B_3)$ such that for two sections S_1 and S_2 of B_3 divided by u, for every connected component H of $G - (V(C_1) \cup V(C_2) \cup V(C_3))$, either H does not have a neighbor in $V(S_2) \setminus \{u, v\}$ or H does not have a neighbor in $V(S_1) \setminus \{u, v\}$. Therefore $X = \{u, v\}$ is a vertex cut of G. By Lemma 3.2, $G - X$ has exactly two connected components. For each connected component S of $G - X$, $G[S \cup X]$ contains C_1 or C_2 and so it has more than 5 vertices, which contradicts to Lemma 3.2.

Then there is a $(V(B_i), V(B_{i+1}))$-path Q_i in $G - V(C_{i+2})$ for each $i \in \{1, 2, 3\}$. By Lemma 2.6 (iii), we may assume that Q_1 and Q_2 intersect at a vertex. Then $Q_1 \cup Q_2$ makes a connected subgraph H of $G - v$ such that $V(H) \cap V(C_i) \neq \emptyset$ and $V(H) \cap E(C_i) = \emptyset$ for each $i \in \{1, 2, 3\}$, which is a contradiction to Lemma 2.6 (iv). □

Claim 3.17. *There are no three* 5*-faces such that every two of them intersect at a path of length* 2.

Proof. Suppose that there are three 5-faces C_1, C_2, and C_3 such that every two of them intersect at a path of length 2. Let uvw be the path of length 2 in which C_1 and C_2 intersect. We also let $C_1 : u, v, w, w_1, u_1, u$ and $C_2 : u, v, w, w_2, u_2, u$. Since C_1 and C_3 intersect at a path of length 2, without loss of generality, we may assume that $w \in V(C_3)$. Then it holds that $C_3 : ww_1u_1u_2w_2w$. See Figure 13 for an illustration.

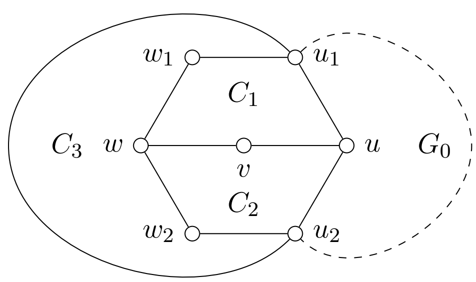


Figure 13: An illustration for Claim 3.17.

Note that G is a plane graph, and C_1, C_2, and C_3 are faces. Then $G_0 = G - \{v, w, w_1, w_2\}$ is 2-connected. As G_0 has no (0 mod 4)-cycle, by the minimality of G, $e(G_0) \leq \frac{3|V(G_0)|-1}{2}$. Then $e(G) = e(G_0) + 6 \leq \frac{3|V(G_0)|-1}{2} + 6 = \frac{3n-1}{2}$, which is a contradiction. □

Note that the Ramsey number $r(3, 3) = 6$. If G has at least six 5-faces, then three of them either pairwise intersect at a vertex or pairwise intersect at a path of length two, contradicting Claim 3.16 or Claim 3.17. □

4 Extremal graphs

In this section, we discuss how to construct extremal 2-connected graphs without (0 mod 4)-cycles. We define the *gap* function $gap(G)$ by $gap(G) = (3n-1) - 2e(G)$ for an n-vertex graph G. Note that our main theorem says that if G has no (0 mod 4)-cycle then $gap(G) \geq 0$.

Proposition 4.1. *Let G' be the graph defined by $(G;x,y) \uplus (F_i;a,b)$ for a graph $(G;x,y)$ and a graph $(F_i;a,b)$ in Figure 1. Then $gap(G') < gap(G)$. More precisely,*

$$gap(G') = \begin{cases} gap(G) - 1 & \text{if } F_i = F_3 \\ gap(G) - 2 & \text{if } F_i \in \{F_4, F_6\} \\ gap(G) - 3 & \text{if } F_i = F_9, \end{cases} \qquad \text{and} \qquad gap(G') \leq \begin{cases} gap(G) - 1 & \text{if } F_i = F_7 \\ gap(G) - 3 & \text{if } F_i = F_8. \end{cases}$$

Proof. Let $|V(G')| - |V(G)| = d_v$ and $e(G') - e(G) = d_e$. Then $gap(G') = gap(G) + 3d_v - 2d_e$. If there is no edge between a and b in F_i, then $3d_v - 2d_e = 3|V(F)| - 2e(F) - 6$. If there is an edge between a and b in both G and F_i, then the edge might not be counted in d_e and so $3|V(F)| - 2e(F) - 6 \leq 3d_v - 2d_e \leq 3|V(F)| - 2e(F) - 4$. Thus the proposition holds. □

Proposition 4.2. *For a graph $(G;x,y)$, let G' be the graph defined by $(G;x,y) \uplus P_4$ or defined by $(G;x,y) \uplus (F_6;b,c)$, where c is a neighbor of b. Then $gap(G') = gap(G)$.*

Proof. If G' is $(G;x,y) \uplus P_4$, then $|V(G')| - |V(G)| = 2$ and $e(G') - e(G) = 3$, and so $gap(G') = gap(G) + 0$. If G' is $(G;x,y) \uplus (F_6;b,c)$, then $|V(G')| - |V(G)| = 4$ and $e(G') - e(G) = 6$, and so $gap(G') = gap(G) + 0$. □

For simplicity, we let $L_i = \{i, i+1\}$ for $i \in \{0,1,2\}$, and $L_3 = \{0,3\}$. We say L_0 and L_1 are matched to each other, and L_2 and L_3 are matched to each other. Recall that (a,b) is an L_3-type in F_6, an L_0-type in F_7, an L_1-type in F_8, and an L_2-type in F_4 and F_9. Moreover, if c is a neighbor of b in F_6, then (b,c) is L_0-type in F_6.

Let $\sigma : G_0, \ldots, G_k$ be a sequence of graphs without (0 mod 4)-cycles. We call σ a *gap-reducing sequence* if $gap(G_s) \geq gap(G_{s+1})$ for each $0 \leq s \leq k-1$, and σ is strict if if $gap(G_s) > gap(G_{s+1})$ for each $0 \leq s \leq k-1$.

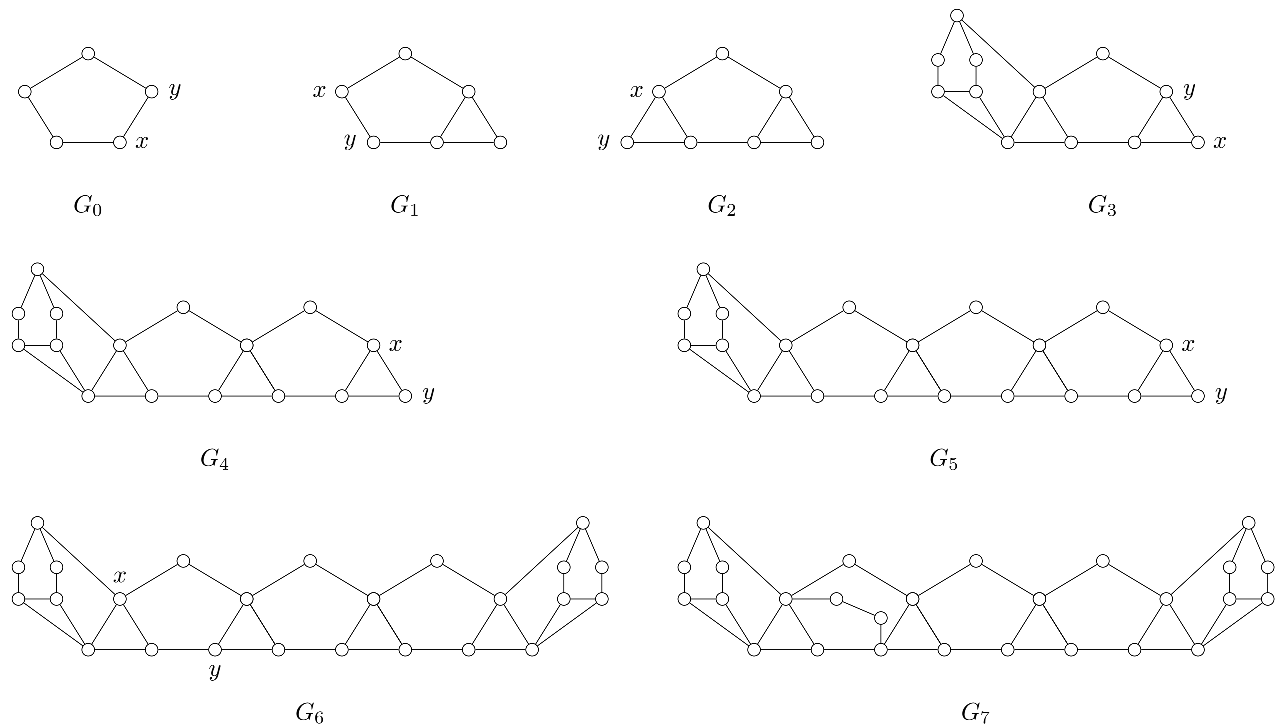
Now we explain a way to obtain a (strict) gap-reducing sequence $\sigma : G_0, \ldots, G_k$ of graphs without (0 mod 4)-cycles. We take a graph G_0 without (0 mod 4)-cycles. When we have G_s, G_{s+1} is obtained from one of the following procedures (R1), (R2), and (R3).

(R1) Find two vertices x and y such that (x,y) is an L_i-type in G for some $i \in \{0,1,2,3\}$. Let L_j be the set matched to L_i. Then we define by $G_{s+1} = (G_s;x,y) \uplus (H;a',b')$, where (a',b') is an L_j-type in H and $H = K_3$ or $H = F_t$ in Figure 1.

(R2) Find two vertices x and y such that x and y are not connected by a (1 mod 4)-path in G. Let L_j be the set matched to L_i. Then we define by $G_{s+1} = (G_s;x,y) \uplus P_4$.

(R3) Find two vertices x and y such that (x, y) is an L_1-type in G. Then we define by $G_{s+1} = (G_s; x, y) \uplus (F_6; b, c)$, where c is a neighbor of b in Figure 1.

By Propositions 4.1 and 4.2, it is clear that any sequence obtained from this way is a gap-reducing sequence of graphs without (0 mod 4)-cycles. If we conduct only (R1), then the resulting sequence is a strict gap-reducing sequence. In addition, if G_0 is 2-connected, then every graph in the sequence is also 2-connected. Figure 14 shows some example, which starts with a 5-cycle G_0.



n	G_0	$\to$	G_1	$\to$	G_2	$\to$	G_3	$\to$	G_4	$\to$	G_5	$\to$	G_6	$\to$	G_7
$gap(G_i)$	4	(R1)	3	(R1)	2	(R1)	1	(R3)	1	(R3)	1	(R1)	0	(R2)	0

Figure 14: A gap-reducing sequence $\sigma : G_0, \ldots, G_7$ by the procedures (R1), (R2) and (R3)

As one can observe from the example in Figure 14, we can construct infinitely many extremal 2-connected graphs without (0 mod 4)-cycles. In addition to (R2) and (R3), we can find other graph constructions that preserve the value of the gap function, and these can be used to produce extremal examples with diverse structures.

In [9], the authors construct an n-vertex graph G without (0 mod 4)-cycles such that $e(G) = \left\lfloor \frac{19(n-1)}{12} \right\rfloor$ as follows. They denote F_8 and $(F_7; a, b) \uplus (F_8; a, b)$ by L_8 and L_{13}, respectively. Extremal examples for general graphs are obtained by performing one-vertex identifications on multiple copies of L_8 and L_{13}. According to our analysis in this paper, their extremal examples are essential in a certain sense. If n is sufficiently large, then

$$\left\lfloor \frac{19(n-1)}{12} \right\rfloor > \left\lfloor \frac{3n-1}{2} \right\rfloor .$$

Thus, for an n-vertex graph G with exactly $\left\lfloor \frac{19(n-1)}{12} \right\rfloor$ edges, if it does not have (0 mod 4)-cycle, then it cannot be 2-connected, and therefore must have a cut-vertex x. If $G-x$ contains a large block B with n_b vertices, then $e(B) \leq \frac{3n_b-1}{2}$, and hence

$$e(G) \leq \frac{3n_b - 1}{2} + \frac{19(n - n_b - 2)}{12} < \left\lfloor \frac{19(n-1)}{12} \right\rfloor.$$

This implies that no block of G can contain a large number of vertices, when G is an n-vertex graph G with exactly $\left\lfloor \frac{19(n-1)}{12} \right\rfloor$ edges such that G does not have (0 mod 4)-cycle.

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B: 양식 통일해주세요. 해당 논문이 인트로에서 말하는 논문이 맞는지 하나씩 다운받아서 확인해주세요. 하는 김에 모든 레퍼런스는 저자이름과 연도로 이름 라벨링해서 우리 오버리프 Reference 폴더에 pdf를 넣어주세요.

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Appendix

We provide a sketch of proof of the Proposition 2.3. Some cases are omitted, as they are straightforward yet tedious to check. Due to the large number of such cases, verification is provided via code. See the supplementary repository: `https://github.com/Homoon-ryu/2con_0mod4-cycle_free`.

Proof of Proposition 2.3. For simplicity, let $t(n)$ be the target function defined as follows:

$$t(n) = \begin{cases} \frac{3n-4}{2} & \text{if } L = \{0,3\} \text{ and } n \leq 6; \\ \frac{3n-3}{2} & \text{if } L = \{0,1\} \text{ and } n \leq 7; \\ \frac{3n-2}{2} & \text{if } L = \{1,2\} \text{ and } n \leq 8; \\ \frac{3n-3}{2} & \text{if } L = \{2,3\} \text{ and } n \leq 9. \end{cases}$$

We have the following Table 2, where each cell in the last four rows gives the target upper bound for $e(H)$. That is, if $t(n)$ is an integer, each cell gives $t(n) - 1$ except for the cases of equality parts, and each cell is a value for $t(n)$ otherwise.

If $n \leq 4$, then one can easy to check the inequality holds. Suppose that $n = 5$ and $e(H) = 6$. If H is Hamiltonian, then the chord of a Hamiltonian cycle makes a cycle of length four. Thus H is not a Hamiltonian, and so the longest cycles of H is a triangle. Since $e(H) = 6$, H has at least two triangles sharing one vertex. Then (x, y) becomes a $\{2, 3, 4\}$-type, which is a contradiction. Thus the column for $n = 5$ works. Now we assume that $n \geq 6$.

Note that (x, y) is an L-type with $|L| = 2$ in H. If there is no (x, y)-path of length (ℓ mod 4) for some $\ell \in L$, then it is easy to see that H is a bipartite, so $e(H) \leq \frac{3n-6}{2}$ by Lemma 2.5 (iii). Therefore we assume that there are both an (x, y)-path of length (ℓ_1 mod 4) and an (x, y)-path of length (ℓ_2 mod 4) in H, when $L = \{\ell_1, \ell_2\}$. To complete the proof, from Table 2, it remains to show the following:

n	3	4	5	6	7	8	9
$\left\lfloor \frac{19(n-1)}{12} \right\rfloor$	3	4	6	7	9	11	12
$L = \{0,3\}$	$\frac{5}{2}$	3	$\frac{11}{2}$	7 $[F_6]$			
$L = \{0,1\}$	2	$\frac{9}{2}$	5	$\frac{15}{2}$	9 $[F_7]$		
$L = \{1,2\}$	$\frac{7}{2}$	4	$\frac{13}{2}$	7	$\frac{19}{2}$	11 $[F_8]$	
$L = \{2,3\}$	2	$\frac{7}{2}$	5	$\frac{15}{2}$	8	$\frac{21}{2}$	12 $[F_9]$

Table 2: The target values $t(n)-1$ or $t(n)$, where the graphs in squared brackets are only tight examples (up to reversing-equivalent relation)

(A) Suppose that $L = \{0,3\}$. For $n = 6$, if $e(H) = 7$ then H is isomorphic to F_6.

(B) Suppose that $L = \{0,1\}$. For $n = 7$, if $e(H) = 9$ then H is reversing-equivalent to F_7.

(C) Suppose that $L = \{1,2\}$. For $n = 8$, if $e(H) = 11$ then H is reversing-equivalent to F_8.

(D) Suppose that $L = \{2,3\}$. For $n = 7$, $e(H) \leq 8$.

(E) Suppose that $L = \{2,3\}$. For $n = 9$, if $e(H) = 12$ then H is reversing-equivalent to F_9.

We use induction on n, that is, all cases of Table 2 hold when $|V(H)| < n$ for some $n \geq 6$. First, we suppose that it is one of the cases (A), (B), and (C). If H has a pendent edge e and $L = \{i, i+1\}$, then e must be adjacent to x or y, say $e = xz$, then $H - x$ has less vertices and (z, x) satisfies the same condition of the proposition so that (z, x) is an $\{i-1, i\}$-type in $H - z$, which implies that $e(H) \leq e(H - z) + 1$. It is a contradiction since $e(H) \geq e(H - z) + 2$ from Table 2. Thus $\delta(H) \geq 2$.

Suppose that x and y is not on a same cycle C. Since $\delta(H) \geq 2$, there is a cycle C_x containing x and a cycle C_y containing y. Since $n \leq 8$, one of C_x or C_y is a triangle and the other has length at most six, say C_x is a triangle. Since (x, y) is an L-type for some $|L| = 2$, C_y must be an even cycle and so C_y has length 6, which is the case (C). But $L = \{1,2\}$, which is impossible. Therefore, x and y are on a same cycle C. It is also easy to see that a longest cycle containing both x and y is not a triangle, and so we assume that $\ell(C) \geq 5$.

Suppose the case (A), that is, $L = \{0,3\}$, $n = 6$ and $e(H) = 7$. If H is Hamiltonian, then it has only one chord, which implies that H is isomorphic to F_6. Suppose that H is not Hamiltonian. Thus C is a 5-cycle. The vertex z not on C has two neighbors in C. In any case, H becomes a Hamiltonian or H has a 4-cycle, which is a contradiction.

Suppose the case (B), that is, $L = \{0,1\}$, $n = 7$ and $e(H) = 9$. Since $L = \{0,1\}$, C must be a 6-cycle. This also implies H is not Hamiltonian. Let $C : v_0, v_1, \ldots, v_5, v_0$. By Table 2, the $H[V(C)]$ has 6 vertices and at most seven edges. For the vertex z in $V(H) - V(C)$, z has at most two neighbors in C, since H is not Hamiltonian and H has no 4-cycle. Thus $H[V(C)]$ has exactly 7 edges, and we may assume that

z is adjacent to v_0, v_3. Since every chord of C must be the form of $v_i v_{i+1}$, the only possible way is that H is isomorphic to F_7.

Suppose the case (C), that is $L = \{1, 2\}$, $n = 8$ and $e(H) = 11$. The cycle C must be a 7-cycle, since (x, y) is a $\{1, 2\}$-type in H. Suppose that x and y are on a 7-cycle C. Then the vertex z not on C has at most two neighbors in C, since any cycle containing both x and y has length 3 or 7. Since $e(H[V(C)]) \leq 9$ by the previous argument, $e(H) = 11$ implies that $e(H[V(C)]) = 9$ and z has exactly two neighbors in C, which implies that H is reversing-equivalent to F_8. (The details could be checked in `https://github.com/Homoon-ryu/2con_0mod4-cycle_free`.)

We suppose that it is one of the cases (D), (E). Then $L = \{2, 3\}$ and so every cycle containing both x and y has length 5, 6, 8, or 9. We omit the proof here due to tedious checking of a large number of cases. See `https://github.com/Homoon-ryu/2con_0mod4-cycle_free` for checking detail. □

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