Model Categories and Derived Functors

§1. Introduction

- Model category:
  - A stage to do abstract Homotopy theories
  - Introduced by Daniel Quillen
  - References: - 《代数K-理论》 by 黎景辉
    - Homotopical Algebra by Daniel Quillen
    - Model categories by Mark Hovey
    - Categorical Homotopy Theory by Emily Riehl
    - (Easy to read) Homotopy Theories and Model Categories by W. G. Dwyer & J. Splanski.

§2 Motivating Examples in Topological Spaces

- Def.: A morphism $f: X \to Y$ in $\mathcal{T}op$ is called a weak homotopy equivalence if $\forall\, n \geq 0$, $f_*: \pi_n(X) \to \pi_n(Y)$ is an isomorphism.

  A --- is called a homotopy equivalence if $\exists\, g: Y \to X$ s.t. $fg \simeq id_Y$, $gf \simeq id_X$.

- A homotopy equivalence is a weak equivalence.

- Prop. ("2 out-of 3"): Let $f: X \to Y$, $g: Y \to Z$ be morphisms in $\mathcal{T}op$. If 2 of f, g, gf are weak-equivalences, then the remaining one is also a ~

Proof.: Only consider the case that f, gf are ~.

Consider the commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} \pi_n(X,x) \arrow[rr, "f_*"] \arrow[dr, "(gf)_*"'] & & \pi_n(Y,fx) \arrow[dl, "g_*"] \\ & \pi_n(Z, gfx) & \end{tikzcd}$$

since f_* and $(gf)_*$ are isomorphisms, g_* is an isomorphism.
However, not every $y \in Y$ are in $f(X)$. But since $f_*: \pi_0(X) \to \pi_0(Y)$ is an isomorphism, $\exists\, x \in X$ and path $\bar{\sigma}: I \to Y$ s.t. $\bar{\sigma}(0) = fx$, $\bar{\sigma}(1) = y$.
We obtain

$$\begin{tikzcd} \pi_n(Y, y) \arrow[r, "h_1"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_n(Z, gy) \arrow[d, "h_3"] \\ \pi_n(Y, fx) \arrow[r, "h_4"'] & \pi_n(Z, gfx) \end{tikzcd}$$

h_4 is an isom. as shown above, h_2, h_3 also isoms. (as inv. of $\bar{\sigma}_*$ and $(g \circ \bar{\sigma})_*$). Hence h_1 is an isom, which is induced by g. $\boxed{\equiv}$

· Many other properties ---

Some concepts

· ~~Recall from~~ <u>algebraic topology</u>:

· <u>Fibrations</u> (Hurewicz): A map $p: X \to Y$ satisfying the <u>Homotopy Lifting Property (HLP)</u> for any topological space W: $\forall$ commutative diagram

$$\begin{tikzcd} W \times \{0\} \arrow[r, "h_0"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ W \times I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd},$$

$\exists$ a continuous map $h: W \times I \to X$ s.t. the new diagram also commutes.

- <u>Emp</u>: A <u>covering</u> $p: E \to B$ is a fibration.

~~· Serre Fibration~~ ---

· <u>Cofibration</u>: A map $f: A \to B$ satisfying the <u>Homotopy Extension Property (HEP)</u> for any topological space Z: $\forall$ commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Z^I \arrow[d, "p_0"] \\ B \arrow[r, "h_0"'] \arrow[ur, dashed, "h"] & Z \end{tikzcd}$$

(Z^I: continuous maps from I to Z, compact-open top.)

$\exists$ a continuous map $h: B \to Z^I$ s.t. the new diagram also commutes, Z

- Emp. The inclusion $S^{n-1} \hookrightarrow D^n$. $\Downarrow$ cofib preserved push-outs & compositions

  $A \hookrightarrow X$ for relative CW complex (A, X).

§2 Definition and Examples of Model Categories

$\mathcal{C}$ arbitrary cat.

· Def.: A morphism f is said to be a retract of g iff $\exists$ a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B \end{tikzcd}$$

Rmk: The name "retract" also originates from algebraic top.

· Def.: Given a commutative diagram of the following form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

if $\forall$ diagram $\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$ a lifting exists

a lifting in the diagram is a map $h: B \to X$ such that the new diagram commutes. i (resp. p) is said to have a left (resp. right) lifting property w.r.t. p (resp. i)

· Def.: A (closed) model category is a category $\mathcal{C}$ with three distinguished classes of maps (i) weak equivalences (ii) fibrations (iii) cofibrations each of which is closed under compositions and contains all identity maps (a map which is both a fibration / cofibration and a weak equivalence is called a trivial fibration / cofibration), satisfying the following axioms:

(MC1) $\mathcal{C}$ admits finite limits and colimits

(MC2) "2-of-3" If two of f, g, gf are weak-equivalences, then the third also is.

(MC3) If f is a retract of g and g is a fibration / cofibration / weak-equivalence, then f also is.

(MC 4) Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd},$$

a lift exists in this diagram in either of the two following situations (i) i is a cofibration and p is a trivial fibration; (ii) i is a trivial cofibration and p is a fibration.

(MC 5) Any morphism f admit decompositions

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd},$$

where p is a fibration, i is a trivial cofibration, j is a cofibration and q is a trivial fibration.

Examples

- Classical Quillen Model structure on $\mathcal{T}op$:
  - weak equilences: weak homotopy equivalences;
  - fibrations: Serre fibrations, which are maps having the right lifting property w.r.t all inclusions of the form $i_0: D^n \hookrightarrow D^n \times I$, $D^n \longmapsto D^n \times \{0\}$;
  - cofibrations: retracts of relative CW complexes;
- Hurewicz (or Strøm Model Structure on $\mathcal{T}op$:
  - weak equivalences: homotopy equivalences;
  - fibrations: Hurewicz fibrations;
  - cofibrations: closed Hurewicz cofibrations;
- R an associative ring with unit, Mod_R the category of left R-modules. Ch_R the category of nonnegatively graded chain complexes of R-modules. Then Ch_R admits a model structure with:
  - weak equivalences: all $f: M \to N$ that induces isomorphisms,

$H_k(M) \longrightarrow H_k(N)$ $(k \geq 0)$;

- fibrations: all $f: M \to N$ s.t. $\forall k \geq 0$ the map $f_k: M_k \to N_k$ is an epi.
- cofibrations: all $f: M \to N$ s.t. $\forall k \geq 0$ the map $f_k: M_k \to N_k$ is a mono with a projective R-module as its cokernel.

- Joyal model structure on sSet / Bergner model structure on sCat
  (many other model str.) — presents $(\infty,1)$-cat; "Quillen equivalent" by Lurie
- Tabuada model structure on dg-cat_k. e.g. Quillen-Kan ~

§ 3 Homotopy & localization

In the beginning, we've mentioned that model categories provide us a stage to do abstract homotopy theories. So we need to define the homotopy concept in an abstract way. However, since I'm not able to finish doing this within the time limit, so I just provide some flavor of it. For the whole story, see Ch 4 and Ch 5 of Dwyer & Splanski.

Recall in Top, in order to define homotopy we need cylinders $W \times I$ and path spaces Z^I. Here we define them in model categories.

- Def: Let $\mathcal{C}$ be a (closed) model category. A and X are objects in $\mathcal{C}$. A cylinder object for A is an object $A \wedge I$ of $\mathcal{C}$ together with a diagram $A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{\sigma} A$ which factors the folding map $A \amalg A \xrightarrow{id_A + id_A} A$ (here i is a cofibration and σ is a weak equivalence)

$$\begin{tikzcd} A \amalg A \arrow[dr, "id_A+id_A"] & A \arrow[l, "i_2"'] \arrow[d, "id_A"] \\ A \arrow[u, "i_1"] \arrow[r, "id_A"'] & A \end{tikzcd}$$

- Rmk: (1) In Top we have

$$\begin{tikzcd} A \amalg A \arrow[d, "\nabla"'] \arrow[dr, "i"] & \\ A & A \times I, \arrow[l, "\sigma"] \end{tikzcd}$$

where i send the first A in $A \amalg A$ to $A \times \{0\}$ and the second to $A \times \{I\}$, σ is the projection from $A \times I$ to A, ∇ is the co-diagonal map

(2) more concepts: good cylinder object, very good cylinder object, left homotopy ...

(3) $A \wedge I$ is just a notation. I is not defined in $\mathcal{C}$.

· <u>Def</u>: A <u>path object</u> for X is an object X^I of $\mathcal{C}$ together with a diagram: $X \xrightarrow{s} X^I \xrightarrow{p} X \times X$, which factors the diagonal map $(id_X, id_X): X \to X \times X$ (here s is a weak equivalence and p is a fibration)

· <u>Rmk</u>: (1) In $\mathcal{T}op$ we have

$$\begin{tikzcd} X^I & X \arrow[l, "s"'] \arrow[d, "\Delta"] \\ & X \times X \arrow[from=1-1, "p"'] \end{tikzcd},$$

where $s(b): I \to X$ sends all $t \in I$ to b, $p(\ell) = (\ell(0), \ell(1))$ for $\ell \in S^I$, Δ the diagonal map.

(2) more concepts: <u>good path object</u>, <u>very good path object</u>, <u>right homotopy</u> ---

(3) X^I just a notation.

(* definitions on page 8.)

· We can then define <u>fibrant and cofibrant objects</u>. Let $f, g: A \to X$ be maps. If A is cofibrant and B is fibrant, then the concepts left homotopy and right homotopy coincides, and then we can def. when f and g are "<u>homotopic</u>" in an abstract manner.

· Using these concepts, one can define the <u>homotopy category</u> $Ho(\mathcal{C})$ of a (closed) model category $\mathcal{C}$. (see Dwyer & Splanski Ch 5) This construction coincides with <u>the localization of $\mathcal{C}$ w.r.t the class of weak equivalences</u> (nothing to do with fibrations & cofibrations! But this doesn't mean they're useless. They can be used to carry out other <u>constructions</u>!)

- <u>fibration sequence</u>, have similarity with distinguished triangles.
- <u>cofibration sequence</u>.

· <u>Rmk</u>: In the book Categorical Homotopy Theory, a more general concept "<u>homotopical categories</u>" is proposed (model categories are just special cases of them). They are categories which admits a special family of morphisms W (also called "weak-equivalences") satisfying

"2-of-6" axiom. The homotopy category of a homotopical category is just defined to be its localization w.r.t W. We'll also mention them in the next section.

§ 4 Derived Functors

For homotopical categories / model categories, a functor between them is regarded as homotopical if it preserves weak equivalences. A generic functor may not be homotopical, but in some cases we can build ~~a~~ universal homotopical approximations of them. These approximations are called "<u>derived functors</u>". We'll show how they relates with the derived functors of abelian categories in the end of this section.

We begin with the following definition:

- <u>Def.</u>: (<u>Kan extension</u>)

Given ~~a~~ functors $F: C \to E$, $K: C \to D$, a <u>left Kan extension</u> of F along K is a functor $\mathrm{Lan}_K F: D \to E$ together with a natural transformation $\eta: F \Rightarrow (\mathrm{Lan}_K F)\circ K$ such that for any other such pair $(G: D \to E, \gamma: F \Rightarrow GK)$, γ factors uniquely through η:

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & E \\ & D \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\overset{\Rightarrow}{\exists!}"] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\varepsilon\Uparrow"] & E \\ & D \arrow[ur, "\mathrm{Ran}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\overset{\Leftarrow}{\exists!}"] & \end{tikzcd}$$

Dually, a <u>right Kan extension</u> of F along K is a functor $\mathrm{Ran}_K F: D \to E$ together with a natural transformation $\varepsilon: \mathrm{Ran}_K F \circ K$ such that for any $(G: D \to E, \delta: GK \Rightarrow F)$, δ factors uniquely through ε.

- <u>Rmk.</u>: In fact, Kan extension is a very generic stuff in category theory, from which one can re-construct the concept of adjunctions, (co)ends, (co)limits, monads --- (See Ch1 of Cat. Homotopy Thy.)

· Def.: Let $\mathcal{C}$ be a model category, $Q: \mathcal{C} \to Ho(\mathcal{C})$ the localization functor. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. Then the right Kan extension $\underbrace{Ran_Q F}_{=:LF}$ is called the <u>left derived functor of F</u>, and the left Kan extension $\underbrace{Lan_Q F}_{=:RF}$ is called the <u>right derived functor</u> of F.

· Kan extensions may not exist for a functor F and so does derived functors. Here we give <u>a criterion of existence</u>.

(* move to page 6)

· Def.: Let $\mathcal{C}$ be a model category, $\emptyset$ an initial and $*$ an final object of it (existence guaranteed by MC1). An object $A \in \mathcal{C}$ is said to be <u>cofibrant</u> if $\emptyset \to A$ is a cofibration and fibrant if $A \to *$ is a fibration. <u>Rmk</u>: In homotopical cats. (co) fibrant objects are defined in a even more abstract way.

· Prop. Let $\mathcal{C}$ be a model category and $F: \mathcal{C} \to \mathcal{D}$ a functor with the property that $F(f)$ is an isomorphism whenever f is a weak ~~eq~~ equivalence between cofibrant objects in $\mathcal{C}$. Then the left derived functor (LF, t) of F exists. and for each cofibrant object X of $\mathcal{C}$ the map $t_X: LF(X) \to F(X)$ is an isomorphism.

· <u>Definition</u>: $F: \mathcal{C} \to \mathcal{D}$ a functor between model cats. $Q_{\mathcal{C}}: \mathcal{C} \to Ho(\mathcal{C})$ and $Q_{\mathcal{D}}: \mathcal{D} \to Ho(\mathcal{D})$ the localization functors. Then the <u>total left derived functor</u> $\mathbb{L}F$ is defined to be $L(Q_{\mathcal{D}} \circ F) = Ran_{Q_{\mathcal{C}}}(Q_{\mathcal{D}} \circ F): Ho(\mathcal{C}) \to Ho(\mathcal{D})$

· Emp.: R an associative ring with unit, Ch_R the model cat. of chain complexes. M an right R-mod. Then $M \otimes -$ is a functor from Ch_R to $Ch_{\mathbb{Z}}$. By the criterion above we have the total left derived functor $\mathbb{L}F$ exists. Let N be a left R-mod (also considered as a chain complex concentrated in degree 0). Then the criterion says that $\mathbb{L}F(N)$ is an isomorphism in $Ho(Ch_{\mathbb{Z}})$ to $F(P)$, where →

P is any cofibrant chain with a weak equivalence $P \to N$. Such a cofibrant chain complex is just a projective resolusion of N. Hence

$$H_i(\mathbb{L}(M\otimes -))(N) \cong \mathrm{Tor}_i^R(M,N),\ i \geq 0.$$

- Rmk: (1) Sometimes the categories we are considering don't admit model structures. In Ch.2 of Cat. Homotopy Thy, derived functors are defined for homotopical categories, which can be more widely applied.
  (2) Sometimes in Top, limits/colimits don't preserve homotopy equivalences. This motivates people to use the derived functors of them, denoted holim/hocolim.

§5 Applications

- K-Theory, see the book 《代数K-理论》
- Specific models of $(\infty, 1)$-categories
  - Joyal's model structure on sSet
  - Bergner's model structure on sCat

  } Quillen equivalent, by Jacob Lurie
- Tabuada's cofibrantly generated model structures on dg-cat$_k$.
  - for applications, see Lectures on dg-categories by Toën Bertrand
- Find analogous theorems in other model cats of theorems in Top.
- Simplicial methods:
  - Bousfield localization;
  - Rational Homotopy Theory;

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