

On Lemma 1.4.2 of Wilfrid Hodges' *A Shorter Model Theory*

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In the (a) \Rightarrow (b) part of the original proof given by the author A is assumed to be $\langle \bar{a} \rangle_A$. However, by unfolding the process I found that there seemed to be a logical gap which couldn't be filled. Hence I would give my proof of this lemma as follow:

Theorem (Diagram Lemma). *Let A and B be L -structures, \bar{c} a sequence of constants, and (A, \bar{a}) and (B, \bar{b}) be $L(\bar{c})$ -structures. Then (a) and (b) are equivalent*

- (a) *for every atomic sentence ϕ of $L(\bar{c})$, if $(A, \bar{a}) \models \phi$ then $(B, \bar{b}) \models \phi$.*
- (b) *there is a homomorphism $f : \langle \bar{a} \rangle_A \rightarrow B$ such that $f\bar{a} = \bar{b}$.*

The homomorphism f in (b) is unique if it exists; it is an embedding if and only if

- (c) *for every atomic sentence ϕ of $L(\bar{c})$, $(A, \bar{a}) \models \phi \iff (B, \bar{b}) \models \phi$.*

Proof.

- (a) \Rightarrow (b):

Consider the construction of $\langle \bar{a} \rangle_A$ according to Theorem 1.2.3's proof

$$Y_0 = \bar{a} \cup \{c^A : c \text{ a constant of } L\} = \text{interpretations of } L(\bar{c})\text{'s constants in } (A, \bar{a})$$

$$Y_{m+1} = Y_m \cup \{F^A(\bar{d}) : \text{for some } n > 0, F \text{ is an } n\text{-ary symbol of } L \text{ and } \bar{d} \text{ is an } n\text{-tuple of elements of } Y_m\}$$

Hence $\langle \bar{a} \rangle_A = \bigcup_{m < \omega} Y_m$ are exactly the set of interpretations of closed terms of $L(\bar{c})$ in (A, \bar{a}) .

By lemma 1.4.1, to prove (b) it suffices to find an $L(\bar{c})$ -homomorphism $f : (\langle \bar{a} \rangle_A, \bar{a}) \rightarrow (B, \bar{b})$. Firstly, we define a map $f : \langle \bar{a} \rangle_A \rightarrow B$. By the discussion above, for any element α in $\langle \bar{a} \rangle_A$, there exists a closed term t of $L(\bar{c})$ such that $t^{(A, \bar{a})} = \alpha$. Let $f(t^{(A, \bar{a})}) = t^{(B, \bar{b})}$. This is a well-defined function: $s^{(A, \bar{a})} = t^{(A, \bar{a})}$ implies $(A, \bar{a}) \models s \equiv t$; so $(B, \bar{b}) \models s \equiv t$ by (a) and hence $s^{(B, \bar{b})} = t^{(B, \bar{b})}$.

Next, we are to prove that the f defined above is indeed a homomorphism from $\langle \bar{a} \rangle_A$ to B . By Lemma 1.4.1, it suffices to show that f is a $L(\bar{c})$ -homomorphism from $(\langle \bar{a} \rangle_A, \bar{a})$ to (B, \bar{b}) (view $\langle \bar{a} \rangle_A$ as a substructure of A):

- for any constant c of $L(\bar{c})$, we have $f(c^{(\langle \bar{a} \rangle_A, \bar{a})}) = c^{(B, \bar{b})}$ by the definition of f .
- for any relation symbol R of $L(\bar{c})$ and n -tuple $\bar{\alpha}$ in $\text{dom}(\langle \bar{a} \rangle_A)$ (n the arity of R), we have $\bar{\alpha} = \bar{t}^{(A, \bar{a})}$ for a n -tuple \bar{t} of closed terms of $L(\bar{c})$. If $\bar{\alpha} \in R^{(\langle \bar{a} \rangle_A, \bar{a})}$, then $\bar{\alpha} \in R^{(A, \bar{a})}$, $(A, \bar{a}) \models R\bar{t}$, hence $(B, \bar{b}) \models R\bar{t}$ by assumption (note that $R\bar{t}$ is an atomic sentence since no variables occur in it). That is, $f(\bar{\alpha}) = \bar{t}^{(B, \bar{b})} \in R^{(B, \bar{b})}$.
- for any function symbol F of $L(\bar{c})$ and n -tuple $\bar{\alpha}$ in $\text{dom}(\langle \bar{a} \rangle_A)$ (n the arity of F), we have $\bar{\alpha} = \bar{t}^{(A, \bar{a})}$ for an n -tuple \bar{t} of closed terms of $L(\bar{c})$. $f(F^{(\langle \bar{a} \rangle_A, \bar{a})}(\bar{\alpha})) = f(F^{(\langle \bar{a} \rangle_A, \bar{a})}(\bar{t}^{(A, \bar{a})})) = f(F^{(A, \bar{a})}(\bar{t}^{(A, \bar{a})})) = f((F(\bar{t}))^{(A, \bar{a})}) = (F(\bar{t}))^{(B, \bar{b})} = F^{(B, \bar{b})}(\bar{t}^{(B, \bar{b})}) = F^{(B, \bar{b})}(f(\bar{t}^{(A, \bar{a})})) = F^{(B, \bar{b})}(f\bar{\alpha})$.

So in conclusion, f is a homomorphism from $(\langle \bar{a} \rangle_A, \bar{a})$ to (B, \bar{b}) , which is a homomorphism from $\langle \bar{a} \rangle_A$ to B mapping \bar{a} to \bar{b} .

- (b) \Rightarrow (a):

Suppose there is a homomorphism $f : \langle \bar{a} \rangle_A \rightarrow B$ sending \bar{a} to \bar{b} . Then for any atomic sentence ϕ of $L(\bar{c})$, $(A, \bar{a}) \models \phi \Rightarrow A \models \tilde{\phi}(\bar{a})$ ($\tilde{\phi}$ the atomic formula obtained by replacing constants in \bar{c} occurring in it with the fresh variable corresponding to \bar{a}). Since $\langle \bar{a} \rangle_A$ can be viewed as a substructure of A and $\bar{a} \in \langle \bar{a} \rangle_A$, by Theorem 1.3.1(c) we have $\langle \bar{a} \rangle_A \models \tilde{\phi}(\bar{a})$. By Theorem 1.3.1(b) we have $B \models \tilde{\phi}(\bar{b})$ and hence $(B, \bar{b}) \models \phi$, which proves (b) \Rightarrow (a).

- Suppose f exists in (b). By Theorem 1.3.1(a), any homomorphism g from $(\langle \bar{a} \rangle_A, \bar{a})$ to (B, \bar{b}) must satisfy $g(t^{(A, \bar{a})}) = g(t^{(\langle \bar{a} \rangle_A, \bar{a})}) = t^{(B, \bar{b})}$ for a closed term t . Hence f is unique from the arguments in (a) \Rightarrow (b).

- (c) \Rightarrow embedding:

- f is injective: otherwise suppose $f(\alpha) = f(\alpha')$. Represent α and α' as $t^{(A, \bar{a})}$ and $t'^{(A, \bar{a})}$. Then $(B, \bar{b}) \models t \equiv t'$, and hence $(A, \bar{a}) \models t \equiv t'$ by assumption. That is, $t^{(A, \bar{a})} = t'^{(A, \bar{a})}$, or equivalently, $\alpha = \alpha'$
- For each $n > 0$, each n -ary relation symbol R of $L(\bar{c})$ and each n -tuple $\bar{\alpha}$ from $(\langle \bar{a} \rangle_A, \bar{a})$, there exists an n -tuple of closed terms \bar{t} of $L(\bar{c})$ such that $\bar{\alpha} = \bar{t}^{(A, \bar{a})}$. $f\bar{\alpha} \in R^{(B, \bar{b})} \Rightarrow \bar{t}^{(B, \bar{b})} \in R^{(B, \bar{b})} \Rightarrow (B, \bar{b}) \models R\bar{t} \Rightarrow (A, \bar{a}) \models R\bar{t} \Rightarrow \bar{t}^{(A, \bar{a})} \in R^{(A, \bar{a})} \Rightarrow \bar{\alpha} \in R^{(A, \bar{a})}$. Since $\bar{\alpha} \in \text{dom}(\langle \bar{a} \rangle_A, \bar{a})$, we have $\bar{\alpha} \in R^{(\langle \bar{a} \rangle_A, \bar{a})}$.

Hence f is an embedding

- embedding \Rightarrow (c):

For each atomic sentence ϕ of $L(\bar{c})$, $(B, \bar{b}) \models \phi \Rightarrow B \models \tilde{\phi}(\bar{b})$ ($\tilde{\phi}$ the atomic formula obtained by replacing constants in \bar{c} occurring in it with the fresh variables corresponding to \bar{b}). By Theorem 1.3.1(c) we have $\langle \bar{a} \rangle_A \models \tilde{\phi}(\bar{a})$ and again by Thm 1.3.1(c) we have $A \models \tilde{\phi}(\bar{a})$ (since $\langle \bar{a} \rangle_A$ is a substructure of A and $\bar{a} \in \langle \bar{a} \rangle_A$). So $(A, \bar{a}) \models \phi$, which proves (c).

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