On Lemma 1.4.2 of Wilfrid Hodges' *A Shorter Model Theory*

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In the (a) $\Rightarrow$ (b) part of the original proof given by the author A is assumed to be $\langle\bar{a}\rangle_A$. However, by unfolding the process I found that there is a logical gap which makes this argument unsustainable. Hence I would give my proof of this lemma as follow:

Theorem (Diagram Lemma). *Let A and B be L-structures, $\bar{c}$ a sequence of constants, and $(A, \bar{a})$ and $(B, \bar{b})$ be $L(\bar{c})$-structures. Then (a) and (b) are equivalent*

(a) for every atomic sentence ϕ of $L(\bar{c})$, if $(A, \bar{a}) \models \phi$ then $(B, \bar{b}) \models \phi$.

(b) there is a homomorphism $f : \langle\bar{a}\rangle_A \to B$ such that $f\bar{a} = \bar{b}$.

The homomorphism f in (b) is unique if it exists; it is an embedding if and only if

(c) for every atomic sentence ϕ of $L(\bar{c})$, $(A, \bar{a}) \models \phi \iff (B, \bar{b}) \models \phi$.

Proof.

- (a) $\Rightarrow$ (b):

  Consider the construction of $\langle\bar{a}\rangle_A$ according to Theorem 1.2.3's proof

  $Y_0 = \bar{a} \cup \{c^A : c \text{ a constant of } L\} = \text{interpretations of } L(\bar{c})\text{'s constants in } (A, \bar{a})$
  $Y_{m+1} = Y_m \cup \{F^A(\bar{d}) : \text{ for some } n > 0,\ F \text{ is an } n\text{-ary symbol of } L \text{ and } \bar{d} \text{ is an } n\text{-tuple of elements of } Y_m\}$

  Hence $\langle\bar{a}\rangle_A = \cup_{m<\omega} Y_m$ are exactly the set of interpretations of closed terms of $L(\bar{c})$ in $(A, \bar{a})$.

  By lemma 1.4.1, to prove (b) it suffices to find an $L(\bar{c})$-homomorphism $f : (\langle\bar{a}\rangle_A, \bar{a}) \to (B, \bar{b})$. Firstly, we define a map $f : \langle\bar{a}\rangle_A \to B$. By the discussion above, for any element α in $\langle\bar{a}\rangle_A$, there exists a closed term t of $L(\bar{c})$ such that $t^{(A,\bar{a})} = \alpha$. Let $f(t^{(A,\bar{a})}) = t^{(B,\bar{b})}$. This is a well-defined function: $s^{(A,\bar{a})} = t^{(A,\bar{a})}$ implies $(A, \bar{a}) \models s \equiv t$; so $(B, \bar{b}) \models s \equiv t$ by (a) and hence $s^{(B,\bar{b})} = t^{(B,\bar{b})}$.

  Next, we are to prove that the f defined above is indeed a homomorphism from $\langle\bar{a}\rangle_A$ to B. By Lemma 1.4.1, it suffices to show that f is a $L(\bar{c})$-homomorphism from $(\langle\bar{a}\rangle_A, \bar{a})$ to $(B, \bar{b})$ (view $\langle\bar{a}\rangle_A$ as a substructure of A):

  - for any constant c of $L(\bar{c})$, we have $f(c^{(\langle\bar{a}\rangle_A,\bar{a})}) = c^{(B,\bar{b})}$ by the definition of f.
  - for any relation symbol R of $L(\bar{c})$ and n-tuple $\bar{\alpha}$ in $dom(\langle\bar{a}\rangle_A)$ (n the arity of R), we have $\bar{\alpha} = \bar{t}^{(A,\bar{a})}$ for a n-tuple $\bar{t}$ of closed terms of $L(\bar{c})$. If $\bar{\alpha} \in R^{(\langle\bar{a}\rangle_A,\bar{a})}$, then $\bar{\alpha} \in R^{(A,\bar{a})}$, $(A, \bar{a}) \models R\bar{t}$, hence $(B, \bar{b}) = R\bar{t}$ by assumption (note that $R\bar{t}$ is an atomic sentence since no variables occur in it). That is, $f(\bar{\alpha}) = \bar{t}^{(B,\bar{b})} \in R^{(B,\bar{b})}$.
  - for any function symbol F of $L(\bar{c})$ and n-buple $\bar{\alpha}$ in $dom(\langle\bar{a}\rangle_A)$ (n the arity of F), we have $\bar{\alpha} = \bar{t}^{(A,\bar{a})}$ for an n-tuple $\bar{t}$ of closed terms of $L(\bar{c})$. $f(F^{(\langle\bar{a}\rangle_A,\bar{a})}(\bar{\alpha})) = f(F^{(\langle\bar{a}\rangle_A,\bar{a})}(\bar{t}^{(A,\bar{a})})) = f(F^{(A,\bar{a})}(\bar{t}^{(A,\bar{a})})) = f((F(\bar{t}))^{(A,\bar{a})}) = (F(\bar{t}))^{(B,\bar{b})} = F^{(B,\bar{b})}(\bar{t}^{(B,\bar{b})}) = F^{(B,\bar{b})}(f(\bar{t}^{(A,\bar{a})})) = F^{(B,\bar{b})}(f\bar{\alpha})$.

  So in conclusion, f is a homomorphism from $(\langle\bar{a}\rangle_A, \bar{a})$ to $(B, \bar{b})$, which is a homomorphism from $\langle\bar{a}\rangle_A$ to B mapping $\bar{a}$ to $\bar{b}$.

- (b) $\Rightarrow$ (a):

  Suppose there is a homomorphism $f : \langle \bar{a} \rangle_A \to B$ sending $\bar{a}$ to $\bar{b}$. Then for any atomic sentence ϕ of $L(\bar{c})$, $(A, \bar{a}) \models \phi \Rightarrow A \models \tilde{\phi}(\bar{a})$ ($\tilde{\phi}$ the atomic formula obtained by replacing constants in $\bar{c}$ occuring in it with the fresh variable corresponding to $\bar{a}$). Since $\langle \bar{a} \rangle_A$ can be viewed as a substructure of A and $\bar{a} \in \langle \bar{a} \rangle_A$, by Theorem 1.3.1(c) we have $\langle \bar{a} \rangle_A \models \tilde{\phi}(\bar{a})$. By Theorem 1.3.1(b) we have $B \models \tilde{\phi}(\bar{b})$ and hence $(B, \bar{b}) \models \phi$, which proves (b) $\Rightarrow$ (a).

- Suppose f exists in (b). By Theorem 1.3.1(a), any homomorphism g from $(\langle \bar{a} \rangle_A, \bar{a})$ to $(B, \bar{b})$ must satisify $g(t^{(A,\bar{a})}) = g(t^{(\langle \bar{a} \rangle_A, \bar{a})}) = t^{(B,\bar{b})}$ for a closed term t. Hence f is unique from the arguments in (a) $\Rightarrow$ (b).

- (c) $\Rightarrow$ embedding:

  - f is injective: otherwise suppose $f(\alpha) = f(\alpha')$. Represent α and α' as $t^{(A,\bar{a})}$ and $t'^{(A,\bar{a})}$. Then $(B, \bar{b}) \models t \equiv t'$, and hence $(A, \bar{a}) \models t \equiv t'$ by assumption. That is, $t^{(A,\bar{a})} = t'^{(A,\bar{a})}$, or equivalently, $\alpha = \alpha'$
  - For each $n > 0$, each n-ary relation symbol R of $L(\bar{c})$ and each n-tuple $\bar{\alpha}$ from $(\langle \bar{a} \rangle_A, \bar{a})$, there exists an n-tuple of closed terms $\bar{t}$ of $L(\bar{c})$ such that $\bar{\alpha} = \bar{t}^{(A,\bar{a})}$. $f\bar{\alpha} \in R^{(B,\bar{b})} \Rightarrow \bar{t}^{(B,\bar{b})} \in R^{(B,\bar{b})} \Rightarrow (B, \bar{b}) \models R\bar{t} \Rightarrow (A, \bar{a}) \models R\bar{t} \Rightarrow \bar{t}^{(A,\bar{a})} \in R^{(A,\bar{a})} \Rightarrow \bar{\alpha} \in R^{(A,\bar{a})}$. Since $\bar{\alpha} \in dom(\langle \bar{a} \rangle_A, \bar{a})$, we have $\bar{\alpha} \in R^{(\langle \bar{a} \rangle_A, \bar{a})}$.

  Hence f is an embedding

- embedding $\Rightarrow$ (c):

  For each atomic sentence ϕ of $L(\bar{c})$, $(B, \bar{b}) \models \phi \Rightarrow B \models \tilde{\phi}(\bar{b})$ ($\tilde{\phi}$ the atomic formula obtained by replacing constants in $\bar{c}$ occuring in it with the fresh variables corresponding to $\bar{b}$). By Theorem 1.3.1(c) we have $\langle \bar{a} \rangle_A \models \tilde{\phi}(\bar{a})$ and again by Thm 1.3.1(c) we have $A \models \tilde{\phi}(\bar{a})$ (since $\langle \bar{a} \rangle_A$ is a substructure of A and $\bar{a} \in \langle \bar{a} \rangle_A$). So $(A, \bar{a}) \models \phi$, which proves (c).

$\square$