

# On Lemma 1.4.2 of Wilfrid Hodges' *A Shorter Model Theory*

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In the (a)  $\Rightarrow$  (b) part of the original proof given by the author  $A$  is assumed to be  $\langle \bar{a} \rangle_A$ . However, by unfolding the process I found that there seemed to be a logical gap which couldn't be filled. Hence I would give my proof of this lemma as follow:

**Theorem (Diagram Lemma).** *Let  $A$  and  $B$  be  $L$ -structures,  $\bar{c}$  a sequence of constants, and  $(A, \bar{a})$  and  $(B, \bar{b})$  be  $L(\bar{c})$ -structures. Then (a) and (b) are equivalent*

(a) *for every atomic sentence  $\phi$  of  $L(\bar{c})$ , if  $(A, \bar{a}) \models \phi$  then  $(B, \bar{b}) \models \phi$ .*

(b) *there is a homomorphism  $f : \langle \bar{a} \rangle_A \rightarrow B$  such that  $f\bar{a} = \bar{b}$ .*

*The homomorphism  $f$  in (b) is unique if it exists; it is an embedding if and only if*

(c) *for every atomic sentence  $\phi$  of  $L(\bar{c})$ ,  $(A, \bar{a}) \models \phi \iff (B, \bar{b}) \models \phi$ .*

*Proof.*

- (a)  $\Rightarrow$  (b):

Consider the construction of  $\langle \bar{a} \rangle_A$  according to Theorem 1.2.3's proof

$Y_0 = \bar{a} \cup \{c^A : c \text{ a constant of } L\} = \text{interpretations of } L(\bar{c})\text{'s constants in } (A, \bar{a})$

$Y_{m+1} = Y_m \cup \{F^A(\bar{d}) : \text{for some } n > 0, F \text{ is an } n\text{-ary symbol of } L \text{ and } \bar{d} \text{ is an } n\text{-tuple of elements of } Y_m\}$

Hence  $\langle \bar{a} \rangle_A = \cup_{m < \omega} Y_m$  are exactly the set of interpretations of closed terms of  $L(\bar{c})$  in  $(A, \bar{a})$ .

By lemma 1.4.1, to prove (b) it suffices to find an  $L(\bar{c})$ -homomorphism  $f : (\langle \bar{a} \rangle_A, \bar{a}) \rightarrow (B, \bar{b})$ . Firstly, we define a map  $f : \langle \bar{a} \rangle_A \rightarrow B$ . By the discussion above, for any element  $\alpha$  in  $\langle \bar{a} \rangle_A$ , there exists a closed term  $t$  of  $L(\bar{c})$  such that  $t^{(A, \bar{a})} = \alpha$ . Let  $f(t^{(A, \bar{a})}) = t^{(B, \bar{b})}$ . This is a well-defined function:  $s^{(A, \bar{a})} = t^{(A, \bar{a})}$  implies  $(A, \bar{a}) \models s \equiv t$ ; so  $(B, \bar{b}) \models s \equiv t$  by (a) and hence  $s^{(B, \bar{b})} = t^{(B, \bar{b})}$ .

Next, we are to prove that the  $f$  defined above is indeed a homomorphism from  $\langle \bar{a} \rangle_A$  to  $B$ . By Lemma 1.4.1, it suffices to show that  $f$  is a  $L(\bar{c})$ -homomorphism from  $(\langle \bar{a} \rangle_A, \bar{a})$  to  $(B, \bar{b})$  (view  $\langle \bar{a} \rangle_A$  as a substructure of  $A$ ):

- for any constant  $c$  of  $L(\bar{c})$ , we have  $f(c^{(\langle \bar{a} \rangle_A, \bar{a})}) = c^{(B, \bar{b})}$  by the definition of  $f$ .
- for any relation symbol  $R$  of  $L(\bar{c})$  and  $n$ -tuple  $\bar{\alpha}$  in  $\text{dom}(\langle \bar{a} \rangle_A)$  ( $n$  the arity of  $R$ ), we have  $\bar{\alpha} = \bar{t}^{(A, \bar{a})}$  for a  $n$ -tuple  $\bar{t}$  of closed terms of  $L(\bar{c})$ . If  $\bar{\alpha} \in R^{(\langle \bar{a} \rangle_A, \bar{a})}$ , then  $\bar{\alpha} \in R^{(A, \bar{a})}$ ,  $(A, \bar{a}) \models R\bar{t}$ , hence  $(B, \bar{b}) \models R\bar{t}$  by assumption (note that  $R\bar{t}$  is an atomic sentence since no variables occur in it). That is,  $f(\bar{\alpha}) = \bar{t}^{(B, \bar{b})} \in R^{(B, \bar{b})}$ .
- for any function symbol  $F$  of  $L(\bar{c})$  and  $n$ -buple  $\bar{\alpha}$  in  $\text{dom}(\langle \bar{a} \rangle_A)$  ( $n$  the arity of  $F$ ), we have  $\bar{\alpha} = \bar{t}^{(A, \bar{a})}$  for an  $n$ -tuple  $\bar{t}$  of closed terms of  $L(\bar{c})$ .  $f(F^{(\langle \bar{a} \rangle_A, \bar{a})}(\bar{\alpha})) = f(F^{(A, \bar{a})}(\bar{t}^{(A, \bar{a})})) = f(F^{(A, \bar{a})}(\bar{t}^{(A, \bar{a})})) = f((F(\bar{t}))^{(A, \bar{a})}) = (F(\bar{t}))^{(B, \bar{b})} = F^{(B, \bar{b})}(\bar{t}^{(B, \bar{b})}) = F^{(B, \bar{b})}(f(\bar{t}^{(A, \bar{a})})) = F^{(B, \bar{b})}(f\bar{\alpha})$ .

So in conclusion,  $f$  is a homomorphism from  $(\langle \bar{a} \rangle_A, \bar{a})$  to  $(B, \bar{b})$ , which is a homomorphism from  $\langle \bar{a} \rangle_A$  to  $B$  mapping  $\bar{a}$  to  $\bar{b}$ .

- (b)  $\Rightarrow$  (a):

Suppose there is a homomorphism  $f : \langle \bar{a} \rangle_A \rightarrow B$  sending  $\bar{a}$  to  $\bar{b}$ . Then for any atomic sentence  $\phi$  of  $L(\bar{c})$ ,  $(A, \bar{a}) \models \phi \Rightarrow A \models \tilde{\phi}(\bar{a})$  ( $\tilde{\phi}$  the atomic formula obtained by replacing constants in  $\bar{c}$  occurring in it with the fresh variable corresponding to  $\bar{a}$ ). Since  $\langle \bar{a} \rangle_A$  can be viewed as a substructure of  $A$  and  $\bar{a} \in \langle \bar{a} \rangle_A$ , by Theorem 1.3.1(c) we have  $\langle \bar{a} \rangle_A \models \tilde{\phi}(\bar{a})$ . By Theorem 1.3.1(b) we have  $B \models \tilde{\phi}(\bar{b})$  and hence  $(B, \bar{b}) \models \phi$ , which proves (b)  $\Rightarrow$  (a).

- Suppose  $f$  exists in (b). By Theorem 1.3.1(a), any homomorphism  $g$  from  $(\langle \bar{a} \rangle_A, \bar{a})$  to  $(B, \bar{b})$  must satisfy  $g(t^{(A, \bar{a})}) = g(t^{(\langle \bar{a} \rangle_A, \bar{a})}) = t^{(B, \bar{b})}$  for a closed term  $t$ . Hence  $f$  is unique from the arguments in (a)  $\Rightarrow$  (b).

- (c)  $\Rightarrow$  embedding:

- $f$  is injective: otherwise suppose  $f(\alpha) = f(\alpha')$ . Represent  $\alpha$  and  $\alpha'$  as  $t^{(A, \bar{a})}$  and  $t'^{(A, \bar{a})}$ . Then  $(B, \bar{b}) \models t \equiv t'$ , and hence  $(A, \bar{a}) \models t \equiv t'$  by assumption. That is,  $t^{(A, \bar{a})} = t'^{(A, \bar{a})}$ , or equivalently,  $\alpha = \alpha'$
- For each  $n > 0$ , each  $n$ -ary relation symbol  $R$  of  $L(\bar{c})$  and each  $n$ -tuple  $\bar{\alpha}$  from  $(\langle \bar{a} \rangle_A, \bar{a})$ , there exists an  $n$ -tuple of closed terms  $\bar{t}$  of  $L(\bar{c})$  such that  $\bar{\alpha} = \bar{t}^{(A, \bar{a})}$ .  $f\bar{\alpha} \in R^{(B, \bar{b})} \Rightarrow \bar{t}^{(B, \bar{b})} \in R^{(B, \bar{b})} \Rightarrow (B, \bar{b}) \models R\bar{t} \Rightarrow (A, \bar{a}) \models R\bar{t} \Rightarrow \bar{t}^{(A, \bar{a})} \in R^{(A, \bar{a})} \Rightarrow \bar{\alpha} \in R^{(A, \bar{a})}$ . Since  $\bar{\alpha} \in \text{dom}(\langle \bar{a} \rangle_A, \bar{a})$ , we have  $\bar{\alpha} \in R^{(\langle \bar{a} \rangle_A, \bar{a})}$ .

Hence  $f$  is an embedding

- embedding  $\Rightarrow$  (c):

For each atomic sentence  $\phi$  of  $L(\bar{c})$ ,  $(B, \bar{b}) \models \phi \Rightarrow B \models \tilde{\phi}(\bar{b})$  ( $\tilde{\phi}$  the atomic formula obtained by replacing constants in  $\bar{c}$  occurring in it with the fresh variables corresponding to  $\bar{b}$ ). By Theorem 1.3.1(c) we have  $\langle \bar{a} \rangle_A \models \tilde{\phi}(\bar{a})$  and again by Thm 1.3.1(c) we have  $A \models \tilde{\phi}(\bar{a})$  (since  $\langle \bar{a} \rangle_A$  is a substructure of  $A$  and  $\bar{a} \in \langle \bar{a} \rangle_A$ ). So  $(A, \bar{a}) \models \phi$ , which proves (c).

□