

# Model Categories and Derived Functors

## § 1. Introduction

### • Model category:

• A stage to do abstract Homotopy theories

• Introduced by Daniel Quillen

• References: - 《代数K-理论》 by 黎景辉

- Homotopical Algebra by Daniel Quillen

- Model categories by Mark Hovey

- Categorical Homotopy Theory by Emily Riehl

(Easy to read)

- Homotopy Theories and Model Categories by W. G. Dwyer & J. Spalinski.

## § 2 Motivating Examples in Topological Spaces

• Def: A morphism  $f: X \rightarrow Y$  in  $\text{Top}$  is called a weak homotopy equivalence if  $\forall n \geq 0$ ,  $f_* = \pi_n(X) \rightarrow \pi_n(Y)$  is an isomorphism.

A --- is called a homotopy equivalence if  $\exists g: Y \rightarrow X$  s.t.  $fg \simeq \text{id}_Y$ ,  $gf \simeq \text{id}_X$ .

• A homotopy equivalence is a weak equivalence.

(2 out of 3)

• Prop: Let  $f: X \rightarrow Y$ ,  $g: Y \rightarrow Z$  be morphisms in  $\text{Top}$ . If 2 of  $f$ ,  $g$ ,  $gf$  are weak equivalences, then the remaining one is also a ~.

Proof: Only consider the case that  $f$ ,  $gf$  are ~.

Consider the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow gf & \swarrow g \\ & Z & \end{array}$$

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{f_*} & \pi_n(Y, fx) \\ & \searrow (gf)_* & \swarrow g_* \\ & \pi_n(Z, gfx) & \end{array}$$

since  $f_*$  and  $(gf)_*$  are isomorphisms,  $g_*$  is an isomorphism. However, not every  $y \in Y$  are in  $f(X)$ . But since  $f_x: \pi_0(X) \rightarrow \pi_0(Y)$  is an isomorphism,  $\exists x \in X$  and path  $\bar{c}: I \rightarrow Y$  s.t.  $\bar{c}(0) = f_x, \bar{c}(1) = y$ . We obtain

$$\begin{array}{ccc} \pi_n(Y, y) & \xrightarrow{h_1} & \pi_n(Z, g(y)) \\ h_2 \downarrow & \cong & \downarrow h_3 \\ \pi_n(Y, f(x)) & \xrightarrow{h_4} & \pi_n(Z, g(f(x))) \end{array}$$

$h_4$  is an isom. as shown above,  $h_2, h_3$  also isom. (as inv. of  $\bar{c}_*$  and  $(g \circ \bar{c})_*$ ). Hence  $h_1$  is an isom, which is induced by  $g$ .  $\square$

Many other properties ---

Some concepts

Recall from algebraic topology:

(Hurewicz)

• Fibrations: A map  $p: X \rightarrow Y$  satisfying the Homotopy Lifting Property (HLP) for any topological space  $W$ :  $\forall$  commutative

diagram

$$\begin{array}{ccc} W \times \{0\} & \xrightarrow{h_0} & X \\ \downarrow i & \nearrow h & \downarrow p \\ W \times I & \xrightarrow{g} & Y \end{array}$$

commutes

$\exists$  a continuous map  $h: W \times I \rightarrow X$  s.t. the new diagram also

• Emp.: A covering  $p: E \rightarrow B$  is a fibration.

~~• Local Fibration~~

• Cofibration: A map  $f: A \rightarrow B$  satisfying the Homotopy Extension Property (HEP) for any topological space  $Z$ :  $\forall$  commutative

diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & Z^I \text{ (continuous maps from } I \text{ to } Z, \text{ compact open top.)} \\ f \downarrow & \nearrow h & \downarrow p_0 \\ B & \xrightarrow{h_0} & Z \end{array}$$

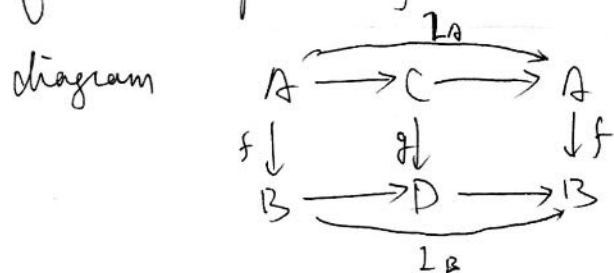
$\exists$  a continuous map  $h: B \rightarrow Z^I$  s.t. the new diagram also commutes.

- Emp. The inclusion  $S^{n-1} \hookrightarrow D^n \hookrightarrow D^n$   $\Rightarrow$  cofibrations push-outs & compositions  
 $A \hookrightarrow X$  for relative CW complex  $(A, X)$ .

## §2 Definition and Examples of Model Categories

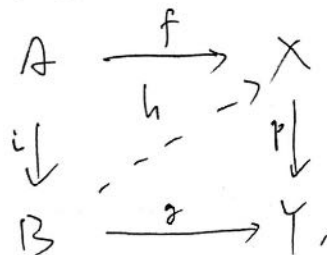
Arbitrary cat.

Def:- A morphism  $f$  is said to be a retract of  $g$  iff  $\exists$  a commutative



Rmk:- The name "retract" also originates from algebraic top.

Def:- Given a commutative diagram of the following form



if  $\forall$  diagram  
 $A \rightarrow X$  a lifting  
 $i \downarrow \quad \downarrow p$  exists  
 $B \rightarrow Y$

a lifting in the diagram is a map  $h: B \rightarrow X$  such that the new  
 diagram commutes.  $i$  is said to have a left lifting property w.r.t.  $p$  (resp. right) (resp.  $i$ )

Def:- A (closed) model category is a category  $C$  with three distinguished  
 classes of maps (i) weak equivalences (ii) fibrations (iii) cofibrations,  
 each of which is closed under compositions and contains all identity  
 maps (a map which is both a fibration/cofibration and a weak  
 equivalence is called a trivial fibration/cofibration), satisfying  
 the following axioms:

(MC1)  $C$  admits finite limits and colimits

(MC2) "2-of-3" If two of  $f, g, gf$  are weak-equivalences, then  
 the third also is.

(MC3) If  $f$  is a retract of  $g$  and  $g$  is a fibration/cofibration/  
 weak-equivalence, then  $f$  also is.

(MC 4) Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{g} & Y, \end{array}$$

a lift exists in this diagram in either of the two following situations

(i)  $i$  is a cofibration and  $p$  is a trivial fibration; (ii)  $i$  is a trivial cofibration and  $p$  is a fibration.

(MC 5) Any morphism  $f$  admit decompositions

$$\begin{array}{ccc} & Z & \\ i \nearrow & & \searrow p \\ X & \xrightarrow{f} & Y \\ j \searrow & & \nearrow q \\ & W & \end{array}$$

where  $p$  is a fibration,  $i$  is a trivial cofibration,  $j$  is a cofibration and  $q$  is a trivial fibration.

### Examples

• Classical Quillen Model structure on  $\text{Top}$ :

- weak equivalences: weak homotopy equivalences;
- fibrations: Serre fibrations, which are maps having the right lifting property w.r.t all inclusions of the form  $i_0: D^n \hookrightarrow D^n \times I$ ,  $D^n \hookrightarrow D^n \times \{0\}$ ;
- cofibrations: retracts of relative CW complexes;

• Hurewicz (or Strøm) Model Structure on  $\text{Top}$ :

- weak equivalences: homotopy equivalences;
- fibrations: Hurewicz fibrations;
- cofibrations: closed Hurewicz cofibrations;

•  $R$  an associative ring with unit,  $\text{Mod}_R$  the category of left  $R$ -modules.

Chk the category of nonnegatively graded chain complexes of  $R$ -modules.

Then  $\text{Chk}$  admits a model structure with:

- weak equivalences:  $\text{all } f: M \rightarrow N \text{ that induces isomorphisms,}$

$$H_k(M) \rightarrow H_k(N) \quad (k \geq 0);$$

- fibrations: all  $f: M \rightarrow N$  s.t.  $\forall k \geq 0$  the map  $f_k: M_k \rightarrow N_k$  is an epi.
- cofibrations: all  $f: M \rightarrow N$  s.t.  $\forall k \geq 0$  the map  $f_k: M_k \rightarrow N_k$  is a mono with a projective  $R$ -module as its cokernel.

Joyal model structure on  $sSet$  / Bergner model structure on  $sCat$

Tabuada model structure on  $dg-cat$  <sup>among other model str.</sup>

§ 3 } Homotopy & localization

↑ presents (co, 1)-cat  
"Quillen equivalent" by Lurie  
e.g. Quillen-Kan ~

In the beginning, we've mentioned that model categories provide us a stage to do abstract homotopy theories. So we need to define the concept in an abstract way. However, since I'm not able to finish doing this within the time limit, so I just provide some flavor of it. For the whole story, see Ch 4 and Ch 5 of Dwyer & Spalinski.

Recall in Top, in order to define homotopy we need cylinders  $W \times I$  and path spaces  $Z^I$ . Here we define them in model categories.

Def: Let  $C$  be a (closed) model category,  $A$  and  $X$  are objects in  $C$ . A cylinder object for  $A$  is an object  $A \amalg I$  of  $C$  together with a diagram  $A \amalg A \xrightarrow{i} A \amalg I \xrightarrow{\bar{c}} A$  which factors the folding map  $A \amalg A \xrightarrow{id_A + id_A} A$  (here  $i$  is a cofibration and  $\bar{c}$  is a weak equivalence)

Remk: (1) In Top we have

$$\begin{array}{ccc} A \amalg A & & \\ \nabla \downarrow & \searrow i & \\ A & \xleftarrow{\bar{c}} & A \times I \end{array}$$

$$\begin{array}{ccc} A \amalg A & \xleftarrow{i_2} & A \\ \uparrow i_1 & \searrow id_A + id_A & \downarrow id_A \\ A & \xrightarrow{id_A} & A \end{array}$$

where  $i$  send the first  $A$  in  $A \amalg A$  to  $A \times \{0\}$  and the second to  $A \times \{1\}$ ,  $\bar{c}$  is the projection from  $A \times I$  to  $A$ ,  $\nabla$  is the <sup>co</sup>diagonal map.

(2) more concepts: good cylinder object, very good cylinder object, left homotopy ---

(3)  $A \amalg I$  is just a notation.  $I$  is not defined in  $C$ .



• Def: A path object for  $X$  is an object  $X^I$  of  $\mathcal{C}$  together with a diagram:  $X \xrightarrow{s} X^I \xrightarrow{p} X \times X$ , which factors the diagonal map  $(\text{id}_X, \text{id}_X): X \rightarrow X \times X$  (here  $s$  is a weak equivalence and  $p$  is a fibration).

• Remark: "In Top we have

$$\begin{array}{ccc} X^I & \xleftarrow{s} & X \\ & \searrow p & \downarrow \Delta \\ & & X \times X \end{array}$$

where  $s(b): I \rightarrow X$  sends all  $t \in I$  to  $b$ ,  $p(l) = (l(0), l(1))$  for  $l \in SI$ ,  $\Delta$  the diagonal map.

(2) more concepts: good path object, very good path object, right homotopy ---

(3)  $X^I$  just a notation.

(X definitions on page 8.)

• We can then define fibrant and cofibrant objects. Let  $f, g: A \rightarrow X$  be maps. If  $A$  is cofibrant and  $B$  is fibrant, then the concepts left homotopy and right homotopy coincide, and then we can def. when  $f$  and  $g$  are "homotopic" in an abstract manner.

• Using these concepts, one can define the homotopy category  $Hoc$  of a (closed) model category  $\mathcal{C}$ . (see Dwyer & Spalinski Ch 5) This construction coincides with the localization of  $\mathcal{C}$  w.r.t the class of weak equivalences (nothing to do with fibrations & cofibrations! But this doesn't mean they're useless. They can be used to carry out other constructions!)

- fibration sequence, have similarity with distinguished triangles.
- cofibration sequence,

• Remark: In the book Categorical Homotopy Theory, a more general concept "homotopical categories" is proposed (model categories are just special cases of them). They are categories which admits a special family of morphisms  $W$  (also called "weak equivalences") satisfying

"2-of-6" axiom. The homotopy category <sup>of a homotopical category</sup> is just defined to be its localization w.r.t  $W$ . We'll also mention them in the next section.

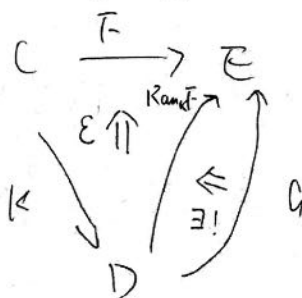
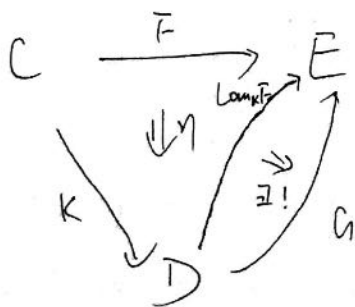
## § 4 Derived Functors

For homotopical categories / model categories, a functor between them is regarded as homotopical if it preserves weak equivalences. A generic functor may not be homotopical, but in some cases we can build a universal homotopical approximations of them. These approximations are called "derived functors". We'll show how they relates with the derived functors of abelian categories in the end of this section.

We begin with the following definition:

• Def<sub>1</sub>: (Kan extension)

Given a functors  $F: C \rightarrow E$ ,  $K: C \rightarrow D$ , a left Kan extension of  $F$  along  $K$  is a functor  $\text{Lan}_K F: D \rightarrow E$  together with a natural transformation  $\eta: F \Rightarrow (\text{Lan}_K F) \circ K$  such that for any other such pair  $(G: D \rightarrow E, \gamma: F \Rightarrow G \circ K)$ ,  $\gamma$  factors uniquely through  $\eta$ .



Dually, a right Kan extension of  $F$  along  $K$  is a functor  $\text{Ran}_K F: D \rightarrow E$  together with a natural transformation  $\epsilon: \text{Ran}_K F \circ K \Rightarrow F$  such that for any  $(G: D \rightarrow E, \delta: G \circ K \Rightarrow F)$ ,  $\delta$  factors uniquely through  $\epsilon$ .

• Remark: In fact, Kan extension is a very generic stuff in category theory, from which one can re-construct the concept of adjunctions, (co)ends, (co)limits, monads --- (See Ch1 of Cat. Homotopy Thg.)

Def: Let  $\mathcal{C}$  be a model category,  $Q: \mathcal{C} \rightarrow H_0(\mathcal{C})$  the localization functor. Let  $\bar{F}: \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then the right Kan extension  $\text{Kan}_{\alpha} \bar{F} \stackrel{=}{=} LF$  is called the left derived functor of  $\bar{F}$ , and the left Kan extension  $\text{Lan}_{\alpha} \bar{F} \stackrel{=}{=} RF$  is called the right derived functor of  $\bar{F}$ .

Kan extensions may not exist for a functor  $\bar{F}$  and so does derived functors. Here we give a criterion of existence.

(\* move to page 6)

Def: Let  $\mathcal{C}$  be a model category,  $\emptyset$  an initial and  $*$  an final object of it (existence guaranteed by MC1). An object  $A \in \mathcal{C}$  is said to be cofibrant if  $\emptyset \rightarrow A$  is a cofibration and fibrant if  $A \rightarrow *$  is a fibration. Remark: In homotopical cats, <sup>(w)</sup> fibrant objects are defined in a even more abstract way.

Prop: Let  $\mathcal{C}$  be a model category and  $F: \mathcal{C} \rightarrow \mathcal{D}$  a functor with the property that  $F(f)$  is an isomorphism whenever  $f$  is a weak equivalence between cofibrant objects in  $\mathcal{C}$ . Then the left derived functor  $(L\bar{F}, t)$  of  $\bar{F}$  exists, and for each cofibrant object  $X$  of  $\mathcal{C}$  the map  $t_X: LF(X) \rightarrow \bar{F}(X)$  is an isomorphism.

Definition:  $\bar{F}: \mathcal{C} \rightarrow \mathcal{D}$  a functor between model cats.  $Q_{\mathcal{C}}: \mathcal{C} \rightarrow H_0(\mathcal{C})$  and  $Q_{\mathcal{D}}: \mathcal{D} \rightarrow H_0(\mathcal{D})$  the localization functors. Then the total left derived functor  $L\bar{F}$  is defined to be  $L(Q_{\mathcal{D}} \circ \bar{F}) = \text{Kan}_{\alpha_{\mathcal{C}}}(Q_{\mathcal{D}} \circ \bar{F}): H_0(\mathcal{C}) \rightarrow H_0(\mathcal{D})$ .

Exmp:  $R$  an associative ring with unit,  $\text{Ch } R$  the model cat. of chain complexes.  $M$  an right  $R$ -mod. Then  $M \otimes -$  is a functor from  $\text{Ch } R$  to  $\text{Ch } R$ . By the criterion above we have the total left derived functor  $L\bar{F}$  exists. Let  $N$  be a left  $R$ -mod (also considered as a chain complex concentrated in degree 0). Then the criterion says that  $L\bar{F}(N)$  is an isomorphism in  $H_0(\text{Ch } R) \hookrightarrow F(P)$ , where  $\rightarrow$



$P$  is any cofibrant chain with a weak equivalence  $P \rightarrow N$ . Such a cofibrant chain complex is just a projective resolution of  $N$ . Hence

$$H_i(\mathbb{L}(M \otimes -))(N) \cong \operatorname{Tor}_i^R(M, N), i \geq 0.$$

- Remark: (1) Sometimes the categories we are considering don't admit model structures. In <sup>Ch. 2 of</sup> Cat. Homotopy Thg, derived functors are defined for homotopical categories, which can be more widely applied.
- (2) Sometimes in Top, limits/colimits don't preserve homotopy equivalences. This motivates people to use the derived functors of them, denoted  $\operatorname{holim}/\operatorname{hocolim}$ .

## § 5 Applications

- K-Theory, see the book <<代数 K-理论>>
- Specific models of  $(\infty, 1)$ -categories
  - Joyal's model structure on  $s\mathcal{G}et$
  - Bergner's model structure on  $s\mathcal{C}at$
 } Quillen equivalent, by Jacob Lurie
- Tabuada's cofibrantly generated model structures on dg-cat  $k$ .
  - for applications, see Lectures on dg-categories by Toen Bertrand
- Find analogous theorems in other model. cats of theorems in Top.
- Simplicial methods:
  - Bousfield localization;
  - Rational Homotopy Theory;

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