1. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $F: \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor to another abelian category $\mathcal{B}$. Suppose there is an exact sequence $0 \to A \to J^0 \to J^1 \to \cdots$, where each J^i is acyclic for F, $i \geq 0$. Then for each $i \geq 0$ there is a natural isomorphism $R^iF(A) \cong H^i(F(J^\cdot))$.

Proof: (From theorem III.1.1A we know that given a short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, there is an associated long exact sequence $\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta_i} R^{i+1}F(A') \to R^{i+1}F(A') \to R^{i+1}F(A'') \to \cdots$ in $\mathcal{B}$.) $\forall$ chain complex $\cdots \to A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots$, let $Z^i(A^\cdot) := \ker d^i$.

From the associated long exact sequence of $0 \to A \to J^0 \to J^1 \to Z^1(J^\cdot) \to 0$ and the fact that $R^iF(J^0) = 0$ for any $i \geq 1$, we get $R^{i-1}F(Z^1(J^\cdot)) \cong R^iF(A)$ for any $i \geq 2$. And from the associated long exact sequence of $0 \to Z^1(J^\cdot) \to J^1 \to Z^2(J^\cdot) \to 0$ and the fact that $R^iF(J^1) = 0$ for any $i \geq 1$ we get $R^{i-2}F(Z^2(J^\cdot)) = R^{i-1}F(Z^1(J^\cdot)) = R^iF(A)$ for any $i \geq 3$ ------ In this way, we get

$$R^i(F(A)) = R^{i-1}F(Z^1(J^\cdot)) = \cdots = R^1F(Z^{i-1}(J^\cdot)), \quad i \geq 1.$$

Moreover, from the associated long exact sequence of $0 \to Z^{i-1}(J^\cdot) \to J^{i-1} \to Z^i(J^\cdot) \to 0$ and the fact that $R^1F(J^{i-1}) = 0$, we get an exact sequence

$$0 \to F(Z^{i-1}(J^\cdot)) \to F(J^{i-1}) \to F(Z^i(J^\cdot)) \to R^1F(Z^{i-1}(J^\cdot)) \to 0,$$

and hence

$$R^1F(Z^{i-1}(J^\cdot)) = F(Z^i(J^\cdot)) / \operatorname{im}(F(J^{i-1}) \to F(Z^i(J^\cdot))).$$

We have an exact sequence

$$0 \to Z^i(J^\cdot) \to J^i \to J^{i+1}$$

Since F is left exact, we have an exact sequence

$$0 \to F(Z^i(J^\cdot)) \to F(J^i) \to F(J^{i+1})$$

and hence

$$F(Z^i(J^\cdot)) = \ker(F(J^i) \longrightarrow F(J^{i+1})),$$
$$\mathrm{im}(F(J^{i-1})) \to F(Z^i(J^\cdot)) = \mathrm{im}(F(J^{i-1}) \to F(J^i)).$$

So

$$R^1F(Z^{i-1}(J^\cdot)) = \ker(F(J^i) \to F(J^{i+1}))/\mathrm{im}(F(J^{i-1}) \to F(J^i))$$
$$= H^i(F(J^\cdot))$$

Therefore $R^iF(A) = H^i(F(J^\cdot))$ for any $i \geq 1$. Using the fact that F is left exact we also have $R^0F(A) = H^0(F(J^\cdot))$. ☒

2. Let $(X, \mathcal{O}_X)$ be a ringed space, $j: U \hookrightarrow X$ an open subspace, with sheaf of rings $\mathcal{O}_U = \mathcal{O}_X|_U$. Recall that we have an "extension by zero"

$$j_!: AB(U) \longrightarrow AB(X).$$

i) Show that $j_!$ is exact.

ii) Show that if $\mathcal{F}$ is an $\mathcal{O}_U$-module, then $j_!(\mathcal{F})$ has a natural $\mathcal{O}_X$-module structure.

iii) Show that $j_!: Mod(\mathcal{O}_U) \longrightarrow Mod(\mathcal{O}_X)$ is left adjoint to $j^*(=j^{-1})$: $Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_U)$

Proof: i) Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on U, then it is exact at stalks. If $x \in U$, then $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ implies that $0 \to j_!(\mathcal{F}')_x \to j_!(\mathcal{F})_x \to j_!(\mathcal{F}'')_x \to 0$ since $j_!(\mathcal{F})_x \cong \mathcal{F}_x$, $j_!(\mathcal{F}')_x \cong \mathcal{F}'_x$, $j_!(\mathcal{F}'')_x \cong \mathcal{F}''_x$ and the diagrams

$$\begin{tikzcd} \mathcal{F}'_x \arrow[r] \arrow[d, "\cong"'] & \mathcal{F}_x \arrow[d, "\cong"] \\ j_!(\mathcal{F}')_x \arrow[r] & j_!(\mathcal{F})_x \end{tikzcd}$$

(and other three) commutes. If $x \notin U$, then the sequence

$$0 \to j_!(\mathcal{F}')_x \to j_!(\mathcal{F})_x \to j_!(\mathcal{F}'')_x \to 0$$ is just $0 \to 0 \to 0 \to 0 \to 0$.

So the sequence $0 \to j_!(\mathcal{F}') \to j_!(\mathcal{F}) \to j_!(\mathcal{F}'') \to 0$ induces exact sequen on stalks, hence is exact.

ii) If $V \subset U$, then $\Gamma(V, j_!^{pre}(\mathcal{F})) \simeq \Gamma(V, \mathcal{F})$ as an $\mathcal{O}_X(V)$-module under the natural isomorphism $\mathcal{O}_X(V) \simeq \mathcal{O}_U(V)$.

If $V \not\subset U$, then $\Gamma(V, j_!^{pre}(\mathcal{F})) = 0$, which is naturally an $\mathcal{O}_X(V)$-module.

Let $j_!^{pre}(\mathcal{F}) \xrightarrow{\theta} j_!(\mathcal{F})$ be the sheafification map, then $\Gamma(V, j_!(\mathcal{F}))$ can be given a natural $\mathcal{O}_X(V)$-module structure via $\theta(V)$: $\forall a \in \Gamma(V, j_!(\mathcal{F}))$, $r \in \mathcal{O}_X(V)$. let $b \in \theta(V)^{-1}(a)$, $r \cdot a := \theta(V)(r \cdot b)$ (this is independent with the choice of b). Moreover, this $\mathcal{O}_X$-action is compatible with restriction maps, so $j_!(\mathcal{F})$ has a natural $\mathcal{O}_X$-module structure.

iii) Since j is the inclusion map of an open subset, it is an open map, $(j^{-1}\mathcal{G})(V) = \mathcal{G}(j(V))$ for sheaf $\mathcal{G}$ on X and $V \subseteq U$. Hence $j^{-1}\mathcal{G}$ is a sheaf for all sheaf $\mathcal{G}$ on X.

$\forall V \underset{open}{\subseteq} X$, $(j_!^{pre} j^{-1}\mathcal{G})(V) = \begin{cases} \mathcal{G}(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$. So there is a canonical morphism from $j_!^{pre} j^{-1}\mathcal{G}$ to $\mathcal{G}$ which induces a natural injection $\varepsilon_{\mathcal{G}}$ from $j_! j^{-1}\mathcal{G}$ to $\mathcal{G}$ (injective on stalks).

On the other hand, $\forall V \underset{open}{\subseteq} U$, $(j^{-1} j_! \mathcal{F})(V) = (j_! \mathcal{F})(j(V)) = \mathcal{F}(V)$ since the restriction of $j_! \mathcal{F}|_U = \mathcal{F}$. Hence we have an isomorphism $\eta_{\mathcal{F}}: \mathcal{F} \to j^{-1} j_! \mathcal{F}$. In addition, $\varepsilon = \{\varepsilon_{\mathcal{G}}\}_{\mathcal{G} \in Mod(\mathcal{O}_X)}$ and $\eta = \{\eta_{\mathcal{F}}\}_{\mathcal{F} \in Mod(\mathcal{O}_U)}$ are natural in $\mathcal{F}$ and $\mathcal{G}$ respectively. So $\eta: id \to j^* j_!$ and $\varepsilon: j_! j^* \to \mathcal{G}$ are natural transformations. (in the category of functors)

Moreover, since the diagram

$$\begin{tikzcd} j_! \arrow[r, "j_!\eta"] \arrow[dr, "id_{j_!}"'] & j_! j^* j_! \arrow[d, "\varepsilon_{j_!}"] \\ & j_! \end{tikzcd}$$

and

$$\begin{tikzcd} j^* \arrow[r, "\eta_{j^*}"] \arrow[dr, "id_{j^*}"'] & j^* j_! j^* \arrow[d, "j^*\varepsilon"] \\ & j^* \end{tikzcd}$$

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$\forall \mathcal{F} \in Mod(\mathcal{O}_U)$ and $\mathcal{G} \in Mod(\mathcal{O}_X)$, we have $j_! \dashv j^*$ with unit η and counit ε. ◻

3. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, $G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{C}$ be additive functors, (F, G, α) be an adjunction such that

$$\alpha_{A,B} : Hom_{\mathcal{C}}(FA, B) \longrightarrow Hom_{\mathcal{D}}(A, GB), \quad A \in Ob(\mathcal{D}), B \in Ob(\mathcal{C})$$

are isomorphism of groups (rather than just bijections). Assume further more that F is exact. Show that G takes injective objects in $\mathcal{C}$ to injective objects in $\mathcal{D}$

Proof: Let A be an injective object in $\mathcal{C}$. Suppose $B' \xhookrightarrow{i} B$ is a monomorphism in $\mathcal{D}$, and there is a morphism $G(A) \xrightarrow{f} B'$. Then there is a corresponding diagram

$$\begin{tikzcd} & F(B) \\ A \arrow[r, "g"] & F(B') \arrow[u, "Fi"'] \end{tikzcd}$$

$\leftarrow$ monomorphism by the exactness of F. By injectiveness of A we can lift g to a morphism $A \xrightarrow{g'} F(B)$ and have a new diagram

$$\begin{tikzcd} & F(B) \\ A \arrow[ur, "g'"] \arrow[r, "g"'] & F(B') \arrow[u, "Fi"'] \end{tikzcd}$$

, which corresponds to a diagram

$$\begin{tikzcd} & B \\ G(A) \arrow[ur, "f'"] \arrow[r, "f"'] & B' \arrow[u, "i"'] \end{tikzcd}$$

. Hence $G(A)$ is also an injective object. ◻