Ex III.5.10: Let X be a projective scheme over a noetherian ring A, and let $\mathcal{F}^1 \xrightarrow{\varphi_1} \mathcal{F}^2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{r-1}} \mathcal{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0 such that for all $n \geq n_0$, the sequence of global sections $\Gamma(X, \mathcal{F}^1(n)) \to \Gamma(X, \mathcal{F}^2(n)) \to \cdots \to \Gamma(X, \mathcal{F}^r(n))$ is exact.

Proof: We have short exact sequences

$$0 \to \ker \varphi_i \to \mathcal{F}^i \xrightarrow{\varphi_i} \operatorname{im} \varphi_i \to 0$$

$$\cancel{0 \to \operatorname{im} \varphi_i \xrightarrow{\iota_i} \mathcal{F}^{i+1} \to \operatorname{coker} \varphi_i \to 0}$$

Twist by n_i and take global sections, we obtain

$$0 \to \Gamma(X, (\ker \varphi_i)(n_i)) \to \Gamma(X, \mathcal{F}^i(n_i)) \to \Gamma(X, (\operatorname{im} \varphi_i)(n_i)) \to 0$$

$$\cancel{0 \to \Gamma(X, (\operatorname{im} \varphi_i)(n_i)) \xrightarrow{\iota_i(X)} \Gamma(X, \mathcal{F}^{i+1}(n_i)) \to \Gamma(X, (\operatorname{coker} \varphi_i)(n_i)) \to 0}$$

by theorem III.5.2 for big enough n_i. Take $n = \max\{n_1, \cdots, n_r\}$, then we have exact sequences (Note that $\ker \varphi_i$ and $\operatorname{im} \varphi_i$ are also quasi-coherent by Prop II.5.7.)

$$0 \to \Gamma(X, (\ker \varphi_i)(n)) \to \Gamma(X, \mathcal{F}(n)) \to \Gamma(X, (\operatorname{im} \varphi_i)(n)) \to 0 \quad \text{for all } i$$

Since $\operatorname{im} \varphi_i = \ker \varphi_{i+1}$, we have $\Gamma(X, (\operatorname{im} \varphi_i)(n)) = \Gamma(X, (\ker \varphi_{i+1})(n))$, so these sequences can be conjuncted into a long exact sequence

$$\Gamma(X, \mathcal{F}^1(n)) \to \Gamma(X, \mathcal{F}^2(n)) \to \cdots \to \Gamma(X, \mathcal{F}^r(n)).$$ ☒

（科目：　　　　）**数　　学　　作　　业　　纸**

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Ex III.8.1: Let $f: X \to Y$ be a continous map of topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geq 0$,

$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$

Proof: Let $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ be an injective resolution of $\mathcal{F}$. Since $R^i f_*(\mathcal{F}) = 0 \ \forall i > 0$, $0 \to f_*\mathcal{F} \to f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to \cdots$ is also exact. Moreover, because f_* preserves injective object, the sequence above is an injective resolution of $f_*(\mathcal{F})$ in Y. By definition, $\Gamma(X, \mathcal{F}) \cong \Gamma(Y, f_*\mathcal{F})$. And because right derived functors are universal δ-functors, we have $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for each i. $\boxtimes$

Ex III.8.2: Let $f: X \to Y$ be an affine morphism schemes with X noetherian, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Show that the hypothesis of Ex III.8.1 are satisfied, and hence that $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for each $i \geq 0$

Proof: Let $\mathfrak{U} = \{U_i\}$ be a base for Y consists of open affines, then each $f^{-1}(U_i)$ is also affine and hence $\{f^{-1}(U_i)\}$ forms an open cover of X. By proposition 1.2, $R^i f_* \mathcal{F} = a[\overbrace{V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})}^{\mathcal{G}:=}]$

So $R^i f_*(\mathcal{F})(U_i) = 0$ since $H^i(\underbrace{f^{-1}(U_i)}_{\text{affine}}, \underbrace{\mathcal{F}|_{f^{-1}(V)}}_{\text{q coh}}) = 0$ $(i > 0)$. Hence $R^i f_*(\mathcal{F}) = 0$ $(\forall i > 0)$ and the conclusion follows. $\boxtimes$

$$\begin{tikzcd} 0 = \mathcal{G}(U_i) \arrow[rr, "\theta'"] \arrow[dr] & & a(\mathcal{G})(U_i) \arrow[dl] \\ & \mathcal{H}(U_i) \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

$\forall \mathcal{H}$ sheaf, so $a(\mathcal{G})(U_i) = 0$

Ex III.8.3: let $f: X \to Y$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the *projection formula*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Proof: WLOG, let $\mathcal{E} = \bigoplus_{fin} \mathcal{O}_Y$ since the question is local on Y. Then

$f^*\mathcal{E} = \bigoplus_{fin} \mathcal{O}_X$. $R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) = a[V \mapsto H^i(f^{-1}(V), \underbrace{\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}}_{= \bigoplus_{fin} \mathcal{F}}|_{f^{-1}(V)})]$

$\cong \bigoplus_{fin} a[H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})]$; $R^i f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} = a[V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) \otimes_{\mathcal{O}_Y} \bigoplus_{fin} \mathcal{O}_Y]$

$\cong \bigoplus_{fin} a[H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})]$. So the isomorphism follows. $\boxtimes$

1. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, with two projections $p_1, p_2 : X \longrightarrow \mathbb{P}^1_k$. For integers a, b, let $\mathcal{O}_X(a,b) = p_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b)$. It is an $\mathcal{O}_X$-module (coherent)

i) Compute the k-dim $h^q(X, \mathcal{O}_X(3,1))$ and $h^q(X, \mathcal{O}_X(-2,0))$ of cohomgroups H^q, $q \in \mathbb{Z}$

ii) Identify the $\mathcal{O}_{\mathbb{P}^1}$-modules $R^q(p_2)_* \mathcal{O}_X(-2,3)$ for $q \in \mathbb{Z}$.

Proof. : Let $\{U_1, U_2\}$ and $\{V_1, V_2\}$ be the standard covering of $\mathbb{P}^1_k$ the firstone and the second one $\to \mathbb{P}^1_k$, respectively, then $\{U_i \times V_j\}_{i,j \in \{1,2\}}$ is an open covering of X. Then the Čech complexes w.r.t $\{U_1, U_2\}$ and $\{V_1, V_2\}$ are

$0 \to \mathcal{O}_{\mathbb{P}^1}(a)(U_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)(U_2) \to \mathcal{O}_{\mathbb{P}^1}(a)(U_1 \cap U_2) \to 0$ and

$0 \to \mathcal{O}_{\mathbb{P}^1}(b)(V_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b)(V_2) \to \mathcal{O}_{\mathbb{P}^1}(b)(V_1 \cap V_2) \to 0$,

whose tensor product gives out the Čech complex w.r.t. $\{U_i \times V_j\}_{i,j=1,2}$ since $(p_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b))(U_i \times V_j) \cong \mathcal{O}_{\mathbb{P}^1}(a)(U_i) \otimes \mathcal{O}_{\mathbb{P}^1}(b)(V_j)$. Hence we have

$$h^q(X, \mathcal{O}_X(a,b)) = \bigoplus_{q_1+q_2=q} \dim_k (H^{q_1}(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^{q_2}(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(b)))$$

So
$$\begin{cases} h^0(X, \mathcal{O}_X(3,1)) = \binom{3+1}{3}\binom{1+1}{1} = 8 \\ h^1(X, \mathcal{O}_X(3,1)) = 0 \\ h^2(X, \mathcal{O}_X(3,1)) = 0 \end{cases} \qquad \begin{cases} h^0(X, \mathcal{O}_X(-2,0)) = 0 \\ h^1(X, \mathcal{O}_X(-2,0)) = \cancel{0}\ 1 \\ h^2(X, \mathcal{O}_X(-2,0)) = 0 \end{cases}$$

$h^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(0)) = 1$

ii) $R^q(p_2)_* \mathcal{O}_X(-2,3) = a[V \mapsto H^q(p_2^{-1}(V), \mathcal{O}_X(-2,3)|_{p_2^{-1}(V)})]$

$q=0$, RHS $=0$ 　　So LHS $=0$, $\forall q$.

$q=1$, RHS $=0$ 　$\mathcal{O}(3)$

$q \geq 2$, RHS $=0$