Hirzebruch-Riemann-Roch as a categorical trace

Abstract

Transcription of the talk https://www.mpim-bonn.mpg.de/node/7032.

In this talk, the speaker gave out a new proof of the Hirzebruch-Riemann-Roch formula using higher categorical tools hoping it can serve as an entry path for the audience to get familiar with the language of higher category theory and derived algebraic geometry.

This talk is based on papers [Mar09] [BZN13] [PV$^+$12] and finally [KP16]. The first part mainly follows the last paper, but in some proofs the speaker replaced the $(\infty, 2)$-categorical language with an 2-categorical one. And although in [KP16] the authors derived Hirzebruch-Riemann-Roch formula derictly from Atiyah-Bott, the speaker gave another approach in Part II and Part III which goes deeper into the machinary of derived algebraic geometry.

In the appendix some background information is given, based on Lurie's talk [Lur].

There may be typos and misunderstanding. Please consult the original videos if necessary. For convenience I've put some time labels.

Contents

1 Part I

Let X be a proper scheme over a field k of characteristic zero. $X \xrightarrow{F} X$ an endomorphism with isolated and transversal(i.e. the differential $dF_x - Id|T_xX$ is nonzero) fixed points. Let $\mathcal{E} \in Coh(X)$, $F^* \longrightarrow \mathcal{E} \Leftrightarrow \mathcal{E} \xrightarrow{\alpha} F_*\mathcal{E}$ be an adjunction. By taking sections we have the following diagram

$$\begin{tikzcd}
\Gamma(X,\mathcal{E}) \arrow[rr, "\alpha"] \arrow[dr, "\alpha"'] & & \Gamma(X,\mathcal{E}) \arrow[dl, "\simeq"] \\
& \Gamma(X, F_*\mathcal{E}) &
\end{tikzcd}$$

Theorem 1 (Atiyah-Bott)**.** *(05:24)*

$$Tr(\alpha, \Gamma(X,\mathcal{E})) = \sum_{x \in X^F} Tr(\alpha, \mathcal{E}_x) \cdot \lambda_x$$

where $\lambda_x = det(id - dF_x)$. Here Tr means graded trace, from even cohomology to odd cohomology.

Example 1. *$X = \mathbb{P}^1$, F rotation by some angle θ, $\mathcal{E} = \mathcal{O}$(the structure sheaf), $F^*\mathcal{E} \longrightarrow \mathcal{E} = id$. Generally this action will have two fixed points*

Then we have $\Gamma(X,\mathcal{E}) = k$, the map $\Gamma(X,\mathcal{E}) \xrightarrow{\alpha} \Gamma(X,\mathcal{E})$ acts as identity on k. $LHS = Tr(id_k) = 1$. $\mathcal{E}_x = k$ for each x, $Tr(\alpha, k) = 1$, hence $RHS = 1 \cdot 1/2 + 1 \cdot 1/2 = 1$.

Example 2. *Again let $X = \mathbb{P}^1$. Now set $\mathcal{E} = \mathcal{O}(n)$ and F be $\begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$. Then we have*

$LHS = (\theta^n + \theta^{n-1} + \cdots + \theta + 1) = \frac{\theta^n}{1-\theta^{-1}} + \frac{1}{1-\theta} = RHS$

(10:38) Now we are to review the notion of trace.

Definition 1. *Let O be a symmetric monoidal category, $o \in O$ is called* dualizable *if there is another object $o^\vee$ with two maps*[1]

$$1 \xrightarrow{unit} o \otimes o^\vee$$
$$o^\vee \otimes o \xrightarrow{counit} 1$$

satisifing the property that the following two compositions

$$o \xrightarrow{unit \otimes id_o} o \otimes o^\vee \otimes o \xrightarrow{id_o \otimes counit} o$$
$$o^\vee \xrightarrow{id_{o^\vee} \otimes unit} o^\vee \otimes o \otimes o^\vee \xrightarrow{counit \otimes id_{o^\vee}} o^\vee$$

are the identity map.

As an exercise, the reader can check that if $o^\vee$ exists, then it is unique up to unique isomorphism(12:50).

If o is dualizable, then there are several properties:

1. $Hom(o \otimes o_1, o_2) \xrightarrow{\sim} Hom(o_1, o^\vee \otimes o_2)$ (from linear algebra).

[1] In [KP16] they are called evaluation and coevaluation.

2. In particular, one can take o_1 to be 1. Then we have $Hom(o, o') \simeq Hom(1, o^\vee \otimes o') \simeq Hom(o'^\vee, o^\vee)$, which gives us a *dual map*.

Return to traces, if o is dualizable, then we have $T \in End(o) \rightsquigarrow Tr(T, o) \in End(1)$

Definition 2. *The composition* $1 \xrightarrow{unit} o \otimes o^\vee \xrightarrow{T\otimes id} o \otimes o^\vee \underset{\sim}{\xrightarrow{Twist}} o^\vee \otimes o \xrightarrow{counit} 1$ *is the* trace $Tr(T, o)$*.(16:25)*

Example 3. *Fix $O = Vect$, $V \in Vect$ is dualizable iff it is finite dimensional. In this case, one can recover the usual notion of trace.*

We can add more structure to O and consider it as a symmetric monoidal 2-category. Let o_1 and o_2 be two objects of it which are dualizable, T_1 and T_2 be two endomorphisms of them, respectively. Let $\phi : o_1 \to o_2$ be a morphism. Then we want to have a diagram commuting up to a 2-morphism which induces the blue arrow between traces

$$\begin{tikzcd}
o_1 \arrow[r, "T_1"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\swarrow_\alpha"] & o_1 \arrow[r, squiggly] \arrow[d, "\phi"] & Tr(T_1, o_1) \in End(1) \arrow[d, "Tr(\phi)"] \\
o_2 \arrow[r, "T_2"'] & o_2 \arrow[r, squiggly] & Tr(T_2, o_2) \in End(1)
\end{tikzcd}$$

Note that $End(1)$ is a category, so we can talk about arrows between its objects.

We need that ϕ admit a *right adjoint*.

What does this mean? We know that functors may have adjoint functors. Simiarly, a morphism in a 2-category may have an *adjoint morphism*. This notion is somewhat like the notion of a dual.

Definition 3. *(20:36) ϕ admit a right adjoint ϕ^R if there are 2-morphisms*

$$id_{o_1} \xrightarrow{unit} \phi^R \circ \phi$$

$$\phi \circ \phi^R \xrightarrow{counit} id_{o_2}$$

if ϕ^R exist then unique up to canonical morphism.

Now we are able to construct the morphism $Tr(\phi)$. Consider the following diagram

$$\begin{tikzcd}
& o_1 \otimes o_1^\vee \arrow[d, "\phi\circ id"] \arrow[dr, "counit"] & \\
1 \arrow[ur, "(T_1\otimes id)\circ unit"] \arrow[dr, "(T_2\otimes id)\circ unit"'] \arrow[r, phantom, "\swarrow"] & o_2 \otimes o_1^\vee \arrow[d, "id\circ(\phi^R)^\vee"] \arrow[r, phantom, "\swarrow"] & 1 \\
& o_2 \otimes o_2^\vee \arrow[ur, "counit"'] &
\end{tikzcd}$$

the outmost arrows are these in Definition 2, and $(\phi^R)^\vee$ is the dual map of ϕ^R.

By composing two 2-morphisms (the one from $1 \longrightarrow o_1 \otimes o_1^\vee \longrightarrow o_2 \otimes o_1^\vee \longrightarrow o_2 \otimes o_2^\vee$ to $1 \longrightarrow o_2 \otimes o_2^\vee$ and the one from $o_1 \otimes o_1^\vee \longrightarrow 1$ to $o_1 \otimes o_1^\vee \longrightarrow o_2 \otimes o_1^\vee \longrightarrow o_2 \otimes o_2^\vee \longrightarrow 1$) we obtain the outmost 2-morphism(the one from $1 \longrightarrow o_1 \otimes o_1^\vee \longrightarrow 1$ to $1 \longrightarrow o_2 \otimes o_2^\vee \longrightarrow 1$), which gives out $Tr(\phi)$.

It only remains to know what these two 2-morphisms are precisely.

The first one comes from the composing of the following three 2-morphisms, whose constructions are canonical.(26:10)

$$\begin{tikzcd}
& o_1 \otimes o_1^\vee \arrow[d, "\phi\otimes id"] \\
1 \arrow[ur, bend left, "T_1"] \arrow[r, bend left, "\phi\circ T_1"] \arrow[r, bend right, "T_2\circ\phi"'] \arrow[dr, bend right, "T_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & o_2 \otimes o_1^\vee \arrow[d, "id\otimes(\phi^R)^\vee"] \\
& o_2 \otimes o_2^\vee
\end{tikzcd}$$

In the diagram above we write T_1 for $(T_1 \otimes id) \circ unit$. Similar for other lefthandside arrows.

Next, by composing the vertical arrows we have the following diagram,

$$\begin{tikzcd}
o_1 \otimes o_1^\vee \arrow[dd, "\phi\otimes(\phi^R)^\vee"'] \arrow[dr, "unit"] & \\
{} \arrow[r, phantom, "\Swarrow"] & 1 \\
o_2 \otimes o_2^\vee \arrow[ur, "counit"'] &
\end{tikzcd}$$

which tells us the second 2-morphism corresponds to $id \longrightarrow \phi^R \circ \phi$, the unit of (ϕ_R, ϕ).

(31:16)Now we are to use it to prove Atiyah-Bott.

In this case we'll let O be a DG-category, which is a particular case of higher categories.

Unlike in an ordinary category, in a higher category the set $Hom(X, Y)$ is replaced by a "space" $Maps(X, Y)$(where X and Y are objects).[2]

Let $\mathcal{C}$ be an ∞-category, one can attach to it an ordinary cateogry $Ho(\mathcal{C})$. The objects in $Ho(\mathcal{C})$ are the same as those in C and for the morphisms we set $Hom_{Ho(C)}(X, Y) = \pi_0(Maps_{\mathcal{C}}(X, Y))$.

There is some subtleties with higher categories. For example, when taking tensor products, one should remember in which setting – the ordinary one or the higher categorical one – it was taken, for the answer may be different. For example, [3] $\pi_0(S_1 \times_S S_2) \neq \pi_0(S_1) \times_{\pi_0(S)} \pi_0(S_2)$.

∞-categories are needed to form limits/colimits. The triangulated category approach of cone is not canonical, but one can form a more complicated colimit.

Let C be a triangulated category, C_n be a simplicial object ($\cdots C_2 \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} C_1 \substack{\rightarrow\\ \leftarrow\\ \rightarrow} C_0$).

We want to take its geometric realization $|C_n| \in^? C$, which is the colimit of the diagram above. We may not be able to do this, or even we can, the result isn't what we want. It is because triangulated categories don't have sufficient structures.

A *DG-category* is an ∞-categorical upgrade of the notion of triangulated categories linear over k.

The mappings between a pair of objects in an ∞-category form a space. It's difficult to specify morphisms.

Example 4. *Let R be a (differential graded) k-algebra. Then all R-mods form a DG-category. (41:10) Its homotopy category $Ho(R\text{-mod})$ is the usual derived category $D(R\text{-mod})$.*
A particular case is when $R = k$, in this case $R\text{-mod} = Vect$

Let $Hom(C_1, C_2)$ be the category of DG functors from C_1 to C_2

[2]In this talk and in [GR17], the speaker tried to present facts of higher categories in a "model free" way. To get familiar with particular models(for example, the word "space" means weak Kan-complex) a useful resourse is [Gro10].

[3]In fact, this property plays a really important role in derived algebraic geometry. See Appendix 1.

Then DG categories constitute of an ordinary category. [4]

We'll assume that our DG categories are cocomplete(i.e. admit arbitrary direct sums) (44:11) More, when talking about DG functors, we assume they commute with arbitrary direct sums.

We denote the resulting category $O = DGCat$.

Observation:$Hom(Vect, C) \simeq C$ by $Vect \xrightarrow{\phi} C \rightsquigarrow \phi(k)$ on generators.

Since Homs on DG-functors themselves form a category, this can be viewed in the 2-categorical setting.(46:23)

In addition, it's also monoidal. The tensor product operation is defined by Lurie. We won't go over the explicit definition. Instead, we have the relation: $(R_1\text{-}mod) \otimes (R_2\text{-}mod) = (R_1 \otimes R_2)\text{-}mod$. (47:24) And the unit element is $Vect$.

The operation $C_1, C_2 \to C_1 \otimes C_2$ is defined on the class of categories having arbitrary direct sums. But it is functorial in each argument only with respect to functor commuting with arbitrary direct sums.

Let C be a dualizable DG-category (example: $C = R\text{-}mod$, then $C^\vee = R^{op}\text{-}mod$) Let $C \xrightarrow{T} C$ be an endofunctor, then we can assign the trace $Tr(T, C) \in Vect$(which is the Hochschild chain $HH_*(T, C)$) to it(50:21). If T is the identity functor, then it is the Hochschild chain on C.

If we have the following kind of diagram of DG-categories and DG-functors

$$\begin{tikzcd}
C_1 \arrow[r, "T_1"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\swarrow_\alpha"] & C_1 \arrow[d, "\phi"] \\
C_2 \arrow[r, "T_2"] & C_2
\end{tikzcd}$$

which means $\phi \circ T_1 \xrightarrow{\alpha} T_2 \circ \phi$, we can assign it a map of vector spaces $Tr(T_1, C_1) \xrightarrow{Tr(\alpha)} Tr(T_2, C_2)$ if ϕ admits a right adjoint that commutes with arbitrary direct sums.

In particular, we can set $C_1 = Vect$, $C_2 = C$, $T_1 = id$, $T_2 = T$. Then the diagram becomes

$$\begin{tikzcd}
k \in Vect \arrow[r, "id"] \arrow[d, mapsto] \arrow[d, "\phi", shift left=6] \arrow[dr, phantom, "\swarrow_\alpha"] & Vect \ni k \arrow[d, mapsto, shift left=6] \arrow[d, "\phi"] \\
c \in C \arrow[r, "T"'] & C \ni c
\end{tikzcd}$$

where $c = \phi(k)$ (In this case we can say C is "weakly equivariant with respect to T"). Then $c \xrightarrow{\alpha} T(c)$ induces $Tr(id) = k \xrightarrow{Tr(\alpha)} Tr(T, c)$. The second map is equivalent to an element in $Tr(T, c)$.We denote it $cl(\alpha) \in Tr(T, c)$, which means it's one of the interpretations of the Chern class.

Definition 4. *$c \in C$ is called* compact *if $Hom(C, -)$ commutes with arbitrary direct sums.*

One can verify that $\phi : Vect \longrightarrow C$ admits a right adjoint that commutes with arbitrary direct sum iff $\phi(k) = c \in C$ is compact. And this adjoint is just the Hom from this object.

As an example, let $C_1 = C_2 = Vect$ and $T = id$. Consider the diagram

$$\begin{tikzcd}
k \in Vect \arrow[r, "id"] \arrow[d, mapsto] \arrow[d, shift left=6] \arrow[dr, phantom, "\swarrow_\alpha"] & Vect \ni k \arrow[d, mapsto, shift left=6] \arrow[d] \\
V \in Vect \arrow[r, "id"'] & Vect \ni V
\end{tikzcd} .$$

[4]DG categories can also form an ∞-category. But we don't need it here. Drinfeld: "what does dg-cats form?"

We obesrve that α corresponds to a map from V to V, that is, and endomorphism of V. And the 2-categorical trace

$$\begin{tikzcd} Tr(Id, Vect) = k \arrow[d, "Tr(\alpha)"] \\ Tr(Id, Vect) = k \end{tikzcd}$$

corresponds to the usual notion of trace in linear algebra.

Moreover, we have a fact that 2-categorical traces compose well: the composition of traces is the trace of composition.

Now we will relate these results to the Atiyah-Bott formula.

We have a diagram

$$\begin{tikzcd} Vect \arrow[r, "id"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\Swarrow\alpha"] & Vect \arrow[d, "\psi"] \\ QCoh(X) \arrow[r, "F_*"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\Swarrow\beta=id"] & QCoh(X) \arrow[d, "\Gamma"] \\ Vect \arrow[r, "id"'] & Vect \end{tikzcd}$$

where $QCoh(X)$ is viewed as a DG-category and ψ sends k to a coherent complex $\mathcal{E}$(which is a compact object).

By composing the vertical arrows we have

$$\begin{tikzcd} Vect \arrow[r, "id"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\Swarrow"] & Vect \arrow[d, "\phi"] \\ Vect \arrow[r, "T"'] & Vect \end{tikzcd}$$

where ψ sends k to $\Gamma(X, \mathcal{E})$. The 2-morphism corresponds to an endomorphism $\Gamma(X, \mathcal{E}) \xrightarrow{\alpha} \Gamma(X, \mathcal{E})$

Note that we have mentioned 2-categorical traces compose well. Hence we have $Tr(\alpha, \Gamma(X, \mathcal{E})) \in k$ can be written as $k \xrightarrow{Tr(\alpha)} Tr(F_*, QCoh(X)) \xrightarrow{Tr(\beta)} k$.

What we will prove next are (1:08:50)

SETP 1 $\Gamma(F_*, QCoh(X)) = \bigoplus_{x \in X^F} k$.[5]

SETP 2 $Tr(\alpha) = \{Tr(\alpha, \mathcal{E}_x), x \in X^F\}$.

SETP 3 $Tr(\beta) = \{det^{-1}(id - dF_x)\}$.

Joke 1. *Speaker: If one wants to carry on such work, what does he need to do first?*
Audience: Clear the blackbord :-)(1:10:00)

First we need to calculate $Tr(F_*, QCoh(X))$

By definition, this can be down using the following diagram

[5] Hirzebruch-Riemann-Roch is exactly the same thing with one difference, that is, we'll replace the discrete fixed point scheme by a derived scheme since we need to deal with non-transversal intersections. See Part II and III.

$$\begin{tikzcd} Vect \arrow[r, "unit"] & QCoh(X) \otimes QCoh(X)^\vee \arrow[d, "F_* \otimes id"] & \\ & QCoh(X) \otimes QCoh(X)^\vee \arrow[r, "counit"] & Vect \end{tikzcd}$$

However, since we don't know what $QCoh(X)^\vee$ is, it's necessary to identify it.

The idea is to use the relation: $QCoh(X_1) \otimes QCoh(X_2) \xrightarrow{\sim} QCoh(X_1 \times X_2)$ (if either X_1 or X_2 is quasi-compact(quasi-separated)).

Lemma 1. *There is a canonical identification of $QCoh(X)^\vee$ with $QCoh(X)$*

Exercise 1.1. *Repeat the following proof to show that $(R\text{-}mod)^\vee = R^{op}\text{-}mod$*

Proof of Lemma 1(Sketch). Let's look at the following diagrams of schemes

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "\Delta"] & \\ pt & & X \times X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "p"] & X \times X \\ pt & \end{tikzcd} \text{ with } unit = \Delta_* \circ p^* \text{ and } \begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\Delta"] & pt \\ X \times X & \end{tikzcd} \text{ with } counit = p_* \circ \Delta^*$$

Using these components we can build the diagram(the blue part is obtained by taking the cartisan product of the two $X \times X$s)

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"] & X \times X \arrow[r, "p\otimes id"] \arrow[d, "\Delta\otimes id"] & X \\ X \times X \arrow[r, "id\otimes\Delta"] \arrow[d, "id\otimes p"] & X \times X \times X & \\ X & & \end{tikzcd}$$

which induces

$$\begin{tikzcd} QCoh(X) \arrow[r, "\Delta_*"] & QCoh(X \times X) \arrow[r, "(p\otimes id)_*"] & QCoh(X) \\ QCoh(X \times X) \arrow[u, "\Delta^*"] \arrow[r, "(id\otimes\Delta)_*"] & QCoh(X \times X \times X) \arrow[u, "(\Delta\otimes id)^*"'] & \\ QCoh(X) \arrow[u, "(id\otimes p)^*"'] & & \end{tikzcd}$$

By going from the lowerleft $QCoh(X)$ to the upperright $QCoh(X)$ along the zigzag (i.e. non-blue part) we obtain an the functor. On the other hand, the uppermost two horizontal arrows compose to be identity. Same for the leftmost two vertical ones.

If we have a base change $(\Delta \otimes id)^* \circ (id \otimes \Delta)_* = \Delta_* \circ \Delta^*$, then everything works as we want. But unfortunately, base change in algebraic geometry doesn't hold here. (1:18:47)

$$\begin{tikzcd}
Y_1 \times_Y Y_2 \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_1 \arrow[d, "f_1"] \\
Y_2 \arrow[r, "f_2"'] & Y
\end{tikzcd}$$

(Suppose in the diagram above f_1 and f_2 are transversal sections. Then the natural transformation $f_1^* \circ f_{2*} \longrightarrow g_{2*} \circ g_1^*$ is not necessarily an isomorphism. Hence base change fails. An example in shown in the following diagram.)

$$\begin{tikzcd}
pt \arrow[r] \arrow[d] & pt \arrow[d] \\
pt \arrow[r] & \mathbb{A}^1
\end{tikzcd}$$

Here is where derived algebraic geometry comes in. If we view these diagrams in the derived sense, then base change will hold.[6] By chasing the $QCoh$ diagram above(add units and counits if necessary) one is able to prove Lemma 1.[7] □

Now we can return to SETP 1 and carry on the remaining calculations.

Lemma 2. *For any F (with isolated transversal fixed points), $Tr(F_*, QCoh(X)) = \Gamma(X_F, O_{X^F})$, where the fixed point locus X^F is the fiber product*

$$\begin{tikzcd}
X^F \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \arrow[r, "Gr_F"] & X \times X
\end{tikzcd}$$

Here Gr_F means the graph of F.

"Form this fiber product in derived algebraic geometry means taking the derived tensor product of structure sheaves."(1:23:14)

Proof (Sketch). One can prove this by chasing the $QCoh$ diagram induced by

$$\begin{tikzcd}
X_F \arrow[rr, "counit"'] \arrow[d, "i"'] & & X \arrow[r, "p"] \arrow[d, "\Delta"] & pt \\
X \arrow[r, "\Delta"] \arrow[d, "p"'] \arrow[rr, bend right, "Gr_F"'] & X \times X \arrow[r, "F \circ id"] & X \times X & \\
pt & & &
\end{tikzcd}$$

□

[6] Note that the derived fiber product of two affine schemes is not the spectrum of the tensor product of their coordinate rings. (1:12:10)

[7] Note that the sections in this case are transversal. So in this computation higher Tors disappear(See Appendix A). But in the proof of Hirzebruch-Riemann-Roch in Part II and III it is not this case and one need to face "truly derived" schemes.

Let $\mathcal{E}$ be a compact object $\in Coh(X)$, $F^*\mathcal{E} \xrightarrow{\alpha} \mathcal{E} \Leftrightarrow \mathcal{E} \longrightarrow F_*\mathcal{E}$. We can attach α the class $cl(\alpha)$ and identify it with an element in $\Gamma(X^F, O_{X^F})$ as follow:

Note that we have $i^*F^*\mathcal{E} \xrightarrow{i^*\alpha} i^*\mathcal{E}$, but $i^*F^*\mathcal{E} \simeq i^*\mathcal{E}$ since X^F is the fixed point locus of F. Hence we obtain an endomorphism $i^*\alpha \in End_{QCoh(X^F)}(i^*\mathcal{E})$. Since $QCoh(X^F)$ is itself a symmetric monoidal cateogry and i^*F is an object in it, one can take the trace of i^*F in 1-categorical sense and obtain an endomorphism of the unit object(i.e the structure sheaf). Hence $cl(\alpha)$ makes sense as an element in $\Gamma(X^F, \mathcal{O}_{X^F})$. (1:29:22)

Lemma 3. *$cl(\alpha)$ equals $Tr(i^*\alpha, i * \mathcal{E})$.*

Note that the former one is a 2-categorical trace and the latter one is a 1-categorical trace.

Proof(Sketch). By diagram chasing. □

From the lemmas above we can easily obtain the results of SETP 1 and SETP 2, that is, $Tr(F_*, QCoh(X)) = \oplus_{x\in X^F} k$(because the conditions guarantee $X^F = \sqcup_{x\in X^F} pt$ as a scheme) and $cl(\alpha) = \{Tr(\alpha, \mathcal{E}_x)\}$.

For STEP 3, we need to calculate the inverse of the determinent.

By the 2-functorality of the trace we have the functor γ gives $\oplus_{x\in X^F} k \underset{\{r_x\}}{\longrightarrow} k$. We need to show $\forall x, r_x = det^{-1}(Id - dF_x)$.

We already know that $Tr(\alpha, \Gamma(X, \mathcal{E})) = \Sigma_{x\in X^F} Tr(\alpha, \mathcal{E}_x) \cdot \gamma_x$ (because 2-functoriality behaves well with respect to compositions). Pick particular point $x \in X$, i.e $pt \xrightarrow{i_x} X$, set $\mathcal{E} = (i_x)_*(k)$. We have $F_*((i_x)_*(k)) \overset{\alpha}{\simeq} (i_x)_*(e)$ and $\Gamma(X, (i_x)_*(k)) = k$. By calculation we know α acts on global sections as $\Gamma(X, \mathcal{E}) \xrightarrow{id} \Gamma(X, \mathcal{E})$

The results above gives $Tr(\alpha, \Gamma(X, (i_x)_*(k))) = \Sigma_{x'\neq x} Tr(\alpha, ((i_x)_*(k))_{x'}) \cdot \gamma_x + Tr(\alpha, (i_*(k))_x) \cdot \gamma_x$. The LHS is just 1. And for the RHS all terms expect the last one are 0(because $((i_x)_*(k))_{x'}$ is zero for $x' \neq x$). Thus we have $1 = Tr(\alpha, (i_*(k))_x) \cdot \gamma_x$.

One may think that $(i_*(k))_x$ is just k, but it is not this case(for if not, then we'll get in trouble with the result). Don't forget that fibers are also taken in the derived sense. For $i_x^* i_{x*}(k)$ the higher Tors appear.[8] So $H^{-i}(i_x^* i_{x*}(k)) = \bigwedge^i(T_x^* X)$ and $Tr(\alpha, (i_*(k))_x) = \Sigma_{i=0}^{dim} Tr(dF_x, \bigwedge^i(T_x^* X)) = det(1 - dF_x)$, which complete the whole proof.

Note that the trick of choosing a particular $\mathcal{E}$ to simplify the calculation doesn't work for Hirzebruch-Riemann-Roch. There's where Todd class appears. In Part II and III we will talk about Todd class.

[8] See Appendix A

2 Part II

In the last part we have $C \xrightarrow{T} C \rightsquigarrow Tr(T,C) \in Vect$ if C is a dualizable DG-catgegory and T is an endofunctor. And

$$\begin{tikzcd}
C_1 \arrow[r, "T_1"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\swarrow\alpha"] & C_1 \arrow[r, rightsquigarrow] \arrow[d, "\phi"] & Tr(T_1, C_1) \arrow[d, "Tr(\phi,\alpha)"] \\
C_2 \arrow[r, "T_2"] & C_2 \arrow[r, rightsquigarrow] & Tr(T_2, C_2)
\end{tikzcd}$$

where ϕ is a DG-functor admitting a right adjoint that commutes with arbitrary direct sums.

A particular case of this is when C_1 is $Vect$ and T_1 is the identity functor $C_2 = C$, $T_2 = T$

$$\begin{tikzcd}
k \in Vect \arrow[r, "id"] \arrow[d, mapsto] \arrow[d, "\phi"] \arrow[dr, phantom, "\swarrow\alpha"] & Vect \ni k \arrow[d, "\phi"'] \arrow[d, mapsto] \\
c \in C \arrow[r, "T"'] & C \ni c
\end{tikzcd}$$

where $c = \phi(k)$. Then $c \xrightarrow{\alpha} T(c)$ induces $Tr(id) = k \xrightarrow{Tr(\alpha)} Tr(T,c)$ and hence gives out an element $cl(\alpha) \in Tr(T,c)$.

Now we turn to Hirzebruch-Riemann-Roch.

Let X be a proper scheme over a character 0 field k, $\mathcal{E} \in Coh(X)$ a coherent complex. Hirzebruch-Riemann-Roch says that the graded dimension(Euler characteristic) $dim\ \Gamma(X,\mathcal{E})$ equals the integral $\int_x ch(\mathcal{E})Td(X)$.[9] We need to interpret each term of this formula in a categorical sense.

A sketch of the proof goes like this: After transcribed the whole Part, check again

Consider a similar diagram as the one in Part 1

$$\begin{tikzcd}
Vect \arrow[r, "id"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\swarrow_{\alpha=id}"] & Vect \arrow[d, "\psi"] \\
QCoh(X) \arrow[r, "id"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\swarrow_{\beta=id}"] & QCoh(X) \arrow[d, "\Gamma"] \\
Vect \arrow[r, "id"'] & Vect
\end{tikzcd}$$

where $QCoh(X)$ is viewed as a DG-category and ψ sends k to a coherent complex $\mathcal{E}$(which is a compact object). The composition of ϕ and Γ sends k to $\Gamma(X,\mathcal{E})$.

(05:22)Now we take traces and obtain the left vertical row in the following diagram. This induces a scalar multiplication by the graded dimension $dim\ \Gamma(X,\mathcal{E})$ in k(see the right vertical arrow), which corresponds to the left hand side of Hirzebruch-Riemann-Roch.

[9]The relative version — Grothendieck-Riemann-Roch — can be proved in a similar way.

$$\begin{tikzcd}
Tr(Id, Vect) \arrow[rr, "\simeq"] \arrow[d] & & k \arrow[dl, "ch(\mathcal{E})"] \arrow[ddd, "dim\ \Gamma(X,\mathcal{E})"] \\
Tr(Id, QCoh(X)) \arrow[r, "\simeq"] \arrow[dr, "\simeq"] \arrow[dd] & \bigoplus_i \Gamma(X, \Omega^i[i]) \arrow[d, "Td(X)"] & \\
& \bigoplus_i \Gamma(X, \Omega^i[i]) \arrow[dr, "\int"] & \\
Tr(Id, Vect) \arrow[rr, "\simeq"] & & k
\end{tikzcd}$$

On the other hand, remember in Atiyah-Bott we've identified the trace of F_* with the global section of the structure sheaf on the fixed point scheme. We'll do the same, but we'll further identify the global section of structure sheaf with something. $Tr(Id, QCoh(X)) \simeq \bigoplus_i \Gamma(X, \Omega^i[i])$(which can be called Hodge cohomology). We will identify $ch(\mathcal{E})$ and $Td(X)$ as elements in this vector space. However, there are two different ways to do this identification, which induces two maps $k \xrightarrow{ch(\mathcal{E})} \bigoplus_i \Gamma(X, \Omega^i[i])$ and $\bigoplus_i \Gamma(X, \Omega^i[i]) \xrightarrow{\int} k$.

We need to say what the "integral" is. In the diagram
$$\begin{tikzcd}
\bigoplus_i \Gamma(X, \Omega^i[i]) \arrow[r] \arrow[rr, bend left, "\int"] & \Gamma(X, \Omega^n[n]) = \Gamma(X, \omega_X) \arrow[r] & k
\end{tikzcd}$$
the first horizontal arrow is the projection to the top component. Then notice that $\Omega^n[n]$ is just the dualizing sheaf ω_X of X. From Serre duality we have the second horizontal arrow. By composing these all we obtain the integral map.

The most mysterious part is the Todd class. It will be identified with the discrepency of the two identification of $Tr(Id, QCoh(X))$ with $\bigoplus_i \Gamma(X, \Omega^i[i])$.

One can combine all these and obtain the Hirzebruch-Riemann-Roch formula.

Unlike the previous proofs of Hirzebruch-Riemann-Roch, this proof is local on X(the Todd class and Chern class are local quantities on X). Maybe Grothendieck would like this proof.

Now we are to give the details. One recollection: $QCoh(X)^\vee \simeq QCoh(X)$, from which we deduce $Tr(F_*, QCoh(X)) = \Gamma(X^F, O_{X^F})$ by diagram chase in the proof of Atiyah-Bott. This is valid for any endomorphism of our X, in particular for the identity. Remember that X^F is the fiber product

$$\begin{tikzcd}
X^F \arrow[r, "i"] \arrow[d, "i"] & X \arrow[d, "\Delta"] \\
X \arrow[r, "Gr_F"] & X \times X
\end{tikzcd}$$

taken in the derived sense.

This is true for any (derived) scheme X, as singular as you want.

Now we plug in for F being the identity map and then obtain $Tr(Id, QCoh(X)) = \Gamma(Inert_X, O_{Inert_X})$ on the following funny gadget called the *Inertia scheme* of X, which is the self intersection of the diagonal.

$$\begin{tikzcd}
Inert_X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

If you take the fiber product in the category of ordinary schemes, you can only recover X itself. But the whole point is that you take it in the derived sense.

Here is a crash course on derived algebraic geometry.[10]

Joke 2. *If you were to teach an introductory course in algebraic , what would you do first?*
Audience: Pray.
Speaker: You pray, then you clean the blackboard...

The first step is to introduce affine (derived)schemes.

The category $dSch^{aff}$ of affine derived schemes is the category opposite to $ComAlg(Vect^{\leq 0})$(take the monoidal category of complex vector spaces that are "connected"(cohomologically zero and below) and take commutative algebras in its symmetric monoidal category)(13: 40).[11]

Be careful that this is down in a higher categorical sense: Take two algebras R_1 and R_2, instead of taking the set $Hom(R_1, R_2)$, we take the space $Maps(R_1, R_2)$. We don't care about the concrete definition of "spaces", what we need to know is just how to do computations. First, we will assume that R_1 is a polynomial algebra, such as $Sym(V), V \in Vect^{\leq 0}$. Compute in this case. By definition, $Maps_{ComAlg}(Sym(V), R) = Maps_{Vect^{\leq 0}}(V, R)$. We are to compute $Maps$ in the ∞-category $Vect^{\leq 0}$. It's not a discrete set.

Example 5. *If V is concentrated in cohomological degree 0, then the i-th homotopy group $\pi_i(Maps(V, R)) = Hom(V, H^{-i}(R))$ (remember that R is an algebra in $Vect^{\leq 0}$ and we forget the algebra structure and view it as a complex).*

This is what you do for polynomial algebras. In general, any R' can be written as a colimit of free algebras. Once you know how to compute $Maps$ from a polynomial algebra, you know how to compute $Maps$ from any R'. It's the limit of corresponding mapping spaces from these free guys. That is the algorithm. So that if you want to compute anything you can just use this procedure instead of the theoretical definition.

In this way we've defined affine derived schemes.

Now we want to define all derived schemes. Some people stick to traditions and define schemes using atlases, like some people are still using iPhone 6 or even 5.[12] This is not how things should be down in 2017. We are to do the following:

We need to mention that derived schemes as opposed to more general geometric gadgets are not as distingushed. In fact you want to define more and more general algebraic geometric notion right away. So you define the notion of a prestack, this is an arbitrary functor from $(dSch^{aff})^{op}$(which is the higher category of commutative algebras) to the (higher)category Spc of spaces, i.e. an arbitrary thing that has the Grothendieck functor of points. Why to spaces instead of to sets? The reason is that even the representable guys take values in spaces because maps from R_1 to R_2 is already a space. So we must go into spaces.

The general principle is that every other algebraic geometric notion you'll encounter is a particluar case of a prestack. Derived schemes will be a full category of prestacks. So from now on we need only put conditions, not extra structures. We won't use atlases. We just need to specify certain conditions.(20:20)

[10] Terms in this talk are a little confusing(for example, an "algebra" is not a single one but a complex). It's better to consult Appendix A and [GR17]'s Chapter 1.10 and Chapter 2.

[11] In Appendix A, affine derived schemes are discussed more concretely and intuitively.

[12] Up to this lecture(given in Jan. 2017), the newest iPhone was iPhone 7. But now they are iPhone XS and iPhone XR, which was just released in Sep. 13, 2018.

Definition 5. *A* derived scheme *is a prestack that admits an open cover by derived affine schemes.(Plus quasi-compact condition? 20:48)*

This definition is incomplete for we need to say what it means for a prestack to admit an open cover. If S is a derived affine scheme, we need to say what it means for a map $S \to \mathcal{Y}$ to be an open cover($\mathcal{Y}$ a prestack). We do it in the same way as ordinary algebraic geometry. We require the map to have certain properties(we do base change and ask the base change morphism to have certain properties).

$$\begin{tikzcd} S \times_{\mathcal{Y}} S' \arrow[r] \arrow[d] & S' \arrow[d] \\ S \arrow[r] & \mathcal{Y} \end{tikzcd}$$

The lower horizontal arrow is an open cover if the upper horizontal arrow is an open cover. In the category of prestacks we can easily form fiber products so $S \times_{\mathcal{Y}} S'$ is also a prestack. We need to define when a map from a prestack to a derived scheme is an open cover. First of all, we require $S \times_{\mathcal{Y}} S'$ itself be representable by a derived affine scheme. Need to define the notion of basic affine covers inside $S = Spec(R)$.

Let $R \in ComAlg(Vect^{\leq 0})$ and $f \in H^0(R)$. We can canonically attach to them the localization $R_f \in ComAlg(Vect)^{\leq 0}$. We define it by the universal property. We say that $Maps(R_f, R')$ is $\{\phi \in Maps(R, R') \mid \phi : R \to R' \ s.t.\ \phi(f) \in H^0(R') \ is\ invertible\}$. It exists and unique up to canonical isomorphism. Set $Spec(R_f) \to Spec(R)$ to be the basic opens, and define more general opens using them. This gives the notion of open maps between derived affine schemes, and hence recover the notion of derived schemes and Zariski descent.

Now we are to talk about quasi-coherent sheaves. Let $\mathcal{Y}$ be an arbitrary prestack, one can define the category $QCoh(\mathcal{Y}) := \lim\limits_{dSch^{aff} \ni S \to \mathcal{Y}} QCoh(S)$. Here $S \to \mathcal{Y}$ means a point in the mapping space $\mathcal{Y}(S)$. Specifying an object $\mathcal{F} \in QCoh(\mathcal{Y})$ is equivalent to specifying for every map $S \to \mathcal{Y}$ the pullback $\mathcal{F}_{S,\mathcal{Y}}$ of $\mathcal{F}$ to S along this map(S is a derived scheme). This is the notion of quasi-coherent sheaf on an arbitrary prestack. In particular it can apply to derived schemes.

Back to Inertia. $Inert_x = X \times_{X \times X} X$. Let S be a topological space and s a point in it. We can form the base loop space $\Omega_s(S) = pt \times_S pt$ in the category of spaces(the fiber product is taken in the homotopical sense). It has a natural structure of a group in the category of spaces(i.e. it belongs to $Grp(Spc)$).[13] In particular we have $\begin{tikzcd} X \arrow[d, shift left, "p_1"] \\ X \times X \arrow[u, shift left, "\Delta"] \end{tikzcd}$ and $Inert_x = \Omega_X(X \times X)$. [14]

Let's go back to our trace construction. Let us also remind the following: In the general situation when we have $\mathcal{E} \in QCoh(X)$, $\mathcal{E} \xrightarrow{\alpha} F_*(\mathcal{E})$ (which is equivalent to $F^*(\mathcal{E}) \xrightarrow{\alpha} \mathcal{E}$). We can attach

[13] Let X be the base scheme(or prestack), for prestack $\mathcal{Y}$, consider $\begin{tikzcd} X \arrow[d, shift left, "s"'] \\ \mathcal{Y} \arrow[u, shift left, "\pi"'] \end{tikzcd}$ where $\pi \circ s = Id$(regard $\mathcal{Y}$ as fibered over X by means of π and this fiberation admits a section s. You can think it as a family of pointed spaces over X. In our special case we will take $\mathcal{Y}$ to be S). Then we can form the relative loop space $\Omega_X(\mathcal{Y}) = X \times_{\mathcal{Y}} X$. It is a group object in the category of derived schemes over X(i.e. in $Grp(dSch/X)$).

[14] Remark: the Inertia is canonically what is called the groupoid over X. You can make it into a group by preferring one of the projections. This turns Inertia into a derived group scheme over X. It's a very important gadget. $Inert_X$ belongs to $Grp(dSch_X)$. In particular you can take its fiber over a given point $x \in X$, $Inert_{X|x} = pt \times_X pt$. It's a derived group whose underlying group is just a point but it has a nontrivial derived structure. You can actually create a lot of mathematics from this object.

to it its class $cl(\mathcal{E},\alpha) \in \Gamma(X^F, \mathcal{O}_{X^F})$. But there is also a lemma which we didn't prove, that is, $cl(\mathcal{E},\alpha) \simeq Tr(i^*(\alpha), i^*(\mathcal{E}))$(by diagram chase).(35:33)

$$\begin{tikzcd} X^F \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\Delta"] \\ X \arrow[r, "Gr_F"'] & X \times X \end{tikzcd}$$

We will then apply this lemma to our situation and see what we get.

$$\begin{tikzcd} Inert_X \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\Delta"] \\ X \arrow[r] & X \times X \end{tikzcd}$$

In this case, α is identity, and the general construction gives us an endomorphism $i^*\mathcal{E} \to i^*\mathcal{E}$ on $Inert_X$. In fact, this is not an identity map and it encodes the Chern class of $\mathcal{E}$. We'll see what this map is exactly in a moment.

Let's go back to the general situation having a $\mathcal{Y}$. Note that $\Omega_X(\mathcal{Y})$ is a group derived scheme over X. Let $\mathcal{E}' \in QCoh(\mathcal{Y})$. And let's consider its pullback by means of s: $s^*(\mathcal{E}') =: \mathcal{E} \in QCoh(X)$. If you look at the construction, you'll find that $\mathcal{E}$ is naturally equivariant with respect to this group. You can think it as a representation of this group, i.e. $\mathcal{E} \in Rep(\Omega_X(\mathcal{Y}))$.

What does it mean to be a representation? Well, in particular, there is a potentially nontrivial isomorphism $i^*\mathcal{E} \to i^*\mathcal{E}$ and it's compatible with the group law. There is a cocycle condition satisified. See the diagram on the right. [15]

$$\begin{tikzcd} \Omega_X(\mathcal{Y}) \times_X \Omega_X(\mathcal{Y}) \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \Omega_X(\mathcal{Y}) \arrow[d, "i"] \\ X \arrow[d, "s"] \\ \mathcal{Y} \end{tikzcd}$$

This is the general situation[16] and now we will turn to our particular situation.

In our situation, $\mathcal{E}' = p_2^*(\mathcal{E})$(remember we prefered one of the projections when thinking of Ω as a derived group scheme over X. And note that $p_2^*(\mathcal{E})$ is not $p_1^*(\mathcal{E})$, which will make it trivial). Then the resulting action of $\Omega_X(X \times X) = Inert_X$ on $S^* \circ p_2^*(\mathcal{E}) = \mathcal{E}$ is our map $i^*\mathcal{E} \longrightarrow i^*\mathcal{E}$.

Let's summarize what happened. We started from $\mathcal{E}$. We created the quasi-coherent sheaf $\mathcal{E}'$ on $X \times X$, and we pulled it back to X and it becames a representation of the Inertia. This representation exactly encodes the map $i^*\mathcal{E} \longrightarrow i^*\mathcal{E}$.

Let's give the interpretation of the class in these terms. We obtain the following: $cl(\mathcal{E},\alpha) \in \Gamma(Inert_x, O_{Inert_X})$ is the character of $\mathcal{E}$ viewed as a representation of $Inert_X$.

The next step is from here to get the Chern characters we know it. So we need to see the differential forms ω^i appearing(? 44:44). And this will happen in a second. This will be the passage between groups and Lie algebras.

Now we'll make a digression to another facet of derived algebraic geometry. In Part I we mentioned that the main reason for the existance for derived algebraic geometry is the ability to form fiber products so that base change takes place. Yet there is one more aspect to make it extremely useful, that is, deformation theory. The slogan is this: if you want to define the cotangent complex — even the cotangent complex of classical schemes — the most conceptual way to define it is to use derived algebraic geometry, as we will now explain.

[15] In fact, $\Omega_X(\mathcal{Y}) \xrightarrow{i} X$ should be two maps, but since $\Omega_X(\mathcal{Y})$ is a group — not a groupoid — these two coincide.

[16] The action of $\Omega_X(\mathcal{Y})$ on $\mathcal{E}$ is trivial if $\mathcal{E}' = \pi^*(\mathcal{E})$ for $\mathcal{E} \in QCoh(X)$. (41:00)

Let $\mathcal{Y}$ be an arbitrary prestack, and $S = Spec\ R$ be a derived affine scheme mapping to it. $S \xrightarrow{y} \mathcal{Y}$, y a point in $\mathcal{Y}(S)$. To this datum we may attempt to attach a cotangent space $T^*_y\mathcal{Y}$. It's not guaranteed to exist(this will be the question of representability of certain functor), but we still attempt to define it as an object in $QCoh(S)^{\leq 0}$(note that $QCoh(S) = R\text{-}Mod$). If this guy exists, then we will say that $\mathcal{Y}$ admits a coconnetcive tangent space at y.

Namely, it's defined by the universal property. We say that $Maps(T^*_y\mathcal{Y}, M) = Maps(Spec(R \oplus \epsilon M), \mathcal{Y}) \times_{Maps(SpecR,\mathcal{Y})} \{y\}$, where ϵ is square zero.[17] If it happens to be corepresentable, we will say that $\mathcal{Y}$ admits a coconnective tangent space at the point y. If you take H^0(? 49:55) of $T^*_y\mathcal{Y}$ you will recover the classical definition of the cotangent space defined using dual numbers.

Example 6. *Let $\mathcal{Y}$ be a classical scheme. Illusie defines $T\mathcal{Y} \in QCoh(\mathcal{Y})^{\leq 0}$. Let's fix a field valued point Spec $k \xrightarrow{y} \mathcal{Y}$ and take the derived fiber $T\mathcal{Y} \otimes^L_{\mathcal{O}_y} k_y$. Now we want to say who this guy is and who their cohomologies are: $Hom(H^{-i}(T\mathcal{Y} \otimes^L_{\mathcal{O}_y} k_y), V) = \pi_0(Maps(Spec(k \oplus \epsilon V), \mathcal{Y}) \times_{Maps(Spec(k),\mathcal{Y})} \{y\})$.*

The left hand side doesn't depend on derived algebraic geometry but we give a expression of it via derived algebraic geometry. V is placed in cohomological degree $-i$. k is cohomological degree zero. $k \oplus \epsilon V$ is a DG-algebra.

This is an expression of something classical in terms of derived algebraic geometry. This interpretation is really helpful for understanding what deformation theory is.

(Some questions by the audiences 53:32)

Definition 6.

1. *A derived affine scheme $Spec(R)$ is said to be* almost of finite type*(a.f.t.) if $H^0(R)$ is of finite type over k and for every i, $H^i(R)$ is finitely generated as a $H^0(R)$-module.*

2. *A derived scheme is* almost of finite type*(a.f.t.) if it can be covered by a.f.t affines.*

3. *Similiar for quasi-coherent sheaves. Let $\mathcal{Y}$ be a.f.t. and $\mathcal{F} \in QCoh(\mathcal{Y})^-$. We say that $\mathcal{F}$ is a.f.t if its individual cohomology groups [18] are coherent sheaves over the underlying classical scheme $\mathcal{Y}^{cl}$ of $\mathcal{Y}$.*

With this digression, we will now state some theorems about the relation between group objects and Lie algebras.

Consider the category $dSch_{aft}/X$ of derived schemes locally of finite type over X. For this discussion we will assume X is smooth to simplify the proposition[19]. We will construct a functor $Grp(dSch_{aft}) \xrightarrow{Lie} LieAlg(QCoh(X))$, [20].

First of all, there is a forgetful functor $LieAlg(QCoh(X)) \xrightarrow{oblv} QCoh(X)$ and we will say what the underlying quasicoherent sheaf is. As we might expect, the underlying quasi-coherent sheaf is the tangent space. But so far we have only talked about the cotangent space, so one needs to be careful.

[17] We are just mimicing the definition of tangent spaces using dual numbers(See Hartshorne). If you take in classical algebraic geometry, you take R to be a field. And the speaker miswrote $T^*_y\mathcal{Y}$ as $T_y\mathcal{Y}$ in the beginning and corrected these typos at 1:02:07.

[18] These cohomology groups are quasi-coherent sheaves in the classical sense, over the underlying classical scheme of $\mathcal{Y}$. We should have said if we have a derived affine scheme $S = Spec(R)$ we can take $H^0(R)$ and it's a classical scheme. We can apply the same operation globally, i.e. if we have a derived scheme, it has an underlying classical scheme by just discarding the lower cohomologies. And we have a complex, it's individual cohomologies are quasi-coherent sheaves on the underlying classical scheme.

[19] This is because we are dealing with quasi-coherent sheaves. We can remove this assumption and let X be arbitrary if we allow ourselves to talk about *Ind-coherent sheaves.*

[20] QCoh(X) is a symmetric monoidal DG-category over a ground field of characteristic 0. If we have any symmetric monoidal DG-cateogry over k we can talk about algebras over any k-linear operad, in particular, the Lie operad. So the notion $LieAlg(QCoh(X))$ makes sense.

What we will proceed is the following. Let $\mathcal{G} \in Grp(dSch_{aff}/X)$, we attach to it the cotangent space $T_e^*\mathcal{G}$ relative to X. [21] $T_e^*\mathcal{G} \in QCoh_{aft}(X)^{\leq 0}$(There is a general lemma that if a derived scheme is a.f.t. then its cotangent complex will be a.f.t. in this definition).

We can consider the category $QCoh_{aft}(X)^{<\infty}$ and $QCoh_{aft}(X)^{>-\infty}$. If X is smooth, then there is a canonical contravariant equivalence $(QCoh_{aft}(X)^{<\infty})^{op} \xrightarrow{\mathbb{D}} QCoh_{aft}(X)^{>-\infty}$among them, and it's just the naive duality. If we have something locally free, we just pass to the dual. But smoothness of spectral sequence converge(? 1:04:00) and it will be a equivalence between these two categories. So finally we define $Lie(\mathcal{G})$ to be the dual $\mathbb{D}(T_e^*\mathcal{G}) \in QCoh_{aft}(X)^{>-\infty}$

Theorem 2.

(a) $Lie(G)$ has a canonical structure of Lie algebra in $QCoh(X)$.

(b) The functor Lie is an equivalence between the following two categories $Grp(Form(dSch_{aft}/X))$[22] and $LieAlg(\mathbb{D}(QCoh_{aft}(X)^{\leq 0})) \subset LieAlg(QCoh(X))$.[23]

Now we are to prove this theorem and show how this give rise to the appearence of Chern characters.

(Part II-1 ends)

(Part II-2)

Below haven't check

The easiest is constructe the functor in one direction. We start with the Lie algebra $\mathcal{L}$ and attach to it the group $\exp(\mathcal{L})$. The object we will consider is the universal enveloping algebra $U(\mathcal{L}) \in CommHopfAlg(QCoh(X))$ of $\mathcal{L}$. If you consider commutative algebras, recall (1) the operation of tensor product is the coproduct in commutative algebras(with unit) in any symmetric monoidal categories($A_1 \otimes A_2 \to B$ is the universal domain of bilinear maps); (2) Tensor product of cocommutative algebras is the product $B \to A_1 \otimes A_2 \iff B \to A_1$ & $B \to A_2$. $Cocomm(Bialg) = AssociativeAlg(CocommCoAlg)$ it admits associative product. It can be intepreted as $Monoids(CocommCoAlg)$. $Cocomm(Bialg) \supseteq CocomHopfAlg = Grp(CocommCoalg) \subseteq Monoids(CocommCoAlg)$.

With these remarks, we are able to construct a functor in one direction. Starting from a Lie algebra and we can construct the corresponding group object in formal schemes over X. We are trying to produce a formal scheme. It's enought to evaluate it $Maps(Spec_X(R), \exp(\mathcal{L}))$ where $R \in CommAlg(Coh(X)^{\leq 0})$. Because we are trying to construct something formal, we don't need to test for all commutative algebras in quasi-coherent sheaves and it's enough to only consider the coherent ones. We will say what this is. This will be down as follows: by definition, $Maps(Spec_X(R), \exp(\mathcal{L})) = Maps_{CocommCoAlg(Coh(X))}(\mathbb{D}(R), U(\mathcal{L}))$. Here we only remember the coconnective structure of $\mathcal{L}$. Because $U(\mathcal{L})$ is a Hopf algebra, therefore is a group like object, hence the $Maps$ also acquire a structure of a group, i.e. $Maps_{CocommCoAlg(Coh(X))}(\mathbb{D}(R), U(\mathcal{L})) \in Grp(Spaces)$. This characterises the functor $U(\mathcal{L})$ uniquely. And this is indeed an equivalence of categories.

[21] It may or may not exist, but for derived schemes it always exist. Take its cotangent space at the unit section relative to X(i.e. by means of
$$\begin{tikzcd} G \\ X \arrow[u, "e"] \end{tikzcd}$$
).

[22] *Form* means formal schemes. If you pass to the underlying classical scheme and kill the nilpotents, then the projection to X is an isomorphism.

[23] In fact we can remove the ugly restrictions and construct a equivalence between $Grp(Form(dSch_{aft}/X))$ and $LieAlg(QCoh(X))$. But we need to enlarge the objects allowed. That is, enlarge derived schemes to *Inf-schemes*.

$$
\begin{array}{ccc}
\mathrm{Cocomm}(\mathrm{BiAlg}) & = & \mathrm{AssociativeAlg}(\mathrm{Cocomm\,CoAlg}) \\
 & & \parallel \\
\cup\!\!| & & \mathrm{Monoids}(\mathrm{Cocomm\,CoAlg}) \\
 & & \cup\!\!| \\
\mathrm{Cocomm\,Hopf}(\mathrm{Alg}) & = & \mathrm{Grp}(\mathrm{Cocomm\,CoAlg})
\end{array}
$$

(Audience asked questions 07:50)

There is an important feature of this construction, namely the exponential map. Let $\mathcal{L}_0$ be the trivial Lie algebra underlying the same object of $QCoh(X)$ as $\mathcal{L}$.

Corollary 1. *The formal derived schemes $exp(\mathcal{L})$ and $exp(\mathcal{L}_0)$ are isomorphic(the isomorphism denotes $\exp(\mathcal{L}_0) \xrightarrow{exp} \exp(\mathcal{L})$)*

Recall that there exists a canonical isomorphism of cocommutative coalgebras: $U(\mathcal{L}) \simeq Sym(\mathcal{L}_0) = U(\mathcal{L}_0)$(Recall the Poincare-Birkhoff Theorem). And as you see, when I wrote my $\exp(\mathcal{L})$ as a functor, I only used the cocommutative colagebra structure, and it's the only group structure used the Hopf algebra structure. So therefor this corollary follows formally from the construction.

Let's just say explicitly what $\exp(\mathcal{L})$ is. $\mathbb{D}(\mathcal{L})$ is a quasi-coherent complex in degree less than or equal to zero. We have $\exp(\mathcal{L}) = Spec(Sym_{\mathcal{O}_X}(\mathbb{D}(\mathcal{L})))^\wedge$($^\wedge$ means completion along the zero section).

A baby case is when X is a point.

Example 7. *$X = pt$, $\mathcal{L} = f.d.$ Lie algebra in cohomological degree zero. So $\exp(\mathcal{L})$ is the corresponding formal group. And there is a canonical isomorphism $\mathcal{L}_0^\wedge \xrightarrow{exp} \exp(\mathcal{L})$. We take our Lie algebra, we forget the Lie algebra structure and view it as an vector space. Complete this vector space at zero and the exponential gives us an isomorphism of the formal schemes. This is an analogue of the assertion above in a relative situation.*

(to be continued)

3 Part III

(to be continued)

A Bezount's theorem and nonabeilan homological algebra

This part is a brief introduction to the basic ideas of derived algebraic geometry, based on Lurie's talk [Lur]

In an elementary algebraic geometry course, we get in touch with the following famous theorem:

Theorem 3 (Bezout's Theorem for transversal intersection)**.** *Let C and C' be projective curves in $\mathbb{CP}^2$ with degree n and m respectively. If C and C' meet transversely(denote $C \perp C'$), then $\#(C \cap C') = nm$.*

Let $[C]$ and $[C']$ be the fundamental classes of C and C' respectively, then their product is nm. $[C \cap C']$ is of dimension 0, i.e. it constitutes of discrete points.

Now suppose $C \not\perp C'$, then this number may be less than nm. In order to obtain nm, one need to consider the intersection multiplicity.

For simplicity, we assume $C, C' \in \mathbb{A}^2 = \mathbb{C}[x, y]$ and they meet transversely. Let $f(x, y) = 0$ and $g(x, y) = 0$ be the defining equation of C and C'. The $\mathbb{C}[x, y]/(f)$ is the ring of functions on C. Similarly, $\mathbb{C}[x, y]/(g)$ is the ring of functions on C'. And we have $\mathbb{C}[x, y]/(f, g)$ is the ring of functions on $C \cap C'$. This is a scheme-theoretical intersection. It remembers more informations than the set-theoretical one.

Example 8. *Let C be $x = 0$, C' be $y = 0$. Then they meet transversely at the origin. We have $\mathbb{C}[x, y]/(x, y) \cong \mathbb{C}$, which is the ring of funcitons on the origin.*

However, in the case of non-transversal intersection, the situation seemed embarassing:

Example 9. *Add Picture Let C be $x = 0$, C' be $x = y^2$. Then $\mathbb{C}[x, y]/(x, x - y^2) \cong C[x, y]/(y^2)$ is not the ring of functions on the origin, which is the intersection of C and C'.*

But by taking intersection multiplicity into consideration the result is corrected:

Theorem 4 (Bezout's Theorem)**.** *Let C and C' be plane (projective) curves, then we have*

$$nm = \sum_{p \in C \cap C'} dim_{\mathbb{C}}(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{C'})_p$$

One may ask: Can this result be generalized to higher dimensional varieties? The answer is: Yes.

To make it precise, let $C, C' \in \mathbb{CP}^{a+b}$ be (projective) varieties of dimension a and b and degree n and m, respectively. When $C \perp C'$, then we also have $\#C \cap C' = nm$ as in Theorem 3.

If C and C' don't meet transversely, the situation turns to be more complicated. Let C be defined by a single equation f and C' be defined by a ring R of functions. We may want to focus on the dimension of $R/(f)$. This can be done by looking at the sequence

$$R \xrightarrow{f} R \longrightarrow R/(f) \; .$$

If R is finite dimensional, then multiplication by f is a linear transformation of vector spaces and can be represented by a matrix. The results should be stable under pertubation(14:08). Addpic.

$$(dim\ coker\ f) - (dim\ ker\ f) = 0$$

when R is finite dimensional("first approximation"(15:33)).

Sometimes R is infinite-dimensional, but the difference of $dim\ coker\ f$ and $dim\ ker\ f$ is finite. We still need to add some corrections.(17:23).

Let A be a commutative ring, M and N be A-mods. The we can define the torsion groups $Tor_i^A(M, N)$, where $Tor_0^A(M, N) = M \otimes_A N$. We consider all of them simutaneously.

Theorem 5 (Serre's Intersection Formula)**.** *Let C and C' be as above. Then we have*

$$nm = \sum_{p \in C \cup C'} (\sum_{i=0}^{\infty} dim\ Tor_i^{\mathcal{O}_{\mathbb{P}^{a+b},p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p})).$$

This formula is given in Serre's book [J.P00].

It's good to have this formula since it allows us to compute the intersection multiplicity for higher dimensional varieties. However, it is not as conceptual as in the case

$$[C] \cap [C'] = [C \cap C']$$

for planar curves.

This formula encodes not only the informations contained in the scheme-theoretical intersection(i.e. ring of functions on it), but also those contained in higher torsion groups. Fortunately, those informations can be encoded by derived algebraic geometry.

Before giving out the general discussion, let's return to Bezout's theorem.

Let L and L' be two lines. Generically they will intersect at a single point. But they may also coincide. In the latter case, the formula

$$[L][L'] = [L \cap L']$$

fails to be true since the dimensions on both sides don't coincide(24:28).

Example 10. *Let L be $x = 0$, L' be $y = 0$. In this case, we have $\mathbb{C}[x, y]/(x, y) \cong \mathbb{C}$, which is as good as one might expect.*

However, if we let both L and L' be $x = 0$. Then we obtain $\mathbb{C}[x, y]/(x, x) \cong \mathbb{C}[y]$. Something goes wrong with the dimension.

In the later case of this example, we observe that the reason for the result to be wrong is $\mathbb{C}[x, y]/(x) \cong \mathbb{C}[x, y]/(x, x)$, i.e. dividing (x) two times does not make any difference with dividing (x) only once. We may expect that there is some modification of the algebraic structure that can "remember" the difference.

In fact, this is achieveable. The key idea is borrowed from homotopy theory.

Consider in the world of sets, Add Pic(27:22). We have generators(0-cell), relations(1-cell), like a CW-complex in homotopy theory.

One can also impose such operations on commutative rings and obtain something like a combination of commutative rings and spaces.

Topological Commutative rings.

Topological space with ring structure, and the addition and multiplication are continuous.

Example 11. *Let R be a commutative ring with discrete topology on its underlying set. Then R is a topological commutative ring.*

There are also topological commutative ring in different branches of mathematics, such as the real numbers and the p-adic numbers. But they don't admit the property we want, since in the case of the real numbers, it is contractivle, hense its homotopy groups are trivial and can not provide us sufficient informations; in the case of p-adic numbers, it is totally discrete.

$R \xrightarrow{equiv} S$ if they have the same homotopy type(i.e. the arrow induces isomorphisms between homotopy groups). The arrow is not necessarily an isomorphism, but it's good enough.

We need only consider these topological rings obtained by imposing operations on polynomial rings.

Give $\mathbb{C}[x, y]$ the discrete topology. Consider in the case that y intersects with itself.

Add Picture $\xrightarrow{\text{generate free commutative ring}}$ something with good formal property (35:05)
We want to explore its homotopy type. (36:41 audience question)
Let's return to the comparison of Bezout and Serre's formulas.
What we need to do is to take tensor product in the world of topological commutative rings.
$\mathbb{C}[x, y] \longrightarrow R$
components
$\pi_0 R = \mathbb{C}[x]$, the result in the ordinary case.
$\pi_1 R = \mathbb{C}[x]$, here we take the loop into consideration.
$\pi_i R = 0, i > 1$.
If we take tensor product in the world of topological commutative rings
$S = \mathcal{O}_C \otimes^L_{\mathcal{O}_{\mathbb{P}^{a+b}}} \mathcal{O}_{C'}$. L is the derived functor in the sense of homotopical algebra.
$\pi_0 S = \mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^{a+b}}} \mathcal{O}_{C'}$, the ordinary tensor product.
$\pi_i S = Tor_i^{\mathcal{O}_{\mathbb{P}^{a+b}}}(\mathcal{O}_C, \mathcal{O}_{C'})$.
Now we are to give out an important definition. Let R be a topological commutative ring.
$\pi_0 R$ the underlying commutative ring of R.
Review the defnintion of a scheme.

Definition 7 (Derived Scheme)**.** *A* (affine) derived scheme *is a topological space X with a sheaf of topological rings $\mathcal{O}_X$ s.t. $(X, \mathcal{O}_X) \cong (Spec\ A, \mathcal{O}_{Spec\ A})$. $A \in TopCommRing$.*

Spec A as a topological space is just *Spec* $\pi_0 A$ in the ordinary sense. What makes it different is the sheaf on it. In ordinary algebraic geometry we have $\mathcal{O}_{Spec\ A}(U_f) = A[f^{-1}]$. In derived algebraic geometry it is also this case, but the operation of adding the inverse of f to A should be taken in the world of topological commutative rings.

Having this definition, we can have Bezout and Serre's formula improved.

Theorem 6. *Let C and C' be smooth projective variety. They intersect in the derived sense. We have $[C] \cup [C'] = [C \cap C']$. Here $[C], [C'], [C \cap C']$ are virtual fundamental classes.*

By going through the construction of virtual fundamental classes in derived geometry, one can find that they are essentially the same as those defined in Gromov-Witten theory.[Toë14] In GW-theory the virtual fundamental classes often have a "wrong" dimension, this is due to the correction of "homotopical terms".

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