LECTURE 5: EXISTENCE AND UNIQUENESS OF SOLUTIONS TO DIFFERENTIAL EQUATIONS

Theorem 1. *(Existence and uniqueness) Suppose that $p(x)$ and $q(x)$ are continuous on an open interval $I = (a, b)$ containing x_0. Then there exists a unique solution y that satisfies the initial value problem*

$$y' + p(x)y = q(x), \;\; y(x_0) = y_0.$$

This essentially means that the solution of a linear differential equation $y' + p(x)y = q(x)$ subject to the initial condition $y(x_0) = y_0$, exists throughout any subinterval of I containing x_0 in which the functions p and q are continuous. The discontinuities of the solution occur exactly where p and q are not continuous.

Example 1. Determine (withour solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

(a) $y' + (\tan t)y = \sin t, \;\; y(\pi) = 0$
(b) $(4 - t^2)y' + 2ty = 3t^2, \;\; y(-3) = 1$
(c) $(4 - t^2)y' + 2ty = 3t^2, \;\; y(1) = -3$
(d) $(t - 3)y' + (\ln t)y = 2t, \;\; y(1) = 2$

Solution. (a) Note that $p(t) = \tan t$ and $q(t) = \sin t$. We see that $p(t)$ is discontinuous at $t = ..., -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, ...$These values are zeros of $\cos t$ on the real line. Thus we have $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is the interval containing π to guarantee the existence and uniqueness of the solution to IVP.

(b) Write the DE in the standard form

$$y' + \frac{2t}{4 - t^2}y = \frac{3t^2}{4 - t^2}$$

Note that both $p(t) = \frac{2t}{4-t^2}$ and $q(t) = \frac{3t}{4-t^2}$ are discontinuous at $t = -2, 2$ where $4 - t^2 = 0$. Thus $(-\infty, -2)$ is the interval in which the solution to IVP is unique.

(c) From part (b), we see that $(-2, 2)$ is the desired interval containing $t_0 = 1$ in which the solution is unique.

(d) Note that the standard form linear DE is

$$y' + \frac{\ln t}{t - 3}y = \frac{2t}{t - 3}$$

Note that $p(t) = \frac{\ln t}{t-3}$ is discontinuous at $t < 0$ and $t = 3$ and $q(t) = \frac{2t}{t-3}$ is discontinuous at $t = 3$. Hence we have $(0, 3)$ is the interval containing $t_0 = 1$ in which it assure the existence and uniqueness of solution to IVP.

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Example 2. Determine how the interval in which the solution to IVP

$$y' + y^3 = 0, \ \ y(0) = y_0$$

exists depends on the initial value y_0

Solution. The DE is separable and it is not easy to obtain that $y(x) = \frac{y_0}{\sqrt{2y_0^2 t+1}}$. The solution exists provided $2y_0^2 t + 1 > 0$. Thus we see that if $y_0 \neq 0$, the solution exists for $t > -\frac{1}{2y_0^2}$. If $y_0 = 0$, then the solution $y(x) = 0$ for all x.