Proofs

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Theorem One

The limiting distribution of the Poisson(λ) distribution as $\lambda \to \infty$ is normal.

Pf. Let $X \sim Poisson(\lambda)$ which has the probablility mass function

$$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, ...$$

and moment generating function

$$\mathrm{M}_X(t) = e^{e^{\lambda(e^t-1)}}$$

We will specifically consider the standardized Poisson random variable X

$$\frac{X-\lambda}{\sqrt{\lambda}}$$

which has the Moment Generating Function

$$\begin{aligned}
\mathrm{M}_{(X-\lambda)/\sqrt{\lambda}}(t) &= \mathrm{E}\left[\exp\left(t * \frac{X-\lambda}{\sqrt{\lambda}}\right)\right] \\
&= \exp(-t\sqrt{\lambda}) * \mathrm{E}\left[\exp\left(\frac{tX}{\sqrt{\lambda}}\right)\right] \\
&= \exp\left(-t\sqrt{\lambda} + \lambda(e^{t/\sqrt{\lambda}} - 1)\right)
\end{aligned}$$

Now, we take the limit as λ approaches ∞ and utilize the taylor series expansion

$$e^{t/\sqrt{\lambda}} = 1 + t\lambda^{-1/2} + \frac{t^2\lambda^{-1}}{2!} + \frac{t^3\lambda^{-3/2}}{3!} + ... + \frac{t^n\lambda^{-n/2}}{n!}$$

Therefore, we have

$$\begin{aligned}
\mathrm{M}_{(X-\lambda)/\sqrt{\lambda}}(t) &= \exp\left(-t\sqrt{\lambda} + \lambda(e^{t/\sqrt{\lambda}} - 1)\right) \\
&= \exp\left(-t\sqrt{\lambda} + \lambda\left[1 + t\lambda^{-1/2} + \frac{t^2\lambda^{-1}}{2!} + \frac{t^3\lambda^{-3/2}}{3!} + ... + \frac{t^n\lambda^{-n/2}}{n!} - 1\right]\right) \\
&= \exp\left(-t\sqrt{\lambda} + t\sqrt{\lambda} + \frac{t^2}{2} + \frac{t^3\lambda^{-1/2}}{6} + ... + \frac{t^n\alpha^{-(n-1)/2}}{n!}\right)
\end{aligned}$$

To find the limiting moment generating function, we take the limit of this moment generating function as $\lambda \to \infty$ which results in

$$\exp\left(\frac{t^2}{2}\right)$$

which is the moment generating function of $\mathrm{N}(0,1)$.