Renewal Processes

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1. Is is true that:

   (a) $N(t) < n$ if and only if $S_n > t$

   Solution: If $N(t) < n$, then by time t, less than n has been observed. If $S_N > t$, then the nth term will be observed at greater than time t. These statements coincide with one another and are thus **true.**

   (b) $N(t) \leq n$ if and only if $S_n \geq t$

   Solution: If by time t, n or less has been observed, then we know nothing about S_n. The nth observation could have been observed at a time before t, so this is **false.**

   (c) $N(t) > n$ if and only if $S_n < t$

   Solution: This statement means that at time t, greater than n has been observed. However, just because the observation of n occurred at time less than t does not mean that by time t, an additional observation has been made. Therefore, this statement is **false.**

   (d) $N(t) > n$ if and only if $S_n \leq t$

   Solution: If at time t, greater than n have been observed, then S_n must be strictly less than t, so this is **false.**

   (e) $N(t) \geq n$ if and only if $S_n < t$

   Solution: If at time t, greater than or equal to n observations have occurred, then S_n can either be equal to t or less than t. This does not match the above statement and so it is **false.**

2. Suppose the interarrival time distribution for a renewal process is Poisson(μ) (so the interarrival times are *discrete* but the renewal process is defined for continuous time).

   (a) Find the distribution of S_n for each n.

   Solution: As these are each the sums of different Poisson variables, we have

   $$S_i \sim \; Pois(i\mu)$$

   (b) Find the distribution of $N(t)$ for all t.

   Solution:

   $$\begin{aligned} P\left(N\left(t\right) \leq n\right) &= P(S_{n+1} > t) \\ &= 1 - P(S_{n+1} \leq t) \\ &= 1 - e^{-(n+1)\mu} \sum_{i=0}^{\lceil t \rceil} \frac{\left((n+1)\mu\right)^i}{i!} \end{aligned}$$

   which is the Erlang CDF for $Erlang(\lfloor t \rfloor + 1, \mu)$ with $(n+1)$ being the variable.

3. Betsy is a consultant. Each time she gets a job to do, it lasts 3 months on average. The time between jobs is exponentially distributed with mean 2 weeks. At what rate does Betsy start new jobs in the long run?

> **Solution:** By the memoryless property of the exponential distribution, the time between new jobs is 3 months and 2 weeks.

4. Let $N_1(\cdot)$ and $N_2(\cdot)$ be independent renewal processes. Let $N(t) := N_1(t) + N_2(t)$ for all t.

   (a) Are the interarrival times of $N(\cdot)$ independent?

   > **Solution:** Let $N_1(t)$ and $N_2(t)$ be processes with interarrival times distributed $Uniform(0, 1)$. Consider that the first arrival X_1 occurs at time $1/2$ and occurs from $N_1(t)$. So, the first process renews, and the expected time until the next renewal from that process is 1. However, the second process has not yet occurred, and from this point is expected to occur at $3/4$. So, the process as a whole has not renewed. Also, it is clear that the second interarrival time will be dependent on which distribution the previous arrival is drawn from. From these results, this process is not independent or identically distributed.

   (b) Are the interarrival times of $N(\cdot)$ identically distributed?

   > **Solution:** No, see (a).

   (c) Is $N(\cdot)$ a renewal process?

   > **Solution:** No, as the interarrivals are not iid.

5. Let $\{U_k : k \in \{1, 2, ...\}\}$ be independent, Uniform[0,1] variables. Define

$$N := \min\{n : \sum_{k=1}^{n} U_k > 1\}$$

Calculate $E[N]$.

> **Solution:** We can treat this as a renewal process where the arrivals S_n are distributed $Uniform(0, 1)$. We can find the expectation of N by computing
>
> $$\mathbb{E}[N] = 1 + \sum_{n=1}^{\infty} nP(N(1) = n)$$
>
> Since we are treating this as a renewal process, we have
>
> $$\begin{aligned} P(N(1) = n) &= P(N(1) > n - 1) - P(N(1) > n) \\ &= P(S_{n-1} \le 1) - P(S_n \le 1) \\ &= P\left(\sum_{i=1}^{n-1} U_k \le 1\right) - P\left(\sum_{i=1}^{n} U_k \le 1\right) \\ &= \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n-1}{n!} \\ \mathbb{E}[N] &= 1 + \sum_{n=1}^{\infty} n\frac{n-1}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e \end{aligned}$$

6. Let $N(\cdot)$ be a renewal process. For a random variable Y, let $\overline{F}_Y(y) := 1 - F_Y(y) = P(Y > y)$. Show that $\overline{F}_{X_{N(t)+1}}(x) \geq \overline{F}_X(x) \forall x$. Compute both sides exactly for a Poisson process of rate λ by conditioning on $N(t)$ and $S_{N(t)}$.

Solution: In order to calculate the distribution of $X_{N(t)+1}$, we condition on the time of the last renewal at $S_{N(t)}$.

$$\begin{aligned} P(X_{N(t)+1} > x \mid S_{N(t)} = t - s) &= P(\text{interarrival time} > x | \text{interarrival time} > s) \\ &= P(\text{interarrival time} > x)/P(\text{interarrival time} > s) \\ &= \frac{\overline{F}(x)}{\overline{F}(s)} \geq \overline{F}(x) \quad \text{As } \overline{F}(s) \in [0,1] \end{aligned}$$

and so if we take the expectation on both sides of

$$P(X_{N(t)+1} > x \mid S_{N(t)} = t - s) \geq \overline{F}(x)$$

we have

$$\begin{aligned} P(X_{N(t)+1} > x) &\geq \overline{F}(x) \\ \overline{F}_{X_{N(t)+1}}(x) &\geq \overline{F}(x) \end{aligned}$$

For a Poisson process, the interarrival times are distributed exponential, and so we have

$$e^{-\lambda X_{N(t)+1}} \geq e^{-\lambda X}$$

7. Let $A(t)$ and $Y(t)$ denote the age and excess at t of a renewal process. Fill in the missing terms: (Refer to page 460)

(a) $A(t) \geq x$ iff 0 events in the interval:

Solution: $[t - x, t]$

(b) $Y(t) > x$ iff 0 events in the interval:

Solution: $[t, t + x]$

(c) $P(Y(t) > x) = P(A(?) \geq ?)$.

Solution: $P(Y(t) > x) = P(A(t + x) \geq x)$

(d) Compute the joint distribution of $A(t)$ and $Y(t)$ for a Poisson process.

Solution: The time since the last renewal and the next renewal for a Poisson process will both be distributed exponentially. Therefore for a Poisson process with rate λ

$$\begin{aligned} P(Y(t) > x) &= P(N(t + x) - N(t) = 0) \\ &= P(N(x) = 0) \quad \text{by independent incrememnts} \qquad = e^{-\lambda x} \\ \implies P(Y(t) \leq x) &= 1 - e^{-\lambda x)} \end{aligned}$$

$$\begin{aligned} P(A(t) \geq x) &= P(N(x) - N(t - x) = 0) \\ &= P(N(x) = 0) \quad \text{By independent increments} \\ &= e^{-\lambda x} \\ \implies P(A(t) < x) &= 1 - e^{-\lambda x} \end{aligned}$$

Again my independent increments, the age and excess will be independent, so the joint distribution will be the product of the two distributions, so we have

$$\begin{aligned} f_{A(t),Y(t)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} P(A(t) < x) \frac{\mathrm{d}}{\mathrm{d}x} P(Y(t) \leq x) \\ &= \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda x}\right) \\ &= \lambda^2 e^{-2\lambda x} \end{aligned}$$

8. Consider a single-server bank in which potential customers arrive at a Poisson rate λ. However, an arrival only enters the bank if the server is free when he or she arrives. Let G denote the service distribution. Note: The system capacity is 1, so any customer who arrives while the server is busy is lost. What fraction of time is the server busy?

9. A ski slope has n skiers that continually and independently climb up and ski down. The time it takes skiers to climb up or ski down are independent of each other and non-lattice [1], but not identically distributed. In fact, the time it takes the ith skier to climb up has distribution F_i each time and the time it takes her to ski down has distribution G_i each time. All F_i and G_i have finite means.
   (a) Let $N_i(t)$ denote the total number of times skier i has skied down the slope by time t. What is $\lim t \to \infty N_i(t)/t$ a.s. and in mean?
   (b) Let $N(t) = \sum_{i=1}^{n} N_i(t)$. What is $\lim t \to \infty N(t)/t$ a.s. and in mean?
   (c) If $U(t)$ denotes the number of skiers climbing the hill as time t, what is $\lim_t \to \infty E[U(t)]$?

10. A system consisting of four components is said to work whenever both at least one of components 1 and 2 work and at least one of components 3 and 4 work. For instance, an airplace with two engines on each wing may work provided at least one engine on each wing functions properly. Suppose component i alternates between a working and a failed state in accordance with a non-lattice alternating renewal process with respective distribution F_i and G_i, $i \in \{1, 2, 3, 4\}$. If these alternating renewal processes are independent, find $\lim t \to \infty P(\text{system is working at time } t)$.

11. A warehouse stores and sells items. Customers arrive according to a Poisson process of rate 2 per day. Each customer demands exactly one item. The warehouse gives an item to a customer when it has one, but turns away the customer otherwise. The warehouse gives an item to a customer when it has one, but turns away the customer otherwise. The warehouse orders A more items from the supplier when the warehouse becomes empty[2], but it takes a random amount of time for the order to arrive; the order time has a mean of 3 days. Each such order costs the warehouse \$50 (regardless of the order size). Each item costs the warehouse \$1 per day to store. The supplier charges \$70 per item, but the warehouse sells each item for \$80.
    (a) What is the long-run profit per day made by the warehouse?
    (b) What value of A maximizes the long-run profit per day?

12. Let $Q(t)$ denote the number of customers in the system of a $M/G/1/2$ queue (that is, customers arrive according to a Poisson process and require independent service times wiht distribution G, and there is one server, and the maximum queue size is 2). Assume G has fine mean and the arrival rate is λ. Show that in the long run, the proportion of time that $Q(t) = 1$ is

$$\frac{\int_0^\infty e^{-\lambda x} \overline{G}(x)\,\mathrm{d}x}{\int_0^\infty \left(e^{-\lambda x} G(x) + \overline{G}(x)\right) \mathrm{d}x}$$

[1] A nonnegative random variable Y is *lattice* if $\exists d \geq 0$ such that $\sum_{i=0}^{\infty} P(Y = nd) = 1$. The largest d with this property is the period of Y. The property is needed to properly apply the key renewal theorem (also called Blackwell's theorem) and in the study of alternating renewal processes. Refer to Ross's *Stochastic Processes*.

[2] Note that reordering items before running out would be a more realistic and likely better policy. However, this scenario is more difficult to analyze.