Time Series Homework 5
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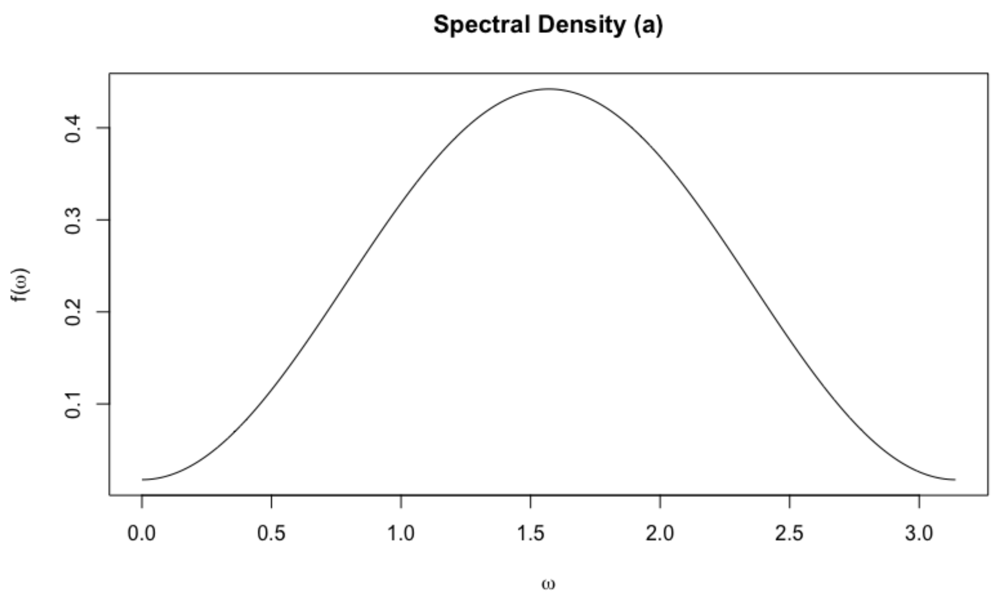
1. Find Spectral density for the time series below. Also use R to plot those spectral densities where $e_t \sim WN(0,1)$

   (a) $X_t = e_t - \frac{2}{3}e_{t-2}$

Solution:

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{2\pi}\left|1 - \frac{2}{3}e^{-2i\omega}\right|^2 \\ &= \frac{\sigma^2}{2\pi}\left|1 - \frac{2}{3}\left(\cos(2\omega) - i\sin(2\omega)\right)\right|^2 \\ &= \frac{\sigma^2}{2\pi}\left(\left(1 - \frac{2}{3}\cos(2\omega)\right)^2 + \left(\frac{2}{3}\sin(2\omega)\right)^2\right) \end{aligned}$$

```
spectrala <- function(x) {
  cos_part <- (1 - (2/3)*cos(2*x))^2
  sin_part <- ((2/3)*sin(2*x))^2
  full <- cos_part + sin_part
  outer <- (1/(2*pi))
  return(outer*full)
}
```

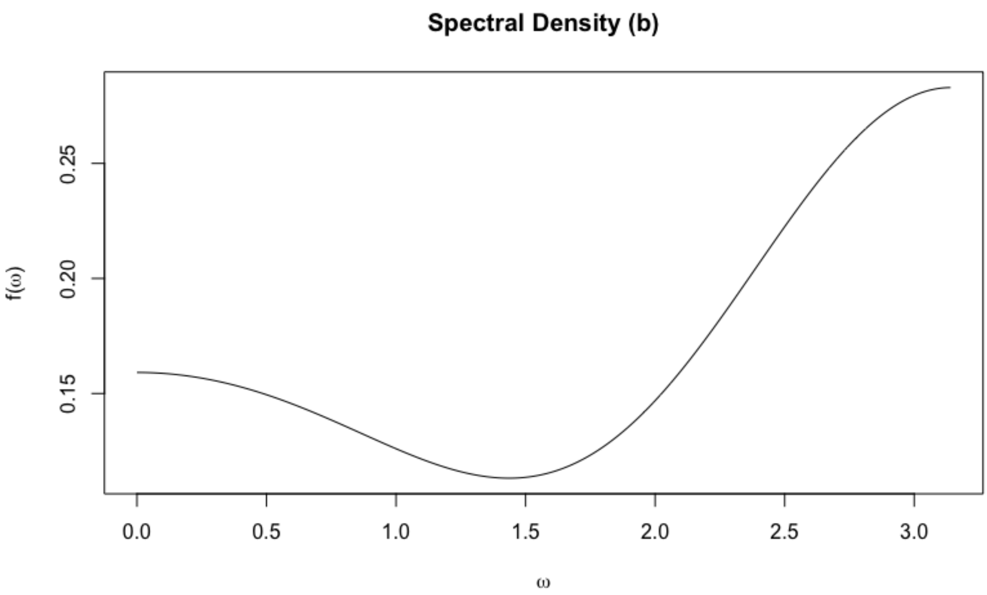


(b) $X_t = e_t - \frac{1}{6}e_{t-1} - \frac{1}{6}e_{t-2}$

Solution:

$$\begin{aligned}
f(\omega) &= \frac{\sigma^2}{2\pi}\left|1 - \frac{1}{6}e^{-i\omega} - \frac{1}{6}e^{-2i\omega}\right|^2 \\
&= \frac{\sigma^2}{2\pi}\left|1 - \frac{1}{6}\left(\cos(\omega) - i\sin(\omega)\right) - \frac{1}{6}\left(\cos(2\omega) - i\sin(2\omega)\right)\right|^2 \\
&= \frac{\sigma^2}{2\pi}\left|1 - \frac{1}{6}\left(\cos(\omega) - \cos(2\omega)\right) - \frac{1}{6}i\left(sin(\omega) + \sin(2\omega)\right)\right| \\
&= \frac{\sigma^2}{2\pi}\left(\left(1 - \frac{1}{6}\left(\cos(\omega) - \cos(2\omega)\right)\right)^2 + \left(\frac{1}{6}\left(sin(\omega) + \sin(2\omega)\right)\right)^2\right)
\end{aligned}$$

```
spectralb <- function(x) {
  cos_part <- (1 - (1/6)*(cos(x) - cos(2*x)))^2
  sin_part <- ((1/6)*(sin(x) + sin(2*x)))^2
  full <- cos_part + sin_part
  outer <- (1/(2*pi))
  return(outer*full)
}
```

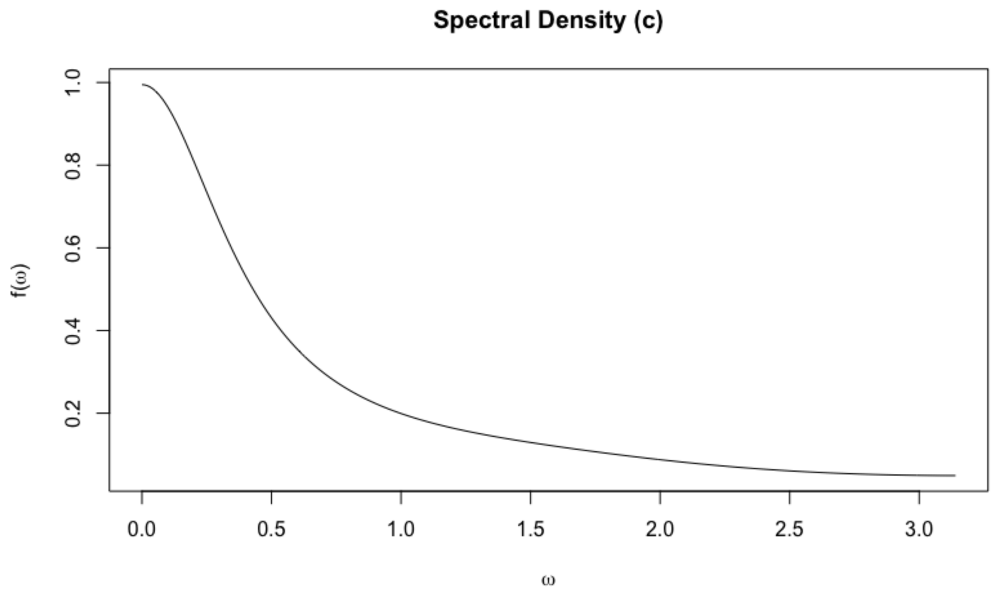


(c) $X_t = 0.7X_{t-1} - 0.1X_{t-2} + e_t$

Solution:

$$\begin{aligned}
f(\omega) &= \frac{\sigma^2}{2\pi}\left|1 - 0.7e^{-i\omega} - 0.1e^{-2i\omega}\right|^{-2} \\
&= \frac{\sigma^2}{2\pi}\left|1 - 0.7\left(\cos(\omega) - i\sin(\omega)\right) - 0.1\left(\cos(2\omega) - i\sin(2\omega)\right)\right|^{-2} \\
&= \frac{\sigma^2}{2\pi}\left|1 - 0.7\cos(\omega) - 0.1\cos(2\omega) + i\left(0.7\sin(\omega) + 0.1\sin(2\omega)\right)\right|^{-2} \\
&= \frac{\sigma^2}{2\pi}\left(\left(1 - 0.7\cos(\omega) - 0.1\cos(2\omega)\right)^2 + \left(0.7\sin(\omega) + 0.1\sin(2\omega)\right)^2\right)^{-1}
\end{aligned}$$

```
spectralc <- function(x) {
  cos_part <- (1 - .7*cos(x) + .1*cos(2*x))^2
  sin_part <- (.7*sin(x) + .1*sin(2*x))^2
  full <- cos_part + sin_part
  outer <- (1/(2*pi))
  return(outer*(full^(-1)))
}
```

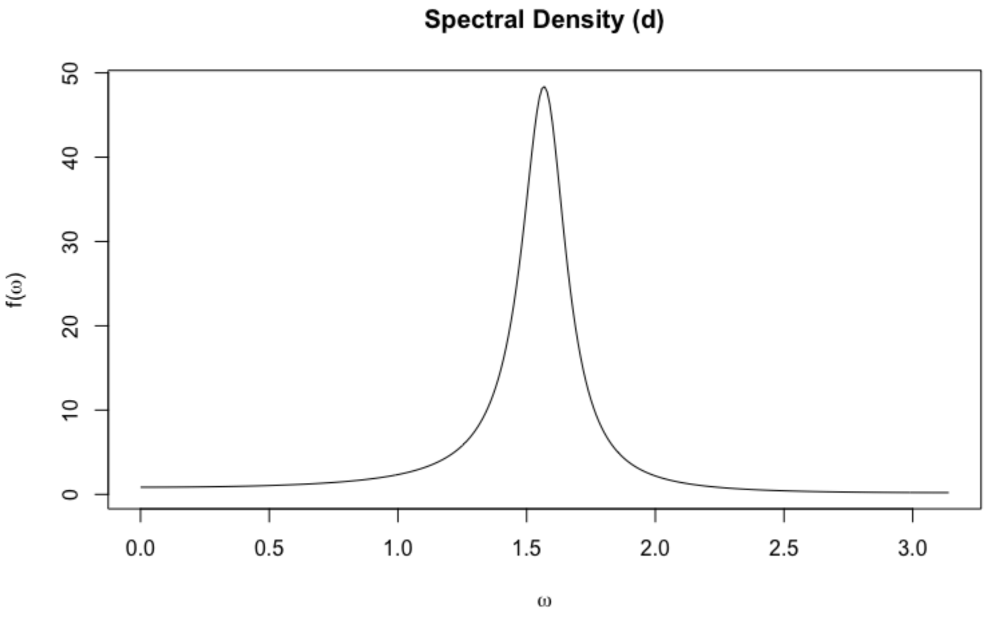


(d) $X_t + 0.81X_{t-2} = e_t + \frac{1}{3}e_{t-1}$

Solution:

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{2\pi}\left(\frac{\left|1+\frac{1}{3}\cos(\omega)-\frac{1}{3}i\sin(\omega)\right|^2}{\left|1+.81\cos(2\omega)-.81i\sin(2\omega)\right|^2}\right) \\ &= \frac{\sigma^2}{2\pi}\left(\frac{\left(1+\frac{1}{3}\cos(\omega)\right)^2+\left(\frac{1}{3}\sin(\omega)\right)^2}{\left(1+.81\cos(2\omega)\right)^2+\left(.81\sin(2\omega)\right)^2}\right) \end{aligned}$$

```
spectrald <- function(x) {
  topcos <- (1 + (1/3)*cos(x))^2
  topsin <- ((1/3)*sin(x))^2
  botcos <- (1 + .81*cos(2*x))^2
  botsin <- (.81*sin(2*x))^2
  full <- (topcos + topsin)/(botcos + botsin)
  outer <- (1/2*pi)
  return(outer*full)
}
```

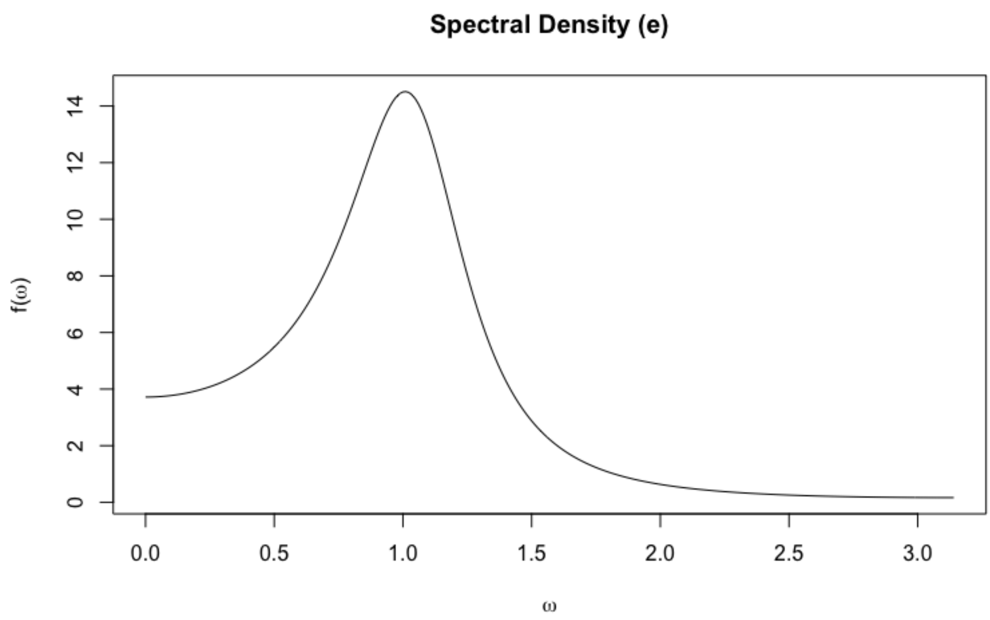


(e) $X_t - \frac{3}{4}X_{t-1} + \frac{9}{16}X_{t-2} = e_t + \frac{1}{4}e_{t-1}$

Solution:

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{2\pi}\left(\frac{\left|1+\frac{1}{4}\cos(\omega) - i\frac{1}{4}\sin(\omega)\right|^2}{\left|1-\frac{3}{4}\cos(\omega)+i\frac{3}{4}\sin(\omega)+\frac{9}{16}\cos(2\omega)-i\frac{9}{16}\sin(2\omega)\right|^2}\right) \\ &= \frac{\sigma^2}{2\pi}\left(\frac{\left(1+\frac{1}{4}\cos(\omega)\right)^2+\left(\frac{1}{4}\sin(\omega)\right)^2}{\left(1-\frac{3}{4}\cos(\omega)+\frac{9}{16}\cos(2\omega)\right)^2+\left(\frac{9}{16}\sin(2\omega)-\frac{3}{4}\sin(\omega)\right)^2}\right) \end{aligned}$$

```
spectrale <- function(x) {
  topcos <- (1 + (1/4)*cos(x))^2
  topsin <- ((1/4)*sin(x))^2
  botcos <- (1 - (3/4)*cos(x) + (9/16)*cos(2*x))^2
  botsin <- ((9/16)*sin(2*x) - (3/4)*sin(x))^2
  full <- (topcos + topsin)/(botcos + botsin)
  outer <- (1/2*pi)
  return(outer*full)
}
```



2. The spectral density of a real-valued time series X_t is defined on $[0, \pi]$ by

$$f(\omega) = \begin{cases} 100, & \text{for } \frac{\pi}{6} - 0.01 < \omega < \frac{\pi}{6} + 0.01 \\ 0, & \text{otherwise} \end{cases}$$

and on $[\pi, 0]$ by $f(\omega) = f(-\omega)$. Evaluate the ACVF of $\{X_t\}$ at lags 0 and 1.

Solution: Utilizing the equation for the autocovariance,

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\omega} f(\omega)\,\mathrm{d}\omega$$

we can solve this.

$$\begin{aligned} \gamma(0) &= \int_{-\pi}^{\pi} e^{0i\omega} 100\,\mathrm{d}\omega \\ &= 100 \int_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} \mathrm{d}\omega \\ &= 100 \left(\frac{\pi}{6} + 0.01 - \frac{\pi}{6} + 0.01 \right) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \gamma(1) &= \int_{-\pi}^{\pi} e^{i\omega} 100\,\mathrm{d}\omega \\ &= 100 \int_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} \cos(\omega)\,\mathrm{d}\omega \\ &\approx 100(.0173202) \\ &\approx 1.73202 \end{aligned}$$