# Chapter Three

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## Exercise One

### Question

The following relationships can be described by generalized linear models. For each one, identify the response variable and the explanatory variables, select a probability distribution for the response (justifying your choice) and write down the linear component.

### Solutions

**(a)**: The effect of age, sex, height, mean daily food intake and mean daily energy expenditure on a person's weight.

*Solution:* The response is the weight, while all other variables are the explanatory. I would assume that the weight is likely distributed approximately normally with a slight right skew. It's far more likely for someone to be perhaps fifty pounds overweight than 50 pounds underwight. For example, if the average weight is 165 lb for some class, one could reasonably weigh 300 pounds but unreasonably weigh 0. In creating the linear component, I will assume general normality on the response. The linear component then would be

$$E[Y_i] = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5}; Y_i \sim \mathrm{N}(\mu_i, \sigma^2)$$

**(b)**: The proportions of laboratory mice that became infected after exposure to bacteria when five different exposure levels are used and twenty mice are exposed at each level.

*Solution:* The levels of exposure are the explanatory variables, while the number infected is the response variable. Being infected or not infected are the possible outcomes so the response is likely distributed binomial. There are two ways the linear component could be created. One could have multiple $x_{ij}$ acting as indicator variables for each level of the exposure, or the $x_{ij}$ could be coded at different levels of exposure (i.e $x = \{1, 2, 3, 4, \text{ or } 5\}$). So,

$$E[Y_i] = \beta_0 + \beta_1 x_{i1}; Y_i \sim \mathrm{Bin}(20, p_i)$$

$$E[Y_i] = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5}; Y_i \sim \mathrm{Bin}(20, p_i)$$

**(c)**: The relationship between the number of trips per week to the super-market for a household and the number of people in the household, the household income, and the distance to the supermarket.

*Solution:* The response here is the number of trips per week to the super-market while the others are explanatory variables. Because these are counts of trips, they are likely distributed Poisson.

$$E[Y_i] = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3}; Y_i \sim \mathrm{Po}(\lambda_i)$$

## Exercise Three

### Question

Show that the following probability density functions belong to the exponential family:

So, show that the following probability density functions can be written as

$$\exp[a(y)b(\theta) + c(\theta) + d(y)]$$

### Solutions

**(a)**: Pareto distribution $f(y_i;\theta) = \theta y^{-\theta-1}\mathbb{1}\{y \geq 1\}$

*Solution:* Note that $e^{\log x} = x$. We will do the same to $f(y_i;\theta)$.

$$\begin{aligned} f(y;\theta) &= \exp[\log(\theta y^{-\theta-1}\mathbb{1}\{y \geq 1\})] \\ &= \exp[\log\theta - \theta\log y - \log y + \log(\mathbb{1}\{y \geq 1\})] \end{aligned}$$

$$a(y) = -\log y \quad b(\theta) = \theta \quad c(\theta) = \log\theta \quad d(y) = -\log y + \log(\mathbb{1}\{y \geq 1\})$$

**(b)**: Exponential distribution $f(y;\theta) = \theta e^{-y\theta}\mathbb{1}\{y \geq 0\}$

*Solution:*

$$\begin{aligned} f(y;\theta) &= \theta e^{-y\theta}\mathbb{1}\{y \geq 0\} \\ &= \exp[\log(\theta e^{-y\theta}\mathbb{1}\{y \geq 0\})] \qquad = \exp[\log\theta - y\theta + \log(\mathbb{1}\{y \geq 0\})] \end{aligned}$$

$$a(y) = -y \quad b(\theta) = \theta \quad c(\theta) = \log\theta \quad d(y) = \log(\mathbb{1}\{y \geq 0\})$$

**(c)**: Negative binomial distribution

$$f(y;\theta) = \binom{y+r-1}{r-1}\theta^r(1-\theta)^y\mathbb{1}\{r \in \mathbb{Z}^+, y \in (r+0, r+1, r+2, ...)\},$$

where $r$ is known.

*Solution:*

$$\begin{aligned} f(y;\theta) &= \binom{y+r-1}{r-1}\theta^r(1-\theta)^y\mathbb{1}\{r \in \mathbb{Z}^+, y \in (r+0, r+1, r+2, ...)\} \\ &= \exp\left[\log\left(\binom{y+r-1}{r-1}\theta^r(1-\theta)^y\mathbb{1}\{r \in \mathbb{Z}^+, y \in (r+0, r+1, r+2, ...)\}\right)\right] \\ &= \exp\left[\log\binom{y+r-1}{r-1} + r\log\theta + y\log(1-\theta) + \log(\mathbb{1}\{r \in \mathbb{Z}^+, y \in (r+0, r+1, r+2, ...)\})\right] \end{aligned}$$

$$a(y) = y \quad b(\theta) = \log(1-\theta) \quad c(\theta) = r\log\theta \quad d(y) = \log\binom{y+r-1}{r-1} + \log(\mathbb{1}\{r \in \mathbb{Z}^+, y \in (r+0, r+1, r+2, ...)\})$$

## Exercise Four

### Question

Use results

$$\mathrm{E}[a(Y)] = -c'(\theta)/b'(\theta) \quad \text{and} \quad \mathrm{var}[a(Y)] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3}$$

to verify the following results:

### Solutions

**(a)**: For $Y \sim \mathrm{Po}(\theta)$, $\mathrm{E}(Y) = \mathrm{var}(Y) = \theta$.

*Solution:* The canonical form of the Poisson distribution is the following:

$$\exp[y \log \theta - \theta - \log y!]$$

where $a(y) = y$, $b(\theta) = \log \theta$, $c(\theta) = -\theta$, and $d(y) = -\log y!$.

Because this is in canonical form, $\mathrm{E}[y] = \mathrm{E}[a(y)]$.

$$\begin{aligned} \mathrm{E}[y] &= \frac{-c'(\theta)}{b'(\theta)} \\ &= \frac{1}{1/\theta} \\ &= \theta \end{aligned} \qquad \begin{aligned} \mathrm{var}[y] &= \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3} \\ &= \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3} \\ &= \frac{1/\theta^2}{1/\theta^3} \\ &= \theta \end{aligned}$$

**(b)**: For $Y \sim \mathrm{N}(\mu, \sigma^2), \mathrm{E}(Y) = \mu$ and $\mathrm{var}(Y) = \sigma^2$.

*Solution:* The canonical form of the normal distribution is

$$\exp\left[-\frac{y^2}{2\sigma^2} + \frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right]$$

where $a(y) = y$, $b(\mu) = \frac{\mu}{\sigma^2}$, $c(\mu) = -\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)$, and $d(y) = -\frac{y^2}{2\sigma^2}$.

Because this is in canonical form, $\mathrm{E}[y] = \mathrm{E}[a(y)]$.

$$\begin{aligned} \mathrm{E}[y] &= \frac{-c'(\mu)}{b'(\mu)} \\ &= \frac{(\mu/\sigma^2)}{(1/\sigma^2)} \\ &= \mu \end{aligned} \qquad \begin{aligned} \mathrm{var}[y] &= \frac{b''(\mu)c'(\mu) - c''(\mu)b'(\mu)}{[b'(\mu)]^3} \\ &= \frac{0 * (-\mu/\sigma^2) - (-1/\sigma^2) * (1/\sigma^2)}{1/\sigma^6} \\ &= \frac{1/\sigma^4}{1/\sigma^6} \\ &= \sigma^2 \end{aligned}$$

**(c)**: For $Y \sim \mathrm{Bin}(n, \pi), \mathrm{E}(Y) = n\pi$ and $\mathrm{var}(Y) = n\pi(1-\pi)$

*Solution:* The canonical form of the Binomial distribution is

$$\exp\left[y \log\left(\frac{\pi}{1-\pi}\right) + n\log(1-\pi) + \log\binom{n}{y}\right]$$

where $a(y) = y, b(\pi) = \log\left(\frac{\pi}{1-\pi}\right), c(\pi) = n\log(1-\pi)$, and $d(y) = \log\binom{n}{y}$.

$$\begin{aligned}
\mathrm{E}[y] &= \frac{-c'(\pi)}{b'(\pi)} \\
&= -\frac{n/(\pi-1)}{1/(\pi-\pi^2)} \\
&= \frac{n(\pi^2-\pi)}{\pi-1} \\
&= \frac{n\pi(\pi-1)}{\pi-1} \\
&= n\pi
\end{aligned}
\qquad
\begin{aligned}
\mathrm{var}[y] &= \frac{b''(\pi)c'(\pi) - c''(\pi)b'(\pi)}{[b'(\pi)]^3} \\
&= \frac{\frac{2\pi-1}{(\pi-1)^2\pi^2}\frac{n}{\pi-1} + \frac{n}{(\pi-1)^2}\frac{1}{\pi-\pi^2}}{\frac{1}{(\pi-\pi^2)^3}} \\
&= \frac{\frac{n-\pi n}{(1-\pi)^3\pi^2}}{\frac{1}{(\pi-\pi^2)^3}} \\
&= \frac{\frac{\pi n-\pi^2 n}{(\pi-\pi^2)^3}}{\frac{1}{(\pi-\pi^2)^3}} \\
&= n\pi - \pi^2 n \\
&= n\pi(1-\pi)
\end{aligned}$$

## Exercise Five

### Questions and Solutions

**(a)**: For a Negative Binomial distribution $Y \sim \mathrm{NBin}(r, \theta)$, find $\mathrm{E}(Y)$ and $\mathrm{var}(Y)$.

*Solution:* The canonical form of the binomial distribution is listed in Exercise Three.

$$\begin{aligned}
\mathrm{E}[y] &= \frac{-c'(\theta)}{b'(\theta)} \\
&= \frac{r/\theta}{1/(1-\theta)} \\
&= \frac{r(1-\theta)}{\theta}
\end{aligned}
\qquad
\begin{aligned}
\mathrm{var}[y] &= \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3} \\
&= \frac{-r\theta^{-1}(1-\theta)^{-2} - r\theta^{-2}(1-\theta)^{-1}}{(-(1-\theta)^{-1})^3} \\
&= \frac{r\theta^{-1}(1-\theta)^{-1}((1-\theta)^{-1} + \theta^{-1})}{(1-\theta)^{-3}} \\
&= \frac{r\theta^{-2}(1-\theta)^{-2}}{(1-\theta)^{-3}} \\
&= \frac{n(1-\theta)}{\theta^2}
\end{aligned}$$

**(b)**: Notice that for the Poisson distribution $\mathrm{E}(Y) = \mathrm{var}(Y)$, for the Binomial distribution $\mathrm{E}(Y) > \mathrm{var}(Y)$, and for the negative Binomial distribution $\mathrm{E}(Y) < \mathrm{var}(Y)$. How might these results affect your choice of a model?

*Solution:* Knowing that these responses can lead to different degrees of scattering from the variance, one could select proper link functions to minimize variance that might negatively affect your model.

## Exercise Seven

### Question

Consider $N$ independent binary random variables $Y_1, ..., Y_N$ with

$$P(Y_i = 1) = \pi_1 \text{ and } P(Y_i = 0) = 1 - \pi_i.$$

The probability function of $Y_i$, the Bernoulli distribution $B(\pi)$, can be written as

$$\pi_i^{y_i}(1 - \pi_i)^{1-y_i},$$

where $y_i = 0$ or 1.

### Solutions

**(a)**: Show that this probability function belongs to the exponential family of distributions.

*Solution:*

$$\begin{aligned}
f(y_i; \pi_i) &= \pi_i^{y_i}(1 - \pi_i)^{1-y_i} \\
&= \exp[\log(\pi_i^{y_i}(1 - \pi_i)^{1-y_i})] \\
&= \exp[y_i \log \pi_i + (1 - y_i)\log(1 - \pi_i)] \\
&= \exp[y_i \log \pi_i + \log(1 - \pi_i) - y_i \log(1 - \pi_i)] \\
&= \exp[y_i \log\left(\frac{\pi_i}{1 - \pi_i}\right) + \log(1 - \pi_i)]
\end{aligned}$$

where $a(y_i) = y_i, b(\pi_i) = \log\left(\frac{\pi_i}{1-\pi_i}\right), c(\pi_i) = \log(1 - \pi_i),$ and $d(y) = 0$

**(b)**: Show that the natural parameter is

$$\log\left(\frac{\pi_i}{1 - \pi_i}\right).$$

This function, the logarithm of the **odds** $\pi_i/(1 - \pi_i)$, is called the **logit** function.

*Solution:*

By definition, $b(\pi_i)$ is the natural parameter and it is shown above.

**(c)**: Show that $\mathrm{E}(Y_i) = \pi_i$.

*Solution:*

$$\begin{aligned}
\mathrm{E}[y] &= \frac{-c'(\pi_i)}{b'(\pi_i)} \\
&= \frac{(1/(1 - \pi_i)}{1/(\pi_i - \pi_i^2)} \\
&= \frac{\pi_i(1 - \pi_i)}{1 - \pi_i} \\
&= \pi_i
\end{aligned}$$

**(d)**: If the link function is

$$g(\pi) = \log\left(\frac{\pi}{1 - \pi}\right) = \vec{x}^T\vec{\beta},$$

show that this is equivalent to modelling the probability $\pi$ as

$$\pi = \frac{e^{\vec{x}^T \vec{\beta}}}{1 + e^{\vec{x}^T \vec{\beta}}}.$$

*Solution:*

$$\begin{aligned}
& \log\left(\frac{\pi}{1-\pi}\right) = \vec{x}^T\vec{\beta} \\
\implies & \frac{\pi}{1-\pi} = \exp[\vec{x}^T\vec{\beta}] \\
\implies & \pi = \exp[\vec{x}^T\vec{\beta}] - \pi(\exp[\vec{x}^T\vec{\beta}]) \\
\implies & \pi + \pi(\exp[\vec{x}^T\vec{\beta}]) = \exp[\vec{x}^T\vec{\beta}] \\
\implies & \pi(1 + \exp[\vec{x}^T\vec{\beta}]) = \exp[\vec{x}^T\vec{\beta}] \\
\implies & \pi = \frac{\exp[\vec{x}^T\vec{\beta}]}{(1 + \exp[\vec{x}^T\vec{\beta}])} \\
\implies & \pi = \frac{e^{\vec{x}^T\vec{\beta}}}{1 + e^{\vec{x}^T\vec{\beta}}}.
\end{aligned}$$

**(e)**: In the particular case where $\vec{x}^T\vec{\beta} = \beta_1 + \beta_2 x$, this gives

$$\pi = \frac{e^{\beta_1 + \beta_2 x}}{1 + e^{\beta_1 + \beta_2 x}},$$

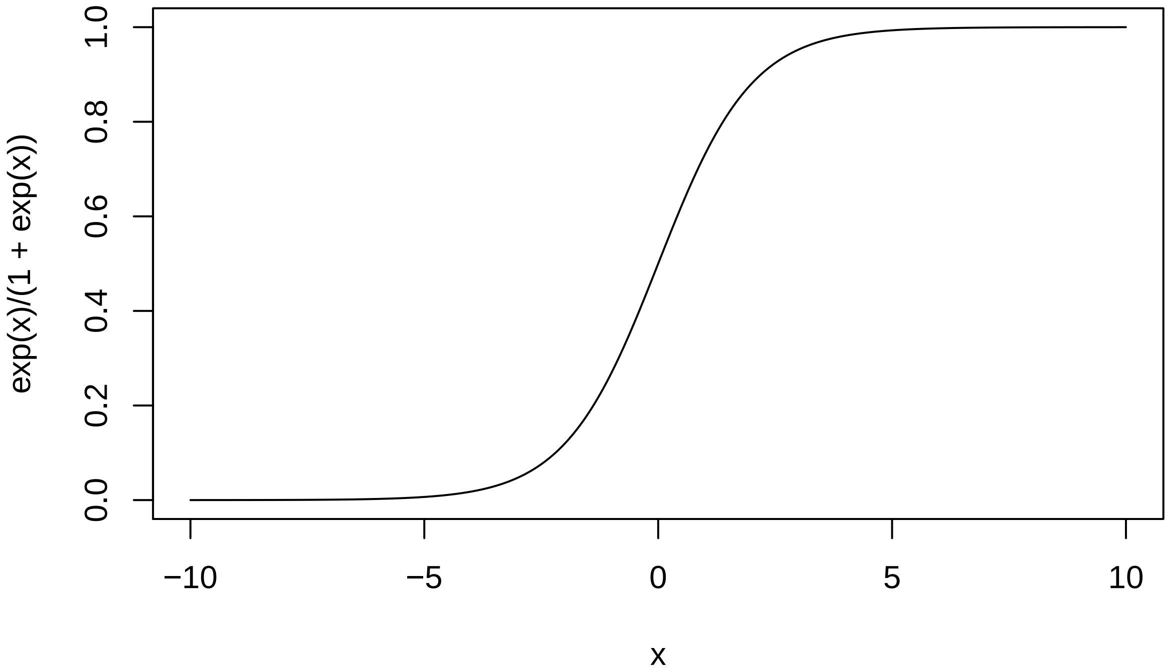
which is the **logistic function**.

Sketch the graph of $\pi$ against $x$ in this case, taking $\beta_1$ and $\beta_2$ as constants. How would you interpret this graph if $x$ is the dose of an insecticide and $\pi$ is the probability of an insect dying?

*Solution:*

I'll "sketch" the graph in R.

```
curve(exp(x)/(1+exp(x)), -10, 10, n = 2000, add = FALSE, type = "l",
      ylab = NULL, log = NULL)
```




Essentially, what this means is that as there is no need for an insane amount of pesticide because there is clearing a ceiling at which given some x, the logistic link is approximately 1. The speed at which this occurs is obviously different for different $\vec{\beta}$. In this case, $\beta_1 = 0, \beta_2 = 1$.

## Exercise Twelve

### Questions and Solutions

*(Note: See Figure 3.3 for more relationships of distributions)*

**(a)**: Show that the Exponential distribution $\mathrm{Exp}(\theta)$ is a special case of the Gamma distribution $\mathrm{G}(\alpha, \beta)$.

*Solution:* The exponential distribution is $\mathrm{Gamma}(1, 1/\theta)$.

$$\begin{aligned}
G[\alpha, \beta] &= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \\
G[1, 1/\theta] &= \frac{1}{\Gamma(1)(1/\theta)^1} x^{1-1} e^{-x/(1/\theta)} \\
&= \frac{1}{\theta^{-1}} x^0 e^{-x\theta} \\
&= \theta e^{-\theta x} \\
&= \mathrm{Exp}[\theta]
\end{aligned}$$

**(b)**: If $X$ has the Uniform Distribution $\mathrm{U}[0, 1]$ that is, $f(x) = 1$ for $0 < x < 1$, show that $Y = -\theta \log X$ has the distribution $\mathrm{Exp}(\theta)$.

*Solution:*

$$\begin{aligned}
\mathrm{Exp}[\theta] &= \frac{1}{\theta} e^{-x/\theta} \\
X \sim \mathrm{U}[0, 1] &= 1 \\
Y &= -\theta \log X \\
\implies X &= e^{-Y/\theta} \\
\frac{dX}{dY} &= \frac{-1}{\theta} e^{-Y/\theta} \\
f_Y(y) &= f_X(X) \left| \frac{dX}{dY} \right| \\
f_Y(y) &= \frac{1}{\theta} e^{-Y/\theta}
\end{aligned}$$

**(c)**: Use the moment generating functions (or other methods) to show

i. $\mathrm{Bin}(n, \pi) \to \mathrm{Po}(\lambda)$ as $n \to \infty$

*Solution:*

ii. $\mathrm{NBin}(r, \theta) \to \mathrm{Po}(r(1 - \theta))$ as $r \to \infty$

*Solution:*

**(d)**: Use the Central Limit Theorem to show

i. $\mathrm{Po}(\lambda) \to \mathrm{N}(\lambda, \lambda)$ for large $\mu$.

*Solution:* $\mathrm{Po}(\lambda)$ can be thought of as the sum of $\lambda$ $\mathrm{Po}(1)$ random variables. For large $\mu$ then, the distributed approaches the normal distribution with mean $\mu$ and standard deviation $\sigma^2/n$ where $\mu = \lambda$ and $\sigma^2 = \lambda^2$ and $n$ being the sample size of $\lambda$, this reduces to $\mathrm{N}(\lambda, \lambda)$.

i. $\text{Bin}(n, \pi) \to \text{N}(n\pi, n\pi(1-\pi))$ for large $n$, provided neither $n\pi$ nor $n\pi(1-\pi)$ is too small.

The binomial distribution can be reduced to a sum of $n$ Bernoulli distributions distributed Bernoulli($p$). The argument is similar to **(a)**.

i. $\text{G}(\alpha, \beta) \to \text{N}(\alpha/\beta, \alpha/\beta^2)$ for large $\alpha$.

The Gamma distribution can be reduced to the sum of $\beta$ Exponential distributions and follows a similar argument to **(c)**.