Time Series Analysis Homework 1

Ryan Honea

February 13, 2018

1. Let X and Y be two random variables with joint pdf $f(x, y) = 3x$ for $0 \leq y \leq x \leq 1$, and zero elsewhere.

   (a) Compute $P(0 < X < 0.5 \cap Y \geq 0.25)$

   Solution:

   $$
   \begin{aligned}
   P(0 < X < 0.5 \cap Y \geq 0.25) &= \int_{0.25}^{0.5} \int_{0.25}^{x} 3x \, \mathrm{d}y \, \mathrm{d}x \\
   &= \int_{0.25}^{0.5} 3xy \, \Big|_{0.25}^{x} \, \mathrm{d}x \\
   &= \int_{0.25}^{0.5} 3x^2 - \frac{3}{4}x \, \mathrm{d}x \\
   &= x^3 - \frac{3}{8}x^2 \, \Big|_{0.25}^{0.50} \\
   &= \left(0.5^3 - \frac{3}{8}0.5^2 \right) - \left(0.25^3 - \frac{3}{8}0.25^2 \right) \\
   &= \frac{5}{128}
   \end{aligned}
   $$

   (b) Compute marginal densities of X and Y

   Solution:

   $$
   \begin{aligned}
   f(x) &= \int_0^x 3x \, \mathrm{d}y & f(y) &= \int_y^1 3x \, \mathrm{d}x \\
   &= 3xy \, \Big|_0^x & &= \frac{3}{2}x^2 \, \Big|_y^1 \\
   &= 3x^2 & &= \frac{3}{2}(1 - y^2)
   \end{aligned}
   $$

2. Suppose X and Y are random variables with joint probability density function of the form $f(x, y) = x + y$, for $0 \le x \le 1$; and $0 \le y \le 1$ and zero elsewhere.

(a) Find the marginal distribution of X and Y.

Solution:

$$f(x) = \int_0^1 x + y \, dy \qquad\qquad f(y) = \int_0^1 x + y \, dx$$

$$= xy + \frac{y^2}{2} \Big|_0^1 \qquad\qquad = \frac{x^2}{2} + xy \Big|_0^1$$

$$= x + \frac{1}{2} \qquad\qquad\qquad = y + \frac{1}{2}$$

(b) Compute $E(X)$, $E(Y)$, $Var(X)$, and $Var(Y)$.

Solution:

$$E[X] = \int_0^1 x \left(x + \frac{1}{2} \right) dx \qquad\qquad E[Y] = \int_0^1 y \left(y + \frac{1}{2} \right) dy$$

$$= \frac{x^3}{3} + \frac{x^2}{4} \Big|_0^1 \qquad\qquad = \frac{y^3}{3} + \frac{y^2}{4} \Big|_0^1$$

$$= \frac{7}{12} \qquad\qquad\qquad\qquad = \frac{7}{12}$$

$$Var[X] = E[X^2] - E[X]^2 \qquad\qquad Var[Y] = E[Y^2] - E[Y]^2$$

$$= \int_0^1 x^2 \left(x + \frac{1}{2} \right) dx - \left(\frac{7}{12} \right)^2 \qquad = \int_0^1 y^2 \left(y + \frac{1}{2} \right) dy - \left(\frac{7}{12} \right)^2$$

$$= \frac{x^4}{4} + \frac{x^3}{6} \Big|_0^1 - \frac{49}{144} \qquad\qquad = \frac{y^4}{4} + \frac{y^3}{6} \Big|_0^1 - \frac{49}{144}$$

$$= \frac{11}{144} \qquad\qquad\qquad\qquad = \frac{11}{144}$$

(c) Compute $Cov(X, Y)$.

Solution:

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E\left[XY - \frac{7}{12}X - \frac{7}{12}Y + \frac{49}{144} \right]$$

$$= E[XY] - \frac{7}{12}(E[X] + E[Y]) + \frac{49}{144}$$

$$= \int_0^1 \int_0^1 x^2 y + xy^2 \, dx \, dy - \frac{49}{144}$$

$$= \int_0^1 \frac{y}{3} + \frac{y^2}{2} \, dy - \frac{49}{144} = \frac{2}{6} - \frac{49}{144}$$

$$= -\frac{1}{144}$$

(d) Compute $E[(2X - Y)^2]$

Solution:

$$E[(2X - Y)^2] = 4E[X^2] - 4E[XY] + E[Y^2]$$

$$= 4 \left(\frac{10}{24} \right) - 4 \left(\frac{2}{6} \right) + \frac{10}{24} \qquad\qquad = \frac{3}{4}$$

3. Suppose X and Y are random variables with joint probability density function of the form

$$f(x,y) = KXe^{-(x+y)/2}, \quad x \geq 0, \quad y \geq 0$$

and zero elsewhere.

(a) Find the value of K.

Solution:

$$1 = \int_0^\infty \int_0^\infty KXe^{-(x+y)/2} \, dy \, dx$$

$$= \int_0^\infty \left[-2Kxe^{-(x+y)/2} \Big|_0^\infty \right] dx \qquad\qquad = 2K \int_0^\infty xe^{-x/2} \, dx$$

$$= 2K \left[\left(-2xe^{-x/2} + 2 \int e^{-x/2} \right) \Big|_0^\infty \right] \qquad = 2K \left[\left(-2xe^{-x/2} - 4e^{-x/2} \right) \Big|_0^\infty \right]$$

$$1 = 8K$$

$$K = .125$$

(b) Compute $Cov(X, Y)$.

Solution: Note that $Cov(X,Y) = E[XY] - E[X]E[Y]$, so first we find $E[XY]$.

$$E[XY] = \frac{1}{8} \int_0^\infty \int_0^\infty x^2 y e^{-(x+y)/2} \, dy \, dx$$

$$= \frac{1}{8} \int_0^\infty x^2 e^{-x/2} \left[\int_0^\infty y e^{-y/2} \, dy \right] dx$$

$$= \frac{1}{8} \int_0^\infty x^2 e^{-x/2} \left[-2ye^{-y/2} - 4e^{-y/2} \right]_0^\infty dx$$

$$= \frac{1}{2} \int_0^\infty x^2 e^{-x/2} \, dx$$

$$= \frac{1}{2} \left[-2x^2 e^{-x/2} + 4 \left(-2xe^{-x/2} - 4e^{-x/2} \right) \right]_0^\infty$$

$$= 8$$

Now, we need the marginal distributions for each of them to find $E[X]$ and $E[Y]$.

$$f_x(x) = \frac{1}{8} xe^{-x/2} \int_0^\infty e^{-y/2} \, dy \qquad\qquad f_y(y) = \frac{1}{8} e^{-y/2} \int_0^\infty xe^{-x/2} \, dx$$

$$= \frac{1}{4} xe^{-x/2} \qquad\qquad\qquad\qquad = \frac{1}{2} e^{-y/2}$$

$$E[X] = \frac{1}{4} \int_0^\infty x^2 e^{-x/2} \, dx \qquad\qquad E[Y] = \frac{1}{2} \int_0^\infty y e^{-y/2}$$

$$= 4 \qquad\qquad\qquad\qquad\qquad\qquad = 2$$

Based on these results, the covariance is

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= 8 - 4 * 2$$

$$= 0$$

(c) Find $E[XY^2]$.

Solution:

$$E[XY^2] = \frac{1}{8} \int_0^\infty x^2 y^2 e^{-(x+y)/2} \, \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{8} \int_0^\infty x^2 e^{-x/2} \left[\int_0^\infty y^2 e^{-y/2} \, \mathrm{d}y \right] \mathrm{d}x$$

Note that the elements within the bracket are similar to the second moment of the exponential distribution, that is, we can write that as 2 times the variance plus the mean squared for $\lambda = 0.5$. For $Exp(\lambda = 1/2)$, the variance is 4 and the mean is 2, so the values within the brackets result in 16.
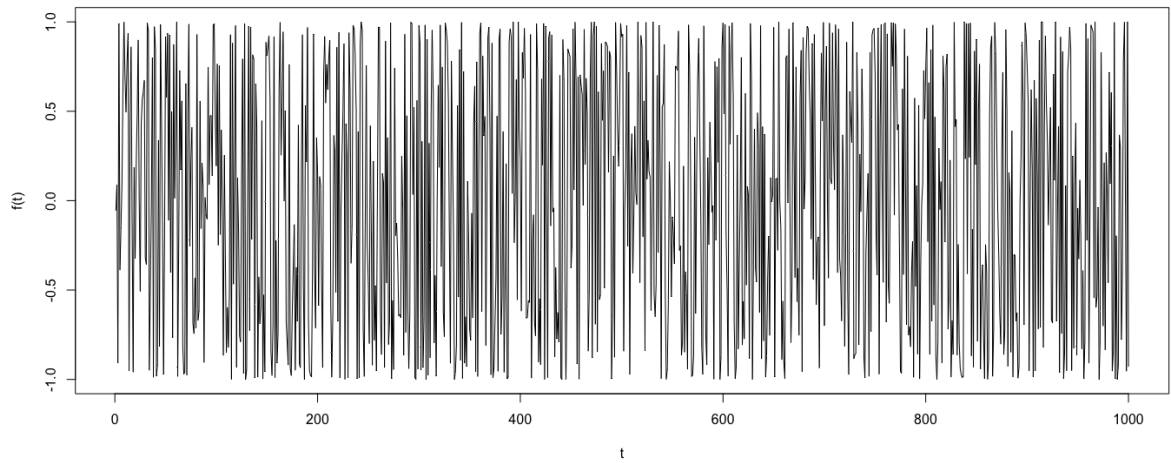
$$E[XY^2] = \frac{1}{8} \int_0^\infty x^2 e^{-x/2} [16] \, \mathrm{d}x$$

$$= 2 \int_0^\infty x^2 e^{-x/2} \, \mathrm{d}x$$

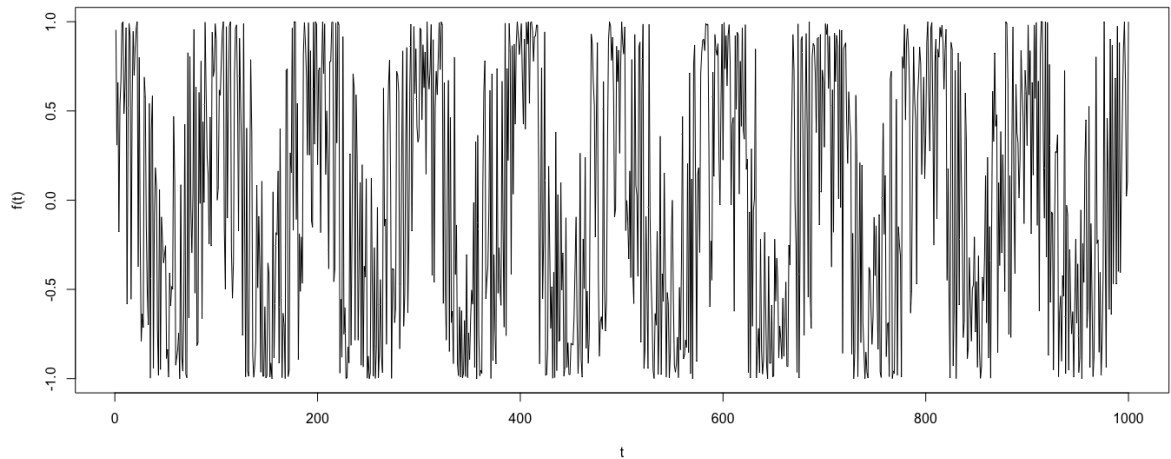The same argument can be made for the above integral and so

$$E[XY^2] = 32$$

4. Consider a random process $X_t = \sin(\frac{2\pi}{100}t + \phi)$.

   (a) Consider $\phi \sim \text{Uniform}(-\pi, \pi)$. Plot the process for $t \in (0, 1000)$

   (b) Consider $\phi \sim \text{Uniform}(0, \pi)$. Plot the process for $t \in (0, 1000)$.

   (c) In terms of mean, what difference do you see in these plots? Explain the difference.

   Solution: The difference in the plot, though slightly in the population mean, is primarily in the means at different neighborhoods. Over time, the mean of (a) appears to be stationary while (b)'s mean seems to fluctuate. This can be shown mathematically. Note first that $E[X_t]$ in (a) is equal to 0 because it is an odd function. For (b), however, this result is different.

   $$E[X_t] = E\left[\sin\left(\frac{2\pi}{100}t + \phi\right)\right]$$

   $$= \int_0^\pi \sin\left(\frac{2\pi}{100}t + \phi\right)\frac{1}{\pi}\,\mathrm{d}\phi = \frac{1}{\pi}\int_0^\pi \sin\left(\frac{2\pi}{100}t + \phi\right)\mathrm{d}\phi$$

   $$= \frac{1}{\pi}\left[-\cos\left(\frac{2\pi}{100}t + \phi\right)\Big|_0^\pi\right]$$

   $$= \frac{1}{\pi}\left[-\cos\left(\frac{2\pi}{100}t + \pi\right) + \cos\left(\frac{2\pi}{100}t + 0\right)\right] = \frac{2}{\pi}\cos\left(\frac{2\pi}{100}t\right)$$

   So clearly, the expected values of (a) and (b) are different which results in the plots observed.