Brownian Motion and Stationary Processes

Ryan Honea

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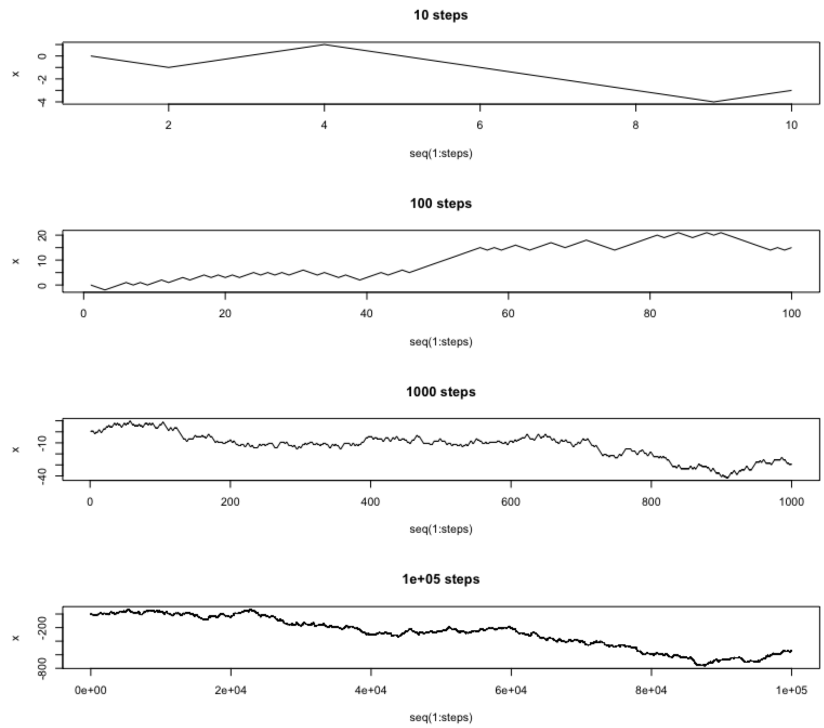
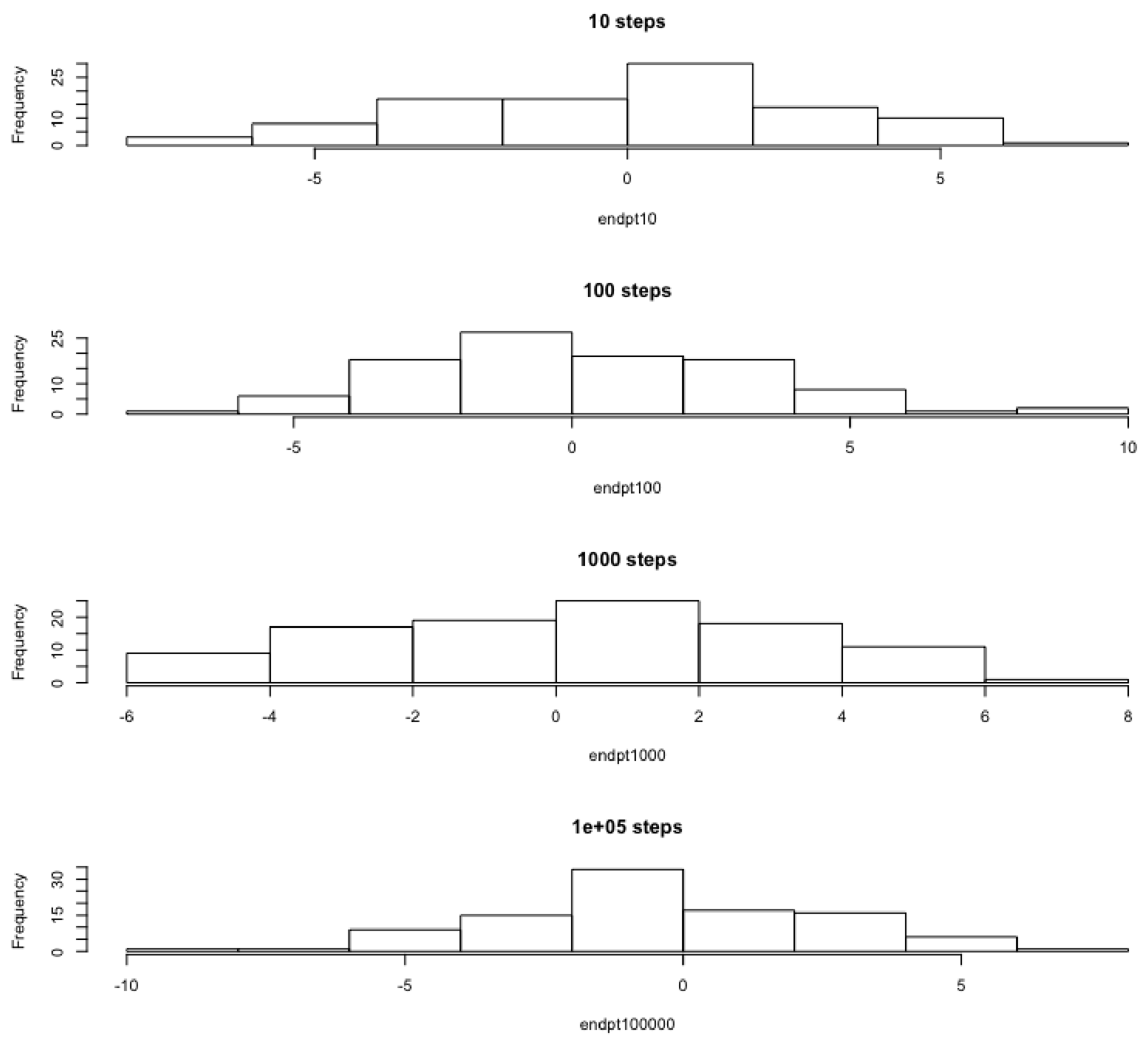
1. Use Excel (or R, or some other software) to simulate a simple random walk in one dimension ($p = 1/2$) for 10 steps, 100 steps, 1000 steps, and 100,000 steps, each one starting at 0. What do you notice about the dimensions of the window drawn around the graph? Recall that we talked about this in class. Also, try simulating each of these random walks 100 times. Make a histogram of the endpoints. Do they appear normally distributed? Explain.

Solution: Utilizing the following code to obtain the plots and histograms...

```
rand_walk <- function(steps) {
  x <- c()
  x[1] <- 0
  for (i in 2:steps) {
    if (runif(1,0,1) < 0.5)
      x[i] <- x[i-1] + 1
    else x[i] <- x[i-1] - 1
  }
  plot(seq(1:steps), x, type = "l",
       main = paste(steps, "steps"))
  return(x[steps])
}
```



The graphs window grows vertically with each increase in number of steps.



And the histograms show that the endpoints are approximately normal because they appear gaussian.

2. Prove that $\text{Cov}(B(s), B(t)) = s \wedge t$

Solution: If $t < s$

$$\begin{aligned} Cov(B(s), B(t)) &= Cov(B(t) + B(s) - B(t), B(t)) \\ &= Cov(B(t), B(t)) + Cov(B(s) - B(t), B(t)) \\ &= Var(B(t)) + 0 \\ &= t \end{aligned}$$

and if $s < t$

$$\begin{aligned} Cov(B(s), B(t)) &= Cov(B(s), B(s) + B(t) - B(s)) \\ &= Cov(B(s), B(s)) + Cov(B(s), B(t) - B(s)) \\ &= Var(B(s)) + 0 \\ &= s \end{aligned}$$

3. Prove that $\{B(ct), t \geq 0\} \stackrel{d}{=} \{c^{1/2}B(t), t \geq 0\}$. Interpret this result using your own words and maybe a graph or two.

Solution: By definition, a Brownian Motion is normal, and so if we can show that they have equal variance and expectation, then they are equal in distribution. Both of have expectation 0 trivially, so we seek to show that they have equal variance.

$$Var\left(B(ct)\right) = ct$$
$$Var\left(c^{1/2}B(t)\right) = cVar(B(t)) = ct$$

so they are equal in distribution.

4. Let $\{X(t), t \geq 0\}$ be a Brownian motion process with drift μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given $X(s) = c$ when

(a) $s < t$

Solution: If we write $X(t) = X(s) + X(t) - X(s)$, then $X(t)$ is distributed as $c + X(t) - X(s)$, which by stationary increments means that this can be written as $c + X(t-s)$, and therefore will be Normal$(c + \mu(t-s), (t-s)\sigma^2)$

(b) $t < s$

Solution: If we write $X(t)$ in the standard Brownian Motion representation, then we have

$$X(s) = \sigma B(s) + \mu t \implies B(s) = \frac{c - \mu s}{\sigma}$$

and can thus use equation 10.4 to find

$$\begin{aligned}
\mathbb{E}\left[X(t)|X(s) = c\right] &= \frac{t}{s}X(s) \\
&= \frac{t}{s}\sigma\frac{c-\mu s}{\sigma} + \mu t \\
&= \frac{(c-\mu s)t}{s} + \mu t \\
Var(X(t)|X(s) = c) &= \frac{\sigma^2 t(s-t)}{s}
\end{aligned}$$

5. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Let $Z(t) = B(t) - tB(1)$. Show that $\{Z(t), t \in [0, 1]\}$ is a Brownian Bridge process. Also, draw a graph or two to explain this.

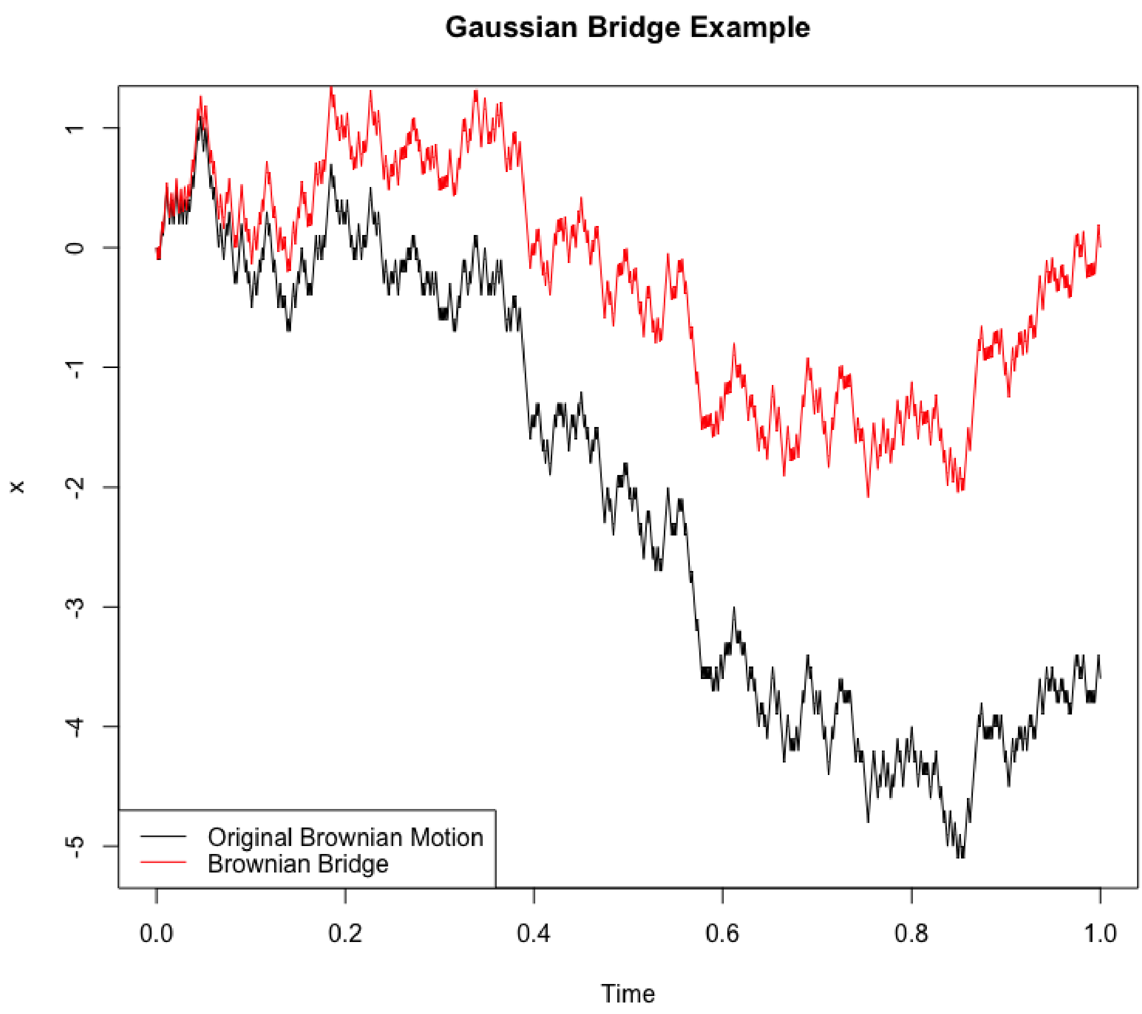
Solution: To show this is a Brownian Bridge process, we must show that $\mathbb{E}\left[Z(t)\right] = 0$ and

$$\text{Cov}\left(Z(t), Z(s)\right) = s(1-t)$$

when $s \leq t$. The first is trivial.

$$\begin{aligned} Cov\left((Z(t), Z(s)\right) &= \text{Cov}\left((B(t), B(s)\right) - s\text{Cov}\left((B(t), B(1)\right) - t\text{Cov}\left((B(s), B(1)\right) + st\text{Cov}\left((B(t), B(1)\right) \\ &= s - st - st + st \\ &= s - st \\ &= s(1-t) \end{aligned}$$

Thus it is a Brownian Bridge. Below is a simulated graph of this Brownian Bridge and it's original function.



It is immediately clear from this graph that the Brownian Bridge process effectively links a brownian motion back to 0 within time [0,1]. That is, by definition, a Brownian Bridge is such that $Z(0) = Z(1) = 0$.

6. Prove that for any time t the Ornstein-Uhlenbeck process $\{V(t), t \geq 0\}$ defined by

$$V(t) = e^{-t} B(e^{2t}), \qquad -\infty < t < \infty$$

has a standard normal distribution.

Solution: We already know that by definition a Brownian Motion is normally distributed with mean 0 and variance $\sigma^2 t$. Since this is a linear combination of a Brownian Motion, it is as well. Therefore, we must show that the variance is 1, and we will have shown that it has standard normal distribution.

$$\begin{aligned} Cov\left(V(t), V(t+s)\right) &= e^{-t} e^{-(t+s)} Cov\left(X\left(e^{2t}\right), X\left(e^{2(t+s)}\right)\right) \\ &= e^{-2t} e^{-s} e^{2t} \qquad \text{By results in Problem 2} \\ &= e^{-s} \end{aligned}$$

When $s = 0$, $Cov\left(V(t), V(t+s)\right) = Var(V(t)) \implies Var(V(t)) = e^0 = 1$.

Therefore, the Ornstein-Uhlenbeck process defined above has a standard normal distribution.

7. A stock is presently (at time 0) selling at a price of \$50 per share. At time 1, its selling price will be (in present value dollars... that is, time 0 dollars) either \$150 or \$25. An option to purchase u units of the stock at time 1 at a strike price of 125 per unit can be purchase for cu at time 0.

   (a) What value of c assures there will be no sure win?

   Solution: Based on the above, we have

   $$\text{value} = \begin{cases} 150y + 25x, & \text{if price is 150} \\ 25y, & \text{if price is 25} \end{cases}$$

   We can set them equal to each other by setting $x = -5y$ which implies the value of our holding at time 1 is $25y$. If the original cost is \$50, then we have

   $$\begin{aligned} \text{Original Cost} &= 50y - 5yc \\ \implies \text{Gain} &= 25y - (50y - 5yc) \\ &= -25y + 5yc = y(-25 + 5c) \end{aligned}$$

   From this, we can see that if $c = 5$, there is no sure win.

   (b) If $c = 4$, how could you guarantee a win?

   Solution: If one buys x units of the options and then $-5x$ units of the stock for $x < 0$.

   (c) If $c = 10$, how would you guarantee a win?

   Solution: If one buys x units of the option and $-5x$ units of the stock, expected profit is $25x$ for $x > 0$.

   (d) Use the arbitrage theorem to verify your answer to part (a).

Solution:

$$\begin{aligned}
0 &= 100p - 25(1-p) \\
25 &= 125p \implies p = \frac{1}{5} \\
E[\text{Return}] &= (25-c)\frac{1}{5} - c\frac{4}{5} \\
&= 5 - c
\end{aligned}$$

which verifies that for $c = 5$, there is no sure win.

8. The current price of stock is 100 per share. Suppose the natural log of the stock price changes according to Brownian Motion with variance parameter $\sigma^2 = 1$. The continually compounded interest rate is 5%. Give the Black-Scholes cost of an option to purchase the stock at time 10 for a cost of

(a) 100 per unit.

Solution:

$$\begin{aligned}
b &= \frac{\alpha t - \sigma^2 t/2 - \log(K/x_o)}{\sigma\sqrt{t}} \\
&= \frac{.5 - 10/2 - \log(100/100)}{\sqrt{10}} \\
&= -4.5\sqrt{10} = -1.423 \\
c &= x_0\phi(\sigma\sqrt{t} + b) - Ke^{-\alpha t}\phi(b) \\
&= 100\phi(\sqrt{10} - 1.423) - 100e^{-.5}\phi(-1.423) \\
&= 100\left(\phi(1.739) - e^{-.5}\left(1 - \phi(1.423)\right)\right) && = 91.21
\end{aligned}$$

(b) 120 per unit.

Solution:

$$\begin{aligned}
b &= \frac{\alpha t - \sigma^2 t/2 - \log(K/x_o)}{\sigma\sqrt{t}} \\
&= \frac{.5 - 10/2 - \log(120/100)}{\sqrt{10}} \\
&= -4.68/\sqrt{10} = -1.481 \\
c &= x_0\phi(\sigma\sqrt{t} + b) - Ke^{-\alpha t}\phi(b) \\
&= 100\phi(\sqrt{10} - 1.481) - 120e^{-.5}\phi(-1.481) \\
&= 100\phi(1.681) - 120e^{-.5}(1 - \phi(1.481)) \\
&= 90.32
\end{aligned}$$

(c) 80 per unit.

Solution:

$$\begin{aligned} b &= \frac{\alpha t - \sigma^2 t/2 - \log(K/x_o)}{\sigma\sqrt{t}} \\ &= \frac{.5 - 10/2 - \log(80/100)}{\sqrt{10}} \\ &= -4.277/\sqrt{10} = -1.352 \\ c &= x_0\phi(\sigma\sqrt{t} + b) - Ke^{-\alpha t}\phi(b) \\ &= 100\phi(\sqrt{10} - 1.352) - 80^{-.5}\phi(-1.352) \\ &= 100\phi(1.81) - 120e^{-.5}(1 - \phi(1.352)) \\ &= 92.21 \end{aligned}$$

9. Let $L_a(r) = \mathbb{E}\left[e^{-rT_a}\right]$ be the Laplace transform of T_a (the first time standard Brownian motion hits a).

(a) Use the fact that the $\{T_a, a \geq 0\}$ process has stationary and independent increments to conclude $L_a(r)L_b(r) = L_{a+b}(r)$.

Solution: Firstly, by independent increments, we know that if $a < b$, then $T_b - T_a = T_{b-a}$. Certainly, $a < a + b$, so we have $T_{a+b} - T_b = T_a$ which implies then that $T_a + T_b = T_{a+b}$. Now, if $L_a(r) = \mathbb{E}\left[e^{-rT_a}\right]$ then,

$$\begin{aligned} L_a(r)L_b(r) &= \mathbb{E}\left[e^{-rT_a}\right]\mathbb{E}\left[e^{-rT_b}\right] \\ &= \mathbb{E}\left[e^{-rT_a}e^{-rT_b}\right] \qquad \text{by independence} \\ &= \mathbb{E}\left[e^{-r(T_a+T_b)}\right] \\ &= \mathbb{E}\left[e^{-rT_{a+b}}\right] \\ &= L_{a+b}(r) \end{aligned}$$

(b) Use the scaling relation you proved in Exercise 3 to show that $T_a \stackrel{d}{=} a^2 T_1$.

Solution: I don't think applies here unless we are suggesting that $aT_1 \stackrel{d}{=} a^2 T_1$. I'm not sure this is true.

A stochastic process $\{X(t), t \geq 0\}$ is said to be a *Martingale process* if for $s < t$

$$\mathbb{E}\left[X(t) \mid X(u) \text{ for } 0 \leq u \leq s\right] = X(s)$$

and

$$\mathbb{E}\left[\left|X(t)\right|\right] < \infty \qquad \text{for all } t$$

10. If $\{X(t), t \geq 0\}$ is a Martingale process, show that $\mathbb{E}\left[X(t)\right] = \mathbb{E}\left[X(0)\right]$.

Solution:

$$\begin{aligned}
\mathbb{E}\left[X(t)\right] &= \mathbb{E}\left[\mathbb{E}\left[X(t) \mid X(u) \text{ for } 0 \leq u \leq s\right]\right] = \mathbb{E}\left[X(s)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X(s) \mid X(u) \text{ for } 0 \leq u \leq s-1\right]\right] = \mathbb{E}\left[X(s-1)\right] \\
&= ... \\
&= \mathbb{E}\left[X(0)\right]
\end{aligned}$$

11. Show that standard Brownian motion is Martingale.

Solution: So, we must show that $\mathbb{E}\left[B(t) \mid B(u) \text{ for } 0 \leq u \leq s\right] = B(s)$.

$$\begin{aligned}
&\mathbb{E}\left[B(t) \mid B(u) \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[B(s) + B(t) - B(s) \mid B(u) \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[B(s) \mid B(u) \text{ for } 0 \leq u \leq s\right] + \mathbb{E}\left[B(t) - B(s) \mid B(u) \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[B(s) \mid B(u) \text{ for } 0 \leq u \leq s\right] + \mathbb{E}\left[B(t) - B(s)\right] \quad \text{by independent increments} \\
=&B(s) + \mathbb{E}\left[B(t)\right] - \mathbb{E}\left[B(s)\right] \\
=&B(s)
\end{aligned}$$

12. Show that when $X(t) = \left(B(t)\right)^2 - t$ then $\{X(t), t \geq 0\}$ is a Martingale process. Also compute $\mathbb{E}\left[X(t)\right]$.

Solution:

$$\begin{aligned}
&\mathbb{E}\left[X(t) \mid X(u) \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[\left(B(t)\right)^2 - t \mid \left(B(u)\right)^2 - u \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[\left(B(t)\right)^2 - t \mid \left(B(u)\right)^2 \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[\left(B(t)\right)^2 - t \mid B(u) \text{ for } 0 \leq u \leq s\right] \mid \left(B(u)\right)^2 - u \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[B(s)^2 - s \mid B(u)^2 \text{ for } 0 \leq u \leq s\right] \qquad \text{follows from problem 11} = \qquad B(s)^2 - s
\end{aligned}$$

which means it is a Martingale as the second property follows from the definition of Brownian motion.

$$\begin{aligned}
\mathbb{E}\left[X(t)\right] &= \mathbb{E}\left[\left(B(t)\right)^2 - t\right] \\
&= \mathbb{E}\left[B(t)^2\right] - t \\
&= Var[B(t)] + \mathbb{E}\left[B(t)\right] - t \\
&= t + 0 - t \\
&= 0
\end{aligned}$$

13. Let x be a real constant. Show that when

$$X(t) = e^{cB(t) - c^2 t/2}$$

then $\{X(t), t \geq 0\}$ is a Martingale process. Also compute $\mathbb{E}\left[X(t)\right]$.

Solution: Using a similar principle to problem 12, we have

$$\begin{aligned}
&\mathbb{E}\left[X(t) \mid X(u) \text{ for } 0 \leq u \leq s\right] \\
=&\mathbb{E}\left[e^{cB(t)-c^2t/2} \mid e^{cB(u)-c^2u/2} \text{ for } 0 \leq u \leq s\right] \\
=&e^{-c^2t/2}\mathbb{E}\left[e^{cB(t)} \mid B(u) \text{ for } 0 \leq u \leq s\right] \\
=&e^{-c^2t/2}\mathbb{E}\left[e^{cB(t)} \mid B(u) \text{ for } 0 \leq u \leq s\right]
\end{aligned}$$

We know from definitions of Brownian motion that given the value of $B(s)$ then $B(t)$ will be normal with mean $B(s)$ and variance $t - s$. Therefore, the above expectation is the moment generating function of a normal random variable and thus we have

$$\begin{aligned}
=&e^{-c^2t/2}\mathbb{E}\left[e^{cB(t)} \mid B(u) \text{ for } 0 \leq u \leq s\right] \\
=&e^{-c^2t/2}e^{cB(s)+(t-s)c^2/2} \\
=&e^{-c^2s/2}e^{cB(s)} = e^{cB(s)-c^2s/2} \\
=&X(s)
\end{aligned}$$

Therefore $X(t)$ is a Martingale. Because $X(t)$ is a martingale, we know that $\mathbb{E}\left[X(t)\right] = \mathbb{E}\left[X(0)\right] = 0$.