Problem Three

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We begin by showing the equivelance, but will first note some terms and identities of importance.

$$\mu_{jk} = \alpha_j + \beta_j x_{jk} \qquad \sum_{k=1}^{K} Y_{jk} = K\overline{Y_j} \qquad \overline{Y_j} = a_j + b_j x_{jk}$$

Now, our final result uses α_j and β_j so we need a term that uses these, and so we add $-\mu_{jk} + \mu_{jk}$ to the left hand side.

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{jk} - \mu_{jk} + \mu_{jk} - a_j - b_j x_{jk})^2, \quad A = Y_{jk} - \mu_{jk}, \quad B = \mu_{jk} - a_j - b_j x_{jk} \\
=&\sum_{j=1}^{J}\sum_{k=1}^{K}(A+B)^2 \\
=&\sum_{j=1}^{J}\sum_{k=1}^{K}A^2 + 2AB + B^2
\end{aligned}$$

Note that A^2 is equal to the first term of the right hand side of the equation (because $A^2 = (Y_{jk} - \mu_{jk})^2 = (Y_{jk} - (\alpha_j + \beta_j x_{jk}))^2$. So, we need to show that $2AB + B^2$ is equal to the remaining two terms.

$$\begin{aligned}
2AB &= 2(Y_{jk} - \mu_{jk})(\mu_{jk} - a_j - b_j x_{jk}) \\
&= 2Y_{jk}\mu_{jk} - 2Y_{jk}a_j - 2Y_{jk}b_j x_{jk} - 2\mu_{jk}^2 + 2\mu_{jk}a_j + 2b_j x_{jk}\mu_{jk}) \\
B^2 &= (\mu_{jk} - a_j - b_j x_{jk})(\mu_{jk} - a_j - b_j x_{jk}) \\
&= \mu_{jk}^2 - a_j\mu_{jk} - b_j x_{jk}\mu_{jk} - a_j\mu_{jk} + a_j^2 + a_j b_j x_{jk} - b_j x_{jk}\mu_{jk} + a_j b_j x_{jk} + b_j^2 x_{jk}^2 \\
&= \mu_{jk}^2 - 2a_j\mu_{jk} - 2b_j x_{jk}\mu_{jk} + 2a_j b_j x_{jk} + a_j^2 + b_j^2 x_{jk}^2 \\
2AB + B^2 &= 2Y_{jk}\mu_{jk} - 2Y_{jk}a_j - 2Y_{jk}b_j x_{jk} - \mu_{jk}^2 + 2a_j b_j x_{jk} + a_j^2 + b_j^2 x_{jk}^2 \\
&= 2Y_{jk}\alpha_j + 2Y_{jk}\beta_j x_{jk} - 2Y_{jk}a_j - 2Y_{jk}b_j x_{jk} - \alpha_j^2 - 2\alpha_j\beta_j x_{jk} - \beta_j^2 x_{jk}^2 + 2a_j b_j x_{jk} + a_j^2 + b_j^2 x_{jk}^2
\end{aligned}$$

Now, looking at the second term, we have

$$\begin{aligned}
&-K\sum_{j=1}^{J}(\overline{Y_j} - \alpha_j - \beta_j\overline{x_j})^2 \\
=&-K\sum_{j=1}^{J}\left[\overline{Y_j}^2 - 2\alpha_j\overline{Y_j} - 2\beta_j\overline{x_j}\,\overline{Y_j} + 2\alpha_j\beta_j\overline{x_j} + \alpha_j^2 + \beta_j^2\overline{x_j}^2\right]
\end{aligned}$$

Note that the two equations are fairly similar. At this point, it is important to apply the two unused identities from the beginning, and we obtain the result

$$= \sum_{j=1}^{J} \sum_{k=1}^{K} (2AB + B^2)$$

$$= \sum_{j=1}^{J} \left[2K\overline{Y_j}\alpha_j + 2K\overline{Y_j}\beta_j\overline{x_j} - 2K\overline{Y_j}a_j - 2K\overline{Y_j}b_j\overline{x_j} - K\alpha_j^2 - 2K\alpha_j\beta_j\overline{x_j} - K\beta_j^2\overline{x_j}^2 + 2Ka_jb_j\overline{x_j} + Ka_j^2 + Kb_j^2\overline{x_j}^2\right]$$

$$= K\sum_{j=1}^{J} \left[2\overline{Y_j}\alpha_j + 2\overline{Y_j}\beta_j\overline{x_j} - 2\overline{Y_j}(\alpha_j + \beta_j\overline{x_j})^2 - \alpha_j^2 - 2\alpha_j\beta_j\overline{x_j} - \beta_j^2\overline{x_j}^2 + (\alpha_j + \beta_j\overline{x_j})^2\right]$$

$$= K\sum_{j=1}^{J} \left[2\overline{Y_j}\alpha_j + 2\overline{Y_j}\beta_j\overline{x_j} - 2\overline{Y_j}^2 - 2\alpha_j\beta_j\overline{x_j} - \beta_j^2\overline{x_j}^2 + \overline{Y_j}^2\right]$$

$$= K\sum_{j=1}^{J} \left[-\overline{Y_j}^2 + 2\overline{Y_j}\alpha_j + 2\overline{Y_j}\beta_j\overline{x_j} - 2\alpha_j\beta_j\overline{x_j} - \beta_j^2\overline{x_j}^2\right]$$

$$= -K\sum_{j=1}^{J} \left[\overline{Y_j}^2 - 2\alpha_j\overline{Y_j} - 2\beta_j\overline{x_j}\overline{Y_j} + 2\alpha_j\beta_j\overline{x_j} + \alpha_j^2 + \beta_j^2\overline{x_j}^2\right]$$

So we have shown that they are equal.

How to obtain the third term, I am completely and absolutely unsure.