Continuous Time Markov Chains

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1. Two machines are maintained by a single repairperson. Machine i functions for an exponentially distributed time with mean μ_i^{-1} and then breaks down, $i \in \{1, 2\}$. All repair times are independent exponential random variables with rate μ, and all the times (repair time, functioning times) are mutually independent. Can we analyze this as a birth-death process? If so, what are the parameters? If not, how can we analyze it?

Solution: This is not a birth-death process because we cannot represent the states in the form of $(0, 1, 2...)$. We can, however represent the states as

$$
\begin{aligned}
S_b &: \text{Both machines are working.} & \nu_{S_b} &= \mu_1 + \mu_2 \\
S_1 &: \text{Only machine one is working.} & \nu_{S_1} &= \mu_1 + \mu \\
S_2 &: \text{Only machine two is working.} & \nu_{S_2} &= \mu_2 + \mu \\
S_{01} &: \text{Both are down, 1 is being worked on.} & \nu_{S_{01}} &= \mu \\
S_{02} &: \text{Both are down, 2 is being worked on.} & \nu_{S_{02}} &= \mu
\end{aligned}
$$

The probabilities are thus:

$$
\begin{aligned}
P_{S_{01},S_1} &= 1 & P_{S_{02},S_2} &= 2 \\
P_{S_2,S_b} &= \frac{\mu}{\mu + \mu_2} & P_{S_1,S_b} &= \frac{\mu}{\mu + \mu_1} \\
P_{S_b,S_2} &= \frac{\mu_2}{\mu_1 + \mu_2} & P_{S_b,S_1} &= \frac{\mu_1}{\mu_1 + \mu_2} \\
P_{S_2,S_{01}} &= \frac{\mu_2}{\mu + \mu_2} & P_{S_1,S_{02}} &= \frac{\mu_1}{\mu + \mu_1}
\end{aligned}
$$

All other probabilities are 0. This model has an important assumption though. Because their are no rules as to which machine that repairperson will be working on, this model assumes that the repairperson starts working on whichever machine goes down first.

2. This is known as the Stochastic SI model. The "SI" stands for "susceptible, infected" and indicates that for any time t, each individual is either susceptible to or infected. There are N individuals in a population, some of who have a certain infection that spreads as follows. Contacts between two members of this population occur according to a Poisson process with rate λ. When a contact occurs, it is equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. If a contact involves an infected and a non-infected individual, then with probability p the non-infected individual becomes infected. Once infected, an individual remains infected throughout. Let $X(t)$ denote the number of infected members of the population at time t.

   (a) Is $\{X(t), t \geq 0\}$ a CTMC?

   Solution: Yes.

   (b) Specify the time of CTMC it is.

   Solution: This can be modeled as a pure birth model where each new infected person is an addition

(c) Starting with a single infected individual, what is the expected time until all members are infected?

Solution: First, we must find the probability that an infected person is coupled with a non-infected person. If any pair can be chosen, then the probability that of the two, only one is infected is distributed $Binomial(2, p_{inf})$ where $p_{inf} = i/N$ and i is the number of currently infected individuals. Then, we must consider the probability p that the non-infected individual becomes infected. Therefore, the rate at which contacts occur and a member is infected is

$$\lambda p \binom{2}{1} \frac{i(N-i)}{N^2}$$

Now, if we let $T = \sum T_i$ where T is the total time until all individuals are infected and T_i is the time until the ith member is infected, then we have

$$\begin{aligned}\mathbb{E}[T] = \mathbb{E}\left[\sum_{i=1}^{N} T_i\right] &= \mathbb{E}\left[\sum_{i=2}^{N} T_i\right] \\ &= \sum_{i=2}^{N} E[T_i] \\ &= \sum_{i=1}^{N-1} \left(\lambda p \binom{2}{1} \frac{i(N-i)}{N^2}\right)^{-1}\end{aligned}$$

3. Consider a Yule (pure birth) process starting with exactly one individual at time 0. That is $X(0) = 1$. Let T_i denote the item it takes the process to go from a population of size i to one of size $i + 1$.

(a) Argue that $T_i, i = 1, 2, ..., j$, are independent exponential random variables with respective rates $i\lambda$.

Solution: Consider the initial population at time 0. We know that the time until they give birth to another member of the process has rate λ. After they give birth, each member gives birth at rate λ and so the population gives birth at rate 2λ. With each birth, this rate increases. The time until the births will be distributed rate $i\lambda$ where i is the population because it is the sum of i independent exponential variables, as the time until the next birth for each member is distributed rate λ.

(b) Let $X_1, X_2, ..., X_j$ denote independent exponential random variables each having rate λ, and interpret X_i as the lifetime of component i. Argue that

$$\max(X_1, X_2, ..., X_j) = \sum_{i=1}^{j} \epsilon_i$$

where the ϵ_i are independent exponential random variables with respective rates $j\lambda, (j-1)\lambda, ..., \lambda$. *Hint:* Interpret ϵ_i as the time between the $i - 1$ and ith failure.

Solution: Similar to the previous part, each part itself has lifetime distributed rate λ. So, when there are j components, the time expected until any given one fails is the sum of j independent exponentials, so it would have rate $j\lambda$. Once a part fails, this drops to $(j-1)\lambda)$ and so forth. Therefore, the maximum lifetime of a part will the lifetime of the initial failure plus the second failure ... + it's own failure or

$$j\lambda + (j-1)\lambda + ... + \lambda = \sum_{i=1}^{j} \epsilon_i = \max(X_1, ..., X_j)$$

(c) Using your answers to the previous two parts, argue that

$$P\left(\sum_{i-1}^{j} T_i \leq t\right) = (1 - e^{-\lambda t})^j$$

Solution: As T_i is the time until the next birth, then let $X_1, X_2, ...X_j$ be the respective time until each person gives birth which has rate λ. Therefore we have

$$\begin{aligned}
P\left(\sum_{i=1}^{j} T_i \leq t\right) &= P\left(\max(X_1, X_2, ..., X_j) \leq t\right) \\
&= P\Big((X_1 \leq t) \cap (X_2 \leq t) \cap ... \cap \big(X_j \leq t\big)\Big) \\
&= P(X_1 \leq t)P(X_2 \leq t) \times ... \times P(X_j \leq t) \\
&= (1 - e^{-\lambda t})^j
\end{aligned}$$

(d) Use the previous part to obtain

$$P_{1j}(t) = (1 - e^{-\lambda t})^{j-1} - (1 - e^{-\lambda t})^j = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}$$

Solution:

$$\begin{aligned}
P_{1j}(t) &= P\left(X(t) = j\right) \\
&= P\left(X(t) \geq j\right) - P\left(X(t) \geq j + 1\right) \\
&= P\left(\sum_{i=1}^{j-1} T_i \leq t\right) - P\left(\sum_{i=1}^{j} T_i \leq t\right) \\
&= (1 - e^{-\lambda t})^{j-1} - (1 - e^{-\lambda t})^j \\
&= e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}
\end{aligned}$$

(e) Finally, conclude that

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-\lambda t i}(1 - e^{-\lambda t})^{j-1}$$

Solution: The results from part (d) show that $P_{1j}(t)$ is distributed geometric with $p = e^{-\lambda t}$. We know from the text that if we start with i population, it is equivalent to having i independent Yule processes, and since that would mean this is the sum of i geometric processes, then we have a Negative Binomial with density

$$\binom{j-1}{i-1} e^{-\lambda t i}(1 - e^{-\lambda t})^{j-1}$$

4. A barbershop operated by a single barber has room for at most two customers (that is, room for one to wait while another gets his/her hair cut). Potential customers arrive according to a Poisson process with rate three per hour, and the successive service times are independent exponential random variables with mean 0.25 per hour.

(a) What is the average number of customers in the shop?

Solution: First, we need to find find P_1, P_2, and P_3. We can find these using the below equations found from the principle that the rate process leaves is equal to the rate the process enters.

$$\begin{aligned} 3P_0 &= 4P_1 \\ 7P_1 &= 4P_2 + 3P_0 \\ 3P_2 &= 4P_1 \\ \sum_{i=0}^{2} P_i &= 1 \end{aligned}$$

Using the above, we find that $P_1 = \frac{3}{4}P_0$ and then that $P_2 = \left(\frac{3}{4}\right)^2 P_0$ which gives

$$\begin{aligned} 1 &= P_0 + P_1 + P_2 \\ &= P_0 + \frac{3}{4}P_0 + \left(\frac{3}{4}\right)^2 P_0 \\ &= P_0 \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right) \\ P_0 &= \frac{16}{37} \\ P_1 &= \frac{12}{37} \\ P_2 &= \frac{9}{37} \end{aligned}$$

From these probabilities, we can find the expected number of customers in the shop with

$$0 * P_0 + 1 * P_1 + 2 * P_2 = \frac{12}{37} + \frac{18}{37} = \frac{30}{37} \approx .8108 \text{ customers}$$

(b) What is the proportion of potential customers that enter the shop?

Solution: As none can enter when the shop has two people, then the proportion of potential customers that enter is

$$1 - P_2 = 1 - \frac{9}{37} = \frac{28}{37}$$

(c) If the barber could work twice as fast, how much more business would he do?

Solution: We must first solve the following system of equations.

$$\begin{aligned} 3P_0 &= 8P_1 \\ 3P_2 &= 8P_1 \\ \sum_{i=0}^{2} P_i &= 1 \end{aligned}$$

which end up giving

$$P_0 = \frac{64}{97} \qquad P_1 = \frac{24}{97} \qquad P_2 = \frac{9}{97}$$

Then we know that if he works at double the speed, the proportion of customers that enter the shop is $\frac{88}{97}$. Therefore, his rate will increase by

$$3\left(\frac{88}{97} - \frac{28}{37}\right) \approx 0.451 \text{ customers per hour}$$

(d) In addition, if the shop is full at time 0, find the distribution of the numbers of customers at the station at time 5 minutes, 30 minutes, 90 minutes, 180 minutes. Are the distributions converging?

Solution: These will be solved using

$$e^{Qt} = \lim_{k\to\infty} \left(I + Qt/k\right)^k$$

in matlab. Note that if we are using the initial conditions,

$$Q = \begin{bmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{bmatrix}$$

Using the following matlab code,

```
t = 5/60;
Q = [-3 3 0;4 -7 3;0 4 -4];
I = eye(3);
Answer = (I + Q*t/1000)^1000;
Answer(3,:)
```

the following distributions are found

$$\begin{aligned}
P_{2j}(5/60) &= [0.0383 \quad 0.2174 \quad 0.7444] &&\implies \mathbb{E}[X(5/60) \mid X(0) = 2] = 1.7061\\
P_{2j}(30/60) &= [0.3223 \quad 0.3371 \quad 0.3406] &&\implies \mathbb{E}[X(30/60) \mid X(0) = 2] = 1.0184\\
P_{2j}(90/60) &= [0.4292 \quad 0.3248 \quad 0.2460] &&\implies \mathbb{E}[X(90/60) \mid X(0) = 2] = 0.8168\\
P_{2j}(180/60) &= [0.4324 \quad 0.3243 \quad 0.2433] &&\implies \mathbb{E}[X(180/60) \mid X(0) = 2] = 0.8108
\end{aligned}$$

It is clear to see that it is converging to the expected number of customers from part (a).

5. Potential customers arrive at a full-service, one pump gas station in accordance with a Poisson process with rate 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponentially distributed with mean five minutes.

(a) What fraction of the attendant's time will be spent service cars?

Solution: Using the same line of reasoning in problem four, we must solve the following system

$$\begin{aligned}
20P_0 &= 12P_1\\
32P_1 &= 20P_0 + 12P_2\\
32P_2 &= 20P_1 + 12P_3\\
12P_3 &= 20P_2\\
P_0 + P_1 + P_2 + P_3 &= 1
\end{aligned}$$

which results in

$$P_0 \approx 0.099 \qquad P_1 \approx 0.1654 \qquad P_2 \approx 0.2757 \qquad P_3 \approx .4956$$

which means that he spends approximately 90% of the time servicing cars.

(b) What fraction of potential customers are lost?

Solution: This is the same as the percentage of the time that there are two cars waiting, so approximately 50% of potential customers are lost.

(c) In addition, if there are no cars at the station at time 0, find the distribution of the numbers of cars at the station at times 1 minute, 3 minutes, 45 minutes, and 180 minutes. Are the distributions converging?

Solution: Using the same procedure as problem four with

$$Q = \begin{bmatrix} -20 & 20 & 0 & 0 \\ 12 & -32 & 20 & 0 \\ 0 & 12 & -32 & 20 \\ 0 & 0 & 12 & -12 \end{bmatrix}$$

So using the same code as in problem 4, the distributions are found to be

$$\begin{aligned} P_{0j}(1/60) &= [.7391 \quad .2213 \quad .0355 \quad .0042] & \implies \mathbb{E}[X(1/60) \mid X(0) = 0] = 0.3047 \\ P_{0j}(3/60) &= [.4576 \quad .3318 \quad .1449 \quad .0556] & \implies \mathbb{E}[X(3/60) \mid X(0) = 0] = 0.7886 \\ P_{0j}(45/60) &= [.0995 \quad .1656 \quad .2757 \quad .4591] & \implies \mathbb{E}[X(45/60) \mid X(0) = 0] = 2.0945 \\ P_{0j}(180/60) &= [.0995 \quad .1656 \quad .2757 \quad .4591] & \implies \mathbb{E}[X(180/60) \mid X(0) = 0] = 2.0945 \end{aligned}$$

They are indeed. They are converging to the expected values.

6. The surface of a bacterium consists of several sites at which foreign molecules – some acceptable and some not – become attached. Consider a particular site and assume that molecules arrive there according to a Poisson process with rate λ. Among these molecules, the proportion which are acceptable is α. Unacceptable molecules stay at the site for a length of time that is exponentially distributed with parameter μ_1, whereas an acceptable molecule remains at the site for an exponential time with rate μ_2. An arriving molecule will become attached only if the site is free of other molecules. What percentage of time is the site occupied with an acceptable (or unacceptable) molecule?

Solution: Let the possible states be 0, G, and B where 0 is no molecules are at the site, G is a good molecule at the site, and B is a bad molecule at the site. This gives the following system of equations.

$$\begin{aligned} \lambda P_0 &= \mu_1 P_B + \mu_2 P_G \\ \mu_2 P_G &= \alpha\lambda P_0 \\ \mu_1 P_B &= (1-\alpha)\lambda P_0 \\ P_1 + P_G + P_B &= 1 \end{aligned}$$

which gives

$$P_G = \frac{\alpha\lambda P_0}{\mu_2} \qquad P_B = \frac{(1-\alpha)\lambda P_0}{\mu_1}$$

which in turn means

$$P_0 = \left(1 + \frac{\alpha\lambda}{\mu_2} + \frac{(1-\alpha)\lambda}{\mu_1}\right)^{-1} = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \alpha\lambda\mu_1 + (1-\alpha)\mu_2\lambda}$$

The proportion of time it is occupied would be therefore

$$1 - P_0 = \frac{\alpha\lambda\mu_1 + (1-\alpha)\lambda\mu_2}{\mu_1\mu_2 + \alpha\lambda\mu_1 + (1-\alpha)\mu_2\lambda}$$

7. We have two machines. We use one of them, and keep the other as a spare. A working machine will function for an exponentially distributed amount of time with rate λ and will then fail. Upon failure, it is immediately replaced by the other machine if that one is in working order, and it goes to the repair facility. The repair facility consists of a single person who takes an exponentially distributed time with rate μ to service the failed machine. At the repair facility, the newly failed machined enters service if the repairperson is free. If the repairperson is busy, it must wait until the other machine is fixed. At that time, the newly repaired machine is put into service and the repair on the other machine begins. If we begin with both machines in good working condition, fine

   (a) the expected value and

   Solution: This can be modeled as a birth-death process where the population is the number of down machines. We know from the derivation in Example 6.7 from the text that the expected time to reach state 2 (or 2 machines down) from state 0 (all machines working) is $\sum_{i=0}^{1} T_i$. So, we have

$$\mathbb{E}[T_0] = \frac{1}{\lambda} \qquad \mathbb{E}[T_1] = \frac{1}{\lambda} + \frac{\mu}{\lambda^2}$$

$$\mathbb{E}[T_{0\to 2}] = \frac{2}{\lambda} + \frac{\mu}{\lambda^2}$$

   (b) variance of the time until both machine are in the repair facility.

   Solution: We know from the derivation in the text that $Var(T_{0\to 2}) = Var(T_0) + Var(T_1)$ and so we have

$$\begin{aligned}
Var(T_0) &= \frac{1}{\lambda^2} \\
Var(T_1) &= \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda} Var(T_0) + \frac{\mu}{\lambda+\mu}\left(\mathbb{E}[T_0] + \mathbb{E}[T_1]\right)^2 \\
&= \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\lambda+\mu}\left(\frac{2}{\lambda} + \frac{\mu}{\lambda^2}\right)^2 \\
Var(T_{0\to 2}) &= \frac{1}{\lambda^2} + \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\lambda+\mu}\left(\frac{2}{\lambda} + \frac{\mu}{\lambda^2}\right)^2
\end{aligned}$$

   (c) In the long-run, what proportion of time is there a working machine?

   Solution: We can find this by finding $P_0 + P_1$ which can both be found by solving the following system of equations.

$$\lambda P_0 = \mu P_1 \qquad \Longrightarrow P_1 = \frac{\lambda}{\mu} P_0$$

$$\mu P_2 = \lambda P_1 \qquad \Longrightarrow P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_0 + P_1 + P_2 = 1$$

   and so therefore

$$P_0 = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}\right)^{-1} = \frac{\mu^2}{\mu^2 + \lambda\mu + \lambda^2} \qquad P_1 = \frac{\lambda}{\mu} P_0 = \frac{\lambda}{\mu}\left(\frac{\mu^2}{\mu^2 + \lambda\mu + \lambda^2}\right)$$

$$P_1 + P_2 = \frac{\mu^2}{\mu^2 + \lambda\mu + \lambda^2} + \frac{\lambda}{\mu}\left(\frac{\mu^2}{\mu^2 + \lambda\mu + \lambda^2}\right) = \frac{1 + (\lambda/\mu)}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2}$$

8. Consider the previous problem. Suppose that when both machines are down that a second repairperson is called in to work on the newly failed machine. Suppose all repair times are independent exponential(μ) random variables. Find the proportion of time at least one machine is working and compare your answer with the one you obtained in the previous problem.

Solution: Solving a similar set of equations...

$$\lambda P_0 = \mu P_1 \qquad \Longrightarrow P_1 = \frac{\lambda}{\mu} P_0$$

$$\mu P_2 = \lambda P_1 \qquad \Longrightarrow P_2 = \frac{\lambda}{\mu} P_1 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_0 + P_1 + P_2 = 1$$

gives

$$P_0 = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2}\right)^{-1} = \frac{2\mu^2}{2\mu^2 + 2\lambda\mu + \lambda^2} \qquad P_1 = \frac{\lambda}{\mu} P_0 = \frac{\lambda}{\mu}\left(\frac{2\mu^2}{2\mu^2 + 2\lambda\mu + \lambda^2}\right)$$

$$\begin{aligned} P_1 + P_2 &= \frac{2\mu^2}{2\mu^2 + 2\lambda\mu + \lambda^2} + \frac{\lambda}{\mu}\left(\frac{2\mu^2}{2\mu^2 + 2\lambda\mu + \lambda^2}\right) \\ &= \frac{1 + (\lambda/\mu)}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2}} \end{aligned}$$

This is almost the exactly same thing as the last except that the final term of the denominator also has a term of $\frac{1}{2}$, which would lead to an increased rate of having a working machine.

9. Consider a taxi station where taxis and customers arrive according to Poisson process with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer who does not find a taxi waiting leaves, never to return. Calculate

(a) the average number of taxis waiting;

Solution: This can be modeled as a birth-death process with entering rate $\lambda = 1$ and leaving rate $\mu = 2$. The general system of equations is

$$\begin{aligned}
\lambda P_0 &= \mu P_1 \\
(\mu + \lambda) P_1 &= \mu P_2 + \lambda P_0 \\
(\mu + \lambda) P_2 &= \mu P_3 + \lambda P_1 \\
(\mu + \lambda) P_3 &= \mu P_4 + \lambda P_2 \\
\ldots &= \ldots \\
(\mu + \lambda) P_i &= \mu P_{i+1} + \lambda P_{i-1}
\end{aligned}$$

which leads to

$$P_0 = \left(\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu} \right)^i \right)^{-1} \qquad P_i = \left(\frac{\lambda}{\mu} \right)^i P_0$$

so in terms of this equation, we have

$$\begin{aligned}
P_0 &= \left(\sum_{i=0}^{\infty} \left(\frac{1}{2} \right)^i \right)^{-1} = \left(\frac{1}{1 - \frac{1}{2}} \right)^{-1} = \frac{1}{2} \\
P_i &= \left(\frac{1}{2} \right)^{i+1}
\end{aligned}$$

which means that the expected number of taxis is

$$\begin{aligned}
\sum_{i=0}^{\infty} i * P_i &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{i}{2^i} \\
&= \frac{1}{2} * 2 \\
&= 1
\end{aligned}$$

(b) the proportion of arriving customers who get taxis.

Solution: This is just the probability that there exists a taxi at the station which is $1 - P_0 = \frac{1}{2}$ so the proportion of arriving customers who get taxis is $\frac{1}{2}$.

10. Four workers share an office that contains four telephones. At any time, each worker is either "working" or "on the phone." Each "working" period of worker i lasts for an exponentially distributed time with rate λ_i and each "on the phone" period lasts for an exponentially distributed amount of time with rate $\mu_i, i \in \{1, 2, 3, 4\}$.

   (a) What proportion of time are all workers "working"?

   Solution: Due to independence of the processes, we can solve a single process and find the proportion of time one worker is working and use that result. Let P_0 be the stationary probability of being on the phone and P_1 be the stationary probability of working.

$$\begin{aligned} \mu_1 P_0 &= \lambda_1 P_1 \\ P_0 + P_1 &= 1 \\ P_0 &= \left(1 + \frac{\mu_1}{\lambda_1}\right)^{-1} \\ &= \frac{\lambda_1}{\lambda_1 + \mu_1} \implies P_1 = \frac{\mu_1}{\lambda_1 + \mu_1} \end{aligned}$$

   Therefore the probability that all are working is

$$\frac{\mu_1 \mu_2 \mu_3 \mu_4}{(\mu_1 + \lambda_1)(\mu_2 + \lambda_2)(\mu_3 + \lambda_3)(\mu_4 + \lambda_4)}$$

   (b) Let $X_i(t)$ equal 1 if worker i is working at time t, and let it be 0 otherwise. Let $X(t)$ denote the random vector $(X_1(t), X_2(t), X_3(t), X_4(t))$. Argue that $\{X_t, t \geq 0\}$ is a CTMC and give its infinitesimal transition rates.

   Solution: Each is its own independent continuous time Markov Chain and so the process as a whole is also a CTMC. The infinitesimal transition ares are going to be a combination of the individual infinitesimal rates. So, letting (i, j, k, l) describe the current activity of the 1st, 2nd, 3rd, and 4th worker, then I set the following variables for ease of displaying the matrix.

a = (0,0,0,0)	b = (0,0,0,1)	c = (0,0,1,0)	d = (0,0,1,1)
e = (0,1,0,0)	f = (0,1,0,1)	g = (0,1,1,0)	h = (0,1,1,1)
i = (1,0,0,0)	j = (1,0,0,1)	k = (1,0,1,0)	l = (1,0,1,1)
m = (1,1,0,0)	n = (1,1,0,1)	o = (1,1,1,0)	p = (1,1,1,1)

   Therefore Q, which is the matrix of the infinitesimal transitions is

$$Q = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \\ o \\ p \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p \end{array} \\ \begin{bmatrix} - & \mu_4 & \mu_3 & 0 & \mu_2 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_4 & - & 0 & \mu_3 & 0 & \mu_2 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_3 & 0 & - & \mu_4 & 0 & 0 & \mu_2 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_3 & \lambda_4 & - & 0 & 0 & 0 & \mu_2 & 0 & 0 & {} & \mu_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 & - & \mu_4 & \mu_3 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \lambda_4 & - & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & \lambda_3 & 0 & - & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & \lambda_3 & \lambda_4 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 \\ \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & \mu_4 & \mu_3 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & - & 0 & \mu_3 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & - & \mu_4 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & \lambda_4 & - & 0 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & - & \mu_4 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & \lambda_4 & - & 0 & \mu_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_2 & 0 & \lambda_3 & 0 & - & \mu_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_2 & 0 & \lambda_3 & \lambda_4 & - \end{bmatrix} \end{array}$$

   The $-$ character in the diagonal is a place holder for $-\nu_z$ with $z \in \{a, b, ..., o, p\}$ in order for the matrix to fit on page.

(c) Is $\{X(t)\}$ time reversible? Why or why not?

Solution: Problem 28 of the text says to show that for a process that contains independent processes, if those processes are time reversible, then the CTMC as a whole is time reversible. As they are all similar, and each trivially time reversible, then the whole CTMC is time reversible.

(d) Suppose that one of the phones is broken, and that a worker who is about to use a phone but finds them all being used begins a new "working" period. What proportion of time are all workers "working"?

Solution: This is almost the exact same matrix except that moving to state a is impossible. So to find the probability that all workers are "working", we can use the following

$$\begin{aligned}
P(\text{all working}|1\text{ phone broken}) &= \frac{P(\text{all working})}{1-P(a)} \\
&= \frac{\frac{\mu_1\mu_2\mu_3\mu_4}{(\mu_1+\lambda_1)(\mu_2+\lambda_2)(\mu_3+\lambda_3)(\mu_4+\lambda_4)}}{1-\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\mu_1+\lambda_1)(\mu_2+\lambda_2)(\mu_3+\lambda_3)(\mu_4+\lambda_4)}} \\
&= \frac{\prod_{i=1}^{4}\mu_i/(u_i+\lambda_i)}{1-\prod_{i=1}^{4}\lambda_i/(u_i+\lambda_i)}
\end{aligned}$$

11. Consider a time-reversible CTMC with infinitesimal transition rates q_{ij} and limiting probabilities $\{P_i\}$. Let A denote a set of states for this chain, and consider a new CTMC with transition rates q^*_{ij} given by

$$q^*_{ij} = \begin{cases} cq_{ij}, & \text{if } i \in A, j \notin A \\ q_{ij}, & o/w \end{cases}$$

where c is an arbitrary positive number. show that this chain remains time reversible, and find its limiting probabilities.

Solution: Letting P^*_i be the new limiting probabilities, we must find a solution that satisfies

$$P^*_i q^*_{ij} = P^*_j q^*_{ji}$$

or particularly

$$\begin{cases} P^*_i cq_{ij} = P^*_j q_{ji}, & \text{if } i \in A, j \notin A \\ P^*_i q_{ij} = P^*_j q_{ji}, & o/w \end{cases}$$

So if we can find limiting probabilities to this problem, then the chain remains time reversible. My initial suspicion was that we could define

$$P^*_i = \begin{cases} P_i/c, & i \in A, j \notin A \\ P_i, & o/w \end{cases}$$

but this doesn't satisfy $\sum P^*_i = 1$ if $\sum P_i = 1$ unless $c = 1$. So, their must be some normalizing constant α that makes $\sum P^*_i = 1$, therefore let

$$P^*_i = \begin{cases} \alpha P_i/c, & i \in A, j \notin A \\ \alpha P_i, & o/w \end{cases}$$

and we can find α by solving

$$\alpha\left(\sum_{i\in A} P_i/c + \sum_{i\notin A} P_i\right) = 1 \qquad\qquad \implies \alpha = \left(\sum_{i\in A} P_i/c + \sum_{i\notin A} P_i\right)^{-1}$$

and so with this solution of α, $\sum P_i = 1$ and the process is time reversible.

12. Let Y denote an exponential random variable with parameter λ that is independent of the CTMC $\{X(T)\}$ and let

$$\overline{P}_{ij} = P\left(X\left(Y\right) = j \mid X\left(0\right) = i\right)$$

(a) Show that

$$\overline{P}_{ij} = \frac{1}{\nu_i + \lambda} \sum_k q_{ik} \overline{P}_{kj} + \frac{\lambda}{\nu_i + \lambda} \mathbb{1}\{i = j\}$$

Solution: If we let T_i be the time it takes until we transition out of i, then we can require

$$P\left(X\left(Y\right) = j \mid X\left(0\right) = i\right)$$

as

$$P\left(X(Y) = j \mid X(0) = i, Y > T_i\right) P(Y > T_i) + P\left(X(Y) = j \mid X(0) = i, Y \leq T_i\right) P(Y \leq T_i)$$

So, if a transition did indeed occur, i.e. $Y > T_i$, then we know from the text that $P_{ij} = \sum_k P_{ik} P_{kj}$. If a transition did not occur, then the probability of $X(y) = j$ given that $X(0) = i$ is only possible if $i = j$. So, we have

$$\sum_k \overline{P}_{ik} \overline{P}_{kj} P(Y > T_i) + \mathbb{1}\{i = j\} P(Y \leq T_i)$$

The remaining probabilities are two exponential random variables being compared which we found in Chapter 5 to be $P(X > Y) = \lambda_X / (\lambda_X + \lambda_Y)$ so our final result is

$$\begin{aligned}
\overline{P}_{ij} &= \sum_k \overline{P}_{ik} \overline{P}_{kj} P(Y > T_i) + \mathbb{1}\{i = j\} P(Y \leq T_i) \\
&= \sum_k \overline{P}_{ik} \overline{P}_{kj} \frac{\nu_i}{\nu_i + \lambda} + \mathbb{1}\{i = j\} \frac{\lambda}{\nu_i + \lambda} \\
&= \sum_k q_{ik} \overline{P}_{kj} \frac{1}{\nu_i + \lambda} + \mathbb{1}\{i = j\} \frac{\lambda}{\nu_i + \lambda} \qquad \text{because } q_{ik} = P_{ik}\nu_i \\
&= \frac{1}{\nu_i + \lambda} \sum_k q_{ik} \overline{P}_{kj} + \frac{\lambda}{\nu_i + \lambda} \mathbb{1}\{i = j\}
\end{aligned}$$

(b) Show the solution of the preceding set of equations is given by

$$\overline{P} = (I - R/\lambda)^{-1}$$

where $\overline{P}$ is the matrix of element $\overline{P}_{ij}$, I is the identity matrix, and R is the matrix of elements

$$r_{ij} = q_{ij} \mathbb{1}\{i \neq j\} + -\nu_i \mathbb{1}\{i = j\}.$$

Solution: Drawing from the previous results, we have

$$\begin{aligned}
(\nu_i + \lambda) \overline{P}_{ij} &= \sum_k q_{ik} \overline{P}_{kj} + \lambda \mathbb{1}\{i = j\} \\
-\lambda \mathbb{1}\{i = j\} &= \sum_k q_{ik} \overline{P}_{kj} - \nu_i \overline{P}_{ij} - \lambda \overline{P}_{ij} \\
&= \sum_k r_{ik} \overline{P}_{kj} - \lambda \overline{P}_{ij} \\
-\lambda I &= R\overline{P} - \lambda \overline{P} \\
-\lambda I &= (R - \lambda) \overline{P} \\
\overline{P} &= -\lambda (R - \lambda)^{-1} \\
&= \left(I - R/\lambda\right)^{-1}
\end{aligned}$$

(c) Suppose now that $Y_1, ..., Y_n$ are independent exponentials with rate λ and independent of $\{X(t)\}$. Show that

$$P\left(X\left(\sum_{k=1}^{n} Y_k\right) = j \;\middle|\; X(0) = i\right)$$

is equal to the element in row i, column j of the matrix $\overline{P}^n$.

Solution: We will consider the case where $n = 2$ for initial simplicity, and use Baye's rule.

$$\begin{aligned}
&P\left(X\left(Y_1 + Y_2\right) = j \mid X(0) = i\right) \\
&= \sum_k P\left(X\left(Y_1 + Y_2\right) = j \mid X(0) = i, X(Y_1) = k\right) P(X(Y_1) = k \mid X(0) = i) \\
&= \sum_k P\left(X\left(Y_1 + Y_2\right) = j \mid X(Y_1) = k\right) P\left(X(Y_1) = k \mid X(0) = i\right) \quad \text{by Markov Prop} \\
&= \sum_k P\left(X\left(Y_2\right) = j \mid X(0) = k\right) P\left(X(Y_1) = k \mid X(0) = i\right) \quad \text{by Markov Prop} \\
&= \sum_k P_{kj} P_{ik}
\end{aligned}$$

which is the two-step transition probability. This can be extended by iteration by applying Baye's rule on additional Y_is which will lead to some n step transition formula.

(d) Explain the relationship of the preceding to the approximation

$$P(t) = e^{Rt} \approx \left(I - R\frac{t}{n}\right)^{-n}$$

Solution: We showed that for one random exponential variable in part (b) that it results in

$$\left(I - \frac{R}{\lambda}\right)^{-1}$$

and similarly for the nth step, it will have a factor of n. In this case, $\frac{t}{n}$ is essentially $\frac{1}{\lambda}$ which results in the approximation above and given in the text.

13. Find the stationary distribution for the number of customers in the $M/M/s$ queue. What is the stability condition? Use this to calculate the expected number of customers in the queue in steady state.

Solution: We can find the stationary solution by solving the system of equations if we let λ be the entry rate and μ be the exit rate.

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ (\lambda+\mu)P_1 &= \lambda P_0 + 2\mu P_2 \\ (\lambda+2\mu)P_2 &= \lambda P_1 + 3\mu P_3 \\ \ldots &= \ldots \\ (\lambda+s\mu)P_s &= \lambda P_{s-1} + s\mu P_{s+1} \\ (\lambda+s\mu)P_{s+1} &= \lambda P_s + s\mu P_{s+2} \end{aligned}$$

which leads to

$$P_i = \begin{cases} \dfrac{(\lambda/\mu)^i}{i!}P_0, & 0 \le i \le s \\[2ex] \dfrac{(\lambda/\mu)^i}{s^i s!}P_0, & i > s \end{cases} \implies P_0 = \left(\sum_{i=0}^{s-1} \frac{\lambda^i}{i!\mu^i} + \sum_{i=s}^{\infty} \frac{\lambda^i}{s^{s-i}s!\mu^i}\right)^{-1}$$

We can simplify this to

$$\begin{aligned} P_0 &= \left(\sum_{i=0}^{s-1} \frac{\lambda^i}{i!\mu^i} + \sum_{i=s}^{\infty} \frac{\lambda^i}{s^{s-i}s!\mu^i}\right)^{-1} \\ &= \left(\sum_{i=0}^{s-1} \frac{\lambda^i}{i!\mu^i} + \frac{\lambda^s}{s!\mu^s}\sum_{i=0}^{\infty} \left(\frac{\lambda}{s\mu}\right)^i\right)^{-1} \\ &= \left(\sum_{i=0}^{s-1} \frac{\lambda^i}{i!\mu^i} + \frac{\lambda^s}{s!\mu^s}\frac{1}{1-\frac{\lambda}{s\mu}}\right)^{-1} \end{aligned}$$

and so the series only converges, and thus is stable, if $\frac{\lambda}{s\mu} < 1$. Therefore, the expected number of customers in the queue if it is in steady state is

$$\sum_{i=0}^{\infty} iP_i = \left(\sum_{i=0}^{s} \frac{(\lambda/\mu)^i}{(i-1)!} + \sum_{i=s+1}^{\infty} i\frac{(\lambda/\mu)^i}{s^i s!}\right)\left(\sum_{i=0}^{s-1} \frac{\lambda^i}{i!\mu^i} + \frac{\lambda^s}{s!\mu^s}\frac{1}{1-\frac{\lambda}{s\mu}}\right)^{-1}$$

14. *Simple Exclusion Process.* Suppose a particle takes a random walk on a 100×100 checkerboard in the following way. After an exponential time with rate 1, it attempts to move up, down, left, or right – each with probability 1/4. If the attempted move would take the particle off the board, it stays put instead. Then, after an exponential time with rate 1, it tries to move again, and on and on. What is the stationary distribution of the particle's position.

Now suppose there are 1278 such particles on the board moving independently, and multiple particles can occupy the same squares. What is the stationary distribution for the number of particles on each square? You might want to think of your state space as consisting of all the 100×100 arrays, where the number in the (i, j) position in the array corresponds to the number of particles there.

Finally answer the previous question when the 1278 particles are only allowed to move to empty squares. That is, each square can only accomodate one particle. Now the state space would be all the 100×100 arrays of 1s and 0s with exactly 1278 1s. *Hint: Use Proposition 24 from the notes.*

Solution:
Part One: I'm going to make an initial guess at the stationary distribution and say that each square has an equal likelihood of being occupied, i.e. $P_i = .0001, \quad \forall i$. I will check this result for the three types of spaces: a corner space, an edge space, and an interior space. These will be denoted by the state space $S = \{C, E, I\}$ respectively. They each have the following rates

$$\nu_C = \frac{1}{2} \qquad \nu_E = \frac{1}{3} \qquad \nu_I = \frac{1}{4}$$

. Note that the probability of moving from a corner space to an interior space is 0, so we shall verify the following.

$$P_C q_{CE} = P_E q_{EC} \quad \text{and} \quad P_I q_{IE} = P_E q_{EI}$$

Note that these all assume, for example, that an edge is directly by a corner or an edge is directly by an interior space. Checking the probability of moving edge to edge or interior to interior is redundant. Now, there are two edges that can enter a corner, and so

$$q_{EC} = \frac{1}{4} + \frac{1}{4} = q_{CE}$$

which verifies our first time reversibility requirement. For any given edge, there is only one way to enter from an interior and therefore

$$q_{IE} = \frac{1}{4} = q_{EI}$$

as there is only one entry from an edge to an interior which verifies the second time reversibility requirement since $P_I = P_C = P_E$ under our assumption that all $P_i = .0001$.

Part Two: As the stationary probability for each square is .0001, then the expected number of particles within each square is $1278 * .0001$. That is, the expected number of particles in each square is .1278.

Part Three: This is really exciting, because it doesn't change! Let $P(NO)$ =Probability that the space being transition into is not occupied $= 1 - \frac{1278}{10000}$. As all of the rates will be multiplied by this same constant, (that is both sides of the equation will be multiplied by this constant), the time reversibility equations remain solved and the expected number of particles ine each square remains .1278.