Ausgangspunkt:

$$C_{lm} = t\,\delta_{lm} + (1-t^2)\left\langle h^{\times}_{(l,0)}, h^{\circ}_{(m+1,0)} \right\rangle$$

$$\delta_{lm} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \; e^{-ik_1(l-m)}$$

$$\left\langle h^{\times}_{(l,0)} h^{\circ}_{(m+1,0)} \right\rangle = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \; e^{-ik_1(l-(m+1))}\,\frac{h(k_1,k_2)}{L(k_1,k_2)}$$

$$h(k_1,k_2) = \mathcal{E}(-k_1)\,\mathcal{E}(k_2)\,\mathcal{E}(-k_2) - \mathcal{E}(k_2) - \mathcal{E}(-k_2)$$

$$L(k_1,k_2) = (1+t^2)^2 - 2t(1-t^2)(\cos(k_1)+\cos(k_2))$$

$$\mathcal{E}(k) = 1 + t e^{ik}$$

Vereinfachungen:

$$\mathcal{E}(k)\,\mathcal{E}(-k) = 1+t^2+2t\cos(k) = |\mathcal{E}(k)|^2$$

$$\mathcal{E}(k)+\mathcal{E}(-k) = 2(1+t\cos(k)) = 2\,\mathrm{Re}(\mathcal{E}(k))$$

$$\Rightarrow h(k_1,k_2) = (1+te^{-k_1})(1+2t\cos(k_2)+t^2) - 2(1+t\cos(k_2))$$

$$= te^{-ik_1}(1+2t\cos(k_2)+t^2) + (t^2-1)$$

$$\Rightarrow (1-t^2)e^{ik_1}h(k_1,k_2) = 2t(1-t^2)\cos(k_2) + (1-t^2)t(1+t^2) - (1-t^2)^2 e^{ik_1}$$

$$= 2t^2(1-t^2)\cos(k_2) + t(1-t^4) - (1-t^2)^2 e^{ik_1}$$

$$\Rightarrow t\,L(k_1,k_2) + (1-t^2)e^{ik_1}h(k_1,k_2)$$

$$= t(1+t^2)^2 - 2t^2(1-t^2)(\cos(k_1)+\cos(k_2)) + 2t^2(1-t^2)\cos(k_2) + t(1-t^4) - (1-t^2)^2 e^{ik_1}$$

$$= t(1+t^2)^2 - 2t^2(1-t^2)\cos(k_1) + t(1-t^4) - (1-t^2)^2 e^{ik_1}$$

$$\left\{\begin{array}{l} t(1+t^2)^2 = t(1+2t^2+t^4) = t+2t^3+t^5 \\ \underline{+\; t(1-t^4)} = t - t^4 \\ \qquad = 2t+2t^3 = 2t(1+t^2) \end{array}\right.$$

$$= 2t(1+t^2) - 2t^2(1-t^2)\cos(k_1) - (1-t^2)^2 e^{ik_1}$$

$$= 2t(1+t^2) - t^2(1-t^2)(e^{ik_1}+e^{-ik_1}) - (1-t^2)^2 e^{ik_1}$$

$$= 2t(1+t^2) - t^2(1-t^2)\,e^{-ik_1} - (t^2-t^4+1-2t^2+t^4)\,e^{ik_1}$$

$$= 2t(1+t^2) - t^2(1-t^2)\,e^{-ik_1} - (1-t^2)e^{ik_1}$$

conclusio :

$$c_{em} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} t e^{-ik_1(e-m)} + (1-t^2) e^{-ik_1(e-(m+1))} \frac{h(k_1,k_2)}{\angle(k_1,k_2)} \, dk_1 \, dk_2$$

$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-ik_1(e-m)} \left(\frac{t \cdot \angle(k_1,k_2) + (1-t^2) e^{ik_1} h(k_1,k_2)}{\angle(k_1,k_2)} \right) dk_1 \, dk_2$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dk_1 \, e^{-ik_1(e-m)} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2t(1+t^2) - t^2(1-t^2) e^{-ik_1} - (1-t^2) e^{ik_1}}{(1+t^2)^2 - 2t(1-t^2)(\cos(k_1) + \cos(k_2))} \, dk_2}_{f(t,k_1)}$$

Lösen des Parameter integrals $f(t,k_1)$:

$\mathfrak{K} := (1+t^2)^2 > 1$ falls $t \in [0,1]$

$\mathcal{K} := 2t(1-t^2) < 1$ falls $t \in [0,1]$

$\tilde{f}(t,k_1) := 2t(1+t^2) - t^2(1-t^2) e^{-ik_1} - (1-t^2) e^{ik_1}$

$\Omega := \mathfrak{K} - \mathcal{K} \cos(k_1)$

$$\Rightarrow f(t,k_1) = \frac{\tilde{f}}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{\Omega - \mathcal{K}\cos(k_2)} = \frac{\tilde{f}}{2\pi \Omega} \int_{-\pi}^{\pi} \frac{dk_2}{1 - \frac{\mathcal{K}}{\Omega}\cos(k_2)}$$

Es muss gezeigt werden: $\frac{\mathcal{K}}{\Omega} < 1$ für $t \in (0,1)$, $k_1 \in [-\pi, \pi]$

$$\frac{\mathcal{K}}{\Omega} = \frac{\mathcal{K}}{\mathfrak{K} - \mathcal{K}\cos(k_1)} = \frac{1}{\frac{\mathfrak{K}}{\mathcal{K}} - \cos(k_1)} < 1 \quad \text{denn} \quad \frac{\mathfrak{K}}{\mathcal{K}} = \frac{(1+t^2)^2}{2t(1-t^2)} > 2$$

$$\frac{\mathfrak{K}}{\mathcal{K}} = \frac{(1+t^2)^2}{2t(1-t^2)} > 2 \iff (1+t^2)^2 - 2 \cdot 2t(1-t^2) > 0$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{1 - a\cos(k_2)} \overset{(M)}{=} \frac{1}{\sqrt{1-a^2}}$$ mit Mathematica, $a = \frac{\mathcal{K}}{\Omega} < 1$

$$\Rightarrow f(t,k_1) = \frac{\tilde{f}}{\Omega} \frac{1}{\sqrt{1 - \frac{\mathcal{K}^2}{\Omega^2}}} = \frac{\tilde{f}}{\sqrt{\Omega^2 - \mathcal{K}^2}}$$

$$\Omega^2 - \mathcal{K}^2 = (\mathfrak{K} - \mathcal{K}\cos(k_1))^2 - \mathcal{K}^2 = \mathfrak{K}^2 - \mathcal{K}^2 - 2\mathcal{K}\mathfrak{K}\cos(k_1) + \mathcal{K}^2\cos(k_1)^2$$

$$= \mathfrak{K}^2 - 2\mathcal{K}\mathfrak{K}\cos(k_1) - \mathcal{K}^2\sin(k_1)^2$$

Zielgröße:

$$e^{2i\delta(\omega)} = \frac{(tt^* e^{i\omega} - 1)(te^{i\omega} - t^*)}{(e^{i\omega} - tt^*)(t^* e^{i\omega} - t)} \qquad \text{mit } t^* = \frac{(1-t)}{(1+t)}$$

Umformung der Zielgröße:

$$e^{2i\delta(\omega)} = \frac{(1 - tt^* e^{i\omega})(t - t^* e^{-i\omega})}{(1 - tt^* e^{-i\omega})(t - t^* e^{i\omega})} \cdot \frac{e^{i\omega}}{e^{i\omega}}$$

$$= \frac{(1 - tt^* e^{i\omega})^2 (t - t^* e^{-i\omega})^2}{(1 - tt^* e^{-i\omega})(1 - tt^* e^{i\omega})(t - t^* e^{i\omega})(t - t^* e^{-i\omega})}$$

$$= \frac{(1 - tt^* e^{i\omega})^2 (t - t^* e^{-i\omega})^2}{(1 - tt^*(e^{i\omega} + e^{-i\omega}) + (tt^*)^2)(t^2 - tt^*(e^{i\omega} + e^{-i\omega}) + (t^*)^2)}$$

$$= \frac{(1 - tt^* e^{i\omega})^2 (t - t^* e^{-i\omega})^2}{((1 + (tt^*)^2) - 2tt^* \cos(\omega))((t^2 + (t^*)^2) - 2tt^* \cos(\omega))}$$

$$= \frac{(1 - tt^* e^{i\omega})^2 (t - t^* e^{-i\omega})^2}{(1 + (tt^*)^2)(t^2 + (t^*)^2) - 2tt^*(1 + (tt^*)^2 + t^{*2} + t^2)\cos(\omega) + 4(tt^*)^2 \cos(\omega)^2}$$

Faktorisieren von $\tilde{f}(k,t)$

$$\tilde{f}(k,t) = 2t(1+t^2) - t^2(1-t^2)e^{-ik_1} - (1-t^2)e^{ik_1}$$

$$= 2t(1+t^2) - t^2(1-t^2)e^{i\omega} - (1-t^2)e^{-i\omega} \qquad \text{mit } \omega = -k_1$$

$$= (t(1+t) - (1-t)e^{-i\omega})(t(t-1)e^{i\omega} + (1+t))$$

$$= (t - t^* e^{-i\omega})(-tt^* e^{i\omega} + 1) \cdot (1+t)^2$$

$$= (1 - tt^* e^{i\omega})(t - t^* e^{-i\omega}) \cdot (1+t)^2$$

$$\Rightarrow \frac{\tilde{f}(k,t)^2}{(1+t)^4} = (1 - tt^* e^{i\omega})^2 (t - t^* e^{-i\omega})^2$$

Wollen Zeigen

$$e^{2i\delta(\omega)} \stackrel{!}{=} \frac{\tilde{f}(-k_1,t)^2}{[\Omega^2 - \lambda^2](-k_1,t)}$$

Umformungen $\Omega^2 - \varkappa^2 = \mu^2 - \varkappa^2 - 2\varkappa\mu\cos(k_1) + \varkappa^2\cos(k_1)^2$

$$\varkappa^2 = (2t(1-t^2))^2 = 4t^2(1-t)^2(1+t)^2 = 4t^2t^{*2}(1+t)^4$$

$$2\varkappa\mu = 2\cdot 2t(1-t^2)(1+t^2)^2 = 4t(1+t^2)(1-t)(1+t)(1+t^2)$$
$$= 4t(1+t^2)\,t^*(1+t^2)(1+t)^2$$
$$= 2tt^*(1+t^2)(1+t^{*2})(1+t)^4 \qquad \frac{2(1+t^2)}{(1+t)^2} = \left(1 + \frac{(1-t)^2}{(1+t)^2}\right) = (1+t^{*2})$$

$$\mu^2 - \varkappa^2 = (1+t^2)^4 - (2t(1-t^2))^2$$
$$= 1 + 14t^4 + t^8$$
$$= \frac{1+14t^4+t^8}{(1+t)^4}(1+t)^4 = (1+(tt^*)^2)(t^2+t^{*2})(1+t)^4$$

$$\Rightarrow e^{2i\delta(\omega)} = \frac{\tilde{f}(-k_1,t)^2\,(1+t)^4}{[\Omega^2-\varkappa^2](-k_1)(1+t)^4} = \frac{\tilde{f}^2(-k_1,t)}{[\Omega^2-\varkappa^2](-k_1)}$$

$$\Rightarrow e^{i\delta(\omega)+n\pi i} = \frac{\tilde{f}(-k_1)}{\sqrt{\Omega^2-\varkappa^2}}$$

da $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\delta(\omega)+n\pi i}\,d\omega \rightarrow 1$ für $T \rightarrow 0$

muss gelten das $n \in 2\mathbb{N}$, denn sonst $e^{i\delta(\omega)} \xrightarrow[T\to 0]{} -1$

$$\Rightarrow f(k_1,T) = e^{i\delta(\omega)}$$

$$\Rightarrow \langle h^{*}_{(0,0)} h^{0}_{(m+\tau,0)} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega m}\, e^{i\delta(\omega)}\, d\omega$$