Functional renormalization group approach for interacting Dirac fermions

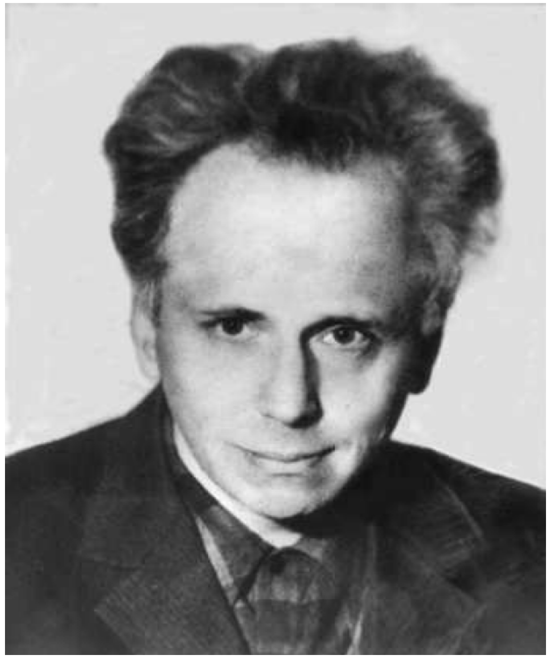
presented by

Pomper, Schreiner, Veider

Overview

- **Graßmann numbers**
- **Fermionic path integrals**
- **Wetterich equation for fermions**
- **The Gross-Neveu-Model**
- **Local-potential approximation and flow equation**
- **Fixpoints and solution of the flow equation**
- **Summary**

Introduction to Graßmann Algebra



Felix Berezin

Hermann Graßmann

Graßmann Algebra

Associative algebra with unity

$$(\underbrace{\mathcal{A}, +, \cdot}_{\text{Vector space}}, \underbrace{\wedge}_{\text{Associative product}})$$

Graßmann variables

Generated by abstract objects $\eta^1, \eta^2, \eta^3, \ldots, \eta^n$

$$\boxed{\eta^i \wedge \eta^j = -\eta^j \wedge \eta^i}$$

Graßmann monomials

Form monomials from variables η^i

$$\omega_p = \eta^{i_1} \wedge \cdots \wedge \eta^{i_p}$$

Degree of the polynomial

Combinations of different variables without permutations

$$\binom{n}{p}$$

First important notes

Monomials of degree p and q

Graded commutation law:

$$\omega_p \wedge \omega'_q = (-1)^{pq}\, \omega'_q \wedge \omega_p$$

Products with two identical variable vanish

$$\eta^i \wedge \eta^i = 0$$

Graßmann numbers

The general form of $\omega \in \mathcal{A}$

$$\omega = a_0 + \sum_{i=1}^{n} a_i\,\eta^i + \sum_{1 \le i_1 < i_2 \le n} a_{i_1 i_2}\,\eta^{i_1} \wedge \eta^{i_2} + \cdots + a_{12\ldots n}\,\eta^1 \wedge \eta^2 \wedge \cdots \wedge \eta^n$$

$$= \sum \Big(\textbf{even}\ \text{Monomial}\Big) \quad + \sum \Big(\textbf{odd}\ \text{Monomial}\Big)$$

$$= \Big(\text{Monomial}\ \textbf{independent of}\ \ \eta_i\Big) \ + \ \eta_i \wedge \Big(\text{Monomial}\ \textbf{independent of}\ \ \eta_i\Big)$$

Invertible Graßmann numbers

Graßmann number is **invertible** $\Leftrightarrow$ $a_0 \neq 0$

Inverse is given by Neumann series

$$\eta^{-1} = a_0^{-1} \sum_{k=0}^{n} (1-\eta)^k$$

A- and C-numbers

We can decompose

$$\mathcal{A} = \mathcal{A}^+ \bigoplus \mathcal{A}^-$$

C - numbers

$\sum$ (**even** Monomial)

- Contains the 1
- Commute with all elements of $\mathcal{A}$

A - numbers

$\sum$ (**odd** Monomial)

- Never invertible
- Anti commute with elements of $\mathcal{A}^-$
- Contains all possible generators

Graßmann (left) derivatives

We can define left derivatives

$$\omega = \omega_0 + \eta^i \omega' \xrightarrow{\frac{\partial}{\partial \eta^i}} \boxed{\frac{\partial \omega}{\partial \eta^i} := \omega'}$$

Derivatives anti commute

$$\boxed{\frac{\partial^2 \omega}{\partial \eta^i \partial \eta^j} = -\frac{\partial^2 \omega}{\partial \eta^j \partial \eta^i}}$$

Also

$$\mathcal{A}^+ \xrightarrow{\frac{\partial}{\partial \eta^i}} \mathcal{A}^- \qquad \mathcal{A}^- \xrightarrow{\frac{\partial}{\partial \eta^i}} \mathcal{A}^+$$

Berezin - integration

The integration turns out to be the same as the differentiation

$$\int \mathrm{d}\eta^i \omega = \frac{\partial \omega}{\partial \eta^i}$$

Multi-integrals are defined by

$$\int \underbrace{\mathrm{d}\eta^{i_1} \cdots \mathrm{d}\eta^{i_p}}_{} \omega := \int \mathrm{d}\eta^{i_1} \left(\int \cdots \left(\int \mathrm{d}\eta^{i_p} \omega \right) \right)$$

The order is important and
denoted in the "measure"

Fermionic path integrals

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \mathrm{e}^{-S[\psi,\bar{\psi}]+\int \bar{\eta}\psi-\int \bar{\psi}\eta}$$

Graßmann valued fields.

A Graßmann valued field is defined by

- I Discrete index set
- S space-time

Graßmann algebra

generated by the $\psi_i(x)$

$$\psi : I \times S \to \mathcal{A}$$
$$(i, x) \mapsto \psi_i(x)$$

- Discrete index

- Space-time point used as index

Independent Graßmann variables indexed by space-time points

Graßmann Path integral

Path integrals introduced by limiting process

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \;:= \lim_{N\to\infty} \int \prod_{i=1}^{N} \prod_{\mu=0}^{3} \mathrm{d}\bar{\psi}_\mu(x_i)\,\mathrm{d}\psi_\mu(x_i)$$

Space-time grid becomes finer and finer!

Derivatives commute with even number of integration measures

Functional Graßmann derivatives

Define derivatives of generators ψ (Graßmann valued fields)

$$\frac{\partial \psi_\mu(x)}{\partial \psi_\nu(y)} = \delta_{\mu,\nu}\ \delta(x-y)$$

$$\frac{\partial \bar{\psi}_\mu(x)}{\partial \psi_\nu(y)} = 0$$

Reflects independence of fields

Extend definition of derivative by linearity

Generating functionals

The generating functionals are defined **analogous** to the **bosonic case**

$$\ln(W[\eta,\bar{\eta}]) := \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \mathrm{e}^{-S[\psi,\bar{\psi}]+\int \bar{\eta}\psi - \int \bar{\psi}\eta}$$

$$\Gamma[\psi,\bar{\psi}] := \int \bar{\eta}\psi - \int \bar{\psi}\eta \ - W[\eta,\eta]$$

Fermionic Wetterich equation

Graßmann valued
matrix operators

$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{\left(\Gamma_k^{(2)} + R\right)^{-1}\partial_t R\right\}$$

Additional minus

Spinor fields used in our project

Three types of indices :

Discrete spinor indices	$\alpha \in \{1, \dots, d_\gamma\} =: I_\gamma$
Discrete flavor indices	$a \in \{1, \dots, N\} =: I_N$
Continuous space-time points	$x \in \mathbb{R}^d$

Graßmann valued fields :

$$\psi : I_\gamma \times I_N \times \mathbb{R}^d \to \mathcal{A}$$
$$(\alpha, a, x) \mapsto \psi^a_\alpha(x)$$

$$\bar{\psi} : I_\gamma \times I_N \times \mathbb{R}^d \to \mathcal{A}$$
$$(\alpha, a, x) \mapsto \bar{\psi}^a_\alpha(x)$$

Definition of $\Gamma_k^{(2)}[\psi, \bar{\psi}]$

Second derivative is "Hessian matrix"

$$\mathbf{\Gamma}_k^{(2)}[\bar{\psi}, \psi](x', x) = \begin{pmatrix} \underbrace{\begin{matrix} \dfrac{\delta^2 \Gamma_k}{\delta\psi(x')\delta\psi(x)} \\ \dfrac{\delta^2 \Gamma_k}{\delta\psi(x')\delta\bar{\psi}(x)} \end{matrix}}_{Nd_\gamma \times Nd_\gamma \text{ - Matrix}} & \overbrace{\begin{matrix} \dfrac{\delta^2 \Gamma_k}{\delta\bar{\psi}(x')\delta\psi(x)} \\ \dfrac{\delta^2 \Gamma_k}{\delta\bar{\psi}(x')\delta\bar{\psi}(x)} \end{matrix}}^{\text{Left derivatives only}} \end{pmatrix}$$

Definition of Regulator R_k

Fields have multiple components $\Rightarrow$ Regulator is matrix

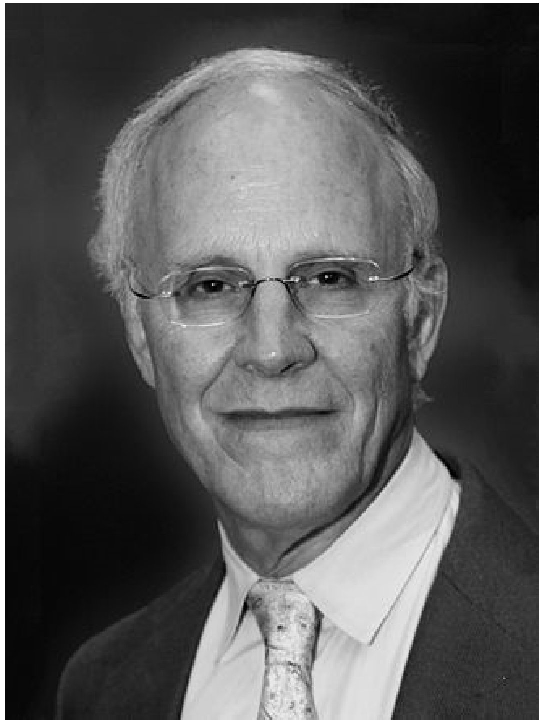
Regulator can be chosen anti-diagonal in momentum space

Dirac momentum matrix

$$\boldsymbol{R}_k(p,p') = \begin{pmatrix} 0 & -\not{p}\, r_k(-p) \\ \not{p}\, r_k(p) & 0 \end{pmatrix} (2\pi)^d \delta(p+p')$$

Regulator function

Gross-Neveu model (1974)



David J. Gross

André Neveu

Bare action

Quantum field theory of N **massless** fermion flavors in d-spacetime.

Toy model to investigate dynamical chiral symmetry breaking

$$\Gamma = \int d^d x \{\bar{\psi} i D \psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2\}$$

$$\mathrm{D} = \mathbb{1}_N \otimes \not{\partial}$$

$$\bar{\psi}\psi \equiv \sum_{a=1}^{N} \sum_{\alpha=1}^{d_\gamma} \bar{\psi}_\alpha^a \psi_\alpha^a$$

Symmetries

$$\psi^a \mapsto e^{i\alpha_a}\psi^a$$

$$\bar{\psi}^a \mapsto \bar{\psi}^a e^{-i\alpha_a}$$

Invariance under global U(N) transformation
for each flavor indices $a = 1, ..., N$

Terms are invariant under transformation

$$\Gamma = \int d^d x \{\bar{\psi} i D\psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2\}$$

$\Rightarrow$ Existence of conserved current for each flavor

Symmetries

Invariance under discrete $\mathbb{Z}_2^5 = \{1, \gamma_5\}$ transformation

$$\psi \mapsto \gamma_5 \psi$$
$$\bar{\psi} \mapsto -\bar{\psi}\gamma_5$$

Terms are invariant under transformation

$$\Gamma = \int d^d x \{\bar{\psi} i D \psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2\}$$

Only invariant due to square

$\Rightarrow$ Does not allow mass term $m\bar{\psi}\psi$ in action

$$\bar{\psi}\psi \mapsto -\bar{\psi}\psi$$

Characteristics d = 2

Toy model for QCD, used in nuclear and particle physics

Asymptotically free theory

Dynamical symmetry breaking and mass generation

Four-vertex interaction $\frac{1}{2}\lambda(\bar{\psi}\psi)^2$ renormalizable

Generalizations: NJL model Γ_{NJL}

Upgrade discrete symmetry into continuous chiral symmetry

$$\Gamma = \int d^d x \{ \bar{\psi} i D \psi + \frac{1}{2} \lambda (\left[\bar{\psi} \psi \right]^2 - \left[\bar{\psi} \gamma_5 \psi \right]^2) \}$$

$$\mathbb{Z}_2^5 = \{1, \gamma_5\} \rightarrow \; U(1)_{\mathrm{chiral}}$$

Requires analysis of IR divergences in 2D, Fierz transformations…

Precurser to our analysis and to GN overall (1961)

Effective action $\Gamma \rightarrow \Gamma_k$

How to make the transition

$$\Gamma = \int d^d x \{\bar{\psi} i D \psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2\} \rightarrow \Gamma_k = ?$$

Motivated by analytical calculability

Split into potential and kinetic term analysis

$$\Gamma_k = \Gamma_{k,kin} + \Gamma_{k,pot}$$

$$\Gamma_{k,pot} = \int d^d x V_k[\bar{\psi}, \psi],\ \Gamma_{k,kin} = \int d^d x T_k[\bar{\psi}, \psi]$$

Local potential approximation $\Gamma_k = \Gamma_{k,kin} + \Gamma_{k,pot}$

Decompose the potential into a Taylor expansion

Coupling factors

Point of field expansion

$$V_k[\bar{\psi}, \psi] = \sum_n \frac{V_k^{(n)}}{n!} (\bar{\psi}\psi - \bar{\psi}_0\psi_0)^n$$

For simplicity $\bar{\psi}_0\psi_0 = 0$, can be important four certain theories

Local potential approximation $\Gamma_k = \Gamma_{k,kin} + \Gamma_{k,pot}$

Symmetry requirement only allows even contributions

Only keep up to second order, with $V_k^{(2)} = \tilde{\lambda}_k$

$$V_k(x) = \frac{1}{2}\tilde{\lambda}_k(\bar{\psi}\psi)^2 + \mathcal{O}\left((\bar{\psi}\psi)^4\right) \approx \frac{1}{2}\tilde{\lambda}_k(\bar{\psi}\psi)^2$$

End up with our scale-dependent four-interaction coupling

Derivative expansion

$$\Gamma_k = \boxed{\Gamma_{k,kin}} + \Gamma_{k,pot}$$

Kinetic term rewritten in terms of derivatives

$$T_k[\bar{\psi}, \psi] = \boxed{Z_k^{\bar{\psi}\psi}}\, \bar{\psi} i D \psi + \boxed{\mathcal{O}((\bar{\psi} i D \psi)^2)}$$

Wave-function renormalization

Higher order derivative terms

$Z_k^{\bar{\psi}\psi}$ in general scale and field dependent quantity

Derivative expansion

$$\Gamma_k = \boxed{\Gamma_{k,kin}} + \Gamma_{k,pot}$$

Flow equation in the point-like limit would be $\partial_t Z_k^{\bar{\psi}\psi} = 0$

We can set $Z_k^{\bar{\psi}\psi} = 1$, and so

$$\boxed{T_k[\bar{\psi}, \psi] \approx \bar{\psi} i D \psi}$$

Effective action - summary

$$\Gamma_k = \Gamma_{k,kin} + \Gamma_{k,pot}$$

-Analytically solvable (choice of regulator)

-Effective theory for QCD

-Simple as possible

$$\Gamma_k[\bar{\psi}, \psi] = \int d^d x \{ \bar{\psi} i D \psi + \frac{1}{2} \tilde{\lambda}_k (\bar{\psi}\psi)^2 \}$$

Setting up the Flow Equation

$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

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$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

Observation 1:

$$\Gamma_k = \int \left(\bar\psi i D \psi + \frac{1}{2}\tilde\lambda_k (\bar\psi\psi)^2 \right)$$

Set fields constant

Kinetic term info lost

$$\Rightarrow \partial_t \Gamma_k \sim \partial_t \tilde\lambda_k (\bar\psi\psi)^2$$

Diagonal expression in momentum space

- Aim: Single out parts $\sim (\bar\psi\psi)^2$ on the left hand side and truncate rest

Setting up the Flow Equation

$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

Calculating Supertrace:

$$\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$
$$= \int \mathrm{d}p \int \mathrm{d}p' \, \mathrm{Tr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}(p, p') \; \delta(p + p')$$

- Biggest challenge: calculating the inverse (contains Graßmann fields)
- Idea: separate matrix into easier to invert parts

Setting up the Flow Equation

$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

Observation 2:

$$\mathrm{Tr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

$$= \mathrm{Tr}\left\{(\mathbb{1} + M)^{-1} M_0^{-1}\partial_t R\right\} \sim \mathrm{Tr}\left\{(\mathbb{1} + M)^{-1}\right\}$$

Contains terms $\sim \bar{\psi}\psi$

Easy to handle

Proportional to $\mathbb{1}$

Setting up the Flow Equation

$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

Observation 3:

- Only interested in terms $\sim (\bar{\psi}\psi)^2$
- M is **the only** term still containing fields
- M contains degree 2 field polynomials

$$\mathrm{Tr}\left\{(\mathbb{1} + M)^{-1}\right\}$$

$$= \mathrm{Tr}\left\{\sum_{k=0}^{N}(-M)^k\right\} = \mathrm{Tr}\left\{\mathbb{1}\right\} - \mathrm{Tr}\left\{M\right\} + \mathrm{Tr}\left\{M^2\right\} + \dots$$

$$\sim \tilde{\lambda}_k^2(\bar{\psi}\psi)^2$$

Setting up the Flow Equation

Putting everything together:

$$\partial_t \Gamma_k = -\frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R)^{-1}\partial_t R\right\}$$

Collecting terms $\sim (\bar{\psi}\psi)^2$ $\longrightarrow$

$$\partial_t \tilde{\lambda}_k = \mathrm{const}\ \tilde{\lambda}_k^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2}\frac{\partial_t r_k}{(1+r_k)^3}$$

Litim Regulator $\longrightarrow$

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2}\frac{\partial_t r_k}{(1+r_k)^3} \sim \text{Volume of d-ball}$$

Setting up the Flow Equation

Putting everything together:

$$\partial_t \tilde{\lambda}_k = -b \; k^{d-2} \tilde{\lambda}_k^2$$

Mass dimension $2-d$

Rescaling $\lambda_k = k^{d-2} \tilde{\lambda}_k$ **:**

$$\boxed{\partial_t \lambda_k = (d-2)\lambda_k - b\lambda_k^2}$$

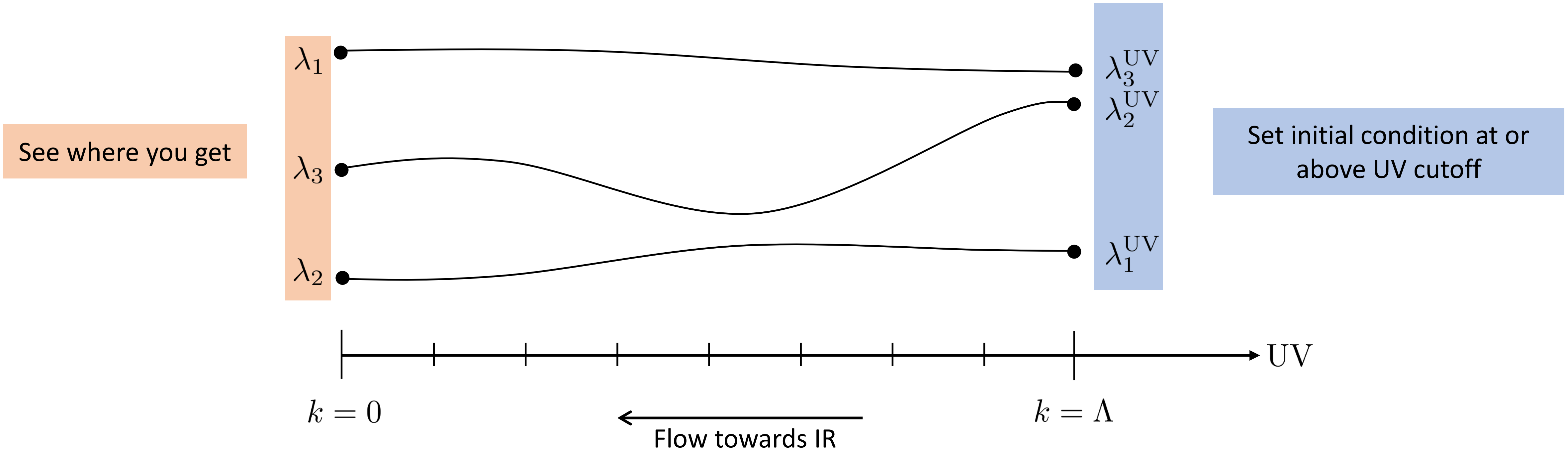
Inspecting the Flow Equation

$$\partial_t \lambda_k = (d-2)\lambda_k - b\lambda_k^2$$

- First order ODE: one initial condition
- Specify this at the UV cutoff (bare coupling)
- Study flow towards the IR

Inspecting the Flow Equation

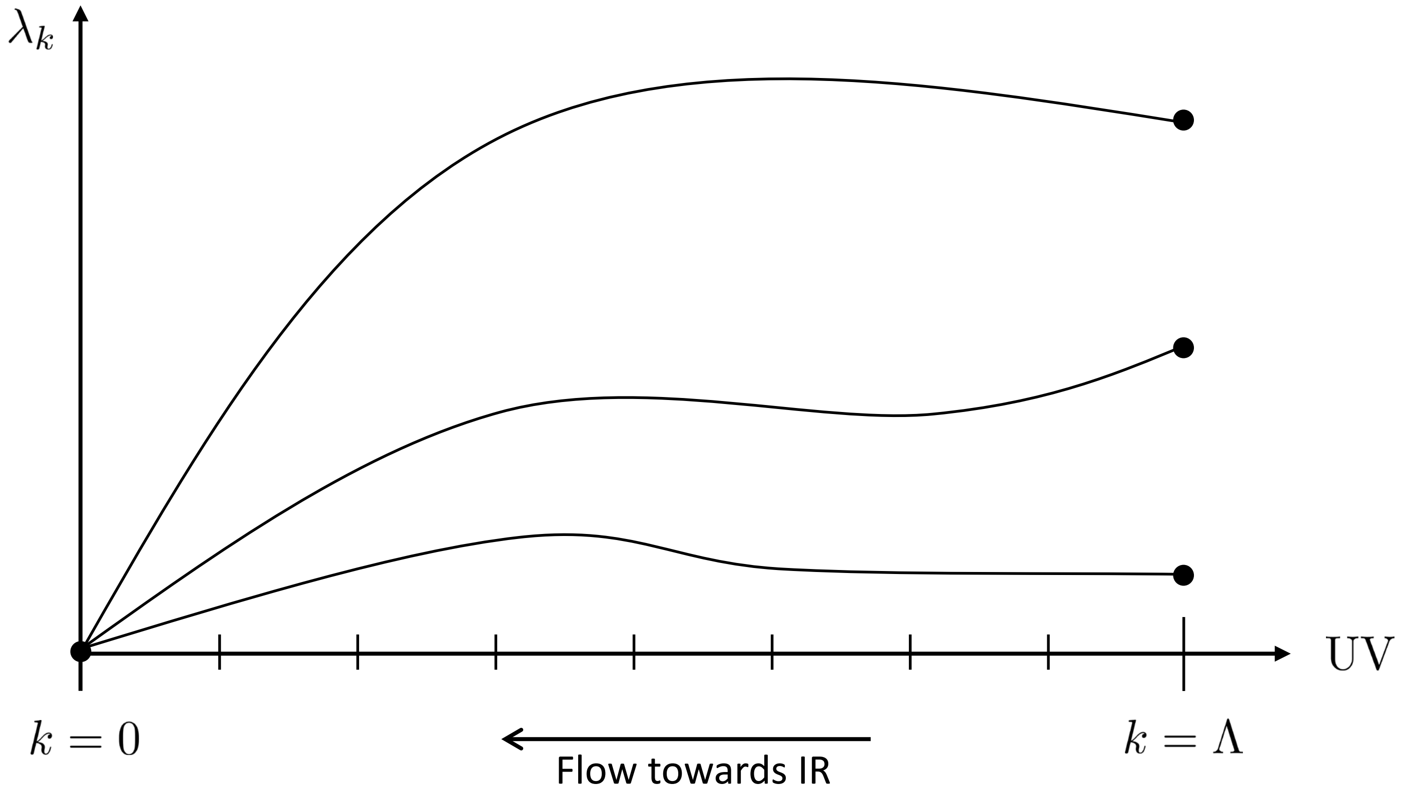
What does flow towards the IR mean?



Inspecting the Flow Equation

And what exactly are IR fixed points?

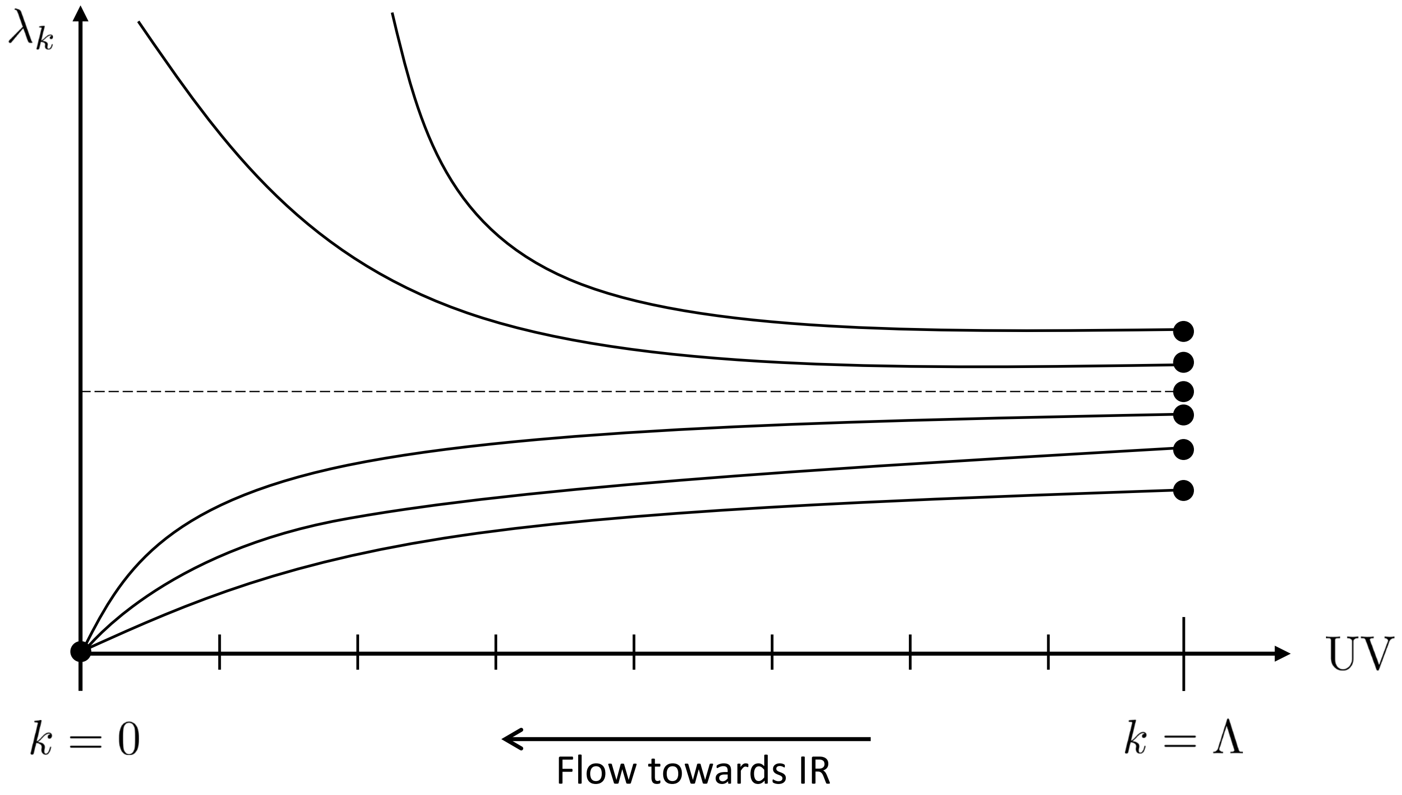
- Coupling approaches the same value independent of its initial value
- Multiple such points can exist
- For vanishing couplings the fixed point is called **Gaussian fixed point**



Inspecting the Flow Equation

And what exactly are UV fixed points?

- Couplings run away from that fixed point even if they're arbitrarily close in the beginning
- Existence of UV fixed point: **asymptotic safety**
- If it's a Gaussian: **asymptotic freedom**



Inspecting the Flow Equation

$$\partial_t \lambda_k = (d-2)\lambda_k - b\lambda_k^2$$

What we can learn from the equation:

$$\partial_t \lambda_k = \boxed{k} \frac{\mathrm{d}\lambda_k}{\mathrm{d}k}$$

Non-negative

- $\partial_t \lambda_k > 0 \Rightarrow \lambda_k$ gets bigger as we increase *k*
  $\Leftrightarrow \lambda_k$ gets smaller as we decrease *k*
- We consider flow towards the IR (i.e. decrease *k*)
- Our flow equation has the structure of a parabola

Inspecting the Flow Equation

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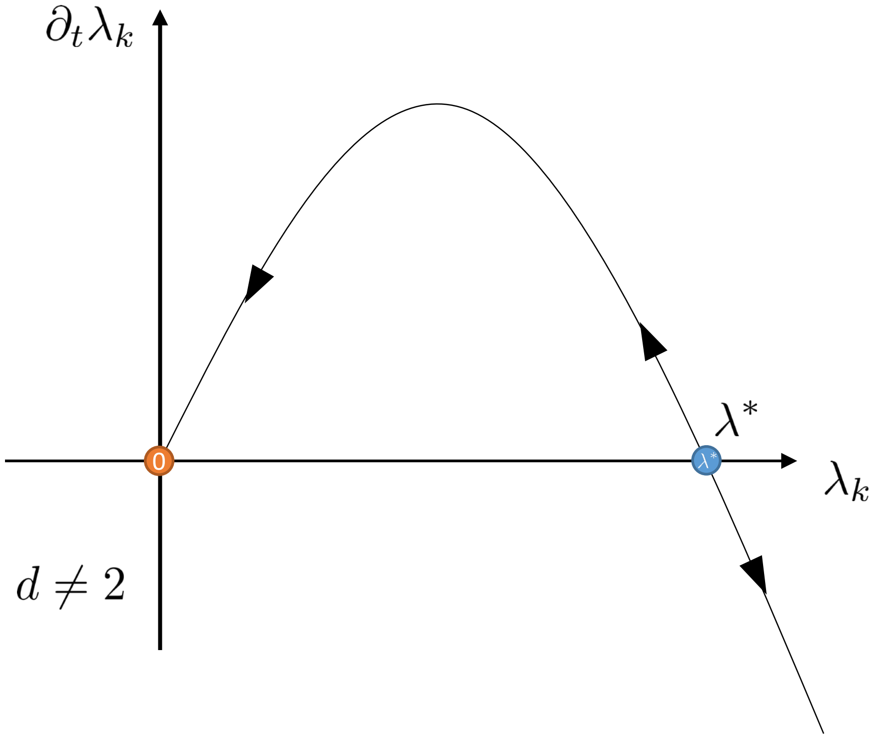
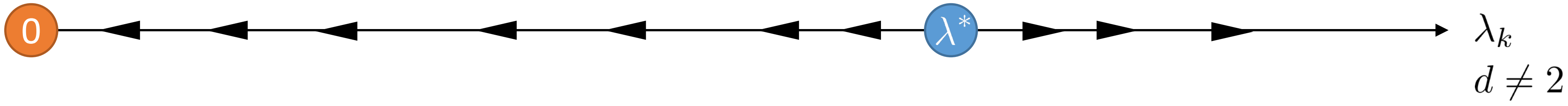
- We consider flow towards the IR (i.e. decrease *k*)
- Our flow equation has the structure of a parabola
- Fixpoints are roots of $\partial_t \lambda_k$



Inspecting the Flow Equation

$$\partial_t \lambda_k = (d-2)\lambda_k - b\lambda_k^2$$

What we can learn from the equation:



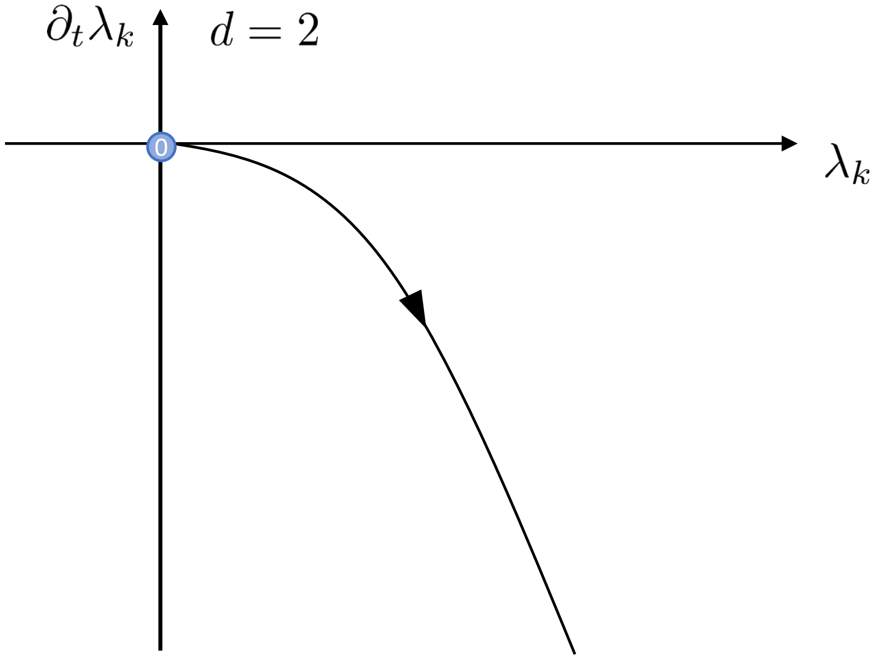
IR fixed point (Gaussian)
(coupling vanishes at low momentum scales)

UV fixed point
(asymptotically safe)

Inspecting the Flow Equation

$$\partial_t \lambda_k = -b\lambda_k^2$$

What we can learn from the equation:



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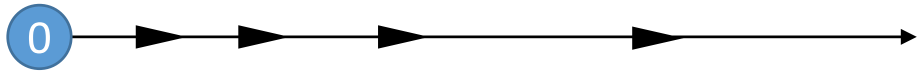

λ_k

$d = 2$

UV fixed point (Gaussian)
(asymptotically free)

Summary

- Graßmann numbers are polynomials in Graßmann variables
- Splitting in a- and c-numbers
- Fermion fields as maps from space-time to Graßmann variables

- Hessian and regulator become matrices
- Symmetries to guide effective action ansatz
- Break it down for analyticity

- Use constant fields to project out local potential couplings
- Litim regulator permits analytic solution
- Most information obtained by analyzing structure of flow equation