W10 H6 Hong Nguyen

1. a) Yes, all the training data will be correctly clasified because set of contraint are actually criterion function of logistic regression. (In case of 2-class classification)

b)

$$L(w, w_0, \lambda) = \frac{1}{2}\|\underline{w}\|^2 - \sum_{i=1}^{N} \lambda_i [z_i(\underline{w}^T \underline{u}_i + w_0) - 1]$$

$$\left.\begin{array}{l} \lambda_i \geq 0 \quad \forall i \\ \lambda_i [z_i(\underline{w}^{*T}\underline{u}_i + w_0^*) - 1] = 0 \quad \forall i \\ z_i(w^T u_i + w_0) - 1 \geq 0 \quad \forall i \end{array}\right\} \text{KKT conditions}$$

c)

$$\left[\begin{array}{l} \nabla_w L(w,w_0,\lambda) = 0 \Leftrightarrow \frac{1}{2}.2\underline{w} - \sum_{i=1}^{N} \lambda_i z_i \underline{u}_i = 0 \\ \qquad \Leftrightarrow \underline{w} = \sum_{i=1}^{N} \lambda_i z_i \underline{u}_i \\ \frac{d}{dw_0} L(w,w_0,\lambda) = 0 \Leftrightarrow 0 - \sum_{i=1}^{N} \lambda_i z_i = 0 \end{array}\right.$$

$$L(w,w_0,\lambda) = \frac{1}{2}\underline{w}^T\underline{w} - \sum_{i=1}^{r} \lambda_i z_i (w^T u_i + w_0) + \sum_{i=1}^{N} \lambda_i$$

Substitude w and Σ to b) we have

$$L_D = \frac{1}{2}\sum_{i=1}^{N} \lambda_i z_i \underline{u}_i^T . \sum_{j=1}^{N} \lambda_j z_j \underline{u}_j - 0 + \sum_{i=1}^{N} \lambda_i$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j z_i z_j \underline{u}_i^T u_j + \sum_{i=1}^{N} \lambda_i$$

with respect to a subject to the constaint

$$\left.\begin{array}{l} \lambda_i \geq 0 \\ \sum_{i=1}^{N} x_i z_i = 0 \end{array}\right\} \text{new KKT condition}$$

② $$L_D(\underline{\lambda}, \mu) = \sum_{i=1}^{3} x_i - \frac{1}{2}\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i \lambda_j z_i z_j u_i^T u_j\right) + \mu\left(\sum_{i=1}^{3} \lambda_i z_i\right)$$

⊕ $$\frac{dL_D}{d\mu} = \sum_{i=1}^{3} \lambda_i z_i = 0$$

$$\Leftrightarrow z_1 \lambda_1 + z_2 \lambda_2 + z_3 \lambda_3 + 0\mu = 0$$

⊕ $$\frac{dL_D}{d\lambda_1} = 1 - \frac{1}{2}\left(2\lambda_1 z_1^2 u_1^T u_1 + 2\lambda_2 z_1 z_2 u_1^T u_2 + 2\lambda_3 z_1 z_3 u_1^T u_3\right) + \mu z_4 = 0$$

$$\Leftrightarrow z_1^2 u_1^T u_1 \lambda_1 + z_2 z_1 u_1^T u_2 \lambda_2 + z_1 z_3 u_1^T u_3 \lambda_3 - z_1 \mu = 0$$

⊕ $$\frac{dL_D}{d\lambda_2} = z_2^2 u_2^T u_2 \lambda_2 + z_1 z_2 u_2^T u_1 \lambda_1 + z_2 z_3 u_2^T u_3 \lambda_3 - z_2 \mu = 0$$

⊕ $$\frac{dL_D}{d\lambda_3} = z_3^2 u_3^T u_3 \lambda_3 + z_3 z_1 u_3^T u_1 \lambda_1 + z_3 z_2 u_3^T u_2 \lambda_2 - z_3 \mu = 0$$

$$A = \begin{bmatrix} z_1^2 u_1^T u_1 & z_1 z_2 u_1^T u_2 & z_1 z_3 u_1^T u_3 & -z_1 \\ z_2 z_1 u_2^T u_1 & z_2^2 u_2^T u_2 & z_2 z_3 u_2^T u_3 & -z_2 \\ z_3 z_1 u_3^T u_1 & z_3 z_2 u_3^T u_2 & z_3^2 u_3^T u_3 & -z_3 \\ z_1 & z_2 & z_3 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^T$$

Check for $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S_1$ $\quad z_1 = z_2 = 1$

$u_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S_2$ $\quad z_3 = -1$

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

hw6

April 1, 2022

[64]:
```
import numpy as np
import matplotlib.pyplot as plt
```

[65]:
```
u1 = np.array([1,2])
u2 = np.array([2,1])
u3 = np.array([0,0])
u3_prime = np.array([1,1])
u3_prime2 = np.array([0,1.5])
z1 = 1
z2 = 1
z3 = -1
```

0.0.1 (b)-(e) Write code in Python to solve lambda_vec for and µ, calculate ∗ and ’ , and check the KKT conditions

(b) Use NumPy to invert your matrix, and to calculate the resulting values for labda_vector and µ

[66]:
```
def question2(z1, z2, z3, u1, u2, u3):
    A = np.array([[z1*z1*np.inner(u1,u1), z1*z2*np.inner(u1,u2), z1*z3*np.
 ↪inner(u1,u3), -z1],
                  [z2*z1*np.inner(u2,u1), z2*z2*np.inner(u2,u2), z2*z3*np.
 ↪inner(u2,u3), -z2],
                  [z3*z1*np.inner(u3,u1), z3*z2*np.inner(u3,u2), z3*z3*np.
 ↪inner(u3,u3), -z3],
                  [                   z1,                    z2,                   ␣
 ↪  z3,   0]])
    b = np.array([1,1,1,0])

    # (i) Use NumPy to invert your matrix, and to calculate the resulting␣
 ↪values for lambda_vector and µ_vector.
    rho = np.dot(np.linalg.inv(A),b.reshape(4,1))

    # (ii) Check KKT conditions for lambda.
    lambda_vector = rho[0:3]
    miu = rho[3]
    KKT_condition = np.sum(lambda_vector.reshape(1,3)*[z1, z2, z3]) < 1e-15
    # print('KKT condition of dual Largrange fomular is:', KKT_condition)

    # (iii) Calculate the optimal (nonaugmented) weight vector * by using your␣
↪result from Problem 1(c)(i).
    # And, find the optimal bias term ' using one of the KKT conditions from␣
↪Problem 1(b).
    w_star = lambda_vector[0]*z1*u1 + lambda_vector[1]*z2*u2 +␣
↪lambda_vector[2]*z3*u3
    w0 = 1/z1 - np.inner(w_star,u1)

    # (iv) Check that the resulting  and ' satisfy the KKT conditions on  and␣
↪ ' of Pr.1(c).
    # Condition 1:
    KKTcondition1 = lambda_vector >= 0
    # Condition 2:
    KKTcondition2 = lambda_vector[0]*(z1*(np.inner(w_star,u1) + w0)-1) < 1e-15␣
↪and\
        lambda_vector[1]*(z2*(np.inner(w_star,u2) + w0)-1) < 1e-15 and\
        lambda_vector[2]*(z3*(np.inner(w_star,u3) + w0)-1) < 1e-15
    return KKT_condition, KKTcondition1, KKTcondition2, w_star, w0,␣
↪lambda_vector, miu
```

(c) Run your code on the given dataset

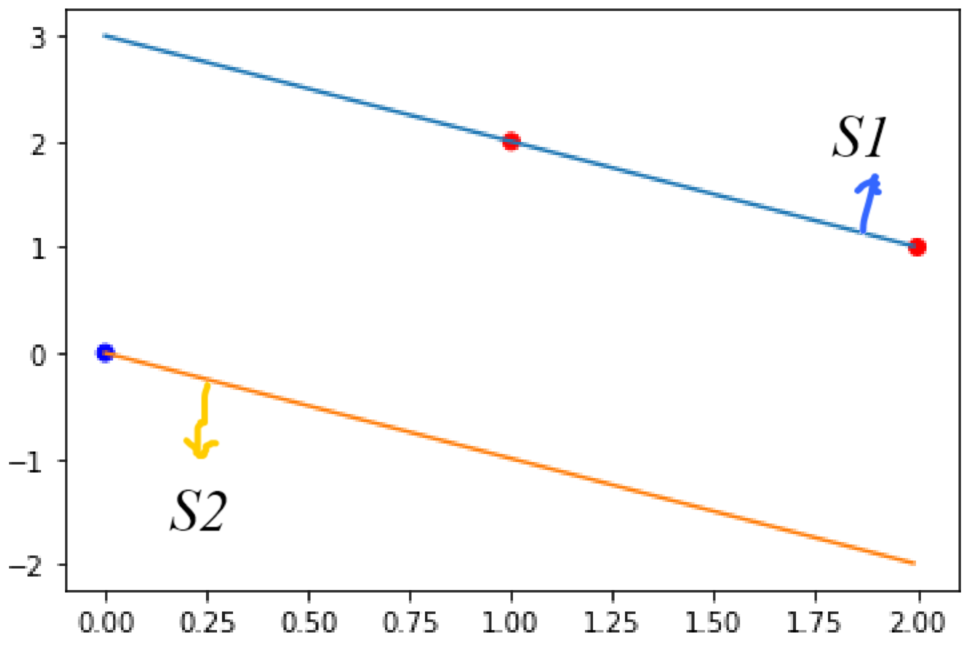
[67]:
```
KKT_condition, KKTcondition1, KKTcondition2, w_star, w0, lambda_vector, miu =␣
 ↪question2(z1, z2, z3, u1, u2, u3)
print('lambda1 = %f, lambda2 = %f, lambda3 = %f, miu = %f' % (lambda_vector[0],␣
 ↪lambda_vector[1], lambda_vector[2], miu))
print('KKT condition of primal Largrange fomular is:', KKT_condition)
print('w_star = ', w_star, ', w0 = ', w0)
print('KKT check on w_star and w0:', int(sum((KKTcondition1.astype(int)))) ==␣
 ↪len(KKTcondition1) and KKTcondition2)
```

```
lambda1 = 0.222222, lambda2 = 0.222222, lambda3 = 0.444444, miu = 1.000000
KKT condition of primal Largrange fomular is: True
w_star =  [0.66666667 0.66666667] , w0 =  -0.9999999999999996
KKT check on w_star and w0: [ True]
```

(d) Plot in 2D nonaugmented feature ( ) space: the data points showing their class labels, the decision boundary defined by $*$ and ', and an arrow showing which side of the boundary is class 1

[68]:
```
x = np.arange(0.0, 2.0, 0.01)
y1 = (1-w0-w_star[0]*x)/w_star[1]
y2 = (-1-w0-w_star[0]*x)/w_star[1]

plt.scatter([u1[0],u2[0],u3[0]],[u1[1],u2[1], u3[1]],c=['r','r','b'])
plt.plot(x,y1)
plt.plot(x,y2)
plt.show()
```



(e) Yes, the decision boundary correctly classify the training data. And it look like maximum-margin boundary because the boudary is lie on closest data point by seeing the figure. By see the largrange formula, the KKT condition is also criteriion of max-margin with parametter lambda.

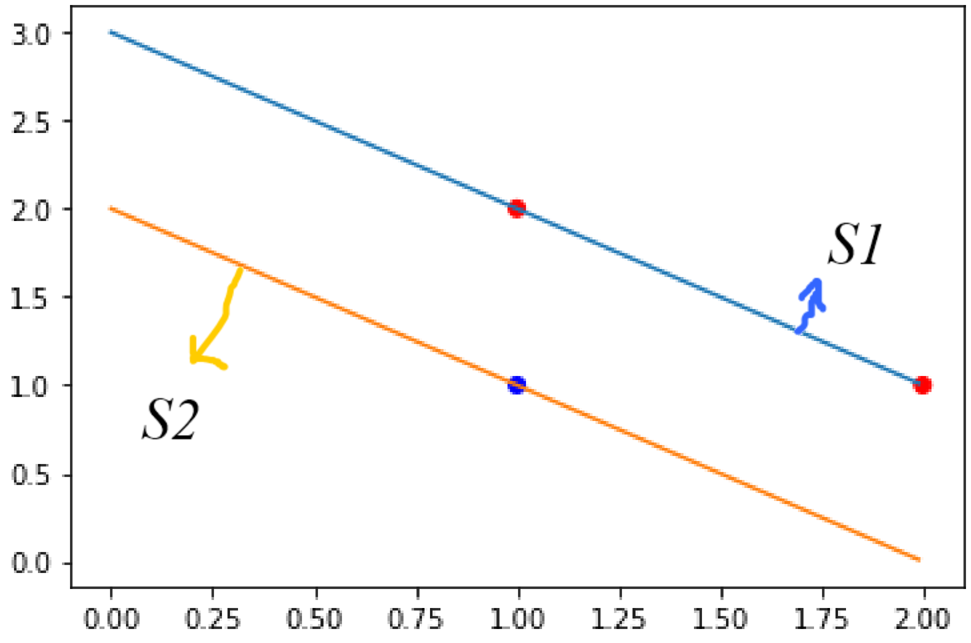
(f) Repeat parts (c) – (e) except for the following dataset:

[69]:
```
KKT_condition, KKTcondition1, KKTcondition2, w_star, w0, lambda_vector, miu =␣
 ↪question2(z1, z2, z3, u1, u2, u3_prime)
print('lambda1 = %f, lambda2 = %f, lambda3 = %f, miu = %f' % (lambda_vector[0],␣
 ↪lambda_vector[1], lambda_vector[2], miu))
print('KKT condition of primal Largrange fomular is:', KKT_condition)
print('w_star = ', w_star, ', w0 = ', w0)
print('KKT check on w_star and w0:', int(sum((KKTcondition1.astype(int)))) ==␣
 ↪len(KKTcondition1) and KKTcondition2)

x = np.arange(0.0, 2.0, 0.01)
y1 = (1-w0-w_star[0]*x)/w_star[1]
y2 = (-1-w0-w_star[0]*x)/w_star[1]

plt.scatter([u1[0],u2[0],u3_prime[0]],[u1[1],u2[1],␣
 ↪u3_prime[1]],c=['r','r','b'])
plt.plot(x,y1)
plt.plot(x,y2)
plt.show()
```

```
lambda1 = 2.000000, lambda2 = 2.000000, lambda3 = 4.000000, miu = 5.000000
KKT condition of primal Largrange fomular is: True
w_star =  [2. 2.] , w0 =  -5.0
KKT check on w_star and w0: [ True]
```



The decision boudary is narrow down compare with figure in (d). This is because the data point of S2 to closer to data point of S1, which make the gap is narrow down.

(g) How do you think the boundary will change (relative to (f)) if we instead use the following data

[70]:
```
KKT_condition, KKTcondition1, KKTcondition2, w_star, w0, lambda_vector, miu =␣
 ↪question2(z1, z2, z3, u1, u2, u3_prime2)
print('lambda1 = %f, lambda2 = %f, lambda3 = %f, miu = %f' % (lambda_vector[0],␣
 ↪lambda_vector[1], lambda_vector[2], miu))
print('KKT condition of primal Largrange fomular is:', KKT_condition)
print('w_star = ', w_star, ', w0 = ', w0)
print('KKT check on w_star and w0:', int(sum((KKTcondition1.astype(int)))) ==␣
 ↪len(KKTcondition1) and KKTcondition2)
```

```
lambda1 = 2.222222, lambda2 = -0.444444, lambda3 = 1.777778, miu = 3.000000
KKT condition of primal Largrange fomular is: True
w_star =  [1.33333333 1.33333333] , w0 =  -3.0
KKT check on w_star and w0: False
```

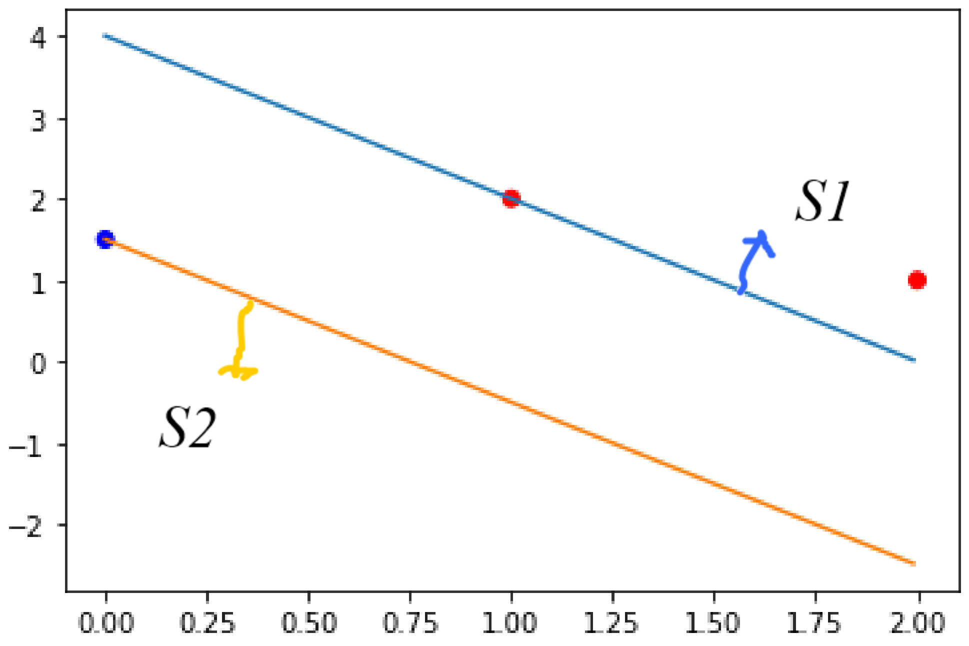
```
[75]: # Condition 1 is not satisfied at lambda2 < 0 ==> Set lambda2 = 0 we have
      A_later = np.array([[z1*z1*np.inner(u1,u1),  z1*z3*np.inner(u1,u3_prime2), -z1],
                    [z3*z1*np.inner(u3_prime2,u1),  z3*z3*np.
       ↪inner(u3_prime2,u3_prime2), -z3],
                    [                   z1,                         z3,    0]])
      b_later = np.array([1,1,0])

      # (i) Use NumPy to invert your matrix, and to calculate the resulting values␣
       ↪for lambda_vector and μ_vector.
      rho_later = np.dot(np.linalg.inv(A_later),b_later.reshape(3,1))
      # (ii) Check KKT conditions for lambda.
      lambda_vector = np.array([float(rho_later[0]) , 0, float(rho_later[1])])
      miu = rho_later[2]
      KKT_condition = np.sum(lambda_vector.reshape(1,3)*[z1, z2, z3]) < 1e-15
      # (iii) Calculate the optimal (nonaugmented) weight vector  * by using your␣
       ↪result from Problem 1(c)(i).
      # And, find the optimal bias term  ' using one of the KKT conditions from␣
       ↪Problem 1(b).
      w_star = lambda_vector[0]*z1*u1 + lambda_vector[1]*z2*u2 +␣
       ↪lambda_vector[2]*z3*u3_prime2
      w0 = 1/z1 - np.inner(w_star,u1)
      # (iv) Check that the resulting   and  ' satisfy the KKT conditions on   and  '␣
       ↪of Pr.1(c).
      # Condition 1:
      KKTcondition1 = lambda_vector >= 0
      # Condition 2:
      KKTcondition2 = lambda_vector[0]*(z1*(np.inner(w_star,u1) + w0)-1) < 1e-15 and\
          lambda_vector[1]*(z2*(np.inner(w_star,u2) + w0)-1) < 1e-15 and\
          lambda_vector[2]*(z3*(np.inner(w_star,u3_prime2) + w0)-1) < 1e-14
      print('lambda1 = %f, lambda2 = %f, lambda3 = %f, miu = %f' % (lambda_vector[0],␣
       ↪lambda_vector[1], lambda_vector[2], miu))
      print('KKT condition of primal Largrange fomular is:', KKT_condition)
      print('w_star = ', w_star, ', w0 = ', w0)
      print('KKT check on w_star and w0:', int(sum((KKTcondition1.astype(int))) ==␣
       ↪len(KKTcondition1) and KKTcondition2)
      x = np.arange(0.0, 2.0, 0.01)
      y1 = (1-w0-w_star[0]*x)/w_star[1]
      y2 = (-1-w0-w_star[0]*x)/w_star[1]

      plt.scatter([u1[0],u2[0],u3_prime2[0]],[u1[1],u2[1],␣
       ↪u3_prime2[1]],c=['r','r','b'])
      plt.plot(x,y1)
      plt.plot(x,y2)
      plt.show()
```

```
lambda1 = 1.600000, lambda2 = 0.000000, lambda3 = 1.600000, miu = 2.200000
KKT condition of primal Largrange fomular is: True
```

```
w_star =  [1.6 0.8] , w0 =  -2.200000000000002
KKT check on w_star and w0: True
```



Because we cant express lambda > 0 on largrange formular, it is best to track KKT condition. When the condition is not satisfied, re-calculate the w_star which make the plot is different. Now the max-margin plot can only put 1 nearest point on the boundary.