AALTO UNIVERSITY

Homework #2

ELEC-E8130 - Nonlinear Control Design and Analysis D

Professor: *Shankar A. Deka* TA: *Jianqiang Ding*

Due date: *October 7th (Monday)*

Aalto University
School of Electrical
Engineering

YOUR NAME [YOUR STUDENT NUMBER]

1. Report guidelines

For this course, your homework will usually have three parts: theory, programming, and writing. The design of each assignment aims to help you get a better grasp of what you've learned in class and how to apply it in your own problems. You are expected to break down complex problems into a sub-problems to solve them.

We provide this LaTeX template to make it easier for you to submit your solutions. Aalto is providing free Overleaf Professional accounts for all students. We recommend you upload this template to overleaf.com if you don't have latex installed. Please check this link for short math guide for latex. Additionally, We have a minimum standard for the format of your submissions and strongly recommend that you verify each report against the following criteria before submission:

- Unless specified otherwise, reports should be a minimum of 2 pages and a maximum of 5 pages in length.
- Include title and author information.
- Ensure that solutions are understandable without the need to refer to external sources.
- Provide detailed yet clear and concise solutions.
- Solutions should be reproducible based on the information provided in the report.
- Maintain a consistent structure and language that is easy to follow.
- Every figure and table must be referred to in the text and have a caption.

Although the primary focus of this course is not on programming, we uphold basic standards for clarity and conciseness in your code:

- Your code should be able to reproduce the results you wrote in your report.
- Each function, including the main one, should be short and to the point. Ideally, the length of each function should be about 25 lines, and no more than 35. (max 120 characters each line.)
- Put comments on each function to explain what it does. Make sure the function names make sense and match the comments. Add comments where needed in the main function.

For the final group project, submit a signed confirmation letter of each member's contributions and their weight and show it as the second slide of the final presentation. Your personal score of the final project will be the group score multiplied by your confirmed contribution weight.

Finally, **remember that plagiarism is never tolerated at Aalto.** However, collaboration among your peers is welcome. Always follow good academic practices when writing your assignments. Turnitin is there to help you check your work for originality.

2. Questions

Problem 1 (20 points): Determine the equilibrium points of the autonomous system

$$\dot{x} = (x-1)(x+y)$$
$$\dot{y} = y - x^2$$

Which of these points are stable? Which of them are asymptotically stable? **Hint:** Obtain the jacobian matrix A at the equilibrium point(s), then apply Lyapunov's indirect method (Theorem 3.7 in Khalil textbook).

Answer 1: Solving the system by having individual differential equations equal 0, like:

$$(x-1)(x+y) = 0$$
$$y - x^2 = 0$$

We obtain (1,1), (0,0), and (-1,1) as equilibrium points. Using Lyapunov's indirect method, we can compute and check whether the Jacobian matrix A is Hurwitz at equilibrium points or not. The Jacobian matrix is:

$$A = \begin{bmatrix} 2x+y-1 & x-1 \\ -2x & 1 \end{bmatrix}$$

For (1,1):

$$A = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}$$
$$\lambda_1 = 1$$
$$\lambda_2 = 2$$

A is not Hurwitz => Not asymptotically stable at (1,1). For (-1,1):

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$
$$\lambda_1 = 1$$
$$\lambda_2 = -2$$

A is not Hurwitz => Not asymptotically stable at (-1,1). For (0,0):

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

A is not Hurwitz => Not asymptotically stable at (0,0). So system has no asymptotically stable equilibrium points.

Problem 2 (20 points): Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 - e^{-2t}x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}.$$

Determine whether the equilibrium point at 0 is stable or not? **Hint:** Use the following Lyapunov function $V(x,t) = x_1^2 + \left(1 + e^{-2t}\right)x_2^2$. See Example 3.20 in Khalil textbook.

Answer 2: Choose $g(t) = e^{-}2t$ g(t) is continuously differentiable, and:

$$\begin{aligned}\dot{g}(t) &= -2e^{-}2t \leq e^{-}2t \\ 0 \leq g(t) &\leq k, \forall t \geq 0\end{aligned}$$

Taking $V(t,x) = x_1^2 + [1 + g(t)]x_2^2$ as a Lyapunov function candidate, it can be easily seen that

$$x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + (1+k)x_2^2, \forall x \in R^2$$

Hence, V(t) is positive definite, decrescent, and radially unbounded. The derivative is:

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

Using the inequality:

$$2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2$$

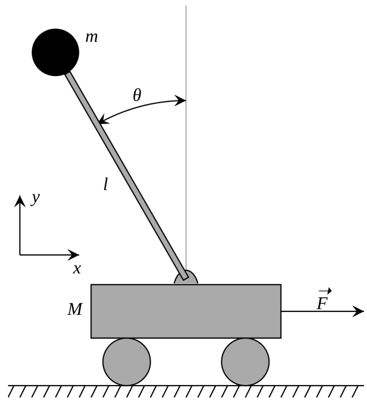
We obtain

$$\dot{V}(t,x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x^T Q x$$

Problem 3 (40 points): Consider the following Cart-Pole system. Let the state be $q := [q_1, q_2, q_3, q_4]^\top = [x, \dot{x}, \theta, \dot{\theta}]^\top$ and u is the control input that represents the force applied on the cart.

$$\dot{q} = f(q, u) = \begin{bmatrix} \dot{x} \\ \frac{u + m \sin\theta (l\dot{\theta}^2 - g\cos\theta)}{M + m \sin^2\theta} \\ \dot{\theta} \\ \frac{u\cos\theta + ml\dot{\theta}^2 \cos\theta \sin\theta - (M+m) g \sin\theta}{l(M + m\sin^2\theta)} \end{bmatrix}$$

with $M = 3, m = 2, l = 1, g = 9.8065$.



[Image credit: 10.3390/electronics8020198]

Write a script (Python/MATLAB) to answer the following questions:

- Consider zero control input i.e., $u = 0$. Notice that $q_e = [0, 0, 0, 0]^T$ is an equilibrium point of the uncontrolled system $\dot{q} = f(q, 0)$, corresponding to the upright position of the pole. Is q_e stable or unstable? **Hint:** Compute $A = \frac{\partial}{\partial q} f(q, 0)$ at $q = q_e$ and check if A is a Hurwitz matrix. Directly apply Theorem 3.7 from Khalil textbook.

- Now we consider a linear feedback controller $u(q) = -K(q - q_e)$. The point $q_e = [0, 0, 0, 0]^T$ is still an equilibrium point for the closed-loop system $\dot{q} = f(q, u(q))$. For $K = [1, 3.371, 1.781, 1.891]$, determine if the equilibrium q_e is now stable or unstable. **Hint:** Use the same steps as before. Start by computing $A = \frac{\partial}{\partial q} f(q, u(q))$ at $q = q_e$.

- Using the A matrix from above, compute a quadratic Lyapunov function of the form $V(q) = q^T P q$ by solving the Lyapunov equation $A^T P + PA = -Q$ with symmetric matrix $Q \succ 0$. You can choose any $Q \succ 0$ you like. **Hint:** Use can dirtectly use the packages like SciPy or Python Control Systems Library to solve the Lyapunov equation.

- Optional bonus question [Extra 10 points]: Set the initial condition for x and $\dot{x}$ to be zero. Through simulations, estimate the region of attraction of q_e in the $\theta - \dot{\theta}$ plane. Plot $\dot{V}(q)$ in the $\theta - \dot{\theta}$ by fixing $x = 0$, $\dot{x} = 0$. Notice that $\dot{V}(q)$ has negative values in some small neighborhood of q_e. **Hint:** For different values of q_3, q_4, start from $(0, 0, q_3, q_4)$ and check if system converges to q_e.

NOTE: Please use this template for programming, and if you are unsure how to share online script, please submit the report and script packaged into a compressed '.zip' file.

Answer 3: YOUR ANSWER HERE.