1: 20 points

MIMO Poles and Zeros

U output direction

(a) Consider a system with plant model

V input direction
$$G(s) = \frac{1}{s^2 + \frac{a}{50}s + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}$$

These dynamics model the motion of a cylindrical satellite spinning at constant rate Ω about the z-axis with $a = (1 - I_{zz}/I_{xx})\Omega$. The controls affect the spin rates about x and y; as a becomes large, the system becomes increasingly sensitive to uncertainty. Assume $a = 100$. Find the poles and zeros of the system including their directions.

(b) Zeros found through the solution to the generalized eigenvalue problem are known as *invariant* zeros because they are unchanged by state feedback $u = Kx$. To see this, show that

$$\text{rank}\left(\begin{bmatrix} A + BK - sI & B \\ C + DK & D \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right).$$

2: 20 Points

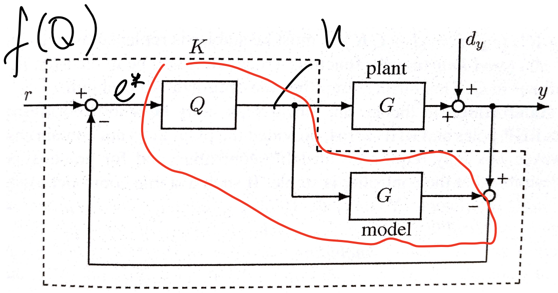
Internal Stability

(a) The control structure shown below is called Internal Model Control (IMC). Show that the IMC structure is internally unstable if Q is unstable.

$u = Q(r - y + Gu)$
e

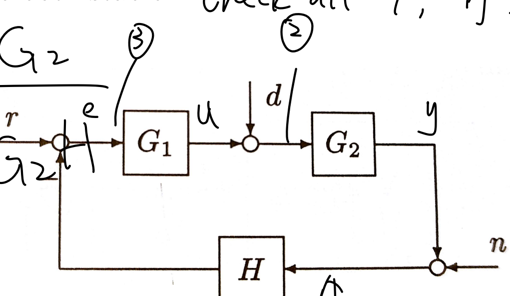
$k = f(Q)$

$\frac{u}{e} = TF$



(b) Recalling that a feedback system is internally stable $\iff$ all closed loop transfer functions are stable, find the conditions for internal stability of the feedback system shown. How do those simplify if $H(s)$ and $G_1(s)$ ~~are both stable~~? Check all 9, if H, G stable, check < 9

$$\frac{G_1G_2}{1+G_1G_2}$$



① = n + G_2(d + G_1 ·

r

$$\frac{y}{d} = (I + G_2G_1H)^{-1}G_2$$

$$\frac{e}{d} = -(I + HG_2G_1)^{-1}HG_2$$

r→1 d→1 n→1

r→2 d→2 n→2

$r \to 3$ $d \to 3$ $n \to 3$

3: 10 points

Control Limitations

Consider the plant model

$$G(s) = \frac{s-1}{s(s-2)}.$$

Based on the RHP poles and zeros of the system, and assuming a conventional loop shape, what limitations are posed on the system bandwidth? Note that your answer might be somewhat nonsensical. What is the minimum peak of the sensitivity function?

1-(a) $$G = \frac{1}{s^2 + 2s + 10000}\begin{bmatrix} s-10000 & 100(s+1) \\ -100(s+1) & s-10000 \end{bmatrix}$$

$$\text{poles} = \frac{-2 \pm \sqrt{4 - 4 \cdot 10000}}{2}$$

$$= \frac{-2 \pm \sqrt{4 - 40000}}{2}$$

$$= \frac{-2 \pm \sqrt{-39996}}{2}$$

$$= -1 + \frac{1}{2}\sqrt{39996}\,j, \quad -1 - \frac{1}{2}\sqrt{39996}\,j$$

$$\therefore \text{poles} = -1 + 99.995j$$
$$-1 - 99.995j$$

For pole 1 and pole 2, any entry of $G(s)$ has a pole at p_1 or p_2, $\therefore$ they are poles of $G(s)$.

Zeros: zeros $= 100j$, $-100j$

V: input direction, U output direction

Direction of Pole 1, $s = -1 + 99.995j$,

$$[u, v] = \begin{bmatrix} -0.707j & -0.707 \\ 0.707 & -0.707j \end{bmatrix}$$

pole 2: $s = -1 - 99.995j$

$$[u, v] = \begin{bmatrix} 0.707j & -0.707 \\ -0.707 & 0.707j \end{bmatrix}$$

zero 1, $s = 100j$

$$[u, v] = \begin{bmatrix} -0.707 & 0.707 \\ 0.707j & -0.707j \end{bmatrix}$$

zero 2, $s = -100j$: $[u, v] = \begin{bmatrix} -0.707 & 0.707 \\ -0.707j & 0.707j \end{bmatrix}$

This script demonstrates the computation of MIMO poles and zeros and their directions.

```
s = tf('s');
a = 100;
G = 1/(s^2+(a/50)*s+a^2) * [s-a^2 a*(s+1); -a*(s+1) s-a^2];
ep = 1e-6;  %Move off of poles by this amount.  Used to compute directions for the SVD approach.
```

Now I'll use the tzero command to compute the zeros of the transfer function. The use of minreal ensures that we get only the transmission zeros.

```
tzero(minreal(G)) %zeros
```

```
ans = 2x1 complex
     0.0000 +100.0000i
     0.0000 -100.0000i
```

```
p = eig(minreal(G)) %poles
```

```
p = 4x1 complex
   -1.0000 +99.9950i
   -1.0000 -99.9950i
   -1.0000 +99.9950i
   -1.0000 -99.9950i
```

Zeros:

s = 100i:

```
G1 = evalfr(G,100i+ep);
[U,S,V] = svd(G1);
S
```

```
S = 2x2
   100.0049         0
          0    0.0000
```

```
U
```

```
U = 2x2 complex
  -0.0071 - 0.7071i  -0.7071 - 0.0071i
   0.7071 - 0.0071i  -0.0071 + 0.7071i
```

```
V
```

```
V = 2x2 complex
  -0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 - 0.7071i   0.0000 - 0.7071i
```

s = -100i:

```
G2 = evalfr(G,-100i+ep);
[U,S,V] = svd(G2);
S
```

```
S = 2x2
   100.0049         0
          0    0.0000
```

```
U
```

```
U = 2x2 complex
  -0.0071 + 0.7071i  -0.7071 + 0.0071i
   0.7071 + 0.0071i  -0.0071 - 0.7071i
```

```
V
```

```
V = 2x2 complex
  -0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 + 0.7071i   0.0000 + 0.7071i
```

Poles:

s = -1+99.995i:

```
G3 = evalfr(G, p(1)+ep);
[U,S,V] = svd(G3);
S
```

```
S = 2x2
10^8 x
    1.0001         0
         0    0.0050
```

```
U
```

```
U = 2x2 complex
  -0.0035 - 0.7071i  -0.7070 - 0.0106i
   0.7071 - 0.0035i  -0.0106 + 0.7070i
```

```
V
```

```
V = 2x2 complex
  -0.7071 + 0.0000i  -0.7071 + 0.0000i
   0.0000 - 0.7071i   0.0000 + 0.7071i
```

s = -1-99.995i:

```
G4 = evalfr(G, p(2)+ep);
[U,S,V] = svd(G4);
S
```

```
S = 2x2
10^8 x
    1.0001         0
         0    0.0050
```

```
U
```

```
U = 2x2 complex
  -0.0035 + 0.7071i  -0.7070 + 0.0106i
   0.7071 + 0.0035i  -0.0106 - 0.7070i
```

```
V
```

```
V = 2x2 complex
  -0.7071 + 0.0000i  -0.7071 + 0.0000i
   0.0000 + 0.7071i   0.0000 - 0.7071i
```

(b) Determinant of matrix A

$$(A+BK-sI)\cdot D - B(C+DK)$$

$$= AD + BKD - sID - BC - BDK$$

$$= AD - sID - BC$$

Determinant of matrix B

$$(A - sI)D - BC$$

$$= AD - sID - BC$$

$$\mathrm{Det}(A) = \mathrm{Det}(B)$$

$$\therefore \quad R(A) = R(B)$$

2 (a) $u = Q(r - y + G_{model} u)$, $r - y = e$

$u = Qe + QG_{model} u$

$u - QG_{model} u = Qe$

$(1 - QG_{model})u = Qe$

$K_{eq} = \frac{u}{e} = \frac{Q}{1 - QG_{model}}$ is the controller

$K_{eq}(1 - QG_{model}) = Q$

$K_{eq} - K_{eq} Q G_{model} = Q$

$K_{eq} = (1 + K_{eq} G_{model}) Q$

$Q = \frac{K_{eq}}{1 + K_{eq} G_{model}} = K_{eq} S$

if Q is unstable, $K_{eq}S$ is unstable.

2b $r \to e$ $r \to u$ $r \to y$

$d \to e$ $d \to u$ $d \to y$

$n \to e$ $n \to u$ $n \to y$

$S = \frac{y}{d} = \frac{G_2}{1 + G_1 G_2 H}$, when G_1, H stable, S is stable.

$① = n + G_2[d + G_1(r - H①)]$

$① = (I+G_2G_1H)^{-1}n + (I+G_2G_1H)^{-1}G_2 d + (I+G_2G_1H)^{-1}G_2G_1 r$

$① = \frac{Sn}{G_2} + Sd + SG_1 r$

$② = d + G_1[r - H(G_2 ② + n)]$

$② = (I+G_1HG_2)^{-1}[d + G_1(r - Hn)]$

$② = [I - HG_1(I+G_2G_1H)^{-1}G_2][d + G_1(r-Hn)]$

$= (I - HG_1S)[d + G_1(r-Hn)]$

$③ = r - H[n + G_2(③G_1 + d)]$

$(I + HG_2G_1)③ = r - H(n + G_2 d)$

$③ = (I + HG_2G_1)^{-1}[r - H(n + G_2 d)]$

$= [I - H(I+G_2G_1H)^{-1}G_2G_1]r - H(I+G_2G_1H)^{-1}n + H(I+G_2G_1H)G_2 d$

$③ = (I - HSG_1)r - (I - HSG_1)Hn + HSd$

We just need to know stability of $\frac{y}{d}$, if S is unstable, other transfer functions are all unstable.

3. $P = 0, 2 \quad Z = 1$

$$\omega_{BT} >> p = 4\,\text{rad/s}$$

$$\omega_{BW} < \frac{1}{2} z = 0.5\,\text{rad/s}$$

Sensitivity Peak: $\|S\|_\infty = \prod_{i=1}^{N_p} \frac{|z+p_i|}{|z-p_i|}$

$$\|S\|_\infty = \frac{|1+0|}{|1-0|} \cdot \frac{|1+2|}{|1-2|}$$

$$\|S\|_\infty = 3$$

Minimum peak is 3