$r - y - n = e$ $\qquad$ $r = e + y + n$

$y = r - n - e$ $\qquad$ $e = uC^{-1}$ $\qquad$ $u = -(1-Q)^{-1}(C+QM)y$

$e = -(1-Q)^{-1}(C+QM)yC^{-1}$

1: 20 Points

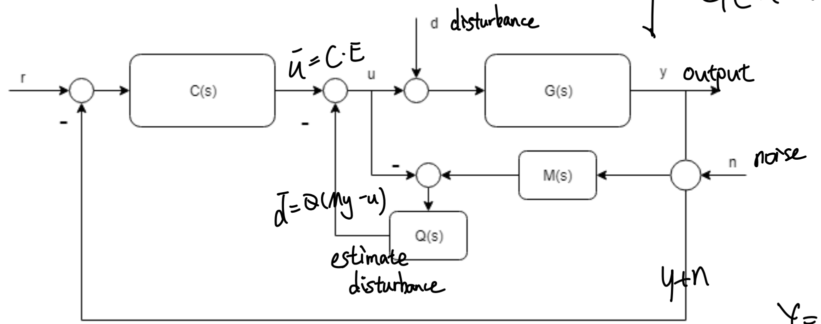
Consider the block diagram shown. This configuration is known as a disturbance observer and is used to cancel the effects of disturbances entering at the plant input node. You can assume that G is invertible throughout for simplicity.

$y = G(U+D)$

$u = \bar{u} - \bar{d}$

$u = C \cdot E - Q(My - u)$

$E = r - y - n$



$Y = G(u+d)$

$R = e + Y + n$

(a) Find the closed loop transfer function $T \triangleq \frac{Y}{R}$ and the sensitivity function $S \triangleq \frac{E}{R}$.

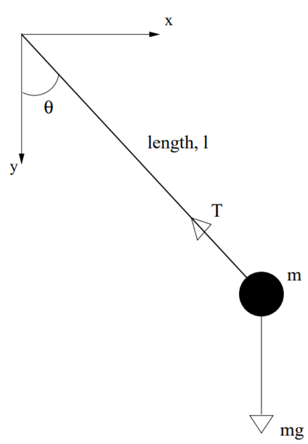
(b) Find the input sensitivity function $\frac{Y}{D}$. What happens when $Q = I$ and $M = G^{-1}$? What function does Q perform in the design?

2: 30 points

Consider the simple pendulum system shown below with a motor at the hinge that produces torque τ. The system dynamics are easy to derive from first principles.

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{\tau}{ml^2}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta - \frac{\tau}{ml^2} = 0$$



$C(r-y) - Q(My - u) = C - Q$

(a) Linearize the differential equation about equilibrium point $\theta = \frac{\pi}{2}$. Analyze the stability of the system in this configuration.

(b) Assume the mass is 1 kg and the length is 4 m. Design a PD controller (maps angle error to motor torque) that places the closed loop poles such that the closed loop system nominally has a 25% overshoot and a 2 second 2% settling time.

$y = G(d+u)$

$u = C - Q[My - u]$

$E = r - y$

(c) Construct the nonlinear dynamics in Simulink and apply the controller designed in part (b). Plot the initial condition response starting at rest from $\theta_0 = 45^o$. For what range of initial conditions does the PD controller stabilize the nonlinear system? No need to derive this analytically - you can use the simulation to answer this. Be sure to include an image of your Simulink model in the solution.

3: 20 points

Consider the LTI system given by

$$\dot{x} = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 2 \end{bmatrix} x.$$

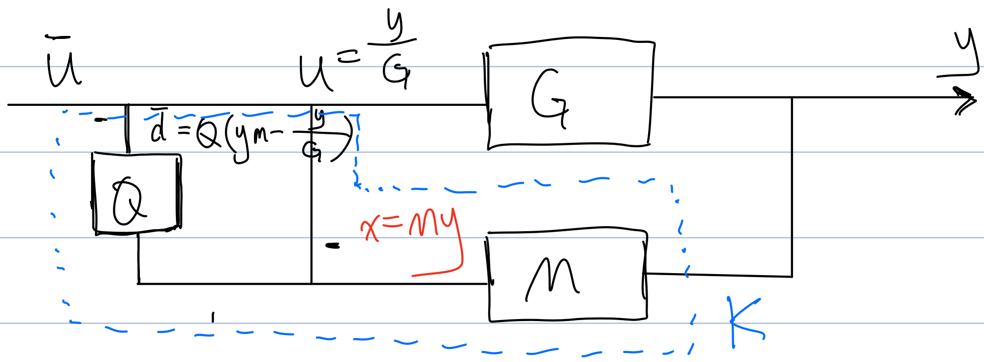
(a) Is the system controllable? Observable?

(b) Find a minimal realization. Design a controller for the minimal realization that achieves a time constant of 1 millisecond.

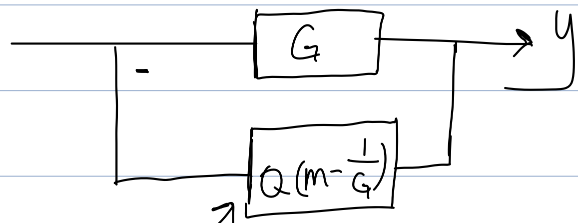
$P = [B \;\; AB]$

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$

1(a) Simplified

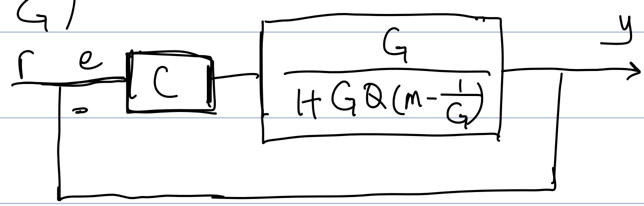


$$\bar{d} = y \cdot K$$



$$K = \frac{\bar{d}}{y}$$

$$K = Q\left(M - \frac{1}{G}\right)$$



$$T = \frac{\frac{CG}{1+GQ(M-\frac{1}{G})}}{1+\frac{CG}{1+GQ(M-\frac{1}{G})}} = \frac{CG}{1+GQ(M-\frac{1}{G})+CG}$$

$$= \frac{CG}{1-Q+MGQ+CG}$$

$$S = \frac{E}{R} = \frac{1}{1+\frac{CG}{1+GQ(M-\frac{1}{G})}} = \frac{1-Q+MGQ}{1-Q+MGQ+GC}$$

(b)
$$\left.\begin{aligned} u &= \bar{u} - \bar{d} \\ \bar{d} &= Q(My - u) \end{aligned}\right\} \begin{aligned} &(4.1) \\ &(4.2) \end{aligned}$$

$\therefore u = \bar{u} - Q(My - u)$

$u = \bar{u} - QMy + Qu$ (4.3) combine eqn 4.1, 4.2

$y = G(u+d)$ (4.4)

$\bar{u} = C \cdot e$, $e = r - y = -y$, $r = 0$, zero reference input

$\therefore \bar{u} = -C \cdot y$ (4.5)

$u = -Cy - QMy + Qu$ combine 4.3 4.5

$u = -(1-Q)^{-1}(C + QM)y$ 4.6

combine 4.4 and 4.6

$y = G\left[-(1-Q)^{-1}(C+QM)y + d\right]$

$\therefore \frac{Y}{D} = \frac{y}{d} = \frac{G}{1+(1-Q)^{-1}(CG+QMG)} = \frac{G}{1+\frac{CG+QMG}{1-Q}}$

$= \frac{(1-Q)G}{1-Q+CG+QMG}$

$\frac{Y}{D} = \frac{G-GQ}{1+(MG-1)Q+CG} = \frac{G-GQ}{1-Q+MGQ+CG}$

if $Q = I$, $M = G^{-1}$

$$\frac{Y}{D} = \frac{G}{1 + (1-I)^{-1}(CG + I)}$$

$$= \frac{(1-I)G}{1 - I + CG + I} = \frac{G - IG}{1 + CG} = 0$$

* The disturbance observer became useless.

Q is a low pass filter to eliminate the noise and obtain filtered disturbance estimate

2.(a) $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{\tau}{ml^2} - \frac{g}{l}\sin\theta \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta = \frac{\pi}{2}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 - \lambda & 1 \\ 0 & 0 - \lambda \end{bmatrix}$$

$$(0-\lambda)^2 - 0 = 0$$

$$\lambda_{1,2} = 0$$

all $\|\lambda_i\| = 0$,

check defectiveness , $\det(A) = 0$

$\therefore$ it is defective

$\therefore$ unstable

(b) Transfer function

$$G(s) = C(sI - A)^{-1} B \ , A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

$$G(s) = \frac{1}{mL^2 s^2}$$

$T_s = 2s$ within 2%

$$T_s = \frac{4}{\xi \omega_n} \ , T_s = 2$$

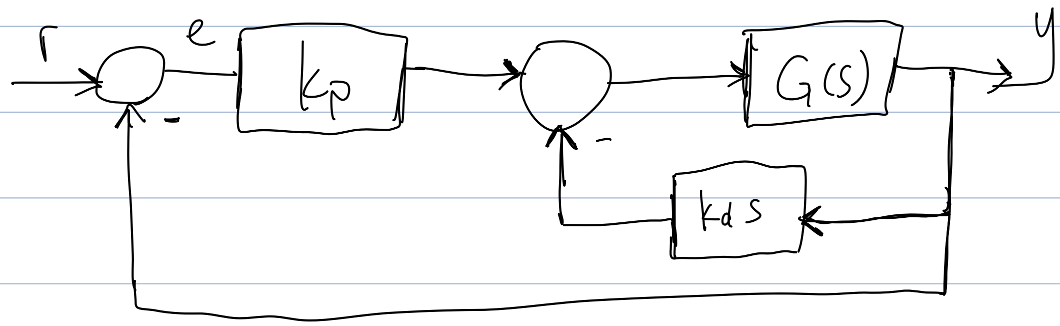
$$\xi \omega_n = 2$$

$$P.O = 100 e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \le 25$$

$$\xi = 0.404$$

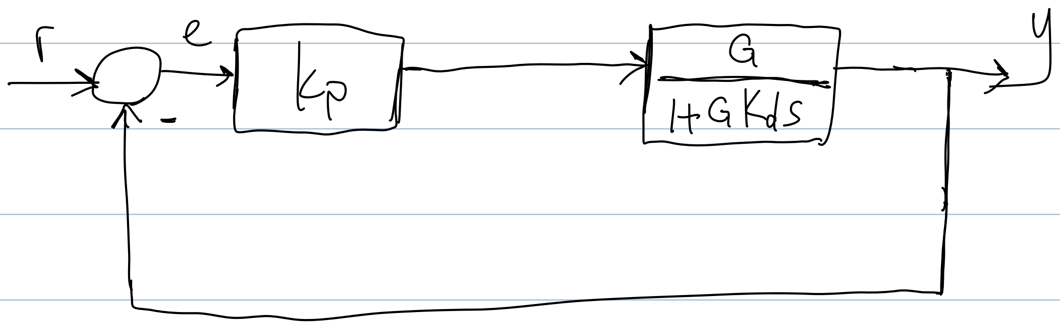
$$\therefore \quad \omega_n = 2 \div 0.404 = 4.95$$

Design the PD controller

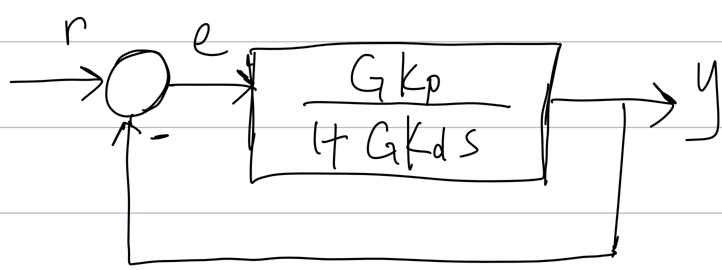


PD structure

$$\Downarrow$$



$$\Downarrow$$



$$G(s) = \frac{1}{ml^2 s^2} = \frac{1}{16 s^2}$$

$\therefore$ In the Block: $\dfrac{\frac{1}{16s^2} k_p}{1 + \frac{1}{16s^2} k_d s}$

$$= \frac{k_p}{16s^2 + k_d s}$$

$\therefore$ CLTF: $\dfrac{\frac{k_p}{16s^2 + k_d s}}{1 + \frac{k_p}{16s^2 + k_d s}}$

$$= \frac{k_p}{16s^2 + k_d s + k_p}$$

$$= \frac{\frac{k_p}{16}}{s^2 + \frac{k_d}{16} s + \frac{k_p}{16}}$$

$$\frac{k_p}{16} = 4.95^2$$

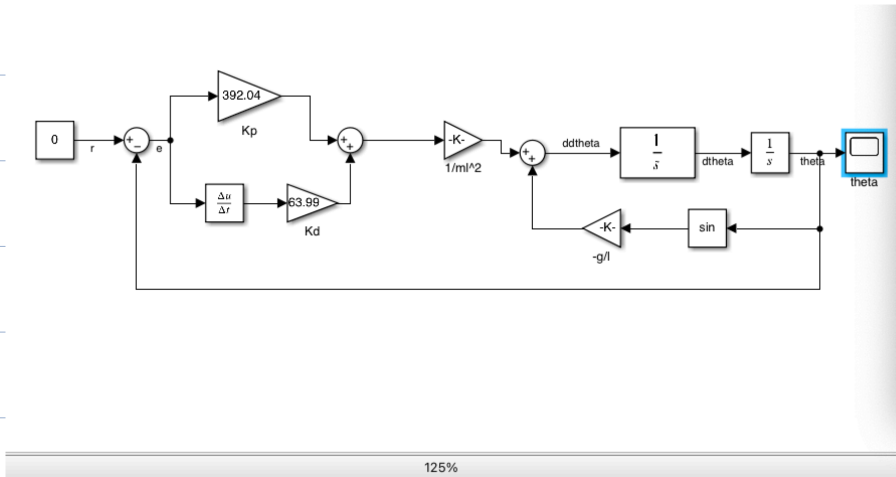
$$k_p = 392.04$$

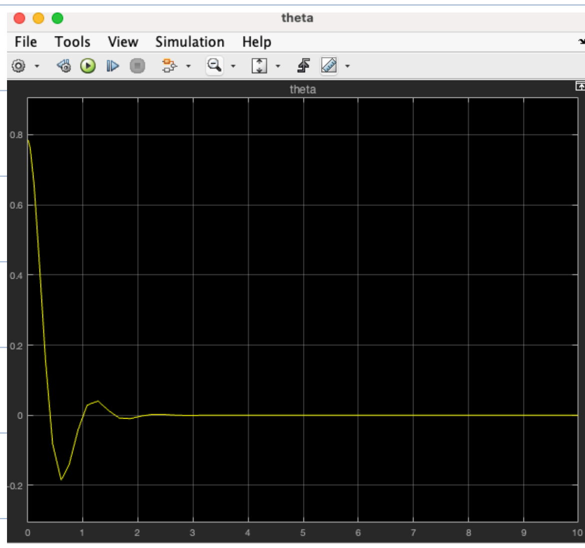
$$2 \times 4.95 \times 0.404 = \frac{k_d}{16}$$

$$k_d = 63.99$$

$$\begin{cases} K_p = 392.04 \\ K_d = 63.99 \end{cases}$$

(c)





can be stablized in all range

3. (a) $A = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $AB = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$C = \begin{bmatrix} 4 & 2 \end{bmatrix}$ $CA = \begin{bmatrix} -4 & -2 \end{bmatrix}$

Controllibility matrix $P = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\operatorname{rank}(P) = 1 < n = 2$ $\therefore$ uncontrollable

observability matrix $Q = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}$ $\operatorname{rank}(Q) = 1 < n$ unobservable

(b) Minimal realization

$$P = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = M^{-1}AM = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} -1 & 0.5 \\ 0 & -0.25 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 6 & -4 \end{bmatrix}$$

$$\dot{\hat{x}}_c = \begin{bmatrix} -1 & 0.5 \\ 0 & -0.25 \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 6 & -4 \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_c \end{bmatrix}$$

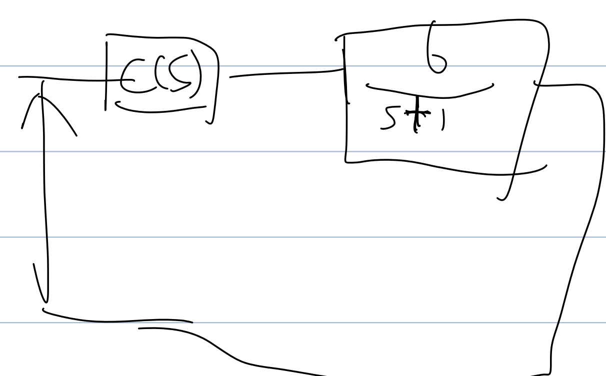
$A_c = -1 \qquad C_c = 6 \qquad B_c = 1$

$Q_c = [C_c] = 6 \quad r(Q_c) = 1$ full rank

$\therefore \dot{\hat{x}}_{co} = -1\hat{x}_{co} + u$

$y = 6\hat{x}_{co}$

$$G(s) = \frac{6}{s+1}$$



Make new $T = \dfrac{\frac{6C}{s+1}}{1 + \frac{6C}{s+1}}$ have $\tau = 0.001$

$$T = \frac{6C}{s+1+6C}$$

$$T = \frac{6C}{s+(6C+1)}$$

$$\frac{1}{6C+1} = 0.001$$

$$C = 166.5$$