1: 30 points

Norms and SVD

(a) Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$. Compute the spectral radius, the Frobenius norm, the 1-norm, the 2-norm, and the ∞-norm of A and B.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) A generalized matrix norm is norm on a matrix that satisfies all properties other than $\|AB\| \leq \|A\|\|B\|$. Show that $\|A\|_{\max} = \max_{i,j} |a_{ij}|$ is a generalized matrix norm by demonstrating that it satisfies the four fundamental properties and giving examples that show $\|A\|_{\max} < \rho(A)$ and $\|AB\|_{\max} > \|A\|_{\max}\|B\|_{\max}$.

(c) For a SISO system, show that the H_2 and H_∞ norms are invariant to time delays and all-pass filters, i.e. show $\|QG\|_2 = \|G\|_2$ and $\|QG\|_\infty = \|G\|_\infty$ for $Q = e^{-sT}$ and $Q = \frac{s-a}{s+a}$ with $a > 0$.

2: 20 points

MIMO Design

Consider the 2×2 transfer function matrix

$$G(s) = \begin{bmatrix} \frac{10(s+2)}{s^2+0.2s+100} & \frac{1}{s+5} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}.$$

In this problem you will design two DIDO controllers for this system using different approaches.

(a) Dynamic Decoupling. Find a proper approximation to G^{-1}. Use this to design a dynamic decoupling-based controller that achieves (approximately) loop shapes of $L = \frac{100}{s}$ for each of the diagonal elements. Form the loop transfer function $L = GK$ and use the margin command to show the performance of $L_{1,1}$ and plot the Bode magnitude of the 2×2 sensitivity function.

(b) Mixed Sensitivity Synthesis. Using first order weights for W_p and W_T and a constant actuator weight corresponding to a maximum control usage of 100 at each input, design a mixed sensitivity controller that

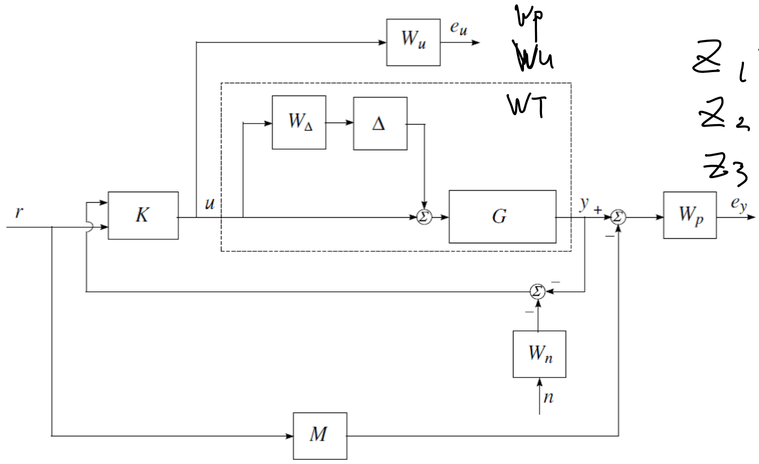
- Maximizes the bandwidth such that $\gamma < 1$.
- Rejects steady state disturbances by a factor of 1000
- Rejects high frequency noise by a factor of 1000
- Has a sensitivity peak of no more than 2 and a complementary sensitivity peak of no more than 1.5
- Has a complementary sensitivity crossover frequency at most $3\times$ the sensitivity crossover frequency.

Plot the magnitudes of W_pS, W_TT, and W_UKS for your final design.

3: 30 points

Generalized Plants

(a) For the block diagram shown below: 1.) Find the generalized plant P, and 2.) Find $N = F_l(P, K)$. For simplicity, feel free to give the system inside the dashed line a name - say $G^{\star}$.



Wp
Wu
WT

$z_1 = W_p(y - rM)$
$z_2 = W_u U$
$z_3 =$

(b) Use the Matlab command *sysic* to generate the generalized plant for the mixed sensitivity problem in Problem 2.b. NOTE: Use the block diagram from class - it is very different from the one in 3.a. Run the command K = *hinfsyn(P,2,2)* for your generalized plant and compare the controller to the one you found in Problem 2.b.

(c) The state space description of a transfer function matrix can be written as an LFT. Find H such that

$$F_l\left(H, \frac{1}{s}\right) = C\,(sI - A)^{-1}\,B + D.$$

1. $A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$ $\lambda_1 = 1 \quad \lambda_2 = -3$

$\therefore$ S radius $= 1$

F norm: $\|A\|_F = \sqrt{0^2 + 1^2 + 3^2 + 2^2}$

$= \sqrt{14}$

1-norm: $\|A\| = \max \begin{bmatrix} |0| & |1| \\ + & + \\ |3| & |-2| \end{bmatrix}$

max col sum: $\|A\|_1 = \max \begin{bmatrix} 3 & 3 \end{bmatrix}$

$= 3$

2-norm: $\|A\|_2 \equiv \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ -2 & 13 \end{bmatrix}$

$\lambda_{max} = 7 + 2\sqrt{10}$

$\therefore \|A\|_2 = \sqrt{7 + 2\sqrt{10}}$

$= 3.6503$

$\|A\|_\infty =$ max row sum

$= \max \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$= 5$

$B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ $\lambda_1 = \sqrt{3}$ $\lambda_2 = -\sqrt{3}$

S. radius $= \sqrt{3}$

F-norm $= \|B\| = \sqrt{0^2 + 1^2 + 3^2 + 0^2} = \sqrt{10}$

1 norm $= \|B\|_1 = \max \begin{bmatrix} 3 & 1 \end{bmatrix}$

$= 3$

2 norm $= \|B\|_2 = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$\lambda = 1, 9$

$\lambda_{max} = 9$

$\therefore \|B\|_2 = \sqrt{9} = 3$

Inf norm $= \|B\|_\infty = \max \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= 3$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & \cdot & & \\ \vdots & & \ddots & \\ a_{j1} & & & a_{ij} \end{bmatrix}$$

(b) $\|A\|_{max} = \max_{i,j} |a_{ij}|$

① $\|e\|$ must bigger or equal to zero because it is maximum abs value of A.

② $\|e\| = 0$ if $e = 0$ $\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\|A\|_{max} = 0$

③ $\|\alpha e\| = |\alpha| \|e\|$, $|\alpha|$ is always positive, which it's a scaling factor of e

$$\max_{i,j} \begin{bmatrix} |\alpha a_{11}| & |\alpha a_{12}| & \cdots & |\alpha a_{1j}| \\ |\alpha a_{21}| & \cdots & & \vdots \\ \vdots & & & \\ |\alpha a_{j1}| & \cdots & \cdots & |\alpha a_{ij}| \end{bmatrix} = |\alpha| \left| \begin{bmatrix} |a_{11}| & |a_{12}| & \cdots & |a_{1j}| \\ |a_{21}| & \cdots & & \vdots \\ \vdots & & & \\ |a_{j1}| & \cdots & \cdots & |a_{ij}| \end{bmatrix} \right|$$

$|\alpha||a_{ij}| = |\alpha a_{ij}|$ assume a_{ij} is max abs value.

④ $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$, Assume a_{ij}, b_{ij} is maximum abs value in e_1 and e_2. Assume $a_{ij} > 0$, $b_{ij} < 0$, $|a_{ij} + b_{ij}| < |a_{ij}| + |b_{ij}|$

$\therefore \|e_1 + e_2\| < \|e_1\| + \|e_2\|$

When $a_{ij} > 0$, $b_{ij} > 0$, $|a_{ij} + b_{ij}| = |a_{ij}| + |b_{ij}|$, $\therefore \|e_1 + e_2\| = \|e_2\|$

$\therefore \|e_1 + e_2\| \leq \|e_2\|$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$\|A\|_{max} = 4 \quad \rho(A) = 4.56 \quad \|A\|_{max} < \rho(A)$$

$$\|AB\|_{max} = \left\| \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \right\|_{max} = \left\| \begin{bmatrix} 0 & 4 \\ -2 & 12 \end{bmatrix} \right\|_{max}$$

$$= 12$$

$$\|A\|_{max} \|B\|_{max} = 8$$

$$\therefore \|AB\|_{max} > \|A\|_{max} \|B\|_{max}$$

(c) G: scalar: $\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega \right)^{\frac{1}{2}}$

With time delay $\|G\theta\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) e^{-Tj\omega}|^2 d\omega \right)^{\frac{1}{2}}$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)[\cos T\omega - j\sin T\omega]|^2 d\omega \right)^{\frac{1}{2}}$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 (\cos^2 T\omega + \sin^2 T\omega) d\omega \right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 \cdot 1 \right]^{\frac{1}{2}} d\omega$$

$$= \|G\|_2$$

equal to the part without time delay

$$\|GQ\|_\infty = \max_w \left| G(jw)\, e^{-jwT} \right|$$

$$= \max_w \sqrt{G(jw) \cdot (\sin^2 Tw + \cos^2 Tw)}$$

$$= \max_w \sqrt{G(jw) \cdot 1}$$

$$= \|G\|_\infty$$

Now, set $G = \frac{s+d}{s^2+bs+c}$, $Q = \frac{s-a}{s+a}$

$$\|GQ\|_2 = \left\| G(jw) \cdot \frac{jw-a}{jw+a} \right\|_2$$

$$= \left(\frac{1}{2}\pi \int_{-\infty}^{\infty} G(jw) \cdot \frac{w^2+a^2}{w^2+a^2} \right)^{\frac{1}{2}}$$

$$= \left(\frac{\pi}{2} \int_{-\infty}^{\infty} G(jw) \cdot 1 \right)^{\frac{1}{2}}$$

$$= \|G\|_2$$

$$\|GQ\|_{max} = \max_w \left(G(jw) \cdot \frac{jw-a}{jw+a} \right)^{\frac{1}{2}}$$

$$= \max_w \left(G(jw) \cdot \frac{w^2+a^2}{w^2+a^2} \right)^{\frac{1}{2}}$$

$$= \max_w \left[G(jw) \right]^{\frac{1}{2}}$$

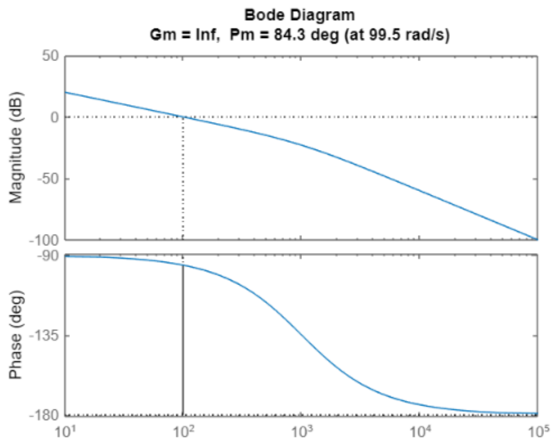
$$= \|G\|_{max}$$

2.

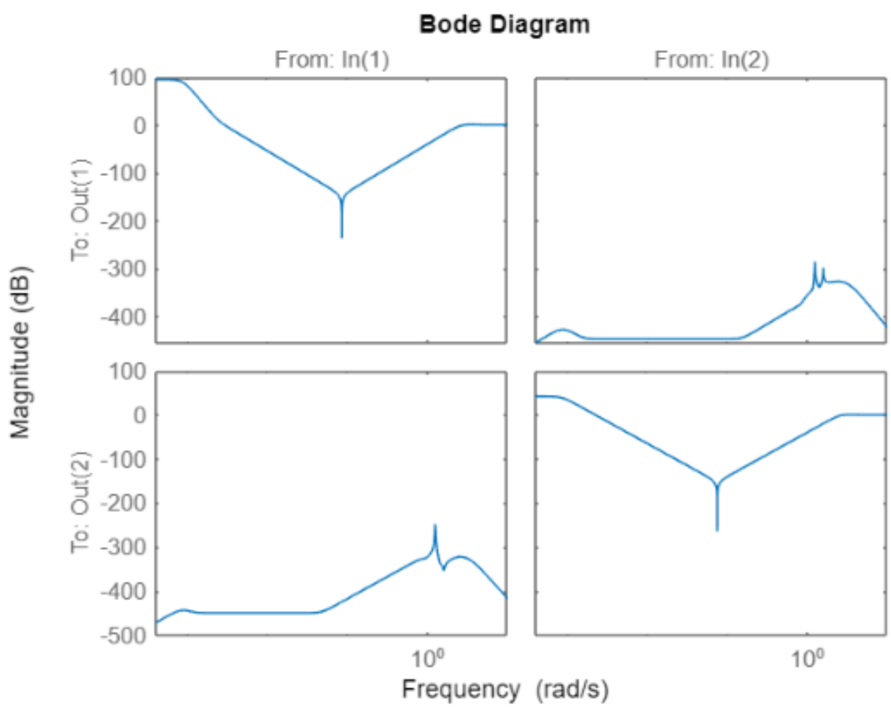
HW4 Q2

(a) Dynamic Decoupling. Find a proper approximation to G−1. Use this to design a dynamic decoupling-based controller that achieves (approximately) loop shapes of L = 100/s for each of the diagonal elements. Form the loop transfer function L = GK and use the margin command to show the performance of L(1,1) and plot the Bode magnitude of the 2x2 sensitivity function.

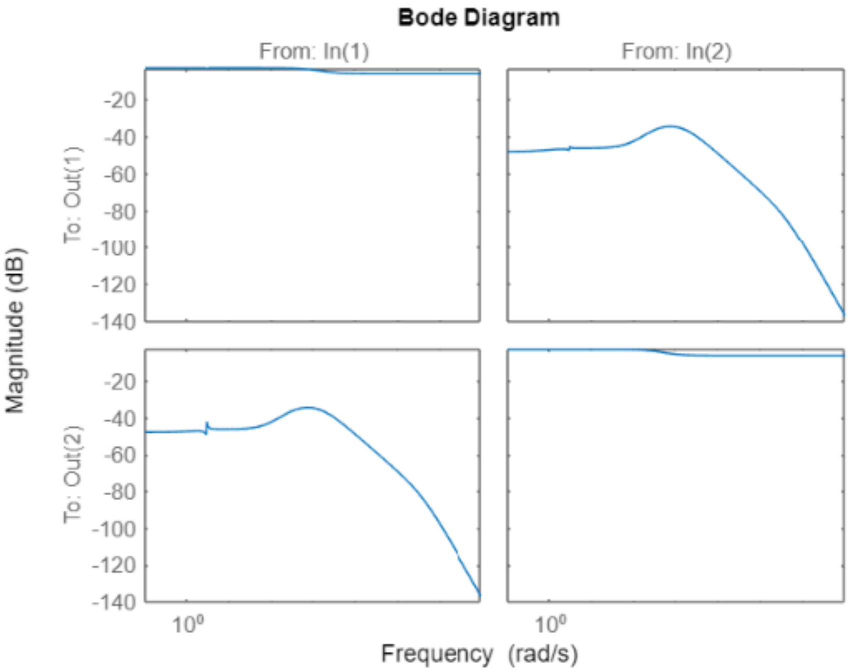
```
s = tf('s');
G = [10*(s+2)/(s^2+0.2*s+100) 1/(s+5);(s+2)/(s^2+0.1*s+10) 5*(s+1)/(s+2)/(s+3)];
Ginv = inv(G)*1000/(s+1000)*eye(2);
K_inv = Ginv*[100/s 0;0 100/s];
L_inv = minreal(G*K_inv, 0.5);
T_inv = feedback(L_inv, eye(2));
S_inv = eye(2)-T_inv;
margin(L_inv(1,1))
```



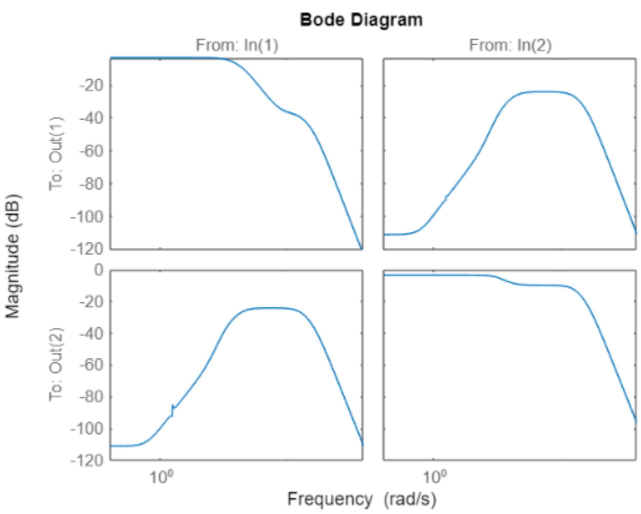

```
bodemag(S_inv)
```



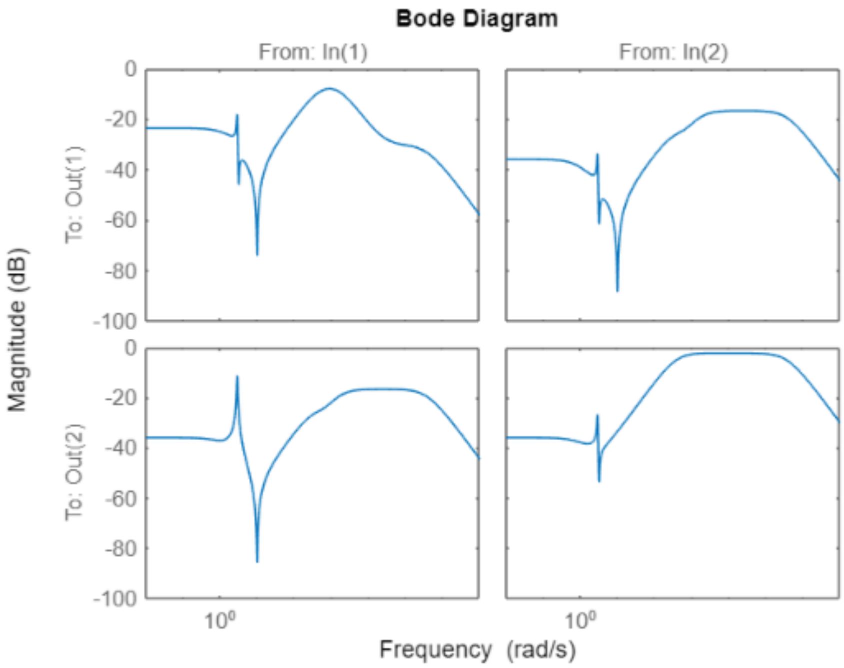
```
s = tf('s');
G = [10*(s+2)/(s^2+0.2*s+100) 1/(s+5); (s+2)/(s^2+0.1*s+10) 5*(s+1)/(s+2)/(s+3)]; %2x2 matrix
M = 2;  % Maximum sensitivity peak
A = 1000;  % Steady state disturbance attenuation
Mt = 1.5;  % Maximum complementary sensitivity peak
At = 0.001;  % High frequency noise attenuation
GAM = 0;  % Dummy value to start
BW = 10; % Trivial low value for bandwidth
BW_step = 5;
Wp = [0 0; 0 0]; % Performance weight
Wu = [1/100 0; 0 1/100];
K = [0 0;0 0];
while GAM<1
    Wp_old = Wp;
    Wt_old = Wt;
    Wp = [((s/M)+BW)/(s+BW/A) 0; 0 ((s/M)+BW)/(s+BW/A)];
    Wt = makeweight(1/Mt,3*BW,1/At) * eye(2,2);  %The 3x bandwidth is to give some separation between
    K_old = K;
    [K,CL,GAM,info] = mixsyn(G,Wp,Wu,Wt);  %This is the magic synthesis command.  Much of this class
    BW = BW+BW_step;  %Bisection would be cleaner, but this works!
end
L_ms = G * K_old;
S_ms = inv(eye(2,2)+L_ms);
T_ms = eye(2,2) - S_ms;
figure
% bodemag(inv(Wp_old),S_ms) %Compare with performance
bodemag(Wp_old * S_ms) %Compare with performance
```



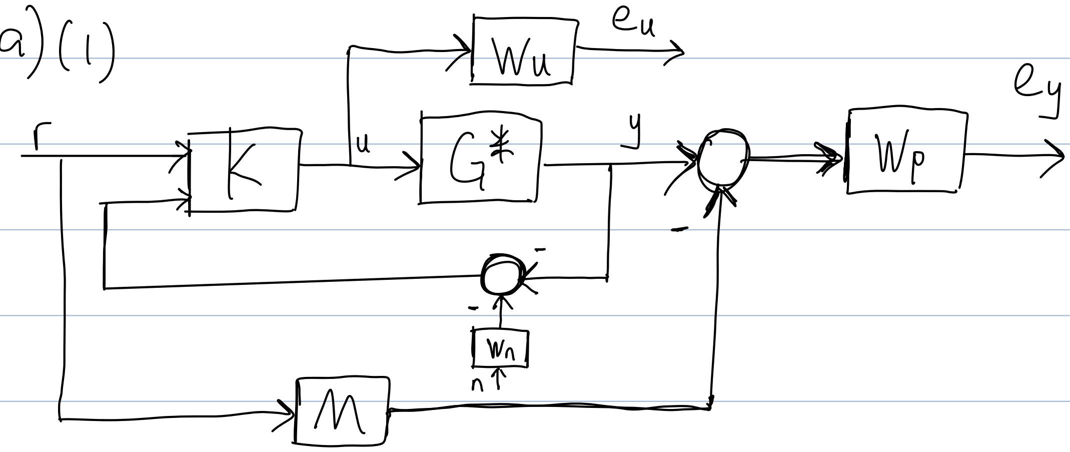

```
figure
bodemag(Wt*T_ms); %Compare with control robustness
```



```
figure
bodemag(Wu*K_old*S_ms);
```



3. (a) (1)



$$G^* = 1 + W_\Delta \Delta$$

Generalized Output $z = \begin{bmatrix} (uG^* - Mr)W_p \\ uW_u \end{bmatrix}$

Controller Input $v = \begin{bmatrix} r \\ -W_n n - uG^* \end{bmatrix}$

Generalized Input $w = \begin{bmatrix} r \\ n \end{bmatrix}$

Controller output $u = u$

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} (y - Mr)W_p \\ uW_u \\ r \\ -W_n n - uG^* \end{bmatrix}$$

$$\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} r \\ n \\ u \end{bmatrix}$$

$$
\begin{array}{l} z \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ v \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array}
\begin{bmatrix} (uG^* - Mr)W_p \\ uW_u \\ r \\ -W_n n - uG^* \end{bmatrix}
=
\left[\begin{array}{cc|c} -W_p M & 0 & W_p G^* \\ 0 & 0 & W_u \\ \hline I & 0 & 0 \\ 0 & -W_n & -G^* \end{array}\right]
\begin{bmatrix} r \\ n \\ u \end{bmatrix}
\begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} w \\ \left.\begin{array}{c} \\ \end{array}\right\} u \end{array}
$$

(2) $F_L(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$

$$
P_{11} = \begin{bmatrix} -W_p M & 0 \\ 0 & 0 \end{bmatrix} \qquad P_{12} = \begin{bmatrix} W_p G^* \\ W_u \end{bmatrix} \qquad P_{21} = \begin{bmatrix} I & 0 \\ 0 & -W_n \end{bmatrix}
$$

$$
P_{22} = \begin{bmatrix} 0 \\ -G^* \end{bmatrix}
$$

```
systemnames = 'G Wp Wu Wt'; %Block name only
```

Now we have to define the inputs and outputs to the system by giving values to *inputvar* and *outputvar*.

```
inputvar = '[r{2};u{2}]';
outputvar = '[Wp;Wu;Wt;r-G]'; %Strangely, the system outputs are just the name
```

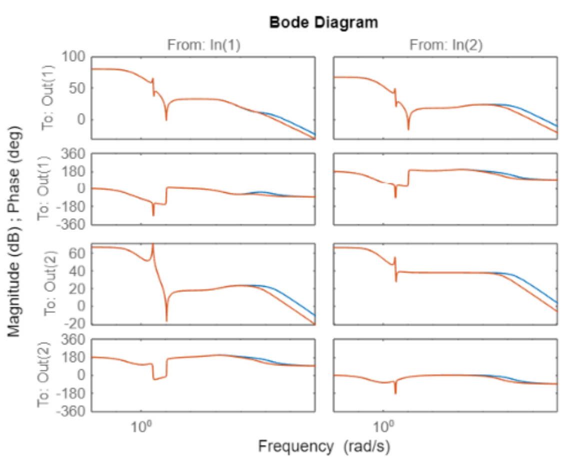
Here the variables are partitioned as before, i.e. the input variables are listed as $[w, u]^T$. Now we need to give the inputs to each system.

```
input_to_G = '[u]';
input_to_Wp = '[r-G]';
input_to_Wu = '[u]';
input_to_Wt = '[G]';
```

That's the requirements! Now we put it in action.

```
cleanupsysic = 'yes';  %This drops all the useless variables from workspace
P = sysic
```

```
[Kms,CL,GAM] = mixsyn(G,Wp_old,Wu,Wt);
[Khinf,CL,GAM] = hinfsyn(P,2,2);
bode(Kms,Khinf)  %They're on top!
```



(c) $F_l(H, \frac{1}{s}) = C(sI - A)^{-1}B + D$

Reorder : $\dfrac{CB}{sI - A} + D$

$$\frac{P_{12}P_{21}}{\frac{I}{K} - P_{22}} + P_{11}$$

$P_{11} = D$ $\qquad P_{12}P_{21} = CB$ $\qquad \frac{I}{K} - P_{22} = sI - A$

$A = P_{22}$ $\quad B = P_{21}$ $\qquad C = P_{12}$ $\quad D = P_{11}$

$A = \begin{bmatrix} 0 \\ -G^* \end{bmatrix}$ $\quad B = \begin{bmatrix} I & 0 \\ 0 & -W_n \end{bmatrix}$ $\quad C = \begin{bmatrix} W_p G^* \\ W_u \end{bmatrix}$ $\quad D = \begin{bmatrix} -W_p n & 0 \\ 0 & 0 \end{bmatrix}$