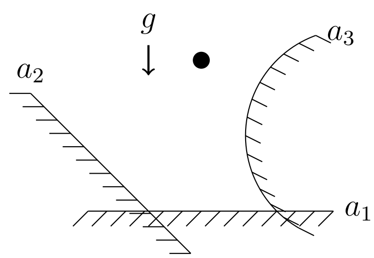
Homework 9: Hybrid Systems

24-760 Robot Dynamics & Analysis
Fall 2019

Name: ______________________________

Problem 1) Falling Ball



Consider a point particle that can make plastic frictionless impact with several constraints. Assume the particle is mass 1 and gravity is 9.8. Let the constraints be $a_1(x,y) = y$, $a_2(x,y) = x + y + 1$, and $a_3(x,y) = (x-2)^2 + (y-1)^2 - 2$.

1.1) What is the hybrid dynamical system for this problem? That is, what are all of the components of $\mathcal{H} = (\mathcal{J}, \mathit{\Gamma}, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$? Consider both impact (IV complementarity) and liftoff (FA complementarity) transitions. You may limit the hybrid system to only the feasible transitions ($\tilde{\mathit{\Gamma}}$ instead of $\mathit{\Gamma}$). For simplicity, assume the particle does not impact multiple constraints at once from the unconstrained mode.

Solution: (40 points) The contact modes for this system are defined by

$$\mathcal{J} = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$$

The feasible transitions $\tilde{\mathit{\Gamma}}$ are given by

$$\tilde{\mathit{\Gamma}} = \{(\{\},\{1\}), (\{\},\{2\}), (\{\},\{3\}), (\{3\},\{\}), (\{1\},\{2\}), (\{2\},\{1\}), (\{1\},\{1,3\})\}.$$

Transitions out of $\{1,2\}$ or $\{1,3\}$ are not feasible because the velocity in these modes must be zero, and transitions from mode 1 or 2 into $\{\}$ are not considered because the dynamics of this (passive) system will never allow such a transition.

The domain for our hybrid system is given by

$$\mathcal{D} = \left\{ \begin{array}{l} \mathcal{D}_{\{\}} = \{q, \dot{q} \colon a_1(q) \geq 0, a_2(q) \geq 0, a_3(q) \geq 0\} \\ \mathcal{D}_1 = \{q, \dot{q} \colon a_1(q) = 0, A_1\dot{q} = 0, a_2(q) \geq 0, a_3(q) \geq 0\} \\ \mathcal{D}_2 = \{q, \dot{q} \colon a_1(q) \geq 0, a_2(q) = 0, A_2\dot{q} = 0, a_3(q) \geq 0\} \\ \mathcal{D}_3 = \{q, \dot{q} \colon a_1(q) \geq 0, a_2(q) \geq 0, a_3(q) = 0, A_3\dot{q} = 0\} \\ \mathcal{D}_{12} = \{q, \dot{q} \colon a_1(q) = 0, A_1\dot{q} = 0, a_2(q) = 0, A_2\dot{q} = 0, a_3(q) \geq 0\} \\ \mathcal{D}_{13} = \{q, \dot{q} \colon a_1(q) = 0, A_1\dot{q} = 0, a_2(q) \geq 0, a_3(q) = 0, A_3\dot{q} = 0\} \end{array} \right\}$$

The flow $\mathcal{F}$ for our hybrid system describes the dynamics in each domain of the system. However, the only component of the dynamics that changes with the domain is the A matrix, so we can write

$$\begin{aligned}
\mathcal{F} &= \{\mathcal{F}_I \colon \bar{M}\ddot{q} + \bar{N}(q,\dot{q}) + A_I^T\lambda = 0\}, \quad I \in \mathcal{J} \\
\bar{M} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\bar{N} &= \begin{bmatrix} 0 \\ 9.81 \end{bmatrix} \\
A &= \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2\,x-4 & 2\,y-2 \end{bmatrix}
\end{aligned}$$

There are no Coriolis or applied forces to $\bar{C} = \Upsilon = 0$.

The guard sets for the hybrid system are given by

$$\mathcal{G} = \left\{ \begin{array}{l}
\mathcal{G}_{\{\},1} = q,\dot{q} \in \mathcal{D}_{\{\}} : a_1(q) = 0, A_1\dot{q} < 0 \\
\mathcal{G}_{\{\},2} = q,\dot{q} \in \mathcal{D}_{\{\}} : a_2(q) = 0, A_2\dot{q} < 0 \\
\mathcal{G}_{\{\},3} = q,\dot{q} \in \mathcal{D}_{\{\}} : a_3(q) = 0, A_3\dot{q} < 0 \\
\mathcal{G}_{1,2} = q,\dot{q} \in \mathcal{D}_1 : a_2(q) = 0, A_2\dot{q} < 0 \\
\mathcal{G}_{2,1} = q,\dot{q} \in \mathcal{D}_2 : a_1(q) = 0, A_1\dot{q} < 0 \\
\mathcal{G}_{1,13} = q,\dot{q} \in \mathcal{D}_1 : a_3(q) = 0, A_3\dot{q} < 0 \\
\mathcal{G}_{3,\{\}} = q,\dot{q} \in \mathcal{D}_3 : U(\lambda_3) < 0
\end{array} \right\}$$

The reset map for each of the edges is given by

$$\mathcal{R} = \left\{ \begin{array}{l}
\mathcal{R}_{\{\},1}(q,\dot{q}^-) = (q,\dot{q}^+) \in \mathcal{D}_1 \\
\mathcal{R}_{\{\},2}(q,\dot{q}^-) = (q,\dot{q}^+) \in \mathcal{D}_2 \\
\mathcal{R}_{\{\},3}(q,\dot{q}^-) = (q,\dot{q}^+) \in \mathcal{D}_3 \\
\mathcal{R}_{1,2}(q,\dot{q}^-) = (q,\dot{q}^+) \in \mathcal{D}_2 \\
\mathcal{R}_{2,1}(q,\dot{q}^-) = (q,\dot{q}^+) \in \mathcal{D}_1 \\
\mathcal{R}_{1,13}(q,\dot{q}^-) = (q,\dot{q}^+) \in \mathcal{D}_{13} \\
\mathcal{R}_{3,\{\}}(q,\dot{q}^-) = (q,\dot{q}^-) \in \mathcal{D}_{\{\}}
\end{array} \right\}, \qquad \begin{array}{l} \dot{q}^+ = \dot{q}^- - A_J^{\dagger^T}(q)A_J(q)\dot{q}^- \\ \mathrm{IV}(J,q,\dot{q}^-) \bigwedge \mathrm{FA}(J,q,\dot{q}^-) \end{array} \qquad J \in \mathcal{J}$$

1.2) Simulate the system in Matlab using `ode45` and an event function. The `odefun` should capture the continuous dynamics $\mathcal{F}$, while the event function detects the guard conditions $\mathcal{G}$. Apply the reset function outside of the `ode45` execution. You may want to make separate Matlab functions to calculate $a, A, \dot{A}, \mathcal{F}, \mathcal{R}$, the block matrix inverse, etc. To solve the complementarity problems, CP_{IV} and CP_{FA}, you do not need to use a computationally efficient algorithm, simply check the complementarity conditions for all possible modes (modes in the local scope, $\mathcal{I}$) and return the (hopefully unique) mode that satisfies the constraints. Here are two pages documenting these Matlab features:
https://www.mathworks.com/help/matlab/ref/ode45.html
https://www.mathworks.com/help/matlab/math/ode-event-location.html
Hint: Start with just a single constraint a_1, and then add in a_2 and a_3. If your simulation is missing events, you may want to try using the `MaxStep` option.

Solution: (40 points) See the posted MATLAB code for the details of this algorithm.

1.3) Run four simulations starting at $(0, 5)$, $(-1.5, 5)$, $(1.5, 5)$, and $(1, 5)$ with zero velocity. Run each simulation for 5 seconds. What contact mode transitions occur and at what times?

Solution: (20 points - 5 points per initial condition) See the posted MATLAB code for the details of this algorithm. The following should be the contact mode transitions and transition times for each of the four initial conditions. Note that for the last initial condition, the transition into and out of mode $\{3\}$ is instantaneous.

```
Initial condition: [0  5]'.
Transition to contact mode {1} at time t = 1.0102 s.

Initial condition: [-1.5  5]'.
Transition to contact mode {2} at time t = 0.95831 s.
Transition to contact mode {1} at time t = 1.0595 s.
Transition to contact mode {1  3} at time t = 1.4447 s.

Initial condition: [1.5  5]'.
Transition to contact mode {3} at time t = 0.73916 s.
Transition to contact mode {} at time t = 0.89201 s.
Transition to contact mode {1} at time t = 1.3706 s.
Transition to contact mode {2} at time t = 1.7155 s.
Transition to contact mode {1} at time t = 2.2402 s.
Transition to contact mode {1  3} at time t = 3.7959 s.

Initial condition: [1  5]'.
Transition to contact mode {3} at time t = 0.78246 s.
Transition to contact mode {} at time t = 0.78246 s.
Transition to contact mode {1} at time t = 1.1404 s.
Transition to contact mode {2} at time t = 1.3041 s.
Transition to contact mode {1} at time t = 2.0866 s.
Transition to contact mode {1  3} at time t = 3.1298 s.
```