Homework 2: Motion

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solution

Note: For homework submission, please submit the PDF of the written portion to "Homework 2" and a zipped folder of your Matlab code to "Homework 2 Programming" in Gradescope.

Problem 1) Lunar Motion

Consider a simplified model of the motion of the earth and the moon. Attach a stationary coordinate frame to the center of the earth (s), a frame with the same origin that rotates with the earth (e), and a frame to the moon (m). The axes of rotation are all aligned with each other and pointing in the $+z$ direction of each frame. Assume the moon's orbit around the earth is circular with radius l_m. The earth's radius is r_e and the moon's radius is r_m. The moon rotates about the earth at a rate of 1 revolution per 28 days, and about it's own axis at a rate of 1 revolution per 28 days. The earth rotates about it's own axis at a rate of 1 revolution per day. *Hint: Draw a figure to keep track of the different frames.*

1.1) Just consider the earth's rotation to start. At time t, assume the earth is rotated so that the earth's $+x$ axis is aligned with the stationary $-y$. What is R_{se}? What is g_{se}? Use this configuration for the other parts of this question.

Solution: (6 points) We can write R_{se} and p_{se} directly from the definition of our coordinate frames as

$$R_{se}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p_{se} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where θ defines the angle between the x-axis of each frame. Thus the rigid body transformation at time t is given by

$$R_{se}(-\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g_{se}(-\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2) For a point q on the surface of the earth, $q_e = [0, r_e, 0]^T$, calculate the location of this point in the stationary frame using a rigid body transformation.

Solution: (6 points)

$$q_s = g_{se} q_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r_e \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_e \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

1.3) What is the body velocity of the earth's rotation, V_{se}^b? What is the spatial velocity V_{se}^s?

Solution: (10 points) Calculating the body velocity will require $\dot{R}_{se}$, which is

$$\dot{R}_{se} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}_{se}.$$

We also note that since the origin of the earth frame is coincident with the stationary frame, so $p_{se} = \dot{p}_{se} = 0$. Thus we can calculate the body and spatial velocity as

$$V_{se}^b = \begin{bmatrix} R_{se}^T \dot{p}_{se} \\ (R_{se}^T \dot{R}_{se})^\vee \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^\vee \end{bmatrix} \dot{\theta}_{se}$$

$$= \begin{bmatrix} 0 \\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\vee \end{bmatrix} \dot{\theta}_{se} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$V_{se}^s = \begin{bmatrix} -\dot{R}_{se} R_{se}^T p_{se} + \dot{p}_{se} \\ (\dot{R}_{se} R_{se}^T)^\vee \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^\vee \end{bmatrix} \dot{\theta}_{se}$$

$$= \begin{bmatrix} 0 \\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\vee \end{bmatrix} \dot{\theta}_{se} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with units of $\dot{\theta}_{se} = 1$ rotation per day. This could also be solved by inspection, noting that $V_{se}^b = [v_{se}^b, \omega_{se}^b]^T$ and $V_{se}^s = [v_{se}^s, \omega_{se}^s]^T$.

1.4) Using that body and spatial velocity, what is the instantaneous velocity of the point q_e in the earth's frame, v_{qe}? What is the velocity in the stationary frame, v_{qs}?

Solution: (6 points) By the definition of the body and spatial velocity, we can compute the velocity of this point in each frame,

$$v_{qe} = \hat{V}^b_{se} q_e = \omega^b_{se} \times q_e + v^b_{se} = \begin{bmatrix} -r_e \\ 0 \\ 0 \end{bmatrix}$$

$$v_{qs} = \hat{V}^s_{se} q_s = \omega^s_{se} \times q_s + v^s_{se} = \begin{bmatrix} 0 \\ r_e \\ 0 \end{bmatrix}$$

1.5) Now consider the position of the moon relative to the earth. Assume at time t that the moon is located at $[l_m, 0, 0]^T$ in the stationary frame, with the moon's x axis pointing to the earth. Calculate g_{sm}, then calculate g_{em} based on g_{sm} and g_{se}.

Solution: (6 points) Determining R_{sm} and p_{sm} from our coordinate frame definitions gives

$$g_{sm} = \begin{bmatrix} -1 & 0 & 0 & l_m \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$g_{em} = g_{se}^{-1} g_{sm} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & l_m \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & l_m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.6) What is the body velocity of the moon's motion, V^b_{sm}?

Solution: (10 points) The body velocity of the moon is given by

$$V^b_{sm} = \begin{bmatrix} v^b_{sm} \\ \omega^b_{sm} \end{bmatrix} = \begin{bmatrix} R^T_{sm} \dot{p}_{sm} \\ (R^T_{sm} \dot{R}_{sm})^\vee \end{bmatrix}$$

$\dot{R}_{sm}$ can be found in the same way to $\dot{R}_{se}$ to be

$$\dot{R}_{sm} = \begin{bmatrix} -\sin\theta_m & -\cos\theta_m & 0 \\ \cos\theta_m & -\sin\theta_m & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}_{sm} = \begin{bmatrix} 0 & \frac{1}{28} & 0 \\ -\frac{1}{28} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where θ_m is the angle between the moon frame x-axis and the stationary frame x-axis, and $\dot{\theta}_{sm}$ is in units of rotations per day. The velocity of the origin of the moon frame in the stationary frame, $\dot{p}_{sm}$, can be calculated by inspection, or with $v_{p_{sm}} = \omega^s_{sm_o} \times p_{sm}$ (see Eq. 2.51 in MLS), where $\omega^s_{sm_o}$ is the rate at which the moon orbits the earth. Either method will yield $\dot{p}_{sm} = [0, \frac{l_m}{28}, 0]^T$, and therefore the body velocity is given by

$$V^b_{sm} = \begin{bmatrix} R^T_{sm}\dot{p}_{sm} \\ (R^T_{sm}\dot{R}_{sm})^\vee \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0 \\ -\frac{l_m}{28} \\ 0 \end{matrix} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{28} & 0 \\ -\frac{1}{28} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\vee \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{l_m}{28} \\ 0 \\ 0 \\ 0 \\ \frac{1}{28} \end{bmatrix}.$$

1.7) Calculate the spatial velocity, V^s_{sm}, using an adjoint operation. What is special about v^s_{sm}?

Solution: (10 points) We can calculate the adjoint from its definition and apply to the body velocity to obtain

$$V^s_{sm} = \mathrm{Ad}_{g_{sm}} V^b_{sm} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} V^b_{sm} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -l_m \\ 0 & 0 & 1 & 0 & -l_m & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{l_m}{28} \\ 0 \\ 0 \\ 0 \\ \frac{1}{28} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{28} \end{bmatrix}.$$

This yields $v^s_{sm} = 0$, which means that a point moving with the moon frame and located at the earth's center does not move. This effect explains why we only ever see one side of the moon, since it is orbiting earth and rotating about its own axis at the same rate.

1.8) Based on the rotation of the earth and the orbit of the moon, how long is a lunar day on earth? That is, from the earth's perspective, how long does the moon take to come back over the same spot on the surface of the earth?

Solution: (6 points) We can apply the coordinate transformation given by $V^s_{ac} = V^s_{ab} + \mathrm{Ad}_{g_{ab}} V^s_{bc}$ and the conversion $V^s_{ab} = -V^b_{ba}$ (Lemma 2.16 in MLS) to yield the spatial velocity of the moon's orbit with respect to the earth. Here we will use the m_o frame that rotates with the moon's orbit to get

$$\begin{aligned} V^s_{em} &= V^s_{es} + \mathrm{Ad}_{g_{es}} V^s_{sm} \\ &= -V^b_{se} + \mathrm{Ad}^{-1}_{g_{se}} V^s_{sm} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{28} \end{bmatrix} \end{aligned}$$

We only care about the last row of this equation, which yields

$$\begin{aligned}
\omega^s_{em,z} &= -1 + \frac{1}{28} \\
&= -\frac{27}{28} \\
T &= \frac{1}{|\omega^s_{em,z}|} = \frac{28}{27}.
\end{aligned}$$

Problem 2) Matlab

The Matlab Robotics System Toolbox provides many of the functions needed to calculate rigid body motions. In particular, there are useful tools for converting between different representations of angles and transformations:
https://www.mathworks.com/help/robotics/coordinate-system-transformations.html
https://www.mathworks.com/help/robotics/ug/coordinate-systems-in-robotics.html
However, it does not have all of the functions we want. Create functions for each of the following operations. You may then call and reuse these functions for other sections of this homework or in future homeworks.

You are welcome to check your answers by running `HW2_Grading_Tests`, it should give a report about all the grading tests.

2.1) In folder `studentDefinedFunctions`, complete the functions `angvel2skew(w)` and `skew2angvel(w)`.

`angvel2skew(w)` is a function that maps the 3-vector w to the 3x3 skew-symmetric matrix $\hat{w}$, and `skew2angvel(w)` is the inverse mapping. Write a unit test script that generates random w and checks that the composition of these functions is identity.

Solution: (6 points) See reference solution if your code doesn't pass the unit test.

2.2) In folder `studentDefinedFunctions`, complete the functions `twist2rbvel(x)` and `rbvel2twist(x)`.

`twist2rbvel(x)` is the function that maps the 6-vector twist ξ to the 4x4 rigid body velocity matrix in homogeneous coordinates, $\hat{\xi}$, and `rbvel2twist(x)` is the inverse mapping. Write a unit test script that generates random ξ and checks that the composition of these functions is identity.

Solution: (6 points) See reference solution if your code doesn't pass the unit test.

2.3) In folder `studentDefinedFunctions`, complete the function `tform2adjoint(g)`.

`tform2adjoint(g)` is the function that maps the rigid body transformation g, in homogeneous coordinates, to the transformation adjoint matrix, Ad_g.

Solution: (6 points) See reference solution if your code doesn't give the correct result or pass the unit test for 2.6.

2.4) In folder `studentDefinedTests`, complete the function `compare_rotm()`.

Generate a random *unit* vector `w` (as `axang2rotm` requires a unit vector for the axis) and rotation amount `t`. Compute the rotation matrix `rotm_matlab` generated with Matlab built-in function `axang2rotm`, and `rotm` generated with your function `angvel2skew` and the Matlab function `expm`.

The function should return `w, t, rotm` and `rotm_matlab`, and `rotm` and `rotm_matlab` should be the same.

Solution: (6 points) See reference solution if your code doesn't pass the unit test.

2.5) In folder `studentDefinedTests`, complete the function `compare_tform()`.

Generate a random *unit* vector `w`, rotation amount `t`, and twist pitch `p`. Using `w` and `t`, compute the rigid body transformation matrix `g_w_matlab` generated with `axang2tform`, and `g_w` generated with your function `twist2rbvel` and the Matlab function `expm`.

Then using a pure translation with a velocity of `v = wp` and the amount `t`, compute the rigid body transformation matrix `g_v_matlab` generated with `trvec2tform` and `g_v` generated with your function `twist2rbvel` and the Matlab function `expm`.

Finally, using `w`, `t`, and `v`, compute the rigid body transformation matrix `g` generated with your function `twist2rbvel` and the Matlab function `expm` and `g_matlab` generated with the composition of `axang2tform` and `trvec2tform`.

The function should return `w, t, p, g_w, g_w_matlab, g_v, g_v_matlab, g` and `g_matlab`.

Solution: (10 points) See reference solution if your code doesn't pass the unit tests.

2.6) In folder `studentDefinedTests`, complete the function `compare_twist()`.

Generate a random transformation `g` as before and a random twist `V`. First, treating `V` as a body velocity, compute the conversion to spatial velocity `V_s_Ad_g` using `tform2adjoint`, and the conversion in homogeneous coordinates `V_s_tform` using `twist2rbvel` and `rbvel2twist`.

Repeat the test the other way, treating `V` as a spatial velocity and converting to body velocity and compute `V_b_Ad_g` and `V_b_tform`.

The function should return `V, g, V_s_Ad_g, V_s_tform, V_b_Ad_g` and `V_b_tform`.

Solution: (6 points) See reference solution if your code doesn't pass the unit tests.