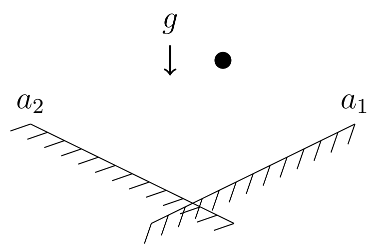
Homework 11: Time-stepping

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solutions

Problem 1) Falling Ball



Consider a point particle that can make plastic frictionless impact with several constraints. Assume the particle is mass 1 and gravity is 9.8. Let the constraints be $a_1(x, y) = 2y - x$ and $a_2(x, y) = 2y + x$.

1.1) Simulate the system with an event-based simulation in Matlab using `ode45` and an event function. You are encouraged to reuse any code that you would like from past homeworks.

Solution: (20 points) See attached code.

1.2) Run a simulation from initial condition $(0.2, 1)$ for 3 seconds. What contact mode transitions occur and at what times? Discuss any problems that arise.

Solution: (20 points) See attached code. The contact mode rapidly fluctuates between $\{1\}$ and $\{2\}$ as the particle loses energy and funnels towards the constraint intersection. With this constraint switching scheme, at each impact some impulse is imparted that breaks contact with the previous constraint. This causes a Zeno paradox, as the particle exponentially loses energy but can never satisfy the conditions to transition into $\{1,2\}$. As the particle loses energy, its post impact velocity approaches zero. Since there is some error in the zero-crossing of the event function, eventually the particle will enter a contact mode very slightly past the constraint ($a(q) = 0 - \epsilon$), without enough post-impact velocity to return to $a(q) = 0$. As a result there are no more impact events, and gravity pulls the particle down the constraint.

1.3) Now simulate the same system with a time-stepping simulation in Matlab using `solve`. At each timestep, pass the full system of equations (including the difference equations for $\dot{q}$ and q as well as all inequality and complementarity constraints) to `solve` in order to find the next $q, \dot{q}$, and λ. The following Matlab resources may help:
https://www.mathworks.com/help/symbolic/solve.html
https://www.mathworks.com/help/symbolic/solve-a-system-of-algebraic-equations.html

Solution: (30 points) See attached code.

1.4) Run a simulation from initial condition $(0.2, 1)$ for 3 seconds with time-step sizes h of $40, 20$, and 10 ms. What contact mode transitions occur and at what times? In time-stepping the "contact mode" can be interpreted as those constraints who provided a positive force or impulse over the time step (as we cannot differentiate between these), though the system may not meet all of the requirements for the domain D_I in the hybrid system sense. Discuss any differences between these and the event-based simulation.

Solution: (30 points) With the above definition of a contact mode, we can identify the contact mode by noting any $U(\lambda) > 0$, and identify contact mode transitions whenever the contact mode changes after resolving the system at the next time step. We could also just plot the $U(\lambda)$ with respect to time and note any times at which $U(\lambda) > 0$ switches. For 40 ms time steps, we get the following contact mode transitions and times:

```
Initialize in mode {}.
Transition from mode {} to mode {1} at t = 0.44.
Transition from mode {1} to mode {1  2} at t = 0.52.
Transition from mode {1  2} to mode {2} at t = 0.56.
Transition from mode {2} to mode {1} at t = 1.04.
Transition from mode {1} to mode {1  2} at t = 1.2.
Terminate in mode {1  2} at t = 3.
```

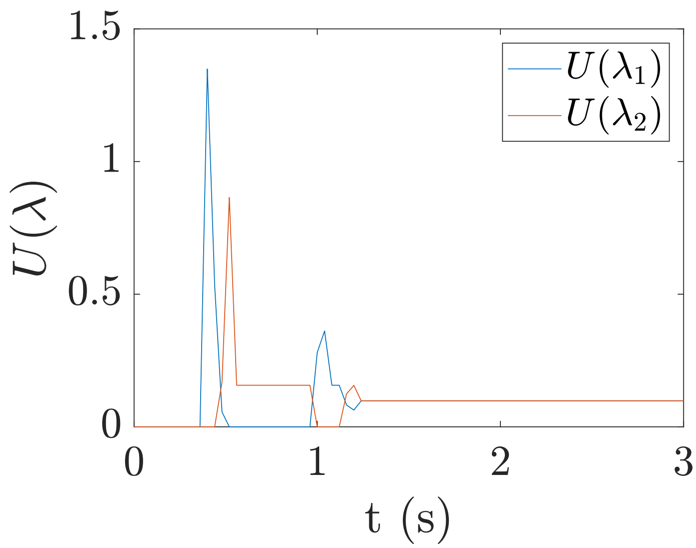


Figure 1: Time-stepping contact mode with 40 ms time step.

For 20 ms time steps, we get:

```
Initialize in mode {}.
Transition from mode {} to mode {1} at t = 0.42.
Transition from mode {1} to mode {2} at t = 0.54.
Transition from mode {2} to mode {1  2} at t = 1.12.
Transition from mode {1  2} to mode {1} at t = 1.14.
```

```
Transition from mode {1} to mode {2} at t = 1.44.
Transition from mode {2} to mode {1  2} at t = 1.56.
Terminate in mode {1  2} at t = 3.
```

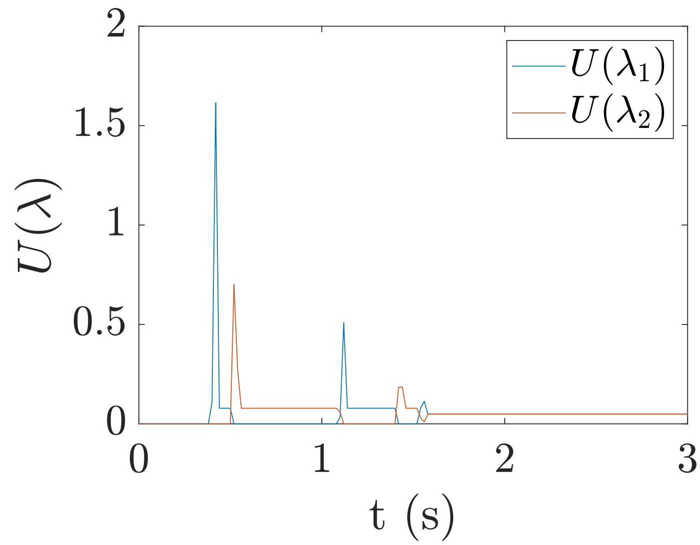


Figure 2: Time-stepping contact mode with 20 ms time step.

And for 10 ms time steps, we get:

```
Initialize in mode {}.
Transition from mode {} to mode {1} at t = 0.43.
Transition from mode {1} to mode {1  2} at t = 0.53.
Transition from mode {1  2} to mode {2} at t = 0.54.
Transition from mode {2} to mode {1} at t = 1.13.
Transition from mode {1} to mode {2} at t = 1.45.
Transition from mode {2} to mode {1} at t = 1.61.
Transition from mode {1} to mode {1  2} at t = 1.68.
Transition from mode {1  2} to mode {2} at t = 1.69.
Transition from mode {2} to mode {1  2} at t = 1.7.
Terminate in mode {1  2} at t = 3.
```

Note that as the time step is reduced, the time-stepping algorithm approaches the Zeno event-based solution, but eventually settles into a steady state in contact mode {1,2}. The event based solution cannot handle this paradox and breaks down. The time-stepping method however does not yield contact modes that fit our standard hybrid system definition - there are transitions from {1,2} to {1} or {2} which would typically be impossible. This arises from our relaxation of the definition of the contact mode to include either impulses of contact forces over a time step.

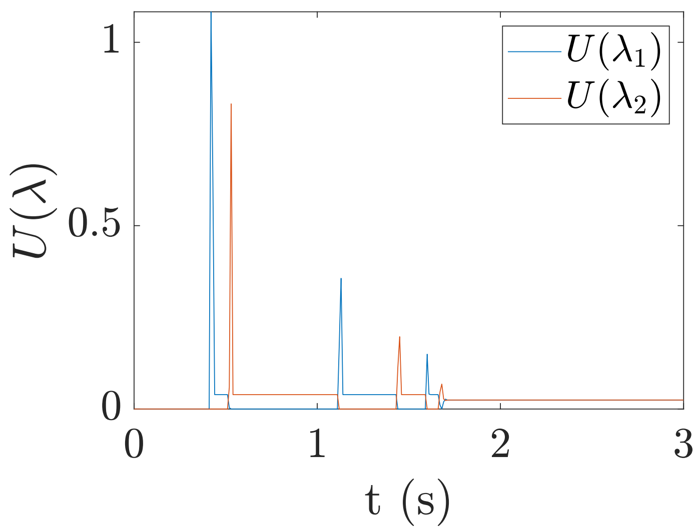


Figure 3: Time-stepping contact mode with 10 ms time step.