Homework 3: Kinematics

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solutions

Problem 1) Two-Link Robot

Consider a planar robot with two revolute joints, with angles q_1 and q_2, and two links of length l_1 and l_2 (Figure 1). The kinematics of this robot was covered in class. In the reading and in class we discussed the spatial and body manipulator Jacobians, J_{st}^s and J_{st}^b. Here we will look at the analytic Jacobian, which we'll call J_{st}^a or simply J^a, which maps joint velocity to the derivative of the local coordinates for the end effector, $[\dot{x}, \dot{y}]^T$, i.e. $[\dot{x}, \dot{y}]^T = J^a \dot{q}$.

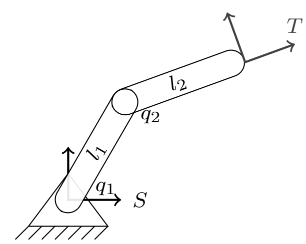


Figure 1: Two link robot.

1.1) Start by writing down the forward kinematic map in local coordinates, $f(q) = [x, y]^T$.

Solution: (5 points) The FK map in local coordinates is

$$f(q) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) \\ l_2 \sin(q_1 + q_2) + l_1 \sin(q_1) \end{bmatrix}$$

1.2) The analytic Jacobian J^a is the Jacobian (or differential) of the function f, sometimes called Df or J_f. Find J^a for our two link robot.

Solution: (10 points) This jacobian is given by

$$J^a = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix} = \begin{bmatrix} -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) & l_2 \cos(q_1 + q_2) \end{bmatrix}$$

1.3) How can we use the analytic Jacobian to move the end effector to move in some desired direction $[\dot{x}^*, \dot{y}^*]^T$?

Solution: (5 points) The analytic Jacobian maps given joint velocities to end effector velocities. We want the inverse of this function, given by

$$\dot{q} = (J^a)^{-1} v = \begin{bmatrix} \frac{\cos(q_1+q_2)}{l_1 \sin(q_2)} & \frac{\sin(q_1+q_2)}{l_1 \sin(q_2)} \\ -\frac{l_2 \cos(q_1+q_2) + l_1 \cos(q_1)}{l_1 l_2 \sin(q_2)} & -\frac{l_2 \sin(q_1+q_2) + l_1 \sin(q_1)}{l_1 l_2 \sin(q_2)} \end{bmatrix} \begin{bmatrix} \dot{x}^* \\ \dot{y}^* \end{bmatrix}$$

1.4) Are there any configurations where the analytic Jacobian has a singularity? What physically do these configurations correspond to?

Solution: (10 points) The analytic Jacobian is singular when its inverse no longer exists - this is clearly the case when $q_2 = k\pi, k \in \mathbb{Z}$. Physically this corresponds to when the two links are aligned. In this configuration, moving either link does not produce an end effector motion normal to the end effector (in the direction of link 2).

1.5) Are the body or spatial Jacobians for this robot singular at these configurations? Can you produce any desired (body or spatial) velocity $[\dot{x}^*, \dot{y}^*]^T$ using the body or spatial Jacobians? What does the rank of the body/spatial Jacobians indicate about the degrees of freedom of the end effector at these locations?

Solution: (10 points) The spatial and body Jacobians for this system are given by

$$J_{st}^s = \begin{bmatrix} 0 & l_1 \sin(q_1) \\ 0 & -l_1 \cos(q_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad J_{st}^b = \begin{bmatrix} l_1 \sin(q_2) & 0 \\ l_2 + l_1 \cos(q_2) & l_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Each of these Jacobians have rank 2 for all q in the indicated configurations and are therefore not singular in any configuration. However not any desired velocity $[\dot{x}^*, \dot{y}^*]^T$ can be produced with either Jacobian (the spatial velocity will be coupled, and the body velocity in the x direction will be unaffected when $q_2 = k\pi, k \in \mathbb{Z}$). Moreover, as both body and spatial Jacobians don't lose rank at the indicated configurations, we know that the end effector still has two degrees of freedom, and the two controllable degrees of freedom are in y and θ. As an additional comment, a two-link planar manipulator will always be unable to produce arbitrary planar motion. To show this more explicitly in our forward kinematics, we could have included θ in addition to x and y, in which case J^a would not have been full rank for any configuration.

Problem 2) ABB IRB 6620

In this problem you will be modeling the kinematics of the ABB IRB 6620, a large industrial robot capable of lifting 150kg[1]. We have included the key datasheet figures on the next page. (One ambiguity in the figure is the location of the 5th axis (E), which is in between the "887" and "200" measurements). A video of the robot in operation can be found here:
https://www.youtube.com/watch?v=LtTTpW_2flw

As always, you may use the functions you defined in the last homework as well as any of the Robotics System Toolbox functions. You will also need to make use of the Symbolic Math Toolbox for some of the calculations, which has some tutorials here if you have not used it before:
https://www.mathworks.com/help/symbolic/getting-started-with-symbolic-math-toolbox.html

Submit solutions as a Matlab script following the given template `HW3_sudent.m` and include all codes in .zip format. Fill in all the `TODO` sections and clearly label sections based on which part they are for. Please use the precise variable names that we define in the template and do not overwrite them in later sections.

Solution: See MATLAB Code, `HW3 Solution.m`

2.1) Begin by writing out by hand the rigid body transformation from the stationary frame to the tool frame in the configuration shown on the next page, `g_st(0)`. Use a stationary base frame with the $x-y$ plane on the floor, the $+x$ axis pointing to the right in the bottom center figure, and the $+z$ axis aligned with Axis 1 in the top figure. Use a tool frame with a $+x$ axis pointing out of the end, the $y-z$ plane flush with the tool plate, and a $+z$ pointing upwards. Submit in the template script that assigns a numerical expression called `gst0`.

Solution: (5 points) The rigid body transformation in the shown configuration is given by:

$$g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & 1407 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1855 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2) Find the general forward kinematics in Matlab. Define a set of rigid body transformations that represent the displacement between each successive link, and include an explanation of how you defined any intermediate frames. The angles should be represented symbolically, e.g. `syms q1 real`. Use these to define $g_{st}(q)$, where $q = [q_1, q_2, q_3, q_4, q_5, q_6]^T$, the forward kinematics, and compare to $g_{st}(0)$ which you found above. Submit in the template script that produces a symbolic expression called `gst`, a numerical expression `gst0_sym` and performs the comparison.

Solution: (10 points) There are multiple ways to define the intermediate frames to achieve this rigid body transformation. One easy way is to maintain the same orientation for each

[1] https://new.abb.com/products/robotics/industrial-robots/irb-6620

frame, and place them immediately after each joint. Another way is to align each $+z$ axis with the axis of rotation for that joint. The full expression for $g_{st}(q)$ is too long to list here (see Matlab solutions script), but can be checked against the above expression for $g_{st}(0)$ and the product of exponentials expression in part 2.3.

2.3) Now use the product of exponentials formula to recalculate the forward kinematics. Does this give the same forward kinematics? Submit in the template script that produces a symbolic expression called `gst_exp` and performs the comparison. (*Hint: You may need to use* `simplify`.)

Solution: (10 points) The twist for each joint is,

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} -680 \\ 0 \\ 320 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} -1655 \\ 0 \\ 320 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \xi_4 = \begin{bmatrix} 0 \\ 1855 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_5 = \begin{bmatrix} -1855 \\ 0 \\ 1207 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \xi_6 = \begin{bmatrix} 0 \\ 1855 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and can be used to compute the rigid body transformation from the stationary frame to the tool frame with,

$$g_{st}(q) = e^{\hat{\xi}_1 q_1} e^{\hat{\xi}_2 q_2} e^{\hat{\xi}_3 q_3} e^{\hat{\xi}_4 q_4} e^{\hat{\xi}_5 q_5} e^{\hat{\xi}_6 q_6} g_{st}(0)$$

This expression should match the $g_{st}(q)$ found in the previous problem.

2.4) Solve the inverse kinematics problem to find a configuration of the robot where the end effector is located $100mm$ in the $+y$ direction from the home configuration, with the same orientation. To do this, define an optimization problem using your forward kinematic map and solve for q. You do not need to consider joint limits or self-collision. Verify the solution by passing it into the forward kinematics $g_{st}(q)$ and comparing to the desired configuration with a tolerance of 0.1. Submit in the template script that produces a numerical vector called `q_sol` and performs the comparison. (*Hint: You may need to use* `matlabFunction` *and* `fminunc`)

Solution: (10 points) To solve the inverse kinematics problem with an optimization algorithm, we must define a scalar cost function that is zero if and only if the joint angles result in the desired end effector position. An example of this would be

$$\text{cost}(q) = \text{sum}(|g_{st}(q) - g^*_{st}|)$$

where g^*_{st} is the desired configuration. Another method to avoid the discontinuous absolute value function would be to square each element of the difference. If 'cost' is a MATLAB function, the IK problem can be solved in MATLAB with `qDes = fminunc(cost, zeros(6,1))`. To confirm that the algorithm worked, check that

$$g_{st}(\texttt{qDes}) = \begin{bmatrix} 1 & 0 & 0 & 1407 \\ 0 & 1 & 0 & 100 \\ 0 & 0 & 1 & 1855 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If each joint frame is defined with the $+z$ axis aligned with the axis of rotation, the desired joint angles to achieve this configuration are

$$\begin{bmatrix} 0.071 \\ 0.00361 \\ -0.00345 \\ -2.07\,10^{-9} \\ -9.01\,10^{-4} \\ 1.97\,10^{-10} \end{bmatrix}$$

2.5) Compute the spatial Jacobian, J^s_{st}, by calculating $\left(\frac{\partial g_{st}}{\partial q_i} g_{st}^{-1}\right)^\vee$ for each joint using the `diff` function. Submit in the template script that produces a symbolic expression called `Js`.

Solution: (10 points) The spatial Jacobian can be calculated directly through

$$J^s = \left[\left(\frac{\partial g_{st}}{\partial q_1} g_{st}^{-1}\right)^\vee \quad \left(\frac{\partial g_{st}}{\partial q_2} g_{st}^{-1}\right)^\vee \quad \left(\frac{\partial g_{st}}{\partial q_3} g_{st}^{-1}\right)^\vee \quad \left(\frac{\partial g_{st}}{\partial q_4} g_{st}^{-1}\right)^\vee \quad \left(\frac{\partial g_{st}}{\partial q_5} g_{st}^{-1}\right)^\vee \quad \left(\frac{\partial g_{st}}{\partial q_6} g_{st}^{-1}\right)^\vee \right]$$

2.6) Compare that Jacobian to one generated using the exponential twists. Submit in the template script that produces a symbolic expression called `Js_exp` and performs the comparison.

Solution: (10 points) Again, the spatial Jacobian can be calculated directly through

$$\begin{aligned} J^s_{st}(q) &= [\xi_1 \quad \xi_2' \quad \xi_3' \quad \xi_4' \quad \xi_5' \quad \xi_6'] \\ \xi_i' &= \mathrm{Ad}_{\left(e^{\hat{\xi}_1 q_1} \ldots e^{\hat{\xi}_{i-1} q_{i-1}}\right)} \xi_i \end{aligned}$$

where ξ_i is the twist associated with the ith joint, as in 2.3. Confirm that this is the same as the Jacobian found in 2.5.

2.7) When the robot is in the initial configuration, what body and spatial velocities correspond to moving the end effector at unit velocity in the tool's $+y$ direction (without rotation)? Can we move the joints in some direction $\dot{q}$ to achieve this velocity? Submit in the template script that produces numerical expressions called `Vb` and `Vs` and performs the singularity check.

Solution: (5 points) We can compute the body and spatial velocity corresponding to this motion by inspection or computation to be

$$V^s_{st} = V^b_{st} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

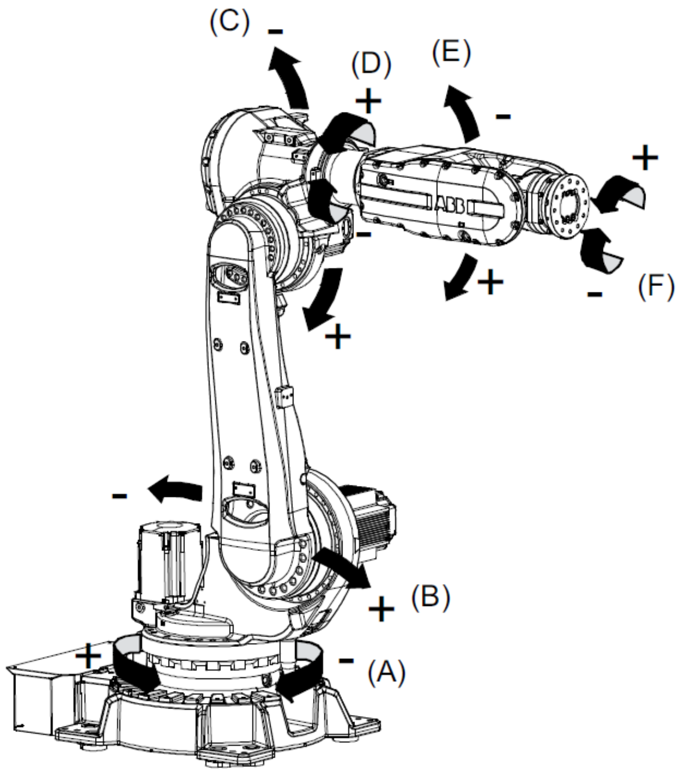
These are the same since the orientation of the tool and stationary frame are the same and the motion is purely translation. The spatial Jacobian in this initial configuration is

$$J_{st}^s = \begin{bmatrix} 0 & -680 & -1655 & 0 & -1855 & 0 \\ 0 & 0 & 0 & 1855 & 0 & 1855 \\ 0 & 320 & 320 & 0 & 1207 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We seek joint velocities $\dot{q}$ such that $V_{st}^s = J_{st}^s \dot{q}$. However, the Jacobian is singular in this configuration, as rotation of the 4th and 6th joints yield the same end effector motion, such that there is no $\dot{q}$ that satisfies this equation (note that these are the only two with a y component but no combination of these or other columns will result in a y velocity with no other motion coupled in). This can be physically seen in the image of the robot, as there is no combination of joint motion that will translate the end effector in the $+y$ direction without rotation.

Manipulator axes

IRB 6620



xx0900000454

Pos	Description	Pos	Description
A	Axis 1	B	Axis 2
C	Axis 3	D	Axis 4
E	Axis 5	F	Axis 6

Dimensions IRB 6620