Homework 9: Hybrid Systems

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solutions

Please turn in a PDF with the answers to the following questions.

Problem 1) Falling Block

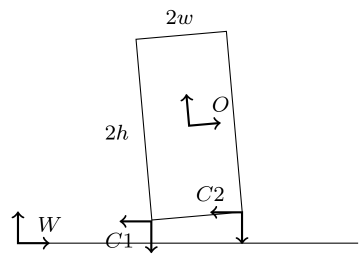

Figure 1: A block.

Consider a planar, rectangular block as shown above and considered in the last homework. The block has mass m, width $2w$, and height $2h$. Assume a tall block, where $h > 2w$. The state of the block in local coordinates is $q = [x, y, \theta]^T$ where each coordinate is expressed relative to the W frame. The gravity vector points in the $-y$ direction in the W frame, and there are no other applied wrenches or friction. There are two contact points on the bottom corners of the block, C_1 and C_2 that can make frictionless contact. Their position constraints are,

$$a_1(q) = y - h\,\cos(\theta) - w\,\sin(\theta)$$
$$a_2(q) = y - h\,\cos(\theta) + w\,\sin(\theta)$$

1.1) What are the possible contact modes, $\mathcal{J}$? Assume the block doesn't tip over (i.e. the only two possible contacts are at C_1 and C_2).

Solution: (5 points) The block can either be in the air, tilted on $C1$, tilted on $C2$, or resting on both $C1$ and $C2$. This gives the set:

$$\mathcal{J} = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$$

1.2) What is the domain, D_I, of each contact mode?

Solution: (10 points) The term $a_i(q)$ represents the position of the contact point with respect to the surface. When the contact point is above the surface, we write $a_i(q) \geq 0$. When the contact point is contacting the surface, we write $a_i(q) = 0$. The term $A_i\dot{q}$ represents the velocity of the contact point with respect to the surface. When the contact point is contacting the surface, we write $A_i\dot{q} = 0$. Using these terms, the domain for each contact mode can be solved for as seen below:

$$\mathcal{D} = \left\{ \begin{array}{l} D_{\{\}} = \{(q,\dot{q}) : a_1(q) \geq 0, a_2(q) \geq 0\} \\ D_{\{1\}} = \{(q,\dot{q}) : a_1(q) = 0, A_1\dot{q} = 0, a_2(q) \geq 0\} \\ D_{\{2\}} = \{(q,\dot{q}) : a_1(q) \geq 0, a_2(q) = 0, A_2\dot{q} = 0\} \\ D_{\{1,2\}} = \{(q,\dot{q}) : a_1(q) = 0, A_1\dot{q} = 0, a_2(q) = 0, A_2\dot{q} = 0\} \end{array} \right\}$$

1.3) Write down the flow for the system, $\mathcal{F}$, i.e. the dynamics of the system in all possible contact modes. Please specify the matrices M, C, N, A, and Υ for the unconstrained contact mode $\{\}$ (i.e. neither C_1 or C_2 touching the ground), then for all other contact modes write down the updated version of any matrices that change. Feel free to use your solutions from HW8 as a start.

Solution: (20 points) The unconstrained mode occurs when the system is in mode $\{\}$. This means there are no contact constraint forces acting on the system, and the only force acting on the system is gravity. Thus, there will be no $A^T\lambda$ term, and Υ will be a zero matrix. Using this information, we can solve for the desired matrices:

$$M\ddot{q} + C\dot{q} + N = \Upsilon$$

Where:

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{3}(h^2 + w^2) \end{bmatrix}$$

$$C = \mathbf{0_{3x3}}$$

$$N = \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

$$\Upsilon = \mathbf{0_{3x1}}$$

The M matrix is solved for using the mass of the block m and the moment of inertial I_o about the blocks center. Using the dimensions of the block, I_o is equal to $(m/12)((2h)^2 + (2w)^2)$, which can then be simplified to $(m/3)(h^2 + w^2)$.

Because the C matrix is solved for by taking partial derivatives of the M matrix with respect to the state variables in the local coordinates q, we know that the C matrix is a zero matrix as M is only dependent on m, h, and w.

The N matrix is solved for by taking the partial derivatives of the potential energy of the system V with respect to the state variables in the local coordinates q. Because the block is just a mass, we know that $V = mgy$. This means that the potential energy of the block is only dependent on the y position of the block, which intuitively makes sense for this system.

For the constrained modes, each of the preceding matrices remains the same, but we now add our $A^T\lambda$ term to account for our generalized constraint forces.

$$M\ddot{q} + C\dot{q} + N + A^T\lambda = \Upsilon$$

We can find the appropriate A matrix by taking the differential of the active constraint(s) in each contact mode.

For contact mode $\{1\}$:

$$\begin{aligned} A &= Da_1 \\ &= [0, 1, h\sin(\theta) - w\cos(\theta)] \end{aligned}$$

For contact mode $\{2\}$:

$$\begin{aligned} A &= Da_2 \\ &= [0, 1, h\sin(\theta) + w\cos(\theta)] \end{aligned}$$

For contact mode $\{1, 2\}$:

$$\begin{aligned} A &= \begin{bmatrix} Da_1 \\ Da_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & h\sin(\theta) - w\cos(\theta) \\ 0 & 1 & h\sin(\theta) + w\cos(\theta) \end{bmatrix} \end{aligned}$$

This is the same way we solved for the A matrix in HW8. The only difference in this problem is that we are considering the block in different contact modes other than {1,2}.

1.4) What are the feasible transitions between contact modes, $\tilde{\Gamma}$, based on the dynamics?

Solution: (15 points) The feasible transitions $\tilde{\Gamma}$ are given by

$$\tilde{\Gamma} = \{(\{\}, \{1\}), (\{\}, \{2\}), (\{\}, \{1, 2\}), (\{1\}, \{2\}), (\{2\}, \{1\}).$$

Because $h > 2w$, we do not have ({1}, {1,2}) or ({2}, {1,2}) contained in the set above. As we saw in the solution to HW8, the transition ({1}, {1,2}) was not possible as it yielded invalid impulses. In this case, we have $h > 2w$, meaning the block is taller than that in HW8, meaning the transitions ({1}, {1,2}) and ({2}, {1,2}) will not be valid for this problem either.

1.5) For each transition above, what are the corresponding guard conditions?

Solution: (20 points) The guard conditions represent the set of conditions that cause the contact modes to change. For this to occur, $a_i(q) = 0$ and $A_i\dot{q} \prec 0$. Respectively these state that contact mode i is contacting the surface and that the contact mode is moving towards the surface when this occurs.

$$\mathcal{G} = \left\{ \begin{array}{l} G_{\{\},1} = \{q, \dot{q} \in D_{\{\}} : a_1(q) = 0, A_1\dot{q} \prec 0\} \\ G_{\{\},2} = \{q, \dot{q} \in D_{\{\}} : a_2(q) = 0, A_2\dot{q} \prec 0\} \\ G_{\{\},\{1,2\}} = \{q, \dot{q} \in D_{\{\}} : a_1(q) = 0, A_1\dot{q} \prec 0, a_2(q) = 0, A_2\dot{q} \prec 0\} \\ G_{1,2} = \{q, \dot{q} \in D_1 : a_2(q) = 0, A_2\dot{q} \prec 0\} \\ G_{2,1} = \{q, \dot{q} \in D_2 : a_1(q) = 0, A_1\dot{q} \prec 0\} \end{array} \right\}$$

1.6) For each transition above, what is the corresponding reset map?

Solution: (15 points) The reset map calculates the new velocity the object has once is transitions into a new contact mode. To do this, we can use the formula $\dot{q}^+ = \dot{q}^- - A_J^{\dagger^T}(q)A_J(q)\dot{q}^-$, which calculates $\dot{q}^+$ (the post-impact velocity) using $\dot{q}^-$ (the pre-impact velocity).

$$\mathcal{R} = \left\{ \begin{array}{l} R_{\{\},1}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_1 \\ R_{\{\},2}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_2 \\ R_{\{\},\{1,2\}}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_{\{1,2\}} \\ R_{1,2}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_2 \\ R_{2,1}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_1 \end{array} \right\}, \qquad \dot{q}^+ = \dot{q}^- - A_J^{\dagger^T}(q)A_J(q)\dot{q}^-$$

1.7) Now, let's drop the assumption that $h > 2w$, and for this problem only assume that $h > w$. Is the transition from $\{1\}$ to $\{1,2\}$ achievable? Is the transition from $\{2\}$ to $\{1,2\}$ achievable? That is, will it ever reach a state where it comes to rest?

Solution: (15 points)

Because we are now using the assumption that $h > w$, we need to check if the block can transition from $\{1\}$ to $\{1,2\}$. So, we need to solve for the maximum height the block can be for this transition to be achievable, then see if it satisfies the $h > w$ assumption.

For this question, we are considering transitioning into $\{1,2\}$ and we assume that both constraints will be active. Therefore, we will use A as:

$$A = \begin{bmatrix} 0 & 1 & h\sin(\theta) - w\cos(\theta) \\ 0 & 1 & h\sin(\theta) + w\cos(\theta) \end{bmatrix}$$

We can symbolically compute $\dot{q}^+$ and $\hat{P}$ using the equation:

$$\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix}$$

First, consider the transition from {1} to {1,2}. In order for the block to be in a position to contact both C_1 and C_2, θ must be equal to 0. Also, $\dot{\theta}$ must be negative in order to impact at C_2. Using this knowledge, we can find $\dot{y}$ in terms of $\dot{\theta}$.

$$\dot{y} = w\dot{\theta}$$

We can substitute these relationships into our equation for $\dot{q}^+$ and $\hat{P}$, and find the following result:

$$\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} \dot{x}^- \\ 0 \\ 0 \\ -\frac{m\dot{\theta}^-}{6w}(h^2 - 2w^2) \\ \frac{m\dot{\theta}^-}{6w}(h^2 + 4w^2) \end{bmatrix}$$

Focusing on the terms for $\hat{P}$, and knowing that $\dot{\theta}$ must be negative to have contact at C_2, we can see that impulse at C_2 must be negative, meaning that the ground is exerting a positive force on the block. Looking at the impulse at C_1, we see that the sign on the impulse depends on h and w. In order for the impulse to not violate our assumptions, the following must be true:

$$\begin{aligned} h^2 - 2w^2 &< 0 \\ h &< \sqrt{2}w \end{aligned}$$

We can use the equation to determine when a transition from $\{2\}$ to $\{1, 2\}$ is possible. Here, we can make the same assumption of θ equal to 0. However, here $\dot{\theta}$ must be positive for this contact to occur. This will also change our $\dot{y}$ to be as follows:

$$\dot{y} = -w\dot{\theta}$$

Substituting these relationships into our equation for $\dot{q}^+$ and $\hat{P}$, we find the following result:

$$\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} \dot{x}^- \\ 0 \\ 0 \\ -\frac{m\dot{\theta}^-}{6w}(h^2 + 4w^2) \\ \frac{m\dot{\theta}^-}{6w}(h^2 - 2w^2) \end{bmatrix}$$

Focusing on the terms for $\hat{P}$, and knowing that $\dot{\theta}$ must be positive to have contact at C_1, we can see that impulse at C_1 must be negative, meaning that the ground is exerting a positive force on the block. Looking at the impulse at C_2, we see that the sign on the impulse depends on h and w, just as before. In order for the impulse to not violate our assumptions, we find the same relationship as before, thus the following must be true:

$$h < \sqrt{2}w$$

In summary, the transitions from $\{1\}$ to $\{1, 2\}$ and $\{2\}$ to $\{1, 2\}$ are achievable if $h < \sqrt{2}w$. Otherwise, the system will alternate between $\{1\}$ and $\{2\}$ an infinite number of times (Zeno).