Homework 6: Lagrangian Dynamics

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solutions

For this homework, please compose everything in a single Matlab script, except helper functions. In the main script, you need to include all reasoning (either type it or insert a picture of your hand written result), calculation, and required output (with the same output variable names given in problem statements). Please fill in all the `TODO` sections and clearly label sections based on which part they are for. Please use the precise variable names that we define in the template and do not overwrite them in later sections. If you used any helper functions, please put them together with the main script in a zip file named as andrewID_24760_HW6.zip, where andrewID is your Andrew ID.

Hint: Look at the Matlab functions `diff`, `gradient`, and `jacobian`.

Problem 1) Unconstrained Lagrangian

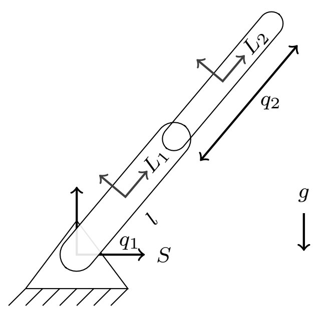

Figure 1: Two link robot. For each frame, the label indicates the x-axis.

Consider the dynamics of the two link robot shown in Figure 2, with one rotational joint and one prismatic joint in the plane. Each link is a rod of length l and mass m with uniform mass distribution and center of mass (COM) at frame L_i. The second joint, q_2, extends the second link from length 0 (fully retracted) to length l (fully extended).

For this question we will consider the generalized coordinates, $q_g = [q_1, q_2]^T$, with no constraints. In the next question we will consider maximal coordinates with constraints. We use subscript g and m to distinguish them (i.e. L_g and L_m). *(HINT: It may be beneficial to create reusable functions that perform Lagrange's equation to differentiate L to get the EOM, or that calculates C from M, etc).*

1.1) What is the kinematic energy $T_g(q_1, q_2, \dot{q}_1, \dot{q}_2)$, potential energy $V_g(q_1, q_2)$, and Lagrangian in generalized coordinates, $L_g(q_1, q_2, \dot{q}_1, \dot{q}_2)$?

Please compute and save them in the symbolic variables `T_g`, `V_g`, and `L_g` respectively in the script.

Solution: (10 points) In order to compute the Lagrangian in generalized coordinates, we must express the kinetic and potential energy as functions of these coordinates. The kinetic energy is described by the linear velocity of the COM of each link and the angular velocity of each link (noting that in these generalized coordinates the angular velocity of each link is the same), by the equation

$$\begin{aligned}T_g &= \frac{1}{2}mv_{sL_1}^{b^2} + \frac{1}{2}mv_{sL_2}^{b^2} + \frac{1}{2}I_o\dot{q}_1^2 + \frac{1}{2}I_o\dot{q}_1^2 \\ &= \frac{1}{2}m\left(\frac{l}{2}\dot{q}_1\right)^2 + \frac{1}{2}m\left(\left(\left(\frac{l}{2}+q_2\right)\dot{q}_1\right)^2 + \dot{q}_2^2\right) + \frac{ml^2\dot{q}_1^2}{12} \\ &= \frac{m}{2}\left(\dot{q}_2^2 + \dot{q}_1^2\left(\frac{l}{2}+q_2\right)^2\right) + \frac{5}{24}ml^2\dot{q}_1^2\end{aligned}$$

The potential energy of the system is defined by the y component of the COM of each link expressed in the S frame, which is given by

$$\begin{aligned}V_g &= mg\frac{l}{2}\sin q_1 + mg\left(\frac{l}{2}+q_2\right)\sin q_1 \\ &= mg\left(l+q_2\right)\sin q_1\end{aligned}$$

We can combine these to yield the Lagrangian for the system,

$$\begin{aligned}L_g &= T_g - V_g \\ &= \frac{m}{2}\left(\dot{q}_2^2 + \dot{q}_1^2\left(\frac{l}{2}+q_2\right)^2\right) + \frac{5}{24}ml^2\dot{q}_1^2 - mg\left(l+q_2\right)\sin q_1\end{aligned}$$

1.2) Compute the applied forces to each generalized coordinate Υ_g and the dynamic equations of motion by using the Lagrange equations to differentiate the Lagrangian in generalized coordinates.

Please compute and save them in the symbolic variables `Y_g` and `EOM_g1` in the script. `EOM_g1` should be a 2 by 1 symbolic matrix that is equal to [0; 0]. It is obtained by subtracting Υ_g on both sides of the equations.

Solution: (10 points) The Lagrange equations are given by

$$\frac{d}{dt}\left(\frac{\partial L_g}{\partial \dot{q}_i}\right) - \frac{\partial L_g}{\partial q_i} = \Upsilon$$

We can use the above expression for the Lagrangian and apply the Lagrange equations to each of our two coordinates. For q_1, we obtain

$$\begin{aligned}\frac{\partial L_g}{\partial \dot{q}_1} &= \frac{5}{12}ml^2\dot{q}_1 + m\dot{q}_1\left(\frac{l}{2}+q_2\right)^2 \\ \frac{d}{dt}\left(\frac{\partial L_g}{\partial \dot{q}_1}\right) &= \frac{5}{12}ml^2\ddot{q}_1 + m\ddot{q}_1\left(\frac{l}{2}+q_2\right)^2 + 2m\dot{q}_1\dot{q}_2\left(\frac{l}{2}+q_2\right) \\ \frac{\partial L_g}{\partial q_1} &= -mg(l+q_2)\cos q_1\end{aligned}$$

Similarly for q_2, we can write

$$\begin{aligned}
\frac{\partial L_g}{\partial \dot{q}_2} &= m\dot{q}_2 \\
\frac{d}{dt}\left(\frac{\partial L_g}{\partial \dot{q}_2}\right) &= m\ddot{q}_2 \\
\frac{\partial L_g}{\partial q_2} &= m\dot{q}_1^2\left(\frac{l}{2}+q_2\right) - mg\sin q_1
\end{aligned}$$

We can also write that the applied forces to each generalized coordinate are

$$\Upsilon = \begin{bmatrix} \tau_1 \\ F_2 \end{bmatrix}$$

where τ_1 is the torque on joint 1 and F_2 is the force on joint 2. Substituting each of these expressions in to the Lagrange equation yields the equations of motion of the system

$$\left(\frac{5}{12}l^2 + \left(\frac{l}{2}+q_2\right)^2\right)m\ddot{q}_1 + 2m\dot{q}_1\dot{q}_2\left(\frac{l}{2}+q_2\right) + mg(l+q_2)\cos q_1 = \tau_1 \tag{1}$$

$$m\ddot{q}_2 - m\dot{q}_1^2\left(\frac{l}{2}+q_2\right) + mg\sin q_1 = F_2 \tag{2}$$

1.3) Re-compute the dynamic equations of motion by directly computing the M_g, C_g, N_g, and Υ_g matrices in the manipulator equation,

$$M_g(q_g)\ddot{q}_g + C_g(q_g,\dot{q}_g)\dot{q}_g + N_g(q_g,\dot{q}_g) = \Upsilon_g$$

Check that you get the same answer as Problem 1.2.

Please compute and save them in the symbolic variables `M_g`, `C_g`, `N_g`, and `EOM_g2` in the script. `EOM_g2` should be a 2 by 1 symbolic matrix that is equal to [0; 0] and should be the same as `EOM_g1` from Problem 1.2.

Solution: (15 points) Rather than bothering with all these partial and time derivatives, we can take advantage of our kinematic formulism for manipulators. In particular, we can calculate the mass matrix, M_g, by noting that

$$M_g = \sum_i J_{sL_i}^{b^T} M_i J_{sL_i}^{b}$$

The generalized inertia matrix, M_i is in a convenient form since the principle axes of each link are aligned with the L_i frame to yield

$$M_1 = M_2 = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$

The body Jacobians for each link can be calculated with previous methods in this course or by inspection to yield

$$J^b_{sL_1} = \begin{bmatrix} 0 & 0 \\ \frac{l}{2} & 0 \\ 1 & 0 \end{bmatrix} \qquad\qquad J^b_{sL_2} = \begin{bmatrix} 0 & 1 \\ \frac{l}{2} + q_2 & 0 \\ 1 & 0 \end{bmatrix}$$

Combining these terms yields

$$M_g = \begin{bmatrix} \frac{5ml^2}{12} + m\left(\frac{l}{2} + q_2\right)^2 & 0 \\ 0 & m \end{bmatrix}$$

This expression matches intuition that the inertia seen by coordinate 1 increases with q_2. We can use this matrix to calculate the Coriolis terms with,

$$C_{ij} = \frac{1}{2}\sum_k \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_k \qquad C_g = \begin{bmatrix} \dot{q}_2\, m\left(\frac{l}{2} + q_2\right) & \dot{q}_1\, m\left(\frac{l}{2} + q_2\right) \\ -\dot{q}_1\, m\left(\frac{l}{2} + q_2\right) & 0 \end{bmatrix}$$

The only nonlinear terms present here are those due to gravity. These can be calculated by

$$N_g(q, \dot{q}) = \frac{\partial V}{\partial q} = \begin{bmatrix} mg\,(l + q_2)\cos(q_1) \\ mg\,\sin(q_1) \end{bmatrix}$$

Problem 2) Constrained Lagrangian

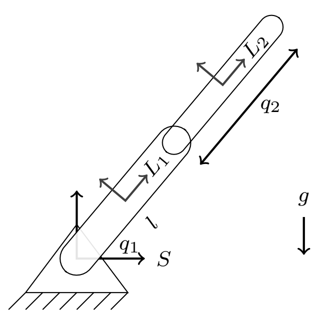

Figure 2: Two link robot. For each frame, the label indicates the x-axis.

In the last question we considered the generalized coordinates, $q_g = [q_1, q_2]^T$, with no constraints. In this question we consider maximal coordinates with constraints.

Maximal coordinates represent the full position and orientation of each link. For each link, use local coordinates for the link frame at the COM, namely L_1 is at (x_1, y_1, ϕ_1) and L_2 is at (x_2, y_2, ϕ_2) with ϕ_i measured counter-clockwise from the S frame. The new combined state is $q_m = [x_1, y_1, \phi_1, x_2, y_2, \phi_2]^T$.

2.1) What position and velocity constraints are there on the system, $a_m(q_m)$ and $A_m\dot{q}_m$?

Please compute and save them in the symbolic variables `a_m`, `A_m` respectively in the script.

Solution: (10 points) We know that the system has two degrees of freedom, so if we choose maximal coordinates to represent the configuration, there must be four constraints (three maximal coordinates for each link means six coordinates in total but only two degrees of freedom). Therefore we know that $a_m(q_m) \in \mathbb{R}^4$. There are a number of ways to write $a_m(q_m)$, but the key is to identify motions that are constrained as a result of the given system (and in particular the nature of the joints). We seek to represent those constraints on the motion as functions of the maximal coordinates such that this function is zero when the constraint is satisfied. An example set of these constraint functions is

$$a_m(q_m) = \begin{bmatrix} x_1 - \frac{L\,\cos(\phi_1)}{2} \\ y_1 - \frac{L\,\sin(\phi_1)}{2} \\ y_2\,\cos(\phi_2) - x_2\,\sin(\phi_2) \\ \phi_2 - \phi_1 \end{bmatrix}$$

From these constraint functions we can compute A_m, the velocity constraint matrix. This matrix defines the velocity constraints $A_m\dot{q}_m = 0$, and is computed by taking the differential of a_m with respect to q_m. This yields

$$A_m = \frac{\partial a_m}{\partial q_m} = \begin{bmatrix} 1 & 0 & \frac{l\,\sin(\phi_1)}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{l\,\cos(\phi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(\phi_2) & \cos(\phi_2) & -x_2\,\cos(\phi_2) - y_2\,\sin(\phi_2) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

A_m has four rows corresponding to each constraint, and six columns corresponding to each maximal coordinate.

2.2) What is the kinematic energy $T_m(q_m, \dot{q}_m)$, potential energy $V_m(q_m)$, and Lagrangian in maximal coordinates, $L_m(q_m, \dot{q}_m)$?

Please compute and save them in the symbolic variables `T_m`, `V_m`, and `L_m` respectively in the script.

Solution: (10 points) Maximal coordinates are very conducive to writing Lagrangians, because both the kinetic and potential energies can be directly written with the coordinates and do not have to be computed. The Lagrangian for this system is

$$L_m(q_m, \dot{q}_m) = \frac{m\,(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2)}{2} + \frac{ml^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right)}{24} - mg\,(y_1 + y_2)$$

2.3) The actuator effort is harder to represent in maximal coordinates. For each link, consider the force or torque applied to a frame at the end of the link (with equal and opposite signs for the joint effort between the links). Calculate Υ_m, the resulting applied force on the two links in maximal coordinates.

Please compute and save it in the symbolic variable `Y_m` in the script.

Solution: (10 points) The actuator forces acting on the system come from a motor that exerts torque between the world and link 1, and a linear actuator between link 1 and link

2. The applied torque τ would not affect any coordinate but ϕ_1, since pure torques do not affect translation, and also since link 2 is not directly attached to link 1 (except by constraint forces, but those aren't considered in Υ). The force F will affect x_1, y_1, x_2, and y_2 because the actuator exerts force on both links. This yields the following expression for Υ:

$$\Upsilon_m = \begin{bmatrix} -F\,\cos(\phi_1) \\ -F\,\sin(\phi_1) \\ \tau \\ F\,\cos(\phi_1) \\ F\,\sin(\phi_1) \\ 0 \end{bmatrix}$$

2.4) Compute the dynamic equations of motion by using the constrained Lagrange equations to differentiate the Lagrangian in maximal coordinates.

Please compute and save it in the symbolic variable `EOM_m1` in the script. `EOM_m1` should be a 6 by 1 symbolic matrix that is equal to [0; 0; 0; 0; 0; 0]. It is obtained by subtracting Υ_m on both sides of the equations.

Solution: (10 points) Using the same methods as in Problem 1.2, we can compute the equations of motion directly from the Lagrangian to obtain

$$m\ddot{x}_1 + \lambda_1 = -F\cos\phi_1 \tag{3}$$
$$m\ddot{y}_1 + mg + \lambda_2 = -F\sin\phi_1 \tag{4}$$
$$\frac{1}{12}ml^2\ddot{\phi}_1 + \frac{l}{2}\sin\phi_1\lambda_1 - \frac{l}{2}\cos\phi_1\lambda_2 - \lambda_4 = \tau \tag{5}$$
$$m\ddot{x}_2 - \sin\phi_1\lambda_3 = F\cos\phi_1 \tag{6}$$
$$m\ddot{y}_2 + mg + \cos\phi_1\lambda_3 = F\sin\phi_1 \tag{7}$$
$$\frac{1}{12}ml^2\ddot{\phi}_2 - x_2\cos\phi_2\lambda_3 - y_2\sin\phi_2\lambda_3 + \lambda_4 = 0 \tag{8}$$

2.5) Re-compute the dynamic equations of motion by directly computing the M_m, C_m, N_m, and Υ_m matrices in the constrained manipulator equation,

$$M_m(q_m)\ddot{q}_m + C_m(q_m,\dot{q}_m)\dot{q}_m + N_m(q_m,\dot{q}_m) + A^T(q_m)\lambda = \Upsilon_m$$

Check that you get the same answer as Problem 2.4.

Please compute and save them in the symbolic variables `M_m`, `C_m`, `N_m`, and `EOM_m2` in the script. `EOM_m2` should be a 6 by 1 symbolic matrix that is equal to [0; 0; 0; 0; 0; 0] and should be the same as `EOM_m1` from Problem 2.4.

Solution: (15 points) We can compute each term in the manipulator equation in exactly the same way we did in Problem 1.3, except the body Jacobian for each link will be different. In particular, we want these Jacobians to map from our local coordinates $\dot{q}_m$ into body velocities $V^b_{sL_i} = J^b_{sL_i}\dot{q}_m$. These velocities would be the same except $\dot{q}_m$ is expressed in the global frame and $V^b_{sL_i}$ is expressed in each link's body frame, so $J^b_{sL_i}$ must be a rotation matrix that rotates vectors from the global frame into the link frame. This is given by

$$J^b_{sL_1} = \begin{bmatrix} \cos(\phi_1) & \sin(\phi_1) & 0 & 0 & 0 & 0 \\ -\sin(\phi_1) & \cos(\phi_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad J^b_{sL_2} = \begin{bmatrix} 0 & 0 & 0 & \cos(\phi_2) & \sin(\phi_2) & 0 \\ 0 & 0 & 0 & -\sin(\phi_2) & \cos(\phi_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

M_m can then be calculated in the same way as in Problem 1, as can C. N_m is calculated in the same way but with a simpler expression for the potential energy, $V = mg(y_1 + y_2)$. Performing these calculations yields the following matrices, which renders the manipulator equation equivalent to Eq. (1) - (6).

$$M_m = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{ml^2}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{ml^2}{12} \end{bmatrix} \quad C_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad N_m = \begin{bmatrix} 0 \\ mg \\ 0 \\ 0 \\ mg \\ 0 \end{bmatrix}$$

2.6) What are the constraint forces?

Please compute and save it in the symbolic variable `lambdaVec` in the script.

Solution: (10 points) The constraint forces can be solved for directly with the following equation (assuming the mass matrix is invertible which is the case for this system's maximal coordinates)

$$\lambda = \left(A_m M_m^{-1} A_m^T\right)^{-1} \left(A_m M_m^{-1} (\Upsilon_m - C_m \dot{q}_m - N_m) + \dot{A}_m \dot{q}_m\right)$$

The only term in this equation we have not yet found is $\dot{A}_m$, which is just the time derivative of A. This can be found a number of ways; the most amenable to MATLAB implementation uses the chain rule:

$$\begin{aligned}\dot{A}_m &= \frac{\partial A_m}{\partial x_1}\dot{x}_1 + \frac{\partial A_m}{\partial y_1}\dot{y}_1 + \frac{\partial A_m}{\partial \phi_1}\dot{\phi}_1 + \frac{\partial A_m}{\partial x_2}\dot{x}_2 + \frac{\partial A_m}{\partial y_2}\dot{y}_2 + \frac{\partial A_m}{\partial \phi_2}\dot{\phi}_2 \\ &= \begin{bmatrix} 0 & 0 & \frac{\dot{\phi}_1 l \cos(\phi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\dot{\phi}_1 l \sin(\phi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dot{\phi}_2 \cos(\phi_2) & -\dot{\phi}_2 \sin(\phi_2) & \begin{matrix} -\dot{\phi}_2 (y_2 \cos(\phi_2) - x_2 \sin(\phi_2)) + \dots \\ \dots - \dot{x}_2 \cos(\phi_2) - \dot{y}_2 \sin(\phi_2) \end{matrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

The full symbolic expression for λ is too long to show here, but it can be calculated with the above equation.

2.7) (Optional) Finally, show that the dynamic equations in maximal coordinates (Problem 2.4 or 2.5) are equivalent to the dynamic equations in generalized coordinates (Problem 1.2 or 1.3) by using the constraint equations and the change of basis between q_g and q_m so that $q_m = h(q_g)$ and $\dot{q}_m = H\dot{q}_g$. The converted EOM in generalized coordinates from maximal coordinates should yield the same results as Problem 2.4 or 2.5.

Please compute and save them in the symbolic variables `h`, `H`, and `EOM_g3` respectively in the script.

Solution: (0 points) To show that the equations of motion are equivalent, we need a change of basis from one set of coordinates to the other to show that they are equivalent. We could define some $\Psi(q_m) = y$ that reduces our maximal coordinates down to our reduced coordinates $y = q_g$. The differential of this map, Y, could then be used to calculate the change of basis matrix H. Alternatively, we could directly determine a map from our reduced coordinates to our maximal coordinates, $q_m = h(q_g)$, and then take the differential of this to directly obtain H. This second method can be executed with

$$h(q_g) = \begin{bmatrix} \frac{l\,\cos(q_1)}{2} \\ \frac{l\,\sin(q_1)}{2} \\ q_1 \\ \left(\frac{l}{2} + q_2\right)\cos(q_1) \\ \left(\frac{l}{2} + q_2\right)\sin(q_1) \\ q_1 \end{bmatrix} \qquad H(q_g) = Dh(q_g) = \begin{bmatrix} -\frac{l\,\sin(q_1)}{2} & 0 \\ \frac{l\,\cos(q_1)}{2} & 0 \\ 1 & 0 \\ -\left(\frac{l}{2} + q_2\right)\sin(q_1) & \cos(q_1) \\ \left(\frac{l}{2} + q_2\right)\cos(q_1) & \sin(q_1) \\ 1 & 0 \end{bmatrix}$$

We can then apply the equations

$$\begin{aligned} \tilde{M}(h(q_g))\ddot{q}_m + \tilde{C}(h(q_g), H(q_g)\dot{q}_g)\dot{q}_g + \tilde{N}(h(q_g)) &= \Upsilon(h(q_g)) \\ \tilde{M} &= H^T \bar{M} H \\ \tilde{C} &= H^T \bar{C} H + H^T \bar{M} \dot{H} \\ \tilde{N} &= H^T \bar{N} \\ \tilde{\Upsilon} &= H^T \Upsilon \end{aligned}$$

to obtain equivalent equations. See the accompanying MATLAB code for these calculations.