Homework 1: Fundamentals

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solution

Note: For homework submission, please upload a readable PDF of your solutions to Gradescope. You will need to label your answer for each question based on the template provided.

Problem 1) Functions

For each of the following real-valued functions, give the domain, image, differentiability class, and if it is one-to-one or onto (consider the codomain for all of these to be $\mathbb{R}$). Use the largest non-complex domain generally considered. For example, $x^3 : \mathbb{R} \to \mathbb{R}$, image is $\mathbb{R}$, it is a member of C^∞, one-to-one, and onto. For any functions that are not C^∞, list a subset of the domain where we can restrict the function to get a C^∞ function.

$$\tan \tag{1}$$

$$\arctan \qquad \text{(Principle branch only)} \tag{2}$$

$$\text{abs} \tag{3}$$

$$\ln \tag{4}$$

$$\text{floor} \tag{5}$$

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \tag{6}$$

$$g(x, y) = x^2 + xy \tag{7}$$

$$f \circ g \tag{8}$$

Solution: 32 points (4 points each)

Domain	Image	Differentiability	Restriction	Properties
$\tan : \mathbb{R} \backslash \left\{ (2k+1)\frac{\pi}{2} : k \in \mathbb{Z} \right\} \to \mathbb{R}$	$\mathbb{R}$	$\tan \in C^\infty$		Onto
$\arctan : \mathbb{R} \to \mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\arctan \in C^\infty$		1-to-1
$\text{abs} : \mathbb{R} \to \mathbb{R}$	$[0, \infty)$	$\text{abs} \in C^0$	$(0, \infty)$ or $(-\infty, 0)$	
$\ln : (0, \infty) \to \mathbb{R}$	$\mathbb{R}$	$\ln \in C^\infty$		1-to-1, Onto
$\text{floor} : \mathbb{R} \to \mathbb{R}$	$\mathbb{Z}$	$\text{floor} \notin C^0$	$(k, k+1) : k \in \mathbb{Z}$	
$f : \mathbb{R} \to \mathbb{R}$	$\mathbb{R}$	$f \in C^1$	$(0, \infty)$ or $(-\infty, 0)$	1-to-1, Onto
$g : \mathbb{R}^2 \to \mathbb{R}$	$\mathbb{R}$	$g \in C^\infty$		Onto
$f \circ g : \mathbb{R}^2 \to \mathbb{R}$	$\mathbb{R}$	$f \circ g \in C^1$	$(0, \infty) \times (0, \infty)$	Onto

Problem 2) Coordinates

Consider a toy train driving around a circular track of radius $r = 1$. At any given time, it's position can be given by $[x(t), y(t)]^T$ and satisfies the position constraint,

$$c(x(t), y(t)) := x^2 + y^2 - r^2 \equiv 0 \tag{9}$$

(Hint: Part of this problem is worked out in the reading for this week.)

2.1) The configuration space is a one dimensional manifold, and so in any local portion of the configuration space we can store the location of the train as just one value, say x. The other coordinate can be recreated implicitly based on the constraint function c. This is only true when $D_y c(x, y)$ is invertible. For what open set of x values is this true?

Solution: (8 points) Differentiating $c(x, y)$ w.r.t y,

$$D_y c(x, y) = 2y$$
$$D_y^{-1} c(x, y) = \frac{1}{2y}$$

therefore $D_y c(x, y)$ is invertible for $y \neq 0$. This holds for the open set $x \in (-1, 1)$.

2.2) Defining coordinates in this way only provides *local* results, i.e. they hold around some nominal point. In this case, that means that we can find some number of functions $g_i(x)$ such that $c(x, g_i(x)) = 0$ for a neighborhood (open set) around an initial $[x_0, g_i(x_0)]^T$. Here, the smallest number of non-overlapping neighborhoods is 2. Write out g_1 and g_2, and for each give an initial point $[x_0, g_i(x_0)]^T$ and the largest open set V_i for which $c(x, g_i(x)) = 0 \; \forall \; [x, g_i(x)]^T \in V_i$ holds.

Solution: (8 points)

$$g_1(x) = \sqrt{r^2 - x^2}, \qquad [x_0, g_1(x_0)]^T = [0, 1]^T \qquad V_1 = \{[x, g_1(x)]^T \mid x \in (-1, 1)\}$$
$$g_2(x) = -\sqrt{r^2 - x^2}, \qquad [x_0, g_2(x_0)]^T = [0, -1]^T \qquad V_2 = \{[x, g_2(x)]^T \mid x \in (-1, 1)\}$$

2.3) Given the limitation in Problem 2.1, if we want to have a local representation at any point we need to consider functions that will represent the location of the train in terms of the other coordinate, y. Add to your list from Problem 2.2 two more functions, g_3 and g_4, such that $c(g_3(y), y) = 0$ and $c(g_4(y), y) = 0$ around some initial $[g_i(y_0), y_0]^T$, as well as their corresponding V_i.

Solution: (8 points)

$$g_3(y) = \sqrt{r^2 - y^2}, \qquad [g_3(y_0), y_0]^T = [1, 0]^T \qquad V_3 = \{[g_3(y), y]^T \mid y \in (-1, 1)\}$$
$$g_4(y) = -\sqrt{r^2 - y^2}, \qquad [g_4(y_0), y_0]^T = [-1, 0]^T \qquad V_4 = \{[g_4(y), y]^T \mid y \in (-1, 1)\}$$

2.4) The velocity at any given time is $[\dot{x}(t), \dot{y}(t)]^T$. What constraints are there on $\dot{x}$ and $\dot{y}$?

Solution: (8 points) Computing the tangent map for $c(x, y)$, we get

$$\dot{c}(x, y, \dot{x}, \dot{y}) = \begin{bmatrix} 2x & 2y \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 2x\dot{x} + 2y\dot{y} = 0$$

2.5) For each of your local coordinates in 2.2 and 2.3, we can also consider the velocity in the local coordinates. What constraints are there on the velocity in local coordinates?

Solution: (8 points) According to the local coordinates in 2.2 and 2.3, define $\phi_i : U_i \to \mathbb{S}^1$ as the map from local coordinate s to the point (x, y) on the track. For example, for $i = 1$, $\phi_1(s) = [s, g_1(s)]^T = [s, \sqrt{r^2 - s^2}]^T$. The velocity in local coordinates is not constrained – from any position s_i in any U_i we can move in any direction (at least for at least a short distance because U_i is an open set). To check this, constraints on the velocity in local coordinates can be found by applying the constraint to the tangent map $Tc \cdot T\phi_i = 0$ (where Tc is evaluated at the point $\phi_i(s)$), this is

$$\begin{aligned} 0 &= \begin{bmatrix} 2x & 2y \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{s}{\sqrt{r^2 - s^2}} \end{bmatrix} \\ &= \begin{bmatrix} 2s & 2\sqrt{r^2 - s^2} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{s}{\sqrt{r^2 - s^2}} \end{bmatrix} \\ &= 2s - 2\sqrt{r^2 - s^2}\frac{s}{\sqrt{r^2 - s^2}} = 0 \end{aligned}$$

Thus by construction the constraint is satisfied for any local coordinate s. The same can be shown for any other $\phi_i(p)$.

Problem 3) Projectile Dynamics

Consider the dynamics of a cannon ball. Assume the ball is shot from the point $(0, 1)$ of the plane at an angle of ϕ degrees up from the positive x axis, with an initial velocity of v. The dynamics are:

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 1 \\ \dot{x}(0) &= v\cos(\phi) \\ \dot{y}(0) &= v\sin(\phi) \\ \ddot{x}(t) &= 0 \\ \ddot{y}(t) &= -g \end{aligned}$$

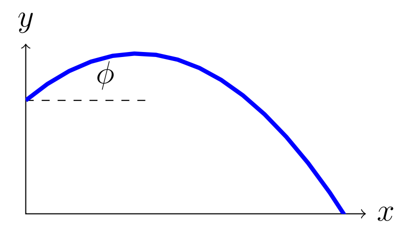

Figure 1: Projectile path.

3.1) Integrate the equations of motion to get the functions $x(t)$, $y(t)$, $\dot{x}(t)$, and $\dot{y}(t)$. When does it hit the ground $(y = 0)$?

Solution: (6 points) By simple integration, we get:

$$\begin{aligned} \dot{x}(t) &= v\cos(\phi) \\ \dot{y}(t) &= v\sin(\phi) - gt \\ x(t) &= vt\cos(\phi) \\ y(t) &= 1 + vt\sin(\phi) - \frac{1}{2}gt^2 \end{aligned}$$

And so,

$$y = 0 \rightarrow t = \frac{v\sin(\phi) + \sqrt{v^2\sin^2(\phi) + 2g}}{g}$$

3.2) Now lets change to polar coordinates, (r, θ), where,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

What is the trajectory of the projectile, $(r(t), \theta(t))$?

Solution: (8 points) Let $f : (x, y) \mapsto (r, \theta)$,

$$\begin{aligned}
f(x,y) &= \left(\sqrt{x^2+y^2}, \arctan(y/x)\right) \\
r(t) &= \pi_1 f(x(t), y(t)) \\
&= \sqrt{v^2t^2\cos^2(\phi) + 1 + 2vt\sin(\phi) - gt^2 + v^2t^2\sin^2(\phi) - vt\sin(\phi)gt^2 + \frac{1}{4}g^2t^4} \\
&= \sqrt{1 + 2vt\sin(\phi) + (v^2-g)t^2 - v\sin(\phi)gt^3 + \frac{1}{4}g^2t^4} \\
\theta(t) &= \arctan\left(\frac{1 + vt\sin(\phi) - \frac{1}{2}gt^2}{vt\cos(\phi)}\right)
\end{aligned}$$

3.3) We now have two ways of determining the velocity in polar coordinates. Assume that $\phi = 0$. First, find the velocity by differentiating the trajectory in polar coordinates. Then, use the tangent map from the change of coordinates to directly compute this velocity from $(\dot{x}(t), \dot{y}(t))$. Show that these two methods yield the same result.

Solution: (14 points) Setting $\phi = 0$ and differentiating the polar coordinate trajectory:

$$\begin{aligned}
r(t) &= \sqrt{1 + (v^2-g)t^2 + \frac{1}{4}g^2t^4} \\
\theta(t) &= \arctan\left(\frac{1 - \frac{1}{2}gt^2}{vt}\right) \\
\dot{r}(t) &= \frac{2t(v^2-g) + g^2t^3}{2\sqrt{1 + (v^2-g)t^2 + \frac{1}{4}g^2t^4}} \\
\dot{\theta}(t) &= \frac{-vgt^2 - (1 - \frac{1}{2}gt^2)v}{v^2t^2\left(1 + \left(\frac{1-\frac{1}{2}gt^2}{vt}\right)^2\right)}
\end{aligned}$$

Computing the tangent map:

$$\begin{aligned}
Df(x,y) &= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{bmatrix} \\
\begin{bmatrix} \dot{r}(t) \\ \dot{\theta}(t) \end{bmatrix} &= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{t(v^2-g)+\frac{1}{2}g^2t^3}{\sqrt{1+(v^2-g)t^2+\frac{1}{4}g^2t^4}} \\ \frac{-vgt^2-(1-\frac{1}{2}gt^2)v}{v^2t^2\left(1+\left(\frac{1-\frac{1}{2}gt^2}{vt}\right)^2\right)} \end{bmatrix}
\end{aligned}$$