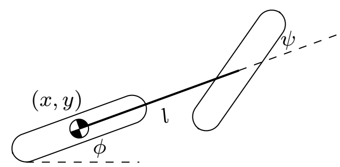
Homework 5: Mobile Robot Kinematics

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solutions

Problem 1) Bicycle



You are building a bicycle. To model the system, consider only the horizontal plane kinematics. The wheels are radius $r = 0.3$m and spaced $l = 1$m apart. Assume the steering turns the front wheel about its center. The state of the system $q = [\psi, x, y, \phi]^T$ consist of the steering angle ψ, the position (x, y) of the rear wheel, and the orientation ϕ of the rear wheel. Consider the bicycle as a kinematic system with input u consisting of forward velocity of the rear wheel v and the rotational velocity of the steering wheel $\dot{\psi}$, so $u = [v, \dot{\psi}]^T$.

1.1) What are the kinematic constraints, $A\dot{q} = 0$, on the velocity of the bicycle states?

Solution: (30 points) There are two kinematic constraints on this system. Assuming that each wheel does not skid along the ground constrains the velocity of each wheel such that the velocity component perpendicular to the wheel is zero. Since the location of the back wheel is the same as the center of mass, the velocity of the center of the wheel is given by $[\dot{x}, \dot{y}]^T$ in the spatial frame. Let's call this spatial frame S, the frame aligned with the back wheel B, and the frame aligned with the front wheel F, and the locations of the back and front wheels p and q respectively. We can transform the velocity of the back wheel in the spatial frame, v_{p_s}, into the B frame to obtain

$$v_{p_b} = R_{sb}^T v_{p_s} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

where v is the forward velocity. The second row of this equation gives our first constraint of $-\dot{x}\sin\phi + \dot{y}\cos\psi = 0$.

The second constraint comes from writing the velocity of the front wheel with respect to the world expressed in the F frame, v_{q_f}. To find this, we can first find the velocity of the front wheel expressed in the B frame, v_{q_b}, and use R_{bf}^T to transform frames. We can find these with

$$v_{q_b} = \omega_{sb}^b \times q_b + v_{sb}^b = \begin{bmatrix} 0 \\ \dot{\phi}l \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ \dot{\phi}l \end{bmatrix}$$

$$v_{q_f} = R_{bf}^T v_{q_b} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} v \\ \dot{\phi}l \end{bmatrix} = \begin{bmatrix} v\cos\psi + \dot{\phi}l\sin\psi \\ -v\sin\psi + \dot{\phi}l\cos\psi \end{bmatrix} = \begin{bmatrix} v_f \\ 0 \end{bmatrix}$$

The second row of the above matrix equation is our second constraint function, $-\dot{x}\cos\phi\sin\psi - \dot{y}\sin\phi\sin\psi + \dot{\phi}l\cos\psi = 0$. Grouping these two matrix equations and factoring out the $\dot{q}$ terms to ensure $A\dot{q} = 0$ gives

$$A(q) = \begin{bmatrix} 0 & -\sin(\phi) & \cos(\phi) & 0 \\ 0 & -\cos(\phi)\,\sin(\psi) & -\sin(\phi)\,\sin(\psi) & l\,\cos(\psi) \end{bmatrix}$$

Alternatively, the second row of $A(q)$ can be found in terms of twists and adjoints. We can solve for V_{wf}^b of the front wheel in terms of V_{wb}^b. V_{wb}^b can be solved for with

$$R_{wb} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \dot{R}_{wb} = \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\phi}.$$

$$\dot{p}_{wb} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}$$

$$V_{wb}^b = \begin{bmatrix} R_{wb}^T \dot{p}_{wb} \\ (R_{wb}^T \dot{R}_{wb})^\vee \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\phi} \right)^\vee \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\vee \end{bmatrix} = \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \\ 0 \\ 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

Now, using the adjoint relationship, V_{wf}^b can be solved for in terms of V_{wb}^b. Additionally, we know that $V_{wbwf}^b = 0$ because the front and back wheels are rigidly connected. So, using the adjoint relationship, we get

$$R_{wf} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p_{wf} = \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}$$

$$V_{wf}^b = \mathrm{Ad}_{g_{wbwf}^{-1}} V_{wf}^b = \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 & 0 & l\sin\psi \\ -\sin\psi & \cos\psi & 0 & 0 & 0 & l\cos\psi \\ 0 & 0 & 1 & 0 & L & 0 \\ 0 & 0 & 0 & \cos\psi & \sin\psi & 0 \\ 0 & 0 & 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \\ 0 \\ 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{x}\cos(\phi+\psi) + \dot{y}\sin(\phi+\psi) + l\dot{\phi}\sin\psi \\ -\dot{x}\sin(\phi+\psi) + \dot{y}\cos(\phi+\psi) + l\dot{\phi}\cos\psi \\ 0 \\ 0 \\ 0 \\ \dot{\phi}+\dot{\psi} \end{bmatrix}.$$

Because we want the y component of the front wheel velocity to equal zero to satisfy the motion constraints, we can use the second row of V_{wf}^b as the second row of our A matrix as seen below to yield an alternate solution

$$A(q) = \begin{bmatrix} 0 & -\sin(\phi) & \cos(\phi) & 0 \\ 0 & -\sin(\phi+\psi) & \cos(\phi+\psi) & l\,\cos(\psi) \end{bmatrix}$$

1.2) What are the kinematic freedoms, $\dot{q} = H(q)u$, i.e. what are the system velocities written in terms of the control input? *(Note that the wheels reading used G instead of H).*

Solution: (30 points) Kinematically the system is constrained such that $\dot{x}, \dot{y}$, and $\dot{\phi}$ are all functions of the forward velocity v, and there are no constraints on the angular velocity of the front wheel. This means we can express the kinematic freedoms as

$$\dot{q} = \begin{bmatrix} \dot{\psi} \\ \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\psi} \\ v\cos\phi \\ v\sin\phi \\ \frac{v\tan\psi}{l} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \\ \frac{\tan\psi}{l} & 0 \end{bmatrix} \begin{bmatrix} v \\ \dot{\psi} \end{bmatrix} = H(q)u$$

Note that $A\dot{q} = AH(q)u = 0$ for any u.

1.3) Show that your kinematic constraints (represented by A) and kinematic freedoms (represented by H) are consistent with each other.

Solution: (20 points)

$$
\begin{aligned}
AH(q)u &= 0 \\
\begin{bmatrix} 0 & -\sin(\phi) & \cos(\phi) & 0 \\ 0 & -\cos(\phi)\sin(\psi) & -\sin(\phi)\cos(\psi) & l\cos(\psi) \end{bmatrix}
\begin{bmatrix} 0 & 1 \\ \cos(\phi) & 0 \\ \sin(\phi) & 0 \\ \frac{\tan(\psi)}{l} & 0 \end{bmatrix}
\begin{bmatrix} v \\ \dot{\psi} \end{bmatrix} &= 0 \\
\begin{bmatrix} 0 & 0 \\ \cos(\psi)\tan(\psi) - \cos^2(\psi)\sin(\psi) - \sin^2(\phi)\sin(\psi) & 0 \end{bmatrix}
\begin{bmatrix} v \\ \dot{\psi} \end{bmatrix} &= 0 \\
\begin{bmatrix} 0 & 0 \\ \sin(\psi) - (\cos^2(\psi) + \sin^2(\psi))\sin(\psi) & 0 \end{bmatrix}
\begin{bmatrix} v \\ \dot{\psi} \end{bmatrix} &= 0 \\
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} v \\ \dot{\psi} \end{bmatrix} &= 0
\end{aligned}
$$

Therefore, $AH(q)u = 0 \quad \forall u \in \mathbb{R}^2$.

1.4) What is the turning radius of the bicycle for a given steering angle ψ? That is, what radius circle will the rear wheel trace out when the front wheel is at an angle of ψ?

Solution: (20 points) The turning radius can be found geometrically by noting that a right triangle is formed by the centers of each wheel and the intersection of the lines perpendicular to each wheel. This right triangle contains a known angle ψ and side lengths l and ρ, where ρ is the turning radius such that

$$
\begin{aligned}
\tan\psi &= \frac{l}{\rho} \\
\rho &= \frac{l}{\tan\psi}
\end{aligned}
$$