Homework 4: Statics & Contact Kinematics

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solution

Problem 1) Statics

Consider a planar robot with two revolute joints, with angles θ_1 and θ_2, and two links of length l_1 and l_2 (Figure 1).

Submit solutions as a Matlab script following the given template `HW4_sudent.m` and include all codes in .zip format. Fill in all the `TODO` sections and clearly label sections based on which part they are for. Please use the precise variable names that we define in the template and do not overwrite them in later sections.

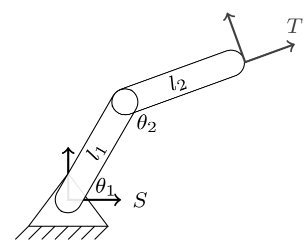

Figure 1: Two link robot.

1.1) (10 points) Assume that our arm is holding a point mass m (located at the origin of the tool frame) against gravity. What body wrench does this point mass apply? Start by computing the forward kinematics `g_st` and then compute the body wrench `F_t` as a function of the configuration of the arm.

Solution: (10 points)

$$F_x = -mg\sin(\theta_1+\theta_2)$$
$$F_y = -mg\cos(\theta_1+\theta_2)$$
$$F_t = [F_x, F_y, 0, 0, 0, 0]^T$$

1.2) (10 points) What joint torques `tau_pm` need to be applied at joint 1 and joint 2 to counteract this body wrench (i.e. hold the arm in place)? Assume massless links. Start by computing the body jacobian `J_b` and then compute `tau_pm`.

Solution: (10 points)

$$J^b_{st} = \begin{bmatrix} l_1\sin(\theta_2) & 0 \\ l_2 + l_1\cos(\theta_2) & l_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\tau_{joint} = -(J^b_{st})^T F_T$$

$$\tau_{joint} = \begin{bmatrix} mg(l_2\cos(\theta_1+\theta_2) + l_1\cos(\theta_1)) \\ mgl_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

1.3) (15 points) For the rest of the problem ignore the point mass from previous sections and assume that our links have masses m_1 and m_2 respectively, with all of the mass located at the geometric center of each link (no inertia). What joint torques `tau_lm` need to be applied to counteract just the weight of the links themselves?

Solution: (10 points)

Determine the wrenches applied at the COM of link1 and COM of link 2:

$$F_{1,COM} = \begin{bmatrix} -m_1 g \sin(\theta_1) \\ -m_1 g \cos(\theta_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_{2,COM} = \begin{bmatrix} -m_2 g \sin(\theta_1 + \theta_2) \\ -m_2 g \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiple ways to do the next part. One would be to find the body Jacobian from the stationary frame to the COM frames. Another that follows more naturly from the statics lecture is to convert the body wrenches to the end of each link and use the body Jacobians from there:

g_1 and g_2 are the transforms from the center of each link to the end

$$g_1 = \begin{bmatrix} 1 & 0 & 0 & l_1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_2 = \begin{bmatrix} 1 & 0 & 0 & l_2/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We then use the adjoint of each matrix to convert the wrenches at the COM to a wrench at the end of the link:

F_1 and F_2 are the applied

$$
\begin{aligned}
F_1 &= Ad^T_{COM,1} F_{1,COM} \\
F_1 &= \begin{bmatrix} -m_1 g \sin(\theta_1) \\ -m_1 g \cos(\theta_1) \\ 0 \\ 0 \\ 0 \\ \frac{m_1 g l_1 \cos(\theta_1)}{2} \end{bmatrix} \\
F_2 &= Ad^T_{COM,2} F_{2,COM} \\
F_2 &= \begin{bmatrix} -m_2 g \sin(\theta_1+\theta_2) \\ -m_2 g \cos(\theta_1+\theta_2) \\ 0 \\ 0 \\ 0 \\ \frac{m_2 g l_2 \cos(\theta_1+\theta_2)}{2} \end{bmatrix}
\end{aligned}
$$

We also need the body jacobian at the end of link1, but that's easy to find if we just pass in $\theta_2 = 0, l_2 = 0$ to J^b_{st}:

$$
J^b_{sL_1} = \begin{bmatrix} 0 & 0 \\ l_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

Now we can find the torque required to resist these weight-induced wrenches:

$$
\begin{aligned}
\tau &= -(J^b_{sL_1})^T F_1 - (J^b_{st})^T F_2 \\
\tau &= \begin{bmatrix} (g(l_1 m_1 \cos(\theta_1) + 2 l_1 m_2 cos(\theta_1) + l_2 m_2 cos(\theta_1+\theta_2)))/2 \\ \frac{m_2 g l_2 \cos(\theta_1+\theta_2)}{2} \end{bmatrix}
\end{aligned}
$$

1.4) (0 points) (This section is **ungraded**, but interesting) Assume both links have unit length and unit mass. What configuration θ minimizes the actuator effort required to maintain? You can use $\tau^T\tau$ as a model for actuator effort where τ is the column vector of torques applied to joint 1 and joint 2 respectively.

Solution: (0 points) We first compute the actuator effort:

$$
\begin{aligned}
effort &= \tau^T\tau \\
&= 96.2361 \cos(\theta_1+\theta_2)^2 + 96.2361(\cos(\theta_1+2\theta_2) + \cos(\theta_1+\theta_2) + \cos(\theta_1))^2
\end{aligned}
$$

To minimize this function, we observe that the function is a sum of quadratic terms, so the minimum possible value is zero. We seek values of θ_1 and θ_2 that make each cosine term in this function zero and since θ_2 never appears alone, we can arbitrarily set $\theta_2 = 0$. Since each term is left with $\cos(\theta_1)$, we know that $\theta_1 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

These results is somewhat intuitive when translated back to the real robot: the minimum effort configuration has the robot standing straight up, so the gravity vector is just applying a normal force to each link, which is resisted by the link material, not the actuators.

Problem 2) Grasp Properties

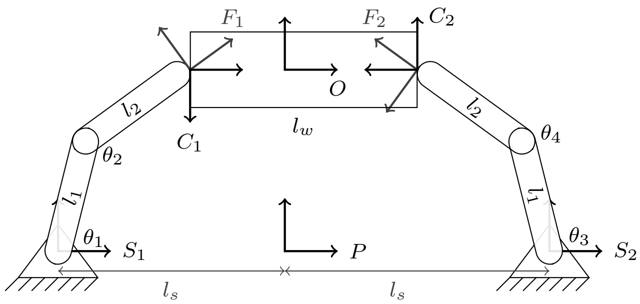

Figure 2: Two fingered robot. For each frame, the label indicates the x-axis.

Consider a planar two finger hand with two links per finger, as shown in Figure 2. The base of each finger is located at $\pm l_s$ along the x-axis, with link lengths of l_1 and l_2. The fingers are holding an object of width l_w and height l_h, with an object frame O at its center. For this planar problem, define the contact frames C_1 and C_2 with the y-axis pointing into the object (normally the C frame is defined with the z-axis pointing inward). For each problem you may solve the problem in SE(3) if you would like and then drop the z direction and rotations out of the plane (which should all be zero). See notes posted under readings on planar kinematics if you have questions.

The object location and orientation relative to the palm is $o = [x_o, y_o, \phi_o]^T$ (*note that the book usually uses the notation x_o instead of o for the whole vector*), and the joint variables are $\theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T$. Assume frictional contact with coefficient μ, and that the contact points are along the object's x-axis (but note that the object frame can otherwise move or rotate from the configuration as drawn). You are encouraged to use Matlab to help with the calculations. Please make sure your answer is consistent in dimensions (planar or spatial).

2.1) (10 points) What is the grasp map, G? What is the combined friction cone, FC?

Solution: (10 points) We can calculate the grasp map with the equation

$$G = \begin{bmatrix} \mathrm{Ad}^T_{g^{-1}_{oc_1}} B_{c_1} & \mathrm{Ad}^T_{g^{-1}_{oc_2}} B_{c_2} \end{bmatrix}.$$

The wrench bases for these planar, frictional contacts are,

Spatial (3D)	Planar (2D)
$B_{c_1} = B_{c_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$B_{c_1} = B_{c_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$

This basis maps the scalar tangential and normal components, $f_{c_i}^1$ and $f_{c_i}^2$, to a wrench in the contact frame. This wrench is then mapped to a wrench in the object frame through $\mathrm{Ad}^T_{g_{oc_i}^{-1}}$. To find this we can use our function for calculating the adjoint from a homogeneous transformation matrix. These matrices are given by
(in 3D)

$$g_{oc_1} = \begin{bmatrix} 0 & 1 & 0 & -\frac{l_w}{2} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{oc_2} = \begin{bmatrix} 0 & -1 & 0 & \frac{l_w}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(in 2D)

$$g_{oc_1} = \begin{bmatrix} 0 & 1 & -\frac{l_w}{2} \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad g_{oc_2} = \begin{bmatrix} 0 & -1 & \frac{l_w}{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Computing the adjoint transpose of the inverse of these matrices and multiplying by the contact wrench bases yields the grasp map for this system,

Spatial (3D)

$$G = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{l_w}{2} & 0 & \frac{l_w}{2} & 0 \end{bmatrix}$$

Planar (2D)

$$G = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ \frac{l_w}{2} & 0 & \frac{l_w}{2} & 0 \end{bmatrix}.$$

The friction cone for this grasp, FC, is given by (No z component)

$$\begin{aligned} FC_1 &= \{f_{c_1} : |f_{c_1}^1| \le \mu f_{c_1}^2, f_{c_1}^2 \ge 0\} \\ FC_2 &= \{f_{c_2} : |f_{c_2}^1| \le \mu f_{c_2}^2, f_{c_2}^2 \ge 0\} \\ FC &= FC_1 \times FC_2 \end{aligned}$$

2.2) (5 points) Is this a force-closure grasp?

Solution: (5 points) A grasp is force-closure if and only if $G(FC) = \mathbb{R}^p$, where $p = 3$ since this system is planar. For this grasp, $\mathrm{rank}(G) = 3$, so the columns of G span $\mathbb{R}^3$. Additionally, our friction cone does not restrict this span, since the proper choice for $f_{c_1}^2, f_{c_2}^2 \ge 0$ can generate any object wrench thanks to the linear dependence of their columns in the grasp map.

2.3) (10 points) In a static scene, if the object mass is m what is the object wrench, F_e due to gravity and what is a feasible vector of contact forces, f_c, that resist this wrench? Write your solution in terms of the object configuration o.

Solution: (10 points) The object wrench due to gravity is given by

Spatial (3D)

$$F_e = \begin{bmatrix} -mg\sin\phi_o \\ -mg\cos\phi_o \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Planar (2D)

$$F_e = \begin{bmatrix} -mg\sin\phi_o \\ -mg\cos\phi_o \\ 0 \end{bmatrix}$$

We can find a feasible vector of contact forces to resist this by finding a solution to the equation $Gf_c = -F_e$ (note that this equation comes from requiring the sum of the forces on the object to be zero, so the object wrench from the contacts is equal and opposite to the external wrench). This equation can be written as

$$Gf_c = -F_e$$

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{l_w}{2} & 0 & \frac{l_w}{2} & 0 \end{bmatrix} \begin{bmatrix} f^1_{c_1} \\ f^2_{c_1} \\ f^1_{c_2} \\ f^2_{c_2} \end{bmatrix} = \begin{bmatrix} mg\sin\phi_o \\ mg\cos\phi_o \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ \frac{l_w}{2} & 0 & \frac{l_w}{2} & 0 \end{bmatrix} \begin{bmatrix} f^1_{c_1} \\ f^2_{c_1} \\ f^1_{c_2} \\ f^2_{c_2} \end{bmatrix} = \begin{bmatrix} mg\sin\phi_o \\ mg\cos\phi_o \\ 0 \end{bmatrix}$$

From this expression we can extract three relations:

$$f^2_{c_1} - f^2_{c_2} = mg\sin\phi_o \tag{1}$$

$$-f^1_{c_1} + f^1_{c_2} = mg\cos\phi_o \tag{2}$$

$$\frac{l_w}{2}(f^1_{c_1} + f^1_{c_2}) = 0. \tag{3}$$

We can solve (2) and (3) to obtain $f^1_{c_1} = -\frac{mg\cos\phi_o}{2}$ and $f^1_{c_2} = \frac{mg\cos\phi_o}{2}$. This leaves (1), which gives a relation between $f^2_{c_1}$ and $f^2_{c_2}$. A feasible set of these forces (given the friction cone constraints) is $f^2_{c_1} = mg + |\frac{mg\cos\phi_o}{2\mu}|$ and $f^2_{c_2} = mg(1-\sin\phi_o) + |\frac{mg\cos\phi_o}{2\mu}|$. If you aren't sure how we got these, read the next section. These forces can be written as a vector,

$$f_c = \begin{bmatrix} f^1_{c_1} \\ f^2_{c_1} \\ f^1_{c_2} \\ f^2_{c_2} \end{bmatrix} = \begin{bmatrix} -\frac{mg\cos\phi_o}{2} \\ mg + |\frac{mg\cos\phi_o}{2\mu}| \\ \frac{mg\cos\phi_o}{2} \\ mg(1-\sin\phi_o) + |\frac{mg\cos\phi_o}{2\mu}| \end{bmatrix}.$$

2.4) (5 points) What are the possible internal forces?

Solution: (5 points) The possible internal forces, f_N, are any vectors in the null space of G and within the friction cone, which is the set of forces:

$$f_N = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \lambda, \quad \lambda \geq 0.$$

2.5) (10 points) What if any is the minimal internal force to hold the object with respect to gravity?

Solution: (10 points) To hold the object with respect to gravity, we require $f_c = f_N + f_P$, where f_P is the particular solution to $Gf_c = -F_e$. In this case the internal forces are necessary to provide the tangential friction forces required to hold the object. The particular solution to this problem is given by

$$f_P = \begin{bmatrix} -\frac{mg\cos\phi_o}{2} \\ \max(mg\sin\phi_o, 0) \\ \frac{mg\cos\phi_o}{2} \\ \max(-mg\sin\phi_o, 0) \end{bmatrix}$$

The friction cone for this system requires that $|f_{c_i}^1| \leq \mu |f_{c_i}^2|$, so to satisfy the friction cone and hold the object with respect to gravity, the internal force must be

$$f_N = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \lambda, \quad \lambda \geq \left| \frac{mg\cos\phi_o}{2\mu} \right|.$$

Therefore the minimal internal force can be found be replacing the inequality above with an equality.

2.6) (10 points) What is the hand Jacobian, J_h?

Solution: (10 points) The hand Jacobian maps the velocities of the joints to the velocity of the contact frame in the directions of the contact constraints, and is calculated by

$$J_h = \begin{bmatrix} B_{c_1}^T \mathrm{Ad}_{g_{s_1c_1}}^{-1} J_{s_1f_1}^s(\theta_{f_1}) & 0 \\ 0 & B_{c_2}^T \mathrm{Ad}_{g_{s_2c_2}}^{-1} J_{s_2f_2}^s(\theta_{f_2}) \end{bmatrix}$$

We have already calculated the spatial Jacobian in class for a two link manipulator, so for each finger we can state

$$J_{s_1f_1}^s = \begin{bmatrix} 0 & l_1\sin(\theta_1) \\ 0 & -l_1\cos(\theta_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad J_{s_2f_2}^s = \begin{bmatrix} 0 & l_1\sin(\theta_3) \\ 0 & -l_1\cos(\theta_3) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

(Drop row 3 to row 5 for 2D version)

Next, we can calculate the rigid body transformations from the finger base frames to the contact frames, $g_{s_i c_i}$. This could be done either by expressing this transformation from the base frame to the palm frame to the object frame to the contact frame, or from the base frame to the finger frame to the contact frame. These will be equivalent because there is an implicit constraint between the configuration of our joints θ and object position o for a given contact.

If using the palm frame, the rigid body transformations will be,

$$g_{s_1 c_1} = \begin{bmatrix} \sin(\phi_o) & \cos(\phi_o) & 0 & x_o + l_s - \frac{l_w}{2}\cos(\phi_o) \\ -\cos(\phi_o) & \sin(\phi_o) & 0 & y_o - \frac{l_w}{2}\sin(\phi_o) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{s_2 c_2} = \begin{bmatrix} -\sin(\phi_o) & -\cos(\phi_o) & 0 & x_o - l_s + \frac{l_w}{2}\cos(\phi_o) \\ \cos(\phi_o) & -\sin(\phi_o) & 0 & y_o + \frac{l_w}{2}\sin(\phi_o) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Drop row 3 and column 3 for 2D version)

Using the finger frame, we already determined the forward kinematics of this manipulator. Thus we only have to add a rotation matrix to this transformation by ϕ_o. Thus,

$$g_{s_1c_1} = \begin{bmatrix} \sin(\phi_o) & \cos(\phi_o) & 0 & l_2 \cos(\theta_1+\theta_2) + l_1 \cos(\theta_1) \\ -\cos(\phi_o) & \sin(\phi_o) & 0 & l_2 \sin(\theta_1+\theta_2) + l_1 \sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{s_2c_2} = \begin{bmatrix} -\sin(\phi_o) & -\cos(\phi_o) & 0 & l_2 \cos(\theta_3+\theta_4) + l_1 \cos(\theta_3) \\ \cos(\phi_o) & -\sin(\phi_o) & 0 & l_2 \sin(\theta_3+\theta_4) + l_1 \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Drop row 3 and column 3 for 2D version)

We can combine all these expressions to yield the hand Jacobian,

$$J_h = \begin{bmatrix} p_{y_1} & p_{y_1} + l_1\cos(\theta_1-\phi_o) & 0 & 0 \\ p_{x_1} & p_{x_1} + l_1\sin(\theta_1-\phi_o) & 0 & 0 \\ 0 & 0 & p_{y_2} & p_{y_2} - l_1\cos(\theta_3-\phi_o) \\ 0 & 0 & p_{x_2} & p_{x_2} - l_1\sin(\theta_3-\phi_o) \end{bmatrix}$$

$$\text{where } p_{x_1} = (x_o+l_s)\sin(\phi_o) - y_o\cos(\phi_o)\ ,\ p_{y_1} = \frac{l_w}{2} - y_o\sin(\phi_o) - (x_o+l_s)\cos(\phi_o)$$

$$p_{x_2} = y_o\cos(\phi_o) - (x_o-l_s)\sin(\phi_o)\ ,\ p_{y_2} = \frac{l_w}{2} + y_o\sin(\phi_o) + (x_o-l_s)\cos(\phi_o)$$

Or,

$$J_h = \left[\begin{matrix} -l_2\cos(\theta_1-\phi_o+\theta_2) - l_1\cos(\phi_o-\theta_1) & -l_2\cos(\theta_1-\phi_o+\theta_2) \\ l_1\sin(\phi_o-\theta_1) - l_2\sin(\theta_1-\phi_o+\theta_2) & -l_2\sin(\theta_1-\phi_o+\theta_2) \\ 0 & 0 \\ 0 & 0 \end{matrix} \right. \cdots$$

$$\cdots \left. \begin{matrix} 0 & 0 \\ 0 & 0 \\ l_2\cos(\theta_3-\phi_o+\theta_4) + l_1\cos(\phi_o-\theta_3) & l_2\cos(\theta_3-\phi_o+\theta_4) \\ l_2\sin(\theta_3-\phi_o+\theta_4) - l_1\sin(\phi_o-\theta_3) & l_2\sin(\theta_3-\phi_o+\theta_4) \end{matrix} \right]$$

2.7) (10 points) What are the possible internal motions?

Solution: (10 points) Internal motions are any joint velocities $\dot{\theta}$ that produce no motion of the contact point in the contact wrench basis, or in other words vectors $\dot{\theta} \in \mathcal{N}(J_h)$.
Some inspection shows that J_h is singular if $\sin\theta_2 = 0$ or $\sin\theta_4 = 0$ (this can be more easily found by inverting and simplifying J_h, which yields terms divided by $\sin\theta_2$ or $\sin\theta_4$). If $\theta_2 = \pi k, k \in \mathbb{Z}$ (or likewise for θ_4), the null space of J_h can be computed to yield

$$\dot{\theta}_N = \begin{bmatrix} -\frac{l_2}{l_1+l_2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \omega,$$

and likewise for θ_4. This configuration occurs when the two links of a finger are aligned, when rotation of either finger joint yields the same motion at the finger tip. Unlike problem 1.5 of HW 3, this is a singular configuration because the hand Jacobian only describes motion of the contact frame along the basis elements of the wrench basis. Since this wrench basis does not include rotation, the hand Jacobian loses rank in such configurations. This means that at this configuration, internal motions as described above are possible.

2.8) (5 points) The robot has 4 degrees of freedom (DOF), two per finger, meaning that in nonsingular configurations there is a 4 dimensional space of motions it can produce. The object in the plane is only 3 DOF. What does the fourth DOF correspond to?

Solution: (5 points) In general, extra degrees of freedom in the robot will correspond to either internal motions or forces. Here, there are no internal motions in nonsingular configurations, but there is in general one degree of internal forces. So the robot can move the object in the plane and independently select the internal squeezing force.