Homework 8: Impact

24-760 Robot Dynamics & Analysis
Fall 2021

Name: Solutions

For this homework, please compose everything in a single Matlab script, except helper functions. In the main script, you need to include all reasoning (either type it or insert a picture of your hand written result), calculation, and required output (with the same output variable names given in problem statements). Please fill in all the TODO sections and clearly label sections based on which part they are for. Please use the precise variable names that we define in the template and do not overwrite them in later sections. If you used any helper functions, please put them together with the main script in a zip file named as andrewID 24760 HW8.zip, where andrewID is your Andrew ID.

Problem 1) Rocking Block

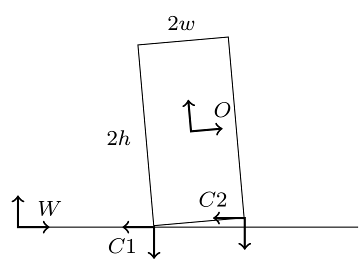

Figure 1: A block rotates about its left corner until impact occurs at the right corner.

Consider a planar, rectangular block as shown in Figure 1. The block has mass m, width $2w$, and height $2h$. The state of the block in local coordinates is $q = [x, y, \theta]^T$ where each coordinate is expressed relative to the W frame. The gravity vector points in the $-y$ direction in the W frame, and there are no other applied wrenches or friction.

The block is initially rotating clockwise with the bottom left corner (frame C_1) in contact with the ground. As the block rotates, the bottom right corner (frame C_2) will eventually come into contact with the ground, and a plastic impact event will occur ($e = 0$). We want to calculate the post-impact motion of this block as a function of the pre-impact states. In other words, given some q and $\dot{q}^-$, we wish to calculate $\dot{q}^+$ as well as the constraint impulse $\hat{P}$.

We can begin by writing the constraint functions $a(q) = [a_1(q), a_2(q)]^T$ associated with the C_1 and C_2 frames. Since there is no friction, there are only two constraints (one constraint for the contact normal of each frame). These functions are given by

$$a(q) = \begin{bmatrix} a_1(q) \\ a_2(q) \end{bmatrix} = \begin{bmatrix} y - h\,\cos(\theta) - w\,\sin(\theta) \\ y - h\,\cos(\theta) + w\,\sin(\theta) \end{bmatrix}$$

1.1) What are the A and M matrices for this system? Explain (in words) why we don't need to consider C, N, or Υ for analyzing impact.

Please compute and save them in the symbolic variables `A` and `M`.

Solution: (30 points) Since the local coordinates $\dot{q} = [\dot{x}, \dot{y}, \dot{\theta}]^T$ already describe the kinetic energy of the system, we can just use the generalized inertia matrix for the block along with $I_b = \frac{1}{12}m\left((2w)^2 + (2h)^2\right)$ to obtain

$$M = M_b = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m(h^2+w^2)}{3} \end{bmatrix}$$

We can obtain the constraint matrix $A(q)$ by taking the differential of $a(q)$:

$$A(q) = \begin{bmatrix} A_1(q) \\ A_2(q) \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1}{\partial q} \\ \frac{\partial a_2}{\partial q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & h\sin(\theta) - w\cos(\theta) \\ 0 & 1 & h\sin(\theta) + w\cos(\theta) \end{bmatrix}$$

Since we assume impact is very short, we find the post impact states by integrating the equations of motions over a time interval and take the limit as the duration of that interval goes to zero. The acceleration terms become change in momentum terms (which is why we still need M), but any other non-impulsive forces or inertial terms (in particular those described by C, N, or Υ) become zero after this limit. The constraint impulses described by $A^T\hat{P}$ are not eliminated by this limit, because they are generated by satisfying a constraint through the impact.

1.2) Consider a wide block of mass $m = 1$ and size $w = 2$, $h = 1$ with pre-impact states $q = [2, 1, 0]^T$ and $\dot{q}^- = [1, -2, -1]^T$. Calculate $\dot{q}^+$ and $\hat{P}$ for a plastic impact into mode $\{C_1, C_2\}$, and verify that both the post-impact constraint velocities and impulses are valid (show that $A\dot{q}^+ \geq 0$ and $\hat{P} \leq 0$).

Please compute and save them in the numerical variables `dq_plus_wide`, `P_hat_wide`, and `A_dq_plus_wide`.

Solution: (20 points) To compute $\dot{q}^+$ and $\hat{P}$ for an impact into mode $\{C_1, C_2\}$, we will assume both constraints will be active ($A = [\frac{\partial a_1}{\partial q}, \frac{\partial a_2}{\partial q}]^T$). We can then calculate both quantities simultaneously with the equation

$$\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix} \tag{1}$$

Substituting the given quantities (which one can check satisfy $a_1(q) = a_2(q) = A_1\dot{q}^- = 0$) yields

$$\dot{q}^+ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{P} = \begin{bmatrix} -0.5833 \\ -1.4167 \end{bmatrix}$$

We can see that both impulses are negative, which means that the ground is exerting a positive force on the block. We can also compute $A\dot{q}^+ = [0, 0]^T \geq 0$, which affirms that the block is not penetrating the ground (and is now sliding across it).

1.3) Now consider a narrow block of mass $m = 1$ and size $w = 1$, $h = 2$ with pre-impact states $q = [1, 2, 0]^T$ and $\dot{q}^- = [2, -1, -1]^T$. Calculate $\dot{q}^+$ and $\hat{P}$ for a plastic impact into mode $\{C_1, C_2\}$, and show that either the post-impact constraint velocities or impulses are invalid.

Please compute and save them in the numerical variables `dq_plus_narrow`, `P_hat_narrow`, and `A_dq_plus_narrow`.

Solution: (20 points) We can apply Eq. (1) to the new narrow block to obtain $\dot{q}^+$ and $\hat{P}$ for an impact into mode $\{C_1, C_2\}$:

$$\dot{q}^+ = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{P} = \begin{bmatrix} 0.3333 \\ -1.3333 \end{bmatrix}$$

The positive impulse $\hat{P}_1$ indicates an adhesive constraint impulse, which violates our assumptions.

1.4) For the same block as in Problem 1.3, find the correct contact mode and recalculate $\dot{q}^+$ and $\hat{P}$. Verify that both the post-impact constraint velocities and impulses are valid.

Please compute and save them in the numerical variables `dq_plus_correct`, `P_hat_correct`, and `A_dq_plus_correct`.

Solution: (30 points) Since the positive constraint impulse corresponded to maintaining contact at C_1, we will check for a transition into contact mode $\{C_2\}$ by removing the A_1 component from A, to obtain

$$A(q) = A_2(q) = \begin{bmatrix} 0 & 1 & h\sin(\theta) + w\cos(\theta) \end{bmatrix}$$

We then reapply Eq. (1) to compute $\dot{q}^+$ and $\hat{P}$ for an impact into mode $\{C_2\}$ to obtain

$$\dot{q}^+ = \begin{bmatrix} 2 \\ 0.25 \\ -0.25 \end{bmatrix} \qquad \hat{P} = [-1.25]$$

Clearly $\hat{P}$ is negative, corresponding to a positive impulse on the robot at C_2, which is valid. There is also some separating velocity at C_1, which we can confirm by calculating

$$\begin{bmatrix} A_1(q)\dot{q}^+ \\ A_2(q)\dot{q}^+ \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

These conditions satisfy our impact assumptions and confirm that the block transitions to contact mode $\{C_2\}$.