Boundary Equilibrium bifurcation in Hybrid systems

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1 Base stone case: SDOF impacting oscillator

For simplicity, we start from a SDOF impact oscillator. For a conservative SDOF impact oscillator, we would like to start from it to know how the Boundary Equilibrium Bifurcation occurs and the condition for its existence and stability. We suppose that there is a external force μ to change the equilibrium for + region to - region.

$$m_0\ddot{X} + c_0\dot{X} + kX = f \tag{1}$$

Dividing both sides of the equation above (in time T) with k and letting $X_{st} = \frac{f}{k}$, we can get

$$\frac{\ddot{X}}{\omega_0^2} + \frac{c_0}{k}\dot{X} + X = X_{st} \tag{2}$$

Meanwhile, take the substitution by $\mathrm{dt} = \omega_0 \mathrm{dT}$,$\bar{X} = \dfrac{X}{X_{st}}$ and $2\xi = \dfrac{c_0\omega_0}{k}$, sometimes called damping ratio, we can get a nondimensionalized equation in a general form

$$\ddot{\bar{X}} + 2\xi\dot{\bar{X}} + \bar{X} = 1 \tag{3}$$

if $f > 0$ and

$$\ddot{\bar{X}} + 2\xi\dot{\bar{X}} + \bar{X} = -1 \tag{4}$$

if $f < 0$. where $\omega_0 = \sqrt{\dfrac{k}{m_0}}$ is the natural frequency of the system. At $\bar{X} = 0$ the velocity will be reset by an impact map.

1.1 $f > 0$

Make translation $U = \bar{X} - 1$, in this case, when the trajectory hits the $\Sigma = \{U + 1 = 0\}$ the $a(x) > 0$ so there is no sticking set.

$$\ddot{U} + 2\xi\dot{U} + U = 0 \tag{5}$$

and now the reset map will take pace at $U = -1$. This model and be depicted by fig.1

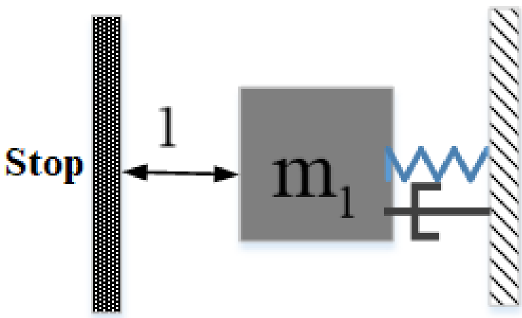

Fig. 1. Sketch of the SDOF impact model

And the general solution of the Eq. 5 will be

$$U = \mathrm{e}^{-\xi t}\left[U_0 \cos\omega_1 t + \frac{\dot{U}_0 + \xi U_0}{\omega_1}\sin\omega_1 t\right] \tag{6}$$

$$\dot{U} = \mathrm{e}^{-\xi t}\left[\dot{U}_0 \cos\omega_1 t - \frac{U_0 + \xi \dot{U}_0}{\omega_1}\sin\omega_1 t\right] \tag{7}$$

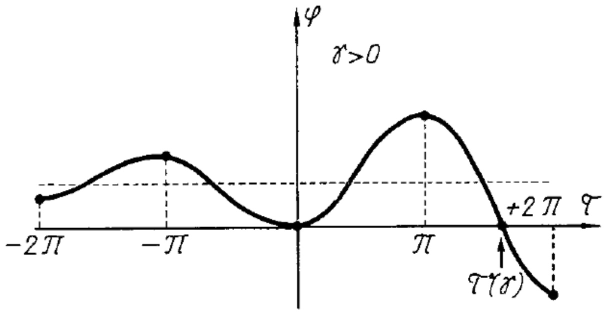

Fig. 2. auxiliary function

If the trajectory's starting point is $(U_0, \dot{U}_0) = (-1, S)$, then

$$U = \mathrm{e}^{-\xi t}\left[-\cos\omega_1 t + \frac{S-\xi}{\omega_1}\sin\omega_1 t\right] \tag{8}$$

$$\dot{U} = \mathrm{e}^{-\xi t}\left[S\cos\omega_1 t - \frac{\xi S - 1}{\omega_1}\sin\omega_1 t\right] \tag{9}$$

Suppose that the trajectory will arrive at the $U = -1$ with velocity $S^{'}$ at time $t_1 = \frac{\tau_1}{\omega_1}$

$$-1 = \mathrm{e}^{-\mu\tau_1}\left[-\cos\tau_1 + \frac{S-\xi}{\omega_1}\sin\tau_1\right] \tag{10}$$

$$S^{'} = \mathrm{e}^{-\mu\tau_1}\left[S\cos\tau_1 - \frac{\xi S-1}{\omega_1}\sin\tau_1\right] \tag{11}$$

Where $\mu = \dfrac{\xi}{\omega_1} = \dfrac{\xi}{\sqrt{1-\xi^2}}$. Thus, we can get S, $S^{'}$ as a function τ_1

$$S = -\frac{\mathrm{e}^{\mu\tau_1} - \cos\tau_1 - \mu\sin\tau_1}{\sqrt{1+\mu^2}\sin\tau_1} \tag{12}$$

$$\begin{aligned}
S^{'} &= \mathrm{e}^{-\mu\tau_1}\left[(\cos\tau_1 - \mu\sin\tau_1)S + \sqrt{1+\mu^2}\sin\tau_1\right] \\
&= \mathrm{e}^{-\mu\tau_1}\left[\frac{-1}{\sqrt{1+\mu^2}\sin\tau_1}(\mathrm{e}^{\mu\tau_1}\cos\tau_1 - \mu\mathrm{e}^{\mu\tau_1}\sin\tau_1 - \cos^2\tau_1 + \mu\cos\tau_1\sin\tau_1 - \mu\cos\tau_1\sin\tau_1 + \mu^2\sin^2\tau_1)\right. \\
&\left. + \sqrt{1+\mu^2}\sin\tau_1\right] \\
&= \frac{\mathrm{e}^{-\mu\tau_1}}{\sqrt{1+\mu^2}\sin\tau_1}\left[-\mathrm{e}^{\mu\tau_1}\cos\tau_1 + \mu\mathrm{e}^{\mu\tau_1}\sin\tau_1 + \cos^2\tau_1 - \mu^2\sin\tau_1 + (1+\mu^2)\sin^2\tau_1\right] \\
&= \frac{\mathrm{e}^{-\mu\tau_1} - \cos\tau_1 + \mu\sin\tau_1}{\sqrt{1+\mu^2}\sin\tau_1}
\end{aligned} \tag{13}$$

we find that

$$\frac{\mathrm{dS}}{\mathrm{d}\tau_1} = -\frac{1 - \mathrm{e}^{\mu\tau_1}(\cos\tau_1 - \mu\sin\tau_1)}{\sqrt{1+\mu^2}\sin^2\tau_1}$$

and

$$\frac{\mathrm{dS}^{'}}{\mathrm{d}\tau_1} = \frac{1 - \mathrm{e}^{-\mu\tau_1}(\cos\tau_1 + \mu\sin\tau_1)}{\sqrt{1+\mu^2}\sin^2\tau_1}$$

We introduce the auxiliary function[**ANDRONOV1966443**]

$$\varphi(\tau,\mu) = 1 - \mathrm{e}^{\mu\tau}(\cos\tau - \mu\sin\tau),\ \partial\varphi/\partial\tau = \mathrm{e}^{\mu\tau}(1+\mu^2)\sin\tau \tag{14}$$

Then we can reorganize the equations above to

$$S = -\frac{e^{\mu\tau_1}(1 - e^{-\mu\tau_1}(\cos\tau_1 + \mu\sin\tau_1))}{\sqrt{1+\mu^2}\sin\tau_1} = -\frac{e^{\mu\tau_1}\varphi(\tau_1, -\mu)}{\sqrt{1+\mu^2}\sin\tau_1} \tag{15a}$$

$$S^{'} = \frac{e^{-\mu\tau_1}(1 - e^{\mu\tau_1}(\cos\tau_1 - \mu\sin\tau_1))}{\sqrt{1+\mu^2}\sin\tau_1} = \frac{e^{-\mu\tau_1}\varphi(\tau_1, \mu)}{\sqrt{1+\mu^2}\sin\tau_1} \tag{15b}$$

$$\frac{\mathrm{dS}}{\mathrm{d}\tau_1} = -\frac{\varphi(\tau_1, \mu)}{\sqrt{1+\mu^2}\sin^2\tau_1} \tag{15c}$$

$$\frac{\mathrm{dS}^{'}}{\mathrm{d}\tau_1} = \frac{\varphi(\tau_1, -\mu)}{\sqrt{1+\mu^2}\sin^2\tau_1} \tag{15d}$$

Thus S is the starting velocity and the $S^{'}$ is the velocity at next intersection point, after some evolution time. Here we consider two different conditions: (1)$f > 0$ (admissible equilibrium); (2)$f < 0$ (pseudo equilibrium).

1. Case I: $\xi > 0$

   For simplicity, we suppose $S > 0$, from fig.2 we know that when $S^{'} = 0$ there will be two roots of τ_1 namely $\tau_1^1 = 0$ and $\tau_1^2 = \tau(\mu) \in (\pi, 2\pi)$. The non-trivial root τ_1^2 stands for the evolution time. Using Eq.15a we can get the critical value for

   $$S_{cr} = -\frac{e^{\mu\tau(\mu)}\varphi(\tau(\mu), -\mu)}{\sqrt{1+\mu^2}\sin\tau(\mu)}$$

   It means that when $S < S_{cr}$ the path will eventually be attracted to the origin and only those cases with larger S will have the chance to hit the boundary $U = -1$ and continue the evolution loop.

   $$S = S_{cr} + \int_{\tau(\mu)}^{\pi} \frac{\mathrm{dS}}{\mathrm{d}\tau_1}\mathrm{d}\tau_1 \tag{16}$$

   According to eq.15c we can get eq.16 and obviously when τ_1 varies from 0 to π, S will vary from 0 to $-\infty$ and vary from 0 to $+\infty$ when τ_1 varies from $\tau(\mu)$ to π; in another word,

   $$\lim_{S\to S_{cr}+} \tau_1 = \tau(\mu)^-, \quad \lim_{S\to+\infty} \tau_1 = \pi^+ \tag{17}$$

   $$\lim_{S\to-0} \tau_1 = 0^+, \quad \lim_{S\to-\infty} \tau_1 = \pi^- \tag{18}$$

   In our case, we assume that the path of this system starts with a positive velocity S and arrives the boundary $U = -1$ with a negative velocity $S^{'}$, and obviously the relationship between the S and $S^{'}$ is implicitly defined by the eq.15. However, based on $|\frac{\mathrm{dS}^{'}}{\mathrm{dS}}| = \frac{\varphi(\tau_1, -\mu)}{\varphi(\tau_1, \mu)} > 0$ and, $\tau_1 \in (\pi, \tau(\mu))$ with initial point $S^{'}(0) = 0$, $S = S_{cr}$. As $S \to +\infty$, there will be an asymptotic result

   $$\lim_{S\to+\infty} |\frac{S^{'}}{S}| = \lim_{\tau_1\to\pi^+} |-e^{-2\mu\tau_1}\frac{\varphi(\tau_1, \mu)}{\varphi(\tau_1, -\mu)}| = \lim_{\tau_1\to\pi^+} |-e^{-2\mu\tau_1}e^{\mu\pi}| = e^{-\mu\pi} \tag{19}$$

   $$\lim_{S\to+\infty} |S^{'}| - e^{-\mu\pi}S = e^{-\mu\pi} \tag{20}$$

   and

   $$\begin{aligned}\frac{|\mathrm{dS}^{'}/\mathrm{dS}|}{\mathrm{dS}} &= \frac{\mathrm{d}\frac{\varphi(\tau_1,-\mu)}{\varphi(\tau_1,\mu)}}{\mathrm{d}\tau_1}\frac{1}{\mathrm{dS}/\mathrm{d}\tau_1} \\ &= \frac{2(1+\mu^2)^{\frac{3}{2}}\sin^3\tau_1(\sinh\mu\tau_1 - \mu\sin\tau_1)}{\phi^3(\tau_1, \mu)} < 0\end{aligned} \tag{21}$$

   Thus we have a map $S_0 \to S_0^{'} = -P_0 * S_0 \to S_1 = -r * S_0^{'} = r * P_0 * S_0 \to S_2^{'} = -P_1 * S_1 \to$ .... We can plot this map fig.3.a. It can be seen that there is no stable limit cycle.

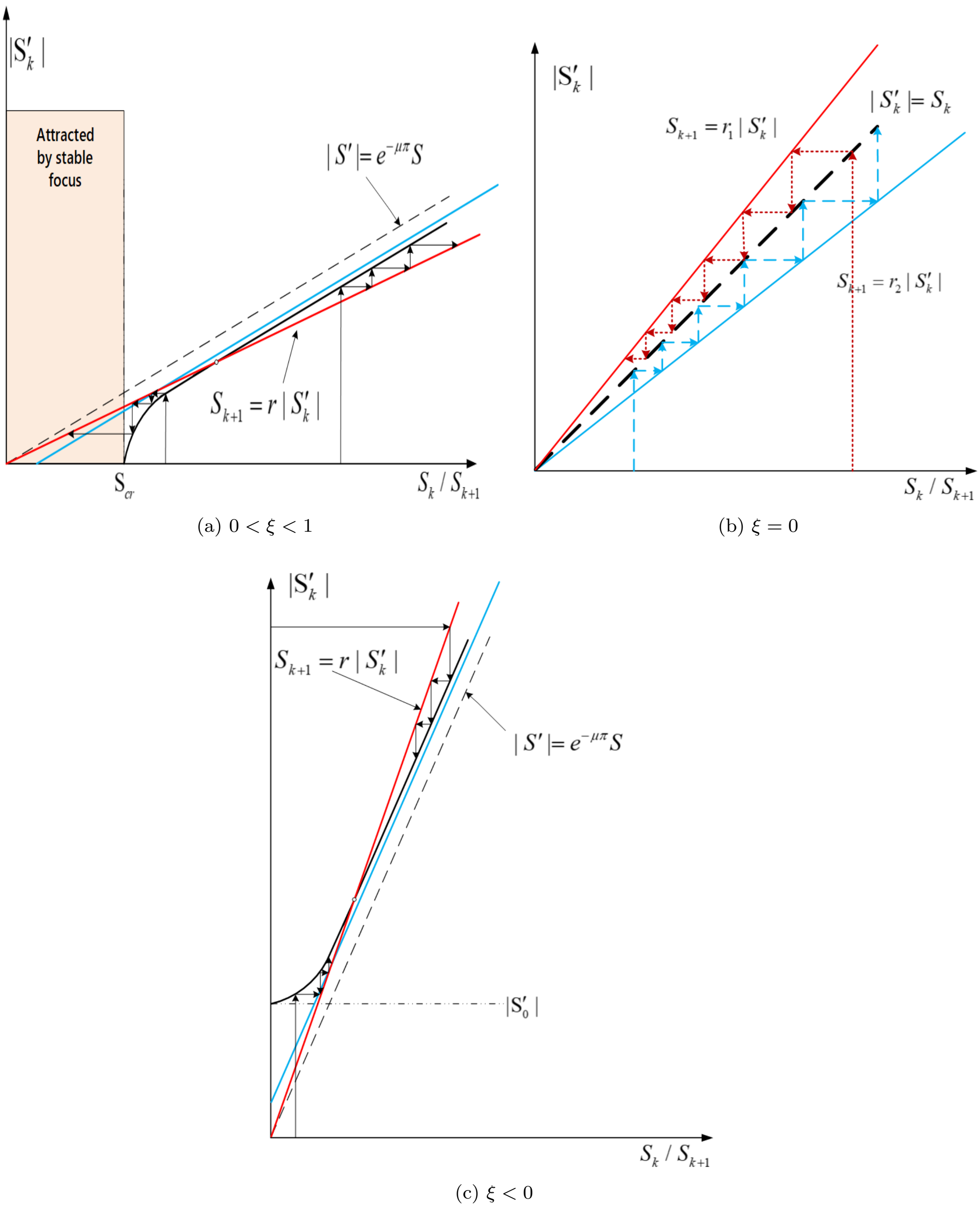

Fig. 3. Three cases : $I:\ 0<\xi<1;\ II:\ \xi=0;\ III:\ \xi<0$

2. Case II: $\xi < 0$

   In this case, given a bounding velocity, the return time τ_1 will be given, $S^{'} > S > 0$ due to the divergent motion. $S = 0$ is the critical value and the $\tau(\mu)$ is solved by eq.15a, which is easily shown the same as the result in the previous case. Substitute the $\tau(\mu)$ into eq.15b we can get $S^{'}_{cr}$. And the asymptotic's slope is the same expression as in eq.20. The impact map is still $S_{k+1} = -rS^{'}_{k}$, and the composed map will be the Fig.3. It shows that if and only if $1/r > \mathrm{e}^{-\mu\pi}$ will there be a limit cycle and even globally stable.

3. Case III: $\xi = 0$

   Interesting enough, in this case, the equilibrium is right located on the boundary, and $r\mathrm{e}^{\mu\pi}$ means there will be LCO with any amplitude, $r < 1$ will lead to convergence to a LCO with amplitude 1 and $r > 1$ will lead to divergent motion.

1.2 $f < 0$

When $f < 0$, the equilibrium is a pseudo type, the steady response will be an equilibrium in sticking set. When f changes from negative to positive, the equilibrium will firstly be a pseudo equilibrium ($f < 0$), then boundary equilibrium ($f = 0$), finally to an unstable admissible equilibrium ($f > 0$). The results can be show in Fig.4 The amplitude's linear dependence on μ

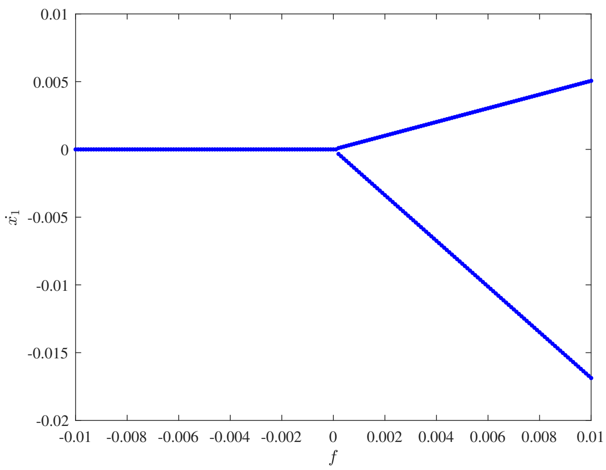

Fig. 4. Boundary equilibrium bifurcation

can be easily explained by rescaling in Eq.3 and Eq.4.

2 3 Dimensional Hybrid System

For a system

$$\begin{cases} \dot{\mathbf{x}} = F(\mathbf{x}) = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{A}\mathbf{x} + \mathbf{f}, & H(\mathbf{x}) > 0 \\ \mathbf{x}^+ = \mathbf{R}(\mathbf{x}^-), & H(\mathbf{x}) = 0 \end{cases} \tag{22}$$

where $H(\mathbf{x})$ is a defined discontinuity manifold and here we treat it as a plane or super-plane in state space.

According to the sign of the velocity $\mathcal{L}_F(H)(x)$ we can define a minus set $\{\Sigma^-|v < 0, H(\mathbf{x}) = 0\}$, a positive set $\{\Sigma^+|v > 0, H(\mathbf{x}) = 0\}$ and tangent set $\{\Sigma^0|v = 0, H(\mathbf{x}) = 0\}$ on the manifold $\{\Sigma|H(\mathbf{x}) = 0\}$. The truth is that the flow will bring every point in positive set to minus set, while the reset map will reset the points in Σ^- to Σ^+. Then there is a composed map defined on both sets and we say a limit cycle exist when the composed map has fix point(s) with period $T^* = \tau_F + \tau_R$, where the evolution time of the flow to get the discontinuity manifold again is τ_F, and τ_R for the reset map respectively. For simplicity but generality, further we define

$H(\mathbf{x}) = C\mathbf{x} = 0$ and $C = [1, 0, \cdots, 0]_{n,1}$ is the normal vector, where we can always transform the coordinates to reach this.

For a 3-D linear ODE system, there are two cases of the eigenvalues: 1) three real eigenvalues; 2) a pair of conjugate complex eigenvalues $(-\alpha \pm \beta i)$ and a real one $-\lambda$. We will focus on the second case which is of more physical meaning. Specifically, the eigenvector of the real eigenvalue λ is $\mathbf{v}_3$. Here we want to give the criteria for existence and stability of the LCO in the 3-D hybrid system. There are different cases in need of careful treatment: Is the equilibrium located on the Σ? If not, Is there any pseudo equilibrium (if yes, how about the stability)? Is the $\mathbf{v}_3$ in parallel to the Σ? So there should be 4 main different cases and we will treat them in turn.

Case I: Equilibrium on the Boundary $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. In this case the equilibrium is on the boundary. And the border distinguishing the positive and minus sets is $\Sigma^0 = \{(x_2, x_3) | a_{12}x_2 + a_{13}x_3 + f_1 = 0\}$, or l_3 with the direction vector $\mathbf{V}_{l3} = [0, \frac{a_{13}}{\sqrt{a_{12}^2+a_{13}^2}}, \frac{-a_{12}}{\sqrt{a_{12}^2+a_{13}^2}}]^\top$. In essence, the reset map will flip the point in minus set and expand/contract along the perpendicular direction to l_3 and make translation along the l_3. Now we construct the reset map as following;

For a state $[0, y_0, z_0]^\top$ in minus set, we know the $r^- = \frac{|a_{12}y_0 + a_{13}z_0 + f_1|}{\sqrt{a_{12}^2+a_{13}^2}}$ and unit vector vertical to the line l_3 as $\mathbf{P}_{l3} = [0, \frac{a_{12}}{\sqrt{a_{12}^2+a_{13}^2}}, \frac{a_{13}}{\sqrt{a_{12}^2+a_{13}^2}}]^\top$, $z^- = [0, y_0, z_0]^\top \mathbf{V}_{l3}$

So the reset map

$$R \circ \begin{bmatrix} 0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \\ z_0 \end{bmatrix} + \phi_1(r^-, z^-)\mathbf{P}_{l3} + \phi_2(r^-, z^-)\mathbf{V}_{l3}$$

(1) $\mathbf{v}_3$ in parallel with the Σ In this case, actually the $\mathbf{v}_3$ is on the Σ and the line l_3 through the origin (equilibrium) is in the derection of $\mathbf{v}_3 = \mathbf{V}_{l3}$, which seperates the discontinuity manifold to Σ^+ and Σ^-.We define the distance between the point of trajectory and the axis as $\mathbf{r}$, the eigen coordinnate along $\mathbf{v}_3$ as z.

$$\begin{bmatrix} r^- \\ z^- \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\alpha\tau_F} & 0 \\ 0 & \mathrm{e}^{-\lambda\tau_F} \end{bmatrix} \begin{bmatrix} r^+ \\ z^+ \end{bmatrix} \tag{23}$$

where $\tau_F = \frac{\pi}{\beta}$ (the discontinuity surface divides the phase evenly) and we define the reset map as

$$\mathbf{R} \circ \begin{bmatrix} r^- \\ z^- \end{bmatrix} = \begin{bmatrix} R_1(r^-, z^-) \\ R_2(r^-, z^-) \end{bmatrix}$$

Suppose we find a LCO starting from $[r_*^+; z_*^+]$ ending at $[r_*^-; z_*^-]$, satisfying the

$$R_1(r_*^-, z_*^-) = \mathrm{e}^{\alpha\tau_F} r_*^-;\ R_2(r_*^-, z_*^-) = \mathrm{e}^{\gamma\tau_F} z_*^-$$

and the Jacobi matrix of the composed map is

$$J = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-\alpha\tau_F} & 0 \\ 0 & \mathrm{e}^{-\lambda\tau_F} \end{bmatrix} \tag{24}$$

where $b_{11} = \frac{\partial R_1}{\partial r}|_{r_*^-}$, $b_{12} = \frac{\partial R_1}{\partial z}|_{z_*^-}$, $b_{21} = \frac{\partial R_2}{\partial r}|_{r_*^-}$, $b_{22} = \frac{\partial R_2}{\partial z}|_{z_*^-}$.

(2) $\mathbf{v}_3$ intersecting with the Σ We ignore the special case where the $\mathbf{v}_3$ is perpendicular to the Σ in which the $\tau_F = 2\pi$ so that the region of the evolution time τ_F is $(\pi, 2\pi)$. There will be two implicit defined functions to describe the relationship.

In this case

Case II: Equilibrium not on the Boundary $\dot{\mathbf{x}} = F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{f}$

(1) $\mathbf{v}_3$ in parallel with the Σ Same with the 1DOF case in finding the condition for $R_1(r)$ map, and only when the condition $R_2(z_*^-) = z_*^- \mathrm{e}^{\tau_F}$ is simultaneously true can a LCO orbit exist.

(2) $\mathbf{v}_3$ intersecting with the Σ Suppose the $\mathbf{v}_3$ intersect with the Σ at point E, and the separating line l_3 will be through this point.

The procedure to construct a desired system:

For a general system $\dot{X} = A(X - X_0)$, we can find its eigenvalues, $\alpha \pm i\beta$, λ and eigenvectors v_i. Especially we define the v_3 is the eigenvector of λ. For the convenience of observation, we can always transform the $\mathbf{v}_3$ to be in parallel with the XoZ plane of the inertial frame, without loss of generality.

We define the relative angle between the $\mathbf{v}_3$ and the normal vector $\mathbf{n} = C^\top$ is θ, and the general rotation matrix

$$\bar{R}(\mathbf{l}, \theta) = \mathbf{I} + \sin(\theta)[\mathbf{l}]_\times + (1 - \cos(\theta))[\mathbf{l}]_\times^2 \tag{25}$$

where

$$[\mathbf{l}]_\times = \begin{bmatrix} 0 & -l_z & l_y \\ l_z & 0 & -l_x \\ -l_y & l_x & 0 \end{bmatrix}$$

and $\mathbf{l}$ is the axis.

1. Transform to canonical forms in align with above 4 lemmas.

   (1) Let $X_0 = 0$ and if $0 < \theta < \frac{\pi}{2}$ and we can rotate the system by $\bar{R}(\mathbf{l}_0, \frac{\pi}{2} - \theta)$ in which $\mathbf{l}_0 = \mathbf{n} \times \mathbf{v}_3$. In the end, we can get a new $\mathbf{v}_3$ satisfying $\mathbf{v}_3 \perp \mathbf{n}$;

   (2) $X_0 = 0$ and $\mathbf{n} \cdot \mathbf{v}_3 \neq 0$; If else we can rotate the system by $\bar{R}(-\mathbf{l}_0, \theta)$;

   (3) Set $X_0 = [-1, y_0, z_0]^\top$ and if $0 < \theta < \frac{\pi}{2}$ and we can rotate the system by $\bar{R}(\mathbf{l}_0, \frac{\pi}{2} - \theta)$ in which $\mathbf{l}_0 = \mathbf{n} \times \mathbf{v}_3$. In the end, we can get a new $\mathbf{v}_3$ satisfying $\mathbf{v}_3 \perp \mathbf{n}$;

   (4) $X_0 = [-1, y_0, z_0]^\top$ and $\mathbf{n} \cdot \mathbf{v}_3 \neq 0$; If else we can rotate the system by $\bar{R}(-\mathbf{l}_0, \theta)$.

2. After the above manipulation, we refresh our matrix A, relative angle θ. We give our special four canonical examples:

   (1) $A = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix}$, $X_0 = \mathbf{0}$

   (2) $A = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$, $X_0 = \mathbf{0}$

   (3) $A = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix}$, $X_0 = [-1, 0, 0]^\top$

   (4) $A = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$, $X_0 = [-1, 0, 0]^\top$

For a system (in the forms given above) like $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with initial condition $X_0 = [x_0, y_0, z_0]^\top$ where

$$A_J = P^{-1}AP = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

and $P = I$(case (1),(3)) or $P = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$(case (2),(4)).

2.1 Virtual equilibrium cases

We focus our attention on the virtual equlibrium cases, where $X_0 = [-1, 0, 0]^\top$.

The solution will be $\mathbf{x} - X_0 = \mathrm{e}^{A_J t}(\mathbf{x}_0 - X_0)$, and we probe into the components of the solution

$$\begin{aligned}
x_1 + 1 = &\left(\cos^2(\theta)\mathrm{e}^{\alpha t}\cos(\beta t) + \sin^2(\theta)\mathrm{e}^{\lambda t}\right)(x_0 + 1) \\
&+ \left(\cos(\theta)\mathrm{e}^{\alpha t}\sin(\beta)\right) y_0 \\
&+ \left(\mathrm{e}^{\alpha t}\cos(\beta t) - \mathrm{e}^{\lambda t}\right)\cos(\theta)\sin(\theta) z_0 \qquad (26)\\
y_1 = &\,\mathrm{e}^{\alpha t}(-\cos(\theta)\sin(\beta t)(x_0 + 1) + \cos(\beta t) y_0 - \sin(\theta)\sin(\beta t) z_0) \qquad (27)\\
z_1 = &\left(\mathrm{e}^{\alpha t}\cos(\beta t) - \mathrm{e}^{\lambda t}\right)\cos(\theta)\sin(\theta)(x_0 + 1) \\
&- \sin(\theta)\mathrm{e}^{\alpha t}\sin(\beta t) y_0 \\
&+ \left(\sin^2(\theta)\mathrm{e}^{\alpha t}\cos(\beta t) + \cos^2(\theta)\mathrm{e}^{\lambda t}\right) z_0 \qquad (28)
\end{aligned}$$

we know the initial point departures from the positive set so $x_0 = 0$ and arrive at $x_1 = 0$.

2.1.1 $\theta = 0$

The previous aolution equations will be the same form in the SDOF case (degenerated case):

$$0 = -1 + \mathrm{e}^{\alpha t_1}\left(\cos(\beta t_1) + \sin(\beta t_1) y_0\right) \qquad (29)$$

$$y_1 = \mathrm{e}^{\alpha t_1}\left(-\sin(\beta t_1) + \cos(\beta t_1) y_0\right) \qquad (30)$$

$$z_1 = \mathrm{e}^{\lambda t_1} z_0 \qquad (31)$$

This case we can solve the semi-Poincaré map ($0 < t_1 < \frac{\pi}{\beta}$)

$$\frac{r^+}{r^-} = \frac{\left(\cos(\beta t_1)\beta - \beta\mathrm{e}^{\alpha t_1} + \alpha\sin(\beta t_1)\right)\mathrm{e}^{\alpha t_1}}{\left(\cos(\beta t_1)\beta - \alpha\sin(\beta t_1)\right)\mathrm{e}^{\alpha t_1} - \beta} \qquad (32)$$

$$\frac{z^+}{z^-} = \mathrm{e}^{\lambda t_1} \qquad (33)$$

with

$$\lim_{t_1 \to \frac{\pi^-}{\beta}} \frac{r^+}{r^-} = \mathrm{e}^{\frac{\pi\alpha}{\beta}}$$

and

$$\lim_{t_1 \to 0^+} \frac{r^+}{r^-} = 1$$

especially, $\frac{\mathrm{d}^2 r^+/r^-}{\mathrm{d}t^2} < 0$. So according to the theorem in previous section, we know we can define a $\phi_1(r^-, z^-) = (1 + R_r)r^-$, $(1 < R_r < \mathrm{e}^{-\frac{\pi\alpha}{\beta}})$ to get a stable LCO, while $\phi_2(r^-, z^-)$ can be independently defined.

2.1.2 $0 < \theta < \frac{\pi}{2}$

In his case we can solve the semi-Poincaré map ($0 < t_1 < \frac{\pi}{\beta}$)

$$\frac{r^+}{r^-} = \frac{a_1 y_0 + a_2 z_0 + a_3}{\mathrm{e}^{\lambda t}\left(1 - \mathrm{e}^{(\alpha-\lambda)t}\cos(\beta t)\right)\left[\cos(\theta)\beta y_0 + (\alpha - \lambda)\cos(\theta)^2 + z_0(\alpha - \lambda)\sin(\theta)\cos(\theta) + \lambda\right]} \qquad (34)$$

$$\frac{z^+}{z^-} = \frac{b_1 y_0 + b_2 z_0 + b_3}{-\cos(\theta)\mathrm{e}^{\lambda t_1}\left(1 - \mathrm{e}^{(\alpha-\lambda)t_1}\cos(\beta t_1)\right)\left(\sin(\theta)(\alpha - \lambda)y_0 - \beta z_0\right)} \qquad (35)$$

where

$$a_1 = \beta\cos(\theta)\mathrm{e}^{2\alpha t_1}\left(1\mathrm{e}^{(\lambda-\alpha)t_1}\cos(\beta t_1)\right)$$

$$a_2 = -\mathrm{e}^{2\lambda t_1}\sin(\theta)\cos(\theta)(\alpha - \lambda)\left(1 - \cos(\beta t_1)\mathrm{e}^{(\alpha-\lambda)t_1}\right)$$

$$a_3 = \mathrm{e}^{\lambda t_1}\left((\alpha-\lambda)\sin^2(\theta)\mathrm{e}^{\lambda t_1}-\alpha\right)\left(1-\mathrm{e}^{(\alpha-\lambda)t_1}\cos(\beta t_1)\right)+\beta\sin(\beta t_1)\mathrm{e}^{\alpha t_1}(\mathrm{e}^{\lambda t}-1)$$

$$b_1 = \sin(\theta)\cos(\theta)\mathrm{e}^{2\alpha t_1}\left(1-\mathrm{e}^{(\lambda-\alpha)t_1}\cos(\beta t_1)\right)$$

$$b_2 = \beta\cos(\theta)\mathrm{e}^{2\lambda t_1}\left(1-\mathrm{e}^{(\alpha-\lambda)t_1}\cos(\beta t_1)\right)$$

$$b_3 = \sin(\theta)\mathrm{e}^{\lambda t_1}\left(\mathrm{e}^{\lambda t}-1\right)\left[(\alpha-\lambda)\sin(\beta t)\mathrm{e}^{(\alpha-\lambda)t_1}-\beta\left(1-\mathrm{e}^{(\alpha-\lambda)t_1}\cos(\beta t_1)\right)\right]$$

and the evolution time t_1 is determined by $y_0,\ ,z_0$ and the equation

$$\cos^2(\theta)\mathrm{e}^{\alpha t_1}\cos(\beta t_1)+\sin^2(\theta)\mathrm{e}^{\lambda t_1}+\cos(\theta)\mathrm{e}^{\alpha t_1}\sin(\beta t_1)y_0-\cos(\theta)\sin(\theta)\mathrm{e}^{\lambda t_1}\left[1-\mathrm{e}^{(\alpha-\lambda)t_1}\cos(\beta t_1)\right]z_0=1 \tag{36}$$

We should notice that when $(0, y_0, z_0)$ is on the line $a_{12}y_0 + a_{13}z_0 + f_1 = C$, which is in parallel with zero velocity line l_3. Specifically, constant C is greater than 0 and determines the distance to the line l_3.

So we can get

$$y_0 = \frac{(-\alpha+\lambda)\cos(\theta)^2-(\alpha-\lambda)\sin(\theta)\cos(\theta)z_0+C-\lambda}{\beta\cos(\theta)} \tag{37}$$

Meanwhile, there is a common term $(\alpha-\beta)$ in above equations, so we discuss two cases:

1. $\alpha = \lambda$

   By substituting Eq.37 to Eq.36 we can get

$$\begin{aligned} z_0 &= -\frac{\beta\mathrm{e}^{-\lambda t}-\cos(\theta)^2\cos(\beta t)\beta+(-C+\lambda)\sin(\beta t)+\cos(\theta)^2\beta-\beta}{\beta\sin(\theta)\cos(\theta)(\cos(\beta t)-1)} \\ &= \cot(\theta)+\frac{\beta(\mathrm{e}^{-\lambda t}-1)+(-C+\lambda)\sin(\beta t)}{\beta\sin(\theta)\cos(\theta)(1-\cos(\beta t))} \end{aligned} \tag{38}$$

$$\begin{aligned} y_0 &= \frac{\sin(\theta)\cos(\theta)\cos(\beta t)z_0-\sin(\theta)\cos(\theta)z_0-\cos(\theta)^2\cos(\beta t)+\cos(\theta)^2-1+\mathrm{e}^{-\lambda t}}{\cos(\theta)\sin(\beta t)} \\ &= -\frac{(1-\cos(\beta t))\sin(\theta)}{\sin(\beta t)}\left(z_0-\frac{\mathrm{e}^{-\lambda t}-1+\cos(\theta)^2(1-\cos(\beta t))}{(1-\cos(\beta t))\sin(\theta)\cos(\theta)}\right) \end{aligned} \tag{39}$$

   and z_0 is anywhere on the plane $x = 0$, so $z_0 \in (-\infty, +\infty)$, $y_0 \in (\frac{-\lambda}{\beta\cos(\theta)}, +\infty)$. Actually, from Eq.38 we can get: when y_0 is a limited positive number

$$\lim_{t\to 0^+} z_0 = -\infty,\ \lim_{t\to\frac{\pi}{\beta}} z_0 = \cot(\theta)-\frac{1-\mathrm{e}^{-\frac{\pi\lambda}{\beta}}}{2\sin(\theta)\cos(\theta)},\ \lim_{t\to\frac{2\pi}{\beta}} z_0 = +\infty$$

   when z_0 is a limited value number and $y_0 > 0$ must be satisfied:

$$\lim_{t\to 0^+} y_0 = \frac{-\lambda}{\beta\cos(\theta)} \text{ for } z_0 \in (-\infty, +\infty)$$

$$\lim_{t\to\frac{\pi}{\beta}^-} y_0 = +\infty \text{ for limited } z_0$$

   Notice that when $y_0 = +\infty$, $0 < t_1 < \frac{\pi}{\beta}$ if $-\infty < z_0 < \cot(\theta) - \frac{1-\mathrm{e}^{-\frac{\pi\lambda}{\beta}}}{2\sin(\theta)\cos(\theta)}$, and $\frac{\pi}{\beta} < t_1 < \frac{2\pi}{\beta}$ if $\cot(\theta) - \frac{1-\mathrm{e}^{-\frac{\pi\lambda}{\beta}}}{2\sin(\theta)\cos(\theta)} < z_0 < +\infty$.
   Meanwhile at the condition z_0 is positive infinite large

$$\lim_{t\to\frac{2\pi}{\beta}^-} y_0 = -\infty$$

so we know that the evolution time $t_1 \in (0, \frac{2\pi}{\beta})$

$$\frac{r^+}{r^-} = -\left[\frac{(-C+\lambda)\mathrm{e}^{\lambda t}+\lambda}{C} + \frac{\sin(\beta t)\beta\left(\mathrm{e}^{\lambda t}-1\right)}{C(\cos(\beta t)-1)}\right] \tag{40}$$

$$\begin{aligned}\frac{z^+}{z^-} &= \mathrm{e}^{\lambda t} + \frac{\sin(\theta)\mathrm{e}^{\lambda t}-\sin(\theta)}{\cos(\theta)z_0} \\ &= 1 + (\frac{\tan(\theta)}{z_0}+1)(\mathrm{e}^{\lambda t}-1) \\ &= 1 + \frac{((\beta\cos(\beta t)+(C-\lambda)\sin(\beta t))\,\mathrm{e}^{\lambda t}-\beta)(\mathrm{e}^{\lambda t}-1)}{(\cos(\theta)^2\cos(\beta t)\beta+(C-\lambda)\sin(\beta t)-\cos(\theta)^2\beta+\beta)\,\mathrm{e}^{\lambda t}-\beta}\end{aligned} \tag{41}$$

Now we should try to know the features of $\frac{r^+}{r^-}$ and $\frac{z^+}{z^-}$:

$$\lim_{t_1\to\frac{2\pi^-}{\beta}}\frac{r^+}{r^-} = +\infty,\quad \lim_{t_1\to\frac{\pi^-}{\beta}}\frac{r^+}{r^-} = \mathrm{e}^{\frac{\pi\lambda}{\beta}} - \frac{\lambda(1+\mathrm{e}^{\frac{\pi\lambda}{\beta}})}{C},\quad \lim_{t_1\to 0^+}\frac{r^+}{r^-} = 1$$

and

$$\lim_{t_1\to\frac{2\pi^-}{\beta}}\frac{z^+}{z^-} = \mathrm{e}^{\frac{2\pi\lambda}{\beta}},\quad \lim_{t_1\to\frac{\pi^-}{\beta}}\frac{z^+}{z^-} = \frac{2\cos(\theta)^2+\mathrm{e}^{\frac{\pi\lambda}{\beta}}-1}{2\cos(\theta)^2+\mathrm{e}^{\frac{-\pi\lambda}{\beta}}-1},\quad \lim_{t_1\to 0^+}\frac{z^+}{z^-} = 1$$

We can know for a LCO with limited z_0 (physical meaningful), we can only define a $\phi_1(r^-, z^-) = (1+R_r)r^-$, $(1 < R_r < \mathrm{e}^{-\frac{\pi\lambda}{\beta}})$ to get a stable LCO, while $\phi_2(r^-, z^-)$ can be independently defined.According to the Eq.41, the z^+/z^- can be always equal to 1 when $z_0 = -\tan(\theta)$.

If we set $0 \le \phi_2(r^-, z^-) \le 1$, in first case, starting from positive value, the coordinates z_0^k will shrink to zero after enough loops k, but we should notice that $\dot{z} = -\beta\sin(\theta)y_0 < 0$, so series z_0 will continue to go down less than 0 and then arrive at $-\tan(\theta)$ where $z^+/z^- = 1$; in another case, starting from value less than $-\tan(\theta)$, the z_0^k will directly shrink to $z = -\tan(\theta)$; in last case, $\tan(\theta) < z_0^0 < 0$, z^+/z^- will always be greater than 1, so the z_0^k will continuously grow in absolute value and converge at $z = -\tan(\theta)$. In a nutshell, the LCO will always hit the boundary with $z_0 = -\tan(\theta)$ in this case, no mater how large the LCO is.

If we set $1 \le \phi_2(r^-, z^-) \le \mathrm{e}^{\frac{-2\pi\lambda}{\beta}}$. The limit cycle oscillation will converge at some value $z_{L2} \le -\tan(\theta)$. There is another critical value for z which is $z_{cr} = \tan(\theta)(\mathrm{e}^{-\lambda\tau}-1)$, τ is controlled by the y_0, z_0. Here if starting point $0 < z_0^k < z_{cr}$, then $z_0^{k+1} < 0$. So, for the $z_{cr} < z_0 < +\infty$, if the $\phi_2(r^-, z^-)$ can't push z away from 0, the coordinate z will always become negative later on (converging at a point or diverging). ■

2. $\alpha \neq \lambda$

By defining $\gamma = \frac{\alpha-\lambda}{\beta}$, $\tau = \beta t$, and substituting Eq.37 to Eq.36 we can get

$$z_0 = \cot(\theta) + \frac{\beta(\mathrm{e}^{-\frac{\lambda}{\beta}\tau}-1)+(-C+\lambda)\mathrm{e}^{\gamma\tau}\sin(\tau)}{\beta\sin(\theta)\cos(\theta)\varphi_2^\gamma(\tau)} \tag{42}$$

$$\begin{aligned}y_0 &= -\frac{\sin(\theta)\mathrm{e}^{-\tau\gamma}\varphi_1^\gamma(\tau)}{\sin(\tau)}\left(z_0 - \frac{(\mathrm{e}^{-\frac{\lambda\tau}{\beta}}-1)+\cos(\theta)^2\varphi_1^\gamma(\tau)}{\sin(\theta)\cos(\theta)\varphi_1^\gamma(\tau)}\right) \\ &= \frac{(C-\lambda)\varphi_1^\gamma(\tau)+\beta\gamma(\mathrm{e}^{-\frac{\lambda\tau}{\beta}}-1)}{\beta\cos(\theta)\varphi_2^\gamma(\tau)}\end{aligned} \tag{43}$$

where $\varphi_1^\gamma(\tau) = 1-\mathrm{e}^{\gamma\tau}\cos(\tau)$, $\varphi_2^\gamma(\tau) = 1-\mathrm{e}^{\tau\gamma}(\cos(\tau)-\gamma\sin(\tau)) = \varphi_1^\gamma + \gamma\mathrm{e}^{\gamma\tau}\sin(\tau)$.

When $\gamma < 0$, $\varphi_1^\gamma(\tau) > 0$ and $\varphi_2^\gamma(\tau) > 0$ for $\tau \in (0, 2\pi)$; Otherwise $\gamma > 0$, $\varphi_1^\gamma(\tau)$ has two non-trivial roots, namely $\tau_c^1(\gamma)_1$and $\tau_c^1(\gamma)_2$, and $\varphi_2^\gamma(\tau)$ has one non-trivial root $\tau_c^2(\gamma)$. $\frac{\partial\varphi_2(\tau)}{\partial\tau} = \mathrm{e}^{\tau\gamma}\sin(\tau)(\gamma^2+1)$

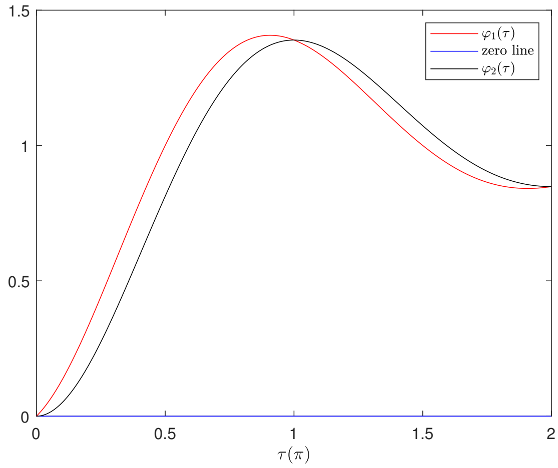

Fig. 5. $\gamma < 0$

(1) $\gamma < 0$

There is no limit of the evolution time for the point on the boundary to return, actually ($C > 0$)

$$\lim_{\tau \to 0^+} z_0 = -\infty, \quad \lim_{\tau \to 0^+} y_0 = +\infty$$

$$\lim_{\tau \to +\infty} z_0 = +\infty, \quad \lim_{\tau \to +\infty} y_0 = -\infty$$

and we can know that

$$\frac{r^+}{r^-} = \frac{1}{C}\left(-\lambda + \beta(\gamma^2+1)\frac{\sin(\tau)\mathrm{e}^{\gamma\tau}(\mathrm{e}^{\frac{\lambda\tau}{\beta}}-1)}{\varphi_2^\gamma(\tau)}\right) + \frac{(C-\lambda)}{C}\frac{\mathrm{e}^{(2\gamma+\frac{\lambda}{\beta})\tau}\varphi_2^{-\gamma}(\tau)}{\varphi_2^\gamma(\tau)} \tag{44}$$

$$\frac{z^+}{z^-} = \frac{(C-\lambda)N_1 + N_2 + N_3}{(C-\lambda)D_1 + D_2 + D_3}$$

where

$$D_1 = \left(-\cos(\theta)^2\gamma\varphi_1^\gamma + \gamma\varphi_3^\gamma\right), \quad \varphi_3^\gamma = 1 - \mathrm{e}^{\tau\gamma}(\cos(\tau) + \frac{\sin(\tau)}{\gamma})$$

$$D_2 = \beta(\gamma^2+1)\left(\mathrm{e}^{-\frac{\lambda\tau}{\beta}} - \sin(\theta)^2\right)$$

$$D_3 = -\beta\cos(\theta)^2\left(1 - \varphi_2^\gamma(\tau) + \gamma^2\mathrm{e}^{\frac{-\lambda\tau}{\beta}}\right)$$

$$N_1 = \mathrm{e}^{(2\gamma+\frac{\lambda}{\beta})\tau}D_1(-\gamma)$$

$$N_2 = -\beta(\gamma^2+1)\cos(\tau)\mathrm{e}^{\gamma\tau}\left(\mathrm{e}^{\frac{\lambda\tau}{\beta}} - \sin(\theta)^2\right)$$

$$N_3 = \beta\cos(\theta)^2\left(\gamma^2\mathrm{e}^{\frac{\lambda\tau}{\beta}}(1-\varphi_3^\gamma(\tau)) + 1\right)$$

Now we should try to know the features of $\frac{r^+}{r^-}$ and $\frac{z^+}{z^-}$:

$$\lim_{\tau \to 0^+} \frac{r^+}{r^-} = 1, \quad \lim_{\tau \to +\infty} \frac{r^+}{r^-} = -\frac{\lambda}{C}$$

$$\lim_{\tau \to 0^+} \frac{z^+}{z^-} = 1, \quad \lim_{\tau \to \tau_z^-} \frac{z^+}{z^-} = -\infty, \quad \lim_{\tau \to \tau_z^+} \frac{z^+}{z^-} = +\infty, \quad \lim_{\tau \to +\infty} \frac{z^+}{z^-} = 0$$

τ_z is the pole of the denominator of $\frac{z^+}{z^-}$, and the bigger C is, the greater τ_z is, but $\tau_z < 2\pi$.

(2) $\gamma > 0$

There is limit of the evolution time $(0 < \tau < \pi < \tau_c^2(\gamma))$ for the point on the boundary to return, actually $(C > 0)$

$$\lim_{\tau \to 0^+} z_0 = -\infty, \quad \lim_{\tau \to 0^+} y_0 = -\infty$$

$$\lim_{\tau \to \tau_c^2(\gamma)^-} z_0 = +\infty, \quad \lim_{\tau \to \tau_c^2(\gamma)^-} y_0 = +\infty$$

Now we should try to know the features of $\frac{r^+}{r^-}$ and $\frac{z^+}{z^-}$:

$$\lim_{\tau \to 0^+} \frac{r^+}{r^-} = 1, \quad \lim_{\tau \to \tau_c^2(\gamma)^-} \frac{r^+}{r^-} = +\infty$$

$$\lim_{\tau \to 0^+} \frac{z^+}{z^-} = 1, \quad \lim_{\tau \to \tau_z(\gamma)^-} \frac{z^+}{z^-} = +\infty, \quad \lim_{\tau \to \tau_z(\gamma)^+} \frac{z^+}{z^-} = -\infty, \quad \lim_{\tau \to +\infty} \frac{z^+}{z^-} = 0$$

we suppose $De_z(\tau)$ is the denominator of $\frac{z^+}{z^-}$ and τ_z is the first non-trivial root of the $De_z(\tau)$, and the τ_z is greater with bigger C. By intermediate value theorem, $0 < \tau_z < \pi < \tau_c^2$ is true based on $De_z(0) = 0$, $\dot{De}_z(0) = -C(1 + \gamma^2 \sin(\theta)^2) < 0$ and

$$\begin{aligned} De_z(\pi) = & \beta\gamma^2 \sin(\theta)^2 \left(1 - \mathrm{e}^{\frac{\lambda\pi}{\beta}}\right) + \beta \left(1 - \sin(\theta)^2 \mathrm{e}^{\frac{\lambda\pi}{\beta}}\right) \\ & + \gamma(C - \lambda)\mathrm{e}^{\frac{\lambda\pi}{\beta}} \sin(\theta)^2 (1 + \mathrm{e}^{\pi\gamma}) + \beta \cos(\theta)^2 \mathrm{e}^{(\gamma + \frac{\lambda}{\beta})\pi} \end{aligned} \tag{45}$$

is obviously positive for every term is positive.

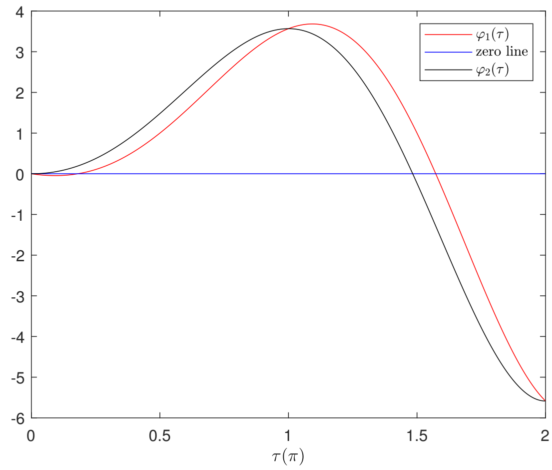

Fig. 6. $\gamma > 0$

For Example,

(1) $\alpha = -0.1, \ \beta = 0.2, \ \gamma = -0.1, \ \theta = 0, \ R_r = 1.5$

(2) $\alpha = -0.1, \ \beta = 0.2, \ \gamma = -0.1, \ \theta = \pi/6, \ R_r = 1.5$

2.1.3 Boundary equilibrium

If there is any pseudo equilibrium, it must be on the line l_3. The sliding vector field will be

$$F_s = (\mathbf{I} - \frac{\mathbf{BCA}}{\mathbf{CBA}})\mathbf{A}(x - X_0) = \mathbf{A_s}(x - X_0) \tag{46}$$

if we set $F_s = 0$, the first equation is just the equation of line l_3, and the rank of $\mathbf{A_s}$ is 2, so the firs equation and any one of the left two to solve the pseudo equilibrium point.

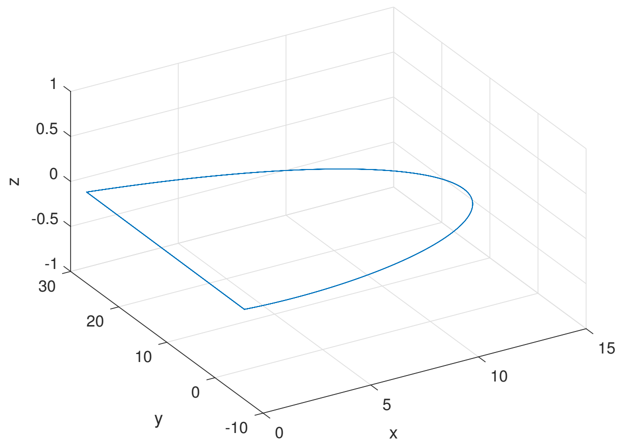

Fig. 7. 3D LCO case3

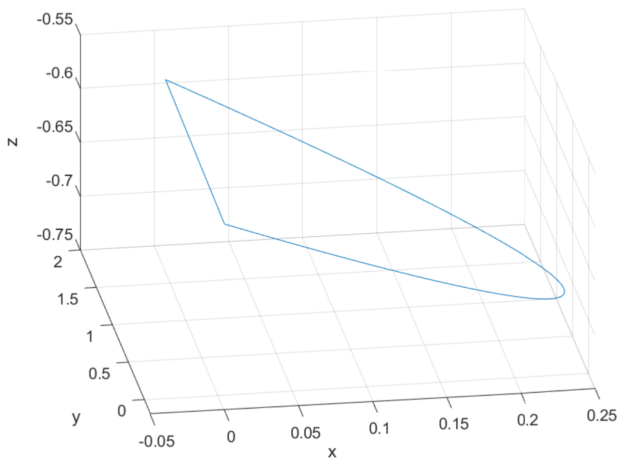

Fig. 8. 3D LCO case4

$$X_{pseudo} = \left[dd\right]$$

and the non-trivial eigenvalue of the $\mathbf{A_s}$ is

$$\lambda_s = \frac{(1+\varphi_r)\left((\gamma+\eta)(1+\sin^2(\theta)\gamma^2) - \gamma\cos^2(\theta)\right) - \varphi_z(\gamma^2+1)\sin(\theta)}{(\gamma^2\sin^2(\theta)+1)(1+\varphi_r)} \tag{47}$$

and we can see that $\lambda_s < 0$ is valid (which means the pseudo equilibrium is stable) if φ_z satisfies

$$\varphi_z > -\frac{\gamma\cos^2(\theta) - (\gamma+\eta)(1+\sin^2(\theta)\gamma^2)}{(\gamma^2+1)\sin(\theta)} \tag{48}$$

2.1.4 Existence of the LCO

For a general first order ODE like $\dot{y} = \mathbf{A}y$, there is a general solution $y = \mathrm{e}^{\mathbf{A}t}\mathbf{y}_0$, if $\mathbf{V}$ and $\mathbf{D}$ are corresponding eigenvector matrix and eigenvalue matrix of $\mathbf{A}$, we can further get another

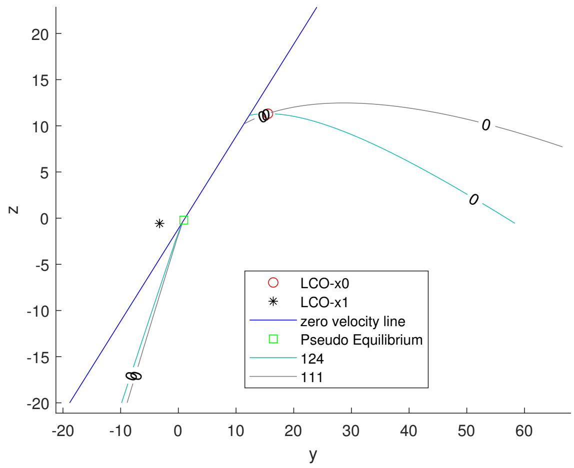

Fig. 9. Lyapunov function discrepancy contour

form

$$\begin{aligned} y &= \mathbf{V}e^{\mathbf{D}t}\mathbf{V}^{-1}\mathbf{y}_0 \\ &\mathbf{V}\ diag(\mathbf{V}^{-1}\mathbf{y}_0)\cdot[e^{\lambda_1 t},\cdots,e^{\lambda_n t}]^\top \quad (49)\\ &= \mathbf{E}\ \mathbf{S}(t) \quad (50)\end{aligned}$$

where $\mathbf{E} = \mathbf{V}\ diag(\mathbf{V}^{-1}\mathbf{y}_0)$ and $\mathbf{S}(t) = [e^{\lambda_1 t},\cdots,e^{\lambda_n t}]^\top$, $\mathbf{y} = \mathbf{x}-\mathbf{x}^0$, and the $\mathbf{x}^0$ is the admissible equilibrium in the region $H(x) < 0$. We have known that there is onset of LCO after the

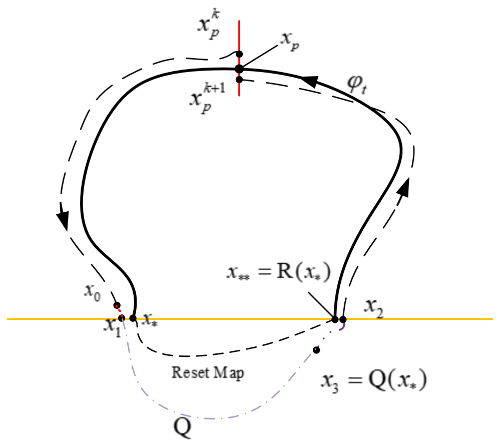

Fig. 10. Poincaré map of the LCO

boundary equilibrium as iluustrated by Fig.10. If we construct a Poincaré map around $\mathbf{x}_p$

$$\Phi\left(\mathbf{x}_p, T^*\right) = \mathbf{x}_p, \quad |\mathbf{x}_p| > 0$$

With $\beta(T) = E(3,:)S(T)$, $\beta(T^*) = \beta_0$ to implicitly solve the smallest positive value T^*.

Recalling the composed map

$$\Phi(\mathbf{x}_0,\tau) = \left(\varphi\circ R\circ\varphi\right)\left(\mathbf{x}_0,\tau(\mathbf{x}_0)\right)$$

, where $\mathbf{R} \to x^+ = x^- + \mathbf{W}\cdot x^- = \mathbf{P}x^-$, $\varphi(x,\tau)$ is flow in linear region. From $\mathbf{x}_p$, the time needed to arrive the impacting surface at $\mathbf{x}_*$ is t_1, and then via the reset map, the evolution time from $\mathbf{x}_{**}$ back to $\mathbf{x}_p$ will be $T-t_1$, so the periodic oribit can be constructed by

$$\varphi(Q(\varphi(\mathbf{x}_p,t_1)),T-t_1) = \mathbf{x}_p \quad (51)$$

where the Q is the modified discontinuity map in [**Bernardo2007**].

If we choose the discontinuity boundary as Poincaré section ($t_1 = 0$), and we say the same equation as Eq.51 will be found as

$$\mathbf{P}\mathrm{e}^{\mathbf{A}T^*}(\mathbf{x}_{**} - \mathbf{x}^0) = \mathbf{x}_{**} - \mathbf{x}^0 = \mathbf{y}_{**} \tag{52}$$

and obviously the $\mathbf{y}_{**}$ is the eigenvector of matrix $\mathbf{P}\mathrm{e}^{\mathbf{A}T^*}$ corresponding to the unit eigenvalue, and $\mu\mathbf{y}_{**}$, where μ is a scalar, is also an eigenvector of the unit eigenvalue, which proves that the ratio between amplitudes of different state variables is the same and linearly dependent to the parameter μ.

In our 3D case, $\mathbf{x}^0 = [-1, 0, 0]^\top$. If we want to find a STABEL LCO, there must a T^* make matrix $\mathbf{P}\mathrm{e}^{\mathbf{A}T^*}$ have a unit eigenvalue with the BIGGEST norm, with eigenvector $\mathbf{y}_{**} = [1, c_1, c_2]^\top$ (normalized by first element), obviously the desired starting point of the LCO is $(0, c_1, c_2)$. For all starting points with given evolution T^* to hit the $H(x) = 0$ again, they must be on a same line, on the discontinuity surface, $l_t = \{(0, d_1, d_2)|[1, 0, 0]\mathrm{e}^{\mathbf{A}T^*}[1, d_1, d_2]^\top = [1, 0, 0]\mathbf{y}_{**} = 1\}$. We want to find a T^* so that $(0, c_1, c_2)$ is on l_t and simultaneously $\mathbf{y}_{**} = [1, c_2, c_3]^\top$ is the eigenvector corresponding to the unit eigenvalue

$$\begin{aligned}&(\mathbf{P}\mathrm{e}^{\mathbf{A}T^*} - \mathbf{I})[1, c_2, c_3]^\top \\ &= \begin{bmatrix} f_1(c_1, c_2, T^*) \\ f_2(c_1, c_2, T^*) \\ f_3(c_1, c_2, T^*) \end{bmatrix} \\ &= \mathbf{0}\end{aligned} \tag{53}$$

If $f_1 = 0$ is valid, the $(0, c_1, c_2)$ is on $l_t(T^*)$;if $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ are all valid then we find our LCO (staring with $(0, c_1, c_2)$), and the it's stable if the biggest norm of eigenvalues is 1, otherwise unstable.

So we now can construct a numerical scheme to justify the existence of the LCO:

(1) define function $MAX(T) > 0$ as the biggest eigenvalue of matrix $\mathbf{P}\mathrm{e}^{\mathbf{A}T^*}$;

(2) use f_2 and f_3 to solve $c_1(T),\ c_2(T)$;

(3) substitute the $c_1(T),\ c_2(T)$ got in step 2 into $F_1(T) = f_1(c_1(T), c_2(T), T)$

(4) plot the function $F_1(T)$ and $MAX(T) - 1$

Judgments:

▷ If there is no T^* let $F_1(T^*) = 0$, there will be no LCO;

▷ If there is T^* let $F_1(T^*) = 0$, there will be a LCO, stable if $MAX(T^*) - 1 = 0$, unstable if $MAX(T^*) - 1 > 0$.

Examples:

(1) $\alpha = -0.1,\ \beta = 0.2,\ \gamma = -0.1,\ \theta = \pi/6,\ R_r = 0.8,\ R_z = 0$

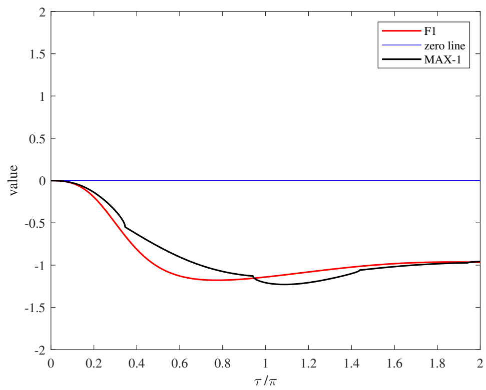

Fig. 11. case 1:No LCO

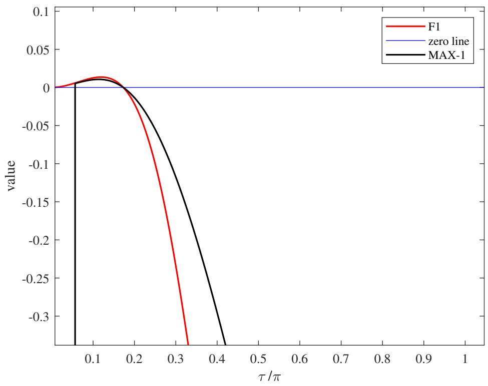

Fig. 12. case 2: LCO

(2) $\alpha = -0.1,\ \beta = 0.2,\ \gamma = -0.1,\ \theta = \pi/6,\ R_r = 1.5,\ R_z = 0$

(3) $\alpha = -0.1,\ \beta = 0.2,\ \gamma = -0.1,\ \theta = \pi/6,\ R_r = 0.8,\ R_z = 4$

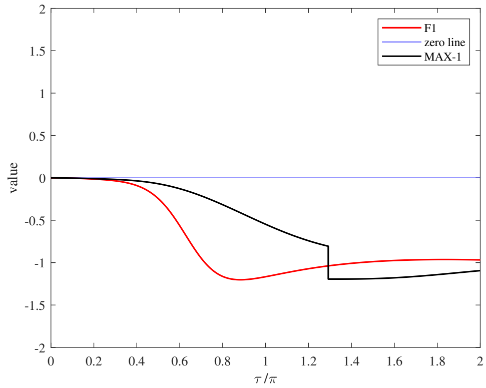

Fig. 13. case 3: No LCO

(4) $\alpha = -0.1,\ \beta = 0.2,\ \gamma = -0.1,\ \theta = \pi/6,\ R_r = 0.8,\ R_z = 8$

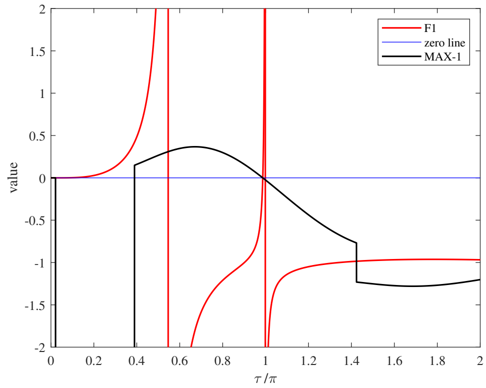

Fig. 14. case 4: LCO

Discussion about saltation matrix:

To prove the stability of the LCO, we need to find the Jacobi matrix J^* around $\mathbf{x}_p$.

$$J^* = \varphi_x(x_{**}, T^* - t_1)\ Q_x(x_*)\ \varphi_x(x_p, t_1)$$

If we denote matrix function $\mathbf{L}(\mathbf{x}_0) = diag(\mathbf{V}^{-1}\mathbf{x}_0)$, and $\mathbf{L}_{i,i} = \sum_{j=1}^{8} \mathbf{V}_{i,j}^{-1} x_0(j)$ with zeros on non-diagonal positions. Then we observe that $\frac{\partial \mathbf{L}_{i,i}}{\partial \mathbf{x}_0^\top} = \mathbf{V}^{-1}(i,:)$, $\mathbf{E}(i,j) = \mathbf{V}(i,:)\mathbf{L}(:,j) = \mathbf{V}(i,j)\mathbf{L}(j,j)$, which can lead to

$$(\mathbf{ES})(i) = \sum_{j=1}^{8} \mathbf{E}(i,j)\mathbf{S}(j) = \sum_{j=1}^{8} \mathbf{V}(i,j)\mathbf{L}(j,j)\mathbf{S}(j)$$

then we can get

$$\frac{\partial(\mathbf{ES})(i)}{\partial(x^*)^\top} = \sum_{j=1}^{8} V(i,j)\frac{\partial L(j,j)}{\partial(x^*)^\top} S(j) \tag{54}$$

furthermore we can write the φ_x as

$$\begin{aligned}\varphi_x &= \left(\mathbf{V} diag(\mathbf{S}(T)) \begin{bmatrix} \frac{\partial \mathbf{L}_{1,1}}{\partial \mathbf{x}_0^\top} \\ \vdots \\ \frac{\partial \mathbf{L}_{2,2}}{\partial \mathbf{x}_0^\top} \end{bmatrix}\right) \\ &= \mathbf{V}\mathrm{e}^{\mathbf{D}T}\mathbf{V}^{-1} \end{aligned} \tag{55}$$

The so-called saltation matrix

$$\begin{aligned}\mathbf{Q}_x(\mathbf{x}_*) &= \mathbf{R}_x(\mathbf{x}_*) + \frac{[F(\mathbf{R}(\mathbf{x}_*)) - \mathbf{R}_x(\mathbf{x}_*)F(\mathbf{x}_*)]H_x(\mathbf{x}_*)}{H_x(\mathbf{x}_*)F(\mathbf{x}_*)} \\ &= \mathbf{P} + \frac{[F(\mathbf{x}_{**}) - \mathbf{P}F(\mathbf{x}_*)]\ C}{C\ F(\mathbf{x}_*)} \end{aligned} \tag{56}$$

Eventually

$$J^* = \mathbf{V}\mathrm{e}^{\mathbf{D}(T^*-t_1)}\mathbf{V}^{-1}\mathbf{Q}_x\mathbf{V}\mathrm{e}^{\mathbf{D}t_1}\mathbf{V}^{-1} \tag{57}$$

$$= \mathbf{V}\mathrm{e}^{\mathbf{D}(T^*-t_1)}\mathbf{V}^{-1}\left(\mathbf{P} + \frac{[F(\mathbf{x}_{**}) - \mathbf{P}F(\mathbf{x}_*)]\ C}{C\ F(\mathbf{x}_*)}\right)\mathbf{V}\mathrm{e}^{\mathbf{D}t_1}\mathbf{V}^{-1} \tag{58}$$