DYNAMICS NEAR A GRAZING SHILNIKOV HOMOCLINIC ORBIT IN IMPACTING SYSTEMS

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Abstract. We describe dynamics and geometry of piecewise smooth flows derived from impacting systems which have a saddle-focus homoclinic orbit.

1. Introduction

What we know at the moment: for our simple system the dynamics can be complicated and the geometry of grazing surfaces is interesting. What we don't know is whether or how these two features interact!

2. A reduced model near an impacting boundary equilrium

Taking an impact system, with normal displacement from the impact surface x we have generally

$$\begin{array}{rrcll} & \dot{x} & = & y & \\ & \dot{y} & = & f(x,y,z) & \\ & \dot{z} & = & g(z,y,z) & \\ \& & y & \mapsto & -ry & \text{impact law on } x=0,\ y<0\ , \end{array} \tag{1}$$

for $x, y \in \mathbb{R}$, $z \in \mathbb{R}^n$, $0 < r \leq 1$, where $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}^n$ are smooth. Denote the impact surface $x = 0$ as Σ.

Assume that

(1) there exists an equilibrium, where $f = g = 0$, at a point $(x, y, z) = (x_0, 0, z_0)$,
(2) the Jacobian $J = \frac{\partial(\dot{x},\dot{y},\dot{z})}{\partial(x,y,z)}$ at $(x_0, 0, z_0)$ is non-singular, and
(3) J has $n-2$ real stable eigenvalues,
(4) J has two complex unstable eigenvalues associated with a 2-dimensional unstable manifold W^u,
(5) W^u lies non-orthogonal to the x and y coordinates,
(6) W^u intersects the impact surface Σ transversally.

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In appendix A we show that this can be reduced, to leading order, to a three dimensional linear impacting system

$$\begin{array}{lrcl} & \dot{x} & = & \rho(x-1)+\omega y \\ & \dot{y} & = & -\omega(x-1)+\rho y+z \\ & \dot{z} & = & 1-z \\ \& & y & \mapsto & -ry\ ,\quad z\mapsto z-cy \qquad \text{impact law on } x=0\ , \end{array} \tag{2}$$

The motivating example comes from a model of chatter in a pressure relief valve [H. Csaba and A. R. Champneys 2016 Grazing bifurcations and chatter in a pressure relief valve model, Physica D, in press]

$$\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -\kappa y-(x+\delta)+z \\ \dot{z} & = & \beta(q-x\sqrt{z}) \end{array} \tag{3}$$

which for $q>0$ has a unique equlibrium at $(p-\delta,0,p)$ where $(p-\delta)\sqrt{p}=q$. In appendix A this is shown to satisfy the conditions above.

3. Basic geometry of the reduced model

I *THINK* I HAVE TAKEN ALL OF THE bs OUT OF THIS SECTION, BUT IT NEEDS CHECKING

From the previous section we choose to work with the system for $x>0$ defined by

$$\begin{array}{rl} \dot{x} & =\rho(x-1)+\omega(y-y_0) \\ \dot{y} & =-\omega(x-1)+\rho(y-y_0)+(z-z_0) \\ \dot{z} & =-\lambda(z-z_0) \end{array} \tag{4}$$

with $\rho,\lambda,\omega>0$ and

$$y_0=-\rho/\omega,\quad z_0=(\rho^2+\omega^2)/(\omega). \tag{5}$$

The first step is to determine some of the geometric properties of these solutions before they intersect $x=0$ after which the generalized impact map is applied.

A solution starting with $\dot{x}>0$ on $x=0$ leaves the impact plane $x=0$ and may eventually return with $\dot{x}\leq 0$. On $x=0$

$$\dot{x}=-\rho+\omega(y-y_0)=\omega y$$

so the condition that $\dot{x}>0$ is simply $y>0$, $\dot{x}<0$ if $y<0$ and the grazing surface on $x=0$ is the z-axis:

$$\text{grazing criterion}:\quad y=0. \tag{6}$$

Since

$$\ddot{x}=\rho\dot{x}+\omega\dot{y}$$

a little manipulation shows that for the choices (5), on the impact surface

$$\ddot{x} = 2\rho\omega y + \omega z$$

and so on the grazing manifold $y = 0$, $\ddot{x} > 0$ if $z > 0$ (invisible folds) and $\ddot{x} < 0$ if $z < 0$ (visible folds).

Finally the genericity condition

$$\frac{d^3}{dt^3}x = \frac{\lambda}{\omega}(\rho^2 + \omega^2) > 0$$

holds at the degenerate point $(y, z) = (0, 0)$ on the grazing manifold on $x = 0$.

An impacting solution will have $\dot{x} \to -r\dot{x}$, or $y \to -ry$ for some $r \in (0, 1)$. This does not fully determine the reset, and we will choose $z \to z + cy$ as the impact reset in the z direction (at least for the moment). So the impact maps is

$$\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} -r & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{7}$$

We now turn our attention the the stationary point $(1, y_0, z_0)$. This has a two-dimensional unstable manifold associated with linearized eigenvalues $\rho \pm i\omega$ and this intersects the impact plane at $z = z_0$. The only dynamically interesting region is where this has an impact, so this is the half-line $z = z_0$ with $y < 0$ which is mapped by the impact map (7) to

$$z = z_0 - \tfrac{c}{r}y, \quad y > 0. \tag{8}$$

The local stable manifold of the stationary point is the half-line

$$\begin{pmatrix} 1 \\ y_0 \\ z_0 \end{pmatrix} + R \begin{pmatrix} \omega \\ -(\rho + \lambda) \\ (\rho + \lambda)^2 + \omega^2 \end{pmatrix}, \quad -\tfrac{1}{\omega} < R < \infty$$

and this intersects the impact surface $x = 0$ when $R = -\frac{1}{\omega b}$ at

$$P_s = \left(0, \frac{\lambda}{\omega}, -\frac{\lambda(\lambda + 2\rho)}{\omega}\right)^T. \tag{9}$$

Thus there is a homoclinic orbit if P_s lies on the image of the impacting part of the unstable manifold of $(1, y_0, z_0)$, (8). This condition is

$$cb\lambda = \rho^2 + \omega^2 + r\lambda(\lambda + 2\rho). \tag{10}$$

PLEASE CHECK THE ABOVE FORMULA I THINK THERE SHOULD BE r MULTIPLYING THE $\rho^2 + \omega^2$ TERM TOO.

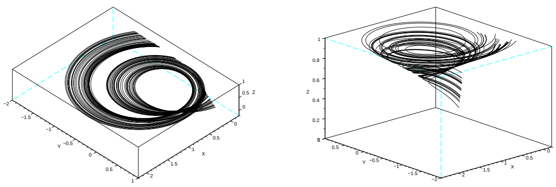

FIGURE 1. Attractors with $\rho = 0.1$, $\omega = 1$. For left-hand figure $\lambda = 0.3$, $r = 0.5$, $c = 0.8$; and in right-hand figure $\lambda = 0.2$, $r = 0.136$, $c = 0.66$.

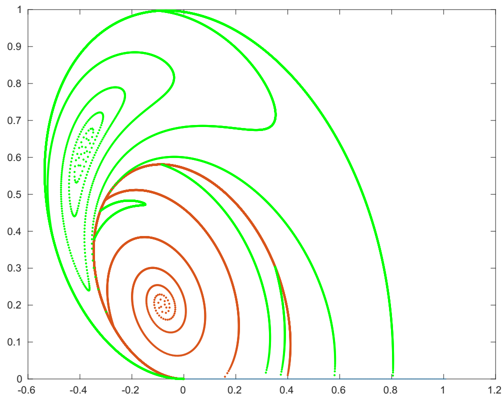

FIGURE 2. G_0, G_1 and G_2 for $\rho = 0.1$, $\omega = 1$, $\lambda = 0.2$, $r = 0.5$, $c = 0.8$.

3.1. **Dynamics and grazing.** Consider Fig. 1. The first panel shows the dynamics at two different parameter values – the solutions were found using a crude scilab programme, and not run for a very long time (30 intersections with the impacting surface only).

The second figure shows the grazing surface and its first two preimages:

3.2. **An implicit return map.** Since it is easy to solve (4) explicitly with initial conditions

$$x(0) = 0, \quad y(0) = Y > 0, \quad z(0) = Z \tag{11}$$

it is possible to write down expressions for the next intersection with the impact plane (if it exists) in terms of a next return time defined by a transcendental equation. In new coordinates

$$u = x - 1, \quad v = y - y_0, \quad z = w - z_0 \tag{12}$$

(4) becomes

$$\begin{aligned} \dot{u} &= \rho u + \omega v \\ \dot{v} &= -\omega u + \rho v + bw \\ \dot{w} &= -\lambda w \end{aligned} \tag{13}$$

with initial conditions from (11)

$$u(0) = -1, \quad v(0) = Y - y_0, \quad w(0) = Z - z_0. \tag{14}$$

Set $\xi = u + iv$, then

$$\begin{aligned} \dot{\xi} &= (\rho - i\omega)\xi + ibw \\ \dot{w} &= -\lambda w. \end{aligned} \tag{15}$$

Solutions are therefore

$$\begin{aligned} \xi &= e^{\rho t}(Ae^{i\omega t} + Be^{-i\omega t}) + De^{-\lambda t} \\ w &= (Z - z_0)e^{-\lambda t} \end{aligned} \tag{16}$$

for complex coefficients A, B and D chosen so that the solution satisfies the first differential equation of (15) and $\xi(0)$ satisfies the initial conditions implied by (14). After some algebra (which needs to be checked!) the solutions to the original equations are found to be

$$\begin{aligned} x(t) &= 1 - \left(1 + \tfrac{\omega(Z-z_0)}{(\lambda+\rho)^2+\omega^2}\right) e^{\rho t}\cos\omega t + \\ &\left(Y - y_0 + \tfrac{(\lambda+\rho)(Z-z_0)}{(\lambda+\rho)^2+\omega^2}\right) e^{\rho t}\sin\omega t + \tfrac{\omega(Z-z_0)}{(\lambda+\rho)^2+\omega^2}e^{-\lambda t}, \end{aligned} \tag{17}$$

$$\begin{aligned} y(t) &= y_0 + \left(Y - y_0 + \tfrac{(\lambda+\rho)(Z-z_0)}{(\lambda+\rho)^2+\omega^2}\right) e^{\rho t}\cos\omega t + \\ &\left(1 + \tfrac{\omega(Z-z_0)}{(\lambda+\rho)^2+\omega^2}\right) e^{\rho t}\sin\omega t - \tfrac{(\lambda+\rho)(Z-z_0)}{(\lambda+\rho)^2+\omega^2}e^{-\lambda t}, \end{aligned} \tag{18}$$

and

$$z(t) = z_0 + (Z - z_0)e^{-\lambda t}. \tag{19}$$

The return map on $x = 0$, $y > 0$ is defined given values (Y_n, Z_n) by finding the small est $t_n > 0$ such that $x(t_n) = 0$ provided it exists, then the next return to the plane $x = 0$ is $(y_n(t_n), z_n(t_n))$ (functions of the initial values (Y_n, Z_n) which then map by the impact map (7) to

$$Y_{n+1} = -ry_n(t_n), \quad Z_{n+1} = cy_n(t_n) + z_n(t_n). \tag{20}$$

3.3. **A codimension two point.** The part of the grazing surface that can be reached potentially by recurrent dynamics is G_0, i.e.

$$y = 0, \quad 0 \le z \le z_0.$$

The left end point of this line segment is the singular point $(0, 0)$ and the right end point is the intersection of the unstable manifold of the fixed point with the grazing surface. Let us look for a grazing homoclinic orbit.

The grazing solution in the unstable manifold intersects $x = 0$ at $(0, 0, z_0)$ and the continuation of this orbit next strikes the impact surface at time τ where, using $(Y, Z) = (0, z_0)$ in (17)-(19),

$$\begin{aligned} 0 &= 1 - e^{\rho\tau}(\cos\omega\tau - y_0 \sin\omega\tau) \\ y^* &= y_0 + e^{\rho\tau}(-y_0\cos\omega\tau + \sin\omega\tau) \\ z &= z_0 \end{aligned} \tag{21}$$

and this will be a homoclinic orbit if

$$\begin{pmatrix} -r & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} y^* \\ z_0 \end{pmatrix} = P_s \tag{22}$$

where P_s is the intersection point of the local stable manifold of the stationary point with the impact surface, (9). A little algebra yields

$$r = -\lambda/(\omega y^*), \quad c = -((\lambda + \rho)^2 + \omega^2)/(\omega b y^*)$$

both of which are positive as expected. Evaluation at the parameter values

$$\rho = 0.1, \quad \lambda = 0.2, \quad \omega = b = 1$$

gives (very) approximate solutions

$$\tau \approx 5.243, \quad y^* \approx -1.4715, \quad r \approx 0.136, \quad c \approx 0.741$$

which explains the choice of parameter in Figure 1.

4. Construction of Poincaré return map

Let us introduce some local co-ordinates and parameters. Let us introduce the codimension-two point $c = c^*$, $r = r^*$ such that the position of the homoclinic orbit within the $W^u \cap \{x = 0\}$ is at $y = y^*$. We seek dynamics that is close to this point. We will do this by considering the dynamics in reverse time. Henceforth then a prime will be used to represent differentiation backwards in time.

We shall introduce a further change of co-ordinates:

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = V \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad \text{where } V = \begin{pmatrix} 1 & 0 & \omega \\ 0 & 1 & -(\rho+\lambda) \\ 0 & 0 & \Delta \end{pmatrix},$$

with $\Delta = (\rho + \lambda)^2 + \omega^2$, such that (ξ, η) are aligned with the 2D stable (in reverse time) manifold of the equilibrium and ζ is in the direction of the 1D unstable manifold. Upon the further introduction of polar co-ordinates in the 2D manifold

$$\xi = \nu\cos\theta, \qquad \eta = \nu\sin\theta,$$

the system (4) becomes, simply

$$\begin{aligned} \nu' &= -\rho\nu, \\ \theta' &= \omega, \\ \zeta' &= \lambda\zeta. \end{aligned} \tag{23}$$

NOTE I HAVE USED ν IN PLACE OF WHAT WAS CALLED R TO AVOID CONFUSION WITH THE RESTITUTION MAP

We introduce three local Poincaré sections. First, let

$$\Sigma_0 = \{(x, y, z) = (0, q + \hat{v}, z_0 + \hat{w}),$$

NOTE v_{in} HAS BECOME q TO SIMPLIFY NOTATION where $q = -\lambda/(\omega r)$ is the y-co-ordinate of the image of p_s and where $\hat{v}$ and $\hat{w}$ are small, with $\hat{v} + q \in [y^*, 0]$. Note that the codimension-two point we are interested in is when $q := q^* = -\lambda/(\omega r^*)$. Next, let

$$\Sigma_1 = \{(\xi_1, \eta_1, \zeta_1) : \zeta_1 = \Delta h, \xi_1, \eta_1 \ll 1\}$$

for some $h > 0$ (which we will be determined later). Finally, we choose

$$\Sigma_2 = \{(x, y, z) = p_s + (0, \hat{v}_2, \hat{w}_2)\},$$

where $\hat{v}_2$ and $\hat{w}_2$ are small.

We construct a Poincaré map from Π_0 back to itself by the composition of

$$\Pi : \Sigma_0 \to \Sigma_0, \quad \begin{pmatrix} \hat{w} \\ \hat{v} \end{pmatrix} \mapsto \begin{pmatrix} \hat{w}_3 \\ \hat{v}_3 \end{pmatrix}; \quad \Pi := R^{-1} \circ \Pi_2 \circ \Pi_1,$$

where R is the restitution map and

$$\Pi_1 : \Sigma_0 \to \Sigma_1 \quad \text{and} \quad \Pi_2 : \Sigma_1 \to \Sigma_2$$

are computed by following the flow. Fixed points of the map Π will correspond to periodic orbits that contain a single impact.

4.1. **The leading-order flow maps.** To construct Π_1 we have to reinterpret the initial conditions in Σ_0 in terms of the eigenvector (polar) co-ordinates (ν, θ, ζ). Let $(\nu_0, \theta_0, \zeta_0)^T$ be the initial conditions in Σ_0 and $v_0 = q + \hat{v}$. Then, a little albegra reveals that $\zeta_0 = \Delta\hat{w}$, and

$$\nu_0^2 = (\omega\hat{w} - 1)^2 + v_0 - (\rho + \lambda)\hat{w})^2, \quad \tan\theta_0 = \frac{(\rho + \lambda)\hat{w} - v_0}{1 - \omega\hat{w}}.$$

That shows that, while $\zeta_0 = O(\hat{w})$, ν_0 and θ_0 are constants to leading order, generically. Specifically we have THESE NEED CHECKING

$$R_0 = \sqrt{1 + q^2} + \frac{1}{1 + q^2}[q\hat{v} - (1 + \lambda + \rho)\hat{w}] + \text{h.o.t.}$$

and

$$\theta_0 = \arctan[-q - \hat{v} + (\rho + \lambda - \omega q]\hat{w} + \text{h.o.t.})$$

Now Π_1 is constructed in the usual way by solving the flow (23). We obtain

$$\xi_1 = R_0 \left(\frac{\hat{w}}{h}\right)^{\delta} \cos\left[\theta_0 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{24}$$

$$\eta_1 = R_0 \left(\frac{\hat{w}}{h}\right)^{\delta} \sin\left[\theta_0 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{25}$$

where $\delta = \rho/\lambda$.

To compute Π_2, note the flow takes a bounded time, so to leading order, under apropriate non-degeneracy conditions we can approximate with an affine map that can be written as a linear map in the appropriate local co-ordinates

$$\begin{pmatrix} \hat{v}_2 \\ \hat{w}_2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} + \text{h.o.t.}. \tag{26}$$

WE NEED TO COMPUTE THE cs EXPLICITLY IF WE CAN. IT IS JUST A PROJECTION.

To leading order, we find

$$\hat{v}_2 = K_1 R_0 \left(\frac{\hat{w}}{h}\right)^{\delta} \cos\left[\Phi_1 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{27}$$

$$\hat{w}_2 = K_2 R_0 \left(\frac{\hat{w}}{h}\right)^{\delta} \cos\left[\Phi_2 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{28}$$

where

$$K_1 = \sqrt{c_1^2 + c_2^2}, \quad K_2 = \sqrt{c_3^2 + c_4^2},$$

$$\Phi_1 = \theta_0 + \arctan(c_2/c_1), \quad \Phi_2 = \theta_0 + \arctan(c_4/c_3) - \pi/2.$$

PLEASE CHECK THE ABOVE

Finally, we have to map this point by R^{-1}. Specifically we have

$$\begin{pmatrix} y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1/r & 0 \\ c/r & 1 \end{pmatrix}, \begin{pmatrix} \frac{\lambda}{\omega} + \hat{v}_2 \\ \frac{\lambda(\lambda+2\rho)}{\omega} + \hat{w}_2 \end{pmatrix}$$

which, after some calcualtion, gives

$$y_3 = q + K_3 R_0 \left(\frac{\hat{w}}{h}\right)^{\delta} \cos\left[\Phi_1 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{29}$$

$$z_3 = \frac{\lambda}{\omega r}(c - c^*) + K_4 R_0 \cos\left[\Phi_4 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{30}$$

where

$$K_3 = \frac{K_1}{r}, \quad K_4 = \ldots, \quad \Phi_4 = \ldots$$

Thus, we have

$$\hat{v}_3 = K_3 R_0 \left(\frac{\hat{w}}{h}\right)^\delta \cos\left[\Phi_1 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{31}$$

$$\hat{w}_3 = \hat{c} + K_4 R_0 \left(\frac{\hat{w}}{h}\right)^\delta \cos\left[\Phi_4 - \frac{\omega}{\lambda}\ln\left(\frac{\hat{w}}{h}\right)\right], \tag{32}$$

where

$$\hat{c} = \frac{\lambda(c - c^*)}{\omega r}.$$

I THINK THERE MAY BE AN ADDITIONAL CONSTANT TERM NECESSARY IN THE V EQUATION BECAUSE OF THE DIFFERENCE BETWEEN y_0 AND q

4.2. **Looking for fixed points.** I AM JUST GOING TO GET ON AND GET THE LOGIC DOWN, WE CAN CHECK DETAILS LATER

Equations (31), (32) define a map $(\hat{v}, \hat{w}) \to (\hat{v}', \hat{w}\prime)$ where, after redefining the constants and setting $\Xi = \omega/\lambda$,

$$\begin{aligned} \hat{v}' &= C_v \hat{w}^\delta \cos(\Phi_v + \Xi \ln \hat{w}) + \text{h.o.t}, \\ \hat{w}' &= \hat{c} + C_w \hat{w}^\delta \cos(\Phi_w + \Xi \ln \hat{w}) + \text{h.o.t}, \end{aligned} \tag{33}$$

valid for $\hat{w} > 0$ and $|\hat{v}|$ sufficiently small and $\hat{v} \in (y_0 - q, -q)$. Fixed points of (33) correspond to periodic orbits with one impact and which pass once through a neighbourhood of the homoclinic orbit. These satisfy $\hat{v}' = \hat{v}$ and $\hat{w}' = \hat{w}$. Since the first of equationa (33) is independent of $\hat{v}$ to lowest order we concentrate on the second. Solutions of the fixed point equation

$$\hat{w} - \hat{c} = C_w \hat{w}^\delta \cos(\Phi_w + \Xi \ln \hat{w}) + \text{h.o.t} \tag{34}$$

in $\hat{w} > 0$ with $\hat{w}$ sufficiently small imply the existence of a fixed point for (33) provided the corresponding solution for $\hat{v}$:

$$\hat{v} = C_v \hat{w}^\delta \cos(\Phi_v + \Xi \ln \hat{w}) + \text{h.o.t}, \tag{35}$$

lies in $(y_0 - q, -q)$ with $\hat{v}$ sufficiently small.

These saddlenode bifurcations occur (asymptotically) at $\hat{w}$ values very close to the maxima and minima of the right hand side of (34, i.e. at values $\hat{w}_n$ with

$$|\cos(\Phi_w + \Xi \ln \hat{w}_n)| \approx 1.$$

This implies that

$$\Xi \ln \hat{w}_n \approx (2n+1)\pi + (-1)^n \frac{1}{2}\pi - \Phi_w \tag{36}$$

endequation and so

$$\Xi \ln \left(\frac{\hat{w}_n}{\hat{w}_{n+1}} \right) \approx \pi.$$

or

$$\left(\frac{\hat{w}_n}{\hat{w}_{n+1}} \right) \approx \exp \left(\frac{\pi \lambda}{\omega} \right). \tag{37}$$

Since the corresponding parameters $\hat{c}_n$ scale like $\hat{w}_n^\delta$ and oscillate in sign this implies that there are sequences of parameters tending to zero from above and below at the (asymptotic) rate

$$\lim_{n \to \infty} \frac{c_{n+2} - c_n}{c_n - c_{n-2}} = \exp \left(-\frac{2\rho}{\omega} \right). \tag{38}$$

This the same as the standard smooth Shilnikov scaling.

Sufficiently small values of $\hat{w}$ satisfying (34) will denerate values of $\hat{v}$ in (35) which have small modulus, so the only way that the standard local analysis can break down is if $\hat{v}$ is outside the interval $(y_0 - q, -q)$. This leads to the possibility of grazing bifurcations which will be considered in the next subsection.

4.3. **Grazings of periodic orbits.** Recall that the grazing surface in $x = 0$ is the line $y = 0$ by construction, and the next impact of the point on the intersection of this line with the unstable manifold of the stationary point, i.e. the point $(0, 0, 0)$, is $(0, y_0, 0)$. Thus the next impact of points on the grazing line close to $(0, 0, 0)$ is a curve and since the flow does not pass close to the stationary point its return map can be expanded as a Taylor series locally:

$$y = y_0 + k_1 \hat{w} + O(|\hat{w}|^2). \tag{39}$$

The trajectory of any point that impacts on this curve must have undergone a graze with the impact surface as it spirals out in the (x, y)-direction. Since $y = q + \hat{v}$ (39) implies that in terms of the local variables the image of the grazing locus is

$$\hat{v} = y_0 - q + k_1 \hat{w} + O(|\hat{w}|^2). \tag{40}$$

A grazing bifurcation of a periodic orbit therefore exists if a solution of the fixed point equations (34), (35) intersects the image of the grazing locus (40). Equating $\hat{v}$ in (35) and (40) gives

$$y_0 - q = C_v \hat{w}^\delta \cos(\Phi_v + \Xi \ln \hat{w}) + \text{h.o.t.} \tag{41}$$

Note that the $k_1 \hat{w}$ term has been incorporated into the higher order terms as $\hat{w} \ll \hat{w}^\delta$ if $\delta < 1$, and so the correction terms become important if the cosine becoes very small, as is the case in the equation (34).

Thus a periodic orbit grazes $x = 0$ if (34) and (41) hold simultaneously. This has a natural interpretation in the two-parameter space defined by $(y_0 - q, \hat{c})$ near $(0, 0)$. Let

$$\sigma(\hat{w}) = \hat{w}^\delta, \quad \Psi(\hat{w}) = \Phi_w + \Xi \ln \hat{w}.$$

Then as $\hat{w} \to 0$, $\sigma \to 0$ and $|\Psi| \to \infty$. The criteria for the existence of a grazing periodic orbit is thus

$$\begin{array}{rl} \hat{c} & \approx -C_w \sigma \cos \Psi \\ y_0 - q & \approx C_v \sigma \cos(\Psi + \psi) \end{array} \tag{42}$$

with $\psi = \Phi_v - \Phi_w$, which as $\sigma \to 0$ and $|\Psi| \to \infty$ describes a spiral tending to the origin $(0, 0)$ in the two parameter space.

4.4. **Two-parameter bifurcation diagram.** The next step is to put together the bifurcations of the simple (single impact, single tour of the homoclinic orbit) periodic orbits discussed in subsections 4.2 and 4.3.

Grazing bifurcations occur on the spiral (42). Successive maxima and minima in $\hat{c}$ have $|\cos \Psi| = 1$, i.e. they occur at

$$\hat{c}_n = (-1)^n C_w \hat{w}_n^\delta \tag{43}$$

where $\hat{w}_n$ is the value of $\hat{w}$ at this turning point, so, since $\Psi_n = (2n + 1)\pi + (-1)^n \frac{1}{2}\pi$,

$$eq : grazwn \Xi \ln \hat{w}_n = (2n + 1)\pi + (-1)^n \frac{1}{2}\pi - \Phi_w. \tag{44}$$

The locus of the saddlenode bifurcations is given by (36) is also $|\cos \Psi| = 1$ (to lowest order) and the parameter values at which these occur are the same as (42), i.e the curve of saddlenode bifurcations terminates on the spiral of grazing bifurcations close to the maxima and minima of this spiral in the $\hat{c}$ direction as shown in Figure 3. This point of intersection is an interesting codimension two bifurcation which accounts for many of the extra bifurcations predicted by the standard Shilnikov analysis.

4.5. **A local codimension two bifurcation.** We work with the approximate normal form

$$x_{n+1} = \begin{cases} c + A x_n & \text{if } x_n \geq 0 \\ c + \gamma \sqrt{|x_n|} - B x_n & \text{if } x_n < 0, \end{cases} \tag{45}$$

with

$$A > 1, \quad B > 0.$$

The parameter c is the standard unfolding parameter for the bifurcation if $\gamma \neq 0$ (since if $c = 0$ then $x = 0$ is a fixed point) and we are

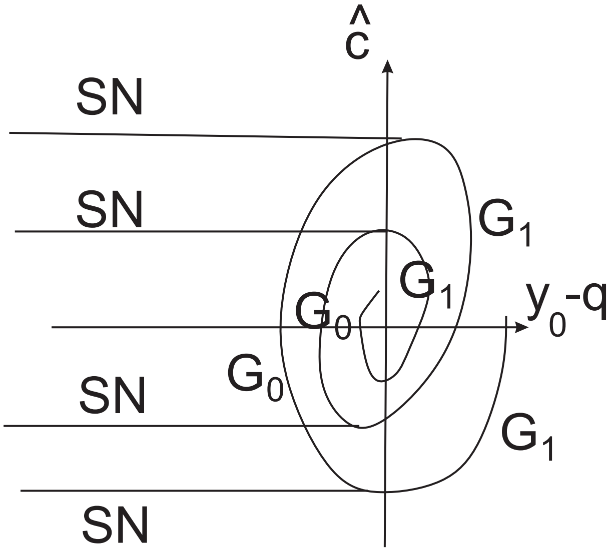

FIGURE 3. Conjectured codimension two bifurcations in the $(\hat{c}, y_0 - q)$-plane. SN represents the locus of saddlenode bifurcations; G_1 the locus of grazing creating a pair of periodic orbits (nonsmooth saddlenode) to wards the origin; G_1 grazing with no change in the number of periodic orbits. The change from G_0 to G_1 occurs at the intersections of the spiral with the locus of saddlenode bifurcations. This local bifurcation is described in section ss:loccod2.

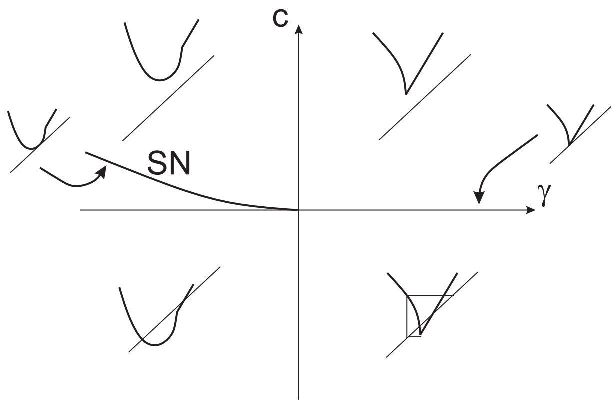

FIGURE 4. Local forms of the map (45) close to $(\gamma, c) = (0, 0)$.

interested in what happens as γ changes sign, i.e. the bifurcations near $(\gamma, c) = (0, 0)$ in parameter space.

Figure 4 shows the form of the map in each of the four quadrants of parameter space. If $c > 0$ and $\gamma > 0$ there is no local recurrence. If $c < 0$ and $\gamma > 0$ with $|x|$ sufficiently small there is an unstable chaotic set conjugate to a full shift on two symbols.

Fixed points of (45) in $x < 0$ satisfy

$$-y^2 = c + \gamma y + By^2$$

where $y = \sqrt{|x|} > 0$, so

$$y_\pm = \frac{1}{2(B+1)}\left(-\gamma \pm \sqrt{\gamma^2 - 4c(B+1)}\right) \tag{46}$$

and hence there is a smooth saddlenode bifurcation at

$$c = \frac{1}{4(B+1)}\gamma^2, \quad \gamma < 0. \tag{47}$$

Note that this is quadratic suggesting that the approximate analysis of the previous section 4.4 which suggested that the curve of saddlenode bifurcations intersects the spiral at the turning points in $\hat{c}$ is correct to lowest order (i,e, these curves are tangential at the point of intersection).

The map in $x < 0$ has a smooth minimum if

$$\frac{1}{2y}\gamma - B = 0$$

where $y = \sqrt{|x|}$ as before, i.e. if $x = -\frac{\gamma^2}{4B^2}$ and hence it is a unimodal map if $\gamma < 0$ with (standard) quadratic minimum. The details of the bifurcations as c decreases depends on the details of the map, but we can prove a simple result that demonstrates the type of behaviour that can occur. The parameter space is sketched in Figure 5.

Lemma 1. *If $B > \frac{A}{A+1}$ then there exists a continuous curve $c = h(\gamma)$ in $\gamma \leq 0$ with $h(0) = 0$ and $\frac{1}{4(B+1)}\gamma^2 > h(\gamma) \geq 0$ such that if $\gamma \leq 0$ and $c \leq h(\gamma)$ then the invariant set of (45) is semi-conjugate to a full shift on two symbols.*

Proof of Lemma 1: The proof relies on the fact that if a unimodal map has a fixed point with positive derivative and does not map the interval this point and its preimage into itself then it has dynamics semi-conjugate to a shift on two symbols. It is a semi-conjugacy because there may be topologically equivalent periodic orbits. These lemma is not in any way optimal – but illustrates one possibility whilst being very simple to prove.

Fix $\gamma \leq 0$ sufficiently small and suppose $c < 0$. Then (45) is a unimodal map with quadratic minimum and has a fixed point with

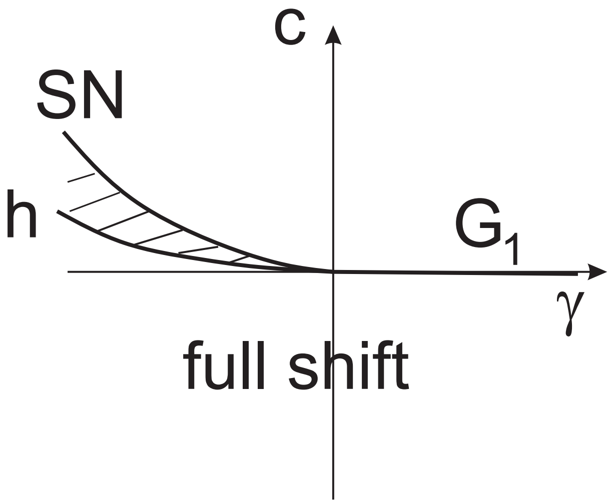

FIGURE 5. Bifurcation regions for (45) close to $(\gamma, c) = (0,0)$ for parameters satisfying the assumptions of Lemma 1. SN is the locus of smooth saddlenode bifurcations, G_1 is the grazing bifurcation creating a full shift on two symbols; h is the graph defined in the proof of Lemma 1. In the shaded region there are complicated sequences of period-doubling bifurcations and chaos.

positive slope at $x_+ = -c/(A-1) > 0$. The turning point is $x = -\frac{\gamma^2}{4B^2} < 0$, so the image of the turning point is

$$x = c - \frac{\gamma^2}{4B} < 0.$$

The image of this point is

$$x_2 = c + \gamma\sqrt{\frac{\gamma^2}{4B} - c} + B(\frac{\gamma^2}{4B} - c)$$

and the invariant set of (45) is semi-conjugate to the full shift on two symbols if $x_2 \geq x_+$. This condition is

$$\gamma\sqrt{\frac{\gamma^2}{4B} - c} + \frac{1}{4}\gamma^2 > c(B - \frac{1}{A-1} - 1) = c(B - \frac{A}{A-1}).$$

The right hand side is clearly positive if $c < 0$ whilst if $B > \frac{A}{A-1}$ then the left hand side is negative. Hence the inequality is automatically satisfied and so there exists $h(\gamma) \geq 0$ such that $x_2 > x_+$ for all $c \leq h(\gamma)$. $\square$

Appendix A. Local normal form

We show here that we can start with the general linear form [equation (50) below] of an impact oscillator, and transform it into equation (64) from our notes, but then rescale to give equation (68) below. The transformation can be written in a shorter / more tidy form, but is written in long form here so it is easier to deconstruct.

There appear to be 4 irremovable parameters in the problem, 2 for the continuous time system (the unstable manifold's complex eigenvalues) and 2 for the impact (the restitution coefficient, and a parameter inherited from the original physical continuous time system).

Let's start with the motivating example of chatter in a pressure relief valve [H. Csaba and A. R. Champneys 2016 Grazing bifurcations and chatter in a pressure relief valve model, Physica D, in press],

$$\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -\kappa y - (x+\delta) + z \\ \dot{z} & = & \beta(q - x\sqrt{z}) \end{array} \tag{48}$$

which for $q > 0$ has a unique equlibrium at $(p-\delta, 0, p)$ where $(p-\delta)\sqrt{p} = q$. If we expand about the equilibrium then

$$\dot{z} = -\beta\sqrt{p}(x-x_0) - \frac{1}{2}\beta\sqrt{p}(z-p) + \mathcal{O}\left((x-x_0)(z-p)\right)$$

where $x_0 = p - \delta$. For typical parameter values of interest (see [**?**]) this has one real negative eigenvalue and two complex eigenvalues with positive real part.

We proceed from a general form that clearly includes this approximation of the valve. Take an impact system with displacement x from the impact surface,

$$\begin{array}{rrcll} & \dot{x} & = & y & \\ & \dot{y} & = & f(x,y,z) & \\ & \dot{z} & = & g(z,y,z) & \\ \& & y & \mapsto & -ry & \quad \text{impact law on } x=0,\ y<0\ , \end{array} \tag{49}$$

for $x, y \in \mathbb{R}$, $z \in \mathbb{R}^n$, $0 < r \leq 1$, where $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}^n$ are smooth. Denote the impact surface $x = 0$ as Σ.

Assume that

(1) there exists an equilibrium, where $f = g = 0$, at a point $(x, y, z) = (x_0, 0, z_0)$,
(2) the Jacobian $J = \frac{\partial(\dot{x},\dot{y},\dot{z})}{\partial(x,y,z)}$ at $(x_0, 0, z_0)$ is non-singular, and
(3) J has $n-2$ real stable eigenvalues,
(4) J has two complex unstable eigenvalues associated with a 2-dimensional unstable manifold W^u,

(5) W^u lies non-orthogonal to the x and y coordinates,
(6) W^u intersects the impact surface Σ transversally.

?? How do we argue reducing z to one dimension?

We than have an impacting system

$$\begin{array}{rrcl} & \dot{x} & = & y \\ & \dot{y} & = & a_1(x-x_0)+a_2y+a_3(z-z_0)+\mathcal{O}(2) \\ & \dot{z} & = & b_1(x-x_0)+b_2y+b_3(z-z_0)+\mathcal{O}(2) \\ \& & y & \mapsto & -ry \qquad \text{impact law on } x=0\ , \end{array} \tag{50}$$

with $\mathcal{O}(2)$ denoting second order terms in x, y, z. Then by assumption the matrix

$$\frac{\partial(\dot{x},\dot{y},\dot{z})}{\partial(x,y,z)} = \begin{pmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \tag{51}$$

has one real stable eigenvalue and two complex unstable eigenvalues, associated with eigenvectors $p=(p_1,p_2,p_3)$ and $s+it=(s_1+s_2+s_3)\pm i(t_1,t_2,t_3)$.

Neglecting higher order terms, the transformation to coordinates

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} c & 0 & 0 \\ -mp_2 & p_1 & 0 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{52}$$

where $q=s\times t$ and where c is a free parameter, yields

$$\begin{array}{rrcl} & \dot{X} & = & C_1(X-X_0)+C_2(Y-Y_0) \\ & \dot{Y} & = & A_1(X-X_0)+A_2(Y-Y_0)+A_3(Z-Z_0) \\ & \dot{Z} & = & B_3(z-z_0) \\ \& & Y & \mapsto & -rY\ , \qquad Z\mapsto Z-(1+r)\frac{q_3}{p_1}Y \qquad \text{impact law on } X=0\ . \end{array} \tag{53}$$

Let's look at this first, then re-scale the constants.

The first row of the transformation matrix preserves the impact surface, so $x=0$ maps to $X=0$.

The zero on the second row preserves the grazing set, as $x=y=0$ maps to $X=Y=0$.

The remaining coefficients transform the tangent spaces of the stable (1D) manifold and unstable (2D) manifold to lie in the $Y=Y_0$ (for $m=1$ only) and $Z=Z_0$ planes respectively:

$$\begin{pmatrix} c & 0 & 0 \\ -mp_2 & p_1 & 0 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} cp_1 \\ p_1p_2(1-m) \\ q.p \end{pmatrix} . \tag{54}$$

and

$$\begin{pmatrix} c & 0 & 0 \\ -mp_2 & p_1 & 0 \\ q_1 & q_2 & q_3 \end{pmatrix} \left\{ \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \pm i \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \right\} = \begin{pmatrix} cs_1 \\ p_1 s_2 - mp_2 s_1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} ct_1 \\ p_1 t_2 - mp_2 t_1 \\ 0 \end{pmatrix} \tag{55}$$

the last row being $q \cdot (s + it) = (s \times t) \cdot (s + it) = 0$.

The Jacobian is now

$$\begin{aligned} \tfrac{\partial(\dot{X},\dot{Y},\dot{Z})}{\partial(X,Y,Z)} &= \begin{pmatrix} C_1 & C_2 & 0 \\ A_1 & A_2 & A_3 \\ 0 & 0 & B_3 \end{pmatrix} \\ &= \tfrac{1}{cp_1q_3} \begin{pmatrix} c & 0 & 0 \\ -mp_2 & p_1 & 0 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} p_1q_3 & 0 & 0 \\ mp_2q_3 & cq_3 & 0 \\ -p_1q_1 - mp_2q_2 & -cq_2 & cp_1 \end{pmatrix} . \end{aligned} \tag{56}$$

so

$$\begin{aligned} &C_1 = \tfrac{mp_2}{p_1} , \qquad C_2 = \tfrac{c}{p_1} , \qquad B_3 = \tfrac{q_2a_3+q_3b_3}{q_3} , \qquad A_3 = \tfrac{p_1a_3}{q_3} , \\ &A_1 = \tfrac{p_1^2(a_1q_3-a_3q_1)+mp_1p_2(a_2q_3-a_3q_2)-m^2p_2^2q_3}{cp_1q_3} , \qquad A_2 = \tfrac{p_1a_2q_3-mp_2q_3-p_1a_3q_2}{p_1q_3} . \end{aligned} \tag{57}$$

The coordinates of the equilibrium are

$$\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} c & 0 & 0 \\ -mp_2 & p_1 & 0 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \\ z_0 \end{pmatrix} . \tag{58}$$

Note that since $C_2 = \frac{c}{p_1}$ the constant term in the first row of (53) vanishes,

$$X_0C_1 + C_2Y_0 = X_0C_1 + C_2Y_0 = cx_0\frac{mp_2}{p_1} + \frac{c}{p_1}(-mp_2x_0) = 0 .$$

We can rescale to simplify the constants and place the cusp at the origin. Let

$$\xi = X/\alpha_1 , \quad \eta = Y/\alpha_2 , \quad \zeta = (Z - Z_0 - X_0\frac{A_1C_2 - A_2C_1}{A_3C_2})/\alpha_3 . \tag{59}$$

This gives after some cancellation

$$\begin{aligned} & \dot{\xi} &=& \; C_1\xi + C_2\tfrac{\alpha_2}{\alpha_1}\eta \\ & \dot{\eta} &=& \; A_1\tfrac{\alpha_1}{\alpha_2}\xi + A_2\eta + A_3\tfrac{\alpha_3}{\alpha_2}\zeta \\ & \dot{\zeta} &=& \; B_3(\zeta - \zeta_0) \\ \& \quad & \eta &\mapsto& \; -r\eta , \qquad \zeta \mapsto \zeta - (1+r)\tfrac{q_3\alpha_2}{p_1\alpha_3}\eta \qquad \text{impact law on } \xi = 0 , \end{aligned} \tag{60}$$

where $\zeta_0 = \frac{A_2C_1 - A_1C_2}{\alpha_3A_3C_2}X_0$.

Then $\dot{\xi} = \ddot{\xi} = 0$ on $\xi = \eta = \zeta = 0$ (i.e. the cusp is at the origin).

To simplify the constants, let $\alpha_2 = k\sqrt{-A_1/C_2}\alpha_1$ for some k, so

$$\begin{aligned} C_2\tfrac{\alpha_2}{\alpha_1} &= kC_2\sqrt{-\tfrac{A_1}{C_2}} = k\sqrt{|A_1C_2|}e^{i\arg C_2}\sqrt{e^{i(\arg A_1-\arg C_2+\pi)}} \\ &= k\sqrt{|A_1C_2|}e^{\frac{1}{2}i(\arg A_1+\arg C_2+\pi)} := k\omega \end{aligned}$$

and

$$\begin{aligned} A_1\tfrac{\alpha_1}{\alpha_2} &= A_1/k\sqrt{-\tfrac{A_1}{C_2}} = \tfrac{\sqrt{|A_1C_2|}e^{i\arg A_1}}{k\sqrt{e^{i(\arg A_1-\arg C_2+\pi)}}} \\ &= \tfrac{1}{k}\sqrt{|A_1C_2|}e^{\frac{1}{2}i(\arg A_1+\arg C_2-\pi)} = -\omega/k. \end{aligned}$$

Let also $\alpha_3 := b\alpha_2/A_3$ and $\lambda = -B_3$, then

$$\begin{array}{rrcl} & \dot{\xi} & = & C_1\xi + \omega k\eta \\ & \dot{\eta} & = & -\omega\xi/k + A_2\eta + b\zeta \\ & \dot{\zeta} & = & -\lambda(\zeta - \zeta_0) \\ \& & \eta & \mapsto & -r\eta\ , \qquad \zeta \mapsto \zeta - (1+r)\tfrac{q_3A_3}{p_1b}\eta \qquad \text{impact law on } \xi = 0\ . \end{array} \tag{61}$$

Now the reason why we introduced m, we have

$$C_1 = \frac{mp_2}{p_1} \qquad \& \qquad A_2 = \frac{p_1a_2q_3 - mp_2q_3 - p_1a_3q_2}{p_1q_3} \tag{62}$$

so

$$C_1 = A_2 := \rho \qquad \text{if} \qquad m = p_1\frac{a_2q_3 - a_3q_2}{2p_2q_3}\ . \tag{63}$$

Thus

$$\begin{array}{rrcl} & \dot{\xi} & = & \rho\xi + \omega k\eta \\ & \dot{\eta} & = & -\omega\xi/k + \rho\eta + b\zeta \\ & \dot{\zeta} & = & -\lambda(\zeta - \zeta_0) \\ \& & \eta & \mapsto & -r\eta\ , \qquad \zeta \mapsto \zeta - (1+r)\tfrac{q_3A_3}{p_1b}\eta \qquad \text{impact law on } \xi = 0\ , \end{array} \tag{64}$$

which is the form in our notes if we set $k = 1$, in terms of constants $\omega, \rho, b, \lambda, \zeta_0, \frac{q_3A_3}{p_1b}$. But we can do better.

Set $\alpha_1 = cx_0$, then

$$\zeta_0 = \frac{A_2C_1 - A_1C_2}{\alpha_3A_3C_2}X_0 = \frac{\rho^2+\omega^2}{b\alpha_2\omega\alpha_1/\alpha_2}cx_0 = \frac{\rho^2+\omega^2}{b\omega}\frac{cx_0}{\alpha_1} = \frac{\rho^2+\omega^2}{b\omega}\ .$$

The equilibrum lies at

$$\xi_0 = \frac{b\omega\zeta_0}{\omega^2+\rho^2} = 1\ , \quad \eta_0 := \frac{-b\rho\zeta_0}{\omega^2+\rho^2} = \frac{-\rho}{\omega}\ , \quad \zeta_0 = \frac{\rho^2+\omega^2}{b\omega}\ . \tag{65}$$

If we set $k = 1$ at this point we have the form in our original notes, and a grant total of 7 constants $(\rho, \omega, b, \lambda, \zeta_0, r, \frac{q_3A_3}{p_1b})$. But we can eliminate a few more of these.

Let $\eta' = k\eta$, $\zeta' = \zeta/\zeta_0$, $k = \omega/(\omega^2+\rho^2) = 1/b\zeta_0$, note $\frac{q_3 A_3}{p_1 b} = \frac{a_3}{b}$ and let $K = \frac{a_3}{bk}$, giving

$$\begin{array}{lrcll} & \dot{\xi} & = & \rho\xi + \omega\eta' & = \rho(\xi-\xi_0) + \omega(\eta'-\eta'_0) \\ & \dot{\eta}' & = & -\omega\xi + \rho\eta' + \zeta' & = -\omega(\xi-\xi_0) + \rho(\eta'-\eta'_0) + (\zeta'-1) \\ & \dot{\zeta} & = & -\lambda(\zeta'-1) & \\ \& & \eta' & \mapsto & -r\eta' \ , \qquad \zeta \mapsto \zeta - (1+r)K\eta' & \text{impact law on } \xi = 0 \ . \end{array} \tag{66}$$

Thus we have the desired form. The equilibrium is now at $(\xi, \eta', \zeta') = (\xi_0, \eta'_0, 1)$ where

$$\xi_0 = \frac{b\omega\zeta_0}{\omega^2+\rho^2} = 1 \ , \quad \eta'_0 := \frac{-\rho}{\omega^2+\rho^2} = \frac{-\rho}{\omega} \ . \tag{67}$$

Note that $\rho\xi_0 + \omega\eta_0 = 0$ and $\omega\xi_0 - \rho\eta_0 - b\zeta_0 = 0$.

This still leaves us two free parameters in b and c, but b no longer appears in the expressions, and the only parameters that depend on c are C_2 and A_1, but their product which defines ω is independent of c. So this freedom does not allow us any simplification of the remaining constants.

If we re-scale time by any of ρ, ω, λ, then we can effectively set one more of these to unity. This gives a minimal form of, for example, (once we rescale coordinates and parameters, including a change to the definition of k to incorporate λ)

$$\begin{array}{lrcll} & \dot{\xi} & = & \rho\xi + \omega\eta & = \rho(\xi-\xi_0) + \omega(\eta-\eta_0) \\ & \dot{\eta} & = & -\omega\xi + \rho\eta + \zeta & = -\omega(\xi-\xi_0) + \rho(\eta-\eta_0) + (\zeta-1) \\ & \dot{\zeta} & = & 1-\zeta & \\ \& & \eta & \mapsto & -r\eta \ , \qquad \zeta \mapsto \zeta - (1+r)K\eta & \text{impact law on } \xi = 0 \ . \end{array} \tag{68}$$

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