1. (a) LIBOR Market Model: $dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t)$

$$\Rightarrow \quad L_i(t) = L_i(0)\, e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i W^{i+1}(t)}$$

$\because$ The forward LIBOR rate $D(t, T_i) = (1 + \Delta_i L_i(T_i, T_{i+1}))\, D(t, T_{i+1})$

$$\therefore \quad L_t(T_i, T_{i+1}) = \frac{1}{\Delta_i} \cdot \frac{D(t, T_i) - D(t, T_{i+1})}{D(t, T_{i+1})}$$

denote $L_i(t) = L_t(T_i, T_{i+1})$, $D_i(t) = D(t, T_i)$

$$\Rightarrow \quad \Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)} .$$

this is a ratio of marketed assets.

If we take the discount bond $D_{i+1}(t)$ as numeraire, then under the martingale measure $\mathbb{Q}^{i+1}$ associated with the numeraire $D_{i+1}(t)$, the process $\Delta_i L_i(t)$ must be a martingale.

(b) The Libor market model is defined as

$$dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t)$$

where $W^{i+1}(t)$ is a standard Browian motion under the risk neutral measure $\mathbb{Q}^{i+1}$, associated to the zero coupon discount bond $D_{i+1}(t) = D(t, T_{i+1})$.

The solution to the Libor market model is given by

$$L_i(T) = L_i(0)\, e^{-\frac{\sigma_i^2}{2}T + \sigma_i W_T^{i+1}}$$

Let V_T denote the value of the financial contract at time t. Under the martingale measure, we have

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{V_T}{D_{i+1}(T)}\right]$$

$$\begin{aligned}
V_0 &= D(0, T_{i+1})\mathbb{E}^{i+1}\left[\Delta_i \sqrt{L_i(T)}\right] \\
&= D(0, T_{i+1})\, \Delta_i\, \mathbb{E}^{i+1}\left[\sqrt{L_i(0)} \cdot e^{-\frac{\sigma_i^2}{4}T + \frac{1}{2}\sigma_i W_T^{i+1}}\right] \\
&= D(0, T_{i+1})\, \Delta_i \sqrt{L_i(0)} \cdot e^{-\frac{\sigma_i^2}{4}T + \frac{1}{8}\sigma_i^2 T} \\
&= D(0, T_{i+1})\, \Delta_i \sqrt{L_i(0)}\, e^{-\frac{1}{8}\sigma_i^2 T}
\end{aligned}$$

(c) $\because V_T = \mathbb{1}_{K_1 \le L_i(T) \le K_2}$

$\therefore V_0 = D(0, T_{i+1}) \mathbb{E}^{i+1}[\mathbb{1}_{K_1 \le L_i(T) \le K_2}]$

Then, we apply CDF: the probability that the forward LIBOR rate $L_i(T_i)$ is within the range $[K_1, K_2]$ under the T_{i+1} forward measure is given by:

$$\mathbb{P}^{T_{i+1}}(K_1 \le L_i(T) \le K_2) = \mathbb{P}^{T_{i+1}}(\ln K_1 \le \ln L_i(T_i) \le \ln K_2)$$

$$= \mathbb{P}^{T_{i+1}}(\ln K_1 \le \ln L_i(0) - \tfrac{1}{2}\sigma_i^2 T + \sigma W_T^{i+1} \le \ln K_2)$$

$$= \mathbb{P}^{T_{i+1}}\left(\frac{\ln K_1 - \ln L_i(0) + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}} \le \frac{\sigma_i W_T^{i+1}}{\sigma_i \sqrt{T}} \le \frac{\ln K_2 - \ln L_i(0) + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}\right)$$

$\because W_T^{i+1}$ is a standard Brownian motion, we recognize

$$\frac{W_T^{i+1}}{\sqrt{T}} = Z \sim N(0,1)$$

$$\therefore \mathbb{P}^{T_{i+1}}(K_1 \le L_i(T) \le K_2) = \mathbb{P}^{T_{i+1}}\left(\frac{\ln \frac{K_1}{L_i(0)} + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}} \le Z \le \frac{\ln \frac{K_2}{L_i(0)} + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}\right)$$

$$= \Phi(d_2) - \Phi(d_1)$$

$$\therefore V_0 = D(0, T_{i+1})[\Phi(d_2) - \Phi(d_1)],$$

where $d_1 = \dfrac{\ln \frac{K_1}{L_i(0)} + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}$

$d_2 = \dfrac{\ln \frac{K_2}{L_i(0)} + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}$

2. (a) The numeraire security associated with the risk-neutral measure $Q^{n+1,N}$ is the present value of a basis point (PVBP) or equivalently the annuity $P_{n+1,N}(t)$, which is the price of zero-coupon bond maturing at T_{n+1}.

(b) $$V_{n,N}^{dig}(0) = P_{n+1,N}(T) S_{n,N}(T) \mathbb{1}_{S_{n,N}(T) > K}$$

$$\frac{V_{n,N}^{dig}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_{n,N}^{dig}(T)}{P_{n+1,N}(T)}\right]$$

$$V_{n,N}^{dig}(0) = P_{n+1,N}(0) \mathbb{E}^{n+1,N}[S_{n,N}(T) \mathbb{1}_{S_{n,N}(T) > K}]$$

$$= P_{n+1,N}(0) \mathbb{E}^{n+1,N}[S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N} W^{n+1,N}(T)} \mathbb{1}_{S_{n,N}(T) > K}]$$

The condition $S_{n,N}(T) > K$ translate to:

$$S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N} W^{n+1,N}(T)} > K$$

$$\Rightarrow \ln S_{n,N}(0) - \frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}\sqrt{T}\, Z > \ln K$$

$$Z > \frac{\ln K - \ln S_{n,N}(0) + \frac{1}{2}\sigma_{n,N} T}{\sigma_{n,N}\sqrt{T}}$$

$$Z > \frac{\ln\left(\frac{K}{S_{n,N}(0)}\right) + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}} = x^*$$

Integral the expectation:

$$\mathbb{E}^{n+1,N}\left[S_{n,N}(T)\,\mathbb{1}_{S_{n,N}>K}\right] = S_{n,N}(0)\int_{x^*}^{\infty} e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}\sqrt{T}x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= S_{n,N}(0)\cdot\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}\sqrt{T}x - \frac{x^2}{2}}\,dx$$

$$= S_{n,N}(0)\cdot\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{1}{2}(x-\sigma_{n,N}\sqrt{T})^2}\,dx$$

$$= S_{n,N}(0)\cdot\left[\Phi(\infty) - \Phi(x^* - \sigma_{n,N}\sqrt{T})\right]$$

$$= S_{n,N}(0)\cdot\left[\Phi(-x^* + \sigma_{n,N}\sqrt{T})\right]$$

$$= S_{n,N}(0)\cdot\Phi\left(\frac{\ln\left(\frac{S_{n,N}(0)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}\right)$$

$$\therefore\ V_{n,N}^{dig}(0) = P_{n+1,N}(0)\, S_{n,N}(0)\, \Phi\left(\frac{\ln\left(\frac{S_{n,N}(0)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}\right)$$

(c) The forward swap rate at a martingale under the annuity measure $Q^{n+1,N}$, not the standard risk-neutral measure Q. Converting between these measures introduces a convexity adjustment due to lognormality of the swap rate. The expected value Q is not simply $S_{n,N}(0)$ but requires an adjustment term term $e^{\frac{1}{2}\sigma_{n,N}^2 T}$.