Project — Part III (Convexity Correction)

Q1: Using the SABR model calibrated earlier, value the following CMS legs:

- **CMS10y received semi-annually over 5 years → PV: *0.20170700161511515***
- **CMS2y received quarterly over 10 years → PV: *0.37817648414075916***

Methodology Summary

- **Discount Factor Interpolation**: Interpolated OIS and LIBOR curves to obtain $D(0,T)$ at arbitrary maturities.
- **SABR Parameter Interpolation**: To interpolate SABR parameters at arbitrary expiry–tenor combinations, we apply 2D bivariate spline interpolation using *"RectBivariateSpline"*. This approach captures the joint dependency between expiry and tenor, addressing the limitations of earlier 1D cubic spline interpolation, which introduced unrealistic curvature and exaggerated convexity effects. The 2D method ensures a smoother and more stable volatility surface, leading to more accurate CMS valuations.
- **SABR Volatility**: Used the full Hagan formula to compute $\sigma_{SABR}(\mathrm{F}, \mathrm{K}, \mathrm{T})$ handling ATM cases carefully. Special handling is implemented when $\mathrm{F} \approx \mathrm{K}$ to avoid numerical issues.
- **Swaption Pricing**: Applied Black76 to price IRR-settled payer and receiver swaptions by using:

$$V_{n,N}(0) = D_0(0,T_n) \cdot IRR\big(S_{n,N}(0)\big) \cdot Black76\big(S_{n,N}(0), K, \sigma_{SABR}, T\big)$$

- **Static Replication:** Computed CMS rates using:

$$\mathrm{E}^T[S_{n,N}(T)] = g(F) + \frac{1}{D_0(0,T)} * \left(\int_0^F h''(K)\ V^{rec}(K)dK + \int_F^\infty h''(K)\ V^{pay}(K)\, dK \right)$$

- **Compute Present Value of CMS Leg:** Summed discounted convexity-adjusted CMS rates over the payment schedule.

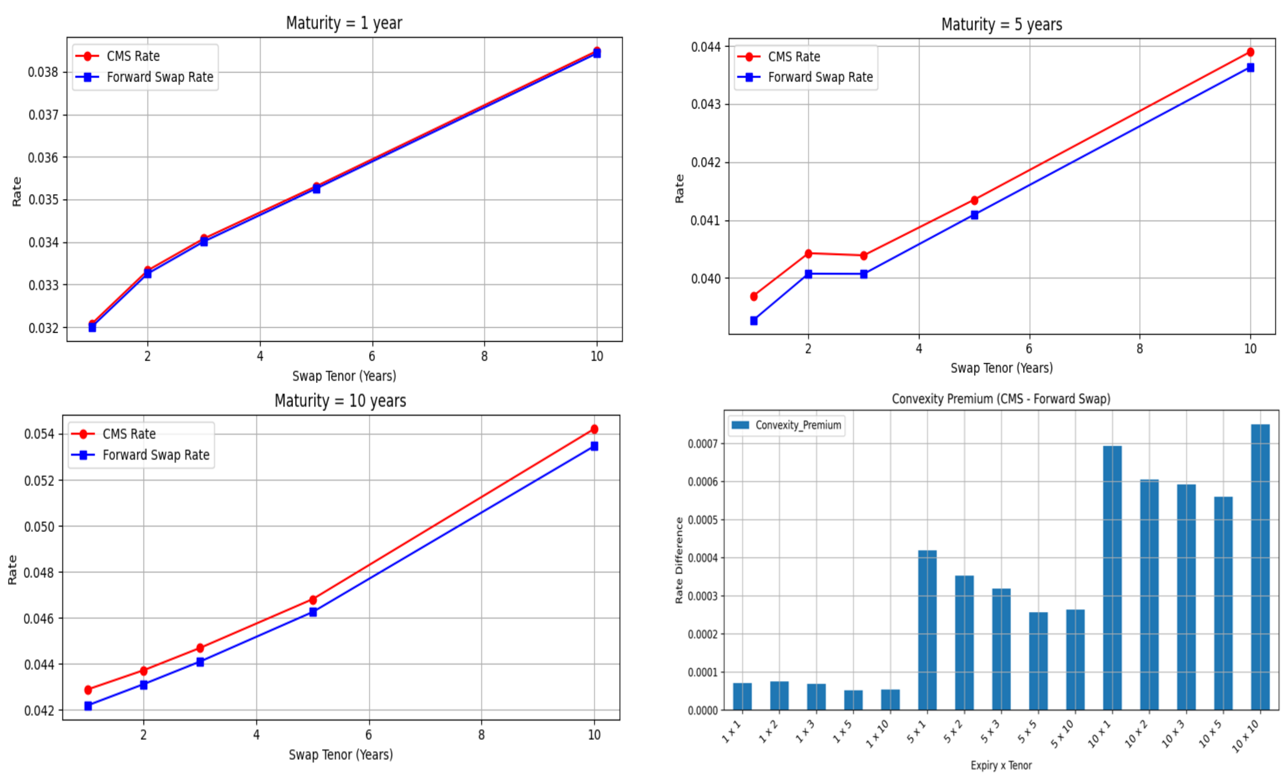
$$PV_{CMS10Y} = D(0,6m) * 0.5 * CMS_{(6m,10y)}(6m) + D(0,1y) * 0.5 * CMS_{(1y,11y)}(1y) + \cdots + D(0,5y) * 0.5 * CMS_{(5y,15y)}(5y)$$

$$PV_{CMS2Y} = D(0,3m) * 0.25 * CMS_{(3m,2y3m)}(3m) + D(0,6m) * 0.5 * CMS_{(6m,2y6m)}(6m) + \cdots + D(0,10y) * 0.25 * CMS_{(10y,12y)}(10y)$$

Q2: Compare the forward swap rates with the CMS rate:

Comparison Between Forward Swap Rate and CMS Rate

Expiry x Tenor	CMS_rate	Forward_Swap_rate
1 x 1	0.032078	0.032007
1 x 2	0.033334	0.033259
1 x 3	0.034080	0.034011
1 x 5	0.035307	0.035255
1 x 10	0.038482	0.038428
5 x 1	0.039693	0.039274
5 x 2	0.040428	0.040075
5 x 3	0.040391	0.040072
5 x 5	0.041350	0.041093
5 x 10	0.043898	0.043634
10 x 1	0.042882	0.042189
10 x 2	0.043721	0.043116
10 x 3	0.044689	0.044097
10 x 5	0.046809	0.046249



The effect of maturity and tenor on convexity correction:

- Convexity correction **increases with expiry** due to rising uncertainty and cumulative volatility.
- It is **less sensitive to tenor**, since the dominant driver is time to exercise, not swap length.
- In practice, always account for convexity when pricing long-expiry CMS — ignoring it can lead to significant mispricing.

Difference between forward swap rates and CMS rates (convexity correction insights):

Convexity correction is the difference between the CMS rate and the corresponding forward swap rate. It arises because the CMS payoff is nonlinear with respect to the underlying swap rate — and this nonlinearity becomes more pronounced under uncertainty (i.e., volatility).

The CMS leg receives a fixed rate based on the future value of a swap rate, which is not a linear function of the underlying rates. Under the risk-neutral measure, we need to take the expectation of a convex function, and due to Jensen's inequality: $E[f(X)] > f(E[X])$ if f is convex. Since the CMS payoff is convex in the swap rate, its expected value is greater than the forward swap rate.