# QF605 Additional Examples
# Session 7: Short-Rate Models and Term Structure

## 1 Questions

1. Consider a stylized interest rate model

$$dr_t = \mu \, dt + \sigma \, dW_t^*,$$

where $W_t^*$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^*$.

(a) Determine the distribution, mean, and variance of the integral

$$\int_t^T r_u \, du.$$

(b) Identify the expressions $A(t,T)$ and $B(t,T)$ in the following expectation:

$$D(t,T) = \mathbb{E}^* \left[ e^{-\int_t^T r_u \, du} \right] = e^{A(t,T) - r_t B(t,T)}.$$

(c) Explain what is an affine interest rate model. Is the short rate model considered above an affine interest rate model?

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

Determine the mean and variance of the integral

$$\int_0^T r_u du,$$

and use this to evaluate the expectation

$$D(0,T) = \mathbb{E}^* \left[ e^{-\int_0^T r_u du} \right].$$

## 2 Suggested Solutions

1. (a) First we integrate the stochastic differential equation from $t$ to $s$ to obtain:

$$r_s = r_t + \mu(s-t) + \int_t^s \sigma \, dW_u^*.$$

Next we integrate $r_s$ from $t$ to $T$ to obtain

$$\begin{aligned}\int_t^T r_s \, ds &= r_t(T-t) + \mu \int_t^T (s-t) \, ds + \int_t^T \int_t^s \sigma \, dW_u^* \, ds \\ &= r_t(T-t) + \mu \left[\frac{s^2}{2} - ts\right]_t^T + \int_t^T \int_u^T \sigma \, ds \, dW_u^* \\ &= r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma(T-u) \, dW_u^*\end{aligned}$$

Hence the mean is given by

$$\mathbb{E}^* \left[\int_t^T r_s \, ds\right] = r_t(T-t) + \frac{\mu}{2}(T-t)^2, \quad \triangleleft$$

and the variance is given by

$$\begin{aligned}V\left[\int_t^T r_s \, ds\right] &= \int_t^T \sigma^2(T-u)^2 du \\ &= \frac{\sigma^2(T-t)^3}{3} \quad \triangleleft\end{aligned}$$

(b) Having identified the mean and variance of the short rate integral, we have

$$\begin{aligned}D(t,T) &= \mathbb{E}^* \left[e^{-\int_t^T r_u \, du}\right] \\ &= e^{-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{1}{2}\frac{\sigma^2(T-t)^3}{3}}\end{aligned}$$

Comparing this against

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

we note that

$$\begin{aligned}A(t,T) &= -\frac{\mu}{2}(T-t)^2 + \frac{\sigma^2(T-t)^3}{6} \quad \triangleleft \\ B(t,T) &= (T-t) \quad \triangleleft\end{aligned}$$

(c) For affine interest rate model, the zero coupon bond prices can be written as

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

for some deterministic functions of $A(t,T)$ and $B(t,T)$ of $t$ and $T$ only. This implies that

$$R(t,T) = \frac{1}{T-t}\Big(-A(t,T) + r_t B(t,T)\Big),$$

i.e. the zero (spot) rates are affine functions of the short rate. $\triangleleft$

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

We can solve this stochastic differential equation by applying Itô's formula to the function $f(r_t, t) = e^{\kappa t} r_t$, and the solution is given by

$$r_t = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-t)}\, dW_u^*$$

Integrating both sides from $0$ to $T$, we have

$$\int_0^T r_t\, dt = \int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt + \underbrace{\int_0^T \int_0^t \sigma e^{\kappa(u-t)}\, dW_u^*\, dt}_{\text{double integral}}.$$

On the right hand side, the first and second integrals can be carried out directly. The double integral can be simplified by exchanging the order of integration (Fubini's Theorem):

$$\begin{array}{llcll} \text{Inner Integral } u: & 0 \le u \le T & \Rightarrow & \text{Inner Integral } t: & u \le t \le T \\ \text{Outer Integral } t: & 0 \le t \le T & & \text{Outer Integral } u: & 0 \le u \le T \end{array}$$

So we have

$$\begin{aligned} \int_0^T \int_0^t \sigma e^{\kappa(u-t)}\, dW_u^*\, dt &= \int_0^T \int_u^T \sigma e^{\kappa(u-t)}\, dt\, dW_u^* \\ &= \int_0^T \left[-\frac{\sigma}{\kappa} e^{\kappa(u-t)}\right]_u^T dW_u^* \\ &= \frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right) dW_u^* \end{aligned}$$

So we can write the overall integral as:

$$\int_0^T r_t\, dt = \int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt + \frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right)\, dW_u^*$$

Taking expectation on both sides gives us the mean of this integral

$$\begin{aligned} \mathbb{E}^*\left[\int_0^T r_t\, dt\right] &= \int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt \\ &= \frac{r_0}{\kappa}\left(1 - e^{-\kappa T}\right) + \theta T - \frac{\theta}{\kappa}\left(1 - e^{-\kappa T}\right). \end{aligned}$$

Taking the variance, we obtain

$V\left[\int_0^T r_t\, dt\right] =$ [redacted]

$$\begin{aligned} &= V\left[\frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right)\, dW_u^*\right] \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T \left(1 - e^{\kappa(u-T)}\right)^2 du \qquad \because \text{ Itô's Isometry} \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T \left(1 - 2e^{\kappa(u-T)} + e^{2\kappa(u-T)}\right) du \\ &= \frac{\sigma^2}{\kappa^2} \left[T - \frac{2}{\kappa}\left(1 - e^{-\kappa T}\right) + \frac{1}{2\kappa}\left(1 - e^{-2\kappa T}\right)\right] \end{aligned}$$

Finally, we can express the discount factor as

$$\begin{aligned} D(0,T) &= \mathbb{E}^*\left[e^{-\int_0^T r_t\,dt}\right] \\ &= \exp\left(\underbrace{-\frac{r_0}{\kappa}\left(1-e^{-\kappa T}\right)-\theta T+\frac{\theta}{\kappa}\left(1-e^{-\kappa T}\right)}_{\text{mean}}+\frac{1}{2}\cdot\underbrace{\frac{\sigma^2}{\kappa^2}\left[T-\frac{2}{\kappa}\left(1-e^{-\kappa T}\right)+\frac{1}{2\kappa}\left(1-e^{-2\kappa T}\right)\right]}_{\text{variance}}\right) \end{aligned}$$