Interest rate is compounded periodically. Derive the following conversion formulas:

Effective Annual Rate (EAR)

$$r_{\text{EAR}} = \left(1 + \frac{r_S}{m}\right)^m - 1$$

Continuously Compounded Rate

$$r_c = m \cdot \ln\left(1 + \frac{r_S}{m}\right)$$

Bond Equivalent Yield (BEY): semi-annual basis

$$r_{\text{BEY}} = \left[\left(1 + \frac{r_S}{m}\right)^{\frac{m}{2}} - 1\right] \cdot 2$$

Let:
r_S:annual interest rate (nominal)
m: number of compounding periods

A coupon bond pays annual coupons (C). Zero rates (r_i) are compounded discretely.

Bond Price

$$B = \sum_{i=1}^{n} \frac{C}{(1+r_i)^i} + \frac{F}{(1+r_n)^n}$$

Par Yield

(=zero rate. A bond trades at par when its price is equal to the face value. B = F)

Derive expression for continuously compounded par yield(y).

c_i is the a fixed value c for all time, the last payment is c_N = c+100

$$B = \sum_{i=1}^{N} c_i e^{-yT_i}$$

Let $T_{i+1} - T_i = \Delta T$, then:

$$B = ce^{-y\Delta T} + \cdots + ce^{-Ny\Delta T} + 100e^{-Ny\Delta T}$$
$$= ce^{-y\Delta T}\left[1 + \cdots + e^{-(N-1)y\Delta T}\right] + 100e^{-Ny\Delta T}$$

Geometric series sum:

$$B = c \cdot \frac{1 - e^{-Ny\Delta T}}{e^{y\Delta T} - 1} + 100e^{-Ny\Delta T}$$

Set $B = 100$:

$$c = 100(e^{y\Delta T} - 1) \quad \Rightarrow \quad y = \frac{1}{\Delta T}\log\left(\frac{c}{100} + 1\right)$$

5-year bond with 11% continuous yield(y), 8% annual coupon(c)

Price of bond

$$B = \sum_{i=1}^{5} c_i e^{-yt_i} + Fe^{-yT} = 8(e^{-0.11} + \cdots + e^{-0.55}) + 100e^{-0.55} = 86.801$$

Modified duration

$$D = -\frac{1}{B}\frac{\partial B}{\partial y} = \frac{1}{B}\left(\sum_{i=1}^{n} t_i c_i e^{-yt_i} + TFe^{-yT}\right) = 4.256$$

Price change if $\Delta y = -0.2\%$

$$\frac{\Delta B}{B} \approx -D \cdot \Delta y, \quad \Delta B \approx -D \cdot \Delta y \cdot B = 0.73885$$

Convexity

$$C = \frac{1}{B}\frac{\partial^2 B}{\partial y^2} = \frac{1}{B}\left(\sum_{i=1}^{n} t_i^2 c_i e^{-yt_i} + T^2 F e^{-yT}\right) = 19.871$$

Price at 10.8% yield

$$B = \sum c_i e^{-0.108t_i} + 100e^{-0.108 \cdot 5} = 87.5434$$

Duration + convexity approximation

$$\Delta B \approx -D \cdot \Delta y \cdot B + \tfrac{1}{2}C(\Delta y)^2 \cdot B = 0.7423, \quad \text{Actual: } 87.5434 - 86.801 = 0.7424$$

Duration and convexity neutral portfolio and adding bonds

Bond	Position	Mod. Duration	Convexity
A	1.5M	3.4	20
B	2.0M	2.8	18

Market Bond	Mod. Duration	Convexity
C	2.9	18
D	1.4	10

$$D_\$(V) = 1.5 \cdot 3.4 + 2.0 \cdot 2.8 = 10.7 \quad C_\$(V) = 1.5 \cdot 20 + 2.0 \cdot 18 = 66$$

Solve:

$$\begin{cases} 10.7 + 2.9B_C + 1.4B_D = 0 \\ 66 + 18B_C + 10B_D = 0 \end{cases} \Rightarrow \quad B_C = -3.8421, \quad B_D = 0.31579$$

Given : LIBOR RATE

The forward LIBOR rate $F(2m, 9m)$.

$$(1 + \Delta_{2m}L_{2m})(1 + \Delta_{7m}F(2m, 9m)) = 1 + \Delta_{9m}L_{9m}$$

$$F(2m, 9m) = \frac{1}{\frac{210}{360}}\left[\frac{1 + \frac{270}{360} \times 0.0155}{1 + \frac{60}{360} \times 0.012} - 1\right] \approx 1.6467\%.$$

(d) What rate would you show for a 2 × 12 FRA (no arbitrage)?

$$(1 + \Delta_{2m}L_{2m})(1 + \Delta_{10m}F(2m, 12m)) = 1 + \Delta_{12m}L_{12m}$$

$$F(2m, 12m) = \frac{1}{\frac{300}{360}}\left[\frac{1 + \frac{360}{360} \times 0.0175}{1 + \frac{60}{360} \times 0.012} - 1\right] \approx 1.85629\%.$$

(e) If your view is that one-month later the spot 1m rate would still remain at 1.15%, how should you trade?

If we think that the 1m spot rate will remain unchanged a month later, we should short the 1 × 2 FRA, since $F(1m, 2m) > 1.15\%$, we can borrow at 1.15% to deposit (lend) at $F(1m, 2m)$ if we were right.

Continuously Compounded Zero Rates:

Given: Zero Rates Table

(a) Calculate the continuously compounded forward rates $F(0y, 1y)$, $F(1y, 2y)$ and $F(2y, 3y$

$$F(0y, 1y) = 4\% \quad \text{(same as observed zero rate)}$$
$$e^{F(0y,1y) \cdot 1} \cdot e^{F(1y,2y) \cdot 1} = e^{0.045 \cdot 2} \Rightarrow F(1y, 2y) = 5\%$$
$$e^{F(0y,1y) \cdot 1} \cdot e^{F(1y,2y) \cdot 1} \cdot e^{F(2y,3y) \cdot 1} = e^{0.0475 \cdot 3} \Rightarrow F(2y, 3y) = 5.25\%$$

(b) Show that the continuously compounded zero rate can be expressed as an arithmetic average of the corresponding forward rates.

$$e^{F(0,1)} \cdot e^{F(1,2)} \cdot \ldots \cdot e^{F(n-1,n)} = e^{r_n \cdot n} \Rightarrow r_n = \frac{F(0,1) + F(1,2) + \cdots + F(n-1,n)}{n}$$

FX Forward Arbitrage:

(a) Spot Exchange rate for USD/SGD is $FX_0 = 1.42$. $D_{\text{SGD}}(0,T) = 0.98$, $D_{\text{USD}}(0,T) = 0.964$. What should be the forward value of the exchange rate at time T?

$$FX_T = FX_0 e^{(r^D - r^F)T}$$

$$FX_T = FX_0 \cdot \frac{D_{\text{USD}}(0,T)}{D_{\text{SGD}}(0,T)} = 1.42 \cdot \frac{0.964}{0.98} = 1.3968$$

(b) If we see that $FX_T = FX_0 = 1.42$, state an arbitrage.

Long 1 unit of USD bond by shorting some SGD bond to get SGD 1.36888 (0.964 × 1.42) today. When USD bond matures, convert \$1 USD → 1.42 SGD. Short SGD bond matures at 1.3968 SGD (1.36888/0.98). Arbitrage profit is 1.42 − 1.3968.

4. Swap Valuation Using Continuous Rates:

Maturity	Zero Rate
3m	1.10%
6m	1.40%
12m	1.75%
18m	1.90%
24m	2.00%

$$D(0, t) = e^{-rt}$$

(a) A 2y fixed leg pays 1.75% semi-annually. What is the PV of this fixed leg?

General formula:

$$\boxed{PV_{\text{fix}} = \sum_{i=1}^{n} D(0, T_i) \cdot \Delta \cdot K} \quad \text{where } K = \text{fixed rate}$$

$$PV_{\text{fix}} = 0.5 \cdot (D(0, 6m) + D(0, 12m) + D(0, 18m) + D(0, 24m)) \cdot 1.75\% = 0.0342$$

(b) A 2y floating leg pays 6m LIBOR rate semi-annually. What is the PV of this floating leg?

General formula:

$$PV_{\text{float}} = \sum_{i=1}^{n} \Delta_i \cdot D(0, T_i) \cdot L(T_{i-1}, T_i) = N \cdot (1 - D(0, T_n)), \text{ where } n = \text{number of periods}$$

$$PV_{\text{flt}} = 1 - D(0, 24m) = 0.03921$$

(c) par swap rate for 2y interest rate swap with semi annual payment?

General formula:

$$\boxed{S_{\alpha,\beta} = \frac{D(0, T_\alpha) - D(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \Delta_i \cdot D(0, T_i)}}$$

if spot swap $(\alpha = 0)$, $D(0, T_\alpha) is 1$

$$S = \frac{1 - D(0, 24m)}{0.5 \cdot (D(0, 6m) + D(0, 12m) + D(0, 18m) + D(0, 24m))} = 2\%$$

Long a receiver at par swap rate above, and 3 months later, we observed:

Maturity	Zero Rate
3m	1.20%
6m	1.50%
12m	1.85%
18m	1.95%
24m	2.05%

what is the value of receiver swap in the portfolio?

General formulas:

Linear interpolation of zero rates:

$$R_t = \frac{T_2 - t}{T_2 - T_1} \cdot R_{T_1} + \frac{t - T_1}{T_2 - T_1} \cdot R_{T_2}$$

From arbitrage-free pricing, the relationship between zero-coupon bonds and forward LIBOR is(to get L(3m,9m)):

$$e^{R_{t_1} \cdot t_1} \cdot (1 + \Delta \cdot L(t_1, t_2)) = e^{R_{t_2} \cdot t_2}$$

Interpolated zero rates:

$$R_{9m} = \frac{R_{6m} + R_{12m}}{2} = 1.675\%, \quad R_{15m} = \frac{R_{12m} + R_{18m}}{2} = 1.90\%, \quad R_{21m} = \frac{R_{18m} + R_{24m}}{2} = 2.00\%$$

Floating leg PV: using current 6m libor rate for the first 3 month pv

$$PV_{\text{flt}} = \Delta \cdot [D(0, 3m) \cdot 1.4\% + L(3m, 9m) \cdot D(0, 9m) + L(9m, 15m) \cdot D(0, 15m) + L(15m, 21m) \cdot$$

Fixed leg PV:

$$PV_{\text{fix}} = 0.5 \cdot [D(0, 3m) + D(0, 9m) + D(0, 15m) + D(0, 21m)] \cdot 2\% = 0.0393$$

Value of receiver swap:

$$V_{\text{rec}} = PV_{\text{fix}} - PV_{\text{flt}} = 0.001 \quad \text{(per \$1 notional)}$$

1. We observe the following instruments in the swap market. All three interest rate swaps have semi-annual payments.

Instrument	Quote
6m LIBOR	2%
1y IRS	2.25%
2y IRS	2.40%
3y IRS	2.50%

(a) Determine the par swap rate for a 1.5y tenor interest rate swap with semi-annual payment.

We need the discount factors $D(0, 6m)$, $D(0, 1y)$, and $D(0, 1.5y)$.

$$D(0, 6m) = \frac{1}{1 + 0.5 \times 2.0\%} = 0.99$$

Using the 1y IRS quote:

$$PV_{\text{fixed}} = \sum_{i=1}^{n} \Delta_i \cdot S \cdot D(0, T_i) = 0.5 \cdot 2.25\% \cdot [D(0, 6m) + D(0, 1y)]$$

$$PV_{\text{float}} = \sum_{i=1}^{n} \Delta_i \cdot L(T_{i-1}, T_i) \cdot D(0, T_i)$$

$$= D(0, 6m) \cdot 0.5 \cdot 2.0\% + D(0, 1y) \cdot 0.5 \cdot L(6m, 12m) = 1 - D(0, 1y)$$

As $PV_{fix} = PV_{float}$: D(0,1y) = 0.9779

Now use the 2y IRS quote(fix=float):

$$0.5 \cdot [D(0, 6m) + D(0, 1y) + D(0, 1.5y) + D(0, 2y)] \cdot 2.4\% = 1 - D(0, 2y)$$

Interpolate:

$$D(0, 1.5y) = \frac{D(0, 1y) + D(0, 2y)}{2}$$

Finally:

$$S = \frac{1 - D(0, 1.5y)}{0.5 \cdot [D(0, 6m) + D(0, 1y) + D(0, 1.5y)]} = 2.335\%$$

(b) A forward starting swap with a 2y tenor starting at $t = 1y$ has the following cashflows:

Time (y)	Pay	Rec
1.5	Par Swap Rate	6m LIBOR
2.0	Par Swap Rate	6m LIBOR
2.5	Par Swap Rate	6m LIBOR
3.0	Par Swap Rate	6m LIBOR

Calculate the par swap rate for this forward starting swap.

We need: $D(0, 1y), D(0, 1.5y), D(0, 2y), D(0, 2.5y), D(0, 3y)$ Interpolate:

$$D(0, 2.5y) = \frac{D(0, 2y) + D(0, 3y)}{2} = 0.4768 + 0.5D(0, 3y)$$

Using 3y IRS quote:

$$PV_{\text{fix}} = 0.5 \cdot [D(0, 6m) + D(0, 1y) + D(0, 1.5y) + D(0, 2y) + D(0, 2.5y) + D(0, 3y)] \cdot 2.5\%$$

$$PV_{\text{flt}} = 1 - D(0, 3y)$$

Solving:

$$D(0, 3y) = 0.928, \quad D(0, 2.5y) = 0.941$$

Now, compute forward swap rate starting at $t = 1y$, i.e., from 1.5y to 3y:

$$S = \frac{D(0, 1y) - D(0, 3y)}{0.5 \cdot [D(0, 1.5y) + D(0, 2y) + D(0, 2.5y) + D(0, 3y)]} = 2.63\%$$

6. Bond Portfolio Immunization

(a) Dollar duration and convexity:

$$D_\$(V) = B_1 \times D_1 + B_2 \times D2$$

$$C_\$(V) = B_1 \times C_1 + B_2 C_2$$

(b) Portfolio value change from 10bp rise (Δy):

$$\Delta V \approx -D_\$(V) \cdot \Delta y + \frac{1}{2}C_\$(V) \cdot (\Delta y)^2 = -13,162,$$

$$V' = 3,500,000 - 13,162 = 3,486,838$$

(c) Immunizing using two more bonds (3 and 4):

$$D_\$(\Pi) = 13.2 + 1.6B_3 + 3.2B_4, \quad C_\$(\Pi) = 76 + 12B_3 + 20B_4,$$

$$B_3 = 3.25 \,\text{mil}, \quad B_4 = -5.75 \,\text{mil}$$

Market Quotes (Uncollateralised). Given Maturity, Instrument (CD, IRS) Rate

Forward LIBOR from zero-coupon discount factors:

Use LIBOR discounting since uncollateralised.

$$D(0, T) = \frac{1}{1 + \Delta_T \cdot L_T}$$

1y IRS (annual payment):

$$IRS(1y) = \frac{1 - D(0, 1y)}{D(0, 1y)} \quad \Rightarrow \quad D(0, 1y)$$

Forward LIBOR:

$$L(6m, 1y) = \frac{1}{\Delta_{6m}} \cdot \frac{D(0, 6m) - D(0, 1y)}{D(0, 1y)}$$

2y IRS (annual payment):

$$0.025 = \frac{1 - D(0, 2y)}{D(0, 1y) + D(0, 2y)} \quad \Rightarrow \quad D(0, 2y$$

Interpolation:

$$D(0, 1y6m) = \frac{1}{2}[D(0, 1y) + D(0, 2y)]$$

Forward LIBOR:

$$L(1y, 1y6m) = \frac{1}{0.5} \cdot \frac{D(0, 1y) - D(0, 1y6m)}{D(0, 1y6m)}$$

Spot LIBOR Rates and OIS Info: Fed-Fund OIS rate = 0.70% flat (30/360), zero basi swap spread. Given: Tenor and Libor Rate

(a) Discount factors for LIBOR and OIS at 3m:

$$D_0(0, t_i) = \left(\frac{1}{1 + \Delta S_0}\right)^{-N} (\text{N} = \text{day}), \quad \Delta S_0 = \frac{0.007}{360}$$

(b) Forward discount factors(3m to 6m):

$$D(t_1, t_2) = \frac{D(0, t_2)}{D(0, t_1)}$$

(c) PV of fixed leg (1y, quarterly, 1.75% fixed):

(i) No collateral (discount using LIBOR):

$$\text{PV}_{\text{fixed}} = \sum_{i=1}^{4} \Delta_i \cdot \widetilde{D}(0, T_i) \cdot 1.75\%$$

(ii) With collateral (discount using OIS):

$$\text{PV}_{\text{fixed}} = \sum_{i=1}^{4} \Delta_i \cdot D_o(0, T_i) \cdot 1.75\%$$

(d) PV of floating leg (3m LIBOR):

(i) No collateral:

$$\text{PV}_{\text{float}} = \widetilde{D}(0, T_0) - \widetilde{D}(0, T_n)$$

(ii) With collateral: discount each forward rate cash flow:

$$L(T_i, T_{i+1}) = \frac{1}{\Delta} \cdot \frac{D(0, T_i) - D(0, T_{i+1})}{D(0, T_{i+1})}$$

$$Since it's quarterly, Find L(3\text{m}, 6\text{m} \quad L(6\text{m}, 9\text{m}), \quad L(9\text{m}, 12\text{m})$$

$$\text{PV}_{\text{float}} = \sum_{i=1}^{n} \Delta_i \cdot L(T_{i-1}, T_i) \cdot D_o(0, T_i)$$

Forward FRA Valuation (3y LIBOR, notional = \$1M). Given: Maturity, Instrument (Spot Libor, Swap), Rate)

(a) Collateralised FRA PV at 3y using OIS

Step 1: Discount factors from OIS at 0.25%:

$$D_o(0, 1\text{y}) \quad D_o(0, 2\text{y}) \quad D_o(0, 3\text{y})$$

Step 2: Solve for forward rates using:

$$\text{PV}_{\text{fixed}} = \text{PV}_{\text{floating}} \quad \Rightarrow \quad \text{solve for } L(1y, 2y), L(2y, 3y)$$

Step 3: PV of Collateralised FRA at pay L(2y,3y) at year 3:

$$\text{PV} = D_o(0, 3\text{y}) \cdot L(2\text{y}, 3\text{y}) \cdot 1{,}000{,}000 \approx 31{,}263.75$$

(b) Uncollateralised FRA PV using LIBOR

From swap rate:

$$0.0225 = \frac{1 - \tilde{D}(0, 2\text{y})}{\tilde{D}(0, 1\text{y}) + \tilde{D}(0, 2\text{y})} \Rightarrow \tilde{D}(0, 2\text{y}) = 0.9563$$

$$0.0255 = \frac{1 - \tilde{D}(0, 3\text{y})}{\tilde{D}(0, 1\text{y}) + \tilde{D}(0, 2\text{y}) + \tilde{D}(0, 3\text{y})} \Rightarrow \tilde{D}(0, 3\text{y}) = 0.927$$

Forward rates:

$$L(1\text{y}, 2\text{y}) = 3.00\%, \quad L(2\text{y}, 3\text{y}) = 3.16\%$$

PV of Uncollateralised FRA at pay L(2y,3y) at year 3:

$$\text{PV} = \tilde{D}(0, 3\text{y}) \cdot L(2\text{y}, 3\text{y}) \cdot 1{,}000{,}000 \approx 29{,}223$$

P: real-world probability. W_t : $\mathbb{P}$-Brownian motion.

StockpriceProcess : $dS_t = \mu S_t dt + \sigma S_t dW_t$. The price of a risk-free bond:

$$dB_t = rB_t dt.$$

Q: risk-neutral measure. risk-free bond: numeraire.

(a) Evaluate $\mathbb{E}^{\mathbb{P}}[S_T]$

By applying Itô's formula: Let $X_t = \log S_t = f(S_t)$

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2$$

$$dX_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

Integrating both sides over $[0, T]$:

$$X_T = X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T$$

Exponentiating both sides:

$$S_T = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right]$$

Taking expectation under $\mathbb{P}$:

$$\mathbb{E}^{\mathbb{P}}[S_T] = S_0 e^{\mu T}$$

(b) Evaluate $\mathbb{E}^{\mathbb{Q}}[S_T]$

Under $\mathbb{Q}$, the process $\frac{S_t}{B_t}$ is a martingale. The Radon-Nikodym derivative is:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right), \quad \text{where } \kappa = \frac{\mu - r}{\sigma}$$

Under $\mathbb{Q}$, define $dW_t^B = dW_t + \kappa dt$. The SDE becomes:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$
$$= \mu S_t\, dt + \sigma S_t \left(dW_t^B - \frac{\mu - r}{\sigma} dt\right)$$
$$= rS_t\, dt + \sigma S_t\, dW_t^B$$

Applying Itô's Formula to $X_t = \log(S_t)$:

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right) T + \sigma W_T^B\right]$$

Taking expectation under $\mathbb{Q}$:

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$$

LIBOR rate follows:

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1},$$

$\mathbb{Q}^{i+1}$: risk-neutral measure, W^{i+1}: a standard BM, $D_{i+1}(t)$: numeraire. Derive the valu formula for a floorlet:

$$F_i(0) = D_{i+1}(0)\Delta_i \mathbb{E}^{i+1}[(K - L_i(T_i))^+]$$

Let $f(L_i(t)) = \log L_i(i)$ and apply Ito's lemma:

$$L_i(t) = L_i(0) \exp\left(-\frac{1}{2}\sigma_i^2 t + \sigma_i W_t^{i+1}\right).$$

Evaluating the expectation, we obtain:

$$F_i(0) = D_{i+1}(0) \cdot \Delta_i \cdot \mathbb{E}^{i+1}[(K - L_i(T_i))^+]$$
$$= D_{i+1}(0) \cdot \Delta_i \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(K - L_i(0) e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i \sqrt{T} x}\right)^+ e^{-\frac{x^2}{2}} dx$$
$$= D_{i+1}(0) \cdot \Delta_i \cdot [K\Phi(-d_2) - L_i(0)\Phi(-d_1)],$$

where

$$d_1 = \frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}, \quad d_2 = d_1 - \sigma_i \sqrt{T}.$$

Suppose the swap rate follows (risk-neutral measure: $\mathbb{Q}^{n+1,N}$, numeraire: $P_{n+1,N}(t)$):

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}.$$

Derive the valuation formula for a receiver swaption:

$$V_{n,N}^{\text{rec}}(0) = P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(K - S_{n,N}(T))^+]$$

$W^{n+1,N}$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$. Therefore:

$$S_{n,N}(T) = S_{n,N}(0) \exp\left(-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N} W_T^{n+1,N}\right).$$

Evaluating the expectation, we obtain:

$$V_{n,N}^{\text{rec}}(0) = P_{n+1,N}(0) \cdot \mathbb{E}^{n+1,N}\left[(K - S_{n,N}(T))^+\right]$$
$$= P_{n+1,N}(0) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(K - S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}\sqrt{T} x}\right)^+ e^{-\frac{x^2}{2}} dx$$
$$= P_{n+1,N}(0) \cdot [K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)],$$

where

$$d_1 = \frac{\log\left(\frac{S_{n,N}(0)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{n,N}\sqrt{T}.$$

Given the stochastic differential equation:

$$dL_i(t) = \sigma_i L_i(t) dW_t^{i+1},$$

where W_t^{i+1} is a Brownian motion under the risk-neutral measure associated to the zer coupon discount bond $D(t, T_{i+1}) = D_{i+1}(t)$.

$$L_i(T) = L_i(0) e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}.$$

A contract paying $L_i(T_i) + L_i^2(T_i)$ at time T_{i+1} has present value:

$$\begin{aligned} V_0 &= D_{i+1}(0)\mathbb{E}^{i+1}[L_i(T_i) + L_i^2(T_i)] \\ &= D_{i+1}(0)(L_i(0)\,\mathbb{E}\left[\exp\left(-\tfrac{1}{2}\sigma_i^2 T_i + \sigma_i W_{T_i}^{i+1}\right)\right] + L_i(0)^2\,\mathbb{E}\left[\exp\left(-\sigma_i^2 T_i + 2\sigma_i W_{T_i}^{i+1}\right)\right]) \\ &= D_{i+1}(0)\left[L_i(0) + L_i^2(0)e^{\sigma_i^2 T_i}\right]. \end{aligned}$$

Given the SDE:

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW_t^{n+1,N},$$

a digital swaption pays:

$$V_{\text{dig}}(T) = P_{n+1,N}(T) \cdot \mathbf{1}_{\{S_{n,N}(T)>K\}}.$$

$$\begin{aligned} \frac{V_{\text{dig}}(0)}{P_{n+1,N}(0)} &= \mathbb{E}^{n+1,N}\left[\frac{V_{\text{dig}}(T)}{P_{n+1,N}(T)}\right] \\ V_{\text{dig}}(0) &= P_{n+1,N}(0) \cdot \mathbb{E}^{n+1,N}\left[\mathbf{1}_{\{S_{n,N}(T)>K\}}\right] \\ &= P_{n+1,N}(0) \cdot \Phi\left(\frac{\log\left(\frac{S_{n,N}(0)}{K}\right) - \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}\right). \end{aligned}$$

The IRR-settled payer and receiver swaptions are:

$$V_{\text{pay}}(t) = D(t,T)\int_K^\infty \text{IRR}(S)(S-K)f(S)dS,$$

$$V_{\text{rec}}(t) = D(t,T)\int_0^K \text{IRR}(S)(K-S)f(S)dS.$$

Using Breeden-Litzenberger on payer swaption V(K) first:

$$\frac{\partial V}{\partial K} = -D(t,T)\int_K^\infty \text{IRR}(S)f(S)\,dS$$

$$\frac{\partial^2 V}{\partial K^2} = D(t,T)\,\text{IRR}(K)f(K)$$

$$\Rightarrow \quad f(K) = \frac{1}{D(t,T)\,\text{IRR}(K)} \cdot \frac{\partial^2 V(K)}{\partial K^2}. \quad \triangleleft$$

Thus:

$$f(K) = \begin{cases} \dfrac{1}{D(0,T)\cdot\text{IRR}(K)} \cdot \dfrac{\partial^2 V_{\text{pay}}(K)}{\partial K^2} & \text{if } K > S_{n,N}(0), \\ \dfrac{1}{D(0,T)\cdot\text{IRR}(K)} \cdot \dfrac{\partial^2 V_{\text{rec}}(K)}{\partial K^2} & \text{if } K < S_{n,N}(0). \end{cases}$$

Use Carr-Madan to integrate by parts twice and derive a static replication formula for a European payoff g(S) on the CMS rate, and apply it to the case g(S)=S.

Suppose we wish to pay a generic function g of the forward swap rate S, i.e., $g(S)$. Based on the static replication approach, let $F = S_{n,N}(0)$, and let $h(K) = \frac{g(K)}{\text{IRR}(K)}$,

$$\begin{aligned} V_0 &= D(0,T)\mathbb{E}[g(S)] \\ &= D(0,T)\int_0^\infty g(K)f(K)dK \\ &= D(0,T)\int_0^\infty g(K)\cdot\frac{1}{D(0,T)}\cdot\frac{1}{\text{IRR}(K)}\cdot\frac{\partial^2 V(K)}{\partial K^2}dK \\ &= \int_0^F h(K)\frac{\partial^2 V^{\text{rec}}(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 V^{\text{pay}}(K)}{\partial K^2}dK \\ &= \left[h(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}\right]_0^F - \int_0^F h'(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}dK \\ &\quad + \left[h(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}\right]_F^\infty - \int_F^\infty h'(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}dK \\ &= h(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - h'(F)V^{\text{rec}}(F) + \int_0^F h''(K)V^{\text{rec}}(K)dK \\ &\quad - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} + h'(F)V^{\text{pay}}(F) + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ &= -h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\ &\quad + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \end{aligned}$$

Integration by parts rule for second derivatives:

$$\int_a^b u(K)\frac{d^2 v(K)}{dK^2}dK = \left[u(K)\frac{dv(K)}{dK}\right]_a^b - \int_a^b u'(K)\frac{dv(K)}{dK}dK$$

The Put-Call Parity for IRR-Settled Swaptions is given by

$$V^{\text{pay}}(K) - V^{\text{rec}}(K) = D(0,T)\text{IRR}(S)(S-K)$$

$$\Rightarrow \frac{\partial V^{\text{pay}}(K)}{\partial K} - \frac{\partial V^{\text{rec}}(K)}{\partial K} = -D(0,T)\text{IRR}(S)$$

Substituting this back:

$$V_0 = D(0,T)g(F) + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK$$

For CMS rate payoff $g(F) = F$ and $V^{\text{pay}}(F) = V^{\text{rec}}(F)$:

$$V_0 = D(0,T)F + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK$$

An FX process observed by the domestic investors:

$$dX_t = (r_D - r_F)X_t\,dt + \sigma X_t\,dW_t^D.$$

Derive the interest rate parity relationship:

$$E_D[X_T] = X_0\,e^{(r_D - r_F)T}.$$

Let $f(X_t) = \ln(X_t)$. Applying Itô's lemma:

$$d\ln(X_t) = f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2 = \left(r_D - r_F - \frac{1}{2}\sigma^2\right)dt + \sigma\,dW^D$$

After integrating:

$$X_T = X_0 \cdot \exp\left[\left(r_D - r_F - \frac{1}{2}\sigma^2\right)T + \sigma W_T^D\right]$$

$$E[e^{\sigma W_T^D}] = e^{\frac{1}{2}\sigma^2 T} \Rightarrow E_D[X_T] = X_0\,e^{(r_D - r_F)T}$$

Show that the foreign investor will see the following SDEs:

$$d\left(\frac{1}{X_t}\right) = (r_F - r_D + \sigma^2)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^D,$$

$$d\left(\frac{1}{X_t}\right) = (r_F - r_D)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

Let $Y_t = \frac{1}{X_t}$. Applying Itô's lemma:

$$dY_t = -\frac{1}{X_t^2}\,dX_t + \frac{1}{2}\cdot\frac{2}{X_t^3}(dX_t)^2$$

$$d\left(\frac{1}{X_t}\right) = \left(r_F - r_D + \frac{1}{2}\sigma^2\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^D. \quad \triangleleft$$

Substituting

$$dW_t = dW_t^F + \frac{r^D - r^F + \sigma^2 - \mu}{\sigma}\,dt$$

into

$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$$

we get:

$$dX_t = (r^D - r^F + \sigma^2)X_t\,dt + \sigma X_t\,dW_t^F$$

Now let $f(X_t) = \frac{1}{X_t}$, and using Itô's formula:

$$d\left(\frac{1}{X_t}\right) = (r^D - r^F)\frac{1}{X_t}\,dt + \sigma\frac{1}{X_t}\,dW_t^F$$

Derive the FX forward price from the foreign investor's perspective and show that its expectation (i.e. $E_F[\frac{1}{X_T}]$)

$$d\log\left(\frac{1}{X_t}\right) = \left(r_F - r_D\right)dt - \sigma\,dW_t^F$$

$$\frac{1}{X_T} = \frac{1}{X_0}\exp\left[\left(r_F - r_D - \frac{1}{2}\sigma^2\right)T - \sigma W_T^F\right]$$

$$E_F\left[\frac{1}{X_T}\right] = \frac{1}{X_0}\,e^{(r_F - r_D)T} \qquad \text{and} \qquad E_F[X_T] = X_0\,e^{(r_D - r_F)T}$$

In a LIBOR-in-arrear contract, the L_i is observed and paid at T_i. Using the LIBOR market model, evaluate $E_i[L_i(T_i)]$ (after performing the single-currency convexity correction)

Under the LIBOR market model, the forward LIBOR $L_i(t)$ is a martingale under the measure associated with the numeraire $D_{i+1}(t)$:

$$L_i(T_i) = L_i(0)\,\exp\left(-\frac{1}{2}\sigma_i^2 T_i + \sigma_i W_{T_i}^{(i+1)}\right)$$

Using Radon-Nikodym derivative to change the measure,

$$\begin{aligned} \mathbb{E}^i\left[L_i(T)\right] &= \mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}L_i(T)\right] \\ &= \mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}L_i(T)\right] \\ &= \frac{1}{1+\Delta_i L_i(0)}\mathbb{E}^{i+1}\left[(1+\Delta_i L_i(T))\cdot L_i(T)\right] \\ &= \frac{1}{1+\Delta_i L_i(0)}\left[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T}\right] \\ &= L_i(0)\cdot\frac{1+\Delta_i L_i(0)e^{\sigma_i^2 T}}{1+\Delta_i L_i(0)} \end{aligned}$$

Derive V_0 for a LIBOR-in-arrear caplet paying $(L_i(T_i) - K)^+$, observed and paid at T_i:

$$\begin{aligned} V_0 &= D_i(0)\mathbb{E}^i\left[(L_i(T) - K)^+\right] \\ &= D_i(0)\mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}(L_i(T) - K)^+\right] \\ &= D_i(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}(L_i(T) - K)^+\right] \\ &= D_{i+1}(0)\mathbb{E}^{i+1}\left[(1+\Delta_i L_i(T))\cdot(L_i(T) - K)^+\right] \\ &= D_{i+1}(0)\left\{\mathbb{E}^{i+1}\left[(L_i(T) - K)^+\right] + \Delta_i\mathbb{E}^{i+1}\left[L_i(T)(L_i(T) - K)^+\right]\right\} \\ &= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) - \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right)\right] \\ &\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{3}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right) - L_i(0)K\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right)\right] \end{aligned}$$

$$\mathbb{E}\left[L_i(T)^2\cdot 1_{L_i(T)>K}\right] \Rightarrow \Phi\left(-x^* + 2\sigma_i\sqrt{T}\right)$$

where $L_i(T)^2 \sim e^{2\sigma_i W_T}$ and exponent shift $= +2\sigma_i\sqrt{T}$

A contract pays $\Delta_i \times L_i(T)$ at $T = T_{i+1}$. Derive using LIBOR market model.

$$dL_i(t) = \sigma_i L_i(t)\,dW^{i+1}(t)$$

where $W^{i+1}(t)$ is a Brownian motion under $\mathbb{Q}^{i+1}$, with numeraire $D_{i+1}(t)$

$$L_i(T) = L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}$$

Let V_0 be the contract value at time 0:

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{V_T}{D_{i+1}(T)}\right] \Rightarrow V_0 = D(0,T_{i+1})\mathbb{E}^{i+1}[\Delta_i L_i(T)]$$

$$\begin{aligned} &= D(0,T_{i+1})\Delta_i\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}\right] \\ &= D(0,T_{i+1})\Delta_i L_i(0)e^{-\frac{1}{2}\sigma_i^2 T}\cdot\mathbb{E}^{i+1}\left[e^{\sigma_i W_T^{i+1}}\right] \\ &= D(0,T_{i+1})\Delta_i L_i(0)e^{-\frac{1}{2}\sigma_i^2 T}\cdot e^{\frac{1}{2}\sigma_i^2 T} = D(0,T_{i+1})\Delta_i L_i(0) \quad \triangleleft \end{aligned}$$

A contract pays $\Delta_i \times L_i(T)$ at $T = T_i$. Derive using LIBOR market model.

$$\begin{aligned} V_0 &= D(0,T_i)\mathbb{E}^i\left[\Delta_i \times L_i(T_i)\right] \\ &= D(0,T_i)\mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}\cdot\Delta_i L_i(T_i)\right] \\ &= D(0,T_i)\Delta_i\mathbb{E}^{i+1}\left[\frac{L_i(T_i) + \Delta_i L_i(T_i)^2}{1+\Delta_i L_i(0)}\right] \\ &= D(0,T_i)\frac{\Delta_i}{1+\Delta_i L_i(0)}\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_{T_i}^{i+1}} + \Delta_i L_i(0)^2 e^{-\sigma_i^2 T + 2\sigma_i W_{T_i}^{i+1}}\right] \\ &= D(0,T_i)\frac{\Delta_i}{1+\Delta_i L_i(0)}\left[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_i}\right] \end{aligned}$$

Let L_i^D be a forward LIBOR rate in the domestic economy (observed at T_i, paid at T_{i+1}), following the LMM with volatility σ_i. Suppose the FX rate satisfies

$$dF_t = \sigma_X F_t\,dW_t^D$$

and is correlated with $W_t^{(i+1)}$ by ρ. Evaluate the expectation from the foreign investor's perspective:

$$E_{i+1,F}\left[L_i^D(T)\right]$$

We have

$$\begin{aligned} dF_t = \sigma_X F_t dW_t^D \quad &\Rightarrow \quad F_T = F_0 e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X W_T^D} \\ dL_i^D(t) = \sigma_i L_i^D(t)dW_t^{i+1} \quad &\Rightarrow \quad L_i^D(T) = L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}} \\ \mathbb{E}^{i+1,F}\left[L_i^D(T)\right] &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{d\mathbb{Q}^{i+1,F}}{d\mathbb{Q}^{i+1,D}}\right] \\ &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_T D_{i+1}^F(T)}{X_0 D_{i+1}^F(0)}\cdot\frac{D_{i+1}^D(0)}{D_{i+1}^D(T)}\right] \\ &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{F_T}{F_0}\right] \\ &= L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T - \frac{1}{2}\sigma_X^2 T}\cdot\mathbb{E}\left[e^{(\sigma_i + \rho\sigma_X)Z_1\sqrt{T} + \sigma_X\sqrt{1-\rho^2}Z_2\sqrt{T}}\right] \\ &= L_i^D(0)e^{\sigma_i\sigma_X\rho T} \quad \triangleleft \end{aligned}$$

Given three correlated Brownian motions W_t^f, W_t^g and W_t^h with correlations

Given three correlated Brownian motions W_t^f, W_t^g and W_t^h with correlations

$$dW_t^f\,dW_t^g = \rho_{fg}\,dt, \quad dW_t^g\,dW_t^h = \rho_{gh}\,dt, \quad dW_t^f\,dW_t^h = \rho_{fh}\,dt,$$

determine the coefficients α_{11}, α_{12}, α_{22}, α_{13}, α_{23} and α_{33} for the Cholesky decomposition

$$dW_t^f = \alpha_{11}dZ_t^1, \quad dW_t^g = \alpha_{12}dZ_t^1 + \alpha_{22}dZ_t^2, \quad dW_t^h = \alpha_{13}dZ_t^1 + \alpha_{23}dZ_t^2 + \alpha_{33}dZ_t^3$$

where Z_t^1, Z_t^2 and Z_t^3 are three mutually independent Brownian motions.

$$\begin{aligned} &\alpha_{11} = 1, && \alpha_{12} = \rho_{fg}, && \alpha_{13} = \rho_{fh} \\ &\alpha_{22} = \sqrt{1-\alpha_{12}^2}, && \alpha_{23} = \frac{\rho_{gh} - \alpha_{12}\alpha_{13}}{\alpha_{22}}, && \alpha_{33} = \sqrt{1-\alpha_{13}^2-\alpha_{23}^2} \end{aligned}$$

Discussion: Siegel's Exchange Rate Paradox

Consider a simplified discrete FX market involving the SGD and USD economies, with a spot FX rate approximately $FX_0 \approx 1.25$. Using a one-step binomial model with parameters

$$u = \frac{6}{5}, \quad d = \frac{5}{6}, \quad p^* = q^* = 0.5,$$

determine the expected forward exchange rate.

From the perspective of a Singapore-based investor, we have:

$$E[FX_T] = \frac{1}{2}\times\frac{6}{5}\times 1.25 + \frac{1}{2}\times\frac{5}{6}\times 1.25 \approx 1.423.$$

From the perspective of a US-based investor, we have:

$$E\left[\frac{1}{FX_T}\right] = \frac{1}{2}\times\frac{5}{6}\times\frac{1}{1.25} + \frac{1}{2}\times\frac{6}{5}\times\frac{1}{1.25} \approx 0.726.$$

Since $1/0.726 \approx 1.377 \neq 1.423$, the same binomial FX model gives rise to two different expectations depending on the denomination.

In the continuous time framework, if the Singapore-based investor observes

$$dX_t = (r_D - r_F)X_t\,dt + \sigma X_t\,dW_t^D,$$

a naive change of numeraire might suggest that the US-based investor sees

$$d\left(\frac{1}{X_t}\right) = (r_F - r_D)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

However, this is incorrect because the SGD money market account is not tradable for the US investor. Instead, applying Itô's formula correctly under the USD numeraire gives

$$d\left(\frac{1}{X_t}\right) = \left(r_F - r_D + \sigma^2\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^D,$$

and then, using the USD money market account as the numeraire,

$$d\left(\frac{1}{X_t}\right) = (r_F - r_D)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

This resolves the paradox.

Short Rate Model:

$$dr_t = \mu\,dt + \sigma\,d\boldsymbol{W}_t, \quad \boldsymbol{W}_t: \text{ Brownian motion under } \mathbb{Q}$$

Find the mean and variance of $\int_t^T r_u\,du$.

Step 1:

$$r_s = r_t + \mu(s-t) + \int_t^s \sigma\,d\boldsymbol{W}_u$$

Step 2:

$$\int_t^T r_s\,ds = r_t(T-t) + \mu\int_t^T(s-t)\,ds + \int_t^T\int_t^s \sigma\,d\boldsymbol{W}_u\,ds$$

$$\int_t^T (s-t)\,ds = \frac{1}{2}(T-t)^2$$

Fubini:

$$\int_t^T\int_t^s \sigma\,d\boldsymbol{W}_u\,ds = \int_t^T \sigma(T-u)\,d\boldsymbol{W}_u$$

Final Form:

$$\int_t^T r_s\,ds = r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma(T-u)\,d\boldsymbol{W}_u$$

Mean:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T r_s\,ds\right] = r_t(T-t) + \frac{\mu}{2}(T-t)^2$$

Variance:

$$\text{Var}\left[\int_t^T r_s\,ds\right] = \frac{\sigma^2}{3}(T-t)^3$$

Find $A(t,T)$, $B(t,T)$ in $D(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du}\right] = e^{A(t,T) - r_t B(t,T)}$

$$D(t,T) = \exp\left(-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3\right)$$

$$A(t,T) = -\frac{\mu}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3, \quad B(t,T) = T-t$$

Is this an affine model?

Yes. Since:

$$D(t,T) = e^{A(t,T) - r_t B(t,T)} \quad \Rightarrow \quad R(t,T) = \frac{-A(t,T) + r_t B(t,T)}{T-t}$$

and $R(t,T)$ is affine in r_t.

Vasicek Model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\,d\boldsymbol{W}_t$$

Compute mean, variance, and $\mathbb{E}\left[\int_0^T r_u\,du\right]$

Solution:

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma\int_0^t e^{-\kappa(t-u)}d\boldsymbol{W}_u$$

$$\int_0^T r_t\,dt = \int_0^T\left[r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})\right]dt + \int_0^T\int_0^t \sigma e^{-\kappa(t-u)}d\boldsymbol{W}_u\,dt$$

$$\int_0^T\int_0^t \sigma e^{-\kappa(t-u)}d\boldsymbol{W}_u\,dt = \frac{\sigma}{\kappa}\int_0^T(1 - e^{-\kappa(T-u)})d\boldsymbol{W}_u$$

Final Expression:

$$\int_0^T r_t dt = \frac{r_0}{\kappa}(1 - e^{-\kappa T}) + \theta\left(T - \frac{1 - e^{-\kappa T}}{\kappa}\right) + \frac{\sigma}{\kappa}\int_0^T(1 - e^{-\kappa(T-u)})d\boldsymbol{W}_u$$

Mean:

$$\mathbb{E}\left[\int_0^T r_t dt\right] = \frac{r_0}{\kappa}(1 - e^{-\kappa T}) + \theta\left(T - \frac{1 - e^{-\kappa T}}{\kappa}\right)$$

Variance:

$$\text{Var}\left[\int_0^T r_t dt\right] = \frac{\sigma^2}{\kappa^2}\left[T - \frac{2}{\kappa}(1 - e^{-\kappa T}) + \frac{1}{2\kappa}(1 - e^{-2\kappa T})\right]$$

Compute $D(0,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}\right]$

$$D(0,T) = \exp\left(-\mathbb{E}\left[\int_0^T r_t dt\right] + \frac{1}{2}\text{Var}\left[\int_0^T r_t dt\right]\right)$$

Ho-Lee model: derive $\theta(t)$ from $D(0,t)$

$$dr_t = \theta(t)\,dt + \sigma\,dW_t^*, \quad W_t^* \text{ under } \mathbb{Q}^*$$

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

$$\int_0^t r_u\,du = r_0 t + \int_0^t \theta(s)(t-s)\,ds + \int_0^t \sigma(t-s)\,dW_s^*$$

Mean:

$$\mathbb{E}\left[\int_0^t r_u\,du\right] = r_0 t + \int_0^t \theta(s)(t-s)\,ds$$

Variance:

$$\text{Var}\left[\int_0^t r_u\,du\right] = \int_0^t \sigma^2(t-s)^2\,ds = \frac{1}{3}\sigma^2 t^3$$

Discount factor:

$$D(0,t) = \exp\left(-r_0 t - \int_0^t \theta(s)(t-s)\,ds + \frac{1}{6}\sigma^2 t^3\right)$$

Differentiate log D:

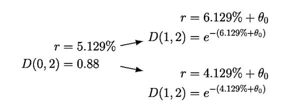
$$\frac{\partial}{\partial t}\log D(0,t) = -r_0 - \int_0^t \theta(s)\,ds + \frac{1}{2}\sigma^2 t^2$$

$$\frac{\partial^2}{\partial t^2}\log D(0,t) = -\theta(t) + \sigma^2 t \Rightarrow \theta(t) = -\frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2 t$$

Ho-Lee Tree: Find θ_0

Given: $D(0,1y) = 0.95$, $D(0,2y) = 0.88$

$$r = -\log(0.95) = 5.129\%$$

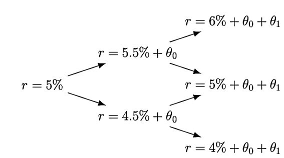


$$0.88 = 0.95 \times \frac{1}{2}\left(e^{-(6.129\%+\theta_0)} + e^{-(4.129\%+\theta_0)}\right) \quad \Rightarrow \theta_0 = 0.0253$$

Ho-Lee Tree: Find θ_0, θ_1

Given: $D(0,1y) = 0.95123$, $D(0,2y) = 0.86936$, $D(0,3y) = 0.78663$

$$D(0,1y) = e^{-R(0,1)} \Rightarrow R(0,1) = 5\% \Rightarrow r_0 = 0.05$$



$$D(0,2y) = D(0,1y)\cdot\tfrac{1}{2}\left(e^{-(\theta_0+0.055)} + e^{-(\theta_0+0.045)}\right) \Rightarrow \theta_0 = 0.04$$

$$D_u(1,3) = e^{-(\theta_0+0.055)}\cdot\tfrac{1}{2}\left(e^{-(\theta_0+\theta_1+0.06)} + e^{-(\theta_0+\theta_1+0.05)}\right)$$

$$D_d(1,3) = e^{-(\theta_0+0.045)}\cdot\tfrac{1}{2}\left(e^{-(\theta_0+\theta_1+0.05)} + e^{-(\theta_0+\theta_1+0.04)}\right)$$

$$D(0,3y) = D(0,1y)\cdot\tfrac{1}{2}(D_u + D_d) \Rightarrow \theta_1 = 0.01$$