SMU
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Session 7
Short Rate Models and Term Structure

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QF605 Fixed Income Securities

Term Structure Models

The **Market Models** and static replication method can handle the pricing of derivatives with **European payoffs**, such as caps, floors and European swaptions.

However, they are not able to handle derivatives with **path-dependent payoffs**, e.g. Bermudan or American option.

To value path-dependent products, we need a model of how the whole **term structure** (not just a single forward rate or bond) evolves.

One set of models specifies dynamics for the short rate under the risk-neutral measure. This then determines prices of zero coupon bonds, and hence, the entire term structure:

$$\mathbb{E}^*_t\left[e^{-\int_t^T r_u\ du}\right] = D(t,T) = e^{-R(t,T)(T-t)}$$

Term Structure Models

A typical **short rate model** will take the following form:

$$dr_t = \mu_t \, dt + \sigma_t \, dW_t^*$$

We begin by considering how two different features of the short rate model affect the spot curve that you obtain from the model:

1. the drift in the short rate (under $\mathbb{Q}^*$)
2. the volatility of the short rate (under $\mathbb{Q}^*$)

1. Drift: Suppose a (simplistic) short rate model specifies

$$dr_t = \mu \, dt,$$

where μ is a constant. The short rate grows linearly over time, and is deterministic. We could also write this as

$$\mu = \frac{dr_t}{dt}.$$

Drift in Short Rate Models

Example We consider a discrete approximation of positive μ. Suppose the initial $3m$ rate (with continuous compounding) is 5%. The next $3m$ rates will be 5.1%, 5.2%, 5.3%, $\cdots$ and so on.

$$\begin{aligned}
D(0,6m) &= e^{-(0.05+0.051)\cdot 0.25} &\Rightarrow\quad R(0,6m) &= 0.0505\\
D(0,9m) &= e^{-(0.05+0.051+0.052)\cdot 0.25} &\Rightarrow\quad R(0,9m) &= 0.051\\
D(0,12m) &= e^{-(0.05+0.051+0.052+0.053)\cdot 0.25} &\Rightarrow\quad R(0,12m) &= 0.0515.
\end{aligned}$$

Based on the calculation, we conclude that the term structure is upward sloping.

If μ is negative, then the term structure will be downward sloping.

Drift in Short Rate Models

Mathematically, we proceed as follows:

- First, we integrate the short rate SDE from 0 to t to obtain an expression for the short rate process:

$$r_t = r_0 + \mu t.$$

- Next, we integrate the short rate process to obtain:

$$\int_t^T r_u\, du = r_0(T-t) + \frac{1}{2}\mu(T^2 - t^2) = r_t(T-t) + \frac{1}{2}\mu(T-t)^2$$

- We can now reconstruct the discount factor as

$$D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right] = e^{-r_t(T-t) - \frac{1}{2}\mu(T-t)^2}.$$

- Therefore, the spot curve in this stylized (simplified) model is given by

$$R(t,T) = -\frac{1}{T-t}\log D(t,T) = \frac{1}{2}\mu(T-t) + r_t.$$

Clearly, if $\mu > 0$, the spot curve is upward sloping, and if $\mu < 0$, the spot curve is downward sloping.

Volatility in Short Rate Models

2. Volatility: Suppose a (simplistic) short rate model specifies

$$dr_t = \sigma dW_t^*$$

where σ is a constant, and W_t^* is a Brownian motion under $\mathbb{Q}^*$.

The short rate follows a random walk without drift under $\mathbb{Q}$, where σ affects the variance of the "error term".

In discrete term, we have

$$r_{t+\Delta t} \approx r_t + \Delta r_t = r_t + \sigma \Delta W_t^*$$

Volatility in Short Rate Models

Example We consider a discrete approximation of this short rate model with small σ. Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are describe by a tree, where each period the short rate can move up or down by 0.5%. The risk-neutral probability of an up/down move is always 0.5.

A 2-period tree looks as follows:

$$r{=}5.5\%$$
$$D(1y{,}2y){=}0.94649$$

$$r{=}5\%$$
$$D(0{,}2y){=}0.90485$$

$$r{=}4.5\%$$
$$D(1y{,}2y){=}0.956$$

Volatility in Short Rate Models

A 3-period tree looks as follows:

$$
\begin{tikzcd}
 & & \begin{matrix} r{=}6\% \\ D(2y{,}3y){=}0.94176 \end{matrix} \\
 & \begin{matrix} r{=}5.5\% \\ D(1y{,}3y){=}0.89585 \end{matrix} \arrow[ur] \arrow[dr] & \\
\begin{matrix} r{=}5\% \\ D(0{,}3y){=}0.86076 \end{matrix} \arrow[ur] \arrow[dr] & & \begin{matrix} r{=}5\% \\ D(2y{,}3y){=}0.95123 \end{matrix} \\
 & \begin{matrix} r{=}4.5\% \\ D(1y{,}3y){=}0.91394 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & & \begin{matrix} r{=}4\% \\ D(2y{,}3y){=}0.96079 \end{matrix}
\end{tikzcd}
$$

From the zero coupon bond prices, we work out the spot rates:

$$
\begin{aligned}
R(0,1y) = 5\%,\ D(0,2y) &= 0.90485 \quad \Rightarrow \quad R(0,2y) = 4.9994\% \\
D(0,3y) &= 0.86076 \quad \Rightarrow \quad R(0,3y) = 4.9979\%
\end{aligned}
$$

Volatility in Short Rate Models

Example We now consider a discrete approximation of the short rate model with large σ. Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are described by a tree, where each period the short rate can move up or down by 2%, and the risk-neutral probability of an up/down move is always $\frac{1}{2}$.

A 2-period tree looks as follows:

$$
\begin{array}{ccc}
 & & \begin{array}{c} r{=}7\% \\ D(1y{,}2y){=}0.93239 \end{array} \\
 & \nearrow & \\
\begin{array}{c} r{=}5\% \\ D(0{,}2y){=}0.90502 \end{array} & & \\
 & \searrow & \\
 & & \begin{array}{c} r{=}3\% \\ D(1y{,}2y){=}0.97045 \end{array}
\end{array}
$$

Volatility in Short Rate Models

A 3-period tree looks as follows:

$$
\begin{tikzcd}
 & & \begin{array}{c} r{=}9\% \\ D(2y{,}3y){=}0.91393 \end{array} \\
 & \begin{array}{c} r{=}7\% \\ D(1y{,}3y){=}0.86953 \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c} r{=}5\% \\ D(0{,}3y){=}0.86157 \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} r{=}5\% \\ D(2y{,}3y){=}0.95123 \end{array} \\
 & \begin{array}{c} r{=}3\% \\ D(1y{,}3y){=}0.94195 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} r{=}1\% \\ D(2y{,}3y){=}0.99005 \end{array}
\end{tikzcd}
$$

From the zero coupon bond prices, we work out the spot rates:

$$
\begin{aligned}
R(0,1y) = 5\%,\ D(0,2y) &= 0.90502 \quad \Rightarrow \quad R(0,2y) = 4.99\% \\
D(0,3y) &= 0.86157 \quad \Rightarrow \quad R(0,3y) = 4.9667\%
\end{aligned}
$$

Volatility in Short Rate Models

Main Conclusions

1. Volatility of the short rate by itself produces a slightly downward sloping spot curve.

2. The higher the volatility, the more negative the slope of the spot curve.

3. This is a consequence of Jensen's inequality and the fact that $f(x) = e^{-x}$ and $f(x) = \frac{1}{1+x}$ are convex in x.

Jensen's inequality states that

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]) \text{ if } f \text{ is convex}$$
$$\mathbb{E}[f(X)] \leq f(\mathbb{E}[X]) \text{ if } f \text{ is concave}$$

Volatility in Short Rate Models

Mathematically, we proceed as follows:

- First, integrate the SDE from 0 to t to obtain the short rate process:

$$r_t = r_0 + \sigma W_t^*, \qquad \text{where } r_t \sim N(r_0, \sigma^2 t)$$

- Next we integrate the short rate process to obtain:

$$\int_t^T r_u \, du = r_0(T-t) + \sigma \int_t^T W_u^* \, du = r_t(T-t) + \sigma \int_t^T (W_u^* - W_t^*) \, du.$$

Recall that in the previous term, we have demonstrated that by applying Itô's formula to the function $X_t = f(t, W_t) = tW_t$, we can write

$$\int_0^T W_u \, du = \int_0^T (T-u) \, dW_u,$$

so that this integral is normally distributed, with mean and variance:

$$\mathbb{E}\left[\int_0^T W_u \, du\right] = 0, \qquad V\left[\int_0^T W_u \, du\right] = \frac{T^3}{3}.$$

Volatility in Short Rate Models

- Applying this results to our integrated short rate process, we note that

$$\mathbb{E}\left[\int_t^T r_u\, du\right] = r_t(T-t)$$
$$V\left[\int_t^T r_u\, du\right] = V\left[\sigma\int_t^T (W_u^* - W_t^*)\, du\right] = \frac{\sigma^2(T-t)^3}{3},$$

  and hence

$$\int_t^T r_u\, du \sim N\left(r_t(T-t), \frac{\sigma^2}{3}(T-t)^3\right).$$

- We can now reconstruct the discount factor as

$$D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right].$$

Volatility in Short Rate Models

- We know how to evaluate the expectation of a lognormal random variable. If $X \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

- Using this, we have

$$D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right] = e^{-r_t(T-t) + \frac{\sigma^2}{6}(T-t)^3}.$$

- Finally, we can express the zero rate $R(t,T)$ as follows:

$$R(t,T) = -\frac{1}{T-t}\log D(t,T) = r_t - \frac{\sigma^2}{6}(T-t)^2.$$

- The further we look ahead (larger $T-t$), the larger the accumulated uncertainty, and hence the lower the corresponding spot rate. Also, the higher σ, the lower all spot rates.

Vasicek Model

The Vasicek model for interest rate is a classic short rate model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*$$

Here, κ is the mean reversion coefficient, θ is the long run mean of the short rate, and σ is the volatility of the short rate. Vasicek model is mean reverting.

Applying Itô's formula to $f(r_t, t) = r_t e^{\kappa t}$, we can show that

$$r_t = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-T)} dW_u^*$$

We conclude that r_t is normally distributed, with a mean of

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)$$

and a variance of

$$V[r_t] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

Vasicek Model

Once again, we can now write the integrated short rate process under Vasicek model as

$$\int_t^T r_u \, du \sim N\left(\mathbb{E}\left[\int_t^T r_u \, du\right], V\left[\int_t^T r_u \, du\right]\right).$$

This in turn allows us to reconstruct the discount factor as follows:

$$D(t,T) = \mathbb{E}\left[e^{-\int_t^T r_u \, du}\right].$$

Vasicek Model

Let $R(t,T)$ denote the zero rate covering the period $[t,T]$, so that

$$D(t,T) = e^{-R(t,T)(T-t)}.$$

After some algebra (see Session 7 Additional Examples Q2), we find that we can write

$$D(t,T) = e^{A(t,T)-B(t,T)r_t},$$

or (equivalently)

$$R(t,T) = \frac{1}{T-t}\Big[-A(t,T) + B(t,T)r_t\Big]$$

where

$$B(t,T) = \frac{1}{\kappa}\Big(1 - e^{-\kappa(T-t)}\Big)$$

$$A(t,T) = \frac{[B(t,T)-(T-t)](\kappa^2\theta - \frac{1}{2}\sigma^2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}$$

Cox-Ingersoll-Ross Model

In any model in which the short rate is normally distributed (including the Vasicek model), there is always a non-zero probability that the short rate is negative.

An alternative model will be the Cox-Ingersoll-Ross (CIR) model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^*$$

However, r_t is non-centrally χ^2-distributed in the CIR model.