Given the stochastic differential equation:

$$dL_i(t) = \sigma_i L_i(t) dW_t^{i+1},$$

where W_t^{i+1} is a Brownian motion under the risk-neutral measure associated to the zero-coupon discount bond $D(t, T_{i+1}) = D_{i+1}(t)$.

$$L_i(T) = L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}.$$

A contract paying $L_i(T_i) + L_i^2(T_i)$ at time T_{i+1} has present value:

$$\begin{aligned} V_0 &= D_{i+1}(0)\mathbb{E}^{i+1}[L_i(T_i) + L_i^2(T_i)] \\ &= D_{i+1}(0)(L_i(0)\mathbb{E}\left[\exp\left(-\tfrac{1}{2}\sigma_i^2 T_i + \sigma_i W_{T_i}^{i+1}\right)\right] + L_i(0)^2\mathbb{E}\left[\exp\left(-\sigma_i^2 T_i + 2\sigma_i W_{T_i}^{i+1}\right)\right]) \\ &= D_{i+1}(0)(L_i(0)e^{-\frac{1}{2}\sigma_i^2 T_i}\cdot\mathbb{E}\left[e^{\sigma_i W_{T_i}^{i+1}}\right] + L_i(0)^2 e^{-\sigma_i^2 T_i}\cdot\mathbb{E}\left[e^{2\sigma_i W_{T_i}^{i+1}}\right]) \\ &= D_{i+1}(0)\left[L_i(0) + L_i^2(0)e^{\sigma_i^2 T_i}\right]. \end{aligned}$$

Given the SDE:

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW_t^{n+1,N},$$

a digital swaption pays:

$$V_{\text{dig}}(T) = P_{n+1,N}(T)\cdot \mathbb{1}_{\{S_{n,N}(T)>K\}}.$$

$$\frac{V_{\text{dig}}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_{\text{dig}}(T)}{P_{n+1,N}(T)}\right]$$

$$\begin{aligned} V_{\text{dig}}(0) &= P_{n+1,N}(0)\cdot\mathbb{E}^{n+1,N}\left[\mathbb{1}_{\{S_{n,N}(T)>K\}}\right] \\ &= P_{n+1,N}(0)\cdot\Phi\left(\frac{\log\left(\frac{S_{n,N}(0)}{K}\right) - \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}\right). \end{aligned}$$

The IRR-settled payer and receiver swaptions are:

$$V_{\text{pay}}(t) = D(t,T)\int_K^\infty \text{IRR}(S)(S-K)f(S)dS,$$

$$V_{\text{rec}}(t) = D(t,T)\int_0^K \text{IRR}(S)(K-S)f(S)dS.$$

Using Breeden-Litzenberger:

$$\frac{\partial V}{\partial K} = -D(t,T)\int_K^\infty \text{IRR}(S)f(S)\,dS$$

$$\frac{\partial^2 V}{\partial K^2} = D(t,T)\,\text{IRR}(K)f(K)$$

$$\Rightarrow \quad f(K) = \frac{1}{D(t,T)\,\text{IRR}(K)}\cdot\frac{\partial^2 V(K)}{\partial K^2}. \quad \triangleleft$$

Thus:

$$f(K) = \begin{cases} \frac{1}{D(0,T)\cdot\text{IRR}(K)}\cdot\frac{\partial^2 V_{\text{pay}}(K)}{\partial K^2} & \text{if } K > S_{n,N}(0), \\ \frac{1}{D(0,T)\cdot\text{IRR}(K)}\cdot\frac{\partial^2 V_{\text{rec}}(K)}{\partial K^2} & \text{if } K < S_{n,N}(0). \end{cases}$$

Derive a static replication formula for a European payoff g(S) on the CMS rate, and apply it to the case g(S)=S.

Suppose we wish to pay a generic function g of the forward swap rate S, i.e., $g(S)$. Based on the static replication approach, let $F = S_{n,N}(0)$ be the expansion point, and let $h(K) = \frac{g(K)}{\text{IRR}(K)}$, then the value of this contract can be written as:

$$\begin{aligned} V_0 &= D(0,T)\mathbb{E}[g(S)] \\ &= D(0,T)\int_0^\infty g(K)f(K)dK \\ &= D(0,T)\int_0^\infty g(K)\cdot\frac{1}{D(0,T)}\cdot\frac{1}{\text{IRR}(K)}\cdot\frac{\partial^2 V(K)}{\partial K^2}dK \\ &= \int_0^F h(K)\frac{\partial^2 V^{\text{rec}}(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 V^{\text{pay}}(K)}{\partial K^2}dK \\ &= \left[h(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}\right]_0^F - \int_0^F h'(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}dK \\ &\quad + \left[h(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}\right]_F^\infty - \int_F^\infty h'(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}dK \\ &= h(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - \cancel{h(0)\frac{\partial V^{\text{rec}}(0)}{\partial K}} - [h'(K)V^{\text{rec}}(K)]_0^F + \int_0^F h''(K)V^{\text{rec}}(K)dK \\ &\quad + \cancel{h(\infty)\frac{\partial V^{\text{pay}}(\infty)}{\partial K}} - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} - [h'(K)V^{\text{pay}}(K)]_F^\infty + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ &= h(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - h'(F)V^{\text{rec}}(F) + \cancel{h'(0)V^{\text{rec}}(0)}^{=0} + \int_0^F h''(K)V^{\text{rec}}(K)dK \\ &\quad - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} - \cancel{h'(\infty)V^{\text{pay}}(\infty)}^{=0} + h'(F)V^{\text{pay}}(F) + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ &= -h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\ &\quad + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \end{aligned}$$

integration by parts rule for second derivatives:

$$\int_a^b u(K)\frac{d^2v(K)}{dK^2}dK = \left[u(K)\frac{dv(K)}{dK}\right]_a^b - \int_a^b u'(K)\frac{dv(K)}{dK}dK$$

The Put-Call Parity for IRR-Settled Swaptions is given by

$$V^{\text{pay}}(K) - V^{\text{rec}}(K) = D(0,T)\mathbb{E}[\text{IRR}(S)(S-K)^+] - D(0,T)\mathbb{E}[\text{IRR}(S)(K-S)^+] = D(0,T)\text{IRR}(S)(S-K)$$

$$\Rightarrow \frac{\partial V^{\text{pay}}(K)}{\partial K} - \frac{\partial V^{\text{rec}}(K)}{\partial K} = -D(0,T)\text{IRR}(S)$$

Substituting this back into the derivation on the previous page, we obtain:

$$\begin{aligned} V_0 &= -h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\ &\quad + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ &= D(0,T)h(F)\text{IRR}(F) + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ &= D(0,T)g(F) + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \end{aligned}$$

For example, for CMS rate, the payoff is $g(F) = F$, and recognizing that $V^{\text{pay}}(F) - V^{\text{rec}}(F) = 0$, we have the following CMS replication formula:

$$V_0 = D(0,T)F + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK$$

1. An FX process observed by the domestic investors $dX_t = (r_D - r_F)X_t dt + \sigma X_t dW_t^D$.

(a) The price of an FX forward (domestic investor). Derive the interest rate parity relationship

$$E_D[X_T] = X_0 e^{(r_D - r_F)T}.$$

First, compute $d\ln(X_t)$. Define $f(X_t) = \ln(X_t)$. Applying Itô's lemma:

$$\begin{aligned} d\ln(X_t) &= f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2 \\ &= \frac{1}{X_t}\left[(r_D - r_F)\,X_t\,dt + \sigma\,X_t\,dW_t^D\right] + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)\cdot\sigma^2 X_t^2\,dt \\ &= \left(r_D - r_F - \frac{1}{2}\sigma^2\right)dt + \sigma\,dW_t^D \end{aligned}$$

Integrating:

$$\ln(X_T) = \ln(X_0) + \left(r_D - r_F - \frac{1}{2}\sigma^2\right)T + \sigma W_T^D$$

Therefore:

$$X_T = X_0\cdot\exp\left[\left(r_D - r_F - \frac{1}{2}\sigma^2\right)T + \sigma W_T^D\right]$$

Taking the expectation(recalling that W_T^D is normally distributed so that $E[e^{\sigma W_T^D}] = e^{\frac{1}{2}\sigma^2 T}$)

$$E_D[X_T] = X_0\,e^{(r_D - r_F)T}.$$

(b) Show that the foreign investor will see the following SDEs:

$$d\left(\frac{1}{X_t}\right) = \left(r_F - r_D + \sigma^2\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^D,$$

$$d\left(\frac{1}{X_t}\right) = \left(r_F - r_D\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

Define $Y_t = \frac{1}{X_t}$ and apply Itô's lemma:

$$dY_t = -\frac{1}{X_t^2}\,dX_t + \frac{1}{2}\cdot\frac{2}{X_t^3}(dX_t)^2.$$

Since

$$dX_t = (r_D - r_F)X_t\,dt + \sigma X_t\,dW_t^D,$$

Final Result:

$$d\left(\frac{1}{X_t}\right) = \left(r_F - r_D + \frac{1}{2}\sigma^2\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^D. \quad \triangleleft$$

On the other hand, starting with B_t^D and we note that $\frac{B_t^D}{X_t}$ is a foreign tradable. Let $Y_t = \frac{B_t^D}{X_t} = f(B_t^D, X_t)$, we again use Itô's formula to derive(Q^D)

$$dY_t = \left(r^D + \sigma^2 - \mu\right)Y_t\,dt - \sigma Y_t\,dW_t.$$

Next, note that $Z_t = \frac{Y_t}{B_t^F} = g(Y_t, B_t^F)$ is a ratio of foreign tradable assets. We use Itô's formula to derive($Q^B -> Q^F$ no drift)

$$\begin{aligned} dZ_t &= \left(r^D - r^F + \sigma^2 - \mu\right)Z_t\,dt - \sigma Z_t\,dW_t \\ &= -\sigma Z_t\left(dW_t - \frac{r^D - r^F + \sigma^2 - \mu}{\sigma}\,dt\right) \\ &= -\sigma Z_t\,dW_t^F. \end{aligned}$$

Substituting

$$dW_t = dW_t^F + \frac{r^D - r^F + \sigma^2 - \mu}{\sigma}\,dt$$

into

$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$$

Then we get new dX_t

$$dX_t = (r^D - r^F + \sigma^2)X_t\,dt + \sigma X_t\,dW_t^F,$$

using Itô's formula to get $d\left(\frac{1}{X_t}\right)$. (let $f(X_t) = \frac{1}{X_t}$)

Derive the FX forward price from the foreign investor's perspective and show that its expectation (i.e. $E_F[\frac{1}{X_T}]$) is consistent with the forward price obtained by the domestic investor.

Solving the stochastic differential equation, we get

$$d\log\left(\frac{1}{X_t}\right) = \left(r_F - r_D\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

Solving this SDE yields

$$\frac{1}{X_T} = \frac{1}{X_0}\exp\left[\left(r_F - r_D - \frac{1}{2}\sigma^2\right)T - \sigma W_T^F\right].$$

Taking expectations (with $E[e^{-\sigma W_T^F}] = e^{\frac{1}{2}\sigma^2 T}$), we have

$$E_F\left[\frac{1}{X_T}\right] = \frac{1}{X_0}e^{(r_F - r_D)T}.$$

Inverting this relationship gives

$$E_F[X_T] = X_0\,e^{(r_D - r_F)T},$$

which is consistent with the domestic investor's forward price.

In a LIBOR-in-arrear contract, the LIBOR rate L_i is observed at time T_i and paid immediately at T_i.

(a) Using the LIBOR market model, evaluate the expectation (after performing the single-currency convexity correction)

$$E_i[L_i(T_i)].$$

Under the LIBOR market model, the forward LIBOR $L_i(t)$ is a martingale under the measure associated with the numeraire $D_{i+1}(t)$:

$$L_i(T_i) = L_i(0)\exp\left(-\frac{1}{2}\sigma_i^2 T_i + \sigma_i W_{T_i}^{(i+1)}\right).$$

Using Radon-Nikodym derivative to change the measure, we obtain

$$\begin{aligned} \mathbb{E}^i[L_i(T)] &= \mathbb{E}^{i+1}\left[\frac{dQ^i}{dQ^{i+1}}L_i(T)\right] \\ &= \mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}L_i(T)\right] \\ &= \frac{1}{1+\Delta_i L_i(0)}\mathbb{E}^{i+1}[(1+\Delta_i L_i(T))\cdot L_i(T)] \\ &= \frac{1}{1+\Delta_i L_i(0)}\left[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T}\right] \\ &= L_i(0)\cdot\frac{1+\Delta_i L_i(0)e^{\sigma_i^2 T}}{1+\Delta_i L_i(0)} \end{aligned}$$

(b) Derive the valuation formula for a LIBOR-in-arrear caplet paying $(L_i(T_i) - K)^+$, observed and paid at T_i.

$$\begin{aligned} V_0 &= D_i(0)\mathbb{E}^i[(L_i(T) - K)^+] \\ &= D_i(0)\mathbb{E}^{i+1}\left[\frac{dQ^i}{dQ^{i+1}}(L_i(T) - K)^+\right] \\ &= D_i(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}(L_i(T) - K)^+\right] \\ &= D_{i+1}(0)\mathbb{E}^{i+1}[(1+\Delta_i L_i(T))\cdot(L_i(T) - K)^+] \\ &= D_{i+1}(0)\left\{\mathbb{E}^{i+1}[(L_i(T) - K)^+] + \Delta_i\mathbb{E}^{i+1}[L_i(T)(L_i(T) - K)^+]\right\} \\ &= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) - \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right)\right] \\ &\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi(-x^* + 2\sigma_i\sqrt{T}) - L_i(0)K\Phi(-x^* + \sigma_i\sqrt{T})\right] \\ &= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) - \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right)\right] \\ &\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{3}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right) - L_i(0)K\Phi\left(\frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right)\right] \end{aligned}$$

$$\mathbb{E}\left[L_i(T)^2\cdot\mathbb{1}_{L_i(T)>K}\right]$$

Now you have $L_i(T)^2 \sim e^{2\sigma_i W_T}$, $\Rightarrow$ exponent has $2\sigma_i W_T \Rightarrow$ shift $= +2\sigma_i\sqrt{T} \Rightarrow$ gives $\Phi\left(-x^* + 2\sigma_i\sqrt{T}\right)$

(a) A contract pays $\Delta_i \times L_i(T)$ at $T = T_{i+1}$. Derive a valuation formula for this contract using the LIBOR market model.

$$dL_i(t) = \sigma_i L_i(t)\,dW^{i+1}(t),$$

where $W^{i+1}(t)$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^{i+1}$, associated to the zero-coupon bond $D_{i+1}(t) = D(t, T_{i+1})$.

$$L_i(T) = L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}.$$

Let V_t denote the value of a financial contract at time t. Under the martingale measure, we have:

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{V_T}{D_{i+1}(T)}\right]$$

$$\begin{aligned} V_0 &= D(0,T_{i+1})\mathbb{E}^{i+1}[\Delta_i L_i(T)] \\ &= D(0,T_{i+1})\Delta_i\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}\right] \\ &= D(0,T_{i+1})\Delta_i L_i(0)e^{-\frac{1}{2}\sigma_i^2 T}\mathbb{E}^{i+1}\left[e^{\sigma_i W_T^{i+1}}\right] \\ &= D(0,T_{i+1})\Delta_i L_i(0)e^{-\frac{1}{2}\sigma_i^2 T}\cdot e^{\frac{1}{2}\sigma_i^2 T} = D(0,T_{i+1})\Delta_i L_i(0) \quad \triangleleft \end{aligned}$$

(b) A contract pays $\Delta_i \times L_i(T)$ at $T = T_i$. Derive a valuation formula for this contract using the LIBOR market model.

$$\begin{aligned} V_0 &= D(0,T_i)\mathbb{E}^i[\Delta_i \times L_i(T_i)] \\ &= D(0,T_i)\mathbb{E}^{i+1}\left[\frac{dQ^i}{dQ^{i+1}}\cdot\Delta_i L_i(T_i)\right] \\ &= D(0,T_i)\Delta_i\mathbb{E}^{i+1}\left[\frac{L_i(T_i) + \Delta_i L_i(T_i)^2}{1+\Delta_i L_i(0)}\right] \\ &= D(0,T_i)\frac{\Delta_i}{1+\Delta_i L_i(0)}\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}} + \Delta_i L_i(0)^2 e^{-\sigma_i^2 T + 2\sigma_i W_T^{i+1}}\right] \\ &= D(0,T_i)\frac{\Delta_i}{1+\Delta_i L_i(0)}\left[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_i}\right] \end{aligned}$$

Let L_i^D be a forward LIBOR rate in the domestic economy, observed at time T_i and paid at T_{i+1}. It follows the LIBOR market model with volatility σ_i. In addition, there is a forward foreign exchange process given (from the domestic investor's perspective) by

$$dF_t = \sigma_X F_t\,dW_t^D.$$

Suppose the Brownian motions W_t^D (for the exchange rate) and $W_t^{(i+1)}$ (for the LIBOR rate) are correlated with correlation ρ. Evaluate the following expectation (from the foreign investor's perspective):

$$E_{i+1,F}\left[L_i^D(T)\right].$$

We have

$$dF_t = \sigma_X F_t\,dW_t^D \quad\Rightarrow\quad F_T = F_0 e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X W_T^D}$$

and

$$dL_i^D(t) = \sigma_i L_i^D(t)\,dW_t^{i+1} \quad\Rightarrow\quad L_i^D(T) = L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}$$

with $W_T^{i+1}\cdot W_T^D = \rho dt$(the LIBOR rate under $Q^{i+1,D}$ and the FX process). We apply multi-currency change of numeraire theorem to evaluate the expectation:

$$\begin{aligned} \mathbb{E}^{i+1,F}\left[L_i^D(T)\right] &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{dQ^{i+1,F}}{dQ^{i+1,D}}\right] \\ &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_T D_{i+1}^F(T)}{X_0 D_{i+1}^F(0)}\cdot\frac{D_{i+1}^D(0)}{D_{i+1}^D(T)}\right] \\ &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_T D_{i+1}^F(T)}{D_{i+1}^D(T)}\cdot\frac{X_0 D_{i+1}^F(0)}{D_{i+1}^D(0)}^{-1}\right] \\ &= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{F_T}{F_0}\right] \\ &= \mathbb{E}^{i+1,D}\left[L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}\cdot e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X W_T^D}\right] \\ &= L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T}e^{-\frac{1}{2}\sigma_X^2 T}\mathbb{E}^{i+1,D}\left[e^{\sigma_i W_T^{i+1} + \sigma_X W_T^D}\right] \\ &= L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T - \frac{1}{2}\sigma_X^2 T}\cdot\mathbb{E}^{i+1,D}\left[e^{\sigma_i Z_1\sqrt{T} + \sigma_X(\rho Z_1\sqrt{T} + \sqrt{1-\rho^2}Z_2\sqrt{T})}\right] \\ &= L_i^D(0)e^{-\frac{1}{2}\sigma_i^2 T - \frac{1}{2}\sigma_X^2 T}\cdot\mathbb{E}\left[e^{(\sigma_i + \rho\sigma_X)Z_1\sqrt{T} + \sigma_X\sqrt{1-\rho^2}Z_2\sqrt{T}}\right] \end{aligned}$$

Given three correlated Brownian motions W_t^f, W_t^g and W_t^h with correlations

$$dW_t^f\,dW_t^g = \rho_{fg}\,dt, \quad dW_t^g\,dW_t^h = \rho_{gh}\,dt, \quad dW_t^f\,dW_t^h = \rho_{fh}\,dt,$$

determine the coefficients α_{11}, α_{12}, α_{22}, α_{13}, α_{23} and α_{33} for the Cholesky decomposition

$$\begin{aligned} dW_t^f &= \alpha_{11}\,dZ_t^1, \\ dW_t^g &= \alpha_{12}\,dZ_t^1 + \alpha_{22}\,dZ_t^2, \\ dW_t^h &= \alpha_{13}\,dZ_t^1 + \alpha_{23}\,dZ_t^2 + \alpha_{33}\,dZ_t^3, \end{aligned}$$

where Z_t^1, Z_t^2 and Z_t^3 are three mutually independent Brownian motions.

Set $\alpha_{11} = 1$. This means:

$$dW_t^f = dZ_t^1$$

Match correlation between dW_t^f and dW_t^g:

$$\text{Cov}(dW_t^f, dW_t^g) = \alpha_{12} \Rightarrow \alpha_{12} = \rho_{fg}$$

Enforce unit variance for dW_t^g:

$$\alpha_{22} = \sqrt{1-\alpha_{12}^2} = \sqrt{1-\rho_{fg}^2}$$

Match correlation between dW_t^f and dW_t^h:

$$\text{Cov}(dW_t^f, dW_t^h) = \alpha_{13} \Rightarrow \alpha_{13} = \rho_{fh}$$

Match correlation between dW_t^g and dW_t^h:

$$\rho_{gh} = \alpha_{12}\alpha_{13} + \alpha_{22}\alpha_{23} \Rightarrow \alpha_{23} = \frac{\rho_{gh} - \alpha_{12}\alpha_{13}}{\alpha_{22}}$$

Enforce unit variance for dW_t^h:

$$\alpha_{33} = \sqrt{1-\alpha_{13}^2 - \alpha_{23}^2}$$

Discussion: Siegel's Exchange Rate Paradox

Consider a simplified discrete FX market involving the SGD and USD economies, with a spot FX rate approximately $FX_0 \approx 1.25$. Using a one-step binomial model with parameters

$$u = \frac{6}{5}, \quad d = \frac{5}{6}, \quad p^* = q^* = 0.5,$$

determine the expected forward exchange rate.

From the perspective of a Singapore-based investor, we have:

$$E[FX_T] = \frac{1}{2}\times\frac{6}{5}\times 1.25 + \frac{1}{2}\times\frac{5}{6}\times 1.25 \approx 1.423.$$

From the perspective of a US-based investor, we have:

$$E\left[\frac{1}{FX_T}\right] = \frac{1}{2}\times\frac{5}{6}\times\frac{1}{1.25} + \frac{1}{2}\times\frac{6}{5}\times\frac{1}{1.25} \approx 0.726.$$

Since $1/0.726 \approx 1.377 \neq 1.423$, the same binomial FX model gives rise to two different expectations depending on the denomination.

In the continuous time framework, if the Singapore-based investor observes

$$dX_t = (r_D - r_F)X_t\,dt + \sigma X_t\,dW_t^D,$$

a naive change of numeraire might suggest that the US-based investor sees

$$d\left(\frac{1}{X_t}\right) = (r_F - r_D)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

However, this is incorrect because the SGD money market account is not tradable for the US investor. Instead, applying Itô's formula correctly under the USD numeraire gives

$$d\left(\frac{1}{X_t}\right) = \left(r_F - r_D + \sigma^2\right)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^D,$$

and then, using the USD money market account as the numeraire,

$$d\left(\frac{1}{X_t}\right) = (r_F - r_D)\frac{1}{X_t}\,dt - \sigma\frac{1}{X_t}\,dW_t^F.$$

This resolves the paradox.

1. **Short Rate Model:**

$$dr_t = \mu\,dt + \sigma\,dW_t, \quad W_t: \text{ Brownian motion under } \mathbb{Q}$$

(a) Find the mean and variance of $\int_t^T r_u\,du$.

Step 1: Integrate from t to s

$$r_s = r_t + \mu(s-t) + \int_t^s \sigma\,dW_u$$

Step 2: Integrate r_s from t to T

$$\int_t^T r_s\,ds = r_t(T-t) + \mu\int_t^T (s-t)\,ds + \int_t^T\int_t^s \sigma\,dW_u\,ds$$

$$\int_t^T (s-t)\,ds = \frac{1}{2}(T-t)^2$$

Using Fubini's Theorem:

$$\int_t^T\int_t^s \sigma\,dW_u\,ds = \int_t^T \sigma(T-u)\,dW_u$$

Final Form:

$$\int_t^T r_s\,ds = r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma(T-u)\,dW_u$$

Mean:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T r_s\,ds\right] = r_t(T-t) + \frac{\mu}{2}(T-t)^2$$

Variance:

$$\text{Var}\left[\int_t^T r_s\,ds\right] = \frac{\sigma^2}{3}(T-t)^3$$

(b) Find $A(t,T)$ and $B(t,T)$ in $D(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du}\right] = e^{A(t,T) - r_t B(t,T)}$

Using mean and variance:

$$D(t,T) = \exp\left(-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3\right)$$

$$A(t,T) = -\frac{\mu}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3, \quad B(t,T) = T-t$$

(c) Explain what is an affine interest rate model. Is the short rate model considered above an affine interest rate model?

An affine interest rate model has zero-coupon bond prices of the form:

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

where $A(t,T)$ and $B(t,T)$ are deterministic functions. Hence, the spot rate $R(t,T)$ is affine in r_t:

$$R(t,T) = \frac{-A(t,T) + r_t B(t,T)}{T-t}.$$

Yes, the given model is affine.

2. **Vasicek Model:**

$$dr_t = \kappa(\theta - r_t)dt + \sigma\,dW_t$$

(a) Compute mean, variance, and expectation of $\int_0^T r_u\,du$

Solution for r_t:

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma\int_0^t e^{-\kappa(t-u)}dW_u$$

Integrate from 0 to T:

$$\int_0^T r_t\,dt = \int_0^T\left[r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})\right]dt + \int_0^T\int_0^t \sigma e^{-\kappa(t-u)}dW_u\,dt$$

Fubini on the last term:

$$\int_0^T\int_0^t \sigma e^{-\kappa(t-u)}dW_u\,dt = \frac{\sigma}{\kappa}\int_0^T (1 - e^{-\kappa(T-u)})dW_u$$

Final form:

$$\int_0^T r_t dt = \frac{r_0}{\kappa}(1 - e^{-\kappa T}) + \theta\left(T - \frac{1 - e^{-\kappa T}}{\kappa}\right) + \frac{\sigma}{\kappa}\int_0^T (1 - e^{-\kappa(T-u)})dW_u$$

Mean:

$$\mathbb{E}\left[\int_0^T r_t dt\right] = \frac{r_0}{\kappa}(1 - e^{-\kappa T}) + \theta\left(T - \frac{1 - e^{-\kappa T}}{\kappa}\right)$$

Variance:

$$\text{Var}\left[\int_0^T r_t dt\right] = \frac{\sigma^2}{\kappa^2}\left[T - \frac{2}{\kappa}(1 - e^{-\kappa T}) + \frac{1}{2\kappa}(1 - e^{-2\kappa T})\right]$$

$D(0,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}\right]$ using moment-generating function

$$D(0,T) = \exp\left(-\mathbb{E}\left[\int_0^T r_t dt\right] + \frac{1}{2}\text{Var}\left[\int_0^T r_t dt\right]\right)$$

3. Ho-Lee model: derive $\theta(t)$ from D(0, t)

$$dr_t = \theta(t)\,dt + \sigma\,dW_t^*, \quad \text{where } W_t^* \text{ is Brownian motion under } \mathbb{Q}^*.$$

Integrating the interest rate model from 0 to t:

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

Then integrate r_u from 0 to t:

$$\int_0^t r_u\,du = r_0 t + \int_0^t \theta(s)(t-s)\,ds + \int_0^t \sigma(t-s)\,dW_s^* \quad \text{(Fubini)}$$

Mean:

$$\mathbb{E}\left[\int_0^t r_u\,du\right] = r_0 t + \int_0^t \theta(s)(t-s)\,ds$$

Variance:

$$\text{Var}\left[\int_0^t r_u\,du\right] = \int_0^t \sigma^2(t-s)^2\,ds = \frac{1}{3}\sigma^2 t^3$$

Discount factor:

$$D(0,t) = \exp\left(-r_0 t - \int_0^t \theta(s)(t-s)\,ds + \frac{1}{6}\sigma^2 t^3\right)$$

Take logs and differentiate:

$$\frac{\partial}{\partial t}\log D(0,t) = -r_0 - \int_0^t \theta(s)\,ds + \frac{1}{2}\sigma^2 t^2$$

$$\frac{\partial^2}{\partial t^2}\log D(0,t) = -\theta(t) + \sigma^2 t$$

$$\Rightarrow \theta(t) = -\frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2 t$$

4. Ho-Lee Tree: Find θ_0

Given: $D(0,1y) = 0.95$, $D(0,2y) = 0.88$

Compute short rate: $r = -\log(0.95) = 5.129\%$

$r = 6.129\% + \theta_0$

$D(1,2) = e^{-(6.129\% + \theta_0)}$

$r = 5.129\%$

$D(0,2) = 0.88$

$r = 4.129\% + \theta_0$

$D(1,2) = e^{-(4.129\% + \theta_0)}$

$$D(0,2y) = D(0,1y)\times D(1y,2y)$$

$$0.88 = 0.95\times 0.5\left(e^{-(6.129\% + \theta_0)} + e^{-(4.129\% + \theta_0)}\right)$$

$$\Rightarrow \theta_0 = 0.0253$$

5. Ho-Lee Tree: Find θ_0, θ_1

Given: $D(0,1y) = 0.95123$, $D(0,2y) = 0.86936$, $D(0,3y) = 0.78663$

$$D(0,1y) = e^{-R(0,1)} \Rightarrow R(0,1) = 5\% \Rightarrow r_0 = 0.05$$

$r = 6\% + \theta_0 + \theta_1$

$r = 5.5\% + \theta_0$

$r = 5\%$

$r = 5\% + \theta_0 + \theta_1$

$r = 4.5\% + \theta_0$

$r = 4\% + \theta_0 + \theta_1$

$$D(0,2y) = D(0,1y)\cdot\mathbb{E}^*[D(1y,2y)] = D(0,1y)\cdot\tfrac{1}{2}\left(e^{-(\theta_0 + 0.055)} + e^{-(\theta_0 + 0.045)}\right)$$

$$\Rightarrow \theta_0 = 0.04$$

$$D_u(1,3) = e^{-(\theta_0 + 0.055)}\cdot\tfrac{1}{2}\left(e^{-(\theta_0 + \theta_1 + 0.06)} + e^{-(\theta_0 + \theta_1 + 0.05)}\right)$$

$$D_d(1,3) = e^{-(\theta_0 + 0.045)}\cdot\tfrac{1}{2}\left(e^{-(\theta_0 + \theta_1 + 0.05)} + e^{-(\theta_0 + \theta_1 + 0.04)}\right)$$

$$D(0,3y) = D(0,1y)\cdot\tfrac{1}{2}(D_u + D_d) \Rightarrow \theta_1 = 0.01$$

1. Interest rate = 5% (quarterly).

(a) Find Effective Annual Rate (EAR)

$$r_{\text{EAR}} = \left(1+\frac{r_S}{m}\right)^m - 1 = \left(1+\frac{5\%}{4}\right)^4 - 1 \approx 5.095\%.$$

(b) Find Bond Equivalent Yield (BEY)

$$r_{\text{BEY}} = \left[\left(1+\frac{r_S}{m}\right)^{\frac{m}{2}} - 1\right]\times 2 = \left[\left(1+\frac{5\%}{4}\right)^2 - 1\right]\times 2 \approx 5.031\%.$$

(c) Find Continuously Compounded Rate r_c

$$e^{r_c} = \left(1+\frac{r_S}{m}\right)^m \implies r_c = m\times\ln\left(1+\frac{r_S}{m}\right) = 4\times\ln\left(1+\frac{5\%}{4}\right) \approx$$

2.

Maturity	Zero Rate
1y	5%
2y	5%
3y	5%

A 3-year coupon bond pays an annual coup of 5. Zero rates are discretely compounded

(a) Find Price of the bond. $B = \frac{5}{1.05} + \frac{5}{1.05^2} + \frac{105}{1.05^3} = 100$

(b) Find Par Yield of the bond. A bond trades at par when its price is equal to the face value. For this bond, the par yield is 5%

3. Derive an expression for the continuously compounded par yield of a coupon bond:

$$B = \sum_{i=1}^{N} c_i e^{-yT_i}$$

Let $T_{i+1} - T_i = \Delta T$. The bond price is:

$$B = ce^{-y\Delta T} + ce^{-2y\Delta T} + \cdots + ce^{-Ny\Delta T} + 100e^{-Ny\Delta T}$$

$$e^x \cdot e^y = e^{x+y}.$$

$$= ce^{-y\Delta T}\left[1 + e^{-y\Delta T} + \cdots + e^{-(N-1)y\Delta T}\right] + 100e^{-Ny\Delta T}$$

The sum of a geometric progression:

$$a_n = ar^{n-1}, \quad S_n = a\frac{1-r^n}{1-r}.$$

$$= c\cdot e^{-y\Delta T}\cdot\frac{1-e^{-Ny\Delta T}}{1-e^{-y\Delta T}} + 100e^{-Ny\Delta T}$$

$$= c\cdot\frac{1-e^{-Ny\Delta T}}{e^{y\Delta T}-1} + 100e^{-Ny\Delta T}$$

Set $B = 100$:

$$c\cdot\frac{1-e^{-Ny\Delta T}}{e^{y\Delta T}-1} = 100\left(1-e^{-Ny\Delta T}\right) \Rightarrow c = 100\cdot\left(e^{y\Delta T}-1\right)$$

Hence, the par yield is: $y = \frac{1}{\Delta T}\log\left(\frac{c}{100}+1\right)$.

4. A 5-year bond with YTM of 11% (continuous) pays 8% annual coupon.

(a) Find Bond price

$$B = 8\times\left(e^{-0.11\cdot 1} + e^{-0.11\cdot 2} + e^{-0.11\cdot 3} + e^{-0.11\cdot 4} + e^{-0.11\cdot 5}\right) + 100\cdot e^{-0.1}$$

(b) Since the bond yield is continuously compounded, Find modified duration.

$$D = \frac{1}{B}\sum_{i=1}^{n} t_i c_i e^{-yt_i}$$

$$D = \frac{1}{86.801}\left(1\cdot 8\cdot e^{-0.11\cdot 1} + 2\cdot 8\cdot e^{-0.11\cdot 2} + 3\cdot 8\cdot e^{-0.11\cdot 3} + 4\cdot 8\cdot e^{-0.11\cdot 4} + 5\cdot 108\cdot e^{-0.11\cdot 5}\right) = 4.256 \quad \triangleleft$$

(c) If the yield drops by $\Delta y = -0.2\%$, Find Change in Bond Price:

$$\frac{\Delta B}{B} \approx -D\cdot\Delta y = -4.256\cdot(-0.002) = 0.85\% \quad \triangleleft$$

$$\Delta B \approx -D\cdot\Delta y\cdot B = -4.256\cdot(-0.002)\cdot 86.801 = 0.73885 \quad \triangleleft$$

(d) Find convexity

$$C = \frac{1}{P}\times\sum_{i=1}^{n}\left(i^2\cdot w_i\cdot e^{-r\cdot t_i}\right)$$

(e) If yield is 10.8%, the new price is:

$$B = 8\cdot\left(e^{-0.108\cdot 1} + e^{-0.108\cdot 2} + e^{-0.108\cdot 3} + e^{-0.108\cdot 4} + e^{-0.108\cdot 5}\right) + 100\cdot e^{-0.108\cdot 5} = 87.5434$$

Estimated change using duration and convexity:

$$\Delta B \approx -D\cdot\Delta y\cdot B + \frac{1}{2}\cdot C\cdot(\Delta y)^2\cdot B$$

$$\approx -4.256\cdot(-0.002)\cdot 86.801 + \frac{1}{2}\cdot 19.871\cdot(0.002)^2\cdot 86.801$$

$$\approx 0.7423 \quad \triangleleft$$

$$87.5434 - 86.801 = 0.7424$$

5. A portfolio holds:

Bond	Position	Mod. Duration	Convexity
A	1.5 million	3.4	20
B	2.0 million	2.8	18

Market bonds:

Bond	Mod. Duration	Convexity
C	2.9	18
D	1.4	10

What positions (A and B) for portfolio with 0 Duration and Convexity?

$$D_\$(V) = 1.5\times 3.4 + 2\times 2.8 = 10.7$$

$$C_\$(V) = 1.5\times 20 + 2\times 18 = 66$$

$$\begin{cases} D_\$(V) + B_C\times 2.9 + B_D\times 1.4 = 0 \\ C_\$(V) + B_C\times 18 + B_D\times 10 = 0 \end{cases}$$

$B_C = -3.8421$ and $B_D = 0.31579$.

Market Quotes (Uncollateralised)

Maturity	Instrument	Rate
6m	CD	1.20%
1y	IRS	2.00%
2y	IRS	2.50%

Calculate forward LIBOR rates:
general formula:

$$L(T_{i-1},T_i) = \frac{1}{\Delta_{i-1}}\cdot\frac{D(0,T_{i-1}) - D(0,T_i)}{D(0,T_i)}$$

$$D(0,6m) = \frac{1}{1+0.5\times 0.012} = 0.994, \quad D(0,1y) = 0.9804 \text{ (from 1y IRS)}$$

$$L(6m,1y) = \frac{1}{0.5}\cdot\frac{0.994-0.9804}{0.9804} \approx 2.77\%$$

$$D(0,2y) = 0.9517 \text{ (from 2y IRS)}, \quad D(0,1y6m) = 0.5(0.9804+0.9517) = 0.9661$$

$$L(1y,1y6m) = \frac{1}{0.5}\cdot\frac{0.9804-0.9661}{0.9661} \approx 2.96\%, \quad L(1y6m,2y) = \frac{1}{0.5}\cdot\frac{0.9661-0.9517}{0.9517} \approx 3.03\%$$

Spot LIBOR Rates and OIS Info:Fed-Fund OIS(overnight index swaps) rate: 0.70% (flat), 30/360 convention, zero index basis swap spread

Tenor	LIBOR Rate
1m	1.15%
2m	1.20%
3m	1.25%
6m	1.40%
9m	1.55%
12m	1.75%

(a) Calculate LIBOR $\tilde{D}(0,3m)$ and OIS $D_o(0,3m)$

$$\prod_{i=1}^{N}\frac{D_0(0,t_{i-1})}{D_0(0,t_i)} = 1+\Delta S_0, \quad D_0(0,t_N) = \left(\frac{1}{1+\Delta S_0}\right)^{t_N}, \quad \Delta S_0 = \frac{0.007}{360} \text{ (n unit is day)}$$

$$\tilde{D}(0,3m) = \frac{1}{1+0.25\times 0.0125} \approx 0.996885$$

$$D_o(0,3m) = \left(1+\frac{0.007}{360}\right)^{-90} \approx 0.99825$$

(b) Calculate $\tilde{D}(3m,6m)$, $D_o(3m,6m)$

$$\tilde{D}(3m,6m) = \frac{1}{1+0.5\times 0.014}\Big/\frac{1}{1+0.25\times 0.0125} \approx 0.996152$$

$$D_o(3m,6m) = \frac{\prod_{i=1}^{180}\left(1+\frac{0.007}{360}\right)^{-1}}{\prod_{i=1}^{90}\left(1+\frac{0.007}{360}\right)^{-1}} \approx 0.99825$$

(c) PV of 1y fixed leg at 1.75%, quarterly

i. No collateral

$$\text{PV}_{\text{fixed}} = \sum_{i=1}^{4}\Delta_{i-1}\cdot\tilde{D}(0,T_i)\cdot 1.75\% = 0.25\times 0.0175\times\left(\frac{1}{1+0.25\times 0.0125} + \frac{1}{1+0.50\times 0.014} + \frac{1}{1+0.75\times 0.0155} + \frac{1}{1+1.00\times 0.0175}\right) \approx 0.01733 \quad \triangleleft$$

ii. With USD collateral

$$\text{PV}_{\text{fixed}} = \sum_{i=1}^{4}\Delta_{i-1}\cdot D(0,T_i)\cdot 1.75\%$$

$$= 0.25\times 0.0175\times\left(D_o(0,3m) + D_o(0,6m) + D_o(0,9m) + D_o(0,12m)\right)$$

$$\approx 0.01742 \quad \triangleleft$$

(d) PV of 1y floating leg paying 3m LIBOR, quarterly

i. No collateral

$$\text{PV}_{\text{float}} = \sum_{i=1}^{4}\Delta_{i-1}\cdot\tilde{D}(0,T_i)\cdot L(T_{i-1},T_i) = \tilde{D}(0,0) - \tilde{D}(0,12m) \approx 0.0172 \quad \triangleleft$$

ii. With USD collateral
Discounting each cashflows one-by-one(like handling a fixed leg):

$$L(T_i,T_{i+1}) = \frac{1}{\Delta_i}\left[\frac{\tilde{D}(0,T_i) - \tilde{D}(0,T_{i+1})}{\tilde{D}(0,T_{i+1})}\right]$$

These are given by:

$$L(3m,6m) = \frac{1}{0.25}\left[\frac{\frac{1}{1+0.25\cdot L(0,3m)} - \frac{1}{1+0.5\cdot L(0,6m)}}{\frac{1}{1+0.5\cdot L(0,6m)}}\right] = \frac{1}{0.25}\left[\frac{\frac{1}{1+0.25\cdot 0.0125} - \frac{1}{1+0.5\cdot 0.014}}{\frac{1}{1+0.5\cdot 0.014}}\right] = 0.01545$$

$$L(6m,9m) = \frac{1}{0.25}\left[\frac{\frac{1}{1+0.5\cdot L(0,6m)} - \frac{1}{1+0.75\cdot L(0,9m)}}{\frac{1}{1+0.75\cdot L(0,9m)}}\right] = \frac{1}{0.25}\left[\frac{\frac{1}{1+0.5\cdot 0.014} - \frac{1}{1+0.75\cdot 0.0155}}{\frac{1}{1+0.75\cdot 0.0155}}\right] = 0.01837$$

$$L(9m,12m) = \frac{1}{0.25}\left[\frac{\frac{1}{1+0.75\cdot L(0,9m)} - \frac{1}{1+1.00\cdot L(0,12m)}}{\frac{1}{1+1.00\cdot L(0,12m)}}\right] = \frac{1}{0.25}\left[\frac{\frac{1}{1+0.75\cdot 0.0155} - \frac{1}{1+1.00\cdot 0.0175}}{\frac{1}{1+1.00\cdot 0.0175}}\right] = 0.02323$$

$$\sum_{i=1}^{4}\Delta_{i-1}D(0,T_i)L(T_{i-1},T_i) = 0.25\times\left(0.0125\,D(0,3m) + 0.01545\,D(0,6m) + 0.01837\,D(0,9m) + 0.02323\,D(0,12m)\right) \approx 0.0173 \quad \triangleleft$$

Maturity	Instrument	Swap Rate
1y	Spot LIBOR	1.5%
2y	Swap	2.25%
3y	Swap	2.55%

A collateralised forward contract pays the forward $L(2y,3y)$ at the end of 3y for a notional of \$1,000,000. Collateral is posted in cash (same currency), and the overnight discount rate:0.25%(flat).

Collateralized: 1.using collateral rate to get D(0,T). 2.let fix pv= floating pv to get $L(T_{i-1},T_i)$. 3. calculating pv.
Uncollateralized: 1.using par swap rate get D(0,T). 2.using D(0,T) to get $L(T_{i-1},T_i)$. 3. calculating pv.

(a) Calculate the PV of this collateralised trade.
First, we evaluate the overnight discount factors (using daily compounding at 0.25%):

$$D_o(0,1y) = \prod_{k=1}^{360}\left(1+\frac{0.0025}{360}\right)^{-1} \approx 0.9975, \quad D_o(0,2y) \approx 0.995, \quad D_o(0,3y) \approx 0.9925.$$

We observe $L(0,1y) = 0.015$.
Using the 2y swap (annual payment) at 2.25%:(fix pv = floating pv)

$$(D_o(0,1y) + D_o(0,2y))\times 0.0225 = D_o(0,1y)\times 0.015 + D_o(0,2y)\times L(1y,2y).$$

Using the 3y swap (annual payment) at 2.55%:

$$(D_o(0,1y) + D_o(0,2y) + D_o(0,3y))\times 0.0255 =$$

$$D_o(0,1y)\times 0.015 + D_o(0,2y)\times L(1y,2y) + D_o(0,3y)\times L(2y,3y).$$

Solving these gives $L(1y,2y)$ and $L(2y,3y)$. Hence, we can evaluate the PV of the collateralised FRA:

$$\text{PV} = D_o(0,3y)\times L(2y,3y)\times 1{,}000{,}000 \approx 31{,}263.75.$$

(b) If collateralisation is not considered, discounting is at LIBOR. What is the PV?
Without collateral, we use $\tilde{D}(0,1y) = \frac{1}{1+0.015} = 0.9852$.
From the 2y swap at 2.25%:

$$0.0225 = \frac{1-\tilde{D}(0,2y)}{\tilde{D}(0,1y) + \tilde{D}(0,2y)} \implies \tilde{D}(0,2y) = 0.9563.$$

From the 3y swap at 2.55%:

$$0.0255 = \frac{1-\tilde{D}(0,3y)}{\tilde{D}(0,1y) + \tilde{D}(0,2y) + \tilde{D}(0,3y)} \implies \tilde{D}(0,3y) = 0.927.$$

Hence the implied LIBOR forward rates are:

$$L(1y,2y) = 0.03, \quad L(2y,3y) = 0.0316.$$

So the PV of the same FRA (now discounted at LIBOR) is:

$$\text{PV} = \tilde{D}(0,3y)\times L(2y,3y)\times 1{,}000{,}000 \approx 29{,}223.$$

1. The spot LIBOR rates are as follow:

Tenor	Libor Rate
1m	1.15%
2m	1.20%
3m	1.25%
6m	1.40%
9m	1.55%
12m	1.75%

Calculate:

(a) The spot 3m discount factor $D(0,3m)$.

Answer:

$$D(0,3m) = \frac{1}{1+0.25\times 0.0125} \approx 0.996885.$$

(b) The forward discount factor $D(3m,6m)$.

Answer:

$$D(3m,6m) = \frac{D(0,6m)}{D(0,3m)} = \frac{\frac{1}{1+0.5\times 0.014}}{\frac{1}{1+0.25\times 0.0125}} \approx 0.996152.$$

(c) The forward LIBOR rate $F(2m,9m)$.

Answer:

$$(1+\Delta_{2m}L_{2m})(1+\Delta_{7m}F(2m,9m)) = 1+\Delta_{9m}L_{9m}$$

$$F(2m,9m) = \frac{1}{\frac{210}{360}}\left[\frac{1+\frac{270}{360}\times 0.0155}{1+\frac{60}{360}\times 0.012} - 1\right] \approx 1.6467\%.$$

(d) What rate would you show for a 2×12 FRA (no arbitrage)?

Answer:

$$(1+\Delta_{2m}L_{2m})(1+\Delta_{10m}F(2m,12m)) = 1+\Delta_{12m}L_{12m}$$

$$F(2m,12m) = \frac{1}{\frac{300}{360}}\left[\frac{1+\frac{360}{360}\times 0.0175}{1+\frac{60}{360}\times 0.012} - 1\right] \approx 1.85629\%.$$

(e) If your view is that one-month later the spot 1m rate would still remain at 1.15%, how should you trade?

Answer:

If we think that the 1m spot rate will remain unchanged a month later, we should short the 1×2 FRA, since $F(1m,2m) > 1.15\%$, we can borrow at 1.15% to deposit (lend) at $F(1m,2m)$ if we were right.

2. Continuously Compounded Zero Rates:

Zero Rates Table

Maturity	Zero Rate
1y	4%
2y	4.5%
3y	4.75%

(a) Calculate the continuously compounded forward rates $F(0y,1y)$, $F(1y,2y)$ and $F(2y,3y)$.

Answer:

$$F(0y,1y) = 4\% \quad \text{(same as observed zero rate)}$$

$$e^{F(0y,1y)\cdot 1}\cdot e^{F(1y,2y)\cdot 1} = e^{0.045\cdot 2} \Rightarrow F(1y,2y) = 5\%$$

$$e^{F(0y,1y)\cdot 1}\cdot e^{F(1y,2y)\cdot 1}\cdot e^{F(2y,3y)\cdot 1} = e^{0.0475\cdot 3} \Rightarrow F(2y,3y) = 5.25\%$$

(b) Show that the continuously compounded zero rate can be expressed as an arithmetic average of the corresponding forward rates.

Answer:

$$e^{F(0,1)}\cdot e^{F(1,2)}\cdot\ldots\cdot e^{F(n-1,n)} = e^{r_n\cdot n} \Rightarrow r_n = \frac{F(0,1) + F(1,2) + \cdots + F(n-1,n)}{n}$$

3. FX Forward Arbitrage:

(a) Spot Exchange rate for USD/SGD is $FX_0 = 1.42$. $D_{\text{SGD}}(0,T) = 0.98$, $D_{\text{USD}}(0,T) = 0.964$. What should be the forward value of the exchange rate at time T?

Answer:

$$FX_T = FX_0 e^{(r^D - r^F)T} \qquad (1)$$

$$FX_T = FX_0\cdot\frac{D_{\text{USD}}(0,T)}{D_{\text{SGD}}(0,T)} = 1.42\cdot\frac{0.964}{0.98} = 1.3968$$

(b) If we see that $FX_T = FX_0 = 1.42$, state an arbitrage.

Answer:

Long 1 unit of USD bond by shorting some SGD bond to get SGD 1.36888 (0.964×1.42) today. When USD bond matures, convert \$1 USD $\to$ 1.42 SGD. Short SGD bond matures at 1.3968 SGD (1.36888/0.98). Arbitrage profit is $1.42 - 1.3968$.

4. Swap Valuation Using Continuous Rates:

Maturity	Zero Rate
3m	1.10%
6m	1.40%
12m	1.75%
18m	1.90%
24m	2.00%

$$D(0,t) = e^{-rt}$$

(a) A 2y fixed leg pays 1.75% semi-annually. What is the PV of this fixed leg?

General formula:

$$\boxed{PV_{\text{fix}} = \sum_{i=1}^{n} D(0,T_i)\cdot\Delta\cdot K} \quad \text{where } K = \text{fixed rate}$$

Answer:

$$PV_{\text{fix}} = 0.5\cdot(D(0,6m) + D(0,12m) + D(0,18m) + D(0,24m))\cdot 1.75\% = 0.0342$$

(b) A 2y floating leg pays 6m LIBOR rate semi-annually. What is the PV of this floating leg?

General formula:

$$PV_{\text{float}} = \sum_{i=1}^{n}\Delta_i\cdot D(0,T_i)\cdot L(T_{i-1},T_i)$$

$$\boxed{PV_{\text{float}} = N\cdot(1 - D(0,T_n))}, \quad \text{where } n = \text{number of periods}$$

Answer:

$$PV_{\text{flt}} = 1 - D(0,24m) = 0.03921$$

(c) par swap rate for 2y interest rate swap with semi annual payment?

General formula:

$$\boxed{S_{\alpha,\beta} = \frac{D(0,T_\alpha) - D(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\Delta_i\cdot D(0,T_i)}} if spotswap(alpha=0), D(0,T_alpha) is 1$$

Answer:

$$S = \frac{1 - D(0,24m)}{0.5\cdot(D(0,6m) + D(0,12m) + D(0,18m) + D(0,24m))} = 2\%$$

(d) Long a receiver at par swap rate above, and 3 months later, we observed:

Maturity	Zero Rate
3m	1.20%
6m	1.50%
12m	1.85%
18m	1.95%
24m	2.05%

what is the value of receiver swap in the portfolio?

General formulas:

Linear interpolation of zero rates:

$$R_t = \frac{T_2 - t}{T_2 - T_1}\cdot R_{T_1} + \frac{t - T_1}{T_2 - T_1}\cdot R_{T_2}$$

From arbitrage-free pricing, the relationship between zero-coupon bonds and forward LIBOR is:

$$e^{R_{t_1}\cdot t_1}\cdot(1 + \Delta\cdot L(t_1,t_2)) = e^{R_{t_2}\cdot t_2}$$

—

Applying to this question:

Interpolated zero rates:

$$R_{9m} = \frac{1.50\% + 1.85\%}{2} = 1.675\%, \quad R_{15m} = \frac{1.85\% + 1.95\%}{2} = 1.90\%, \quad R_{21m} = \frac{1.95\% + 2.05\%}{2} =$$

Forward LIBOR rates:

$$L(3m,9m) = 1.92\%, \quad L(9m,15m) = 2.25\%, \quad L(15m,21m) = 2.263\%$$

Floating leg PV:

$$PV_{\text{flt}} = 0.5\cdot[D(0,3m)\cdot 1.4\% + L(3m,9m)\cdot D(0,9m) + L(9m,15m)\cdot D(0,15m) + L(15m,21m)\cdot D$$

Fixed leg PV:

$$PV_{\text{fix}} = 0.5\cdot[D(0,3m) + D(0,9m) + D(0,15m) + D(0,21m)]\cdot 2\% = 0.0393$$

Value of receiver swap:

$$V_{\text{rec}} = PV_{\text{fix}} - PV_{\text{flt}} = 0.001 \quad \text{(per \$1 notional)}$$

1. We observe the following instruments in the swap market. All three interest rate swaps have semi-annual payments.

Instrument	Quote
6m LIBOR	2%
1y IRS	2.25%
2y IRS	2.40%
3y IRS	2.50%

(a) Determine the par swap rate for a 1.5y tenor interest rate swap with semi-annual payment.

We need the discount factors $D(0,6m)$, $D(0,1y)$, and $D(0,1.5y)$.

$$D(0,6m) = \frac{1}{1+0.5\times 2.0\%} = 0.99$$

Using the 1y IRS quote:

$$PV_{\text{fixed}} = \sum_{i=1}^{n}\Delta_i\cdot S\cdot D(0,T_i)$$

$$PV_{\text{fix}} = 0.5\cdot 2.25\%\cdot[D(0,6m) + D(0,1y)]$$

$$PV_{\text{float}} = \sum_{i=1}^{n}\Delta_i\cdot L(T_{i-1},T_i)\cdot D(0,T_i)$$

$$PV_{\text{flt}} = D(0,6m)\cdot 0.5\cdot 2.0\% + D(0,1y)\cdot 0.5\cdot L(6m,12m)$$

$$= D(0,6m)(1 + 0.5\cdot 2.0\%) - D(0,1y)$$

$$= 1 - D(0,1y)$$

As $\text{PV}_fix = PV_floatSolving$:

$$D(0,1y) = 0.9779$$

Now use the 2y IRS quote:

$$0.5\cdot[D(0,6m) + D(0,1y) + D(0,1.5y) + D(0,2y)]\cdot 2.4\% = 1 - D(0,2y)$$

Interpolate:

$$D(0,1.5y) = \frac{D(0,1y) + D(0,2y)}{2}$$

$$D(0,2y) = 0.9536, \quad D(0,1.5y) = 0.96575$$

Finally:

$$S = \frac{1 - D(0,1.5y)}{0.5\cdot[D(0,6m) + D(0,1y) + D(0,1.5y)]} = 2.335\%$$

(b) A forward starting swap with a 2y tenor starting at $t = 1y$ has the following cashflows:

Time (y)	Pay	Rec
1.5	Par Swap Rate	6m LIBOR
2.0	Par Swap Rate	6m LIBOR
2.5	Par Swap Rate	6m LIBOR
3.0	Par Swap Rate	6m LIBOR

Calculate the par swap rate for this forward starting swap.
We need: $D(0,1y), D(0,1.5y), D(0,2y), D(0,2.5y), D(0,3y)$
Interpolate:

$$D(0,2.5y) = \frac{D(0,2y) + D(0,3y)}{2} = 0.4768 + 0.5D(0,3y)$$

Using 3y IRS quote:

$$PV_{\text{fix}} = 0.5\cdot[D(0,6m) + D(0,1y) + D(0,1.5y) + D(0,2y) + D(0,2.5y) + D(0,3y)]\cdot 2.5\%$$

$$PV_{\text{flt}} = 1 - D(0,3y)$$

Solving:

$$D(0,3y) = 0.928, \quad D(0,2.5y) = 0.941$$

Now, compute forward swap rate starting at $t = 1y$, i.e., from 1.5y to 3y:

$$S = \frac{D(0,1y) - D(0,3y)}{0.5\cdot[D(0,1.5y) + D(0,2y) + D(0,2.5y) + D(0,3y)]} = 2.63\%$$

6. Bond Portfolio Immunization:

(a) Dollar duration and convexity:

Answer:

$$\text{D}_s(V) = B_1D_1 + B_2D_2 = D_s(V) = 3.2\times 10^6 + 4\times 2.5\times 10^6 = 13.2\,\text{million}$$

$$\text{C}_s(V) = B_1C_1 + B_2C_2 = C_s(V) = 16\times 10^6 + 24\times 2.5\times 10^6 = 76\,\text{million}.$$

(b) Portfolio value change from 10bp rise(Delta_y) :

Answer:

$$\Delta V \approx -D\$(V)\cdot\Delta y + \frac{C\$(V)}{2}(\Delta y)^2 = -13{,}162$$

$$V' = 3{,}500{,}000 - 13{,}162 = 3{,}486{,}838$$

(c) Immunizing using two more bonds (3,4):

Answer:

$$\text{D}_s(\Pi) = 13.2\,\text{mil} + D_3B_3 + D_4B_4 = 13.2\,\text{mil} + 1.6B_3 + 3.2B_4$$

$$\text{C}_s(\Pi) = 76\,\text{mil} + 12B_3 + 20B_4$$

$$B_3 = 3.25 \text{ million}, \quad B_4 = -5.75 \text{ million}$$

P: real-world probability. W_t : $\mathbb{P}$-Brownian motion.
$StockpriceProcess: dS_t = \mu S_t dt + \sigma S_t dW_t$. The price of a risk-free bond:

$$dB_t = rB_t dt.$$

Q: risk-neutral measure. risk-free bond: numeraire.

(a) Evaluate $\mathbf{E}^{\mathbb{P}}[S_T]$

By applying Itô's formula: Let $X_t = \log S_t = f(S_t)$

$$df(X_t) = f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2$$

$$dX_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

Integrating both sides over $[0,T]$:

$$X_T = X_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T$$

Exponentiating both sides:

$$S_T = S_0\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right]$$

Taking expectation under $\mathbb{P}$:

$$\mathbf{E}^{\mathbb{P}}[S_T] = S_0 e^{\mu T}$$

(b) Evaluate $\mathbf{E}^{\mathbb{Q}}[S_T]$

Under $\mathbb{Q}$, the process $\frac{S_t}{B_t}$ is a martingale. The Radon-Nikodym derivative is:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right), \quad \text{where } \kappa = \frac{\mu - r}{\sigma}$$

Under $\mathbb{Q}$, define $dW_t^B = dW_t + \kappa dt$. The SDE becomes:

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t \\ &= \mu S_t\,dt + \sigma S_t\left(dW_t^B - \frac{\mu - r}{\sigma}dt\right) \\ &= rS_t\,dt + \sigma S_t\,dW_t^B \end{aligned}$$

Applying Itô's Formula to $X_t = \log(S_t)$:

$$S_T = S_0\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T^B\right]$$

Taking expectation under $\mathbb{Q}$:

$$\mathbf{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$$

LIBOR rate follows:

$$dL_i(t) = \sigma_i L_i(t)dW^{i+1},$$

$\mathbb{Q}^{i+1}$: risk-neutral measure, W^{i+1}: a standard BM, $D_{i+1}(t)$: numeraire. Derive the valuation formula for a floorlet:

$$F_i(0) = D_{i+1}(0)\Delta_i\mathbb{E}^{i+1}[(K - L_i(T_i))^+]$$

Let $f(L_i(t)) = \log L_i(i)$ and apply Ito's lemma:

$$L_i(t) = L_i(0)\exp\left(-\frac{1}{2}\sigma_i^2 t + \sigma_i W_t^{i+1}\right).$$

Evaluating the expectation, we obtain:

$$\begin{aligned} F_i(0) &= D_{i+1}(0)\cdot\Delta_i\cdot\mathbb{E}^{i+1}[(K - L_i(T_i))^+] \\ &= D_{i+1}(0)\cdot\Delta_i\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(K - L_i(0)e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i\sqrt{T}x}\right)^+ e^{-\frac{x^2}{2}}dx \\ &= D_{i+1}(0)\cdot\Delta_i\cdot[K\Phi(-d_2) - L_i(0)\Phi(-d_1)], \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}, \quad d_2 = d_1 - \sigma_i\sqrt{T}.$$

Suppose the swap rate follows (risk-neutral measure: $\mathbb{Q}^{n+1,N}$, numeraire: $P_{n+1,N}(t)$):

$$dS_{n,N}(t) = \sigma_{n,N}S_{n,N}(t)dW^{n+1,N}.$$

Derive the valuation formula for a receiver swaption:

$$V_{n,N}^{\text{rec}}(0) = P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(K - S_{n,N}(T))^+]$$

$W^{n+1,N}$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$. Therefore:

$$S_{n,N}(T) = S_{n,N}(0)\exp\left(-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}W_T^{n+1,N}\right).$$

Evaluating the expectation, we obtain:

$$\begin{aligned} V_{n,N}^{\text{rec}}(0) &= P_{n+1,N}(0)\cdot\mathbb{E}^{n+1,N}\left[(K - S_{n,N}(T))^+\right] \\ &= P_{n+1,N}(0)\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(K - S_{n,N}(0)e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}\sqrt{T}x}\right)^+ e^{-\frac{x^2}{2}}dx \\ &= P_{n+1,N}(0)\cdot[K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)], \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{S_{n,N}(0)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{n,N}\sqrt{T}.$$