Sample Final Exam Paper

Date / Start Time	-
Course	QF605 Fixed Income Securities
Instructor	Tee Chyng Wen

INSTRUCTIONS TO CANDIDATES

1. The time allowed for this examination paper is 3 hours.
2. You can bring in a prepared 2-sided A4 paper as formula sheet.
3. This examination paper contains a total of 10 questions and comprises 11 pages including this instruction sheet.
4. You are allowed to bring along your calculators.
5. You are required to return the full set of question paper and the solutions at the end of the examination.
6. You are encouraged to indicate techniques that may solve questions where you only provide partial answers.
7. Use 30/360 day count convention in your calculation for all questions.
8. Use linear interpolation on discount factors in necessary.

1. We observe the following quotes in the interest rate swap market:

Instrument	Quote
$6m$ LIBOR	2.0%
$1y$ Interest Rate Swap	2.2%
$2y$ Interest Rate Swap	2.5%

The interest rate swaps quoted have semi-annual payment. The overnight interest rate is flat at $f = 0.8\%$.

(a) What is the difference in the value of the forward LIBOR rate $L(1y6m, 2y)$ if the swap market is collateralized *vs.* uncollateralized.

(b) A contract pays $L(1y6m, 2y)$ accrued over $6m$ at $T = 2y$. Calculate the PV of this contract if it is

- collateralized
- uncollateralized

(c) Calculate the OIS discount factor $D_o(1y, 2y)$ and LIBOR discount factor $D(1y, 2y)$.

2. (a) i. State the forward value of a foreign exchange rate at time T, given that we know the spot exchange rate today, and the domestic and foreign discount bonds ($D_d(0,T)$ and $D_f(0,T)$) maturing at time T.

ii. If we observe the following:

Instrument	Quote
FX_0	1.5
FX_T	1.45
$D_f(0,T)$	0.95

What is the discount bond value $D_d(0,T)$ implied from the market?

(b) There are 4 coupon bonds with annual payment from the same issuer:

Maturity	Coupon	Price
$1y$	4.5	101.5
$2y$	5	102
$2y$	5.25	?
$3y$	5.5	102.5

What is the no-arbitrage price of the 3^{rd} bond?

3. Let r_t denote the short rate at time t, which follows the simple stochastic differential equation

$$dr_t = \mu dt + \sigma dW_t^*$$

under $\mathbb{Q}^*$. Let $D(t,T)$ denote the price of a zero coupon bond maturing at time T. Show that

(a)

$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u \, du} \right].$$

(b) The zero rate (or spot rate) is defined as

$$D(t,T) = e^{-R(t,T)(T-t)}.$$

Explain whether the short rate model given above will result in an affine short rate model.

4. Let S_t denote the price of a stock at time t. Suppose we are approached by a client to price a contract which pays

$$\log\left(\frac{\alpha S_T}{\beta}\right)$$

at time T, where α and β are both constants. How should we value this contract if

(a) we assume a Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

under the risk-neutral measure $\mathbb{Q}^*$.

(b) we use a static replication approach starting with

$$e^{-rT}\int_0^\infty g(K)f(K)\,dK,$$

where $g(K) = \log\left(\frac{\alpha K}{\beta}\right)$.

5. Let $\frac{1}{X_t}$ denote the value of one domestic currency in foreign denomination. Let B^F be a risk-free bond in the foreign economy, and B^D be a risk-free bond in the domestic economy. Starting with the following differential equations:

$$\begin{cases} dB_t^F = r^F B_t^F dt \\ d\frac{1}{X_t} = \mu \frac{1}{X_t} dt + \sigma \frac{1}{X_t} dW_t \\ dB_t^D = r^D B_t^D dt \end{cases}$$

where W_t is a standard Brownian motion under the risk world probability measure. Show that from the foreign investor's perspective, the exchange rate follows

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t} dt + \sigma \frac{1}{X_t} dW_t^F,$$

where W_t^F is a standard Brownian motion under the risk-neutral measure associated with the foreign bond B^F numeraire.

6. (a) State the LIBOR market model for the forward LIBOR rate $L_i(t)$.

   (b) What must be the choice of numeraire for the expectation of $L_i(t)$ to be a martingale?

   (c) A contract pays \$100 (in cash) if the LIBOR rate L_i is above 5% on the fixing date T_i. The payment is made at time T_{i+1}. Derive a valuation formula for this contract:

$$V_0 = D_{i+1}(0)\mathbb{E}^{i+1}\left[\$100\mathbb{1}_{L_i(T)>5\%}\right].$$

7. Let L_i^F be a forward LIBOR rate in the foreign economy, observed at time T_i and paid at T_{i+1}. It follows the LIBOR market model in the foreign economy with a volatility of σ_i. There is also have a forward foreign exchange process following

$$d\frac{1}{F_t} = \sigma_X \frac{1}{F_t} dW_t^F$$

from the foreign investor's perspective. Suppose the Brownian motion of the exchange rate process and the LIBOR market model is correlated with ρ, show that from the domestic investor's perspective we have

$$\mathbb{E}^{i+1,D}\left[L_i^F(T)\right] = L_i^F(0)e^{\rho\sigma_X\sigma_i T}.$$

8. Suppose we want to value a contract paying $L_i(T)^2$, observed at time T_i and paid at T_{i+1}.

   (a) Derive the valuation formula for this contract using LIBOR market model.

   (b) Formula the static replication portfolio using Breeden-Litzenberger approach, if we are able to obtain liquid caplet $V^c(K)$ and floorlet $V^f(K)$ prices across a wide range of strikes.

9. Consider the Ho-Lee interest rate model

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where W_t^* is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^*$. Discuss the distribution, mean, and variance of the integral

$$\int_t^T r_u \, du,$$

and use this to find the expressions $A(t,T)$ and $B(t,T)$ in the following expectation:

$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u \, du} \right] = e^{A(t,T) - r_t B(t,T)}.$$

10. Consider the Vasicek interest rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

(a) Explain how does this model produce the mean reversion characteristic.

(b) Solve the stochastic differential equation, and evaluate the following limits

$$\lim_{t\to\infty} \mathbb{E}^*[r_t], \qquad \lim_{t\to\infty} V[r_t].$$