QF605 Additional Examples
Session 4: LIBOR and Swap Market Models

1   Questions

1. Let $\mathbb{P}$ denote the real-world probability measure, and W_t denote a $\mathbb{P}$-Brownian motion. A stock price process follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

In the market, there is also a risk-free bond whose price process follows the differential equation

$$dB_t = r B_t dt.$$

Let $\mathbb{Q}$ denote the risk-neutral probability measure associated with the risk-free bond as numeraire.

   (a) Evaluate $\mathbb{E}^{\mathbb{P}}[S_T]$.
   (b) Evaluate $\mathbb{E}^{\mathbb{Q}}[S_T]$

2. Suppose the LIBOR rate follows the stochastic differential equation

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1},$$

where W^{i+1} is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^{i+1}$ associated with the numeraire $D_{i+1}(t)$. Derive the valuation formula for a floorlet

$$F_i(0) = D_{i+1}(0)\Delta_i \mathbb{E}^{i+1}[(K - L_i(T_i))^+].$$

3. Suppose the swap rate follows the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N},$$

where $W^{n+1,N}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^{n+1,N}$ associated with the numeraire $P_{n+1,N}(t)$. Derive the valuation formula for a receiver swaption

$$V_{n,N}^{rec}(0) = P_{n+1,N}\mathbb{E}^{n+1,N}[(K - S_{n,N}(T))^+].$$

2 Suggested Solutions

1. (a) Solving the stochastic differential equation of the stock price process directly under the real-world probability measure by applying Itô's formula to the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \log(x)$, with $X_t = \log S_t = f(S_t)$, we have

$$S_T = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right].$$

Taking expectation under the $\mathbb{P}$ measure, we obtain

$$\mathbb{E}^{\mathbb{P}}[S_T] = S_0 e^{\mu T} \quad \lhd$$

(b) Under the risk-neutral measure $\mathbb{Q}$, the price process $\frac{S_t}{B_t}$ is a martingale. We are able to identify the Radon-Nikodym derivative that allows us to attain this measure change

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right) \qquad \text{where} \qquad \kappa = \frac{\mu - r}{\sigma},$$

and a $\mathbb{Q}$-Brownian motion W_t^B, where

$$dW_t^B = dW_t + \frac{\mu - r}{\sigma}dt.$$

Substituting, we obtain

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$
$$= \mu S_t dt + \sigma S_t \left(dW_t^B - \frac{\mu - r}{\sigma}dt\right)$$
$$= r S_t dt + \sigma S_t dW_t^B.$$

Solving this stochastic differential equation (again by applying Itô's Formula to $X_t = \log S_t$), we obtain

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^B\right].$$

Taking expectation under the $\mathbb{Q}$ measure, we obtain

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT} \quad \lhd$$

2. The LIBOR rate follows the stochastic differential equation

$$dL_i(t) = \sigma_i L_i(t)dW^{i+1},$$

where W^{i+1} is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^{i+1}$ associated with the numeraire $D_{i+1}(t)$. The solution is given by

$$L_i(t) = L_i(0)e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i W^{i+1}}.$$

Evaluating the expectation, we obtain

$$F_i(0) = D_{i+1}(0) \cdot \Delta_i \cdot \mathbb{E}^{i+1}[(K - L_i(T_i))^+]$$
$$= D_{i+1}(0) \cdot \Delta_i \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(K - L_i(0)e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i \sqrt{T}x}\right)^+ e^{-\frac{x^2}{2}} \, dx$$
$$= D_{i+1}(0) \cdot \Delta_i \cdot [K\Phi(-d_2) - L_i(0)\Phi(-d_1)],$$

where

$$d_1 = \frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}, \quad d_2 = d_1 - \sigma_i \sqrt{T}. \quad \lhd$$

3. The swap rate follows the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N},$$

where $W^{n+1,N}$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$. The solution is given by

$$S_{n,N}(T) = S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 t + \sigma_{n,N} W_t^{n+1,N}}.$$

Evaluating the expectation, we obtain

$$
\begin{aligned}
V_{n,N}^{rec}(0) &= P_{n+1,N}(0) \cdot \mathbb{E}^{n+1,N}[(K - S_{n,N}(T))^+] \\
&= P_{n+1,N}(0) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(K - S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 t + \sigma_{n,N}\sqrt{T}x}\right)^+ e^{-\frac{x^2}{2}} \, dx \\
&= P_{n+1,N}(0) \cdot [K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)],
\end{aligned}
$$

where

$$d_1 = \frac{\log\left(\frac{S_{n,N}(0)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{n,N}\sqrt{T}. \quad \lhd$$