C.3 Stirling's formula

The derivation of Stirling's formula proceeds by using the integral expression for $n!$ in eqn C.1, namely

$$n! = \int_0^\infty x^n \mathrm{e}^{-x}\, \mathrm{d}x. \tag{C.10}$$

We will play with the right-hand side of this integral and develop an approximation for it. We notice that the integrand $x^n \mathrm{e}^{-x}$ consists of a function that increases with x (the function x^n) and a function that decreases with x (the function e^{-x}), and so it must have a maximum somewhere (see Fig. C.3(a)). Most of the integral is due to the bulge around this maximum, so we will try to approximate this region around the bulge. As we are eventually going to take logs of this integral, it is natural to work with the logarithm of this integrand, which we will call $f(x)$. Hence we define the function $f(x)$ by

$$\mathrm{e}^{f(x)} = x^n \mathrm{e}^{-x}. \tag{C.11}$$

This implies that $f(x)$ is given by

$$f(x) = n \ln x - x, \tag{C.12}$$

which is sketched in Fig. C.3(b). When the integrand has a maximum, so will $f(x)$. Hence the maximum of the integrand, and also the maximum of this function $f(x)$, can be found using

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{n}{x} - 1 = 0, \tag{C.13}$$

which implies that the maximum in f is at $x = n$. We can differentiate again and get

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -\frac{n}{x^2}. \tag{C.14}$$

Now we can perform a Taylor expansion[3] around the maximum, so that

$$\begin{aligned} f(x) &= f(n) + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=n} (x-n) + \frac{1}{2!}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)_{x=n} (x-n)^2 + \cdots \\ &= n\ln n - n + 0 \times (x-n) - \frac{1}{2}\frac{n}{n^2}(x-n)^2 + \cdots \\ &= n \ln n - n - \frac{(x-n)^2}{2n} + \cdots \end{aligned} \tag{C.15}$$

The Taylor expansion approximates $f(x)$ by a quadratic (see the dotted line in Fig. C.3) and hence $\mathrm{e}^{f(x)}$ approximates to a Gaussian.[4] Putting this as the integrand in eqn C.1, and removing from this integral the terms that do not depend on x, we have

$$n! = \mathrm{e}^{n\ln n - n} \int_0^\infty \mathrm{e}^{-(x-n)^2/2n + \cdots}\, \mathrm{d}x. \tag{C.16}$$

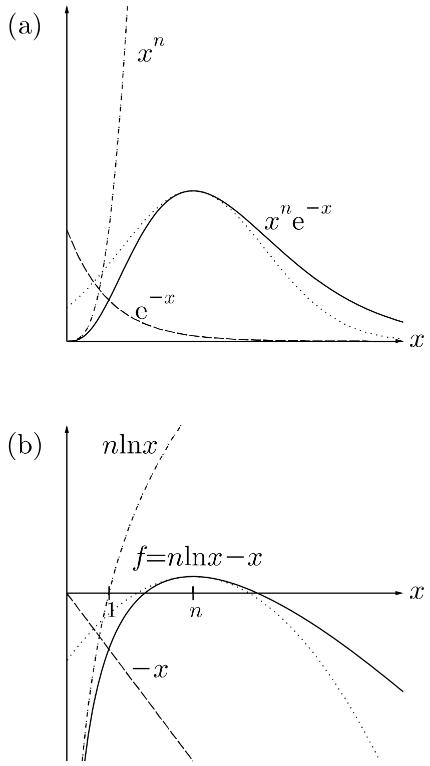

Fig. C.3 (a) The integrand $x^n \mathrm{e}^{-x}$ (solid line) contains a maximum. (b) The function $f(x) = -x + n \ln x$ (solid line), which is the natural logarithm of the integrand. The dotted line is the Taylor expansion around the maximum (from eqn C.15). These curves have been plotted for $n = 3$, but the ability of the Taylor expansion to model the solid line improves as n increases. Note that (b) shows the natural logarithm of the curves in (a).

[3] See Appendix B.

[4] See Appendix C.2.

The integral in this expression can be evaluated with the help of eqn C.3 to be

$$\int_0^\infty e^{-(x-n)^2/2n+\cdots}\,dx \approx \int_{-\infty}^\infty e^{-(x-n)^2/2n}\,dx = \sqrt{2\pi n}. \qquad \text{(C.17)}$$

(Here we have used the fact that it doesn't matter if you put the lower limit of the integral as $-\infty$ rather than 0 since the integrand, $e^{-(x-n)^2/2n}$, is a Gaussian centred at $x = n$ with a width that scales as $\sqrt{n}$ so that the contribution to the integral from the region between $-\infty$ and 0 is vanishingly small as n becomes large.) We have that

$$n! \approx e^{n\ln n - n}\sqrt{2\pi n}, \qquad \text{(C.18)}$$

and hence

$$\boxed{\ln n! \approx n\ln n - n + \tfrac{1}{2}\ln 2\pi n,} \qquad \text{(C.19)}$$

which is one version of **Stirling's formula**. When n is very large, this can be written

$$\boxed{\ln n! \approx n\ln n - n,} \qquad \text{(C.20)}$$

which is another version of Stirling's formula.

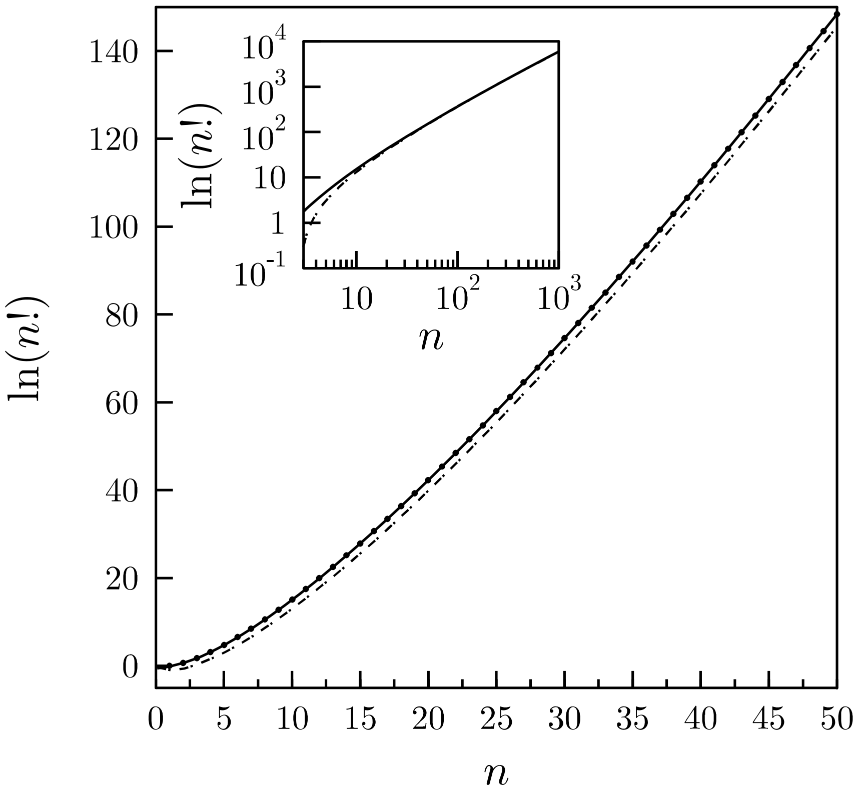

Fig. C.4 Stirling's approximation for $\ln n!$. The dots are the exact results. The solid line is according to eqn C.19, while the dashed line is eqn C.20. The inset shows the two lines for larger values of n and demonstrates that as n becomes large, eqn C.20 becomes a very good approximation.

The approximation in eqn C.19 is very good, as can be seen in Fig. C.4. The approximation in eqn C.20 (the dotted line in Fig. C.4) slightly underestimates the exact result when n is small, but as n becomes large (as is often the case in thermal physics problems) it becomes a very good approximation (as shown in the inset to Fig. C.4).

C.4 Riemann zeta function

The **Riemann zeta function** $\zeta(s)$ is usually defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{C.21}$$

and converges for $s > 1$ (see Fig. C.5). For $s = 1$ it gives a divergent series. Some useful values are listed in Table C.2.

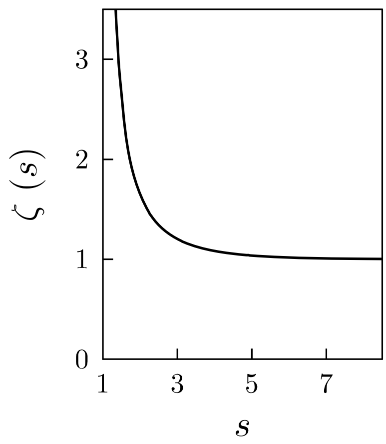

Fig. C.5 The Riemann zeta function $\zeta(s)$ for $s > 1$.

s	$\zeta(s)$
1	∞
$\frac{3}{2}$	≈ 2.612
2	$\pi^2/6 \approx 1.645$
$\frac{5}{2}$	≈ 1.341
3	≈ 1.20206
4	$\pi^4/90 \approx 1.0823$
5	≈ 1.0369
6	$\pi^6/945 \approx 1.017$

Table C.2 Selected values of the Riemann zeta function.

Our reason for introducing the Riemann zeta function is that it is involved in many useful integrals. One such is the **Bose integral** $I_{\mathrm{B}}(n)$ defined by

$$I_{\mathrm{B}}(n) = \int_0^{\infty} \mathrm{d}x \frac{x^n}{\mathrm{e}^x - 1}. \tag{C.22}$$

We can evaluate this as follows:

$$\begin{aligned} I_{\mathrm{B}}(n) &= \int_0^{\infty} \mathrm{d}x\, \frac{x^n \mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} \\ &= \int_0^{\infty} \mathrm{d}x\, x^n \sum_{k=0}^{\infty} \mathrm{e}^{-(k+1)x} \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}} \int_0^{\infty} \mathrm{d}y\, y^n \mathrm{e}^{-y} \\ &= \zeta(n+1)\, \Gamma(n+1). \end{aligned} \tag{C.23}$$

Thus we have that

$$\boxed{I_{\mathrm{B}}(n) = \int_0^{\infty} \mathrm{d}x \frac{x^n}{\mathrm{e}^x - 1} = \zeta(n+1)\, \Gamma(n+1).} \tag{C.24}$$