Chapter 9 Quiz

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1) $\sum_{n=1}^{\infty} \frac{4^{n+1}+6^{n+3}}{7^{n+2}} = \frac{3904}{147}$ geometric series

$= \sum_{n=0}^{\infty} \frac{4^{n+2}+6^{n+4}}{7^{n+3}} = \sum_{n=0}^{\infty} \left(\frac{4^2\cdot 4^n}{7^3\cdot 7^n} + \frac{6^4\cdot 6^n}{7^3\cdot 7^n}\right)$

$= \sum_{n=1}^{\infty} \left(\frac{4^2}{7^3}\right)\cdot\left(\frac{4}{7}\right)^n + \sum_{n=1}^{\infty}\left(\frac{6^4}{7^3}\right)\left(\frac{6}{7}\right)^n$

$= \frac{\frac{4^2}{7^3}}{1-\frac{4}{7}} + \frac{\frac{6^4}{7^3}}{1-\frac{6}{7}}$

$= \frac{4^2}{3\cdot 7^2} + \frac{6^4}{1\cdot 7^2}$

$= \frac{4^2+6^4\cdot 3}{7^2\cdot 3} = \frac{3904}{147}$

$\because \left|\frac{4}{7}\right| < 1$

$\therefore \sum_{n=1}^{\infty}\left(\frac{4^2}{7^3}\right)\left(\frac{4}{7}\right)^n$ converges

$\because \left|\frac{6}{7}\right| < 1$

$\therefore \sum_{n=1}^{\infty}\left(\frac{6^4}{7^3}\right)\left(\frac{6}{7}\right)^n$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{4^{n+1}+6^{n+3}}{7^{n+2}}$ converges absolutely.

2) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \frac{\pi^2}{8}$

$= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$\because n \geq 1$

$\therefore 0 < 2n-1 < 2n$

$0 < (2n-1)^2 < (2n)^2$

$\frac{1}{(2n-1)^2} > \frac{1}{(2n)^2} > 0$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{(2n-1)^2}}{\frac{1}{(2n)^2}}$

$= \lim_{n\to\infty}\frac{(2n)^2}{(2n-1)^2} = 1 > 0$

$\because \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-serie, $p=2>1$

$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$ converges.

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ converges absolutely

$\because 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$+\; 1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots = \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$

$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{6}$

$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \left(1-\frac{1}{4}\right)\cdot\frac{\pi^2}{6}$

$= \frac{3}{4}\cdot\frac{\pi^2}{6}$

$= \frac{\pi^2}{8}$

3) $\sum_{n=0}^{\infty}\frac{1}{n^2+4n+3} = \frac{3}{4}$

$= \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+3)}$

$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n+1} - \sum_{n=0}^{\infty}\frac{1}{n+3}\right]$

$= \frac{1}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots - \frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\cdots\right]$

$= \frac{1}{2}\left[1+\frac{1}{2}\right]$

$= \frac{1}{2}+\frac{3}{2}$

$= \frac{3}{4}$

$\frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3} \Rightarrow \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$

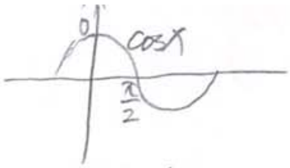
$\frac{1}{2} = \frac{1}{-1+3} = A$

$-\frac{1}{2} = \frac{1}{-3+1} = B$

$\because \sum_{n=0}^{\infty}\frac{1}{n^2+4n+3} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n+1} - \sum_{n=0}^{\infty}\frac{1}{n+3}\right]$ is a telesoping serie

$\therefore \sum_{n=0}^{\infty}\frac{1}{n^2+4n+3}$ converges absolutely.

4) $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(\frac{1}{n})$



$\because n \geq 1$

$\therefore 0 < \frac{1}{n} \leq 1$

$\Rightarrow 0 < \cos(\frac{1}{n}) \leq 1$

$\Rightarrow \frac{1}{n^2} \geq \frac{1}{n^2}\cos(\frac{1}{n})$

$\because \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-serie, $p=2>1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}\cos(\frac{1}{n})$ converges absolutely.

5) $\sum_{n=1}^{\infty} (\frac{2n}{5n-1})^n$

$\because n \geq 1$

$\therefore \frac{2n}{5n-1} > 0$

$\Rightarrow |\frac{2n}{5n-1}| = \frac{2n}{5n-1}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|^n} = \lim_{n\to\infty} \sqrt[n]{(\frac{2n}{5n-1})^n}$$
$$= \lim_{n\to\infty} |\frac{2n}{5n-1}|$$
$$= \lim_{n\to\infty} (\frac{2n}{5n-1})$$
$$= \frac{2}{5} < 1$$

$\therefore \lim_{n\to\infty} (\frac{2n}{5n-1})^n$ converges absolutely.

6) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^3-6n^2+5}$

$= \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{4n^3-6n^2+5}$

$\frac{n^{\frac{1}{2}}}{n^3} = \frac{1}{n^{\frac{5}{2}}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ is a p-serie. $p = \frac{5}{2} > 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ converges.

$$\lim_{n\to\infty} \frac{\frac{n^{\frac{1}{2}}}{4n^3-6n^2+5}}{\frac{1}{n^{\frac{5}{2}}}} = \lim_{n\to\infty} \frac{n^3}{4n^3-6n^2+5} = \frac{1}{4} > 0$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^3-6n^2+5}$ converges absolutely.

7) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

$\because n \geq 1$

$\therefore \ln(n+2) > \ln(n+1) > 0$

$\Rightarrow 0 < \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$ converges conditional.

$\sum_{n=1}^{\infty} |\frac{(-1)^{n+1}}{\ln(n+1)}| = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

$\ln(n+1) < n$

$\frac{1}{\ln(n+1)} > \frac{1}{n}$

$\frac{1}{n}$ is a harmonic series and diverges.

$\Rightarrow \frac{1}{\ln(n+1)}$ diverges.

8) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$

$= \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$\because n+\sqrt{n} > n$

$\therefore \frac{1}{n+\sqrt{n}} < \frac{1}{n}$

$\because \sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic serie

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\lim_{n\to\infty} \frac{a_n}{b_n}$

$= \lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 1 > 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ diverges.

9) $\sum_{n=1}^{\infty} \frac{2^n}{(n+6)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+7)!} \cdot \frac{(n+6)!}{2^n}\right| = \lim_{n\to\infty} \left|\frac{2}{n+7}\right| = \lim_{n\to\infty} \frac{2}{n+7} = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \frac{2^n}{(n+6)!}$ converges absolutely.

10). Find power series centered at 2 for $f(x) = \frac{1}{3x-2}$

$f(x) = \frac{1}{3x-2} = \frac{1}{3x-6+4} = \frac{1}{4+3(x-2)} = \frac{1}{4} \cdot \frac{1}{1-(-\frac{3}{4})(x-2)} = \frac{a}{1-r}$

$a = \frac{1}{4}, \quad r = -\frac{3}{4}(x-2)$

$\Rightarrow f(x) = \frac{1}{3x-2} = \sum_{n=0}^{\infty} (\frac{1}{4})[-\frac{3}{4} \cdot (x-2)]^n$

$= \sum_{n=0}^{\infty} \frac{(-3)^n}{4 \cdot 4^n} (x-2)^n$

$= \sum_{n=0}^{\infty} \frac{(-3)^n}{4^{n+1}} (x-2)^n$

11) Find the radius and interval of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n\,3^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-3)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^{n+1}(x-3)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(-1)\cdot(x-3)}{3} \cdot \frac{n}{n+1}\right|$

$= \frac{1}{3} \lim_{n\to\infty} |x-3| = \frac{1}{3}|x-3| < 1$

$|x-3| < 3$ ← radius of convergence

$-3 < x-3 < 3$

$0 < x < 6$

when $x=0$. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)\cdot(-1)^n \cdot (-1)^n \cdot 3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)(-1)^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ ⇐ har se

$\Rightarrow$ diverges.

when $x=6$ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ | $n+1 > n > 0$

$0 < \frac{1}{n+1} < \frac{1}{n}$ $\Rightarrow$ converges.

$\lim \frac{1}{n} = 0$

12) Find the number of terms necessary to approximate $\sum_{n=0}^{\infty}\left(-\frac{e}{\pi}\right)^n$ to within an error of less than 0.001.

$|S-S_N| = |R_N| \le a_{N+1}$ $\qquad \int_k^{\infty}\left(-\frac{e}{\pi}\right)^x dx < 0.001$

$|R_N| \le a_{N+1} = \left(-\frac{e}{\pi}\right)^{N+1} = (-1)^{N+1}\left(\frac{e}{\pi}\right)^{N+1} \le 0.001$

$\left(\frac{e}{\pi}\right)^{N+1} \le \frac{1}{1000} \Rightarrow (N+1)\ln\left(\frac{e}{\pi}\right) \geqslant \ln\frac{1}{1000}$

$N+1 \geqslant \frac{\ln 1 - \ln 10^3}{\ln e - \ln \pi}$

$N \geqslant \frac{-3\ln 10}{1-\ln\pi} - 1 \approx 47$

13) Find the first 4 non-zero terms in the Taylor series about $x=-2$ for $y=f(x)=\arctan x. = \arctan(-2) + \frac{(x+2)}{5} + \frac{2}{25}(x+2)^2 + \frac{11}{375}(x+2)^3 + \frac{6}{625}(x+2)^4$

$\because x=-2$

$f(x) = \arctan x \qquad f(-2) = \arctan(-2) \qquad C_0 = \arctan(-2)$

$f'(x) = \frac{1}{1+x^2} \qquad f'(-2) = \frac{1}{5} \qquad C_1 = \frac{1}{5}$

$f''(x) = -(1+x^2)^{-2}\cdot 2x \qquad f''(-2) = 4\cdot\frac{1}{5^2} = \frac{4}{25} \qquad C_2 = \frac{1}{2!}\cdot\frac{4}{25} = \frac{2}{25}$

$f'''(x) = -(1+x^2)^{-2}\cdot 2 - 2x(-2)(1+x^2)^{-3}\cdot 2x \qquad f'''(-2) = -2\cdot\frac{1}{5^2} + 2^5\cdot\frac{1}{5^3} = \frac{2^5-10}{5^3} = \frac{22}{125} \qquad C_3 = \frac{1}{3!}\cdot\frac{22}{125} = \frac{11}{375}$

$= -2(1+x^2)^{-2} + 2^3x^2(1+x^2)^{-3}$

$f^{(4)}(x) = 4\cdot 2x(1+x^2)^{-3} + 2^3x^2(-3)(1+x^2)^{-4}\cdot 2x + (1+x^2)^{-3}\cdot 2^3\cdot 2x \qquad f^{(4)}(-2) = 8(-2)5^{-3} + 2^7\cdot 3\cdot 5^{-4} - 2^5\cdot 5^{-3} \qquad C_4 = \frac{1}{4!}\cdot\frac{144}{625} = \frac{6}{625}$

$= 8x(1+x^2)^{-3} + 2^4x^3(-3)(1+x^2)^{-4} + 2^4x(1+x^2)^{-3} \qquad = \frac{-16}{125} + \frac{384}{625} - \frac{32}{125} = \frac{144}{625}$

14) a) Find the Maclaurin Series for $y=f(x)=\cos^2 x$

$C=0.$

$f(0) = \cos^2 x = 1 \qquad C_0 = 1$

$f'(0) = 2\cos x(-\sin x) = -2\sin x\cos x = 0 \qquad C_1 = 0$

$f''(0) = -2[\sin x\cdot(-\sin x) + \cos x\cos x] = -2[\cos^2 x - \sin^2 x] = -2\cos 2x = -2 \qquad C_2 = \frac{-2}{2!}$

$C_3 = 0$

$f'''(0) = -2(-\sin 2x)\cdot 2 = 4\sin 2x = 0$

$C_4 = \frac{8}{4!}$

$f^{(4)}(0) = 4\cos 2x\cdot 2 = 8\cos 2x = 8$

$C_5 = 0$

$f^{(5)}(0) = -16\sin 2x = 0$

$C_6 = \frac{-32}{6!}$

$f^{(6)}(0) = -32\cos 2x = -32$

$f(x) = \cos^2 x = 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \frac{2^5}{6!}x^6 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}\cdot 2^{2n-1}}{(2n)!}$

14> b> Find the Maclauim series for $y=f(x)=x^3e^{2x}$.

$$e^z=1+\frac{z}{1}+\frac{z}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots$$

$$e^{2x}=1+\frac{(2x)}{1}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^3}{4!}+\cdots$$

$$x^3e^{2x}=x^3+\frac{x^3(2x)}{1}+\frac{x^3(2x)^2}{2!}+\frac{x^3(2x)^3}{3!}+\frac{x^3(2x)^3}{4!}+\cdots$$

$$f(x)=x^3e^{2x}=\sum_{n=0}^{\infty}\frac{2^n x^{n+3}}{n!}$$

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$$\sum_{n=0}^{\infty} \frac{2^n(x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}(x-4)^{n-1}}{n-1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^n(x-4)^n}{n}\cdot\frac{n-1}{2^{n-1}(x-4)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|2\cdot(x-4)\cdot\frac{n-1}{n}\right|$$

$$= 2|x-4| < 1$$

$$-\frac{1}{2} < x-4 < \frac{1}{2}$$

$$\frac{7}{2} < x < \frac{9}{2}$$

when $x=\frac{7}{2}$ $\sum_{n=1}^{\infty} \frac{2^{n-1}(\frac{7}{2}-4)^{n-1}}{n-1} = \sum_{n=1}^{\infty} \frac{2^{n-1}(-\frac{1}{2})^{n-1}}{n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n-1} \Rightarrow$ converges.

$\because a_n = \frac{1}{n-1}$ is continuous and decring.

$\lim_{n\to\infty}\frac{1}{n-1} = 0$

when $x=\frac{9}{2}$. $\sum_{n=1}^{\infty} \frac{2^{n-1}(\frac{9}{2}-4)^{n-1}}{n-1} = \sum_{n=1}^{\infty}\frac{1}{n-1} \Rightarrow$ diverges.

$\therefore x \in [\frac{7}{2}, \frac{9}{2})$.