APPENDIX C

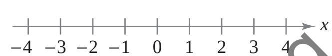
Precalculus Review

APPENDIX C.1 Real Numbers and the Real Number Line

Real Numbers and the Real Number Line • Order and Inequalities • Absolute Value and Distance

Real Numbers and the Real Number Line

Real numbers can be represented by a coordinate system called the **real number line** or x-axis (see Figure C.1). The real number corresponding to a point on the real number line is the **coordinate** of the point. As Figure C.1 shows, it is customary to identify those points whose coordinates are integers.

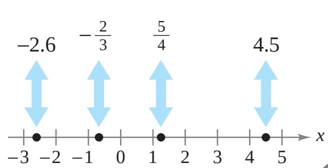


The real number line
Figure C.1

The point on the real number line corresponding to zero is the **origin** and is denoted by 0. The **positive direction** (to the right) is denoted by an arrowhead and is the direction of increasing values of x. Numbers to the right of the origin are **positive.** Numbers to the left of the origin are **negative.** The term **nonnegative** describes a number that is either positive or zero. The term **nonpositive** describes a number that is either negative or zero.

Each point on the real number line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real number line. This type of relationship is called a **one-to-one-correspondence.**

Each of the four points in Figure C.2 corresponds to a **rational number**—one that can be written as the ratio of two integers. $\left(\text{Note that } 4.5 = \frac{9}{2} \text{ and } -2.6 = -\frac{13}{5}.\right)$ Rational numbers can be represented either by *terminating decimals* such as $\frac{2}{5} = 0.4$, or by *repeating decimals* such as $\frac{1}{3} = 0.333 \ldots = 0.\overline{3}$.



Rational numbers
Figure C.2

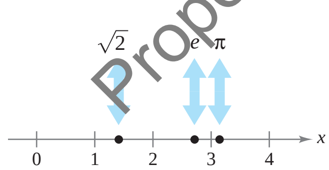
Real numbers that are not rational are **irrational.** Irrational numbers cannot be represented as terminating or repeating decimals. In computations, irrational numbers are represented by decimal approximations. Here are three familiar examples.

$$\sqrt{2} \approx 1.414213562$$

$$\pi \approx 3.141592654$$

$$e \approx 2.718281828$$

(See Figure C.3.)



Irrational numbers
Figure C.3

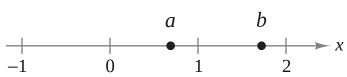
Order and Inequalities

One important property of real numbers is that they are **ordered.** If a and b are real numbers, a is **less than** b if $b - a$ is positive. This order is denoted by the **inequality**

$$a < b.$$

This relationship can also be described by saying that b is **greater than** a and writing $b > a$. When three real numbers a, b, and c are ordered such that $a < b$ and $b < c$, you say that b is **between** a and c and $a < b < c$.

Geometrically, $a < b$ if and only if a lies to the *left* of b on the real number line (see Figure C.4). For example, $1 < 2$ because 1 lies to the left of 2 on the real number line.



$a < b$ if and only if a lies to the left of b.
Figure C.4

The following properties are used in working with inequalities. Similar properties are obtained if < is replaced by ≤ and > is replaced by ≥. (The symbols ≤ and ≥ mean **less than or equal to** and **greater than or equal to,** respectively.)

Properties of Inequalities

Let a, b, c, d, and k be real numbers.

1. If $a < b$ and $b < c$, then $a < c$.	Transitive Property
2. If $a < b$ and $c < d$, then $a + c < b + d$.	Add inequalities.
3. If $a < b$, then $a + k < b + k$.	Add a constant.
4. If $a < b$ and $k > 0$, then $ak < bk$.	Multiply by a positive constant.
5. If $a < b$ and $k < 0$, then $ak > bk$.	Multiply by a negative constant.

NOTE Note that you *reverse the inequality* when you multiply the inequality by a negative number. For example, if $x < 3$, then $-4x > -12$. This also applies to division by a negative number. So, if $-2x > 4$, then $x < -2$.

A **set** is a collection of elements. Two common sets are the set of real numbers and the set of points on the real number line. Many problems in calculus involve **subsets** of one of these two sets. In such cases, it is convenient to use **set notation** of the form $\{x\colon \text{condition on } x\}$, which is read as follows.

$$\overbrace{\{}^{\text{The set of all}}\quad \overbrace{x}^{x}\quad \overbrace{:}^{\text{such that}}\quad \overbrace{\text{condition on } x}^{\text{a certain condition is true.}}\}$$

For example, you can describe the set of positive real numbers as

$\{x\colon x > 0\}.$ Set of positive real numbers

Similarly, you can describe the set of nonnegative real numbers as

$\{x\colon x \geq 0\}.$ Set of nonnegative real numbers

The **union** of two sets A and B, denoted by $A \cup B$, is the set of elements that are members of A *or* B *or both*. The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements that are members of A *and* B. Two sets are **disjoint** if they have no elements in common.

The most commonly used subsets are **intervals** on the real number line. For example, the **open** interval

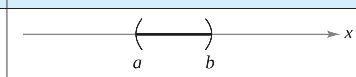
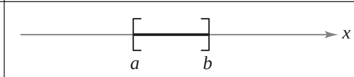
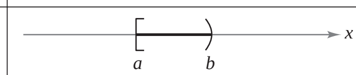
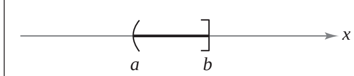
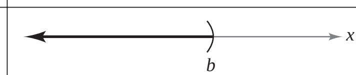
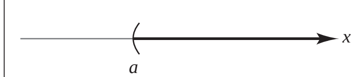
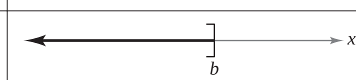
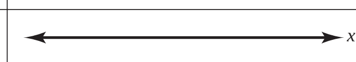
$(a, b) = \{x: a < x < b\}$ Open interval

is the set of all real numbers greater than a and less than b, where a and b are the **endpoints** of the interval. Note that the endpoints are not included in an open interval. Intervals that include their endpoints are **closed** and are denoted by

$[a, b] = \{x: a \le x \le b\}.$ Closed interval

The nine basic types of intervals on the real number line are shown in the table below. The first four are **bounded intervals** and the remaining five are **unbounded intervals.** Unbounded intervals are also classified as open or closed. The intervals $(-\infty, b)$ and (a, ∞) are open, the intervals $(-\infty, b]$ and $[a, \infty)$ are closed, and the interval $(-\infty, \infty)$ is considered to be both open *and* closed.

Intervals on the Real Number Line

	Interval Notation	Set Notation	Graph
Bounded open interval	(a, b)	$\{x: a < x < b\}$	
Bounded closed interval	$[a, b]$	$\{x: a \le x \le b\}$	
Bounded intervals (neither open nor closed)	$[a, b)$ $(a, b]$	$\{x: a \le x < b\}$ $\{x: a < x \le b\}$	 
Unbounded open intervals	$(-\infty, b)$ (a, ∞)	$\{x: x < b\}$ $\{x: x > a\}$	 
Unbounded closed intervals	$(-\infty, b]$ $[a, \infty)$	$\{x: x \le b\}$ $\{x: x \ge a\}$	 
Entire real line	$(-\infty, \infty)$	$\{x: x$ is a real number$\}$	

NOTE The symbols ∞ and $-\infty$ refer to positive and negative infinity, respectively. These symbols do not denote real numbers. They simply enable you to describe unbounded conditions more concisely. For instance, the interval $[a, \infty)$ is unbounded to the right because it includes *all* real numbers that are greater than or equal to a.

EXAMPLE 1 Liquid and Gaseous States of Water

Describe the intervals on the real number line that correspond to the temperature x (in degrees Celsius) for water in

a. a liquid state. **b.** a gaseous state.

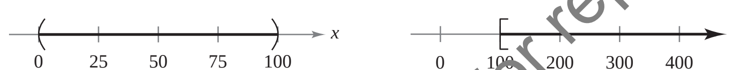
Solution

a. Water is in a liquid state at temperatures greater than 0°C and less than 100°C, as shown in Figure C.5(a).

$$(0, 100) = \{x\colon\ 0 < x < 100\}$$

b. Water is in a gaseous state (steam) at temperatures greater than or equal to 100°C, as shown in Figure C.5(b).

$$[100, \infty) = \{x\colon\ x \geq 100\}$$



(a) Temperature range of water (in degrees Celsius)

(b) Temperature range of steam (in degrees Celsius)

Figure C.5

A real number a is a **solution** of an inequality if the inequality is **satisfied** (is true) when a is substituted for x. The set of all solutions is the **solution set** of the inequality.

EXAMPLE 2 Solving an Inequality

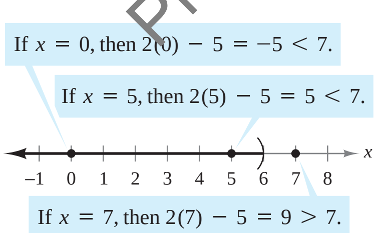
Solve $2x - 5 < 7$.

Solution

$2x - 5 < 7$	Write original inequality.
$2x - 5 + 5 < 7 + 5$	Add 5 to each side.
$2x < 12$	Simplify.
$\dfrac{2x}{2} < \dfrac{12}{2}$	Divide each side by 2.
$x < 6$	Simplify.

The solution set is $(-\infty, 6)$.

NOTE In Example 2, all five inequalities listed as steps in the solution are called **equivalent** because they have the same solution set.



Checking solutions of $2x - 5 < 7$
Figure C.6

Once you have solved an inequality, check some x-values in your solution set to verify that they satisfy the original inequality. You should also check some values outside your solution set to verify that they *do not* satisfy the inequality. For example, Figure C.6 shows that when $x = 0$ or $x = 5$ the inequality $2x - 5 < 7$ is satisfied, but when $x = 7$ the inequality $2x - 5 < 7$ is not satisfied.

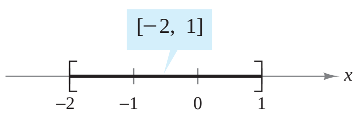
EXAMPLE 3 Solving a Double Inequality

Solve $-3 \le 2 - 5x \le 12$.

Solution

$$\begin{aligned}
-3 \le\ & 2 - 5x \le 12 && \text{Write original inequality.}\\
-3 - 2 \le\ & 2 - 5x - 2 \le 12 - 2 && \text{Subtract 2 from each part.}\\
-5 \le\ & -5x \le 10 && \text{Simplify.}\\
\frac{-5}{-5} \ge\ & \frac{-5x}{-5} \ge \frac{10}{-5} && \text{Divide each part by } -5 \text{ and reverse both inequalities.}\\
1 \ge\ & x \ge -2 && \text{Simplify.}
\end{aligned}$$

The solution set is $[-2, 1]$, as shown in Figure C.7.



Solution set of $-3 \le 2 - 5x \le 12$
Figure C.7

The inequalities in Examples 2 and 3 are **linear inequalities**—that is, they involve first-degree polynomials. To solve inequalities involving polynomials of higher degree, use the fact that a polynomial can change signs *only* at its real **zeros** (the x-values that make the polynomial equal to zero). Between two consecutive real zeros, a polynomial must be either entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into **test intervals** in which the polynomial has no sign changes. So, if a polynomial has the factored form

$$(x - r_1)(x - r_2) \cdot \cdot \cdot (x - r_n), \qquad r_1 < r_2 < r_3 < \cdot \cdot \cdot < r_n$$

the test intervals are

$$(-\infty, r_1), \ (r_1, r_2), \ . \ . \ . \ , \quad (r_{n-1}, r_n), \quad \text{and} \quad (r_n, \infty).$$

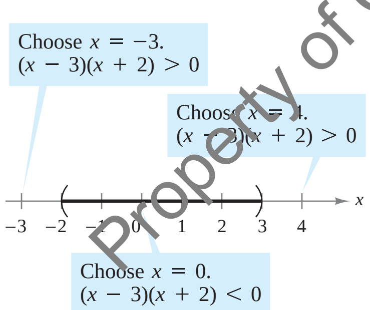
To determine the sign of the polynomial in each test interval, you need to test only *one value* from the interval.

EXAMPLE 4 Solving a Quadratic Inequality

Solve $x^2 < x + 6$.

Solution

$$\begin{aligned}
x^2 &< x + 6 && \text{Write original inequality.}\\
x^2 - x - 6 &< 0 && \text{Write in general form.}\\
(x - 3)(x + 2) &< 0 && \text{Factor.}
\end{aligned}$$



Testing an interval
Figure C.8

The polynomial $x^2 - x - 6$ has $x = -2$ and $x = 3$ as its zeros. So, you can solve the inequality by testing the sign of $x^2 - x - 6$ in each of the test intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. To test an interval, choose any number in the interval and compute the sign of $x^2 - x - 6$. After doing this, you will find that the polynomial is positive for all real numbers in the first and third intervals and negative for all real numbers in the second interval. The solution of the original inequality is therefore $(-2, 3)$, as shown in Figure C.8.

Absolute Value and Distance

If a is a real number, the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

The absolute value of a number cannot be negative. For example, let $a = -4$. Then, because $-4 < 0$, you have

$$|a| = |-4| = -(-4) = 4.$$

Remember that the symbol $-a$ does not necessarily mean that $-a$ is negative.

Operations with Absolute Value

Let a and b be real numbers and let n be a positive integer.

1. $|ab| = |a|\,|b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

3. $|a| = \sqrt{a^2}$ **4.** $|a^n| = |a|^n$

NOTE You are asked to prove these properties in Exercises 73, 75, 76, and 77.

Properties of Inequalities and Absolute Value

Let a and b be real numbers and let k be a positive real number.

1. $-|a| \leq a \leq |a|$

2. $|a| \leq k$ if and only if $-k \leq a \leq k$.

3. $|a| \geq k$ if and only if $a \leq -k$ or $a \geq k$.

4. *Triangle Inequality:* $|a + b| \leq |a| + |b|$

Properties 2 and 3 are also true if $\leq$ is replaced by $<$.

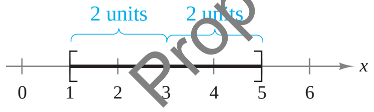
EXAMPLE 5 Solving an Absolute Value Inequality

Solve $|x - 3| \leq 2$.

Solution Using the second property of inequalities and absolute value, you can rewrite the original inequality as a double inequality.

$$\begin{aligned} -2 \leq\ & x - 3 \leq 2 && \text{Write as double inequality.} \\ -2 + 3 \leq\ & x - 3 + 3 \leq 2 + 3 && \text{Add 3 to each part.} \\ 1 \leq\ & x \leq 5 && \text{Simplify.} \end{aligned}$$

The solution set is $[1, 5]$, as shown in Figure C.9.



Solution set of $|x - 3| \leq 2$
Figure C.9

EXAMPLE 6 A Two-Interval Solution Set

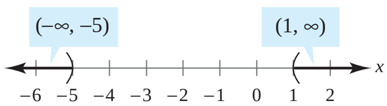
Solve $|x + 2| > 3$.

Solution Using the third property of inequalities and absolute value, you can rewrite the original inequality as two linear inequalities.

$$x + 2 < -3 \quad \text{or} \quad x + 2 > 3$$

$$x < -5 \qquad\qquad x > 1$$

The solution set is the union of the disjoint intervals $(-\infty, -5)$ and $(1, \infty)$, as shown in Figure C.10.

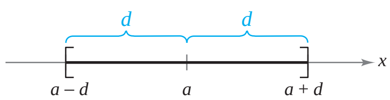


Solution set of $|x + 2| > 3$
Figure C.10

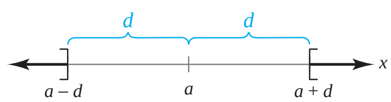
Examples 5 and 6 illustrate the general results shown in Figure C.11. Note that if $d > 0$, the solution set for the inequality $|x - a| \le d$ is a *single* interval, whereas the solution set for the inequality $|x - a| \ge d$ is the union of *two* disjoint intervals.

The **distance between two points** a and b on the real number line is given by

$$d = |a - b| = |b - a|.$$



Solution set of $|x - a| \le d$



Solution set of $|x - a| \ge d$
Figure C.11

The **directed distance from a to b** is $b - a$ and the **directed distance from b to a** is $a - b$, as shown in Figure C.12.

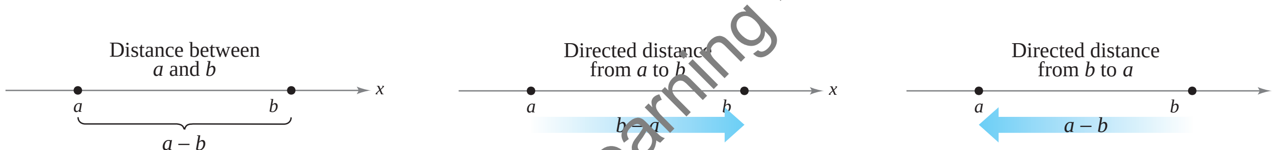


Figure C.12

EXAMPLE 7 Distance on the Real Number Line

a. The distance between -3 and 4 is

$$|4 - (-3)| = |7| = 7 \quad \text{or} \quad |-3 - 4| = |-7| = 7.$$

(See Figure C.13.)

b. The directed distance from -3 to 4 is $4 - (-3) = 7$.

c. The directed distance from 4 to -3 is $-3 - 4 = -7$.

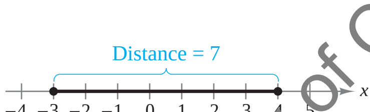


Figure C.13

The **midpoint** of an interval with endpoints a and b is the average value of a and b. That is,

$$\text{Midpoint of interval } (a, b) = \frac{a + b}{2}.$$

To show that this is the midpoint, you need only show that $(a + b)/2$ is equidistant from a and b.

EXERCISES FOR APPENDIX C.1

In Exercises 1–10, determine whether the real number is rational or irrational.

1. 0.7
2. -3678
3. $\dfrac{3\pi}{2}$
4. $3\sqrt{2} - 1$
5. $4.3\overline{451}$
6. $\frac{22}{7}$
7. $\sqrt[3]{64}$
8. $0.\overline{8177}$
9. $4\frac{5}{8}$
10. $(\sqrt{2})^3$

In Exercises 11–14, write the repeating decimal as a ratio of two integers using the following procedure. If $x = 0.6363\ldots$, then $100x = 63.6363\ldots$. Subtracting the first equation from the second produces $99x = 63$ or $x = \frac{63}{99} = \frac{7}{11}$.

11. $0.\overline{36}$
12. $0.3\overline{18}$
13. $0.\overline{297}$
14. $0.\overline{9900}$

15. Given $a < b$, determine which of the following are true.

(a) $a + 2 < b + 2$ (b) $5b < 5a$

(c) $5 - a > 5 - b$ (d) $\dfrac{1}{a} < \dfrac{1}{b}$

(e) $(a - b)(b - a) > 0$ (f) $a^2 < b^2$

16. Complete the table with the appropriate interval notation, set notation, and graph on the real number line.

Interval Notation	Set Notation	Graph
		
$(-\infty, -4]$		
	$\{x: 3 \le x \le \frac{11}{2}\}$	
$(-1, 7)$		

In Exercises 17–20, verbally describe the subset of real numbers represented by the inequality. Sketch the subset on the real number line, and state whether the interval is bounded or unbounded.

17. $-3 < x < 3$
18. $x \ge 4$
19. $x \le 5$
20. $0 \le x < 8$

In Exercises 21–24, use inequality and interval notation to describe the set.

21. y is at least 4.
22. q is nonnegative.
23. The interest rate r on loans is expected to be greater than 3% and no more than 7%.
24. The temperature T is forecast to be above 90°F today.

In Exercises 25–44, solve the inequality and graph the solution on the real number line.

25. $2x - 1 \ge 0$
26. $3x + 1 \ge 2x + 2$
27. $-4 < 2x - 3 < 4$
28. $0 \le x + 3 < 5$
29. $\dfrac{x}{2} + \dfrac{x}{3} > 5$
30. $x > \dfrac{1}{x}$
31. $|x| < 1$
32. $\dfrac{x}{2} - \dfrac{x}{3} > 5$
33. $\left|\dfrac{x - 3}{2}\right| \ge 5$
34. $\left|\dfrac{x}{2}\right| > 3$
35. $|x - a| < b,\ b > 0$
36. $|x + 2| < 5$
37. $|2x + 1| < 5$
38. $|3x + 1| \ge 4$
39. $\left|1 - \dfrac{2}{3}x\right| < 1$
40. $|9 - 2x| < 1$
41. $x^2 \le 3 - 2x$
42. $x^4 - x \le 0$
43. $x^2 + x - 1 \le 5$
44. $2x^2 + 1 < 9x - 3$

In Exercises 45–48, find the directed distance from a to b, the directed distance from b to a, and the distance between a and b.

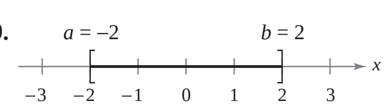
45. 

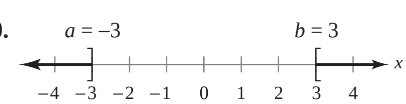
46. 

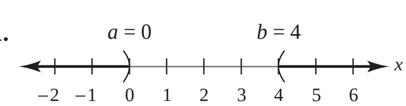
47. (a) $a = 126,\ b = 75$
(b) $a = -126,\ b = -75$

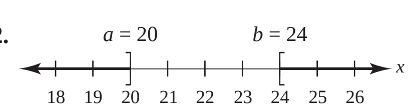
48. (a) $a = 9.34,\ b = -5.65$
(b) $a = \frac{16}{5},\ b = \frac{112}{75}$

In Exercises 49–54, use absolute value notation to define the interval or pair of intervals on the real number line.

49. 

50. 

51. 

52. 

53. (a) All numbers that are at most 10 units from 12.

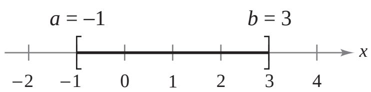
(b) All numbers that are at least 10 units from 12.

54. (a) y is at most two units from a.

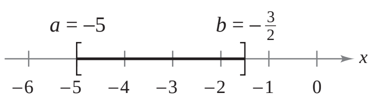
(b) y is less than δ units from c.

In Exercises 55–58, find the midpoint of the interval.

55.



56.



57. (a) $[7, 21]$

(b) $[8.6, 11.4]$

58. (a) $[-6.85, 9.35]$

(b) $[-4.6, -1.3]$

59. ***Profit*** The revenue R from selling x units of a product is

$$R = 115.95x$$

and the cost C of producing x units is

$$C = 95x + 750.$$

To make a (positive) profit, R must be greater than C. For what values of x will the product return a profit?

60. ***Fleet Costs*** A utility company has a fleet of vans. The annual operating cost C (in dollars) of each van is estimated to be

$$C = 0.32m + 2300$$

where m is measured in miles. The company wants the annual operating cost of each van to be less than \$10,000. To do this, m must be less than what value?

61. ***Fair Coin*** To determine whether a coin is fair (has an equal probability of landing tails up or heads up), you toss the coin 100 times and record the number of heads x. The coin is declared unfair if

$$\left|\frac{x - 50}{5}\right| \geq 1.645$$

For what values of x will the coin be declared unfair?

62. ***Daily Production*** The estimated daily oil production p at a refinery is

$$|p - 2{,}250{,}000| < 125{,}000$$

where p is measured in barrels. Determine the high and low production levels.

In Exercises 63 and 64, determine which of the two real numbers is greater.

63. (a) π or $\frac{355}{113}$

(b) π or $\frac{22}{7}$

64. (a) $\frac{224}{151}$ or $\frac{144}{97}$

(b) $\frac{73}{81}$ or $\frac{6427}{7132}$

65. ***Approximation—Powers of 10*** Light travels at the speed of 2.998×10^8 meters per second. Which best estimates the distance in meters that light travels in a year?

(a) 9.5×10^5 (b) 9.5×10^{15}

(c) 9.5×10^{12} (d) 9.6×10^{16}

66. ***Writing*** The accuracy of an approximation to a number is related to how many significant digits there are in the approximation. Write a definition for significant digits and illustrate the concept with examples.

True or False? **In Exercises 67–72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. The reciprocal of a nonzero integer is an integer.

68. The reciprocal of a nonzero rational number is a rational number.

69. Each real number is either rational or irrational.

70. The absolute value of each real number is positive.

71. If $x < 0$, then $\sqrt{x^2} = -x$.

72. If a and b are any two distinct real numbers, then $a < b$ or $a > b$.

In Exercises 73–80, prove the property.

73. $|ab| = |a||b|$

74. $|a - b| = |b - a|$

$[Hint\text{: } (a - b) = (-1)(b - a)]$

75. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0$

76. $|a| = \sqrt{a^2}$

77. $|a^n| = |a|^n, \quad n = 1, 2, 3, \ldots$

78. $-|a| \leq a \leq |a|$

79. $|a| \leq k$, if and only if $-k \leq a \leq k, \quad k > 0.$

80. $|a| \geq k$ if and only if $a \leq -k$ or $a \geq k, \quad k > 0.$

81. Find an example for which $|a - b| > |a| - |b|$, and an example for which $|a - b| = |a| - |b|$. Then prove that $|a - b| \geq |a| - |b|$ for all a, b.

82. Show that the maximum of two numbers a and b is given by the formula

$$\max(a, b) = \tfrac{1}{2}\big(a + b + |a - b|\big).$$

Derive a similar formula for $\min(a, b)$.

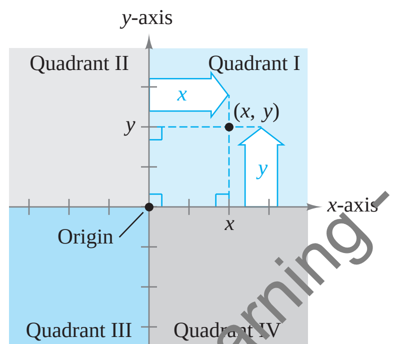
APPENDIX C.2 The Cartesian Plane

The Cartesian Plane • The Distance and Midpoint Formulas • Equations of Circles

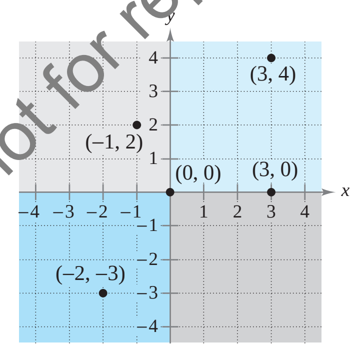
The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes.

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure C.14. The horizontal real number line is usually called the ***x*-axis,** and the vertical real number line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin.** The two axes divide the plane into four parts called **quadrants.**



The Cartesian plane
Figure C.14



Points represented by ordered pairs
Figure C.15

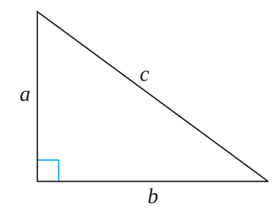
Each point in the plane is identified by an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The number x represents the directed distance from the y-axis to the point, and the number y represents the directed distance from the x-axis to the point (see Figure C.14). For the point (x, y), the first coordinate is the ***x*-coordinate** or **abscissa,** and the second coordinate is the ***y*-coordinate** or **ordinate.** For example, Figure C.15 shows the locations of the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$ in the Cartesian plane.

NOTE The signs of the coordinates of a point determine the quadrant in which the point lies. For instance, if $x > 0$ and $y < 0$, then the point (x, y) lies in Quadrant IV.

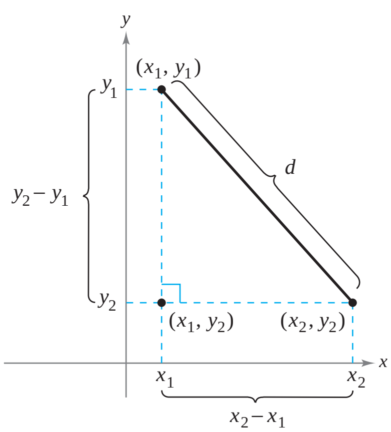
Note that an ordered pair (a, b) is used to denote either a point in the plane *or* an open interval on the real number line. This, however, should not be confusing—the nature of the problem should clarify whether a point in the plane or an open interval is being discussed.

The Distance and Midpoint Formulas

Recall from the Pythagorean Theorem that, in a right triangle, the hypotenuse c and sides a and b are related by $a^2 + b^2 = c^2$. Conversely, if $a^2 + b^2 = c^2$, then the triangle is a right triangle (see Figure C.16).



The Pythagorean Theorem:
$a^2 + b^2 = c^2$
Figure C.16



The distance between two points
Figure C.17

Suppose you want to determine the distance d between the two points (x_1, y_1) and (x_2, y_2) in the plane. If the points lie on a horizontal line, then $y_1 = y_2$ and the distance between the points is $|x_2 - x_1|$. If the points lie on a vertical line, then $x_1 = x_2$ and the distance between the points is $|y_2 - y_1|$. If the two points do not lie on a horizontal or vertical line, they can be used to form a right triangle, as shown in Figure C.17. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, it follows that

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

Replacing $|x_2 - x_1|^2$ and $|y_2 - y_1|^2$ by the equivalent expressions $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$ produces the following result.

Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is given by

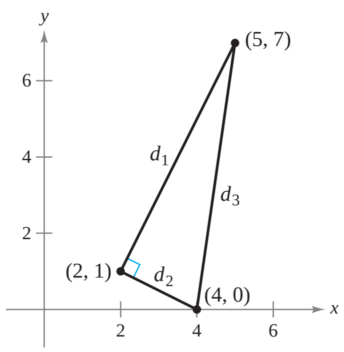
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1 Finding the Distance Between Two Points

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Solution

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= \sqrt{5^2 + 3^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \\ &\approx 5.83 \end{aligned}$$



Verifying a right triangle
Figure C.18

EXAMPLE 2 Verifying a Right Triangle

Verify that the points (2, 1), (4, 0), and (5, 7) form the vertices of a right triangle.

Solution Figure C.18 shows the triangle formed by the three points. The lengths of the three sides are as follows.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$
$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$
$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

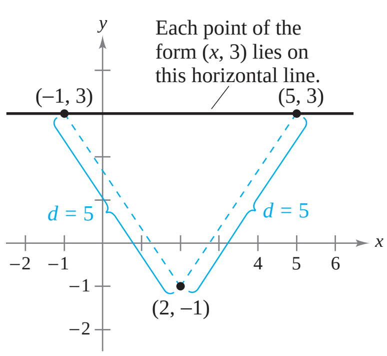
Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 \qquad \text{Sum of squares of sides}$$

and

$$d_3^2 = 50 \qquad \text{Square of hypotenuse}$$

you can apply the Pythagorean Theorem to conclude that the triangle is a right triangle.



Given a distance, find a point.
Figure C.19

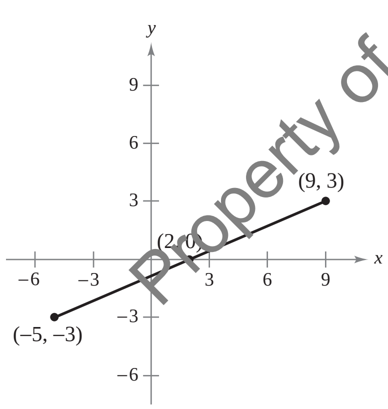
EXAMPLE 3 Using the Distance Formula

Find x such that the distance between $(x, 3)$ and $(2, -1)$ is 5.

Solution Using the Distance Formula, you can write the following.

$$5 = \sqrt{(x-2)^2 + [3-(-1)]^2} \qquad \text{Distance Formula}$$
$$25 = (x^2 - 4x + 4) + 16 \qquad \text{Square each side.}$$
$$0 = x^2 - 4x - 5 \qquad \text{Write in general form.}$$
$$0 = (x-5)(x+1) \qquad \text{Factor.}$$

Therefore, $x = 5$ or $x = -1$, and you can conclude that there are two solutions. That is, each of the points (5, 3) and $(-1, 3)$ lies five units from the point $(2, -1)$, as shown in Figure C.19.



Midpoint of a line segment
Figure C.20

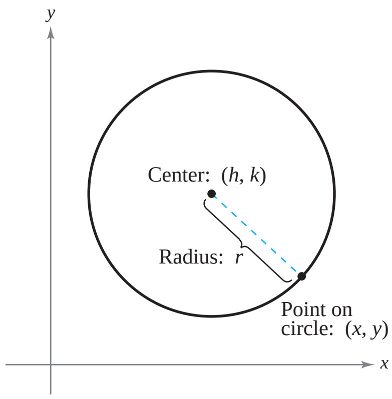
The coordinates of the **midpoint** of the line segment joining two points can be found by "averaging" the x-coordinates of the two points and "averaging" the y-coordinates of the two points. That is, the midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \qquad \text{Midpoint Formula}$$

For instance, the midpoint of the line segment joining the points $(-5, -3)$ and (9, 3) is

$$\left(\frac{-5+9}{2}, \frac{-3+3}{2}\right) = (2, 0)$$

as shown in Figure C.20.



Definition of a circle
Figure C.21

Equations of Circles

A **circle** can be defined as the set of all points in a plane that are equidistant from a fixed point. The fixed point is the **center** of the circle, and the distance between the center and a point on the circle is the **radius** (see Figure C.21).

You can use the Distance Formula to write an equation for the circle with center (h, k) and radius r. Let (x, y) be any point on the circle. Then the distance between (x, y) and the center (h, k) is given by

$$\sqrt{(x-h)^2+(y-k)^2}=r.$$

By squaring each side of this equation, you obtain the **standard form of the equation of a circle.**

Standard Form of the Equation of a Circle

The point (x, y) lies on the circle of radius r and center (h, k) if and only if

$$(x-h)^2+(y-k)^2=r^2.$$

The standard form of the equation of a circle with center at the origin, $(h, k) = (0, 0)$, is

$$x^2+y^2=r^2.$$

If $r = 1$, the circle is called the **unit circle.**

EXAMPLE 4 Writing the Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure C.22. Write the standard form of the equation of this circle.

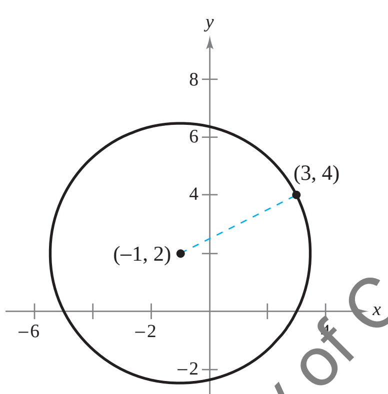


Figure C.22

Solution The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$r=\sqrt{[3-(-1)]^2+(4-2)^2}=\sqrt{16+4}=\sqrt{20}$$

You can write the standard form of the equation of this circle as

$$[x-(-1)]^2+(y-2)^2=\left(\sqrt{20}\right)^2$$

$$(x+1)^2+(y-2)^2=20. \qquad \text{Write in standard form.}$$

By squaring and simplifying, the equation $(x-h)^2+(y-k)^2=r^2$ can be written in the following **general form of the equation of a circle.**

$$Ax^2+Ay^2+Dx+Ey+F=0, \quad A\neq 0$$

To convert such an equation to the standard form

$$(x-h)^2+(y-k)^2=p$$

you can use a process called **completing the square.** If $p > 0$, the graph of the equation is a circle. If $p = 0$, the graph is the single point (h, k). If $p < 0$, the equation has no graph.

EXAMPLE 5 Completing the Square

Sketch the graph of the circle whose general equation is

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0.$$

Solution To complete the square, first divide by 4 so that the coefficients of x^2 and y^2 are both 1.

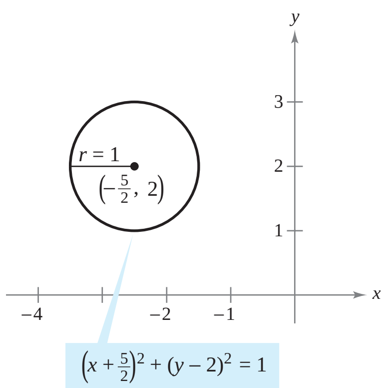
$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \qquad \text{Write original equation.}$$

$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0 \qquad \text{Divide by 4.}$$

$$(x^2 + 5x + \quad) + (y^2 - 4y + \quad) = -\frac{37}{4} \qquad \text{Group terms.}$$

$$\left(x^2 + 5x + \underbrace{\frac{25}{4}}_{(\text{half})^2}\right) + (y^2 - 4y + \underbrace{4}_{(\text{half})^2}) = -\frac{37}{4} + \frac{25}{4} + 4 \qquad \text{Complete the square by adding } \left(\tfrac{5}{2}\right)^2 = \tfrac{25}{4} \text{ and } \left(\tfrac{4}{2}\right)^2 = 4 \text{ to each side.}$$

$$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1 \qquad \text{Write in standard form.}$$



A circle with a radius of 1 and center at $\left(-\frac{5}{2}, 2\right)$
Figure C.23

Note that you complete the square by adding the square of half the coefficient of x *and* the square of half the coefficient of y to each side of the equation. The circle is centered at $\left(-\frac{5}{2}, 2\right)$ and its radius is 1, as shown in Figure C.23.

You have now reviewed some fundamental concepts of *analytic geometry*. Because these concepts are in common use today, it is easy to overlook their revolutionary nature. At the time analytic geometry was being developed by Pierre de Fermat and René Descartes, the two major branches of mathematics—geometry and algebra—were largely independent of each other. Circles belonged to geometry and equations belonged to algebra. The coordination of the points on a circle and the solutions of an equation belongs to what is now called analytic geometry.

It is important to become skilled in analytic geometry so that you can move easily between geometry and algebra. For instance, in Example 4, you were given a geometric description of a circle and were asked to find an algebraic equation for the circle. So, you were moving from geometry to algebra. Similarly, in Example 5 you were given an algebraic equation and asked to sketch a geometric picture. In this case, you were moving from algebra to geometry. These two examples illustrate the two most common problems in analytic geometry.

1. Given a graph, find its equation.

$\boxed{\text{Geometry}} \Rightarrow \boxed{\text{Algebra}}$

2. Given an equation, find its graph.

$\boxed{\text{Algebra}} \Rightarrow \boxed{\text{Geometry}}$

In the next section, you will review other examples of these two types of problems.

EXERCISES FOR APPENDIX C.2

In Exercises 1–6, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(2, 1), (4, 5)$
2. $(-3, 2), (3, -2)$
3. $\left(\frac{1}{2}, 1\right), \left(-\frac{3}{2}, -5\right)$
4. $\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{5}{6}, 1\right)$
5. $\left(1, \sqrt{3}\right), (-1, 1)$
6. $(-2, 0), \left(0, \sqrt{2}\right)$

In Exercises 7–10, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

7. $x = -2$ and $y > 0$
8. $y < -2$
9. $xy > 0$
10. $(x, -y)$ is in Quadrant II.

In Exercises 11–14, show that the points are the vertices of the polygon. (A rhombus is a quadrilateral whose sides are all of the same length.)

	Vertices	*Polygon*
11.	$(4, 0), (2, 1), (-1, -5)$	Right triangle
12.	$(1, -3), (3, 2), (-2, 4)$	Isosceles triangle
13.	$(0, 0), (1, 2), (2, 1), (3, 3)$	Rhombus
14.	$(0, 1), (3, 7), (4, 4), (1, -2)$	Parallelogram

15. ***Number of Stores*** The table shows the number y of Target stores for each year x from 1998 through 2007. *(Source: Target Corp.)*

Year, x	1998	1999	2000	2001	2002
Number, y	851	912	977	1053	1147

Year, x	2003	2004	2005	2006	2007
Number, y	1225	1308	1397	1488	1591

Select reasonable scales on the coordinate axes and plot the points (x, y).

16. ***Conjecture*** Plot the points $(2, 1)$, $(-3, 5)$, and $(7, -3)$ on a rectangular coordinate system. Then change the sign of the x-coordinate of each point and plot the three new points on the same rectangular coordinate system. What conjecture can you make about the location of a point when the sign of the x-coordinate is changed? Repeat the exercise for the case in which the sign of the y-coordinate is changed.

In Exercises 17–20, use the Distance Formula to determine whether the points lie on the same line.

17. $(0, -4), (2, 0), (3, 2)$
18. $(0, 4), (7, -6), (-5, 11)$
19. $(-2, 1), (-1, 0), (2, -2)$
20. $(-1, 1), (3, 3), (5, 5)$

In Exercises 21 and 22, find x such that the distance between the points is 5.

21. $(0, 0), (x, -4)$
22. $(2, -1), (x, 2)$

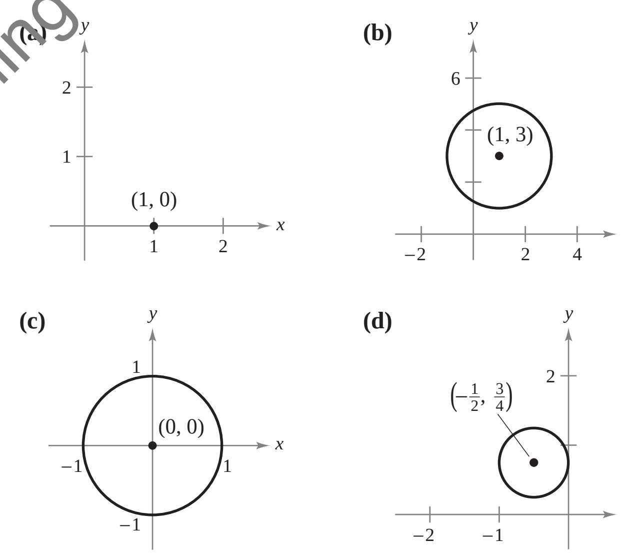
In Exercises 23 and 24, find y such that the distance between the points is 8.

23. $(0, 0), (3, y)$
24. $(5, 1), (5, y)$

25. Use the Midpoint Formula to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four equal parts.

26. Use the result of Exercise 25 to find the points that divide the line segment joining the given points into four equal parts.
(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

In Exercises 27–30, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



27. $x^2 + y^2 = 1$
28. $(x - 1)^2 + (y - 3)^2 = 4$
29. $(x - 1)^2 + y^2 = 0$
30. $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{4}\right)^2 = \frac{1}{4}$

In Exercises 31–38, write the general form of the equation of the circle.

31. Center: $(0, 0)$
Radius: 3
32. Center: $(0, 0)$
Radius: 5
33. Center: $(2, -1)$
Radius: 4
34. Center: $(-4, 3)$
Radius: $\frac{5}{8}$

35. Center: $(-1, 2)$
Point on circle: $(0, 0)$

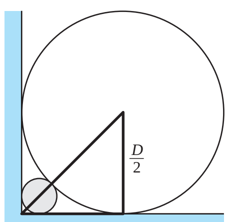
36. Center: $(3, -2)$
Point on circle: $(-1, 1)$

37. Endpoints of diameter: $(2, 5), (4, -1)$

38. Endpoints of diameter: $(1, 1), (-1, -1)$

39. ***Satellite Communication*** Write the standard form of the equation for the path of a communications satellite in a circular orbit 22,000 miles above Earth. (Assume that the radius of Earth is 4000 miles.)

40. ***Building Design*** A circular air duct of diameter D is fit firmly into the right-angle corner where a basement wall meets the floor (see figure). Find the diameter of the largest water pipe that can be run in the right-angle corner behind the air duct.



In Exercises 41–48, write the standard form of the equation of the circle and sketch its graph.

41. $x^2 + y^2 - 2x + 6y + 6 = 0$

42. $x^2 + y^2 - 2x + 6y - 15 = 0$

43. $x^2 + y^2 - 2x + 6y + 10 = 0$

44. $3x^2 + 3y^2 - 6y - 1 = 0$

45. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$

46. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$

47. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$

48. $x^2 + y^2 - 4x + 2y + 3 = 0$

In Exercises 49 and 50, use a graphing utility to graph the equation. Use a square setting. (*Hint:* It may be necessary to solve the equation for y and graph the resulting two equations.)

49. $4x^2 + 4y^2 - 4x + 24y - 63 = 0$

50. $x^2 + y^2 - 8x - 6y - 11 = 0$

In Exercises 51 and 52, sketch the set of all points satisfying the inequality. Use a graphing utility to verify your result.

51. $x^2 + y^2 - 4x + 2y + 1 \le 0$

52. $(x - 1)^2 + \left(y - \frac{1}{2}\right)^2 > 1$

53. Prove that

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

is one of the points of trisection of the line segment joining (x_1, y_1) and (x_2, y_2). Find the midpoint of the line segment joining

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

and (x_2, y_2) to find the second point of trisection.

54. Use the results of Exercise 53 to find the points of trisection of the line segment joining the following points.

(a) $(1, -2), (4, 1)$ (b) $(-2, -3), (0, 0)$

True or False? **In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. If $ab < 0$, the point (a, b) lies in either Quadrant II or Quadrant IV.

56. The distance between the points $(a + b, a)$ and $(a - b, a)$ is $2b$.

57. If the distance between two points is zero, the two points must coincide.

58. If $ab = 0$, the point (a, b) lies on the x-axis or on the y-axis.

In Exercises 59–62, prove the statement.

59. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

60. The perpendicular bisector of a chord of a circle passes through the center of the circle.

61. An angle inscribed in a semicircle is a right angle.

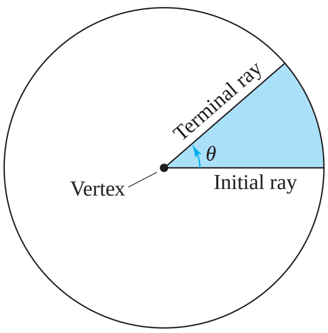
62. The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

APPENDIX C.3 Review of Trigonometric Functions

Angles and Degree Measure • Radian Measure • The Trigonometric Functions • Evaluating Trigonometric Functions • Solving Trigonometric Equations • Graphs of Trigonometric Functions

Angles and Degree Measure



Standard position of an angle
Figure C.24

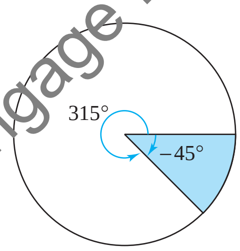
An **angle** has three parts: an **initial ray,** a **terminal ray,** and a **vertex** (the point of intersection of the two rays), as shown in Figure C.24. An angle is in **standard position** if its initial ray coincides with the positive x-axis and its vertex is at the origin. It is assumed that you are familiar with the degree measure of an angle.* It is common practice to use θ (the Greek lowercase *theta*) to represent both an angle and its measure. Angles between 0° and 90° are **acute,** and angles between 90° and 180° are **obtuse.**

Positive angles are measured *counterclockwise,* and negative angles are measured *clockwise*. For instance, Figure C.25 shows an angle whose measure is $-45°$. You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved. For example, Figure C.25 shows that the angle measuring $-45°$ has the same terminal ray as the angle measuring 315°. Such angles are **coterminal.** In general, if θ is any angle, then

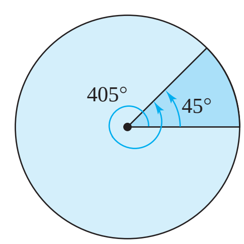
$$\theta + n(360), \quad n \text{ is a nonzero integer}$$

is coterminal with θ.

An angle that is larger than 360° is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure C.26. You can form an angle whose measure is less than $-360°$ by revolving a terminal ray more than one full revolution clockwise.



Coterminal angles
Figure C.25



Coterminal angles
Figure C.26

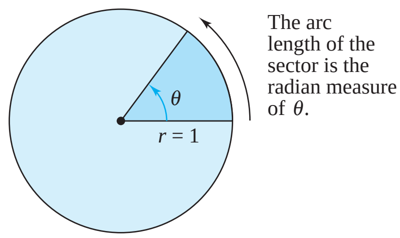
NOTE It is common to use the symbol θ to refer to both an *angle* and its *measure*. For instance, in Figure C.26, you can write the measure of the smaller angle as $\theta = 45°$.

For a more complete review of trigonometry, see Precalculus, 7th edition, by Larson and Hostetler (Boston, Massachusetts: Houghton Mifflin, 2007).

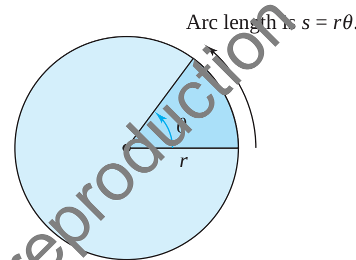
Radian Measure

To assign a radian measure to an angle θ, consider θ to be a central angle of a circle of radius 1, as shown in Figure C.27. The **radian measure** of θ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a **unit circle** (of radius 1) is 2π. This implies that the radian measure of an angle measuring 360° is 2π. In other words, $360° = 2\pi$ radians.

Using radian measure for θ, the length s of a circular arc of radius r is $s = r\theta$, as shown in Figure C.28.

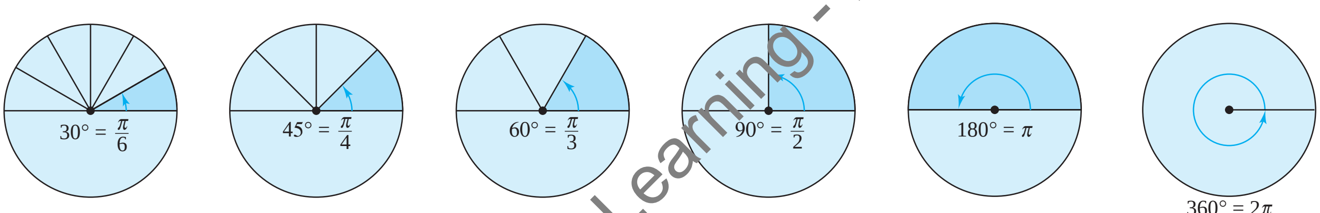


Unit circle
Figure C.27



Circle of radius r
Figure C.28

You should know the conversions of the common angles shown in Figure C.29. For other angles, use the fact that 180° is equal to π radians.



Radian and degree measure for several common angles
Figure C.29

EXAMPLE 1 Conversions Between Degrees and Radians

a. $40° = (40\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = \dfrac{2\pi}{9}$ radian

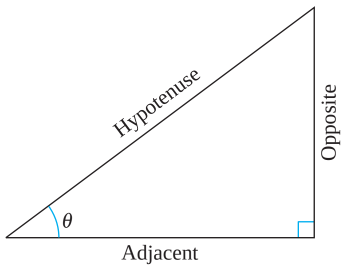
b. $540° = (540\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = 3\pi$ radians

c. $-270° = (-270\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = -\dfrac{3\pi}{2}$ radians

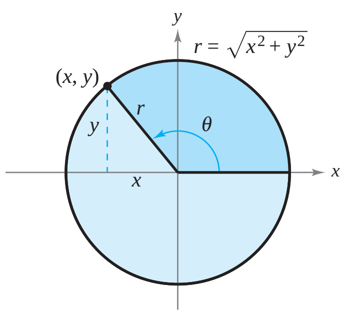
d. $-\dfrac{\pi}{2}$ radians $= \left(-\dfrac{\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = -90°$

e. 2 radians $= (2\ \cancel{\text{rad}})\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = \dfrac{360}{\pi} \approx 114.59°$

f. $\dfrac{9\pi}{2}$ radians $= \left(\dfrac{9\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = 810°$



Sides of a right triangle
Figure C.30



An angle in standard position
Figure C.31

The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal side of an angle in standard position. The six trigonometric functions, **sine, cosine, tangent, cotangent, secant,** and **cosecant** (abbreviated as sin, cos, etc.), are defined below from both viewpoints.

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$ (see Figure C.30).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Circular function definitions, where θ *is any angle* (see Figure C.31).

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

$$\csc\theta = \frac{r}{y},\ y \neq 0 \qquad \sec\theta = \frac{r}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

The following trigonometric identities are direct consequences of the definitions. (ϕ is the Greek letter *phi*.)

Trigonometric Identities

[*Note that* $\sin^2\theta$ *is used to represent* $(\sin\theta)^2$.]

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Reduction Formulas:

$$\sin(-\theta) = -\sin\theta \qquad \sin\theta = -\sin(\theta - \pi)$$
$$\cos(-\theta) = \cos\theta \qquad \cos\theta = -\cos(\theta - \pi)$$
$$\tan(-\theta) = -\tan\theta \qquad \tan\theta = \tan(\theta - \pi)$$

Sum or Difference of Two Angles:

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Half–Angle Formulas:

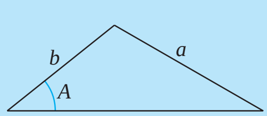
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

Double–Angle Formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities:

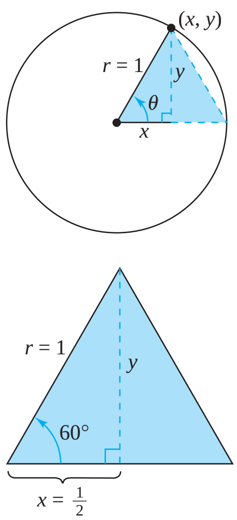
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions: (1) decimal approximations with a calculator and (2) exact evaluations using trigonometric identities and formulas from geometry. When using a calculator toevaluate a trigonometric function, remember to set the calculator to the appropriate mode—*degree* mode or *radian* mode.

EXAMPLE 2 Exact Evaluation of Trigonometric Functions

Evaluate the sine, cosine, and tangent of $\dfrac{\pi}{3}$.



The angle $\pi/3$ in standard position
Figure C.32

Solution Begin by drawing the angle $\theta = \pi/3$ in standard position, as shown in Figure C.32. Then, because $60° = \pi/3$ radians, you can draw an equilateral triangle with sides of length 1 and θ as one of its angles. Because the altitude of this triangle bisects its base, you know that $x = \frac{1}{2}$. Using the Pythagorean Theorem, you obtain

$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Now, knowing the values of x, y, and r, you can write the following.

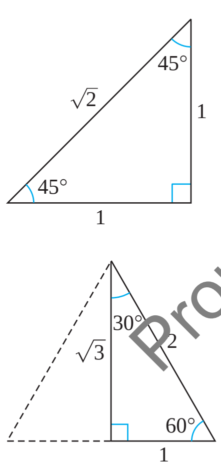
$$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

NOTE All angles in this text are measured in radians unless stated otherwise. For example, when $\sin 3$ is written, the sine of 3 radians is meant, and when $\sin 3°$ is written, the sine of 3 degrees is meant.

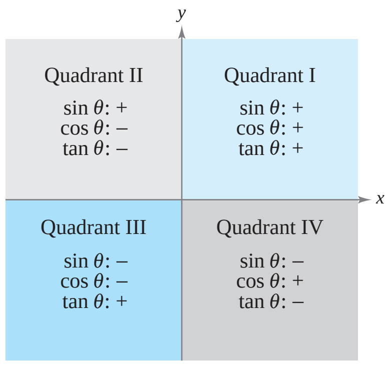
The degree and radian measures of several common angles are shown in the table below, along with the corresponding values of the sine, cosine, and tangent (see Figure C.33).



Common angles
Figure C.33

Common First Quadrant Angles

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined



Quadrant signs for trigonometric functions
Figure C.34

The quadrant signs for the sine, cosine, and tangent functions are shown in Figure C.34. To extend the use of the table on the preceding page to angles in quadrants other than the first quadrant, you can use the concept of a **reference angle** (see Figure C.35), with the appropriate quadrant sign. For instance, the reference angle for $3\pi/4$ is $\pi/4$, and because the sine is positive in Quadrant II, you can write

$$\sin \frac{3\pi}{4} = +\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, because the reference angle for 330° is 30°, and the tangent is negative in Quadrant IV, you can write

$$\tan 330° = -\tan 30° = -\frac{\sqrt{3}}{3}.$$

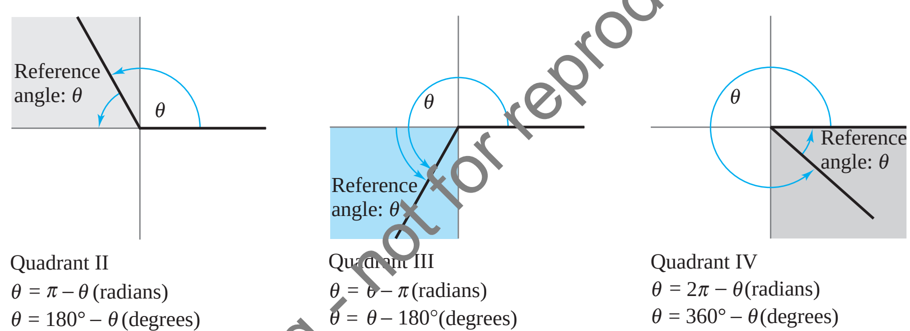


Figure C.35

EXAMPLE 3 Trigonometric Identities and Calculators

Evaluate each trigonometric expression.

a. $\sin\left(-\frac{\pi}{3}\right)$ **b.** sec 60° **c.** cos(1.2)

Solution

a. Using the reduction formula $\sin(-\theta) = -\sin\theta$, you can write

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

b. Using the reciprocal identity $\sec\theta = 1/\cos\theta$, you can write

$$\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2.$$

c. Using a calculator, you obtain

$$\cos(1.2) \approx 0.3624.$$

Remember that 1.2 is given in *radian* measure. Consequently, your calculator must be set in *radian* mode.

Solving Trigonometric Equations

How would you solve the equation $\sin\theta = 0$? You know that $\theta = 0$ is one solution, but this is not the only solution. Any one of the following values of θ is also a solution.

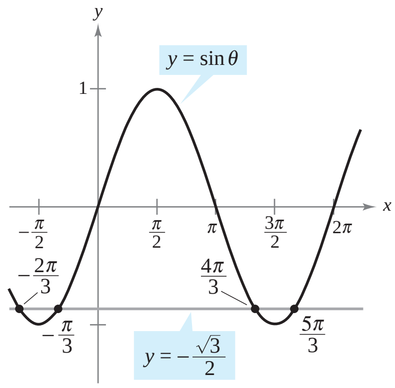
$$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

You can write this infinite solution set as $\{n\pi: n \text{ is an integer}\}$.

EXAMPLE 4 Solving a Trigonometric Equation

Solve the equation

$$\sin\theta = -\frac{\sqrt{3}}{2}.$$



Solution points of $\sin n = -\frac{\sqrt{3}}{2}$
Figure C.36

Solution To solve the equation, you should consider that the sine is negative in Quadrants III and IV and that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

So, you are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$. In the interval $[0, 2\pi]$, the two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

By adding integer multiples of 2π to each of these solutions, you obtain the following general solution.

$$\theta = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

See Figure C.36.

EXAMPLE 5 Solving a Trigonometric Equation

Solve $\cos 2\theta = 2 - 3\sin\theta$, where $0 \le \theta \le 2\pi$.

Solution Using the double-angle identity $\cos 2\theta = 1 - 2\sin^2\theta$, you can rewrite the equation as follows.

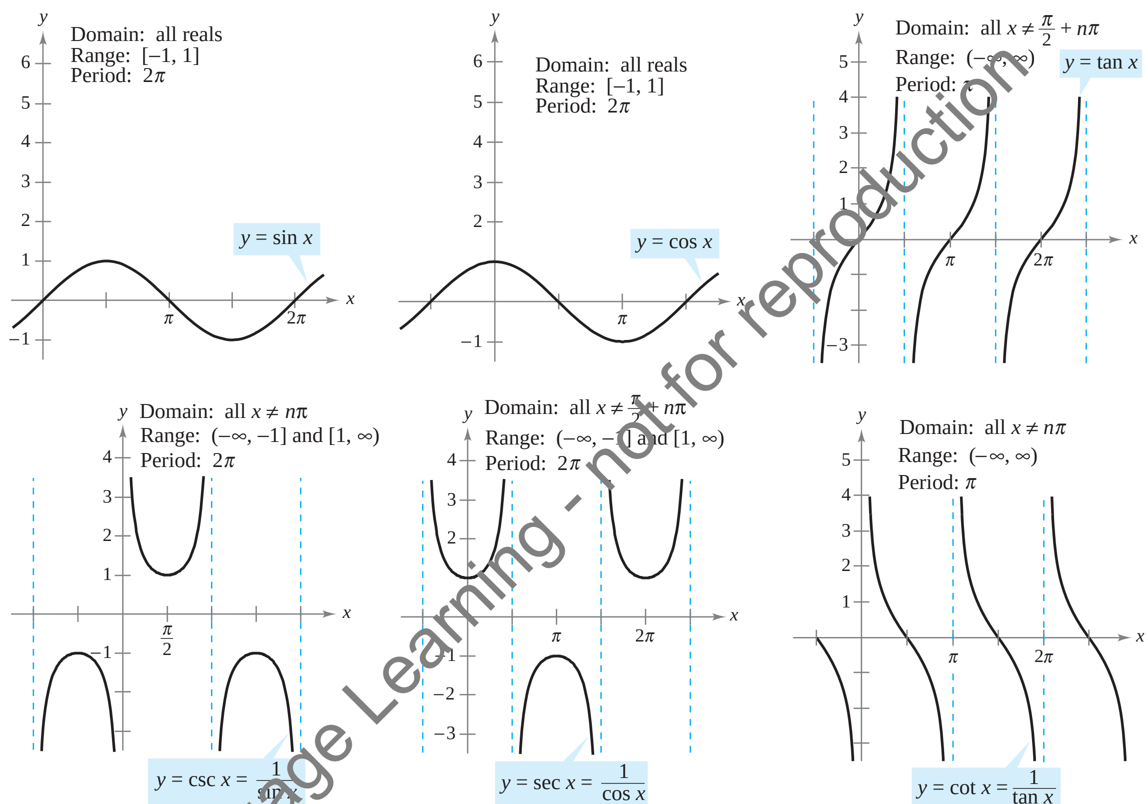
$$\cos 2\theta = 2 - 3\sin\theta \qquad \text{Write original equation.}$$
$$1 - 2\sin^2\theta = 2 - 3\sin\theta \qquad \text{Trigonometric identity}$$
$$0 = 2\sin^2\theta - 3\sin\theta + 1 \qquad \text{Quadratic form}$$
$$0 = (2\sin\theta - 1)(\sin\theta - 1) \qquad \text{Factor.}$$

If $2\sin\theta - 1 = 0$, then $\sin\theta = 1/2$ and $\theta = \pi/6$ or $\theta = 5\pi/6$. If $\sin\theta - 1 = 0$, then $\sin\theta = 1$ and $\theta = \pi/2$. So, for $0 \le \theta \le 2\pi$, there are three solutions.

$$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \text{or} \quad \frac{\pi}{2}$$

Graphs of Trigonometric Functions

A function f is **periodic** if there exists a nonzero number p such that $f(x + p) = f(x)$ for all x in the domain of f. The smallest such positive value of p (if it exists) is the **period** of f. The sine, cosine, secant, and cosecant functions each have a period of 2π, and the other two trigonometric functions, tangent and cotangent, have a period of π, as shown in Figure C.37.



The graphs of the six trigonometric functions
Figure C.37

Note in Figure C.37 that the maximum value of $\sin x$ and $\cos x$ is 1 and the minimum value is -1. The graphs of the functions $y = a \sin bx$ and $y = a \cos bx$ oscillate between $-a$ and a, and so have an **amplitude** of $|a|$. Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a \sin bx$ and $y = a \cos bx$ each have a period of $2\pi/|b|$. The table below summarizes the amplitudes and periods for some types of trigonometric functions.

Function	Period	Amplitude
$y = a \sin bx$ or $y = a \cos bx$	$\frac{2\pi}{\lvert b \rvert}$	$\lvert a \rvert$
$y = a \tan bx$ or $y = a \cot bx$	$\frac{\pi}{\lvert b \rvert}$	Not applicable
$y = a \sec bx$ or $y = a \csc bx$	$\frac{2\pi}{\lvert b \rvert}$	Not applicable

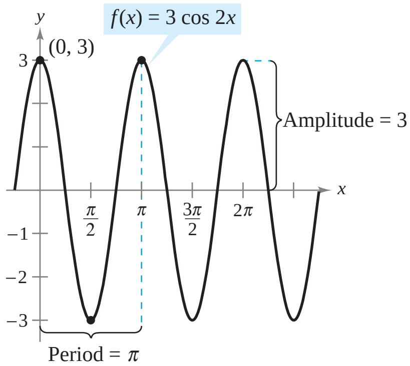


Figure C.38

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Sketch the graph of $f(x) = 3 \cos 2x$.

Solution The graph of $f(x) = 3 \cos 2x$ has an amplitude of 3 and a period of $2\pi/2 = \pi$. Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

Maximum: $(0, 3)$

Minimum: $\left(\dfrac{\pi}{2}, -3\right)$

Maximum: $(\pi, 3)$

By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure C.38.

Horizontal shifts, vertical shifts, and reflections can be applied to the graphs of trigonometric functions, as illustrated in Example 7.

EXAMPLE 7 Shifts of Graphs of Trigonometric Functions

Sketch the graph of each function.

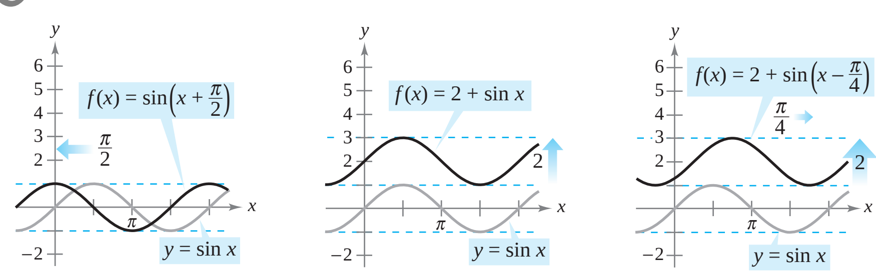
a. $f(x) = \sin\left(x + \dfrac{\pi}{2}\right)$ **b.** $f(x) = 2 + \sin x$ **c.** $f(x) = 2 + \sin\left(x - \dfrac{\pi}{4}\right)$

Solution

a. To sketch the graph of $f(x) = \sin(x + \pi/2)$, shift the graph of $y = \sin x$ to the left $\pi/2$ units, as shown in Figure C.39(a).

b. To sketch the graph of $f(x) = 2 + \sin x$, shift the graph of $y = \sin x$ upward two units, as shown in Figure C.39(b).

c. To sketch the graph of $f(x) = 2 + \sin(x - \pi/4)$, shift the graph of $y = \sin x$ upward two units and to the right $\pi/4$ units, as shown in Figure C.39(c).

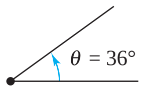


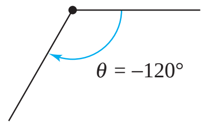
(a) Horizontal shift to the left (b) Vertical shift upward (c) Horizontal and vertical shifts

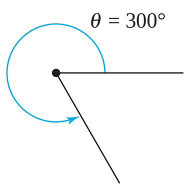
Transformations of the graph of $y = \sin x$
Figure C.39

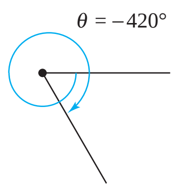
EXERCISES FOR APPENDIX C.3

In Exercises 1 and 2, determine two coterminal angles in degree measure (one positive and one negative) for each angle.

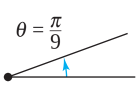
1. (a)


(b)


2. (a)


(b)


In Exercises 3 and 4, determine two coterminal angles in radian measure (one positive and one negative) for each angle.

3. (a)


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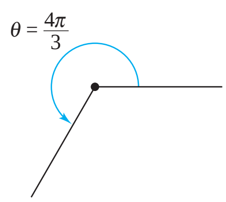
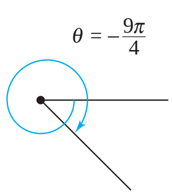


4. (a)


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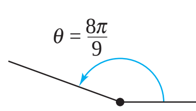


In Exercises 5 and 6, rewrite each angle in radian measure as a multiple of π and as a decimal accurate to three decimal places.

5. (a) $30°$ (b) $150°$ (c) $315°$ (d) $120°$

6. (a) $-20°$ (b) $-240°$ (c) $-270°$ (d) $144°$

In Exercises 7 and 8, rewrite each angle in degree measure.

7. (a) $\frac{3\pi}{2}$ (b) $\frac{7\pi}{6}$ (c) $-\frac{7\pi}{12}$ (d) -2.367

8. (a) $\frac{7\pi}{3}$ (b) $-\frac{11\pi}{30}$ (c) $\frac{11\pi}{6}$ (d) 0.438

9. Let r represent the radius of a circle, θ the central angle (measured in radians), and s the length of the arc subtended by the angle. Use the relationship $s = r\theta$ to complete the table.

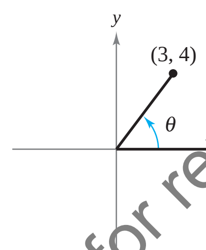
r	8 ft	15 in.	85 cm		
s	12 ft			96 in.	8642 mi
θ		1.6	$\frac{3\pi}{4}$	4	$\frac{2\pi}{3}$

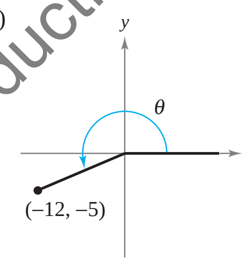
10. ***Angular Speed*** A car is moving at the rate of 50 miles per hour, and the diameter of its wheels is 2.5 feet.

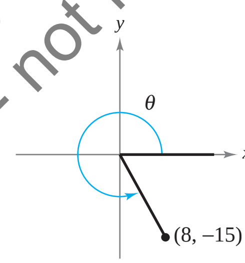
(a) Find the number of revolutions per minute that the wheels are rotating.

(b) Find the angular speed of the wheels in radians per minute.

In Exercises 11 and 12, determine all six trigonometric functions for the angle θ.

11. (a)


(b)


12. (a)


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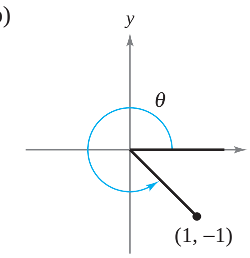


In Exercises 13 and 14, determine the quadrant in which θ lies.

13. (a) $\sin\theta < 0$ and $\cos\theta < 0$

(b) $\sec\theta > 0$ and $\cot\theta < 0$

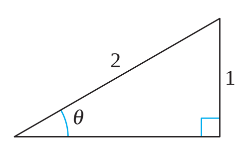
14. (a) $\sin\theta > 0$ and $\cos\theta < 0$

(b) $\csc\theta < 0$ and $\tan\theta > 0$

In Exercises 15–18, evaluate the trigonometric function.

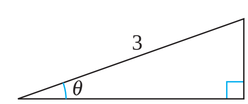
15. $\sin\theta = \frac{1}{2}$

$\cos\theta =$



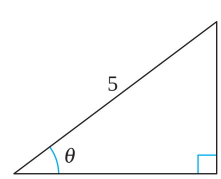
16. $\sin\theta = \frac{1}{3}$

$\tan\theta =$



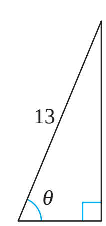
17. $\cos\theta = \frac{4}{5}$

$\cot\theta =$



18. $\sec\theta = \frac{13}{5}$

$\csc\theta =$



In Exercises 19–22, evaluate the sine, cosine, and tangent of each angle *without* using a calculator.

19. (a) $60°$ (b) $120°$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$

20. (a) $-30°$ (b) $150°$ (c) $-\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

21. (a) $225°$ (b) $-225°$ (c) $\frac{5\pi}{3}$ (d) $\frac{11\pi}{6}$

22. (a) $750°$ (b) $510°$ (c) $\frac{10\pi}{3}$ (d) $\frac{17\pi}{3}$

In Exercises 23–26, use a calculator to evaluate each trigonometric function. Round your answers to four decimal places.

23. (a) $\sin 10°$ (b) $\csc 10°$

24. (a) $\sec 225°$ (b) $\sec 135°$

25. (a) $\tan \frac{\pi}{9}$ (b) $\tan \frac{10\pi}{9}$

26. (a) $\cot(1.35)$ (b) $\tan(1.35)$

In Exercises 27–30, find two solutions of each equation. Give your answers in radians $(0 \le \theta < 2\pi)$. Do not use a calculator.

27. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ (b) $\cos \theta = -\frac{\sqrt{2}}{2}$

28. (a) $\sec \theta = 2$ (b) $\sec \theta = -2$

29. (a) $\tan \theta = 1$ (b) $\cot \theta = -\sqrt{3}$

30. (a) $\sin \theta = \frac{\sqrt{3}}{2}$ (b) $\sin \theta = -\frac{\sqrt{3}}{2}$

In Exercises 31–38, solve the equation for θ $(0 \le \theta < 2\pi)$.

31. $2\sin^2 \theta = 1$

32. $\tan^2 \theta = 3$

33. $\tan^2 \theta - \tan \theta = 0$

34. $2\cos^2 \theta - \cos \theta = 1$

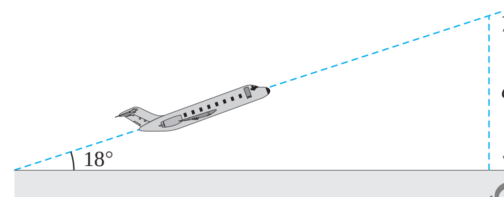
35. $\sec \theta \csc \theta = 2 \csc \theta$

36. $\sin \theta = \cos \theta$

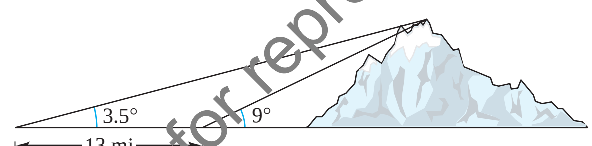
37. $\cos^2 \theta + \sin \theta = 1$

38. $\cos \frac{\theta}{2} - \cos \theta = 1$

39. ***Airplane Ascent*** An airplane leaves the runway climbing at an angle of $18°$ with a speed of 275 feet per second (see figure). Find the altitude a of the plane after 1 minute.



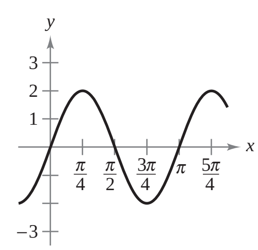
40. ***Height of a Mountain*** In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is $3.5°$. After you drive 13 miles closer to the mountain, the angle of elevation is $9°$. Approximate the height of the mountain.



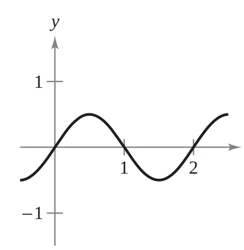
Not drawn to scale

In Exercises 41–44, determine the period and amplitude of each function.

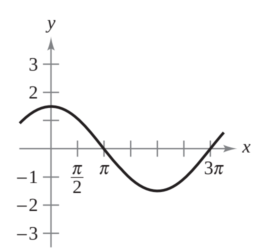
41. (a) $y = 2 \sin 2x$



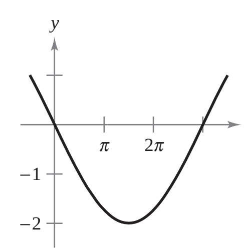
(b) $y = \frac{1}{2} \sin \pi x$



42. (a) $y = \frac{3}{2} \cos \frac{x}{2}$



(b) $y = -2 \sin \frac{x}{3}$



43. $y = 3 \sin 4\pi x$

44. $y = \frac{2}{3} \cos \frac{\pi x}{10}$

In Exercises 45–48, find the period of the function.

45. $y = 5 \tan 2x$ **46.** $y = 7 \tan 2\pi x$

47. $y = \sec 5x$ **48.** $y = \csc 4x$

Writing **In Exercises 49 and 50, use a graphing utility to graph each function f in the same viewing window for $c = -2$, $c = -1$, $c = 1$, and $c = 2$. Give a written description of the change in the graph caused by changing c.**

49. (a) $f(x) = c \sin x$
(b) $f(x) = \cos(cx)$
(c) $f(x) = \cos(\pi x - c)$

50. (a) $f(x) = \sin x + c$
(b) $f(x) = -\sin(2\pi x - c)$
(c) $f(x) = c \cos x$

In Exercises 51–62, sketch the graph of the function.

51. $y = \sin \dfrac{x}{2}$ **52.** $y = 2 \cos 2x$

53. $y = -\sin \dfrac{2\pi x}{3}$ **54.** $y = 2 \tan x$

55. $y = \csc \dfrac{x}{2}$ **56.** $y = \tan 2x$

57. $y = 2 \sec 2x$ **58.** $y = \csc 2\pi x$

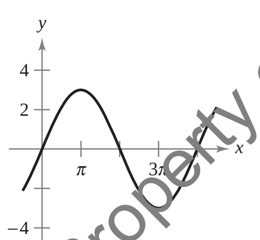
59. $y = \sin(x + \pi)$ **60.** $y = \cos\left(x - \dfrac{\pi}{3}\right)$

61. $y = 1 + \cos\left(x - \dfrac{\pi}{2}\right)$

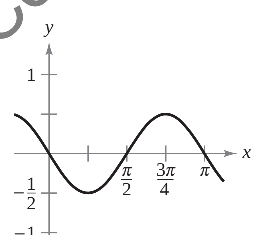
62. $y = 1 + \sin\left(x + \dfrac{\pi}{2}\right)$

Graphical Reasoning **In Exercises 63 and 64, find a, b, and c such that the graph of the function matches the graph in the figure.**

63. $y = a \cos(bx - c)$



64. $y = a \sin(bx - c)$



65. ***Think About It*** Sketch the graphs of $f(x) = \sin x$, $g(x) = |\sin x|$, and $h(x) = \sin(|x|)$. In general, how are the graphs of $|f(x)|$ and $f(|x|)$ related to the graph of f?

66. ***Think About It*** The model for the height h of a Ferris wheel car is

$$h = 51 + 50 \sin 8\pi t$$

where t is measured in minutes. (The Ferris wheel has a radius of 50 feet.) This model yields a height of 51 feet when $t = 0$. Alter the model so that the height of the car is 1 foot when $t = 0$.

67. ***Sales*** The monthly sales S (in thousands of units) of a seasonal product are modeled by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where t is the time (in months) with $t = 1$ corresponding to January. Use a graphing utility to graph the model for S and determine the months when sales exceed 75,000 units.

68. ***Investigation*** Two trigonometric functions f and g have a period of 2, and their graphs intersect at $x = 5.35$.

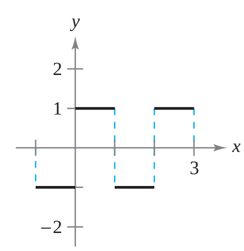
(a) Give one smaller and one larger positive value of x where the functions have the same value.

(b) Determine one negative value of x where the graphs intersect.

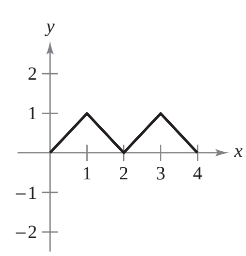
(c) Is it true that $f(13.35) = g(-4.65)$? Give a reason for your answer.

Pattern Recognition **In Exercises 69 and 70, use a graphing utility to compare the graph of f with the given graph. Try to improve the approximation by adding a term to $f(x)$. Use a graphing utility to verify that your new approximation is better than the original. Can you find other terms to add to make the approximation even better? What is the pattern? (In Exercise 69, sine terms can be used to improve the approximation and in Exercise 70, cosine terms can be used.)**

69. $f(x) = \dfrac{4}{\pi}\left(\sin \pi x + \dfrac{1}{3} \sin 3\pi x\right)$



70. $f(x) = \dfrac{1}{2} - \dfrac{4}{\pi^2}\left(\cos \pi x + \dfrac{1}{9} \cos 3\pi x\right)$



APPENDIX D

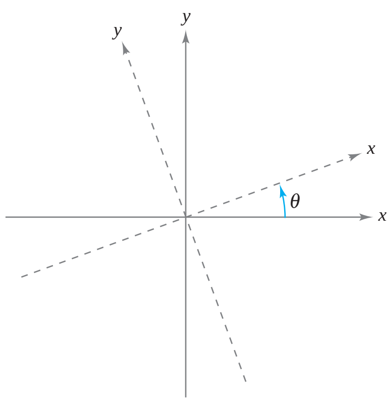
Rotation and the General Second-Degree Equation

Rotation of Axes • Invariants Under Rotation

Rotation of Axes

Previously, you learned that equations of conics with axes parallel to one of the coordinate axes can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \qquad \text{Horizontal or vertical axes}$$



After rotation of the x- and y-axes counterclockwise through an angle θ, the rotated axes are denoted as the x'-axis and y'-axis.
Figure D.1

Here you will study the equations of conics whose axes are rotated so that they are *not* parallel to either the x-axis or the y-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy\text{-plane}$$

To eliminate this xy-term, you can use a procedure called **rotation of axes.** You want to rotate the x- and y-axes until they are parallel to the axes of the conic. (The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure D.1.) After the rotation has been accomplished, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \qquad \text{Equation in } x'y'\text{-plane}$$

Because this equation has no $x'y'$-term, you can obtain a standard form by completing the square.

The following theorem identifies how much to rotate the axes to eliminate an xy-term and also the equations for determining the new coefficients A', C', D', E', and F'.

THEOREM D.1 Rotation of Axes

The general second-degree equation of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B \neq 0$, can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

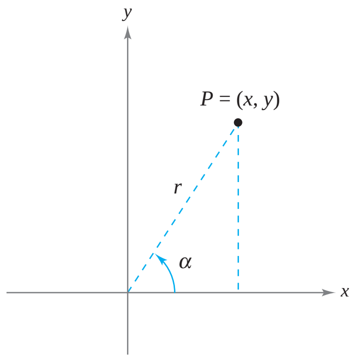
by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

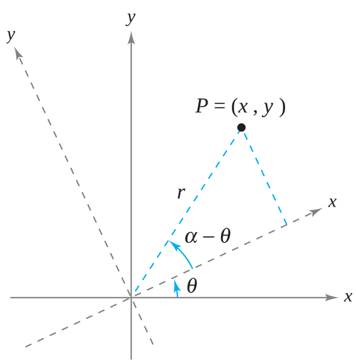
The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta.$$



Original: $x = r\cos\alpha$
$y = r\sin\alpha$



Rotated: $x' = r\cos(\alpha - \theta)$
$y' = r\sin(\alpha - \theta)$

Figure D.2

Proof To discover how the coordinates in the xy-system are related to the coordinates in the $x'y'$-system, choose a point $P = (x, y)$ in the original system and attempt to find its coordinates (x', y') in the rotated system. In either system, the distance r between the point P and the origin is the same, and so the equations for x, y, x', and y' are those given in Figure D.2. Using the formulas for the sine and cosine of the difference of two angles, you obtain

$$\begin{aligned} x' &= r\cos(\alpha - \theta) \\ &= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\ &= x\cos\theta + y\sin\theta \\ y' &= r\sin(\alpha - \theta) \\ &= r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \\ &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta \\ &= y\cos\theta - x\sin\theta. \end{aligned}$$

Solving this system for x and y yields

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta.$$

Finally, by substituting these values for x and y into the original equation and collecting terms, you obtain the following.

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ C' &= A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F \end{aligned}$$

Now, in order to eliminate the $x'y'$-term, you must select θ such that $B' = 0$, as follows.

$$\begin{aligned} B' &= 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \\ &= (C - A)\sin 2\theta + B\cos 2\theta \\ &= B(\sin 2\theta)\left(\frac{C - A}{B} + \cot 2\theta\right) \\ &= 0, \quad \sin 2\theta \neq 0 \end{aligned}$$

If $B = 0$, no rotation is necessary, because the xy-term is not present in the original equation. If $B \neq 0$, the only way to make $B' = 0$ is to let

$$\cot 2\theta = \frac{A - C}{B}, \quad B \neq 0.$$

So, you have established the desired results.

EXAMPLE 1 Rotation of a Hyperbola

Write the equation $xy - 1 = 0$ in standard form.

Solution Because $A = 0$, $B = 1$, and $C = 0$, you have (for $0 < \theta < \pi/2$)

$$\cot 2\theta = \frac{A - C}{B} = 0 \quad \Longrightarrow \quad 2\theta = \frac{\pi}{2} \quad \Longrightarrow \quad \theta = \frac{\pi}{4}.$$

The equation in the $x'y'$-system is obtained by making the following substitutions.

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$

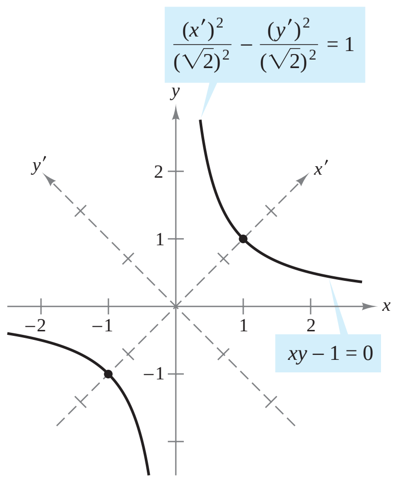
Substituting these expressions into the equation $xy - 1 = 0$ produces

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1. \qquad \text{Write in standard form.}$$

This is the equation of a hyperbola centered at the origin with vertices at $(\pm\sqrt{2}, 0)$ in the $x'y'$-system, as shown in Figure D.3.



Vertices:
$(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$ in $x'y'$-system
$(1, 1)$, $(-1, -1)$, in xy-system
Figure D.3

EXAMPLE 2 Rotation of an Ellipse

Sketch the graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$.

Solution Because $A = 7$, $B = -6\sqrt{3}$, and $C = 13$, you have (for $0 < \theta < \pi/2$)

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} \quad \Longrightarrow \quad \theta = \frac{\pi}{6}.$$

Therefore, the equation in the $x'y'$-system is derived by making the following substitutions.

$$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

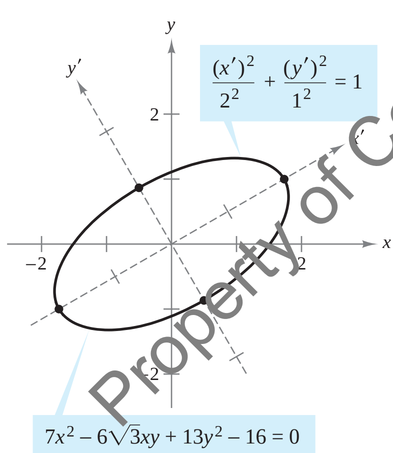
$$y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}$$

Substituting these expressions into the original equation eventually simplifies (after considerable algebra) to

$$4(x')^2 + 16(y')^2 = 16$$

$$\frac{(x')^2}{2^2} + \frac{(y')^2}{1^2} = 1. \qquad \text{Write in standard form.}$$

This is the equation of an ellipse centered at the origin with vertices at $(\pm 2, 0)$ and $(0, \pm 1)$ in the $x'y'$-system, as shown in Figure D.4.



Vertices:
$(\pm 2, 0)$, $(0, \pm 1)$ in $x'y'$-system
$(\pm\sqrt{3}, \pm 1)$, $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ in xy-system
Figure D.4

In Examples 1 and 2, the values of θ were the common angles 45° and 30°, respectively. Of course, many second-degree equations do not yield such common solutions to the equation

$$\cot 2\theta = \frac{A - C}{B}.$$

Example 3 illustrates such a case.

EXAMPLE 3 Rotation of a Parabola

Sketch the graph of $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$.

Solution Because $A = 1$, $B = -4$, and $C = 4$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 4}{-4} = \frac{3}{4}.$$

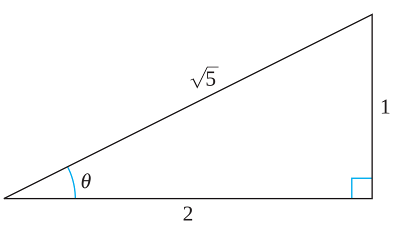


Figure D.5

The trigonometric identity $\cot 2\theta = (\cot^2 \theta - 1)/(2 \cot \theta)$ produces

$$\cot 2\theta = \frac{3}{4} = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

from which you obtain the equation

$$6 \cot \theta = 4 \cot^2 \theta - 4 \quad \Rightarrow \quad 4 \cot^2 \theta - 6 \cot \theta - 4 = 0$$
$$(2 \cot \theta - 4)(2 \cot \theta + 1) = 0.$$

Considering $0 < \theta < \pi/2$, it follows that $2 \cot \theta = 4$. So,

$$\cot \theta = 2 \quad \Rightarrow \quad \theta \approx 26.6°.$$

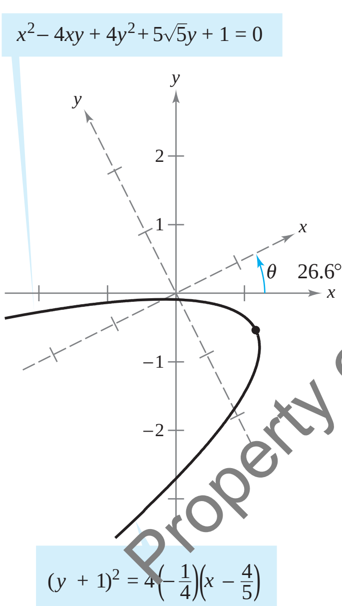
From the triangle in Figure D.5, you obtain $\sin \theta = 1/\sqrt{5}$ and $\cos \theta = 2/\sqrt{5}$. Consequently, you can write the following.

$$x = x' \cos \theta - y' \sin \theta = x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) = \frac{2x' - y'}{\sqrt{5}}$$

$$y = x' \sin \theta + y' \cos \theta = x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) = \frac{x' + 2y'}{\sqrt{5}}$$

Substituting these expressions into the original equation produces

$$\left(\frac{2x' - y'}{\sqrt{5}}\right)^2 - 4\left(\frac{2x' - y'}{\sqrt{5}}\right)\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 4\left(\frac{x' + 2y'}{\sqrt{5}}\right)^2 +$$
$$5\sqrt{5}\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 1 = 0$$



Vertex:
$\left(\frac{4}{5}, -1\right)$ in $x'y'$-system
$\left(\frac{13}{5\sqrt{5}}, -\frac{6}{5\sqrt{5}}\right)$ in xy-system

Figure D.6

which simplifies to

$$5(y')^2 + 5x' + 10y' + 1 = 0.$$

By completing the square, you obtain the standard form

$$5(y' + 1)^2 = -5x' + 4$$

$$(y' + 1)^2 = 4\left(-\frac{1}{4}\right)\left(x' - \frac{4}{5}\right). \qquad \text{Write in standard form.}$$

The graph of the equation is a parabola with its vertex at $\left(\frac{4}{5}, -1\right)$ and its axis parallel to the x'-axis in the $x'y'$-system, as shown in Figure D.6.

Invariants Under Rotation

In Theorem D.1, note that the constant term is the same in both equations—that is, $F' = F$. Because of this, F is said to be **invariant under rotation.** Theorem D.2 lists some other rotation invariants. The proof of this theorem is left as an exercise (see Exercise 34).

THEOREM D.2 Rotation Invariants

The rotation of coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$
2. $A + C = A' + C'$
3. $B^2 - 4AC = (B')^2 - 4A'C'$

You can use this theorem to classify the graph of a second-degree equation *with* an xy-term in much the same way you do for a second-degree equation *without* an xy-term. Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$B^2 - 4AC = -4A'C' \qquad \text{Discriminant}$$

which is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Because the sign of $A'C'$ determines the type of graph for the equation

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

the sign of $B^2 - 4AC$ must determine the type of graph for the original equation. This result is stated in Theorem D.3.

THEOREM D.3 Classification of Conics by the Discriminant

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is, except in degenerate cases, determined by its discriminant as follows.

1. *Ellipse or circle* $\quad B^2 - 4AC < 0$
2. *Parabola* $\quad B^2 - 4AC = 0$
3. *Hyperbola* $\quad B^2 - 4AC > 0$

EXAMPLE 4 Using the Discriminant

Classify the graph of each equation.

a. $4xy - 9 = 0$

b. $2x^2 - 3xy + 2y^2 - 2x = 0$

c. $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

d. $3x^2 + 8xy + 4y^2 - 7 = 0$

Solution

a. The graph is a hyperbola because

$$B^2 - 4AC = 16 - 0 > 0.$$

b. The graph is a circle or an ellipse because

$$B^2 - 4AC = 9 - 16 < 0.$$

c. The graph is a parabola because

$$B^2 - 4AC = 36 - 36 = 0.$$

d. The graph is a hyperbola because

$$B^2 - 4AC = 64 - 48 > 0.$$

EXERCISES FOR APPENDIX D

In Exercises 1–12, rotate the axes to eliminate the xy-term in the equation. Write the resulting equation in standard form and sketch its graph showing both sets of axes.

1. $xy + 1 = 0$
2. $xy - 4 = 0$
3. $x^2 - 10xy + y^2 + 1 = 0$
4. $xy + x - 2y + 3 = 0$
5. $xy - 2y - 4x = 0$
6. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
7. $5x^2 - 2xy + 5y^2 - 12 = 0$
8. $2x^2 - 3xy - 2y^2 + 10 = 0$
9. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
10. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$
11. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
12. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

In Exercises 13–18, use a graphing utility to graph the conic. Determine the angle θ through which the axes are rotated. Explain how you used the graphing utility to obtain the graph.

13. $x^2 + xy + y^2 = 10$
14. $x^2 - 4xy + 2y^2 = 6$
15. $17x^2 + 32xy - 7y^2 = 75$
16. $40x^2 + 36xy + 25y^2 = 52$
17. $32x^2 + 50xy + 7y^2 = 52$
18. $4x^2 - 12xy + 9y^2 + (4\sqrt{13} + 12)x - (6\sqrt{13} + 8)y = 91$

In Exercises 19–26, use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola.

19. $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
20. $x^2 - 4xy - 2y^2 - 6 = 0$
21. $13x^2 - 8xy + 7y^2 - 45 = 0$
22. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
23. $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
24. $36x^2 - 60xy + 25y^2 + 9y = 0$
25. $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$
26. $x^2 + xy + 4y^2 + x + y - 4 = 0$

In Exercises 27–32, sketch the graph (if possible) of the degenerate conic.

27. $y^2 - 4x^2 = 0$
28. $x^2 + y^2 - 2x + 6y + 10 = 0$
29. $x^2 + 2xy + y^2 - 1 = 0$
30. $x^2 - 10xy + y^2 = 0$
31. $(x - 2y + 1)(x + 2y - 3) = 0$
32. $(2x + y - 3)^2 = 0$

33. Show that the equation $x^2 + y^2 = r^2$ is invariant under rotation of axes.
34. Prove Theorem D.2.