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Lecture 23:

Flow Estimation

Key points for today:
- Brightness constancy equation
- Aperature problem
- Lucas-Kanade algorithm

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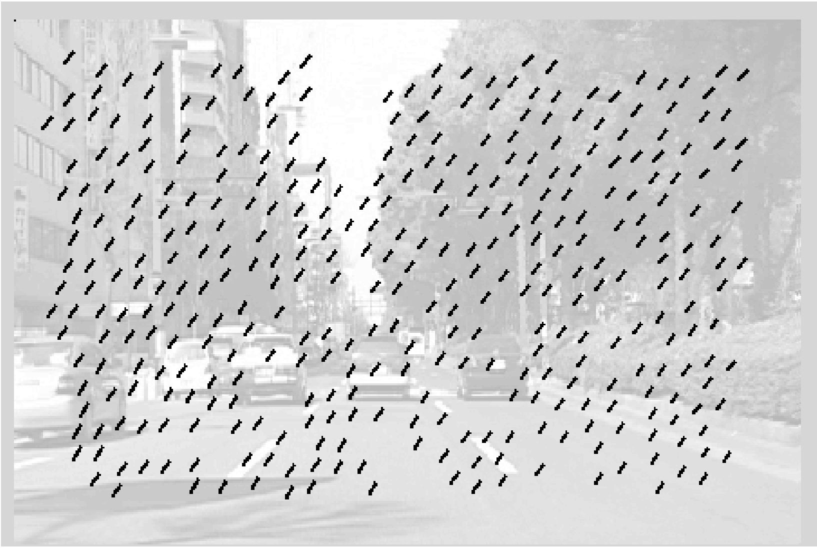
Review: Flow Due to Self-Motion

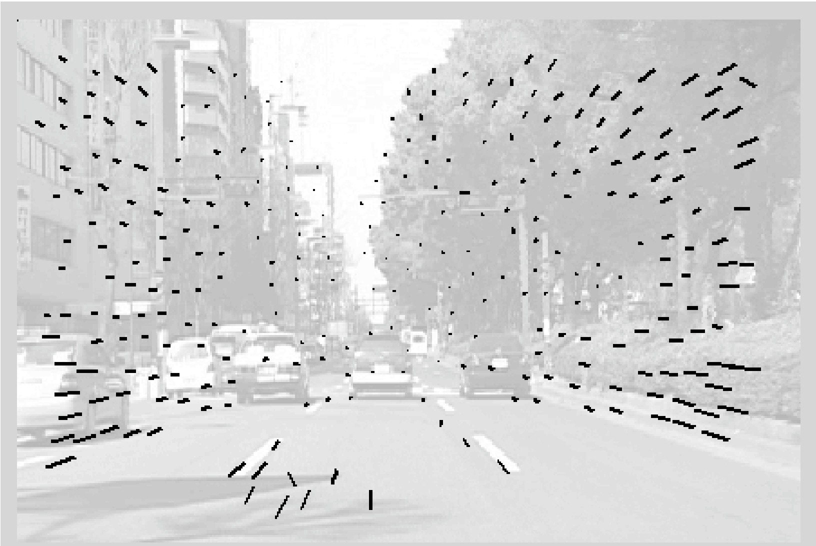
Flow:

$$
\begin{aligned}
u' - u &= \boxed{f\omega_Y - v\omega_Z + \frac{u^2}{f}\omega_Y - \frac{uv}{f}\omega_X} + \boxed{f\frac{T_X}{Z} - u\frac{T_Z}{Z}} \\
v' - v &= \boxed{-f\omega_X + u\omega_Z - \frac{v^2}{f}\omega_X + \frac{uv}{f}\omega_Y} + \boxed{f\frac{T_Y}{Z} - v\frac{T_Z}{Z}}
\end{aligned}
$$

Rotation **Translation**



Rotation component



Translation component

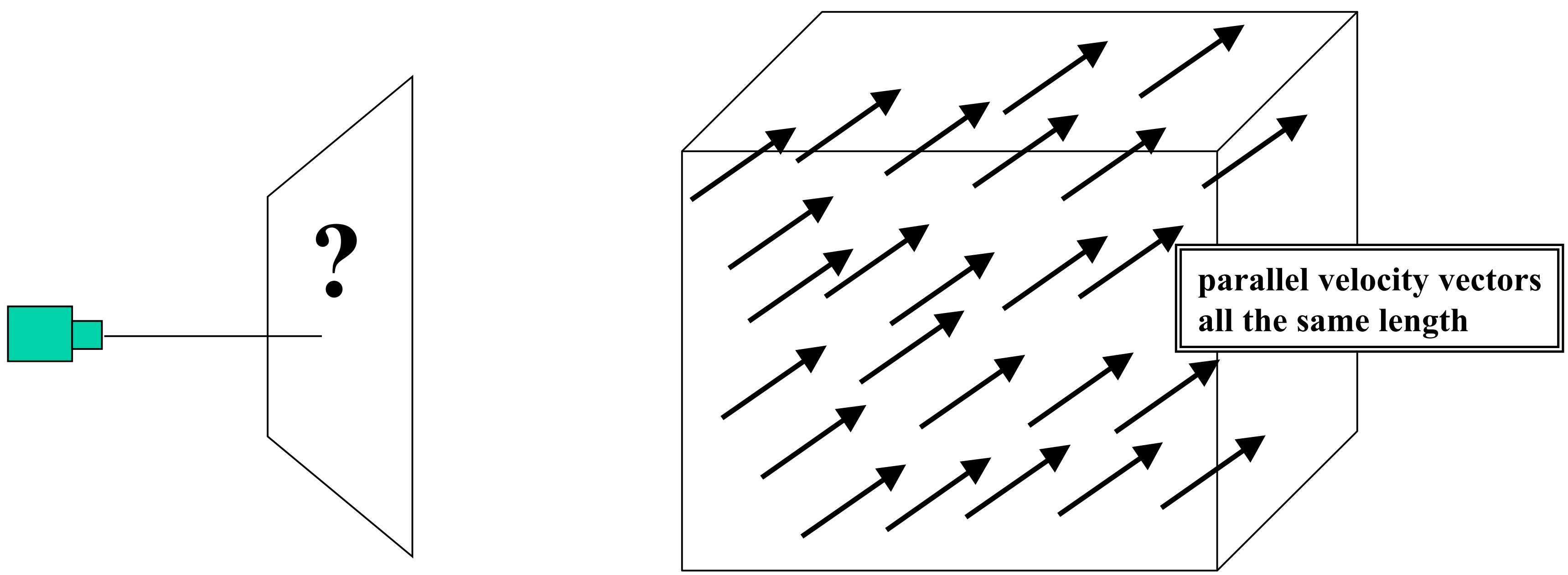
rotation component does not depend on scene structure.

translational component does vary as scene *Z*-value varies. That is, it exhibits motion <u>parallax.</u>

Special Case: Pure Translation

To better understand what flow fields look like, let's just consider the case of pure translational motion.

Then, the flow is formed by projection of parallel velocity vectors in the scene



the 2D image

the 3D scene

Special Case I: Pure Translation

$$\omega = 0 \quad \Rightarrow \quad \begin{aligned} v_x &= \frac{T_z x - T_x f}{Z} \\ v_y &= \frac{T_z y - T_y f}{Z} \end{aligned}$$

Assume $T_z \neq 0$

Define:

$$p_o = \begin{bmatrix} x_o \\ y_o \\ f \end{bmatrix} = \begin{bmatrix} \frac{fT_x}{T_z} \\ \frac{fT_y}{T_z} \\ f \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} v_x &= \frac{T_z x - T_z x_o}{Z} = (x - x_o)\frac{T_z}{Z} \\ v_y &= \frac{T_z y - T_z y_o}{Z} = (y - y_o)\frac{T_z}{Z} \end{aligned}$$

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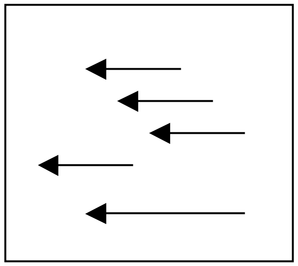
Special Case I: Pure Translation

$$\omega = 0 \quad \Rightarrow \quad v_x = \frac{T_z x - T_x f}{Z} \qquad v_y = \frac{T_z y - T_y f}{Z}$$

What if $T_z = 0$?

$$\Rightarrow \quad v_x = -f\frac{T_x}{Z} \qquad v_y = -f\frac{T_y}{Z}$$



All motion field vectors are parallel to each other and inversely proportional to depth !

TIE IN WITH SIMPLE STEREO!

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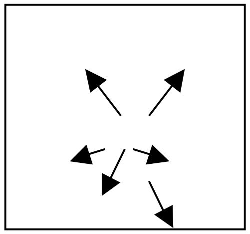
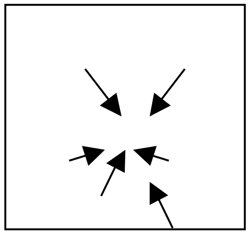
Special Case I: Pure Translation

$$v_x = (x - x_o)\frac{T_z}{Z}$$
$$v_y = (y - y_o)\frac{T_z}{Z}$$



$T_z > 0$



$T_z < 0$

The motion field in this case is RADIAL:

- It consists of vectors passing through $p_o = (x_o, y_o)$
- If:
  - $T_z > 0$, (camera moving towards object)
    - the vectors point away from p_o
    - p_o is the POINT OF EXPANSION
  - $T_z < 0$, (camera moving away from object)
    - the vectors point towards p_o
    - p_o is the POINT OF CONTRACTION

Pure Translation: Properties of the MF

- If $T_z \neq 0$ the MF is RADIAL with all vectors pointing towards (or away from) a single point p_o. If $T_z = 0$ the MF is PARALLEL.

- The length of the MF vectors is inversely proportional to depth Z. If $T_z \neq 0$ it is also directly proportional to the distance between p and p_o.

Pure Translation: Properties of the MF

- p_o is the vanishing point of the direction of translation.

- p_o is the intersection of the ray parallel to the translation vector and the image plane.

Motion Field vs Optic Flow

Motion Field: projection of 3D relative velocity vectors onto the 2D image plane.

Optic Flow: observed 2D displacements of brightness patterns in the image.

Motion field is what we want to know. Optic flow is what we can estimate.

Optic Flow ≠ Motion Field

Consider a moving light source:

MF = 0 since the points on the scene are not moving

OF ≠ 0 since there is a moving pattern in the images

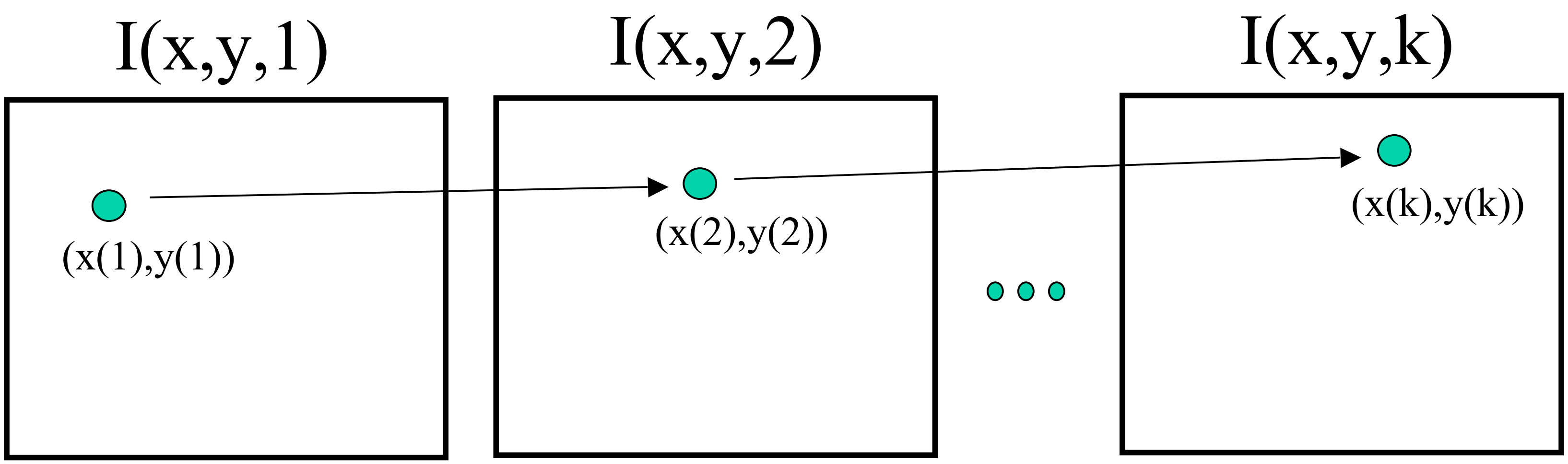
Approximating MF with OF

Nevertheless, we will estimate OF (since MF cannot really be observed!).

To avoid apparent flow due to changing illumination, assume that the apparent brightness of moving objects remains constant.

Brightness Constancy Equation

consider a scene point moving through an image sequence



claim: its brightness/color will remain the same (that's partly how we can recognize that it IS the same point)

$$\boxed{I\,(x(t), y(t), t) = Constant}$$

Brightness Constancy Equation

$$I\left(x(t), y(t), t\right) = Constant$$

Take derivative of both sides wrt time:

$$\frac{d\, I\left(x(t), y(t), t\right)}{dt} = 0$$

(using chain rule)

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Brightness Constancy Equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$\dfrac{\partial I}{\partial x}, \dfrac{\partial I}{\partial y}$ (spatial gradient; we can compute this!)

$\dfrac{dx}{dt}, \dfrac{dy}{dt}$ = (u, v) (optical flow, what we want to find)

$\dfrac{\partial I}{\partial t}$ (derivative across frames. Also known, e.g. frame difference)

Brightness Constancy Equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Becomes:

$$\boxed{(\nabla I)^T . v + I_t = 0}$$

Optical Flow is CONSTRAINED to be on a line !
(equation has for a u + b v + c = 0)

What is the practical implication of this?

Brightness Constancy Equation

$$I_x u + I_y v + I_t = 0$$

Known (spatial and temporal gradients)

Unknowns (components of flow vector)

Optical Flow is CONSTRAINED to be on a line !
(equation has for a u + b v + c = 0)

What is the practical implication of this?

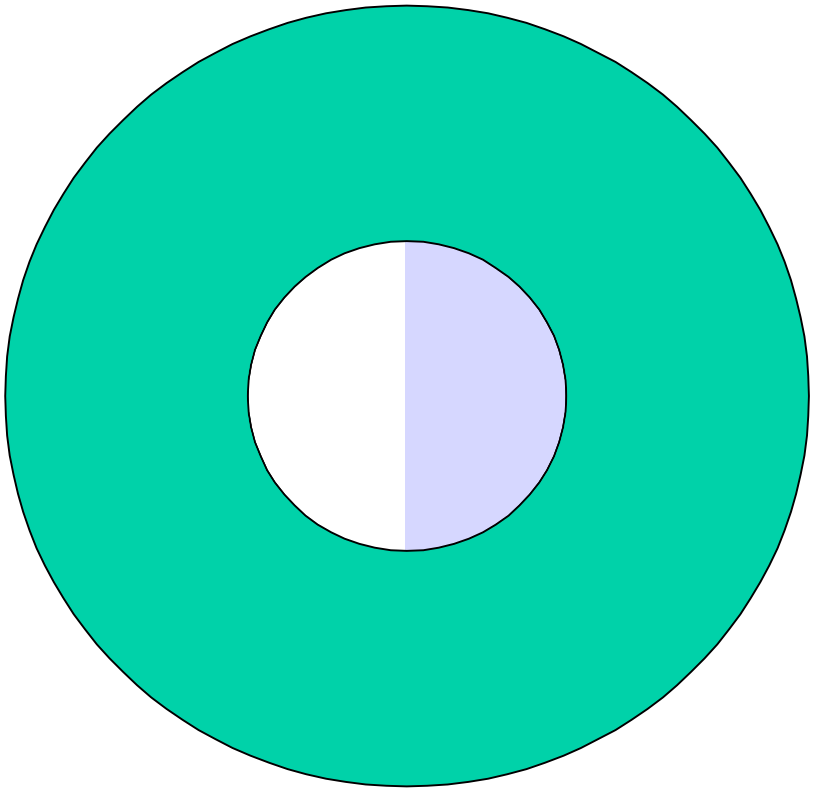
Implications

$$I_x u + I_y v + I_t = 0$$

- Q: how many unknowns and equations per pixel?

- Intuitively, this constraint means that
  - The component of the flow in the gradient direction is determined **(called Normal Flow)**
  - The component of the flow parallel to an edge is unknown

The Aperture Problem



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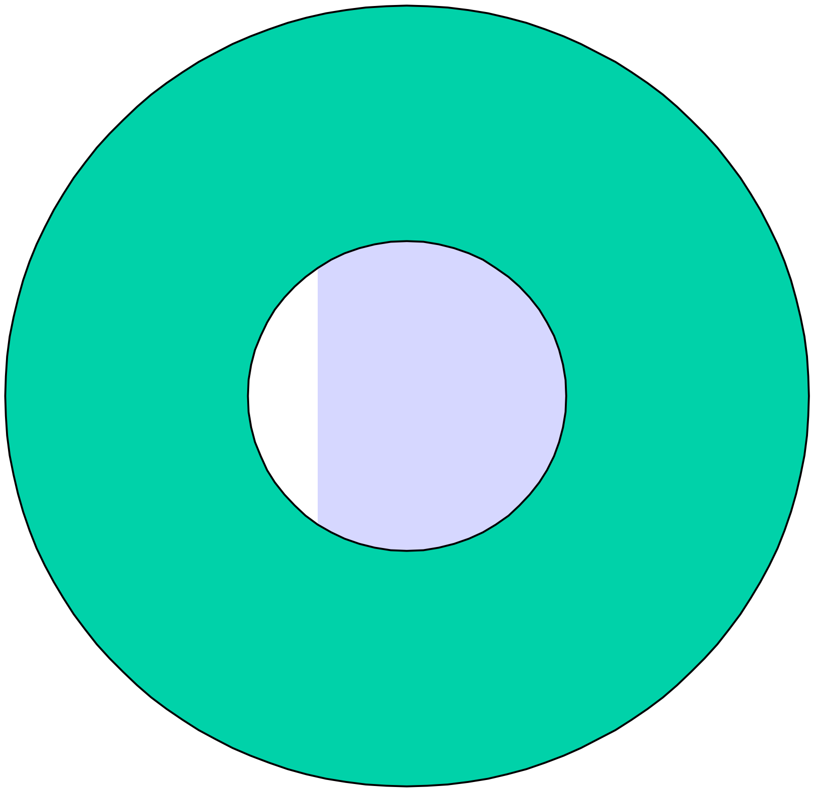
The Aperture Problem



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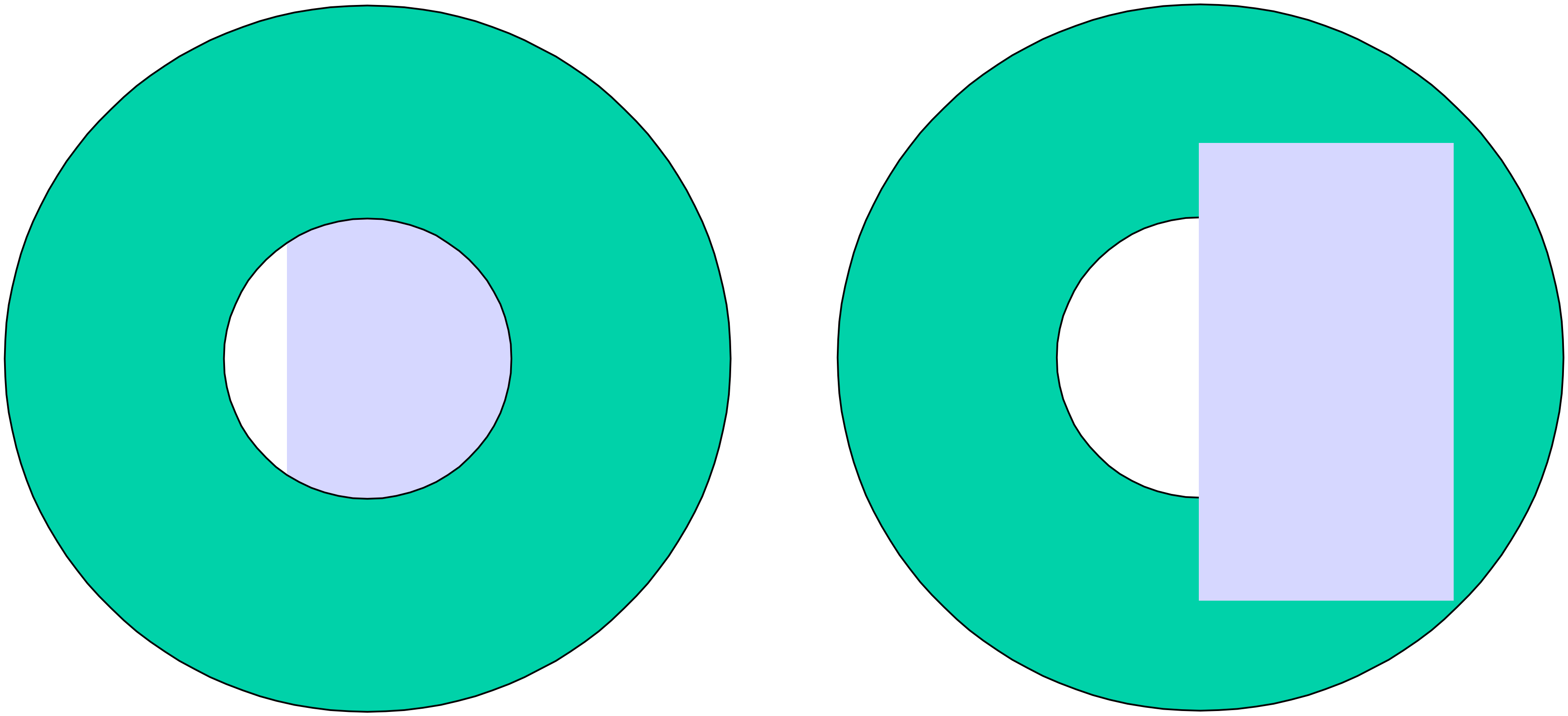
The Aperture Problem



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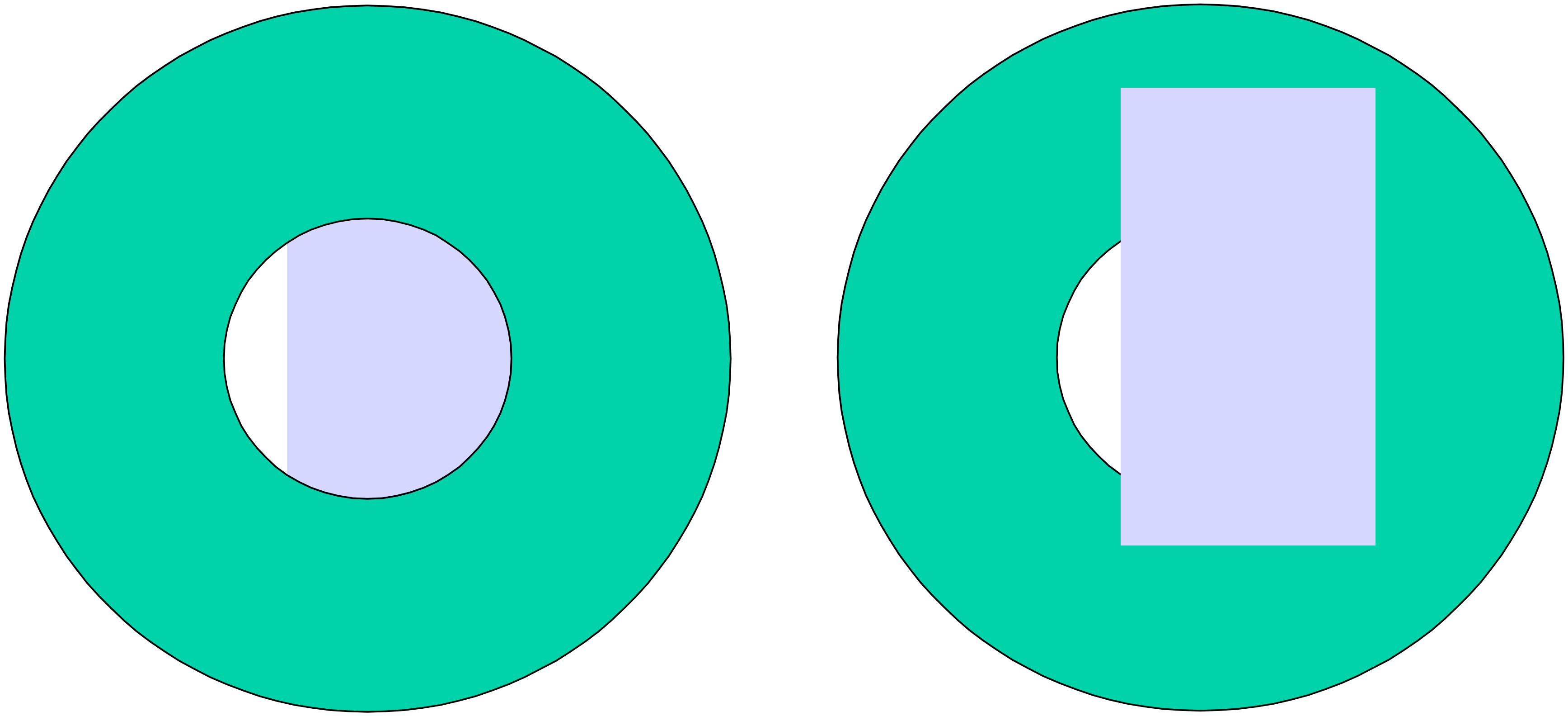
The Aperture Problem



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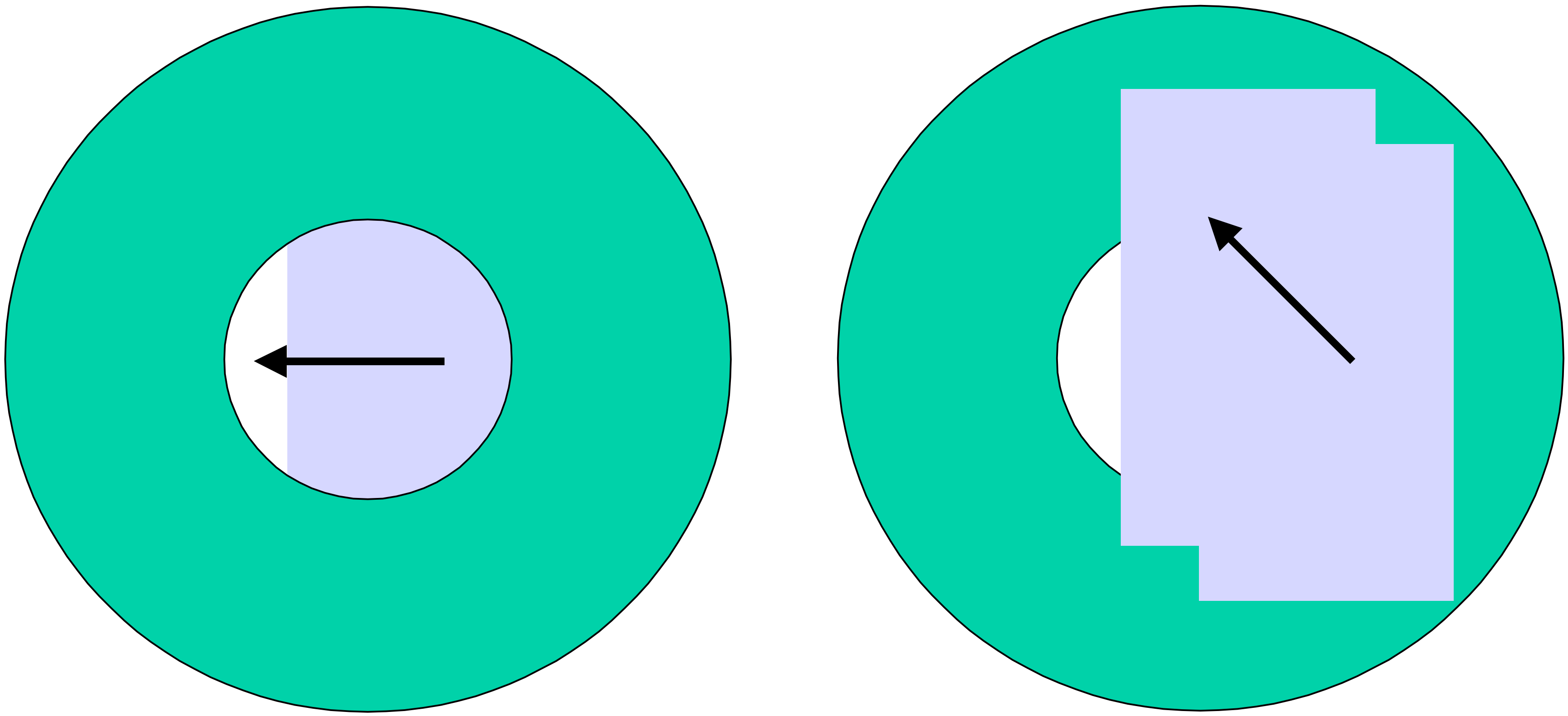
The Aperture Problem



The Image Brightness Constancy Assumption only provides the OF component in the direction of the spatial image gradient

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Aperature Problem

- Another example is the barber-pole illusion.

Consider a point

Optical Flow ≠ Motion Flow

- The aperature problem reminds us once again that optical flow is not the same thing as motion flow.
  - near an edge, we can only observe (measure) the component of flow perpendicular to the edge
  - cannot measure the component of flow parallel to the edge.
  - another case of non-observability of optical flow is in areas of constant intensity. No flow is observed.

Computing Optical Flow

- Algorithms for computing OF are of two types:
  - Differential Techniques
    - Based on spatial and temporal variations of the image brightness at all pixels.
    - Used to compute DENSE flow.
  - Matching Techniques
    - Similar to stereo feature matching, compute disparities.
    - Used to compute SPARSE flow.

A Differential Technique:
Constant Flow, aka Lucas-Kanade

Lucas-Kanade Motivation

$$I_x u + I_y v + I_t = 0$$

- Q: how many unknowns and equations per pixel?

- We need two or more pixels to solve.

- Due to aperature problem, we would like to include pixels with different gradient directions.

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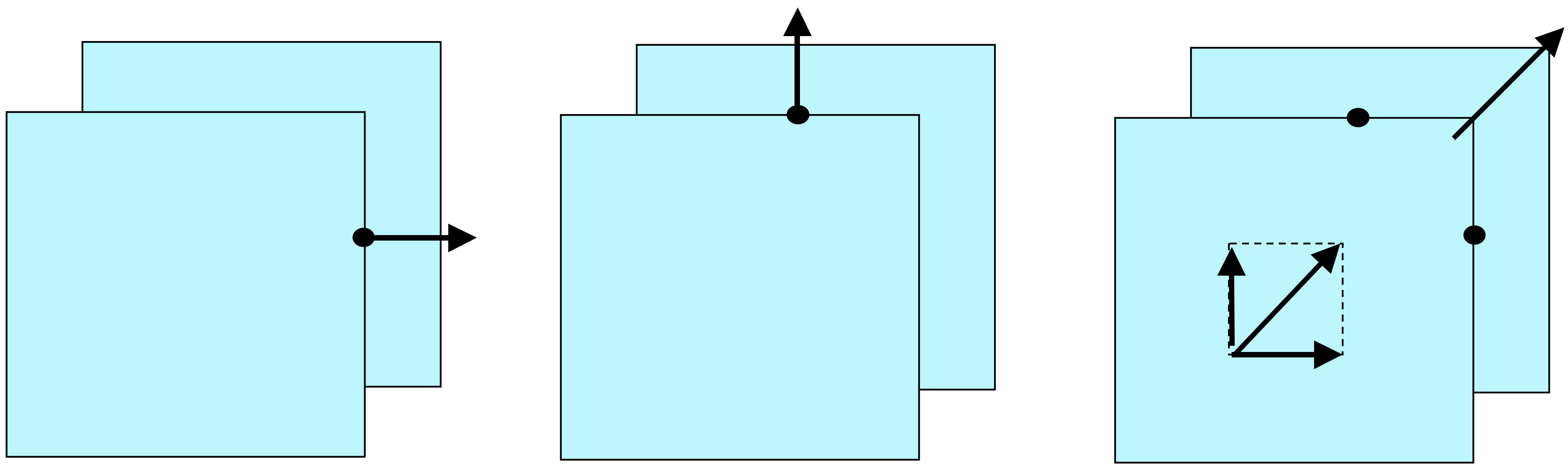
Solving the Aperature Problem

- Using gradients with two or more directions



aperature problem with single gradient direction

can infer correct solution using two or more directions

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Solving the aperture problem

recall: $I_x u + I_y v + I_t = 0$

- How to get more equations for a pixel?
  - Basic idea: impose additional constraints
    - most common is to assume that the flow field is smooth locally
    - e.g. pretend the pixel's neighbors have same displacement (u,v)
      - If we use a 5x5 window, that gives us 25 equations per pixel!

$$\underset{25\times 2}{\underbrace{\begin{bmatrix} I_x(\mathbf{p}_1) & I_y(\mathbf{p}_1) \\ I_x(\mathbf{p}_2) & I_y(\mathbf{p}_2) \\ \vdots & \vdots \\ I_x(\mathbf{p}_{25}) & I_y(\mathbf{p}_{25}) \end{bmatrix}}_{A}} \underset{2\times 1}{\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{d}} = - \underset{25\times 1}{\underbrace{\begin{bmatrix} I_t(\mathbf{p}_1) \\ I_t(\mathbf{p}_2) \\ \vdots \\ I_t(\mathbf{p}_{25}) \end{bmatrix}}_{b}}$$

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Lucas-Kanade flow

- We have more equations than unknowns: solve least squares problem. This is given by:

$$\underset{25\text{x}2}{A}\ \underset{2\text{x}1}{d} = \underset{25\text{x}1}{b} \longrightarrow \text{minimize } \|Ad - b\|^2$$

$$\overset{2\text{x}2}{(A^T A)}\ \overset{2\text{x}1}{d} = \overset{2\text{x}1}{A^T b}$$

$$\underbrace{\begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}}_{A^T A} \begin{bmatrix} u \\ v \end{bmatrix} = - \underbrace{\begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}}_{A^T b}$$

– Summations over all pixels in the KxK window

$A^T A$

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Conditions for solvability

– Optimal (u, v) satisfies Lucas-Kanade equation

$$\underbrace{\begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}}_{A^T A} \begin{bmatrix} u \\ v \end{bmatrix} = - \underbrace{\begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}}_{A^T b}$$

When is This Solvable?

- **A^{T}A** should be invertible
- **A^{T}A** should not be too small due to noise
  – eigenvalues λ_1 and λ_2 of **A^{T}A** should not be too small
- **A^{T}A** should be well-conditioned
  – λ_1/ λ_2 should not be too large (λ_1 = larger eigenvalue)

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Does $A^T A$ seem familiar?

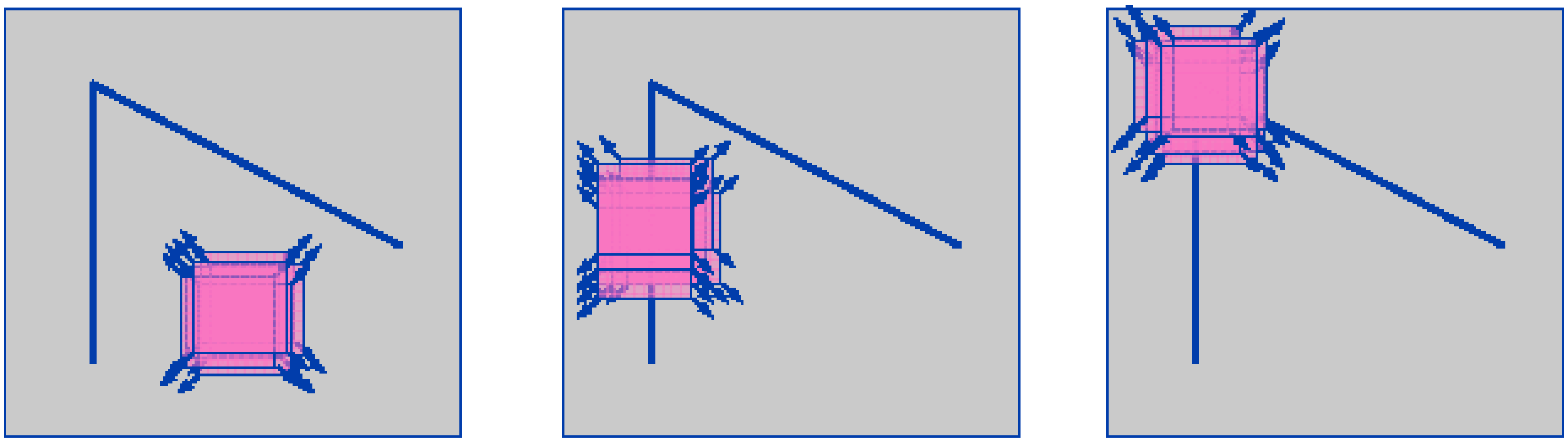
Look at the matrix:

$$\begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$

Harris Corner
Detector Matrix!

Review: Harris Corner Detector

In Lecture 10 we derived the Harris corner detector by considering SSD of shifted intensity patches…



"flat" region:
no change in
all directions

"edge":
no change along
the edge direction

"corner":
significant change
in all directions

Tie in Harris with Motion Analysis

$$E(u,v) = \sum_{x,y} w(x,y)\left[I(x+u,y+v) - I(x,y)\right]^2$$

Window function

Shifted intensity

Intensity

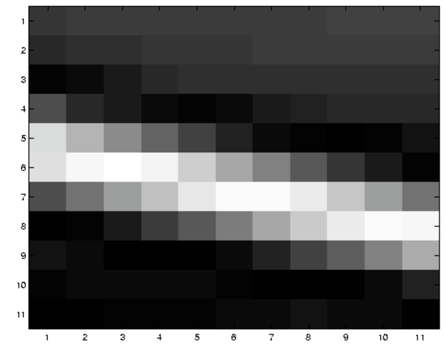
Current frame

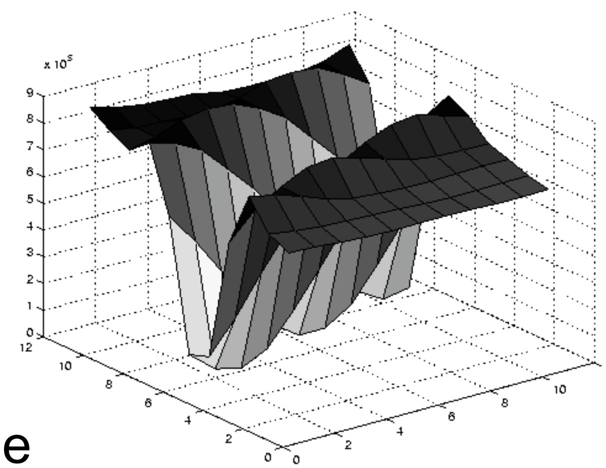
Previous frame

Was spatial shift in single frame. Now is displacement across two frames in time.

Tie in with Harris

Edge





$$\sum \nabla I (\nabla I)^T$$

– large gradients, all the same

– large λ_1, small λ_2

Will have aperature problem

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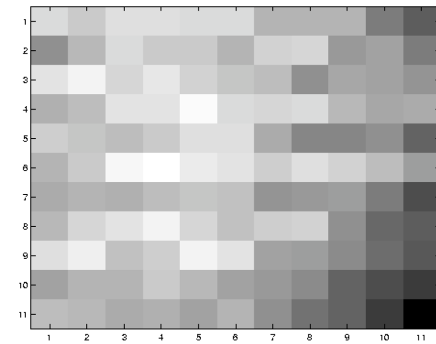
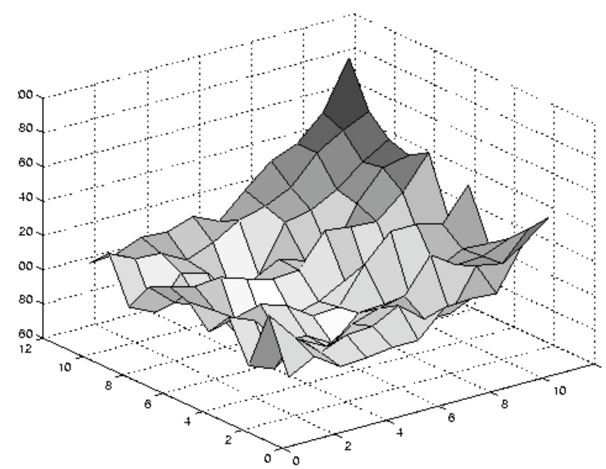
Tie in with Harris

Low Texture Region





$$\sum \nabla I (\nabla I)^T$$

– gradients have small magnitude
– small λ_1, small λ_2

Ill-conditioned matrix. Likely to compute garbage answer.

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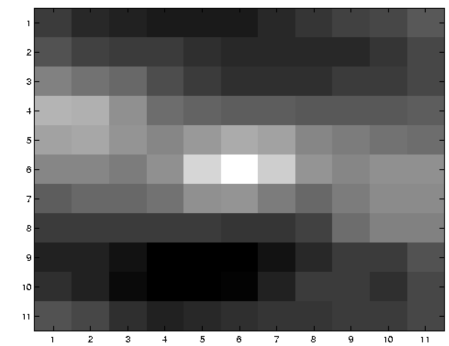
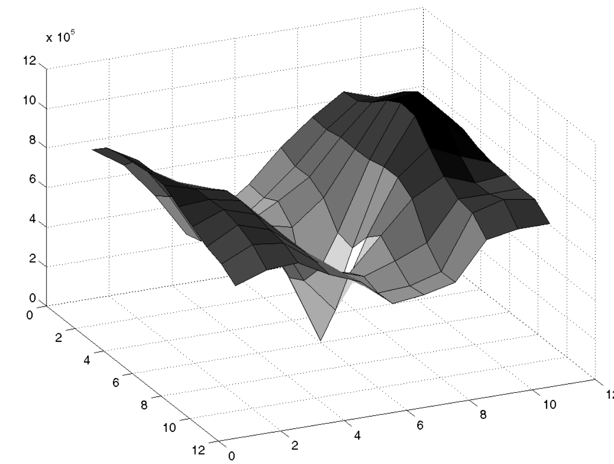
Tie in with Harris

Corner





$$\sum \nabla I (\nabla I)^T$$

- gradients are different, large magnitudes
- large λ_1, large λ_2

Good corner feature. Also good place to estimate flow!

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Implications

- Corners are when $\lambda 1$, $\lambda 2$ are big; this is also when Lucas-Kanade works best.
- Corners are regions with two different directions of gradient (at least).
- Aperture problem disappears at corners.
- Corners are good places to compute flow!

Feature-based Techniques

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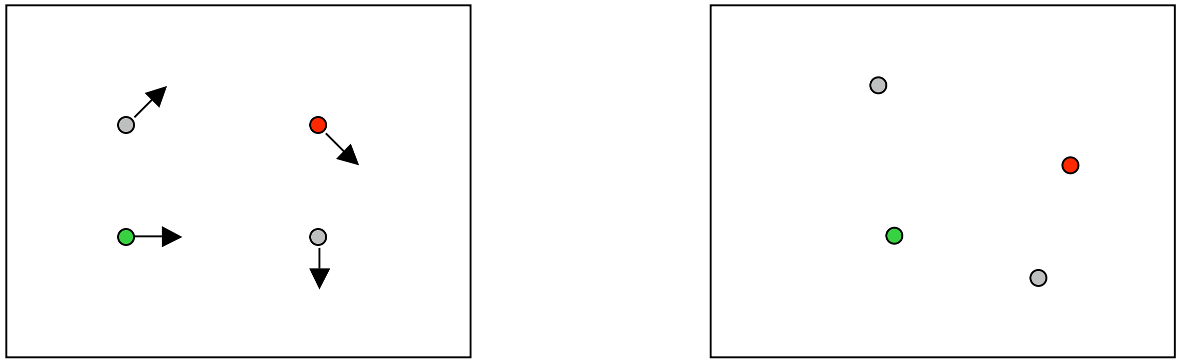
Feature Matching for Sparse Flow



I(x,y,t) I(x,y,t+1)

General idea:

Find Corners in one image (because these are good areas to estimate flow)

Search for Corresponding intensity patches in second image.

KLT Algorithm

Public-domain code by Stan Birchfield.
Available from

http://www.ces.clemson.edu/~stb/klt/

Tracking corner features through 2 or more frames

KLT Algorithm

1. Find corners in first image
2. Extract intensity patch around each corner
3. Use Lucas-Kanade algorithm to estimate constant displacement of pixels in patch

Niceties

1. Iteration and multi-resolution to handle large motions
2. Subpixel displacement estimates (bilinear interp warp)
3. Can track feature through a whole sequence of frames
4. Ability to add new features as old features get “lost”

Correlation-Based Matching

- Another approach is to match intensity patches using correlation or NCC.
  - We talked about this for stereo as well:

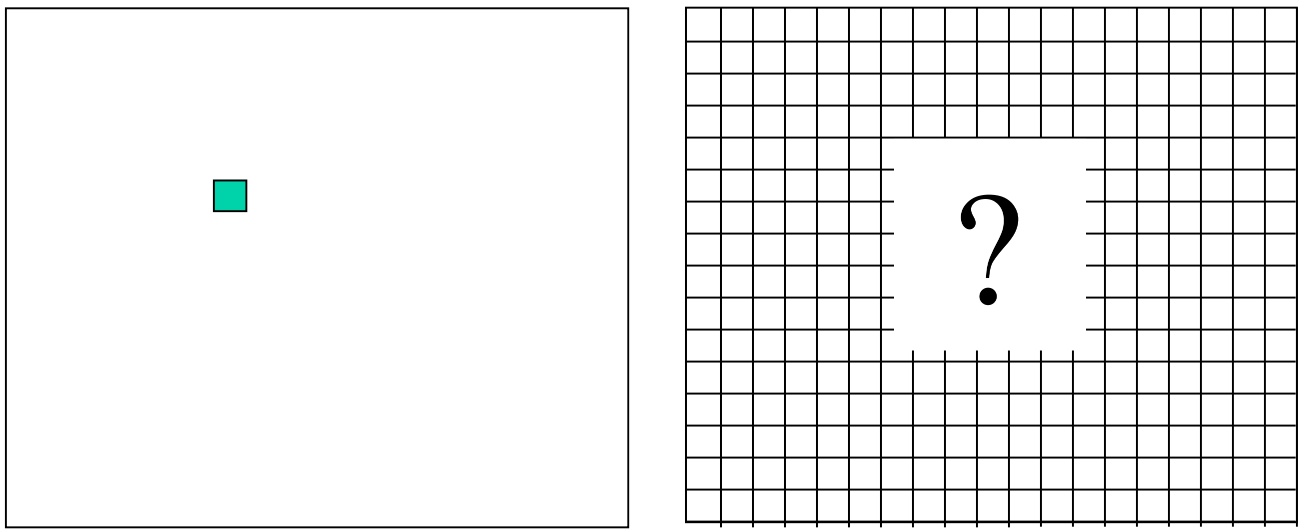
Correlation-based Matching

1. Find corners in first image
2. Extract intensity patch around each corner
3. Use NCC to compute match scores within the search window in second image
4. Identify highest score (best match)

Important Note on Efficiency

Given image patch in one image

We don’t want to search everywhere
in the second image for a match.



Important Note on Efficiency

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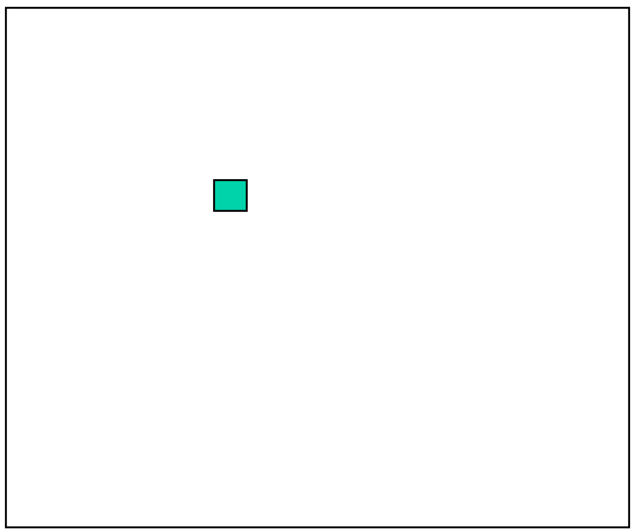
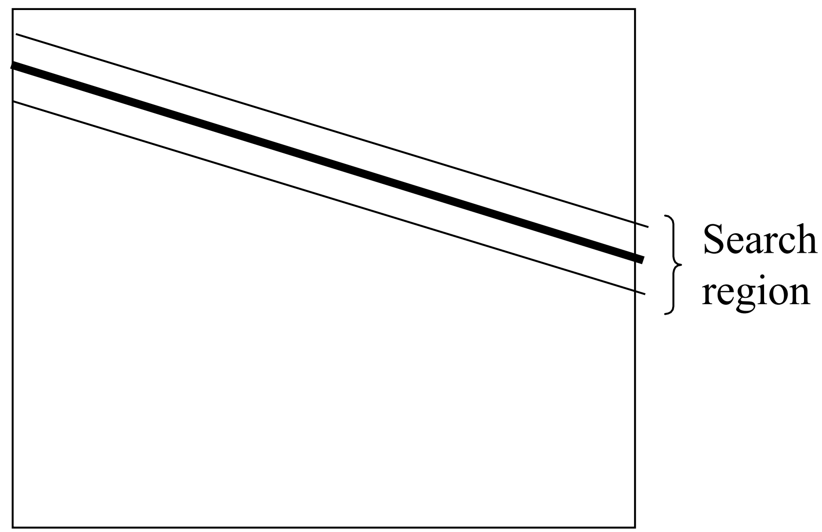


We don't want to search everywhere in the second image for a match.

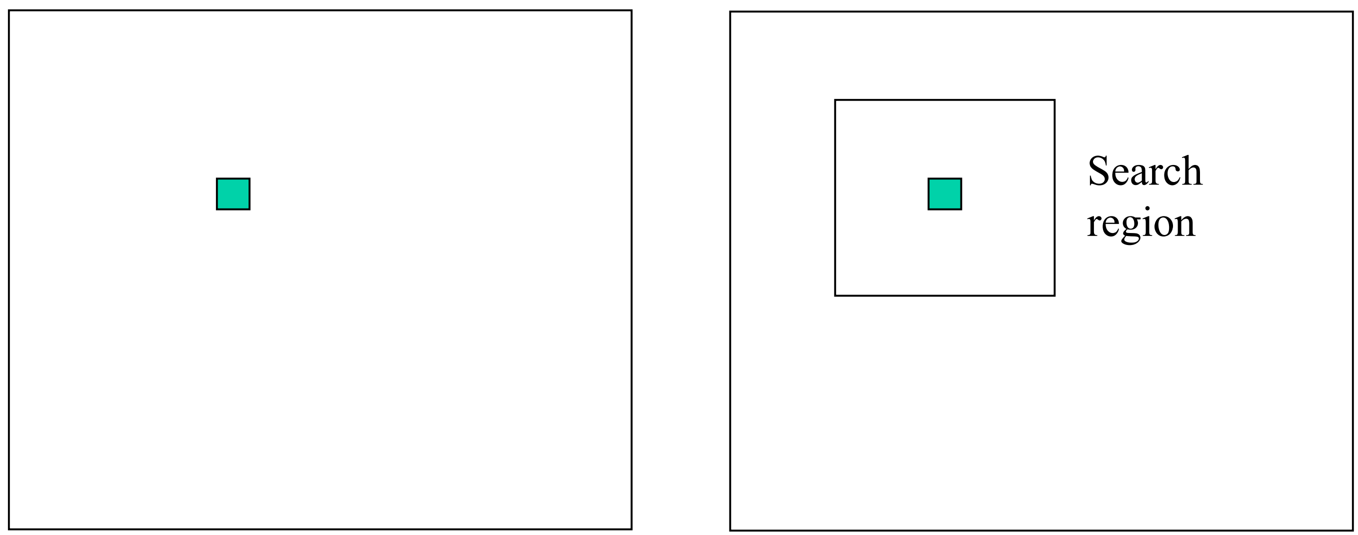


With stereo, we had an epipolar line constraint.

Important Note on Efficiency

Given image patch in one image

We don’t want to search everywhere in the second image for a match.



We don’t know the relative R,T here, so no epipolar constraint.

But… motion is known to be “small”, so can still bound the search region.

Note on using Normalized Correlation

If we use normalized correlation to match feature patches, we can relax the constant brightness assumption!

We thus remove some potential error sources
(changes in illumination or camera gain)

Another Correlation-based Algorithm

Review:

Due to David Nister, "Visual Odometry", CVPR 2004

Observation: corners in one image tend to stay corners over short periods of time.

Therefore, we only need to match corners to corners.

Nister's Algorithm

1. Find corners in first image
2. Extract intensity patch around each corner
3. Find corners in second image
4. Extract intensity patch around each of them
5. Find matching pairs (c1,c2) such that
    - C1 is a corner patch from image 1
    - C2 is a corner patch from image 2
    - C2 is the best match for C1
    - C1 is the best match for C2

Nister's Algorithm

1. Find corners in first image
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5. Find matching pairs (c1,c2) such that
   - C1 is a corner patch from image 1
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   - C2 is the best match for C1
   - C1 is the best match for C2

Question: Why do this too?

Answer: Approx solution to the linear assignment problem -- you get better matches.

Linear Assignment Problem

Aka the "Marriage Problem"

Simplest form:
1) given k boys and k girls
2) ask each boy to rank the girls in order of desire
3) ask each girl to also rank order the boys
4) The marriage problem determines the pairing of boys and girls to maximize sum of overall pairwise rankings.

Note: in general you may not get your most desireable spouse, but on average you rarely get the least.

Linear Assignment Problem

The optimal solution to the LAP is too computationally intensive for real-time feature matching/tracking.

Therefore, we use a heuristic solution where features can pair up only if each is the best match for the other (i.e. boy-girl pairs that each listed each other as #1)

This throws away a lot of potential point matches, but the ones that are left are usually pretty good.