The Kirchhoff-Love thin plate problem

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1 Introduction

This document describes the strong- and associated weak form of the linear thin plate problem based on Kirchhoff-Love plate theory, which is included in the linear elasticity application of *IFEM*. Refer to the class `KirchhoffLovePlate` of the source code for the actual integrand implementation.

2 Strong form

Given a distributed transverse load $p(x,y)$ defined over a domain $\Omega \subset \mathbb{R}^2$, a bending moment $\bar{M}(x,y)$ and twist moment $\bar{T}(x,y)$ defined over the boundary $\partial\Omega_m$, a transverse shear force $\bar{Q}(x,y)$ defined over the boundary $\partial\Omega_q$, and two functions $\bar{w}(x,y)$ and $\bar{\theta}(x,y)$ defined over the boundaries $\partial\Omega_w = \partial\Omega \setminus \partial\Omega_q$ and $\partial\Omega_\theta = \partial\Omega \setminus \partial\Omega_m$, respectively, find the scalar function $w(x,y) \in \mathcal{W}(\Omega)$ satisfying

$$\left.\begin{aligned}
\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} &= -p \\
m_{xx} &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\
m_{yy} &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\
m_{xy} &= -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}
\end{aligned}\right\} \quad \forall \quad \{x,y\} \in \overline{\Omega} \tag{1}$$

$$\left.\begin{aligned}
m_{xx}n_x + m_{xy}n_y &= \bar{M} \\
m_{xy}n_x + m_{yy}n_y &= \bar{T}
\end{aligned}\right\} \quad \forall \quad \{x,y\} \in \partial\Omega_m \tag{2}$$

$$\left(\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y}\right)n_x + \left(\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y}\right)n_y = \bar{Q} \quad \forall \quad \{x,y\} \in \partial\Omega_q \tag{3}$$

$$w = \bar{w} \quad \forall \quad \{x,y\} \in \partial\Omega_w \tag{4}$$

$$\frac{\partial w}{\partial x}n_x + \frac{\partial w}{\partial y}n_y = \bar{\theta} \quad \forall \quad \{x,y\} \in \partial\Omega_\theta \tag{5}$$

where $D = \frac{Et^3}{12(1-\nu^2)}$ is the plate stiffness parameter composed of the Young's modulus E, the Poisson's ratio ν, and the plate thickness t. The two terms inside the parentheses of Equation (3) equals the transverse shear forces, q_x and q_y, respectively, and n_x and n_y are the components of the outward-directed unit normal vector on $\partial\Omega$. The solution space $\mathcal{W}(\Omega)$ defines the set of all admissible solution functions w on the domain Ω, with the additional constraints $w = \bar{w}(x,y)\ \forall\{x,y\} \in \partial\Omega_w$ and $\frac{\partial w}{\partial x}n_x + \frac{\partial w}{\partial y}n_y = \bar{\theta}(x,y)\ \forall\{x,y\} \in \partial\Omega_\theta$. Assuming the plate stiffness D is constant, the Equations (1) can be combined into the following fourth-order partial differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \quad \forall \quad \{x,y\} \in \overline{\Omega} \tag{6}$$

Similarly, Equation (3) can be transformed by substituting the expressions for m_{xx}, m_{yy} and m_{xy} from Equation (1), resulting in

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)n_x + \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)n_y = -\frac{\bar{Q}}{D} \quad \forall \quad \{x,y\} \in \partial\Omega_q \tag{7}$$

3 Weak form

To develop the weak form, it is convenient to introduce the shortened notation for partial differentiation $(\cdot)_{,\alpha} := \frac{\partial(\cdot)}{\partial x_\alpha}$, where α is a running index over the coordinate directions ($\alpha = 1 \ldots 2$), and assuming Einsteins summation convention over repetitive indices, e.g., $a_{\alpha\alpha} := a_{11} + a_{22}$. With $\bar{M}_1 := \bar{M}$ and $\bar{M}_2 := \bar{T}$, the Equations (6), (2) and (7), respectively, can then be written in the following compact form

$$
\begin{aligned}
D\, w_{,\alpha\alpha\beta\beta} &= p \quad \forall \quad x_\alpha \in \overline{\Omega} && (8)\\
m_{\alpha\beta} n_\beta &= \bar{M}_\alpha \quad \forall \quad x_\alpha \in \partial\Omega_m && (9)\\
D\, w_{,\alpha\alpha\beta} n_\beta &= -\bar{Q} \quad \forall \quad x_\alpha \in \partial\Omega_q && (10)
\end{aligned}
$$

The weak form is obtained by multiplying Equation (8) by a test function $v(x_\alpha) \in \mathcal{V}(x_\alpha)$ and then integrating over the domain Ω, viz.

$$D \int_\Omega w_{,\alpha\alpha\beta\beta}\, v \,\mathrm{d}A = \int_\Omega p\, v \,\mathrm{d}A \tag{11}$$

The test space $\mathcal{V}(x_\alpha)$ is the same as $\mathcal{W}(x_\alpha)$, except that their functions $v(x_\alpha)$ have the constraint $v = 0 \;\forall\; x_\alpha \in \Omega_w$ and $v_{,\alpha} n_\alpha = 0 \;\forall\; x_\alpha \in \partial\Omega_\theta$ instead of those of $\mathcal{W}(x_\alpha)$.

By applying the Green's identity (integration by parts), Equation (11) is transformed to

$$-D \int\limits_\Omega w_{,\alpha\alpha\beta}\, v_{,\beta} \,\mathrm{d}A + D \int\limits_{\partial\Omega} w_{,\alpha\alpha\beta}\, n_\beta\, v \,\mathrm{d}S = \int\limits_\Omega p\, v \,\mathrm{d}A \tag{12}$$

The boundary integral of the second term can be further transformed by using that $v(x_\alpha) = 0 \;\forall\; x_\alpha \in \Omega_q$ and substituting Equation (10), resulting in

$$-D \int\limits_\Omega w_{,\alpha\alpha\beta}\, v_{,\beta} \,\mathrm{d}A = \int\limits_\Omega p\, v \,\mathrm{d}A + \int\limits_{\partial\Omega_q} \bar{Q}\, v \,\mathrm{d}S \tag{13}$$

The left-hand-side term is next transformed by applying Green's identity a a second time, resulting in

$$D \int\limits_\Omega w_{,\alpha\beta}\, v_{,\alpha\beta} \,\mathrm{d}A - D \int\limits_{\partial\Omega} w_{,\alpha\beta}\, n_\alpha\, v_{,\beta} \,\mathrm{d}S = \int\limits_\Omega p\, v \,\mathrm{d}A + \int\limits_{\partial\Omega_q} \bar{Q}\, v \,\mathrm{d}S \tag{14}$$

or simply

$$a(w, v) \;=\; l(v) \tag{15}$$

where we introduce the bilinear form $a(w, v)$ and the linear functional $l(v)$ as

$$
\begin{aligned}
a(w, v) &:= D \int\limits_\Omega w_{,\alpha\beta}\, v_{,\alpha\beta} \,\mathrm{d}A \;=\; - \int\limits_\Omega m_{\alpha\beta}\, v_{,\alpha\beta} \,\mathrm{d}A && (16)\\
l(v) &:= \int\limits_\Omega p\, v \,\mathrm{d}A + \int\limits_{\partial\Omega_q} \bar{Q}\, v \,\mathrm{d}S + \int\limits_{\partial\Omega_m} \bar{M}_\alpha\, v_{,\alpha} \,\mathrm{d}S && (17)
\end{aligned}
$$

4 Energy norms

The computed finite element (FE) solution can be assessed by evaluating norm. For a given FE solution w^h, we therefore define its energy norm as

$$U^h = \|w^h\|_E := \sqrt{a(w^h, w^h)} \tag{18}$$

and the corresponding external energy is

$$U^h_{ext} = \sqrt{l(w^h)} \tag{19}$$

The FE implementation can therefore be verified by always asserting that $U^h = U^h_{ext}$ for any problem setup. Notice that in the evaluation of Equation (19), we do not use the primary solution, w^h, but only the secondary solution, $m^h_{\alpha\beta}$, through Equation (17).

5 Error estimates

An estimate of the discretization error in the FE solution can be obtained by projecting the discontinuous secondary solution, $m^h_{\alpha\beta}$ onto a continuous basis of higher order (or same order as the primary solution, w^h), resulting in the recovered solution, $m^*_{\alpha\beta}$. In *IFEM*, several projection methods for obtaining $m^*_{\alpha\beta}$ are available, e.g., discrete and continuous global L_2-projection.

With $m^*_{\alpha\beta}$ available, the discretization error is estimated by

$$\eta^{RES} = \|w^* - w^h\|_E + \sqrt{\sum_{k=1}^{n_{\rm el}} \left\{ h_k^2 \|R^*\|^2_{L_2(\Omega_k)} + \frac{h_k}{2} \left(\|J^*_m\|^2_{L_2(\partial\Omega_{m\,k})} + \|J^*_q\|^2_{L_2(\partial\Omega_{q\,k})} \right) \right\}} \tag{20}$$

where h_k denotes some characteristic size of element k (typically the length of the longest diagonal). R^*, J^*_m, J^*_q are, respectively, the interior residual of the recovered solution $m^*_{\alpha\beta}$, and the jump (boundary residual) between the recovered solution and the prescribed edge moment $\bar{M}_\alpha$ and shear force $\bar{Q}$. The notation $\|\cdot\|_{L_2(\Omega_k)}$ denotes the L_2-norm of the quantity $(\cdot)$ over the sub-domain Ω_k of element k, i.e.

$$\|\cdot\|^2_{L_2(\Omega_k)} := \int\limits_{\Omega_k} (\cdot)^2 \,\mathrm{d}A \quad \text{and} \quad \|\cdot\|^2_{L_2(\partial\Omega_k)} := \int\limits_{\partial\Omega_k} (\cdot)^2 \,\mathrm{d}S \tag{21}$$

The interior residual is computed by inserting $m^*_{\alpha\beta}$ into Equation $(1)_1$:

$$R^* = m^*_{\alpha\beta,\alpha\beta} - p \tag{22}$$

The jump terms are computed from the Neumann boundary conditions, Equations (9) and (10), respectively: