Homework 1

Problem 1.1. *Read "Exposition of a New Theory on the Measurement of Risk" by Daniel Bernoulli, and then "Risk Aversion in the Small and in the Large" by John W. Pratt.*

Solution 1.1. Done.

Problem 1.2. *Prove that an agent is risk-averse, ie. inequality*

$$E[u(w+\tilde{z})] \leq u(w+E[\tilde{z}])$$

holds for all w and $\tilde{z}$, if and only if $u()$ is concave.

Solution 1.2. Jensen's inequality states that, for any real-valued function ϕ, Jensen's inequality states that $E\phi(\tilde{y}) \leq \phi(E[\tilde{y}])$ for any random variable $\tilde{y}$ if and only if ϕ is concave. Apply this with the random variable $\tilde{y} = w + \tilde{z}$, and $\phi = u$, we get the desired conclusion.

Problem 1.3. *Show that when $u(w) = -\exp(-aw)/a$ and $\tilde{w}$ is normally distributed with mean μ and variance σ^2, the Arrow-Pratt approximation is exact and indeed,*

$$E[u(\tilde{w})] = u\left(\mu - \frac{1}{2}a\sigma^2\right)$$

Solution 1.3. Well, the Arrow-Pratt approximation states

$$\Pi \approx \frac{1}{2}\sigma^2 A(w) \quad \text{where} \quad A(w) := -\frac{u''(w)}{u'(w)}$$

so we need to prove that in this case $\Pi = \frac{1}{2}\sigma^2 A(w)$. In fact it's trivial from simple calculus that $A(w) = a$ for the supplied utility function, so what we must show is that $\Pi = a\sigma^2/2$. To show this, we go back to the definition of Π, which defines Π as the solution to the equation $E[u(w+\tilde{z})] = u(w - \Pi)$. Plugging in the utility function, this equation becomes

$$-a^{-1}\exp(-a(w-\Pi)) = -a^{-1}E[\exp(-a(w+\tilde{z}))]$$

which we must solve for Π. Let's first simplify by canceling the constants and taking logs:

$$\Pi = w + a^{-1}\log E[\exp(-a(w+\tilde{z}))]$$

We know how to calculate the mean of a lognormal random variable. (You need this to derive Black-Scholes!) so the rest is just algebra from here.

Problem 1.4. *Describe a trading strategy which looks good from a purely mean-variance (Sharpe ratio) perspective, but which has a left tail that is scary enough that it would be foolish to invest in the strategy. Provide some calculations to justify your claims.*

Solution 1.4. Suppose we sold a deep out-of-the-money put on every stock in the Russell 3000 expiring in about 30 days, and we held all of these options to maturity, at which point we just do the same thing again. In calm market conditions, most of the puts expire worthless and you collect a premium, and in a sequence of calm months the Sharpe ratio would presumably be quite good, but only a great fool would espouse this as a viable strategy.

More generally, when evaluating a strategy, I usually ask myself whether the manager appears to have an informational advantage, a speed advantage, or both. A relatively weak form of market efficiency holds that if a strategy appears to have a high Sharpe ratio without meaningful drawdowns, even though the managers aren't doing anything particularly clever or sophisticated, then you've missed something, such as a hidden tail risk (or a Ponzi scheme – for example the options strategy which Madoff *claimed* was responsible for his profits wasn't particularly clever).

One could think of this as a venn diagram – the three-way intersection of ObviousIdea and EasyToImplement and HighSharpeLowDrawdown is the empty set. When evaluating strategies, it's good to be open-minded, but not so open that your brain falls out.

Problem 1.5. *Consider a function $v(.)$ such that $v(x) = a + bu(x)$ for all x, for some pair of scalars a and b, where $b > 0$. Show that a decision-maker with utility function $v()$ makes the same decisions and has the same certainty-equivalents as a decision maker with utility function $u()$.*

Solution 1.5. The certainty-equivalent e of a lottery is defined by

$$E[u(w + \tilde{z})] = u(w + e)$$

Replacing u with v one has

$$a + bE[u(w + \tilde{z})] = a + bu(w + e)$$

due to the linearity of the expectation. This holds if and only if the first equation holds, so the two agents have the same certainty-equivalent payoff for every lottery.

Now consider an agent comparing two lotteries, $\tilde{z}_1$ and $\tilde{z}_2$. The certainty-equivalents e_i are defined by $E[u(w+\tilde{z}_i)] = u(w+e_i)$ for $i = 1, 2$. It follows that the agent assigns higher utility to lottery 1 than lottery 2 (and hence prefers lottery 1) if and only if $e_1 > e_2$. Hence, if two agents assign the same certainty equivalents to all lotteries, they also have the same preference relation when comparing lotteries.