The vanna - volga method for derivatives pricing.

Agnieszka Janek

Wroclaw University of Technology

July 2011

THE VANNA-VOLGA METHOD FOR DERIVATIVES PRICING

Agnieszka Janek

supervisor: dr hab. Rafał Weron, prof. PWr

Contents

Chapter 1

Introduction

The Foreign Exchange (FX) market is the most liquid over-the-counter (OTC) financial market in the world. At the same time, this is also the largest market for options. Traded derivatives range from single plain vanilla options and first-generation exotics (such as barrier options) to second- or even third-generation exotics. The most complex exotics have no closed-form formulas and some of them are hybrid products. Since after the credit crunch simple and less risky instruments are in high demand, it is very important to provide a fast and accurate pricing methodology for first-generation exotics.

It is possible to calculate analytically the values of vanillas or barrier options using the Black-Scholes model, however, they are far from market quotations. This is because the model is based on an unrealistic assumption that both currency risk-free rates and the volatility remain constant throughout the lifetime of the option. Thus the Black-Scholes model becomes insufficient in the highly volatile world of FX derivatives, in which the model-implied volatilities (see definition A.1 in Appendix A) for different strikes and maturities of options tend to be smile shaped or skewed.

Fore these reasons researchers have tried to find extensions of the model, that could explain this empirical fact. More realistic models assumed that the foreign/domestic interest rates and/or the volatility followed stochastic processes (Bossens et al., 2010). However, assuming constant interest rates for short-dated options (typically up to 1 year) does not normally lead to significant mispricing. As in this thesis we deal mostly with this kind of FX options, we further assume constant interest rates.

Stochastic volatility models can explain the smile shape but their main drawback is that they are computationally demanding and they require relatively lots of market data in order to find the value of parameters that allow the model to reproduce the market dynamics (Bossens et al., 2010). This fact has led to introducing an alternative method, which gives faster results and requires only three available volatility quotes for a given maturity. The *vanna-volga method*, also known as the *trader's rule of thumb*, is intuitive and easy to implement, however not necessarily accurate for each instrument. Nevertheless, it produces reasonable estimates of the market prices of plain vanilla options and, suitably adjusted, of first-generation exotics (Fisher, 2007).

In a nutshell, the vanna-volga (VV) method is based on adding an analytically derived correction to the Black-Scholes price of the instrument. The method constructs a hedging portfolio that zeroes out the Black-Scholes greeks that measure option's sensitivity with

respect to the volatility, i.e. the vega, vanna and volga of the option. In this way the VV method produces smile-consistent values.

The first appearance of the VV approach in the literature dates back to Lipton and McGhee (2002) and Wystup (2003), who consider the vanna-volga method as an empirical adjustment for the pricing of double-no touch options or one-touch options, respectively. However, their analyses are rather informal. The first systematic formulation of the VV method with its justification was proposed by Castagna and Mercurio (2007). Fisher (2007) and Bossens et al. (2010) introduce a number of corrections to handle the pricing inconsistencies of the first-generation exotics. Lastly, a more theoretical justification of the vanna-volga method extended to deal with the stochasticity in interest rates is given by Shkolnikov (2009).

This thesis highlights some basic features and applications of the vanna-volga method and its accuracy when pricing plain vanillas and simple barrier options. It is structured as follows. Chapter 2 introduces the vanna-volga method. In Section 2.1 we make basic assumptions about the market behaviour and the FX option that are further utilized in the justification for the vanna-volga method. Next, in Section 2.2 we construct the replicating portfolio whose hedging costs are added to the Black-Scholes option premium. This Section is central to our further considerations. Further, Chapter 3 develops on the VV method and its variants. In Sections 3.1 and 3.2 we derive formulas for premiums of vanilla FX options using two versions of the vanna-volga method – the exact vanna-volga method and the simplified vanna-volga method. In Section 3.3 we review a very common vanna-volga variation used to price the first-generation exotics. Section 3.4 is devoted to the application of the vanna-volga method to construct the implied volatility surface.

Chapter 4 considers possible alternative approaches and extensions to the vanna-volga method. In Section 4.1 we describe a simple adaptation that allows the vanna-volga method to produce prices of the first-generation exotics reasonably in line with those quoted in the market. This is an extension of the method described in Section 3.3. In Section 4.2 we briefly discuss a popular stochastic volatility model that aims to take the *smile* (see defnition A.2 in Appendix A) effect into account – the Heston model. Its accuracy and efficiency is further compared with that of the vanna-volga method.

Finally, Chapter 5 is devoted to calibration results. Firstly, in Section 5.1 we review market data adjustments necessary for calibrations. In Section 5.2 we move on to the pricing of plain vanilla options using the vanna-volga method. Then, in Section 5.3 we show that the smile of vanilla options can be reproduced using the vanna-volga method. In Sections 5.2 and 5.3 the results obtained by the exact vanna-volga method, the simplified vanna-volga method and the Heston model are compared. Section 5.4 investigates the accuracy of the vanna-volga method applied to barrier options. This Section compares the results provided by the three methods described in Sections 3.3 and 4.1.

The final Chapter 6 brings to a conclusion the main issues raised in the thesis but also considers possible applications of the vanna-volga method in other than FX markets. The plots and graphs in this thesis were produced by programs implemented by the author in *Matlab*. The programs are available on request. Last but not least, all the definitions and market conventions necessary for a full understanding of the thesis have been collected in Appendix A.

Chapter 2

Justification for the vanna-volga method

2.1 Basic assumptions for the FX options

Before we present a justification for the vanna-volga method we need to make a couple of assumptions about the market and the option itself:

- The option is maturing at time $t = T$ and this is a European style option, which means that T is the only possible time for exercising the option.

- The considered underlying S_t is an FX rate quoted in FOR/DOM (foreign/domestic) format, i.e. one unit of the *foreign* (also called *base*) currency costs FOR/DOM units of the *domestic* (also called *numeraire* by Castagna, 2009) currency. It needs to be stressed that the expressions *foreign* and *domestic* do not refer to the location of the counterparty. For example EUR/USD = 1.39000 means that 1 EUR is worth 1.39000 USD and in this case EUR is the *foreign* currency and USD is the *domestic* one, even for the trader who is based in Europe and his national currency is EUR. Hence he would prefer to know how much EUR he must pay for 1 USD, which is equal to the USD/EUR = 0.7194 quote. Convention of the underlying exchange rate quotation is described in detail by Wystup (2006).

- S_t is modelled via geometric Brownian motion (GBM)

$$dS_t = (r_d - r_f)S_t dt + \sigma_t S_t dB_t. \tag{2.1}$$

  Applying Itô's lemma (see definition A.4 in Appendix A) to S_t follows that the process is log-normally distributed at $\tau = T - t$ such as $\ln S_t \sim \mathcal{N}\left(\ln S_0 + r_d - r_f - \frac{1}{2}\sigma_t^2, \sigma_t^2 t\right)$.

- Variable σ_t can be treated as a real time estimate of the spot implied volatility (see definition A.1 in Appendix A) for all t before maturity T. Thus we can consider σ_t as a stochastic process, obtained from the market at time t.

  After Shkolnikov (2009) it will be further called the *fair value implied volatility*. For simplicity we will assume that this is the ATM volatility (see definition A.9 in

Appendix A). We refer to Shkolnikov (2009) for another possible choice of the fair value implied volatility, which is obtained statistically from vanilla options. What is worth mentioning is that if option maturities T_i are not the same, then more than one fair value implied volatility σ_i is available at time t.

2.2   Construction of the replicating portfolio

The first mathematical justification for the vanna-volga method was presented by Castagna and Mercurio (2007). In the original paper, it was only applied to vanilla contracts. Shkolnikov (2009) extended this proof to other options, including exotics.

Our aim is to value an arbitrary option contract O by constructing a replicating portfolio that is vega-neutral in the Black-Scholes (flat-smile) world. The equivalent of the Black-Scholes model in the FX setting is the Garman-Kohlhagen (see definition A.6 in Appendix A), so whenever we use in this thesis the expression *Black-Scholes model*, we mean the Garman-Kohlhagen model.

We assume that the option price can be described by the Black-Scholes PDE (see definition A.5 in Appendix A) with a *flat but stochastic implied volatility*[1] (Castagna and Mercurio, 2007). Our risk-neutral replicating portfolio Π^{BS} consists of a long position in O and two short positions in Δ_t units of the underlying asset S_t and x_i units of three pivot European vanilla calls C_i (or puts) maturing at T or later, that are quoted in the market: 75Δ call (or 25Δ put), ATM call (or ATM put), 25Δ call (or 75Δ put), see A.9 in Appendix A for the explanation of the convention. Their corresponding strikes are further denoted by K_i, $i = 1, 2, 3$, $K_1 < K_2 < K_3$ (which is equivalent to writing $K_1 = K_{25P}$, $K_2 = K_{ATM}$, $K_3 = K_{25C}$) and the market-implied volatilities associated with K_i are denoted by σ_i.

The change $d\Pi^{BS}$ of the value of the portfolio Π^{BS} in a small time interval dt and in the Black-Scholes world is given by the equation:

$$d\Pi^{BS} = dO^{BS}(t) - \Delta_t\, dS_t - \sum_{i=1}^{3} x_i\, dC_i^{BS}. \tag{2.2}$$

---

[1]Shkolnikov (2009) names it a *random but strike-independent implied volatility*.

Applying Itô's lemma to equation (2.2) we get:

$$d\Pi^{BS} = \frac{\partial O^{BS}}{\partial S_t}dS_t + \frac{\partial O^{BS}}{\partial t}dt + \frac{\partial O^{BS}}{\partial \sigma_t}d\sigma_t$$

$$+\frac{1}{2}\left\{\frac{\partial^2 O^{BS}}{\partial S_t^2}(dS_t)^2 + \frac{\partial^2 O^{BS}}{\partial t^2}(dt)^2 + \frac{\partial^2 O^{BS}}{\partial \sigma_t^2}(d\sigma_t)^2\right\}$$

$$+\left\{\frac{\partial^2 O^{BS}}{\partial S_t \partial t}dS_t dt + \frac{\partial^2 O^{BS}}{\partial S_t \partial \sigma_t}dS_t d\sigma_t + \frac{\partial^2 O^{BS}}{\partial t \partial \sigma_t}dt d\sigma_t\right\}$$

$$-\Delta_t dS_t - \sum_{i=1}^{3}x_i\frac{\partial O^{BS}}{\partial S_t}dS_t - \sum_{i=1}^{3}x_i\frac{\partial C_i^{BS}}{\partial t}dt - \sum_{i=1}^{3}x_i\frac{\partial C_i^{BS}}{\partial \sigma_t}d\sigma_t$$

$$-\frac{1}{2}\left\{\sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial S_t^2}(dS_t)^2 + \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial t^2}(dt)^2 + \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial \sigma_t^2}(d\sigma_t)^2\right\}$$

$$-\left\{\sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial S_t \partial t}dS_t dt + \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial S_t \partial \sigma_t}dS_t d\sigma_t + \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial t \partial \sigma_t}dt d\sigma_t\right\}. \tag{2.3}$$

Applying the rules of stochastic calculus (see Weron and Weron, 2005) terms of equation (2.3) with $dS_t dt$, $dt d\sigma_t$ and $(dt)^2$ vanish.

We calculate $(dS_t)^2$ with $\mu = r_d - r_f$ as follows:

$$(dS_t)^2 = (\mu S_t dt + \sigma S_t dB_t)(\mu S_t dt + \sigma S_t dB_t) =$$
$$\mu^2 S_t^2(dt)^2 + 2\mu\sigma_t S_t^2 dt dB_t + \sigma^2 S_t^2(dB_t)^2 = \sigma_t^2 S_t^2 dt. \tag{2.4}$$

The result (2.4) comes from the above-mentioned product rules for stochastic differentials together with the fact that $(dB_t)^2 = dt$.

Therefore equation (2.3) can be simplified as follows:

$$d\Pi^{BS} = \left[\frac{\partial O^{BS}}{\partial S_t} - \Delta_t - \sum_{i=1}^{3}x_i\frac{\partial O^{BS}}{\partial S_t}\right]dS_t$$

$$+\left[\left(\frac{\partial O^{BS}}{\partial t} - \sum_{i=1}^{3}x_i\frac{\partial C_i^{BS}}{\partial t}\right) + \frac{1}{2}\sigma_t^2 S_t^2\left(\frac{\partial^2 O^{BS}}{\partial S_t^2} - \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial S_t^2}\right)\right]dt$$

$$+\left[\frac{\partial O^{BS}}{\partial \sigma_t} - \sum_{i=1}^{3}x_i\frac{\partial C_i^{BS}}{\partial \sigma_t}\right]d\sigma_t + \frac{1}{2}\left[\partial^2 O^{BS}\partial\sigma_t^2 - \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial \sigma_t^2}\right](d\sigma_t)^2$$

$$+\left[\frac{\partial^2 O^{BS}}{\partial S_t \partial \sigma_t} - \sum_{i=1}^{3}x_i\frac{\partial^2 C_i^{BS}}{\partial S_t \partial \sigma_t}\right]dS_t d\sigma_t. \tag{2.5}$$

On the other hand, the following parity is true, based on the no-arbitrage principle in the Black-Scholes world:

$$d\Pi^{BS} = r_d\,\Pi^{BS}\,dt. \tag{2.6}$$

To construct a locally hedging portfolio we choose Δ_t and $\mathbf{x} = (x_1, x_2, x_3)^T$ in such way that they zero out the coefficients of dS_t, $d\sigma_t$, $(d\sigma_t)^2$ and $dS_t d\sigma_t$. The last three partial derivatives in equation (2.5), i.e. $\frac{\partial O}{\partial \sigma_t}$, $\frac{\partial^2 O}{\partial \sigma_t^2}$ and $\frac{\partial^2 O}{\partial S_t \partial \sigma_t}$, are called *vega, volga* and *vanna*,

respectively. If O is a European vanilla, the second term in the coefficient standing by dt will be automatically zeroed out due to the relationship between gamma and vega of the option. However, this is not true for every contract O. Nevertheless, as stated by Shkolnikov (2009), this can be considered as irrelevant for Black-Scholes standard hedging arguments.

By construction of the hedging portfolio Π^{BS} we get rid of the risk associated with the fluctuations of the spot price and volatility and hence our portfolio is now locally risk-free at time t, i.e. no stochastic terms appear in its differential. Δ_t and the coefficient vector $\mathbf{x}$ are calculated from the equations:

$$\Delta_t = \frac{\partial O^{BS}}{\partial S_t} - \sum_{i=1}^{3} x_i \frac{\partial O^{BS}}{\partial S_t}, \tag{2.7}$$

$$\mathbf{w_O} = \mathbf{V}\mathbf{x}, \tag{2.8}$$

where

$$\mathbf{w_O} = \begin{pmatrix} Vega(O) \\ Vanna(O) \\ Volga(O) \end{pmatrix} = \begin{pmatrix} \frac{\partial O^{BS}}{\partial \sigma_t} \\ \frac{\partial^2 O^{BS}}{\partial S_t \partial \sigma_t} \\ \frac{\partial^2 O^{BS}}{\partial \sigma_t^2} \end{pmatrix}, \tag{2.9}$$

$$\mathbf{V} = \begin{pmatrix} Vega_1 & Vega_2 & Vega_3 \\ Vanna_1 & Vanna_2 & Vanna_3 \\ Volga_1 & Volga_2 & Volga_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial C_1^{BS}}{\partial \sigma_t} & \frac{\partial C_2^{BS}}{\partial \sigma_t} & \frac{\partial C_3^{BS}}{\partial \sigma_t} \\ \frac{\partial^2 C_1^{BS}}{\partial S_t \partial \sigma_t} & \frac{\partial^2 C_2^{BS}}{\partial S_t \partial \sigma_t} & \frac{\partial^2 C_3^{BS}}{\partial S_t \partial \sigma_t} \\ \frac{\partial^2 C_1^{BS}}{\partial \sigma_t^2} & \frac{\partial^2 C_2^{BS}}{\partial \sigma_t^2} & \frac{\partial^2 C_3^{BS}}{\partial \sigma_t^2} \end{pmatrix}. \tag{2.10}$$

All the pivot vanilla options C_i are calculated with respective strikes K_i and greeks (see definition A.8 in Appendix A) $Vega(O)$, $Vanna(O)$ and $Volga(O)$ and with strike K, generally not equal to K_i. As mentioned by Shkolnikov (2009) pivots can have different expiry times T_i, but in this thesis we investigate only pivots with the same maturities equal to T ($T = T_1 = T_2 = T_3$). Then the unique solution of the system (2.8) is given by:

$$x_1 = \frac{Vega(O) \ln \frac{K_2}{K} \ln \frac{K_3}{K}}{Vega_1 \ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}}$$

$$x_2 = \frac{Vega(O) \ln \frac{K}{K_1} \ln \frac{K_3}{K}}{Vega_2 \ln \frac{K_2}{K_1} \ln \frac{K_3}{K_2}}$$

$$x_3 = \frac{Vega(O) \ln \frac{K}{K_1} \ln \frac{K}{K_2}}{Vega_3 \ln \frac{K_3}{K_1} \ln \frac{K_3}{K_2}}. $$

$$\tag{2.11}$$

In particular, if $K \in (K_1, K_2, K_3)$ then $x_i = 1$ for i such that $K = K_i$ and the remaining x_j for $j \neq i$ are equal to zero.

Chapter 3

Pricing FX options and volatility smile construction

3.1 The exact vanna-volga method

This method is also called the *modified vanna-volga method* (for example, by Fisher (2007) and Carr, Hogan and Verma, 2006). It can be shown (for a detailed proof see Shkolnikov, 2009) that the following proposition is true for any contract O.

PROPOSITION 3.1 *Under the assumption that S_t follows geometric Brownian motion with stochastic but strike-independent implied volatility there exists a unique self-financing portfolio $\Pi^{MK} = O^{MK} - \Delta_t^{MK} S_t - \sum_{i=1}^3 x_i C_i^{MK}$ such that $\Pi^{MK} = \Pi^{BS}$ for any $0 \leq t \leq T$. It follows that the vanna-volga price is given by:*

$$O_{VV}^{MK} = O^{BS} + \sum_{i=1}^3 x_i(C_i^{MK} - C_i^{BS}). \tag{3.1}$$

Coefficient vector $\mathbf{x}$ is determined from equation (2.8) and depends on t. What is worth noting is the fact that pivot calls and pivot puts can be used interchangeably due to the put–call parity (see definition A.7 in Appendix A). Using puts instead of calls changes the value of delta Δ_t but does not have the affect on x_i values. The term

$$O^{VV} = \sum_{i=1}^3 x_i(C_i^{MK} - C_i^{BS}) \tag{3.2}$$

will be further called either *vanna-volga correction* or *adjustment*, or *overhedge*.

Equation (3.1) provides the solution for the vanna-volga option price, however, in the literature it is more common to see it written in terms of three instruments traded in the market: delta-neutral straddles (known as ATM), 25-delta risk reversals (RR) and 25-delta butterflies (BF), see definitions A.10 in Appendix A. They carry respectively mainly vega, volga and vanna risks. In order to get a new form of equation (2.8) we need

to transform the coordinate system into:

$$\tilde{x}_{ATM} = x_1 + x_2 + x_3$$
$$\tilde{x}_{RR} = \frac{1}{2}(x_3 - x_1)$$
$$\tilde{x}_{BF} = x_1 + x_3.$$
$$(3.3)$$

Then the following system of equations needs to be solved to obtain to the vanna-volga price.

$$\mathbf{w_O} = \mathbf{A}\tilde{\mathbf{x}}, \tag{3.4}$$

where

$$\mathbf{w_O} = \begin{pmatrix} Vega(O) \\ Vanna(O) \\ Volga(O) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} Vega(ATM) & Vega(RR) & Vega(BF) \\ Vanna(ATM) & Vanna(RR) & Vanna(BF) \\ Volga(ATM) & Volga(RR) & Vanna(BF) \end{pmatrix},$$

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_{ATM} \\ \tilde{x}_{RR} \\ \tilde{x}_{BF} \end{pmatrix}.$$

Similarly to equation (3.1), the vanna-volga option price in terms of ATM, risk reversals and butterflies is calculated according to the formula:

$$O_{VV}^{MK} = O^{BS} + \tilde{x}^T Y, \tag{3.5}$$

with

$$\mathbf{Y} = \begin{pmatrix} ATM^{MK} - ATM^{BS} \\ RR^{MK} - RR^{BS} \\ BF^{MK} - BF^{BS} \end{pmatrix}.$$

Hence equation (3.5) is equal to:

$$O_{VV}^{MK} = O^{BS} + \tilde{x}_{ATM}(ATM^{MK} - ATM^{BS}) + \tilde{x}_{RR}(RR^{MK} - RR^{BS}) + \tilde{x}_{BF}(BF^{MK} - BF^{BS}).$$
$$(3.6)$$

Combining systems of equations (3.4) and (3.5) we get:

$$O_{VV}^{MK} = O^{BS} + w_O^T \left(A^T\right)^{-1} Y = O^{BS} + Vega(O)\,\Omega_{vega} + Vanna(O)\,\Omega_{vanna} + Volga(O)\,\Omega_{volga}, \tag{3.7}$$

where
$$\Omega = \begin{pmatrix} \Omega_{vega} \\ \Omega_{vanna} \\ \Omega_{volga} \end{pmatrix} = \left(A^T\right)^{-1} Y.$$

Thus we see from the matrix representation (3.7) that the vector Ω can be interpreted as a vector of market prices of vega, vanna and volga. The quantities Ω_i correspond to the premiums attached to these greeks in order to adjust the Black-Scholes prices of the ATM, RR and BF instruments to their market values.

3.2   The simplified vanna-volga method

This version of the vanna-volga method can be found in many publications: Wystup (2006), Bossens et al. (2010) and Wystup (2008), just to mention a few. In many papers, such as, for example, Castagna and Mercurio (2006), Bossens et al. (2010) or Carr, Hogan and Verma (2006), professor Uwe Wystup is mentioned as one of the pioneers of the vanna-volga method. He was first to formulate this version of the vanna-volga method thus it will be further called either *Wystup's* or *standard vanna-volga method* as, for instance, in Carr, Hogan and Verma (2006).

The standard formulation of the vanna-volga method is given by:

$$O_{VV}^{MK} = O^{BS} + \frac{Vanna(O)}{Vanna(RR)}RR_{cost} + \frac{Volga(O)}{Volga(BF)}BF_{cost}, \qquad (3.8)$$

where

$$
\begin{aligned}
RR_{cost} &= \left[Call\left(K_c, \sigma(K_c)\right) - Put\left(K_p, \sigma(K_p)\right)\right] & (3.9) \\
&\quad - \left[Call\left(K_c, \sigma_{ATM}\right) - Put\left(K_p, \sigma_{ATM}\right)\right], \\
BF_{cost} &= \frac{1}{2}\left[Call\left(K_c, \sigma(K_c)\right) + Put\left(K_p, \sigma(K_p)\right)\right] & (3.10) \\
&\quad - \frac{1}{2}\left[Call\left(K_c, \sigma_{ATM}\right) + Put\left(K_p, \sigma_{ATM}\right)\right],
\end{aligned}
$$

and the Black-Scholes price of the option (O^{BS}) and the greeks of O are calculated with *ATM* volatility.

Bossens et al. (2010) explain that the rationale behind equation (3.8) follows from the fact that both strategies: BF and RR are liquid FX instruments and they carry respectively mainly volga and vanna risks which are added to the corresponding Black-Scholes price to construct smile-consistent values. The weighting factors in equation (3.8) standing by RR_{cost} and BF_{cost} can be treated as the amount of RR to replicate the vanna of the option and as the amount of BF to replicate the volga of the option, respectively.

It is worth noting that Wystup's approach does not take into account a small but non-zero fraction of volga carried by RR and a small fraction of vanna carried by BF. The risk associated with vega is also neglected in formula (3.8) in comparison to the exact solution (3.6).

3.3   Pricing first-generation exotics

Equations (3.1), (3.6) and (3.8) give a reasonably good approximation of the market price of a vanilla option. However, this does not hold any more for the exotics. It is because the vanna-volga overhedge is not needed for an option that can knock out once it knocked out.

Thus the common practice is to rescale the vanna-volga adjustment (3.2) by a factor $p \in [0, 1]$, often called the *survival probability*. There is no general choice for p as it depends on the product to be priced and many traders have different views on this factor

and measure it differently. The most popular choice for p is the *domestic risk-neutral no-touch probability* (Carr, Hogan and Verma (2006), Fisher (2007) Wystup, 2008). Hence the vanna-volga adjusted value of the exotic is given by:

$$O_{VV}^{MK} = O^{BS} + p \sum_{i=1}^{3} x_i (C_i^{MK} - C_i^{BS}), \tag{3.11}$$

or

$$O_{VV}^{MK} = O^{BS} + p\left[Vega(O)\,\Omega_{vega} + Vanna(O)\,\Omega_{vanna} + Volga(O)\,\Omega_{volga}\right], \tag{3.12}$$

for the exact vanna-volga method and

$$O_{VV}^{MK} = O^{BS} + p\left[\frac{Vanna(O)}{Vanna(RR)}RR_{cost} + \frac{Volga(O)}{Volga(BF)}BF_{cost}\right], \tag{3.13}$$

for Wystup's method. Vannila options are correctly priced via equations (3.1), (3.6) and (3.8) because $p = 1$ for vanilla options. The risk-neutral no-touch probability p for a *knock-out*[2] barrier option is equal to:

$$p = 1 - q = 1 - \mathbb{P}[\tau_B \le T], \tag{3.14}$$

where q is the *risk-neutral probability of knocking out* and τ_B is the *first hitting time* (see definitions A.12 in Appendix A).

Using formulas (3.11), (3.12) and (3.13) with p from equation (3.14) one can find the vanna-volga price for any standard knock-out barrier option. *Knock-in*[3] options are calculated using the relationship:

$$Knock\text{-}Out + Knock\text{-}In = Vanilla. \tag{3.15}$$

As analytical formulas are available for every barrier option O – see equations (A.24) - (A.27) for a single barrier option) – one can find its $Vega(O)$, $Vanna(O)$ and $Volga(O)$ and hence obtain the VV price. Formulas for the most common greeks such as delta, gamma or vega can be found in Hakala, Perissé and Wystup (2002), Wystup (2002) or Haug (2007). Note that there is a mistake, included in the Errata, in the first two papers where component D of a barrier option should be exchanged with the formula for the delta Δ_D.

It should be stated that the vanna-volga method is just an approximation technique and thus it is not free of limitations. The corrected price can be sometimes out of the logical bounds. According to Shkolnikov (2009), the main issue that can cause it is a discontinuity in vega, vanna or volga for some options, mainly barriers. Their delta and consequently vanna become discontinuous at the barrier. It follows from equation (3.1) that O_{VV}^{MK} also becomes discontinuous. There is no universal solution for such cases and they need to be dealt with individually by special adjustment of p.

---

[2]A knock-out option ceases to exit when the underlying asset price reaches a certain barrier level.

[3]A knock-in option comes into existence only when the underlying asset price reaches a barrier level.
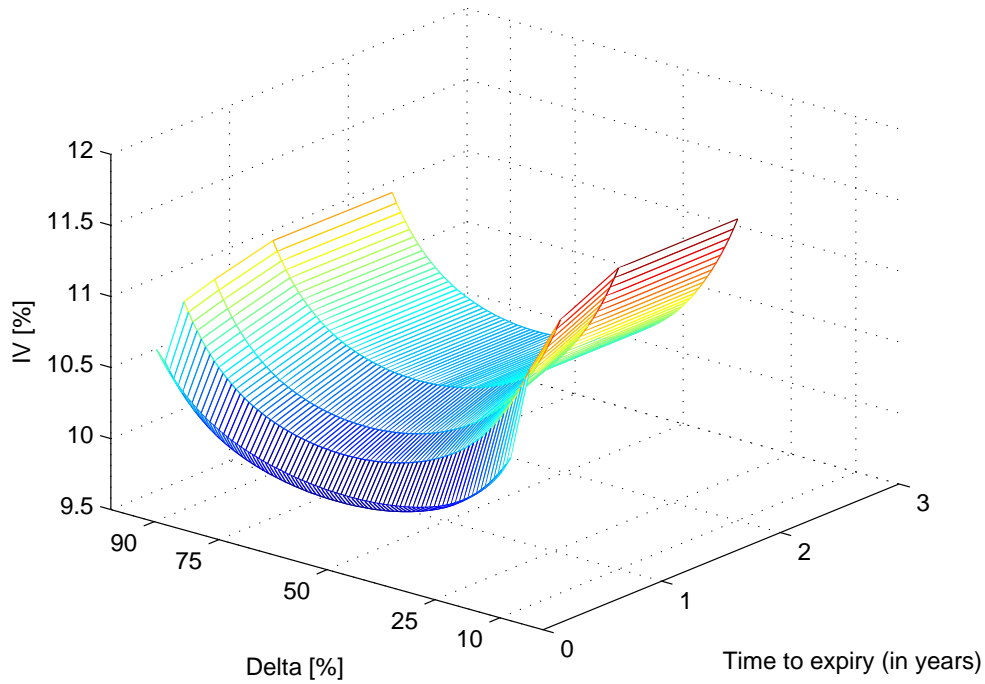
Figure 3.1: Sample implied volatility surface obtained from the VV approach for option values, observed on July 1, 2004.

3.4 Implied volatility smile construction

The vanna-volga method is widely used not just for pricing derivatives but also to construct implied volatility smiles. The VV implied volatility curve $\Delta \to \sigma(\Delta)$ can be straightforwardly retrieved from equations (3.1), (3.6) and (3.8) for each considered Δ, through the formula (A.1), included in Appendix A. Since $\sigma(\Delta_i) = \sigma_i$, formula (3.1), by construction yields an interpolation/extrapolation tool for the market implied volatilities.

Alternatively, Wystup's vanna-volga method has not such a characteristic, therefore inverting formula (3.8) will give just an approximation of the smile for all possible Δ values, also market implied volatilities. Related sources covering the IV smile construction topic can be found in Reiswich and Wystup (2010) and Castagna and Mercurio (2007) and a sample implied volatility surface with respect to delta and time to maturity is plotted in Figure 3.1.

Chapter 4

Extensions and alternative approaches

4.1   Variations on the survival probability

As it was already mentioned in Section 3.3, there is a lot of dispute about the right choice of the value of p. Different approaches to the problem have been proposed in the literature. Wystup (2008) introduces empirically chosen weights of the overhedge, justifying it by the fact that for at the money strikes the long time to maturity should be weighted higher and for lower strikes the short time should be also weighted higher. Bossens et al. (2010) introduce special probabilities for two components: vanna and volga, denoted by p_{vanna} and p_{volga}, respectively. Special conditions are imposed on functions p_{vanna} and p_{volga} and the mathematical explanation of possible values of these factors is rather complicated. For more details we refer to the original paper. Definitely less complex and more intuitive adjustment is proposed by Fisher (2007). Similarly to the solution proposed by Bossens et al. (2010), the vanna-volga price of a barrier option consists of special attenuation factors p_{vega} for vega, p_{vanna} for vanna and p_{volga} for volga components in equation (3.7):

$$O_{VV}^{MK} = O^{BS} + p_{vega} Vega(O)\, \Omega_{vega} + p_{vanna} Vanna(O)\, \Omega_{vanna} + p_{volga} Volga(O)\, \Omega_{volga}. \quad (4.1)$$

However, instead of taking the domestic risk neutral no-touch probability as a starting point, as it is done in equation (3.14), they take the average of the domestic and foreign risk-neutral probabilities p_{sym}, which helps to preserve the foreign-domestic symmetry (see definition A.13 in Appendix A) inherent in FX options. Foreign risk neutral no-touch probability can be obtained from this symmetry for barrier options. It means that to calculate the value of a one-touch option in the foreign currency (necessary to calculate the risk-neutral foreign knock-out probability) one needs to replace S_t and the barrier B by their reciprocal values, exchange r_d and r_f and change the sign of η (A.31) in Appendix A).

The adjustment treats vega and volga differently than vanna. The formula for a knock-out barrier option reads:

$$O_{VV}^{MK} = O^{BS} + \left(\frac{1}{2} + \frac{1}{2}p_{sym}\right) Vega_O \, \Omega_{vega}$$
$$+ p_{sym}Vanna_O \, \Omega_{vanna}$$
$$+ \left(\frac{1}{2} + \frac{1}{2}p_{sym}\right) Volga_O \, \Omega_{volga}.$$

(4.2)

The attenuation factors standing by vega (p_{vega}), vanna (p_{vanna}) and volga (p_{volga}) components are based on the market prices of vega (Ω_{vega}), vanna (Ω_{vanna}) and volga (Ω_{volga}). The approach to vega and volga components proposed by Fisher (2007) is a compromise between two observations: one is that vega and volga should be completely unweighted and the other is that vega and volga should be weighted by a function which goes to zero as the spot approaches the barrier. In Section 5.4 we check whether this variation on barrier option survival probability has empirical support in available market data.

Formula (4.2) is justified only for knock-out options, for which the vanna component of the adjustment is zero once the option has knocked out. But how do we price knock-in options? This is a rather problematic characteristic of the vanna-volga overhedge. Normally substituting p_{sym} with $q = 1 - p_{sym}$, which is the probability of hitting the barrier, should give appropriate results. Unfortunately, the results obtained in this way do not satisfy the no-arbitrage condition for barriers presented in equation (3.15). Bearing this in mind, we fix it manually by pricing knock-in options as the difference between the vanna-volga price of a plain vanilla option and the vanna-volga price of a knock-out option calculated using equation (4.2).

4.2   The Heston model[4]

An alternative approach to the volatility smile problem is to allow the volatility to be driven by a stochastic process (not necessarily flat). The pioneering work of Heston (1993) led to a development of stochastic volatility (SV) models. These are multi-factor models with one of the factors being responsible for the dynamics of the volatility coefficient. Different driving mechanisms for the volatility process have been proposed, including geometric Brownian motion (GBM) and mean-reverting Ornstein-Uhlenbeck (OU) type processes.

The Heston model stands out from this class mainly for two reasons. Firstly, the process for the volatility is non-negative and mean-reverting, which is what we observe in the markets. Secondly, there exists a semi-analytical solution for European options. This computational efficiency becomes critical when calibrating the model to market prices and is the greatest advantage of the model over other (potentially more realistic) SV

---

[4]Section is based on the Chapter *FX smile in the Heston model* by Janek et al. (2011), that appeared in the 2nd edition of the book *Statistical Tools for Finance and Insurance*. For more details we refer to the original paper.

models. Its popularity also stems from the fact that it was one of the first models able to explain the smile and simultaneously allow for a front-office implementation and a valuation of many exotics with values closer to the market than the Black-Scholes model.

Following Heston (1993) let us consider a stochastic volatility model with GBM-like dynamics for the spot price:

$$dS_t = S_t \left(\mu \, dt + \sqrt{v_t} dW_t^{(1)} \right), \tag{4.3}$$

and a non-constant instantaneous variance v_t driven by a mean-reverting square root (or CIR[5]) process:

$$dv_t = \kappa(\theta - v_t) \, dt + \sigma \sqrt{v_t} dW_t^{(2)}. \tag{4.4}$$

The stochastic increments of the two processes are correlated with parameter ρ, i.e. $dW_t^{(1)} dW_t^{(2)} = \rho dt$. The remaining parameters – μ, θ, κ, and σ – can be interpreted as the *drift*, the *long-run variance*, the *rate of mean reversion* to the long-run variance, and the volatility of variance (often called the *vol of vol*), respectively.

It can be shown that any value function of a general contingent claim $U(t, v_t, S_t)$ paying $g(S_T) = U(T, v_T, S_T)$ at time T must satisfy the following partial differential equation (PDE):

$$\frac{1}{2} v_t S_t^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma v_t S_t \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 U}{\partial v^2} + (r_d - r_f) S_t \frac{\partial U}{\partial S}$$
$$+ \left\{ \kappa(\theta - v_t) - \lambda(t, v_t, S_t) \right\} \frac{\partial U}{\partial v} - r_d U + \frac{\partial U}{\partial t} = 0, \tag{4.5}$$

where the term $\lambda(t, v_t, S_t)$ is called the *market price of volatility risk*. Heston (1993) assumed it to be linear in the instantaneous variance v_t, i.e. $\lambda(t, v_t, S_t) = \lambda v_t$.

Heston (1993) solved this PDE analytically and using the method of characteristic functions he derived the formula for the price of the European vanilla FX option:

$$
\begin{aligned}
h(\tau) &= \text{HestonVanilla}(\kappa, \theta, \sigma, \rho, \lambda, r_d, r_f, v_t, S_t, K, \tau, \phi) \\
&= \phi \left[e^{-r_f \tau} S_t P_+(\phi) - K e^{-r_d \tau} P_-(\phi) \right], \tag{4.6}
\end{aligned}
$$

where $\phi = \pm 1$ for call and put options, respectively, strike K is in units of the domestic currency, $\tau = T - t$ is the time to maturity, $u_{1,2} = \pm \frac{1}{2}$, $b_1 = \kappa + \lambda - \sigma\rho$, $b_2 = \kappa + \lambda$ and

$$d_j = \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)}, \tag{4.7}$$

$$g_j = \frac{b_j - \rho\sigma\varphi i + d_j}{b_j - \rho\sigma\varphi i - d_j}, \tag{4.8}$$

$$C_j(\tau, \varphi) = (r_d - r_f)\varphi i \tau + \tag{4.9}$$
$$+ \frac{\kappa\theta}{\sigma^2} \left\{ (b_j - \rho\sigma\varphi i + d_j)\tau - 2\log\left(\frac{1 - g_j e^{d_j \tau}}{1 - g_j}\right) \right\},$$

$$D_j(\tau, \varphi) = \frac{b_j - \rho\sigma\varphi i + d_j}{\sigma^2} \left(\frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}}\right), \tag{4.10}$$

---

[5]The CIR process, named after its creators Cox, Ingersoll and Ross (1985), is a Markov process with continuous paths defined by the following SDE: $dr_t = \theta(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t$, where W_t is a standard Wiener process and θ, μ, σ are the parameters corresponding to to the speed of adjustment, the mean and the volatility, respectively. This process is also widely used to model short term interest rates.

$$f_j(x, v_t, \tau, \varphi) = \exp\{C_j(\tau, \varphi) + D_j(\tau, \varphi)v_t + i\varphi x\}, \tag{4.11}$$

$$P_j(x, v_t, \tau, y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left\{\frac{e^{-i\varphi y} f_j(x, v_t, \tau, \varphi)}{i\varphi}\right\} d\varphi, \tag{4.12}$$

where $\Re(x)$ denotes the real part of x.

Note that the functions P_j are the cumulative distribution functions (in the variable $y = \log K$) of the log-spot price after time $\tau = T - t$ starting at $x = \log S_t$ for some drift μ. Finally:

$$P_+(\phi) = \frac{1-\phi}{2} + \phi P_1(x, v_t, \tau, y), \tag{4.13}$$

$$P_-(\phi) = \frac{1-\phi}{2} + \phi P_2(x, v_t, \tau, y). \tag{4.14}$$

Heston's solution is actually semi-analytical. Formulas (4.13)-(4.14) require to integrate functions f_j, which are typically of oscillatory nature. Hence different numerical approaches can be utilized to determine the price of the European vanilla FX option. These include finite difference and finite element methods, Monte Carlo simulations and Fourier inversion of the characteristic function. The latter is discussed in detail by Janek et al. (2011). We also refer to that paper for the analysis of how changing the input parameters, such as v_0, σ, *kappa*, θ and ρ, affects the shape of the fitted smile curve. This helps in reducing the dimensionality of the problem before calibrating the model to market data. We will make use of this knowledge in Sections 5.2 and 5.3.

Chapter 5

Calibration results

5.1 FX market data

Obtaining market data was quite a challenging part of this thesis as volatility matrices or prices of the barrier options are not available freely on the internet. The following examples cover only two currency pairs: EUR/PLN and EUR/USD. The main source of the volatility matrices and deposit (called *depo*) interest rates was *Bloomberg*. EURIBOR and WIBOR interest rates were taken from *www.euribor.org* and *www.bankier.pl*, respectively.

Before calibrating the model to market data we needed to adjust the data so that it was applicable to our cases. Below is a list of issues that were taken into account:

1. As displayed in Figure 5.1, volatility matrices are usually provided in the *bid/ask format* (or bid/offer). For all the data we dealt with, we computed and used in later calculations the so-called *MID volatilities*, i.e. arithmetic averages of the bid and ask quotes. Some of the volatility matrices were provided in terms of risk reversals (σ_{RR}) and butterflies (σ_{BF}), hence the RR and BF quotes were also transformed into σ_{25C} and σ_{25P} using equations (A.22) and (A.23) and into σ_{10C} and σ_{10P} via corresponding equations for 10Δ.

2. There are many delta conventions that are used by practitioners, which can be quite confusing. Therefore we need to clarify the delta convention used in this thesis because, as we see in Figure 5.1, volatilities are quoted in terms of delta, rather than strikes. We assume that the given *at-the money* (ATM) volatility is the volatility for which the strike fulfils the following condition for vanilla call and put *forward delta* (A.8): $\Delta_C = -\Delta_P = 50\%$. According to Bossens et al. (2010), this convention is generally used for all the maturity pillars of the currency pairs that are outside of the so called G11 group (main 11 currency pairs consisting of USD, EUR, JPY, GBP etc.). For currency pairs from developed economies *spot delta (A.7) convention* (i.e. $\Delta_s^C = -\Delta_s^{P\ 6}$) is used up to 1Y maturities and forward delta convention for longer tenors. However, for convenience we used the forward delta convention for both analysed currency pairs. This should not yield too great an underestimation or overestimation of the model, as spot delta and forward delta are very close in value when we deal with short option tenors (up to 1 year) and the

Figure 5.1: Sample volatility matrix in the bid/ask format in terms of ATM, 25Δ and 10Δ butterflies (BF) and risk reversals (RR), observed on August 12, 2009. Source: *Bloomberg.*

foreign interest rate is close to zero. The average 1Y EURIBOR interest rate in all analysed cases amounted to only about 0.8%, which is relatively low in comparison to the average 1Y WIBOR of approximately 3.5%.

3. Following Brigo and Mercurio (2007), the market convention is to quote short-term (up to 1Y maturity) interest rates assuming simple compounding and for all the maturities above one year – annually compounded interest rates. This fact was taken into consideration when calculating the greeks (A.7)-(A.12), Garman-Kohlhagen (A.5) and Heston (4.6) vanilla option prices, knock-out probabilities (3.14) and barrier option prices (A.24)-(A.27). Therefore the continuous discount factors that appear in all the formulas were suitably transformed into simple discount factors for up to 1 year tenors and annually compounded discount factors for longer maturities.

4. According to market conventions, for model calibration containing either EUR or USD currencies we used the *Act*/360 day count convention, which is equal to the exact number of days between two considered dates in a 360 day year, and for the PLN currency the *Act*/365 day count convention. Public holidays were also taken into account when computing the spot, expiry and delivery dates. For details on rules of shifting forward these dates we refer to Castagna (2009).

5. In the vast majority of the academic books and articles on option pricing one can read only about two dates – one for the beginning and one for the end of the contract. As noted by Wystup (2006), reality is slightly more complicated, and instead of two we have to deal with four dates, which are depicted in Figure 5.2. The volatility used in the Black-Scholes, the Garman-Kohlhagen or the Heston model corresponds to the time period between the *trade date* T_t and the *expiry date* T_e. Foreign r_f and domestic r_d interest rates correspond to the time period from the *spot date* T_s to the *delivery date* T_d.

---

[6]Note that this not equal to $\Delta_s^C = -\Delta_s^P = 50\%$ as the absolute value of a put spot delta and call spot delta are not exactly adding up to one, but to a positive number $e^{r_f \tau}$. Hence $\Delta_s^C - \Delta_s^P \approx 1$ if either the time to maturity τ is short or if the foreign interest r_f rate is close to zero.
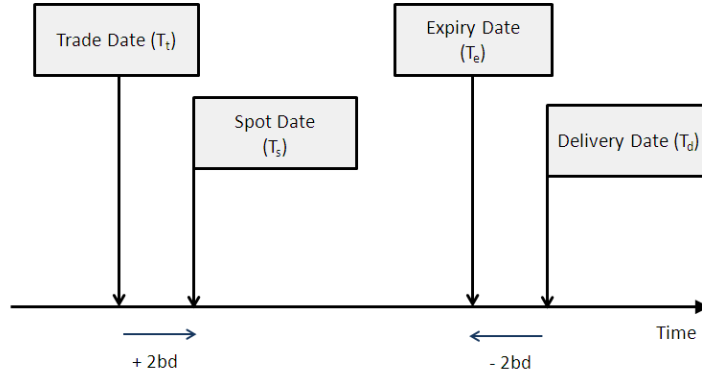
Figure 5.2: Option trading scheme. The spot date is (usually) 2 business days (2bd) after the trade date and the expiry date is 2bd prior to settlement.

Taking into consideration points 3–5, the modified version of the Garman-Kohlhagen price (equation A.5) of a European EUR/PLN option maturing up to 1 year is calculated as follows:

$$O^{BS} = \phi S_0 DF^{EUR} \mathcal{N}(\phi d_+) - \phi DF^{PLN} K \mathcal{N}(\phi d_-), \tag{5.1}$$

where:

- $d_\pm = \frac{1}{\sigma\sqrt{\tau_1}} \ln \frac{S_0 DF^{EUR}}{K DF^{PLN}} \pm \frac{\sigma\sqrt{\tau_1}}{2}$

- $DF^{PLN} = \frac{1}{(1+r^{PLN})^{\tau_2^{PLN}}}$

- $DF^{EUR} = \frac{1}{(1+r^{EUR})^{\tau_2^{EUR}}}$

- $\tau_1 = \frac{Expiry\ Date - Trade\ Date}{365}$

- $\tau_2^{PLN} = \frac{Delivery\ Date - Spot\ Date}{365}$

- $\tau_2^{EUR} = \frac{Delivery\ Date - Spot\ Date}{360}$

Similar formulas with appropriate adjustments listed in points 1–5 were derived for other kinds of options.

Following Castagna (2009), in OTC markets, such as for example FX options market, *sticky delta rule* (see definition A.3 in Appendix A) is used. Thus whenever we analyse some features of the option, such as its premium or implied volatility, we consider it with respect to different levels of delta.

5.2   Calibration of vanilla options

Having derived formulas (3.1), (3.6) or (3.8) for O_{VV}^{MK}, we can apply them to market data in order to obtain prices of vanilla options. Table 5.1 shows the vanna-volga prices (in PLN) of the EUR/PLN call option traded on August 12, 2009, maturing in 1 month for different levels of delta. We used the following market data: $T = 1m = 29/365y$,

Table 5.1: Comparison of the prices of the EUR/PLN call options (maturing in 1 month and traded on August 12, 2009) obtained using the Black-Scholes model (A.5), the exact vanna-volga method (3.1), Wystup's vanna-volga method (3.8) and the Heston model (4.6) for 5 levels of quoted deltas. VV prices that are lower than the corresponding Black-Scholes ones are marked in bold.

Delta	Strike	BS	VV exact	VV Wystup	Heston
10D Call	4.47540	0.00395	0.01125	0.01193	0.01069
25D Call	4.30712	0.02319	0.02989	0.03073	0.02981
ATM	4.16470	0.07128	0.07128	0.07163	0.07138
25D Put	4.04577	0.14350	**0.14165**	**0.14166**	0.14138
10D Put	3.93569	0.23324	0.23332	0.23331	0.23282

$S_O = r_d = r^{PLN} = 3.2291\%$, $r_f = r^{EUR} = 0.52\%$, $\sigma_{ATM} = 15.7025\%$, $\sigma_{25RR} = 2.35\%$, $\sigma_{25BF} = 0.68\%$, $\sigma_{10RR} = 4.105\%$, $\sigma_{10BF} = 2.005\%$, which lead to $\sigma_{10\Delta_C} = 19.76\%$, $\sigma_{25\Delta_C} = 17.5575\%$, $\sigma_{25\Delta_P} = 15.2075\%$, $\sigma_{10\Delta_P} = 15.655\%$ through equations (A.22)–(A.23). Equivalently we can consider this EUR/PLN option in terms of different levels of strikes, because from equation (5.2) we can retrieve strikes from deltas:

$$K = S_0 e^{(r_d - r_f)\tau - \phi\sigma\sqrt{\tau}\mathcal{N}^{-1}(\phi\Delta) + \frac{1}{2}\sigma^2\tau}. \tag{5.2}$$

Strikes corresponding to 5 levels of quoted deltas are presented in Table 5.1. Black-Scholes prices were calculated with σ_{ATM} volatility. Three Heston model parameters were fixed: initial variance $v_0 = (\sigma_{ATM})^2 = 2.4657\%$, mean reversion $\kappa = 1.5$ and market price of the volatility risk $\lambda = 0$ and three were fitted: volatility of variance $\sigma = 89.5146\%$, long-run variance $\theta = 9.3078\%$ and correlation $\rho = 0.3109$. In equation (5.2) continuous discounting is assumed. Therefore for the purpose of calibration, equation (5.2) was suitably adjusted in the way described in Section 5.1. Option prices in the Heston model were obtained using Matlab functions *HestonVanilla.m* and *HestonVanillaFitSmile.m* accompanying the chapter Janek et al. (2011). See this paper also for the explanation of application of the fixed parameters v_0, κ and λ.

From Table 5.1 we see that a larger call delta corresponds to a lower strike. This naturally follows from equation (5.2). VV prices that are lower than the Black-Scholes ones are marked in bold. For some options the vanna-volga overhedge O^{VV} can be negative resulting in lower vanna-volga premium than its Black-Scholes equivalent. This fact is depicted in Figure 5.3, in which the VV correction is skewed and is evidently below zero for Δ_{25P} (or equivalently Δ_{75C}).

What might not be easily visible in Figure 5.3, the VV correction of the call option maturing in 1 month for Δ_{10P} is above zero, which coincides with Table 5.1. In Figure 5.3 it is observable that the longer the time to maturity the greater the absolute value of the VV overhedge. This is a universal feature of vanilla options and other sample calculations with a set of market data from different dates only confirmed it. It is worth noticing that the shape of the overhedge function closely depends on market data and the one from August 12, 2009, presented in Figure 5.3, is not the only possible one. In Figure 5.4 the overhedge is positive for nearly the whole range of delta. Table 5.2, which shows the vanna-volga prices (in USD) of the EUR/USD call option maturing in 1 month and traded on July 1, 2004, only confirms this observation. We used the following EUR/USD market
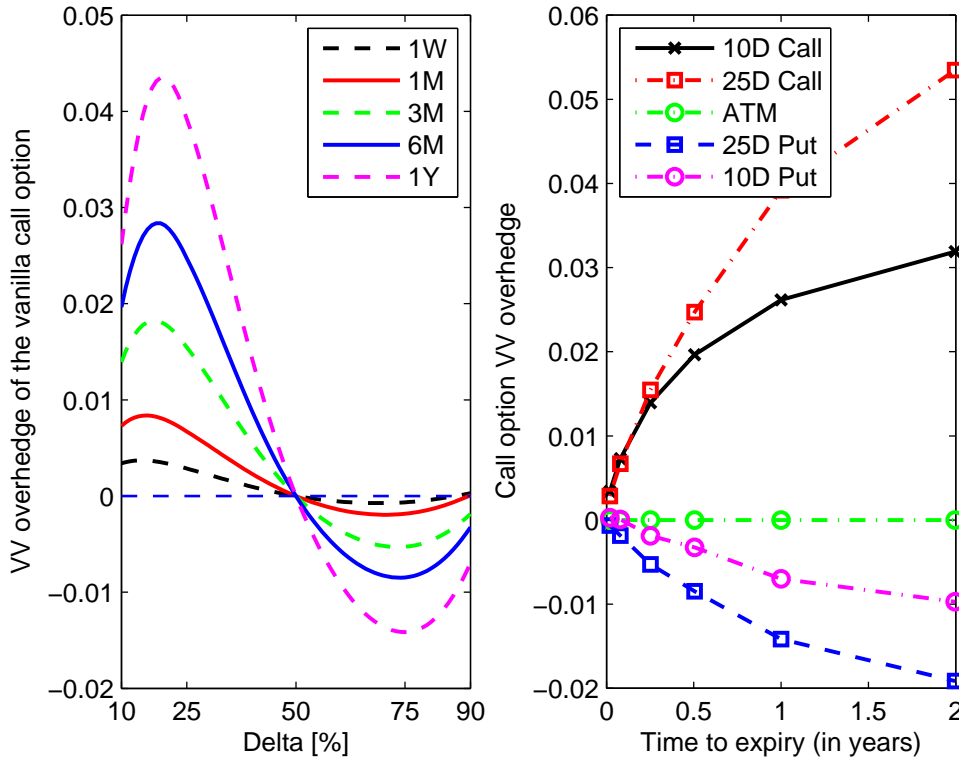
Figure 5.3: The vanna-volga overhedge calculated from equation (3.2) for EUR/PLN vanilla options traded on August 12, 2009, maturing in 1 week, 1, 3, 6 months or 1 year. For the 1 month option the same set of market data as in Table 5.1 was adopted, for the remaining options - quoted market data assigned to their maturities. *Left panel:* The relationship between the overhedge and delta (or equivalently the strike) for each expiry time. *Right panel:* The relationship between the overhedge and time to maturity for main quoted deltas.

data observed on July 1, 2004: $T = 1m = 33/365y$, $S_O = 1.215$, $r_d = r^{USD} = 2.055\%$, $r_f = r^{EUR} = 1.325\%$, $\sigma_{\Delta_{10C}} = 10.65\%$, $\sigma_{\Delta_{25C}} = 10.12\%$, $\sigma_{ATM} = 9.95\%$, $\sigma_{\Delta_{25P}} = 10.12\%$, $\sigma_{\Delta_{10P}} = 10.65\%$. Again, Black-Scholes prices were calculated with σ_{ATM} volatility. Three Heston model parameters were fixed: initial variance $v_0 = (\sigma_{ATM})^2 = 0.99\%$, mean reversion $\kappa = 1.5$ and market price of the volatility risk $\lambda = 0$ and three were fitted: volatility of variance $\sigma = 37.2545\%$, long-run variance $\theta = 2.2462\%$ and correlation $\rho = -0.0062$.

Generally the vanilla option premium closely depends on the fitted implied volatility. Therefore we can conclude that the fitted EUR/USD implied volatilities on July 1, 2004 are above the quoted implied ATM volatility (σ_{ATM}). For an in depth investigation of this proposition we refer to Section 5.3.

The reason why sometimes the VV correction is below zero is the fact that vanna or volga may be negative for some ranges of Δ_C, compare with equations (A.12)–(A.11). As the overhedge comprises the volatility-related greeks (3.7): vega, vanna and volga, it is reasonable to have a look at the range of values of these exposures.

Vega plotted in Figure 5.5 is almost symmetric about the Δ_{ATM} and reaches its maximum for this argument. Similarly to the VV correction, a longer time to maturity implies a higher value of vega. Vega is always positive which follows from equation (A.10).
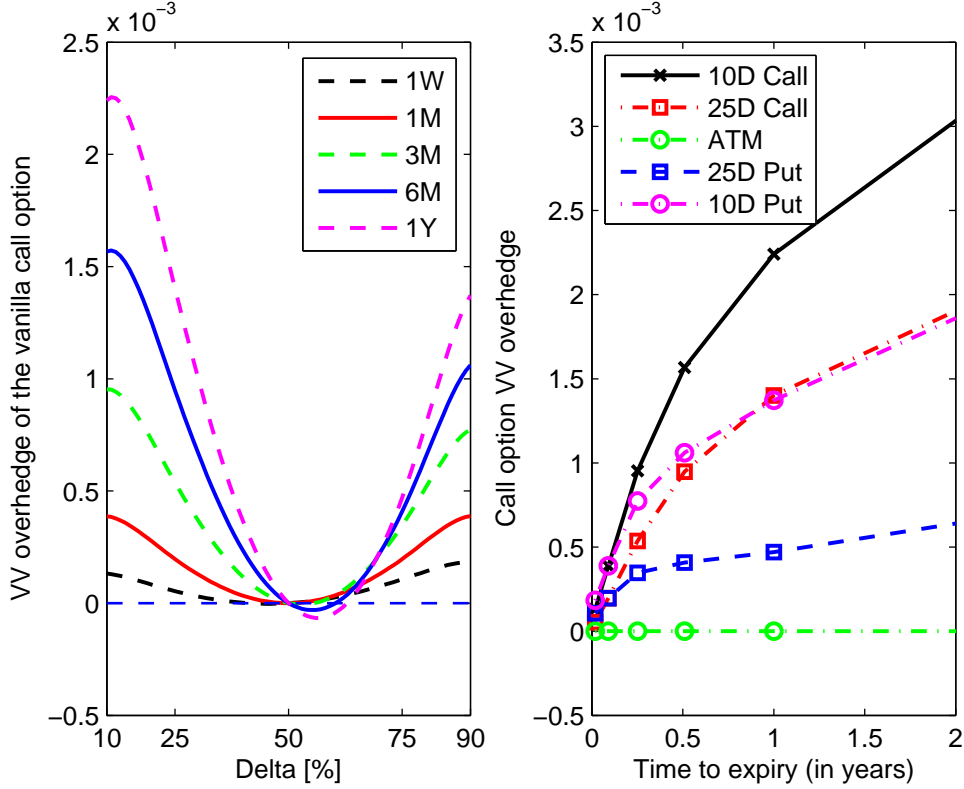
Figure 5.4: The vanna-volga overhedge calculated from equation (3.2) for EUR/USD vanilla options traded on July 1, 2004, maturing in 1 week, 1, 3, 6 months or 1 year. For the 1 month option the same set of market data as in Table 5.2 was adopted, for the remaining options - quoted market data assigned to their maturities. *Left panel:* The relationship between the overhedge and delta (or equivalently the strike) for each expiry time. *Right panel:* The relationship between the overhedge and time to maturity for main quoted deltas.

Vanna is skewed and changes its sign at Δ_{ATM} from positive values for $\Delta_C < \Delta_{ATM}$ to negative values otherwise. This time we cannot say that a longer time to maturity always implies a greater absolute value of the greek parameter, because as it is shown in the right panel of Figure 5.6, for Δ_{10C} and Δ_{10P} we observe an opposite relationship.

Volga in this example is positive for the whole range of Δ_C, but we cannot generalise that this observation is true for the whole range of market data. If d_+ and d_- in equation (A.5) have different signs then we experience negative volga exposure, compare with equation (A.11). Similarly to vega, volga is almost symmetric about Δ_{ATM} and from the definition of ATM volatility (see definition A.9 in Appendix A) and volga (A.11) it follows that d_+^{ATM} is equal to zero and hence volga for Δ_{ATM} is zero. Again, we observe greater values of volga for longer maturities. It is worth to notice that the volga contribution to the VV overhedge is about 5 times larger than the contributions of vega and vanna. The same remark applies to the EUR/USD option traded on July 1, 2004, where the shape of the VV overhedge curve (Figure 5.4) resembles very much the shape of the volga curve because its contribution is far greater than the contributions of the two other greeks.
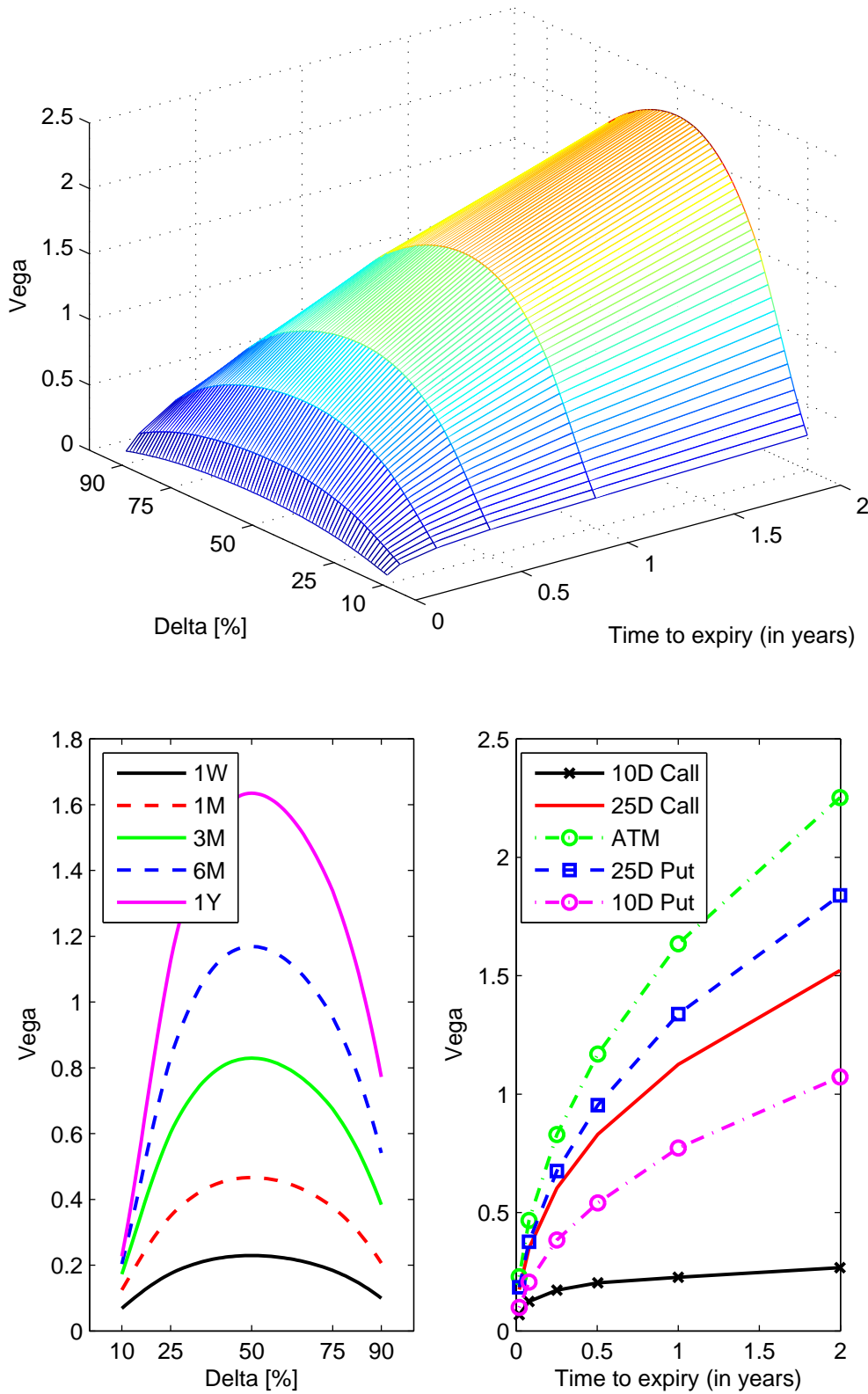
Figure 5.5: *Top panel:* Black-Scholes vega with respect to the time to maturity τ and delta Δ_C calculated from equation (A.10) with adjusted discounting and σ_{ATM} volatility for EUR/PLN vanilla options traded on August 12, 2009, maturing in 1 week, 1, 3, 6 months or 1 year. For the 1 month option the same set of market data as in Table 5.1 was adopted, for the remaining options - quoted market data assigned to their maturities. *Bottom left panel:* The same Black-Scholes vega with respect to delta (or equivalently the strike). *Bottom right panel:* The same Black-Scholes vega with respect to the time to maturity.

Figure 5.6: *Top panel:* Black-Scholes vanna with respect to the time to maturity τ and delta Δ_C calculated from equation (A.10) with adjusted discounting and σ_{ATM} volatility for EUR/PLN vanilla options traded on August 12, 2009, maturing in 1 week, 1, 3, 6 months or 1 year. For the 1 month option the same set of market data as in Table 5.1 was adopted, for the remaining options - quoted market data assigned to their maturities. *Bottom left panel:* The same Black-Scholes vanna with respect to delta (or equivalently the strike). *Bottom right panel:* The same Black-Scholes vanna with respect to the time to maturity.
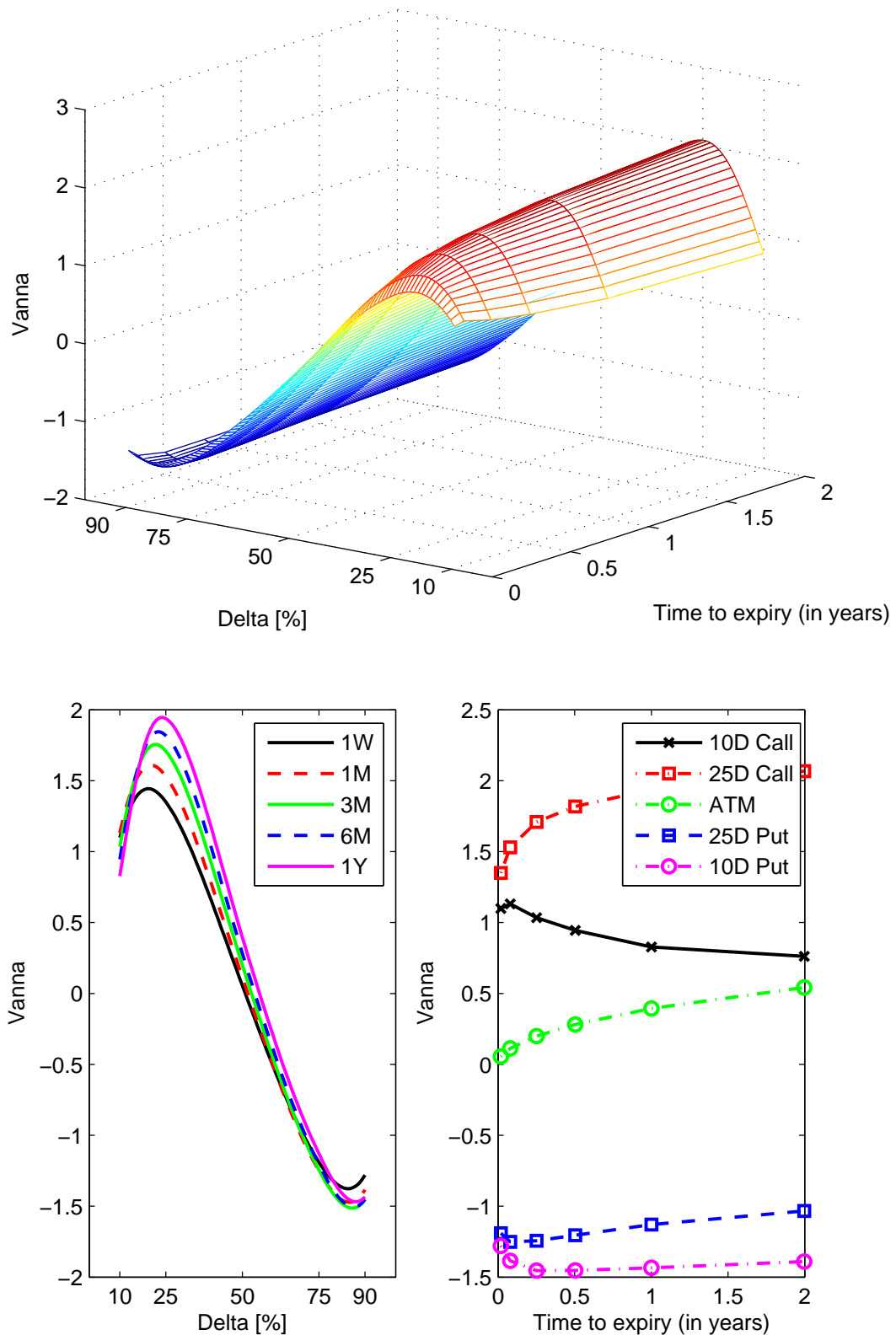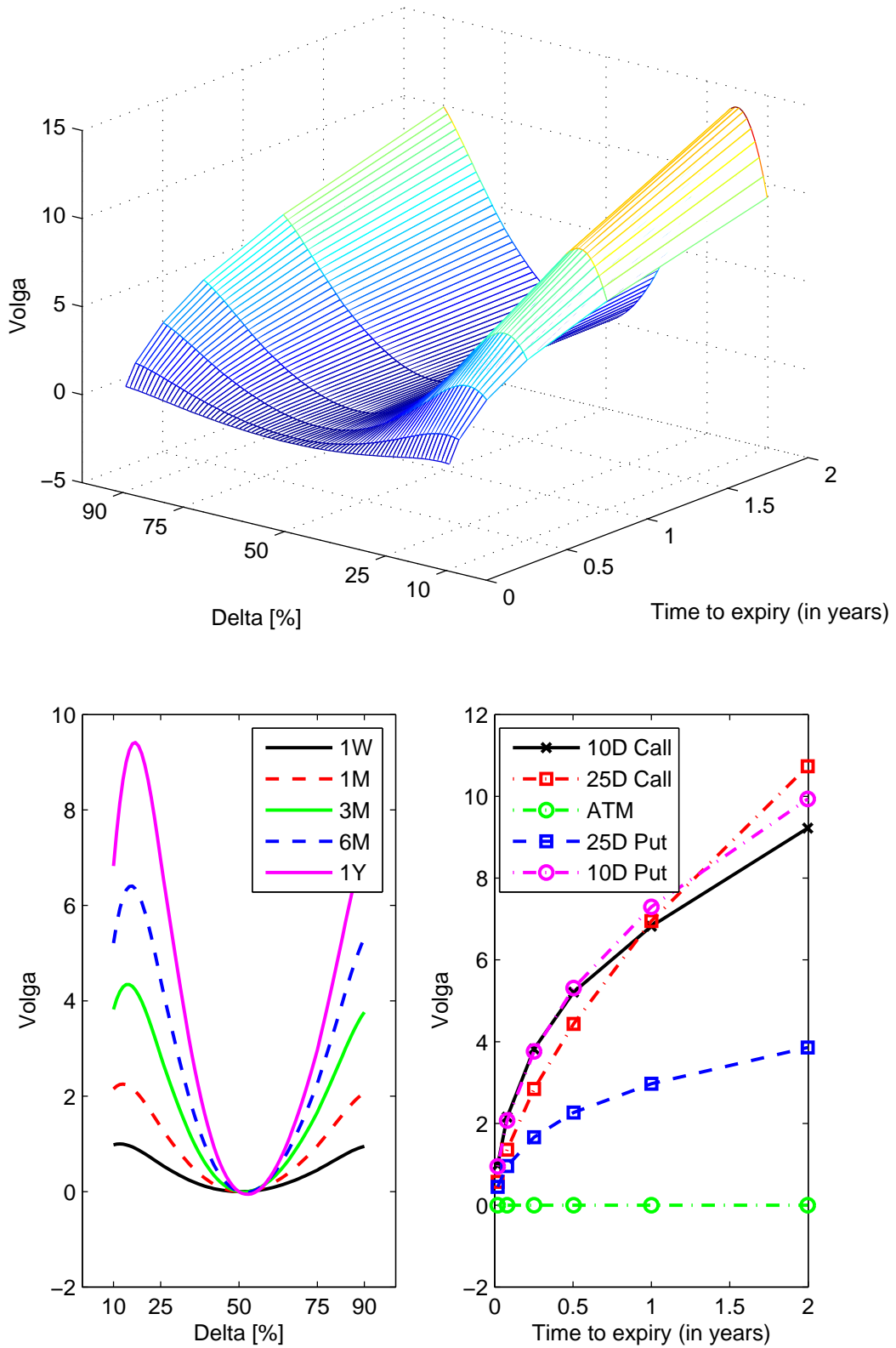
Figure 5.7: *Top panel:* Black-Scholes volga with respect to the time to maturity τ and delta Δ_C calculated from equation (A.10) with adjusted discounting and σ_{ATM} volatility for EUR/PLN vanilla options traded on August 12, 2009, maturing in 1 week, 1, 3, 6 months or 1 year. For the 1 month option the same set of market data as in Table 5.1 was adopted, for the remaining options - quoted market data assigned to their maturities. *Bottom left panel:* The same Black-Scholes volga with respect to delta (or equivalently the strike). *Bottom right panel:* The same Black-Scholes volga with respect to the time to maturity.

Table 5.2: Comparison of the call prices obtained using the Black-Scholes model, the exact vanna-volga method, Wystup's vanna-volga method and the Heston model for 5 levels of quoted deltas.

Delta	Strike	BS	VV exact	VV Wystup	Heston
10D Call	1.26734	0.00139	0.00178	0.00179	0.00181
25D Call	1.24155	0.00523	0.00543	0.00543	0.00544
ATM	1.21631	0.01422	0.01422	0.01422	0.01415
25D Put	1.19162	0.0295	0.0297	0.02969	0.02965
10D Put	1.16748	0.04964	0.05003	0.05002	0.04987

Tables 5.1 and 5.2 illustrate the fact there is no general rule for the VV premium obtained using Wystup's method. It is sometimes greater than the exact vanna-volga and Black-Scholes premiums and sometimes lower. This obviously depends on market data which imply prices of risk reversals and butterflies in equation (3.8). The same remark also applies to the Heston model premiums.

5.3   Implied volatility surface

We have seen in Section 5.2 that there is a close relationship between the premium of a plain vanilla FX option and its implied volatility (IV). Therefore another indirect way of examining the European option premium is to calibrate the volatility smile (see definition A.2 in Appendix A) for this option using a whole range of possible methods. As noted by Castagna and Mercurio (2006) the vanna-volga method is one of them and they propose an approximation of the implied volatility based on the exact vanna-volga method. However, in this thesis, we apply a more straightforward way of calibrating implied volatilities:

1. First, we retrieve strikes K_i from the quoted market implied volatilities $\{\hat{\sigma}_i\}_{i=1}^5$ (for Δ_{10C}, Δ_{25C}, Δ_{ATM}, Δ_{25P} and Δ_{10P}). As we can see from equation (5.2) this requires only calculating the inverse of the standard Gaussian distribution function.

2. Secondly, for the purpose of fitting the Heston model to the volatility smile, following Janek et al. (2011), we fix two parameters (initial variance v_0 and mean reversion κ) and fit the remaining three: volatility of variance σ, long-run variance θ and correlation ρ for a fixed time to maturity and a given vector of IV $\{\hat{\sigma}_i\}_{i=1}^5$. As far as both VV methods are concerned there is no need of fixing or fitting any parameters at this stage.

3. Having fitted strikes K_i and other parameters for the Heston model we calculate the market option prices using the exact vanna-volga method (equation (3.1) or equivalently (3.7)), Wystup's vanna-volga method (3.8) and the Heston model (4.6).

4. Then we retrieve the corresponding Black-Scholes implied volatilities $\{\hat{\sigma}_i\}_{i=1}^5$ from equation (A.1). For this purpose we use Matlab function *fzero.m*, which comprises a combination of bisection, secant and inverse quadratic interpolation methods.

Table 5.3: SSE and SSE^* values calculated for the implied volatility fit of EUR/USD options, maturing in 1 week, 1, 3, 6 months, 1 and 2 years, traded on July 1, 2004, obtained using the exact vanna-volga method (SSE_{VVe}), Wystup's vanna-volga method (SSE_{VVw} and SSE^*_{VVw}) and the Heston model (SSE_H and SSE^*_H). Due to the fact that $SSE^*_{VVe} = SSE_{VVe}$, SSE of the exact vanna-volga fit for $\sigma_{10\Delta C}$ and $\sigma_{10\Delta P}$ was not included in the table.

July 1, 2004	SSE_{VVe}	SSE_{VVw}	SSE_H	SSE^*_{VVw}	SSE^*_H
1W	6.39E-07	5.88E-07	1.64E-07	5.87E-07	2.31E-08
1M	6.25E-07	6.78E-07	1.83E-07	6.67E-07	1.14E-08
3M	7.18E-07	1.08E-06	1.85E-08	1.00E-06	1.68E-09
6M	2.25E-06	3.36E-06	5.60E-09	3.14E-06	1.15E-09
1Y	1.23E-06	3.85E-06	4.46E-08	3.36E-06	1.16E-08
2Y	1.16E-06	4.69E-06	1.17E-07	3.76E-06	3.04E-08

5. This is the end of calibration for both VV methods and we check if the fit is within a reasonable range by calculating the objective function, for example the Sum of Squared Errors (SSE), as suggested by Janek et al. (2011):

$$\text{SSE} = \sum_{i=1}^{5} (\hat{\sigma}_i - \sigma_i)^2. \tag{5.3}$$

6. For the Heston model fit we find the optimal set of parameters κ, θ, σ, ρ, v_0 and implied volatilities $\{\hat{\sigma}_i\}_{i=1}^{5}$ by minimizing another version of the SSE function (this is covered by *fminsearch.m* in Matlab):

$$\text{SSE}(\kappa, \theta, \sigma, \rho, v_0) = \sum_{i=1}^{5} \{\hat{\sigma}_i - \sigma_i(\kappa, \theta, \sigma, \rho, v_0)\}^2. \tag{5.4}$$

The same procedure was applied to both VV methods (exact and Wystup's) and the Heston model. Hence it is possible to compare their accuracy. The results are presented in Figures 5.8, 5.9, 5.10, 5.11 and Tables 5.3 and 5.4.

It is worth stressing that the exact vanna-volga method is by construction an interpolation–extrapolation tool for the market implied volatilities because three fitted implied volatilities σ_{25C}, σ_{ATM} and σ_{25P} are equal to the quoted market implied volatilities. This follows from equation (3.1) and already the mentioned fact that if $K \in \{K_1, K_2, K_3\} = \{K_{25\Delta P}, K_{ATM}, K_{25\Delta C}\}$ then $x_i = 1$ for i such that $K_i = K$ and the remaining coefficients x_j are equal to zero. However, this is not the case for the simplified vanna-volga method and the Heston model. To compare the three methods we introduce another error measure function – the Sum of Squared Errors consisting of just two fitted values: $\sigma_{10\Delta C}$ and $\sigma_{10\Delta P}$, further denoted by SSE^*.

We start our analysis of the implied volatility fit with EUR/USD options quoted on July 1, 2004, already described in Section 5.2. We have carried out the same type of analysis for EUR/USD and EUR/PLN vanilla options traded on 15 different dates. In this thesis we present only 4 of them as they well represent the types of fits of the whole market data at our disposal.

As can be seen in Figure 5.8 the quoted market smile on July 1, 2004 is very symmetric and can be considered as a 'proper' smile with ATM volatility lower than other volatilities. As far as the Heston model fit is concerned, we note that the fit is generally very good for intermediate tenors (from 3 months up to 1 year) and inaccurate for very short (up to 1 month) and very long maturities (2 years and more). However, we cannot say the same about the goodness-of-fit of both vanna-volga methods. They are not as accurate as the Heston fit. This is due the fact that in the Heston model fit we use all five quoted market volatilities $\{\hat{\sigma}_i\}_{i=1}^5$ to estimate all relevant parameters such as σ, θ, κ. Conversely, in both vanna-volga methods we make use of only three implied volatilities: $\hat{\sigma}_{25C}$, $\hat{\sigma}_{ATM}$ and $\hat{\sigma}_{25P}$. Consequently, the vanna-volga fit is never as good as the Heston model fit. Table 5.3 confirms this remark. The sum of squared errors (SSE) and the sum of squared errors limited to the two extreme implied volatilities (SSE^*) for the Heston model are several orders of magnitude smaller than the corresponding errors for the VV model.

On the other hand, both vanna-volga fits are not that bad – SSE values are very small (Table 5.3). Generally Wystup's and the exact vanna-volga fits have similar shapes (Figure 5.8). Both methods underestimate implied volatility σ_{10C} and are extremely accurate in finding σ_{10P} for 1W tenor and, on the contrary, underestimate both implied volatilities σ_{10C} and σ_{10P} for 1M tenor. Both methods overestimate the implied volatility σ_{10C} for tenors from 3M up to 2Y (Wystup's method slightly more than the exact one). The exact method at the same time overestimates the implied volatilities σ_{10P}, whereas the simplified vanna-volga method reproduces almost exactly the input volatilities $\hat{\sigma}_{10P}$ (with the exception of the 6M tenor).

In accordance with intuition, Table 5.3 shows that the overall exact vanna-volga method fit is much better than Wystup's method (compare SSE values), with the only exception for the EUR/USD option with 1 week to maturity. Obviously, it is not without significance that the exact vanna-volga method is an interpolation method within the range $[\sigma_{25C}, \sigma_{25P}]$, resulting in an overall smaller sum of squared errors (SSE) as the three fitted implied volatilities σ_{25C}, σ_{ATM} and σ_{25P} are exactly the same as the input values. These three implied volatilities are quite well matched by Wystup's vanna-volga method but naturally they are not the same as in the exact method. It is worth stressing that the comparison of the SSE^* values for both VV methods is not that favourable for the exact one. Wystup's method universally better fits two implied volatilities σ_{10C} and σ_{10P}, which is consistent with smaller values of SSE^*_{VVw} than SSE_{VVe} for each tenor.

What can be surprising is the fact that both vanna-volga fits are more accurate for shorter tenors (up to 3 months for the exact vanna-volga method and up to 1 month for the simplified vanna-volga method). This is the opposite trend to the one observed for the Heston model. The best exact vanna-volga method fit is observable for the 1M tenor and for Wystup's method it is the 1W tenor. On the other hand, the Heston model fits for these two tenors are the worst ones among all.

Let us now have a look at the smile produced by the other currency pair – EUR/PLN, observed on July 31, 2008 (Figure 5.9). This time the smile is not as symmetric as on July 1, 2004 – it is slightly skewed to the right. Again, the Heston model fit is the most accurate for the reasons already mentioned above. It is exceptionally good for short and intermediate maturities and acceptable for long maturities (1–2 years). As for the vanna-volga methods, it is visible that the exact method better fits the data for each tenor, resulting in smaller SSE and SSE^* values for this method. The simplified vanna-volga

Table 5.4: SSE and SSE^* values calculated for the implied volatilities fit of EUR/PLN options, maturing in 1 week, 1, 3, 6 months, 1 and 2 years, traded on March 23, 2011, obtained using the exact vanna-volga method (SSE_{VVe}), Wystup's vanna-volga method (SSE_{VVw} and SSE^*_{VVw}) and the Heston model (SSE_H and SSE^*_H). Due to the fact that $SSE^*_{VVe} = SSE_{VVe}$, SSE of the exact vanna-volga fit for $\sigma_{10\Delta C}$ and $\sigma_{10\Delta P}$ was not included in the table.

March 23, 2011	SSE_{VVe}	SSE_{VVw}	SSE_H	SSE^*_{VVw}	SSE^*_H
1W	5.28E-03	4.39E-03	8.92E-04	3.98E-03	1.40E-04
1M	5.25E-06	1.24E-05	2.80E-08	1.16E-05	2.09E-09
3M	1.56E-05	4.03E-05	1.44E-06	3.55E-05	1.07E-07
6M	9.78E-05	1.95E-04	3.41E-05	1.83E-04	9.89E-06
1Y	1.54E-04	3.14E-04	6.28E-05	2.88E-04	1.83E-05
2Y	1.01E-05	5.36E-05	2.65E-06	3.79E-05	6.79E-07

method overestimates σ_{10C} and σ_{10P} for tenors up to three months, and σ_{10C} and σ_{25C} for longer tenors. In turn, the exact method always overestimates both extreme implied volatilities σ_{10C} and σ_{10P}, but not as much as Wystup's VV method, as the overall fit of the exact method is better. The best exact vanna-volga method fit is observed for the EUR/PLN option maturing in 2 years, the best Wystup's method fit – for the option with 6 months to expiry and the best Heston model fit – for the option with 1 month to maturity.

Figure 5.10 depicts a very common pattern of the volatility surface alongside the smile, called a skew. In this case we deal with a forward type of the skew (see definition A.2 in Appendix A). Skew fits in Figure 5.10 are implied by the EUR/USD options traded on July 22, 2010. Similarly to the previous examples, the Heston model fit for short 1 week maturity is unsatisfactory, compared to the other tenors. We also observe a relatively poor fit of the exact vanna-volga method – σ_{10C} and σ_{10P} values are underestimated for each tenor except for 2Y, for which σ_{10P} is slightly above the input value $\hat{\sigma}_{10P}$. Wystup's method also underestimates all the implied volatilities for a given set of delta pillars (Δ_{10C}, Δ_{25C}, Δ_{ATM}, Δ_{25P} and Δ_{10P}) for intermediate and long maturities (above 3 months).

Another remark also applies to Wystup's vanna-volga method – the longer the tenor, the poorest the fit. Therefore, the worst fit is observed for the option maturing in 2 years. It is such bad, that this is one of two tenors (together with 1Y) for which the sum of squared errors SSE_{VVw} is greater than the corresponding SSE_{VVe}. This fact might be surprising but it is confirmed by SSE values, omitted in the thesis. As far as SSE^* is concerned, SSE^*_{VVw} is always smaller than the corresponding SSE^*_{VVe}, equal to SSE_{VVe}. Hence, this is the example of the fit when Wystup's vanna-volga method is more accurate than the exact one for most of tenors. As we can see, it depends on market data, which fit is better. This confirms that Wystup's vanna-volga method is not useless when we have the exact vanna-volga method at our disposal – for some examples of non-standard volatility surfaces it can give slightly better results.

Last but not least, we analyse the most recent data and calibrate both vanna-volga methods and the Heston model to the EUR/PLN volatility surface observed on March 23, 2011. The results are shown in Table 5.4 and Figure 5.11. This time the volatility surface

is very non-standard – it changes significantly with maturities. The only accurate fit and two relatively good ones of all three methods are observable for 1M, 3M and 2Y tenors, respectively, when the implied volatility surface shape is more typical and resembles a reverse skew (see definition A.2 in Appendix A).

Even the Heston model does a poor job of reconstructing the input volatilities for 1W, 6M and 1Y tenors for an unusual IV surface shape. However, it is not that bad as for both vanna-volga methods, which significantly under- or overestimate both extreme implied volatilities, σ_{10C} and σ_{10P}, for the above mentioned tenors. Wystup's VV method generally does not fit properly also the middle implied volatilities – σ_{25C}, σ_{ATM}, σ_{25P} – with one exception of a decent fit observed for the option with 1 month to expiry. The most accurate fit for each vanna-volga method and the Heston model is assigned to the 1M tenor, and the worst one – to the 1W tenor. SSE and SSE^* values (see Table 5.4) say it all – they are a couple of magnitudes larger than SSE or SSE^* in Table 5.3. Similarly to the fit from July 1, 2004, the exact vanna-volga method performs better than Wystup's method except for the 1W tenor, for which we observe the opposite relationship. Values of SSE and SSE^* confirm this remark.

To sum up, all analysed examples in this Section show that the exact vanna-volga method in majority of cases fits better the input implied volatilities than the simplified method. This fact is in accordance with the remark made by Carr, Hogan and Verma (2006). Nevertheless, it is not universally true and this might not be the case for certain market data (for example, for those quoted on July 22, 2010). As a general rule, both vanna-volga methods perform substantially better when the input volatility surface shape is more *standard*. They fit very well volatility smiles, especially symmetric ones. They fit the skews quite well, but they are unfortunately unsatisfactory for different volatility surface shapes (see for example Figure 5.11).

Usually the fastest calibrations were made for the simplified vanna-volga method, only slightly faster than for the exact vanna-volga method. It is not a surprise, as there are no time-consuming operations on matrices in equation (3.8). Heston model, because of its complexity, does the best job of reconstructing the input implied volatilities. However, this complexity has a huge influence on the speed of calculations. It is true that both vanna-volga methods yield larger deviations from the quoted market implied volatilities, but, on the other hand, they offer a speed advantage. Producing the same kind of fits by the exact vanna-volga method, Wystup's vanna-volga method and the Heston model, as presented in Figures 5.8, 5.9, 5.10 and 5.11, took on average 150 times longer for the Heston model than in the case of both vanna-volga methods. That would lengthen a great deal more when dealing with portfolios consisting of many instruments. For this reason, the vanna-volga *rule of thumb* may appeal to bankers much more than the Heston model.
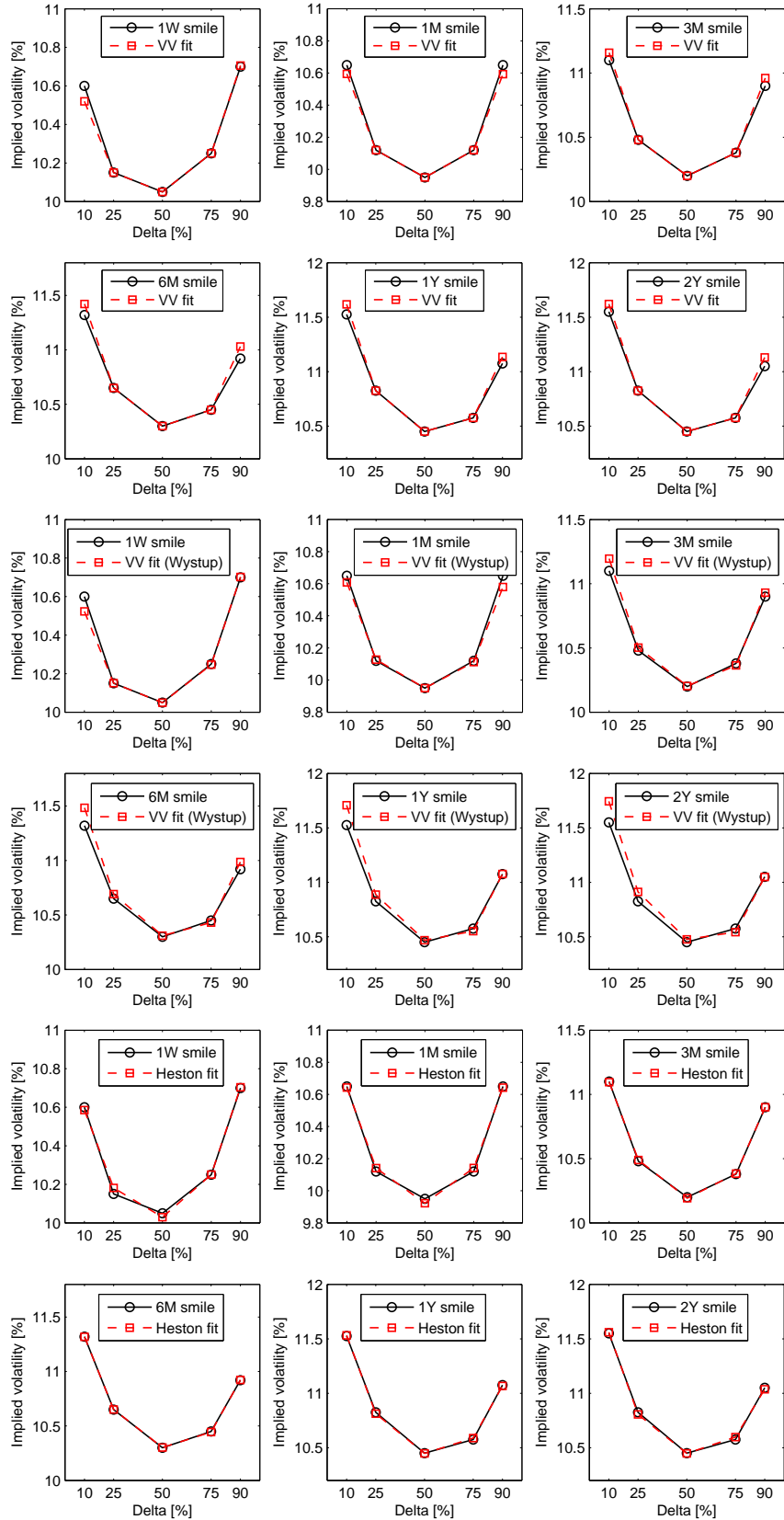
Figure 5.8: The EUR/USD market smile observed on July 1, 2004 and the fit obtained with the exact vanna-volga method (two rows at the top), Wystup's vanna-volga method (two rows in the middle) and the Heston model (two rows at the bottom) for different times to maturity τ: 1 week , 1, 3, 6 months, 1 and 2 years. For the 1 month option the same set of market data as in Table 5.2 was adopted, for the remaining options – quoted market data assigned to their maturities with spot $S_O = 1.215$ EUR/USD.
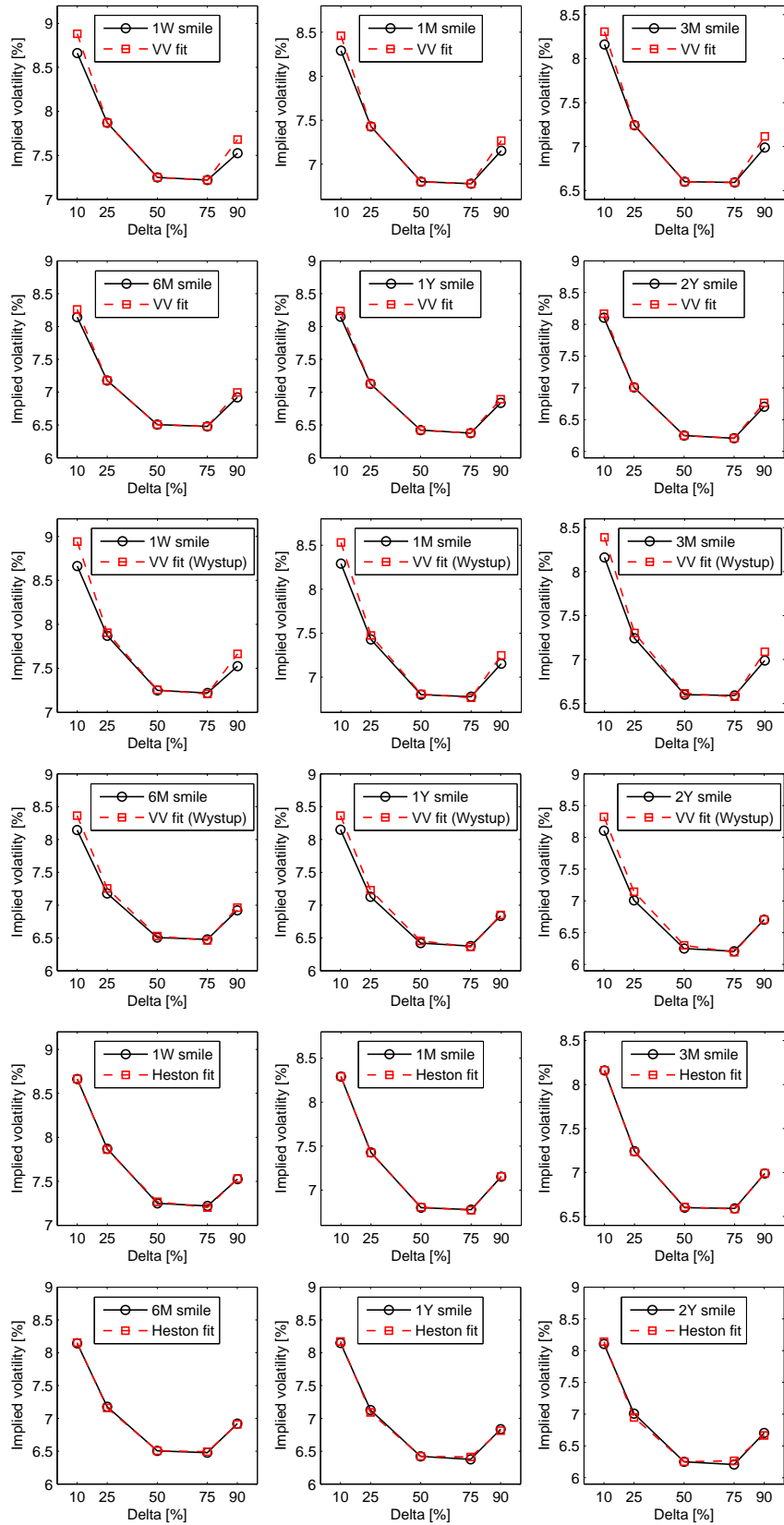
Figure 5.9: The EUR/PLN market smile observed on July 31, 2008 and the fit obtained with the exact vanna-volga method (two rows at the top), Wystup's vanna-volga method (two rows in the middle) and the Heston model (two rows at the bottom) for different times to maturity τ: 1 week , 1, 3, 6 months, 1 and 2 years. Quoted market data assigned to the maturities and spot $S_O = 3.2063$ EUR/PLN were used.
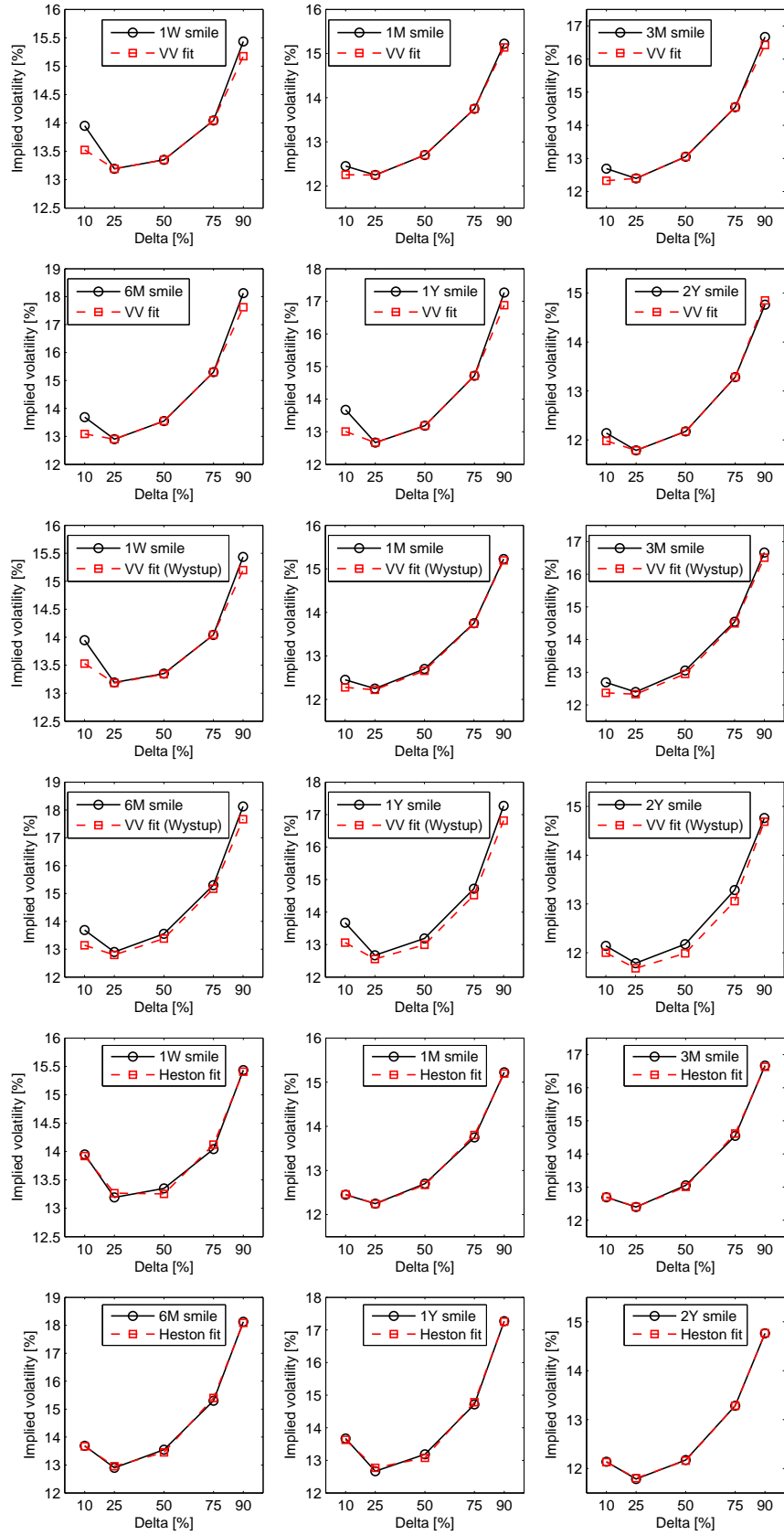
Figure 5.10: The EUR/USD market smile observed on July 22, 2010 and the fit obtained with the exact vanna-volga method (two rows at the top), Wystup's vanna-volga method (two rows in the middle) and the Heston model (two rows at the bottom) for different times to maturity τ: 1 week , 1, 3, 6 months, 1 and 2 years. Quoted market data assigned to the maturities and spot $S_O = 1.2779$ EUR/USD were used.
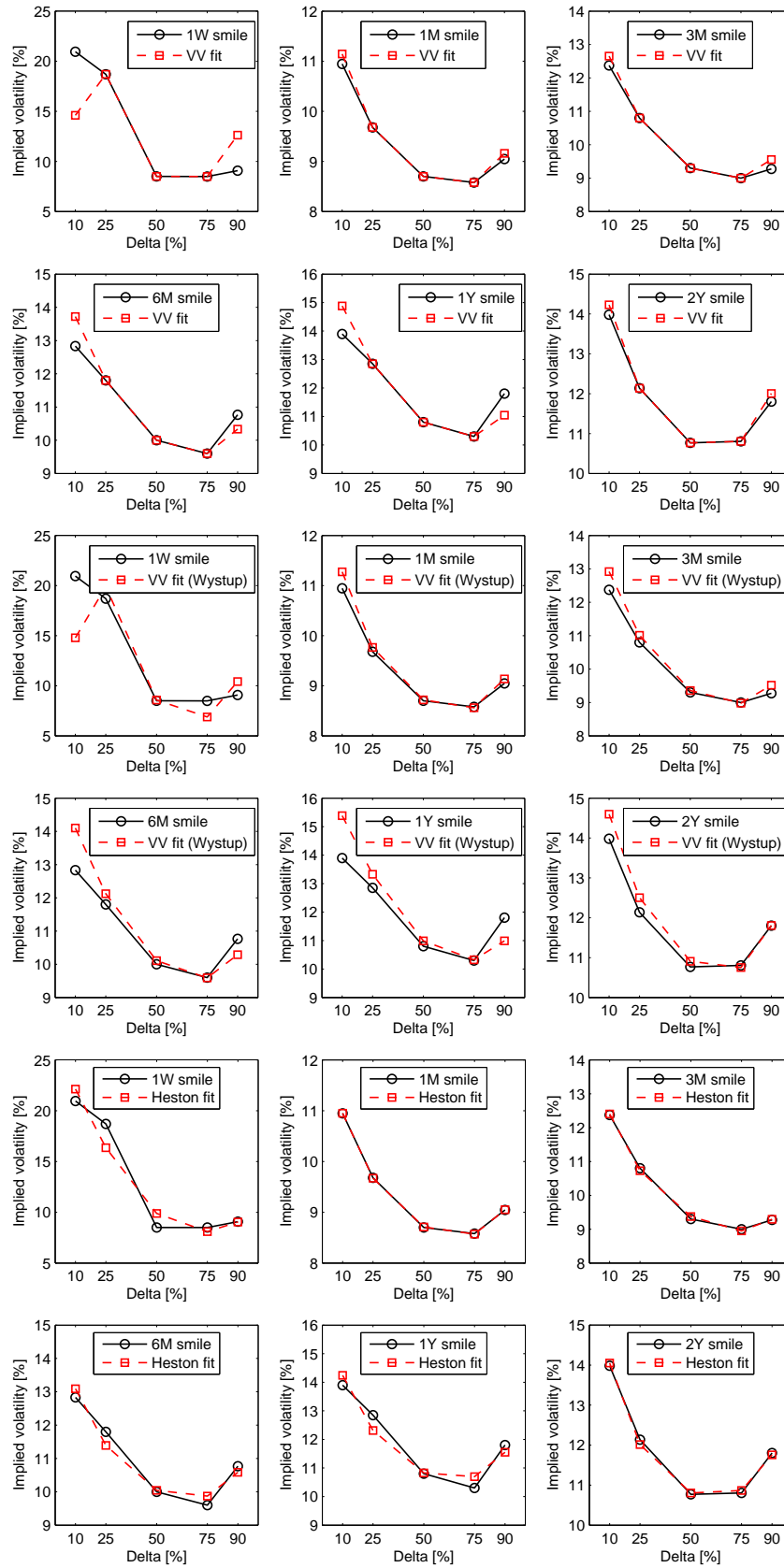
Figure 5.11: The EUR/PLN market smile observed on March 23, 2011 and the fit obtained with the exact vanna-volga method (two rows at the top), Wystup's vanna-volga method (two rows in the middle) and the Heston model (two rows at the bottom) for different times to maturity τ: 1 week , 1, 3, 6 months, 1 and 2 years. Quoted market data assigned to the maturities and spot $S_O = 4.03$ EUR/PLN were used.

5.4 Calibration of barrier options

In Section 5.2 we calibrated the vanna-volga prices of vanilla options. In this Section we move on to more interesting options from a trader's point of view – barrier options. They receive the lion's share of the traded volume. They are so attractive because, being cheaper than the corresponding plain vanillas, they can generate the same profits as Europan options (see Derman and Kani, 1996).

We had at our disposal the market prices of two kinds of barrier options: *down-and-out* calls (DOC) traded on August 12, 2009 and *up-and-out* calls (UOC) traded more recently, on March 23, 2011. *Down-and-out* options can have a positive payoff only if the barrier (which is lower than the current spot price) is never crossed before expiration. As long as the FX spot never crosses the barrier, the barrier knock-out option remains a plain vanilla option. By analogy, an up-and-out option has a barrier that is higher than the spot. In total there are 16 types of barrier options and we refer to Table A.1 to become familiar with all of them.

Both quoted types of barrier options are EUR/PLN calls. As down-and-out calls traded on August 12, 2009 have non-standard expiry times (see Table 5.5), we first interpolate all input data such as implied volatilities and EUR and PLN interest rates, and then, using interpolated values of these, we calculate the vanna-volga prices. Then we compare the results produced by the exact vanna-volga method and Wystup's vanna-volga method applied to barrier options. We do not, however, compare them with barrier option prices obtained from the Heston model, as it is beyond the scope of this thesis. We investigate barrier options features with respect to the FX spot rate rather than delta, because it is easier to observe the effect of reaching the barrier.

The greeks are available in closed form not just for vanilla options (A.7)–(A.12) but also for barrier options. Nevertheless, following Castagna (2009), without substantial loss of computing time or accuracy, it is easier to calculate them numerically. On the other hand, in all the papers covering greeks for barrier options, one can see only formulas for the most popular greeks such as delta, gamma, vega or theta (see for example Wystup, 2002) and the remaining greeks are omitted. Vanna and volga calculations are rather tedious, therefore this was another reason supporting using finite difference methods to calculate the necessary greeks. Figure 5.12, comparing vega for the down-and-out EUR/PLN options traded on August 12, 2009 obtained analytically and numerically, confirms that the approximation is extremely accurate and both plots practically coincide. The vega-related derivatives are approximated as follows:

- Vega

$$\frac{\partial O}{\partial \sigma} = \frac{O^{BS}(S_t, \sigma_{ATM} + \Delta\sigma) - O^{BS}(S_t, \sigma_{ATM} - \Delta\sigma)}{2\Delta\sigma}, \tag{5.5}$$

- Volga

$$\frac{\partial^2 O}{\partial \sigma^2} = \frac{O^{BS}(S_t, \sigma_{ATM} + \Delta\sigma) - 2O^{BS}(S_t, \sigma_{ATM}) + O^{BS}(S_t, \sigma_{ATM} - \Delta\sigma)}{\Delta\sigma^2}, \tag{5.6}$$
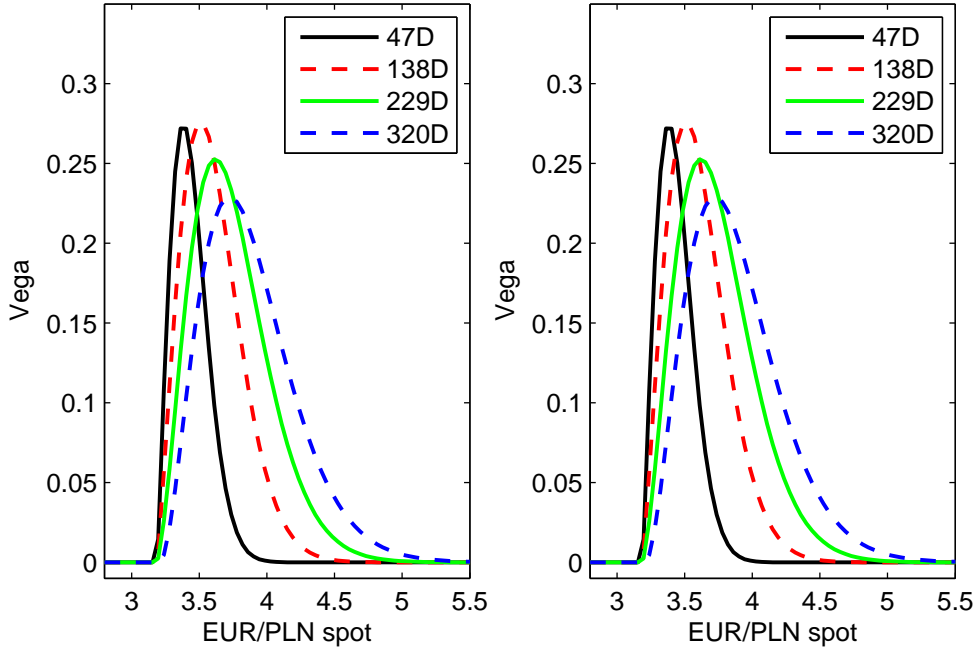
Figure 5.12: *Left panel*: Black-Scholes vega with respect to the EUR/PLN spot of the down-and-out call option traded on August 12, 2009, maturing in 47, 138, 229 or 320 days. *Right panel*: a numerical approximation of vega for the same market data. A numerical approximation and the analytical value are the same up to 8 decimal places.

- Vanna

$$
\frac{\partial^2 O}{\partial \sigma \partial S} = \frac{O^{BS}(S_t + \Delta S, \sigma_{ATM} + \Delta\sigma) - O^{BS}(S_t - \Delta S, \sigma_{ATM} + \Delta\sigma)}{4\Delta\sigma\Delta S} \tag{5.7}
$$
$$
- \frac{O^{BS}(S_t + \Delta S, \sigma_{ATM} - \Delta\sigma) - O^{BS}(S_t - \Delta S, \sigma_{ATM} - \Delta\sigma)}{4\Delta\sigma\Delta S},
$$

with $\Delta\sigma$ and ΔS set at a suitably small level, i.e. $\Delta\sigma = 0.01\%$ and $\Delta S = 0.005 S_t$. This time O^{BS} denotes the Black-Scholes price of the barrier option (see definition A.11 in AppendixA). Similarly to what was done in Section 5.2, we suitably adjust all the relevant formulas to take into consideration simple discounting for options maturing up to 1 year and compound discounting for options with maturities above 1 year.

Table 5.5 presents prices (in PLN) of EUR/PLN down-and-out call options, traded on August 12, 2009, maturing in 47, 138, 229 and 320 days, with FX spot $S_0 = 4.1511$, strike $K = 3.29$ and barrier $B = 3.19$. The premiums are calculated by multiplying PLN currency units times the notional amount (in EUR currency units) equal to $N = 250\,000$. We compare provided market prices of these options with the calibrated results obtained from: the Black-Scholes model for barrier options (A.24)–(A.27), the exact vanna-volga method for barriers with survival probability p equal to the domestic risk-neutral no-touch probability (3.11)–(3.12), Wystup's vanna-volga method for barriers with the same p (3.13) and the exact vanna-volga method for barriers with weighted survival probability (denoted by p^*), proposed in Section 4.1 (4.2).

The penultimate column of Table 5.5 shows that the values of the domestic risk-neutral no-touch probabilities (p) of each down-and-out call option are very high – all above

95%. It is almost certain that 1 week DOC option will not knock out before expiry, as $p \approx 100\%$. The spot (S_0) is far enough from the barrier level (B). This fact obviously has an impact on the prices of the options. Analysed options are quite expensive because of high probability of exercising and being deep in-the-money as strike K is much lower than the FX spot rate. The relationship between survival probability p and spot of the DOC option is plotted in Figure 5.14 for standard maturities (1 month, 3 months, 6 months and 1 year). For shorter tenors the p curve is steeper and faster (i.e. for lower spot rates) approaches value of 1 than for longer tenors. This has its explanation in the fact that sudden moves of the FX rate in the opposite direction are highly unlikely in a short period of time. Obviously when the spot reaches the barrier $B = 3.19$, regardless of the maturity of the option, the survival probability is equal to 0 and the same holds for the payoff of the option.

We also notice in Table 5.5 that premiums calculated using Wystup's vanna-volga method and the exact vanna-volga method with weighted survival probability p^* most closely match the market premiums. In the last column of Table 5.5 one can see relative errors E between the best calibration method, i.e. the exact VV method with p^*, and market prices, calculated in the following way:

$$E = \frac{VV_{p^*} - MK}{MK} * 100\%. \tag{5.8}$$

Relative error values E are on average about -7%, which means that the exact VV prices with weighted survival probability are slightly underestimated. Yet it is still a reasonably good result taking into consideration that we are not quite sure what market data and methods are taken into account when determining market prices of the options.

As far as the DOC option with 320 days to maturity is concerned, we observe that the vanna-volga prices obtained by the three methods give lower prices than the corresponding Black-Scholes ones (see the values marked in bold in Table 5.5). This confirms the point already made in Section 5.2 that the vanna-volga correction can be also negative. We see now that this fact applies but also to barrier options not just to plain vanillas. The contribution of the vanna-volga adjustment of the DOC options (with the same market data as in Table 5.5 except for maturities, which were changed into standard ones so that the results obtained for the DOC and UOC barrier options could be easily compared) with respect to the EUR/PLN spot rate is plotted in Figure 5.13.

We conclude that the VV overhedge curves for each option tenor are similar in shape. They are just shifted in the direction of higher prices. The VV adjustment is not needed any more once the spot reaches the barrier, hence $O^{VV} = 0$ for $S_t < B$. The VV overhedge gains in value with the increase of the spot value, reaching its maximum at some point and then again decreases to zero, which implies equating the VV price with the BS price. It is worth noting that the contribution of the VV adjustment is relatively small compared to the overall VV price, which shows the the vega-related greeks somehow cancel each other out (see Figures 5.17 - 5.19).

Now we will investigate vanna-volga prices of barrier options, traded more recently, on March 23, 2011. Table 5.6 compares premiums of the up-and-out call options obtained using the same methods as in Table 5.5. The main characteristics of the UOC options are as follows: 4 maturities – 1, 3, 6 months or 1 year, spot $S_0 = 4.03$, set of at-the-money forward (ATMF)[7] strikes $K \in \{4.04, 4.05, 4.07, 4.11\}$ corresponding to subsequent tenors and two different upper barriers B corresponding to each strike. The barriers in the first
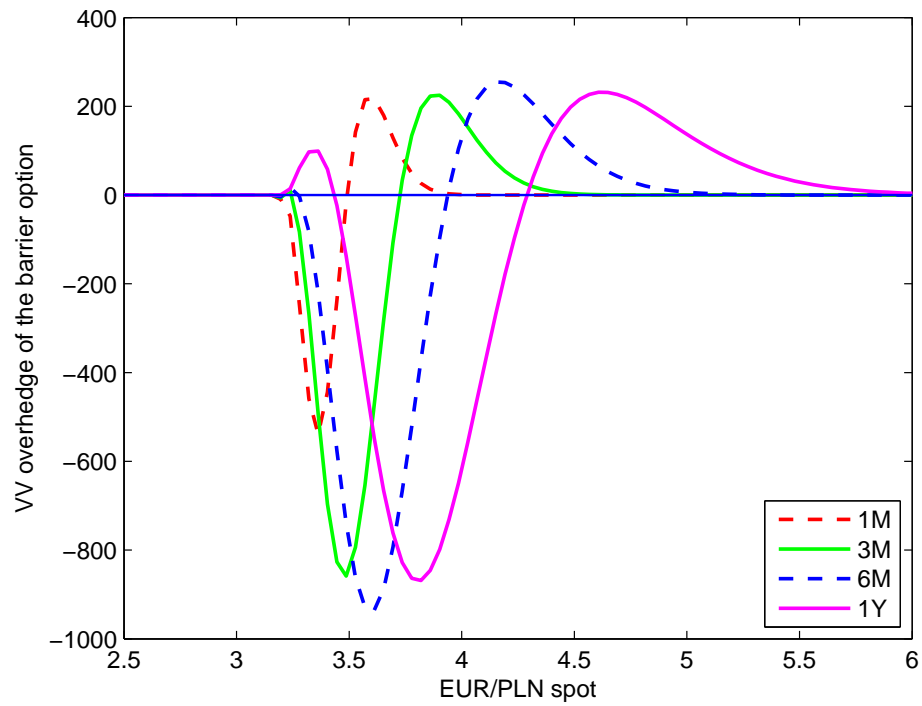
Figure 5.13: The vanna-volga overhedge calculated from equation (4.2) for the EUR/PLN down-and-out call option traded on August 12, 2009, maturing in 1, 3, 6 months or 1 year.
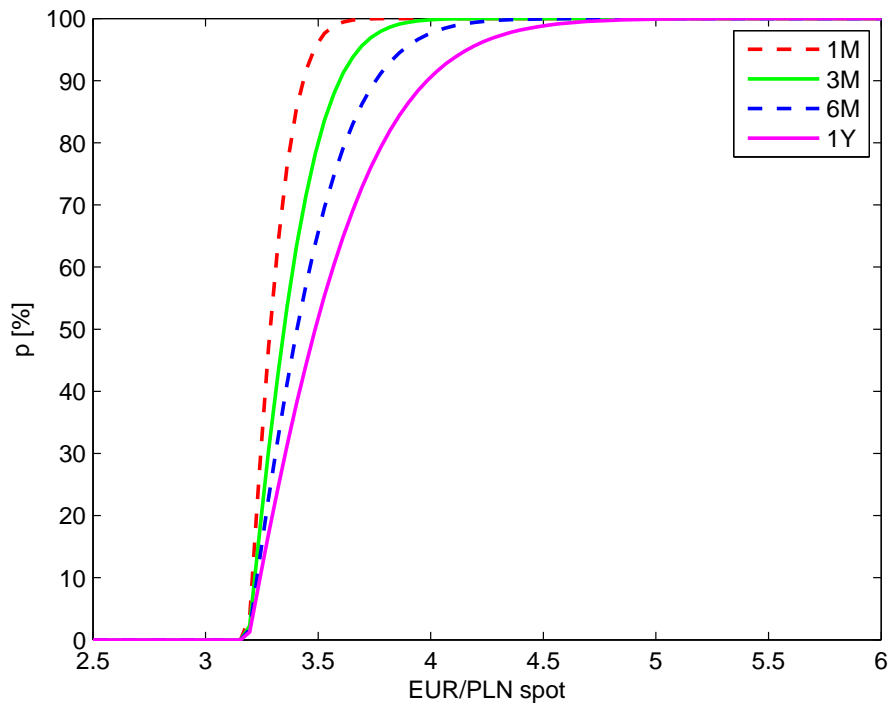

Figure 5.14: Survival probabilities of the EUR/PLN down-and-out call option traded on August 12, 2009, maturing in 1, 3, 6 months or 1 year. Survival probabilities are equal to the risk-neutral no-touch probabilities, calculated from equation (3.14).

4 rows of Table 5.6 are equal to strikes corresponding to Δ_{25C} for vanillas and the barriers in the last 4 rows are equal to the strikes corresponding to Δ_{10C} for vanillas. All prices are expressed in PLN pips[8].

The VV corrections and survival probabilities of the up-and-out call options are plotted in Figures 5.15 and 5.16. Survival probabilities of the UOC options with spot $S_0 = 4.03$ and the first set of barriers $B_{\Delta_{25C}}$ are not that high as in the previous example. It is on average 45% likely that the options will knock out. Indeed, the spot and the barriers are close in values for each option tenor, which implies a quite low survival probability. This fact is not without influence on the premiums – they are relatively cheap. Comparing shapes of the p curves plotted in Figures 5.14 and 5.16, we notice the adverse behaviour of the survival probability of the DOC option and UOC option. When the spot increases, the survival probability decreases as far as the latter option is concerned. This follows from the fact that the spot is approaching the barrier. The survival probability decreases faster in the left panel of Figure 5.16. Due to the fact that in the left panel we deal with lower values of barriers ($B_{\Delta_{10C}}$) than in the right panel. Contrary to Figure 5.14, no monotonic relationship between the FX rate and maturities can be spotted in Figure 5.16. This is because we deal with different strikes and barriers for each option, so they are not that easily comparable.

The survival probabilities of the last four UOC options in Table 5.6 (i.e. the ones with barriers $B_{\Delta_{10C}}$) are on average equal to 85%, which is in accordance with our observations made from analysing both panels of Figure 5.16. Therefore, these options are more expensive than those with barriers $B_{\Delta_{25C}}$.

Again, the exact vanna-volga does the best job of approximating market prices. The range of relative errors E in case of UOC options with barriers $B_{\Delta_{25C}}$ is smaller than for the DOC options considered earlier – on average the VV prices are undervalued or overvalued by 4-5%. Value of the 1 year UOC option with barrier $B = 4.36$ in the Black-Scholes model is almost twice as low as its market price. This fact shows how useful the vanna-volga method is. As far as the UOC options with spot $S_0 = 4.03$ and barriers $B_{\Delta_{25C}}$ are concerned, the VV correction is always positive, meaning that the Black-Scholes prices are undervalued for each option tenor. On the other hand, the same options with barriers $B_{\Delta_{10C}}$ always have a negative VV overhedge (compare both panels of Figure 5.15). The premium approximation obtained by the vanna-volga method that takes into account foreign/domestic symmetry in the FX options, is incredibly good for the UOC options with barriers $B_{\Delta_{10C}}$ and tenors 1M, 3M and 1Y. The relative errors E are very small – around 1.5%. Only the value of VV_{p*} for the option maturing in 6 months is much higher than its market price. The relative error of this approximation is around 22.63%, which is a lot in comparison to the remaining options. This overestimation may by caused by numerical errors or lack of some extra knowledge about the market behaviuor, that was taken into consideration when determining the market price of the option.

The overhedge curves for both sets of barriers $B_{\Delta_{25C}}$ and $B_{\Delta_{10C}}$, plotted in Figure 5.15,

---

[7]At the money forward (ATMF) option is the FX option where the strike is the same as the outright forward foreign exchange rate (notion used by Reiswich and Wystup, 2010) at the time the option is written.

[8]The smallest price change that a given exchange rate can exhibit, which is equal to the change of the last decimal point. For the EUR/PLN pair this is the equivalent of 1/100 of one percent, or one basis point.
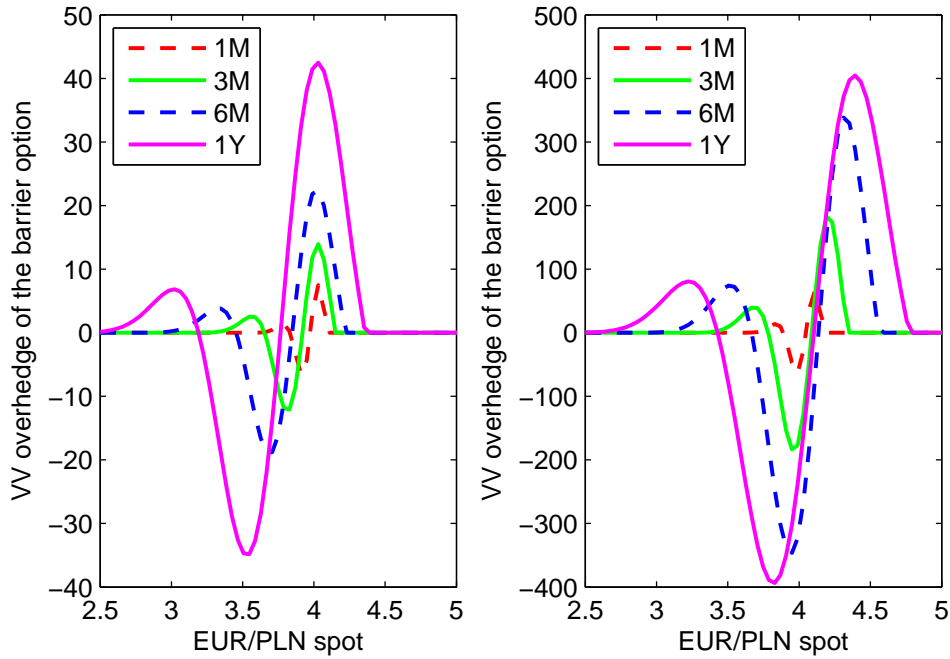
Figure 5.15: *Left panel*: The vanna-volga overhedge of the EUR/PLN up-and-out call options with barriers $B_{\Delta_{25C}}$, traded on March 23, 2011, maturing in 1, 3, 6 months or 1 year. *Right panel*: The vanna-volga overhedge of the EUR/PLN up-and-out call options with barriers $B_{\Delta_{10C}}$, with the same maturities and traded on the same day.

have pretty similar shapes – the only difference is that the curves for particular tenors are shifted. Unlike in the case of DOC options, the range of values of the vanna-volga adjustments for UOC options suggests that their contribution to the overall VV price is significant. For each UOC option the overhedge before hitting the barrier is positive. Therefore, the option, for which the spot is just before reaching the barrier, is more expensive than the same option in the Black-Scholes setting. One of the possible reasons why this happens is that the option is deep in-the-money when the spot rate approaches the barrier and hence the buyer can profit the most in that situation, provided that the barrier will never be hit. When the spot rate of the option is far away from the upper barrier and strike (i.e. $S_t \ll K$) the VV overhedge goes to zero and the VV price of the option is equal to its price in the Black-Scholes world.

Figures 5.17, 5.18 and 5.19 show the vanna-volga overhedge of the down-and-out call option traded on August 12, 2009, decomposed into each of the vega-related greeks. In Figure 5.17 we see a numerical approximation of vega of the DOC option. The vega contribution to the overall VV overhedge (3.7) is about 10 times smaller than the contributions of the two other greeks. This confirms that a suitable weighting of the vega, vanna and volga components does make sense. As long as the spot is far away from the barrier, the vega shape of the DOC option resembles the shape of the same greek of a vanilla call option (compare Figure 5.17 with Figure 5.5). This is reasonable because a DOC option behaves like a plain vanilla option when the FX spot is far away from the barrier.

What is more, vega is almost always positive for each DOC option traded on August 12, 2009, except for the option maturing in 1 year – when spot is close to the barrier the
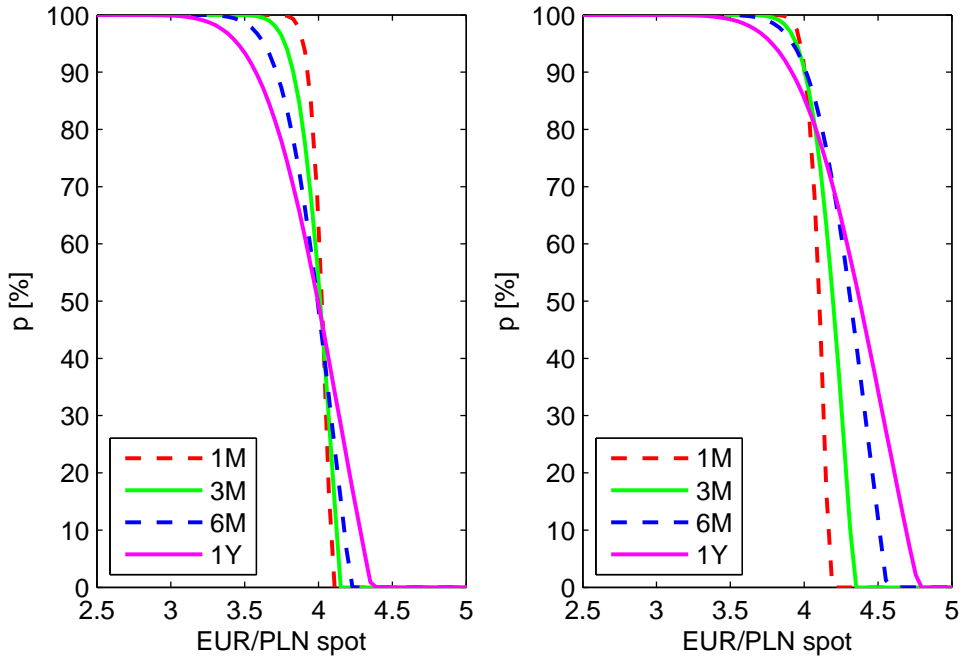
Figure 5.16: *Left panel*: Survival probabilities of the EUR/PLN up-and-out call options with barriers $B_{\Delta_{25C}}$, traded on March 23, 2011, maturing in 1, 3, 6 months or 1 year. *Right panel*: Survival probabilities of the EUR/PLN up-and-out call options with barriers $B_{\Delta_{10C}}$, with the same maturities and traded on the same day. The survival probabilities are equal to the risk-neutral no-touch probabilities.

greek is slightly below zero. Vegas of the UOC options with barriers $B_{\Delta_{10C}}$ are plotted in Figure 5.20. In this case vegas switch sign from positive (on the left) to negative (on the right) when the spot changes.

Vanna of the DOC option is positive just before the spot reaches the barrier. Similarly to vega, this remark does not apply to the option maturing in 1 year - vanna is then also slightly below zero. For spot values near the barrier, vanna and volga behave differently: vanna becomes large, while volga becomes small. Therefore, as noted by Bossens et al. (2010), the conditions imposed on attenuation factors p_{vanna} and p_{volga} should reflect that behaviour. Thus volga of the DOC option is negative just before hitting the barrier. Again, this is true for all tenors except for 1Y. A closer inspection of Figure 5.19 does reveal that volga of the 1Y DOC option is slightly positive for spot close to the barrier $B = 3.19$. A quite similar swinging profile of volga can be observed for the UOC options with barriers $B_{\Delta_{10C}}$ (see Figure 5.20). In this case contribution of volga is much larger than the contributions of the remaining greeks. From Figures 5.17-5.20 we conclude that for very small or very large values of the spot, vega, vanna and volga approach zero, meaning that the vanna-volga overhedge is not needed any more for this kind of options.

To put it briefly, the vanna-volga methods approximate reasonably well market premiums of the DOC and UOC barriers. The exact vanna-volga method with weighted survival probability performs very well, Wystup's vanna-volga method with domestic risk-neutral no-touch probability as the survival probability gives slightly less accurate results and the worst approximation is obtained by the exact vanna-volga method with the same survival probability. This fact shows that a suitable weighting of the survival probabilities assigned to each greek component is crucial to obtain satisfying results.

Table 5.5: Comparison of the EUR/PLN down-and-out call premiums (in PLN), traded on August 12, 2009, obtained using the Black-Scholes model for barriers (A.24)–(A.27), denoted by *BS*, the exact vanna-volga method for barriers with survival probability p equal to the domestic risk-neutral no-touch probability (3.11)–(3.12), denoted by *VV exact*, Wystup's vanna-volga method for barriers with the same p (3.13), denoted by *VV Wystup* and the exact vanna-volga method for barriers with weighted survival probability p^* (4.2), denoted by VV_{p^*} – all calculated for 4 levels of maturities: 47, 138, 229 and 320 days. The table also contains quoted market prices of the options, denoted by MK, domestic risk-neutral no-touch probabilities p for each tenor and relative errors between VV_{p^*} and MK denoted by *E [%]*. The VV prices that are lower than the Black-Scholes equivalents are marked in bold.

August 12, 2009	Strike	Barrier	MK	BS	VV exact	VV Wystup	VV_{p^*}	p_{surv}	E [%]
47D	3.29	3.19	234 822.89	217 879.97	217 882.43	217 882.59	217 882.43	100.00%	-7.21%
138D	3.29	3.19	238 796.06	221 842.71	222 046.86	222 067.55	222 047.53	99.72%	-7.01%
229D	3.29	3.19	242 337.19	225 555.53	225 736.26	225 761.55	225 743.86	98.28%	-6.85%
320D	3.29	3.19	245 472.23	228 827.71	**228 702.80**	**228 716.87**	**228 717.82**	96.01%	-6.83%

Table 5.6: Comparison of the EUR/PLN up-and-out call premiums (in PLN), traded on March 23, 2011, obtained using the Black-Scholes model for barriers, the exact vanna-volga method for barriers with survival probability p equal to the domestic risk-neutral no-touch probability, Wystup's vanna-volga method for barriers with the same p and the exact vanna-volga method for barriers with weighted survival probability p^* – all calculated for 4 levels of maturities: 1, 3, 6 and 1 year. The table also contains quoted market prices of the options, domestic risk-neutral no-touch probabilities p for each tenor and relative errors between VV_{p^*} and MK denoted by E [%]. Note that there are two barriers B for each tenor – the barriers in the first 4 rows of the Table are equal to strikes corresponding to Δ_{25C} for vanillas and the barriers in the last 4 rows of the Table are equal to the strikes corresponding to Δ_{10C} for vanillas.

| March 23, 2011	Strike	Barrier	MK	BS	VV exact	VV Wystup	VV_{p^*}	p_{surv}	E [%]
1M	4.0383	4.0928	20.43	12.04	17.06	17.26	19.60	45.76%	-4.08%
3M	4.0537	4.1505	30.88	17.23	26.53	27.15	31.33	43.97%	1.46%
6M	4.0743	4.2217	46.24	25.63	40.49	41.64	48.38	42.38%	4.63%
1Y	4.1125	4.3596	98.14	49.83	78.51	80.63	92.28	45.41%	-5.97%
1M	4.0383	4.1733	133.88	163.92	135.87	132.71	135.89	83.89%	1.50%
3M	4.0537	4.3409	233.30	367.87	241.59	224.00	239.82	87.09%	2.79%
6M	4.0743	4.5604	333.60	676.31	412.27	368.05	409.11	88.84%	22.63%
1Y	4.1125	4.7745	616.28	781.16	614.40	560.51	620.73	83.56%	0.72%
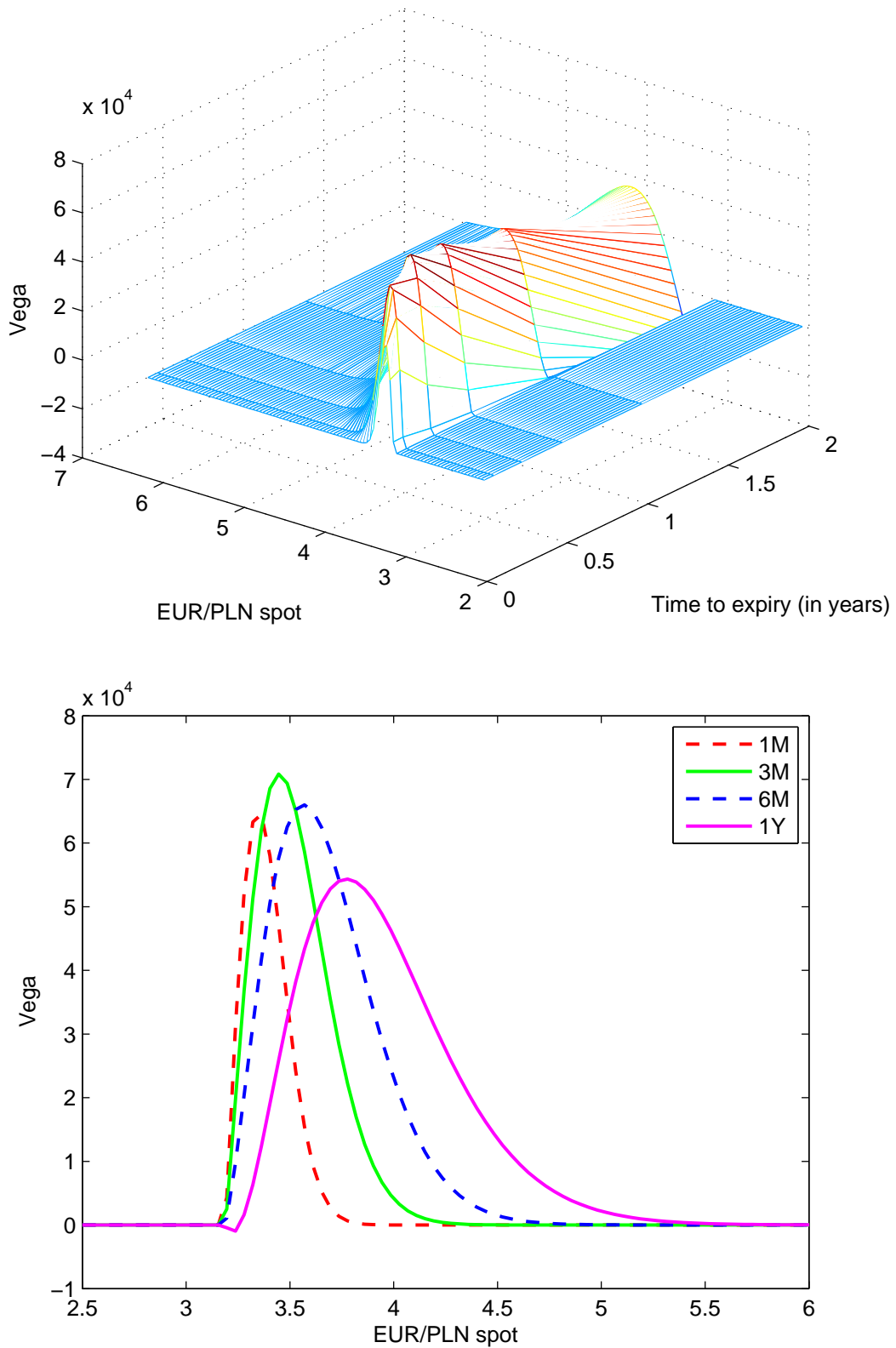
Figure 5.17: *Top panel*: Numerical approximation of the Black-Scholes vega of the down-and-out call option with respect to the EUR/PLN spot and time to maturity T, traded on August 12, 2009, maturing in 1, 3, 6 months or 1 year. *Bottom panel*: The same numerical approximation of the Black-Scholes vega in two dimensions – just with respect to the EUR/PLN spot.

Figure 5.18: *Top panel*: Numerical approximation of the Black-Scholes vanna of the down-and-out call option with respect to the EUR/PLN spot and time to maturity T, traded on August 12, 2009, maturing in 1, 3, 6 months or 1 year. *Bottom panel*: The same numerical approximation of the Black-Scholes vanna in two dimensions – just with respect to the EUR/PLN spot.
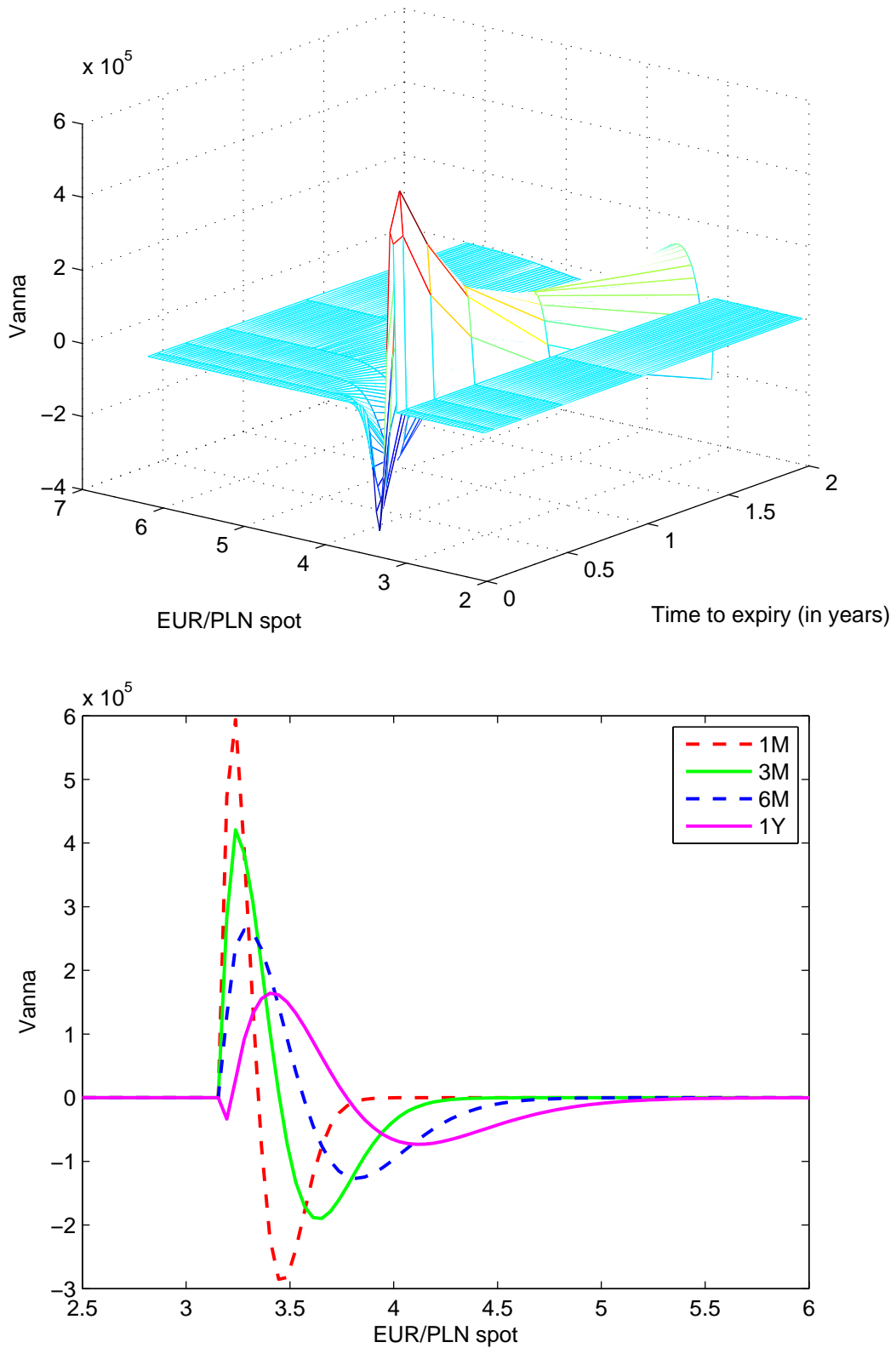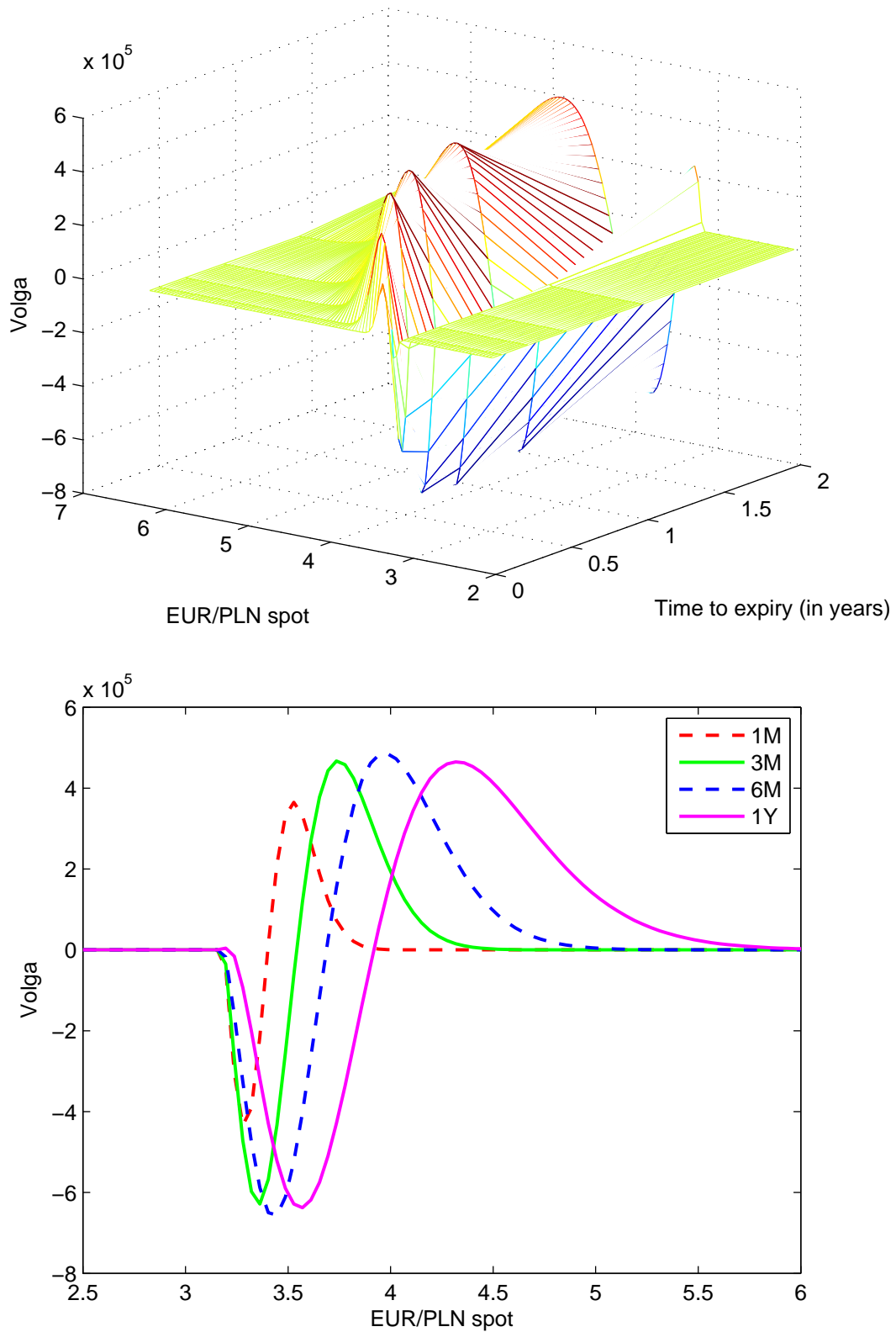
Figure 5.19: *Top panel*: Numerical approximation of the Black-Scholes volga of the down-and-out call option with respect to the EUR/PLN spot and time to maturity T, traded on August 12, 2009, maturing in 1, 3, 6 months or 1 year. *Bottom panel*: The same numerical approximation of the Black-Scholes volga in two dimensions – just with respect to the EUR/PLN spot.
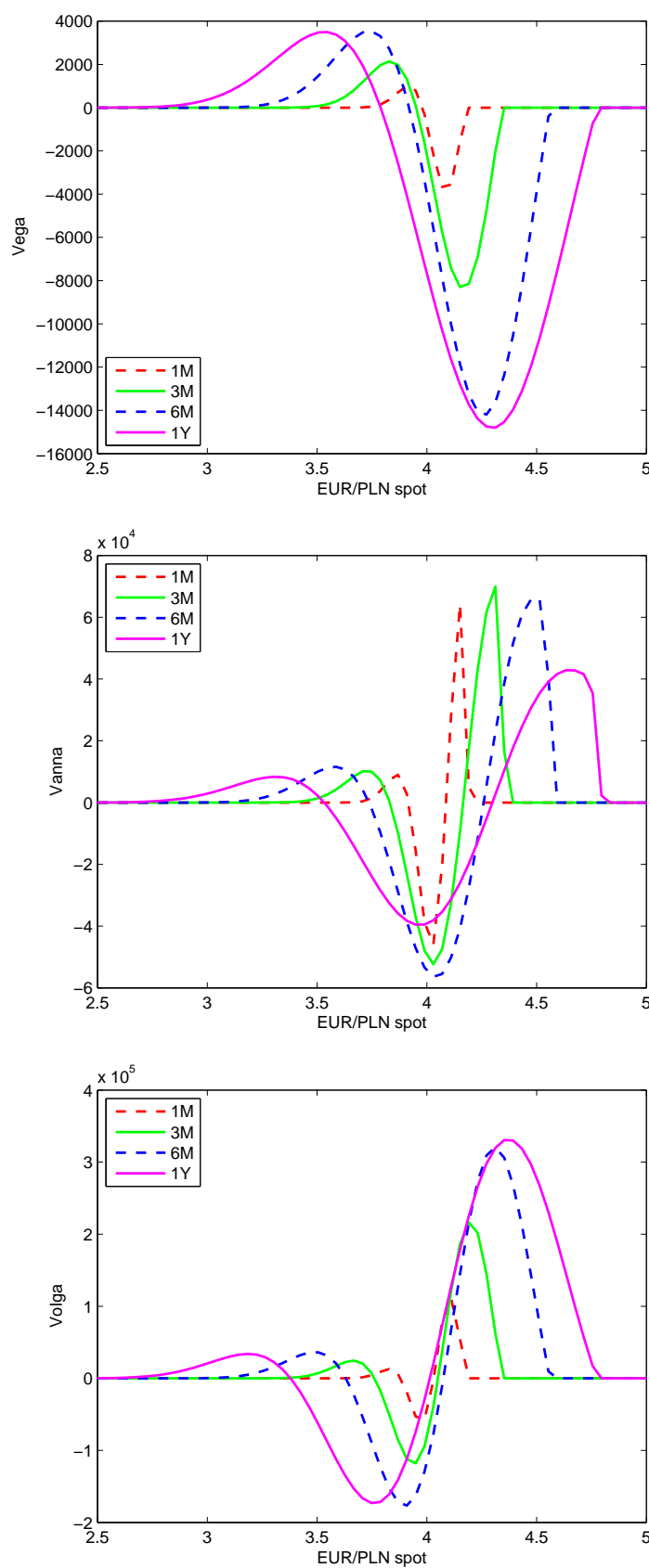
Figure 5.20: Numerical approximation of the Black-Scholes vega (top), vanna (middle) and volga (bottom) of the up-and-out call option with respect to the EUR/PLN spot, traded on August 12, 2009, maturing in 1, 3, 6 months or 1 year.

Chapter 6

Conclusion

In this thesis we have described the vanna-volga approach, i.e. an empirical method that aims to produce market-consistent implied volatilities, prices of plain vanilla options and first-generation exotics. We have proved that, by constructing a locally replicating portfolio whose associated hedging costs are added to the corresponding Black-Scholes prices, we can find the smile-consistent values.

We have implemented in *Matlab* various variants of the vanna-volga method to empirically check the accuracy of each version and indicate the best. The programs were written to obtain the vanna-volga prices of vanilla and barrier options and to produce the vanna-volga implied volatility fits and then compare them with available market data and also with the values implied by the Heston model. For the purpose of implementation we have used the following toolboxes in *Matlab*: Statistic Toolbox, Curve Fitting Toolbox, Optimization Toolbox and Financial Toolbox.

We have compared the implied volatility fits produced by the exact vanna-volga method, Wystup's vanna-volga method and the Heston model. Empirical analysis of almost 15 sets of market data, proved that the latter results were the most accurate. It is due to the fact that while the Heston model uses all (i.e. 5) available volatility quotes for a given maturity to reproduce the market behaviour, both vanna-volga methods use only three available volatility quotes for a given maturity – σ_{25C}, σ_{ATM} and σ_{25P}. Nevertheless, the biggest advantage of both vanna-volga methods over the Heston model is its simplicity and robustness and most of all – the speed of calculations. The Heston model, because of its complexity, requires lots of time-consuming calibrations. As it turns out, it also has problems with reproducing the market prices of exotics (Kucharczyk, 2011).

What is more, the exact vanna-volga method in majority of cases fits better the input implied volatilities than Wystup's method. However, both vanna-volga methods have their limitations. They generally perform well when the input volatility surface shape is more *standard* (symmetric smiles, typical skews etc.). For non-standard volatility surface shapes the fits are unfortunately unsatisfactory.

As for the vanna-volga premiums of barrier options, we have shown that a suitable rescaling of the survival probabilities assigned to each component of vega, vanna and volga is crucial to obtain satisfying results. Therefore, the exact vanna-volga method with weighted survival probability performs incredibly well, Wystup's vanna-volga method with domestic risk-neutral no-touch probability as the survival probability produces slightly less

accurate results and the exact vanna-volga method with the same survival probability gives the least accurate results. The range of errors depends closely on market data and for some data even the first method may lead to large mispricing of barrier options.

It should be emphasized that the vanna-volga approach is not an advanced model, which exactly reproduces volatility matrices or option market prices. It is rather a simple empirical method which can use available data in an efficient manner. The VV method as an approximation technique is not free of limitations.

The vanna-volga pricing scheme for vanillas and barriers and the implied volatility surface construction included in this thesis are general and can be applied not only to FX options, but also in any other market where at least three volatility quotes are available for a given maturity. The VV approach can be also utilized for pricing other exotics than barriers. These valuations often require more adjustments to be made and which are much harder to justify theoretically. For a recent treatment of them we refer to Castagna (2009), as this is beyond the scope of the thesis.

Appendix A

Definitions

For simplicity of notation in the following definitions we assume continuous discounting.

DEFINITION A.1 *Implied volatility*
Implied volatility (IV) is the volatility that, when used in a particular pricing model, yields the theoretical value for the option equal to the current market price of that option. In particular for the Black-Scholes model we have:

$$O^{MK} = O^{BS}(\sigma_{IV}). \tag{A.1}$$

DEFINITION A.2 *Types of volatility skews*

- *Volatility smile*
  A situation when at-the-money (ATM) options have lower implied volatilities than in-the-money (ITM) or out-of-the-money (OTM) options is referred to as a volatility smile, due to the U shape curve it creates, which resembles a smile. It is very common in the Forex market.

- *Volatility forward skew*
  In the forward skew pattern out-of-the-money calls and in-the-money puts are in greater demand compared to in-the-money calls and out-of-the-money puts. This implies lower implied volatilities for ITM calls and OTM puts and greater for OTM calls and ITM puts.

- *Volatility reverse skew*
  The reverse skew pattern suggests that in-the-money calls and out-of-the-money puts are more expensive compared to out-of-the-money calls and in-the-money puts. Therefore, greater implied volatilities are observed for ITM calls and OTM puts and lower for OTM calls and ITM puts.

DEFINITION A.3 *Sticky Delta and Strike Rules*
If sticky delta rule is adopted, then implied volatilities are quoted for each expiry with respect to the delta of the option. Similarly, according to the sticky strike rule implied volatilities are mapped with respect to the strike prices. Sticky delta rule is usually adopted in OTC markets and the latter is used in exchange trading.

DEFINITION A.4 *Itô's lemma*
In its simplest two-dimensional form, the Itô's lemma states that for any process that is modelled via a stochastic differential equation (SDE) of the form: $dX_t = \mu_t\,dt + \sigma_t\,dB_t$ and any twice differentiable function $f(t,X)$ of two real variables X and t we get:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial X} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma_t \frac{\partial f}{\partial X}\,dB_t. \tag{A.2}$$

Equation A.2 is derived from the expansion of $f(t, X)$ in a Taylor series:

$$
\begin{aligned}
df(t, X) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}dt^2 + 2\frac{\partial^2 f}{\partial X \partial t}dX\,dt + \frac{\partial^2 f}{\partial X^2}dX^2 \right) = \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}dX^2,
\end{aligned}
\tag{A.3}
$$

and applying the rules of stochastic calculus (see Weron and Weron, 2005).

DEFINITION A.5 *Black-Scholes PDE*
In the Black-Scholes model, the option value at time t with the payoff $C(T, S_T) = [\phi(S_T - K)]^+$ and time to maturity $\tau = T - t$ can be found as the solution of the Black-Scholes partial differential equation:

$$\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} + (r_d - r_f)\,S_t \frac{\partial C}{\partial S} - r_d C - \frac{\partial C}{\partial \tau} = 0. \tag{A.4}$$

DEFINITION A.6 *Garman-Kohlhagen model*
Garman and Kohlhagen (1983) formulated a model that is equivalent to the Black-Scholes model in the FX setting. In this model a vanilla option value at time t, which is a solution of the Black-Scholes PDE A.4, is given by:

$$O^{BS} = \phi e^{-r_f \tau} S_t \mathcal{N}(\phi d_+) - \phi e^{-r_d \tau} K \mathcal{N}(\phi d_-), \tag{A.5}$$

where:

- S_t - current spot,

- K - strike,

- $\tau = T - t$ - time to maturity (in years),

- $r_{f/d}$ - continuous foreign/domestic interest rates,

- σ - volatility,

- $d_\pm = \frac{\ln \frac{S_t}{K} + \left(r_d - r_f \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$,

- $\phi \in \{1, -1\}$ - *call/put indicator*,

- $\mathcal{N}(x)$ - *cumulative standard normal distribution function.*

DEFINITION A.7 *Put–call parity*
The put–call parity is the relationship:

$$C^{BS}(S_t, K, \tau, \sigma, r_d, r_f) + P^{BS}(S_t, K, \tau, \sigma, r_d, r_f) = S_t e^{-r_f\tau} - Ke^{-r_d\tau}, \qquad (A.6)$$

where C/P denote call/put option, respectively.

DEFINITION A.8 *Greeks*
Greeks such as delta, gamma, vega, vanna and volga are of particular interest in the vanna-volga method. Hence they are explained below:
Spot Delta

$$\Delta_s = \frac{\partial O}{\partial S} = \phi e^{-r_f\tau} \mathcal{N}(\phi d_+). \qquad (A.7)$$

Forward Delta *(also called driftless delta by Reiswich and Wystup (2010) and Wystup, 2006)*

$$\Delta = \frac{\partial O}{\partial v_f} = \phi \mathcal{N}(\phi d_+), \qquad (A.8)$$

where v_f is the value of the forward contract, i.e.
$v_f = e^{-r_d\tau}\left(f(t,T) - K\right) = S_t e^{-r_f\tau} - Ke^{-r_d\tau}$ *and* $f(t,T) = S_t e^{(r_d - r_f)\tau}$ *is a outright forward rate (notation used by Reiswich and Wystup, 2010).*
Gamma

$$\Gamma = \frac{\partial^2 O}{\partial S^2} = e^{-r_f\tau} \frac{n(d_+)}{S_t \sigma\sqrt{\tau}}, \qquad (A.9)$$

where $n(x)$ is the probability density function of the standard normal distribution.
Vega

$$\frac{\partial O}{\partial \sigma} = S_t e^{-r_f\tau} \sqrt{\tau} n(d_+). \qquad (A.10)$$

Volga

$$\frac{\partial^2 O}{\partial \sigma^2} = S_t e^{-r_f\tau} \sqrt{\tau} n(d_+) \frac{d_+ d_-}{\sigma}. \qquad (A.11)$$

Vanna

$$\frac{\partial^2 O}{\partial \sigma \partial S} = -e^{-r_f\tau} \sqrt{\tau} n(d_+) \frac{d_-}{\sigma}. \qquad (A.12)$$

DEFINITION A.9 *At-the-money, 25C and 25P volatility*
At-the-money (ATM) volatilities quoted by brokers can have different interpretations depending on currency pairs. In this thesis we assume that the ATM volatility is the value from the smile curve where the strike is such that the forward delta (A.8) of the call equals, in absolute value, that of the put, which is given by the condition:

$$\Delta_C\left(K_{ATM}, \sigma(K_{ATM})\right) = -\Delta_P\left(K_{ATM}, \sigma(K_{ATM})\right) = 0.5. \qquad (A.13)$$

This immediately leads to:

$$K_{ATM} = S_0 e^{\left(r_d - r_f + \frac{1}{2}\sigma^2_{ATM}\right)\tau}. \tag{A.14}$$

By analogy, σ_{25C} and σ_{25P} correspond to the volatilities at the strikes K_{25C} and K_{25P}, respectively, for which the following conditions are satisfied:

$$\Delta_C\left(K_{25C}, \sigma(K_{25C})\right) = 0.25, \tag{A.15}$$

$$\Delta_P\left(K_{25P}, \sigma(K_{25P})\right) = -0.25. \tag{A.16}$$

DEFINITION A.10 *Strangle, straddle, ATM, butterfly, risk reversal*
In very liquid FX markets these are some of the most traded strategies. They are explained below:

$$Strangle(K_C, K_P) = Call(K_C, \sigma(K_C)) + Put(K_P, \sigma(K_P)), \tag{A.17}$$

$$Straddle(K) = Call(K, \sigma_{ATM}) + Put(K, \sigma_{ATM}), \tag{A.18}$$

$$ATM(K) = \frac{1}{2}Straddle(K), \tag{A.19}$$

$$Butterfly(K_P, K, K_C) = \frac{1}{2}[Strangle(K_C, K_P) - Straddle(K)], \tag{A.20}$$

$$Risk\ Reversal(K_C, K_P) = Call(K_C, \sigma(K_C)) - Put(K_P, \sigma(K_P)). \tag{A.21}$$

Hence
25Δ-Risk-Reversal (RR) volatility:

$$\sigma_{RR25} = \sigma_{25\Delta C} - \sigma_{25\Delta P}, \tag{A.22}$$

25Δ-Butterfly (BF) volatility:

$$\sigma_{BF25} = \frac{1}{2}[\sigma_{25\Delta C} + \sigma_{25\Delta P}] - \sigma_{ATM}. \tag{A.23}$$

Term volatility in case of 25Δ-risk-reversals and 25Δ-butterflies might be confusing here, because σ_{RR25} and σ_{BF25} from equations (A.22) and (A.23), respectively, can be negative for certain values of $\sigma_{25\Delta C}$, $\sigma_{25\Delta P}$ and σ_{ATM}. Obviously volatility cannot be negative, thus this term should be treated just as a convention, not the real value of the volatility of one of these strategies.

DEFINITION A.11 *Single barrier option valuation*
Merton (1973) and Reiner and Rubinstein (1991) developed formulas for the values of single barrier options with a barrier B without a prespecified cash rebate paid out if the option has not been knocked in or has knocked out during its lifetime. These formulas can be found in, for example, Haug (2007) or Weber and Wystup (2009) with slightly changed notation.

$$A = \phi S_t e^{-r_f\tau}\mathcal{N}(\phi x_1) - \phi K e^{-r_d\tau}\mathcal{N}(\phi(x_1 - \sigma\sqrt{\tau})), \tag{A.24}$$

$$B = \phi S_t e^{-r_f\tau}\mathcal{N}(\phi x_2) - \phi K e^{-r_d\tau}\mathcal{N}(\phi(x_2 - \sigma\sqrt{\tau})), \tag{A.25}$$

$$C = \phi \left(\frac{B}{S_t}\right)^{2\lambda-2} \left[S_t e^{-r_f\tau} \left(\frac{B}{S_t}\right)^2 \mathcal{N}(\eta y_1) - K e^{-r_d\tau} \mathcal{N}(\eta(y_1 - \sigma\sqrt{\tau})) \right], \qquad \text{(A.26)}$$

$$D = \phi \left(\frac{B}{S_t}\right)^{2\lambda-2} \left[S_t e^{-r_f\tau} \left(\frac{B}{S_t}\right)^2 \mathcal{N}(\eta y_2) - K e^{-r_d\tau} \mathcal{N}(\eta(y_2 - \sigma\sqrt{\tau})) \right], \qquad \text{(A.27)}$$

where:

- *η is the binary variable describing whether B is a lower barrier ($\eta = 1$) or an upper barrier ($\eta = -1$),*

- *$\theta_\pm \overset{d}{=} \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$,*

- *$\mu \overset{d}{=} \sigma\theta_-$,*

- *$\lambda = 1 + \frac{\mu}{\sigma^2}$,*

- *$x_1 = \frac{ln(\frac{S_t}{K}) + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$,*

- *$x_2 = \frac{ln(\frac{S_t}{B}) + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$,*

- *$y_1 = \frac{ln(\frac{B^2}{S_t K}) + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$,*

- *$y_2 = \frac{ln(\frac{B}{S_t}) + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$.*

If an option pays rebate then two extra components E and F are specified. For formulas we refer to Haug (2007). Different single barrier options are combinations of the above components, as presented in Table A.1.

DEFINITION A.12 *Knock-out probability*
The risk neutral probability of knocking out is given by (see Hakala, Perissé and Wystup (2002) or Weber and Wystup (2009)):

$$\mathbb{P}[\tau_B \leq T] = \mathbb{E}\left[\mathbb{I}_{\{\tau_B \leq T\}}\right] = \frac{1}{R} e^{r_d T} v(0, S_0), \qquad \text{(A.28)}$$

where

- *$\tau_B \overset{d}{=} inf\{t \geq 0 : \eta S_t \leq \eta B\}$ is the first hitting time,*

- *R is the rebate [9], by default in FX options markets equal to 1 unit of the domestic currency,*

---

[9]Note that this is the rebate of a one-touch option, not the same as the one mentioned before in definition A.11 for single barrier options.

Table A.1: Barrier option valuation schemes.

option type	ϕ	η	in/out	reverse	combination
standard up and in call	+1	−1	−1	$K > B$	A
reverse up and in call	+1	−1	−1	$K \leq B$	$B - C + D$
reverse up and in put	−1	−1	−1	$K > B$	$A - B + D$
standard up and in put	−1	−1	−1	$K \leq B$	C
standard down and in call	+1	+1	−1	$K > B$	C
reverse down and in call	+1	+1	−1	$K \leq B$	$A - B + D$
reverse down and in put	−1	+1	−1	$K > B$	$B - C + D$
standard down and in put	−1	+1	−1	$K \leq B$	A
standard up and out call	+1	−1	+1	$K > B$	0
reverse up and out call	+1	−1	+1	$K \leq B$	$A - B + C - D$
reverse up and out put	−1	−1	+1	$K > B$	$B - D$
standard up and out put	−1	−1	+1	$K \leq B$	$A - C$
standard down and out call	+1	+1	+1	$K > B$	$A - C$
reverse down and out call	+1	+1	+1	$K \leq B$	$B - D$
reverse down and out put	−1	+1	+1	$K > B$	$A - B + C - D$
standard down and out put	−1	+1	+1	$K \leq B$	0

- $v(t, S_0)$ *at* $t = 0$ *is the value of a one-touch option paying rebate* R *if barrier* B *is hit at any time before the expiration time. It is calculated as follows:*

$$v(t, S_0) = e^{-\omega r_d \tau} \left[\left(\frac{B}{S_t} \right)^{\frac{\theta_- + v_-}{\sigma}} \mathcal{N}(-\eta e_+) + \left(\frac{B}{S_t} \right)^{\frac{\theta_- - v_-}{\sigma}} \mathcal{N}(-\eta e_-) \right], \qquad (A.29)$$

- $\omega = 0$ *if the rebate is paid at the first hitting time* τ_B *or* $\omega = 1$ *(default in the FX options market) if the rebate is paid at maturity time* T,

- $v_- \overset{d}{=} \sqrt{\theta_-^2 + 2(1 - \omega)r_d}$,

- $e_\pm \overset{d}{=} \frac{\pm ln \frac{S_t}{B} - \sigma v_- \tau}{\sigma \sqrt{\tau}}$.

DEFINITION A.13 *Foreign-domestic symmetry*

The value of an option can be calculated from the domestic and from the foreign perspective. We consider an FX spot rate S_t for the pair XXX/YYY. The vanilla call option paying $(S_T - K)^+$ costs $C(S_t, K, r_d, r_f, T, \sigma, 1)$ YYY units and hence $\frac{1}{S_t} O^{BS}(S_t, K, r_d, r_f, T, \sigma, 1)$ XXX units. On the other hand, this XXX-call option can also be viewed as a YYY-put option with payoff $K(\frac{1}{K} - \frac{1}{S_T})^+$. This option costs $KO^{BS}(\frac{1}{S_t}, \frac{1}{K}, r_f, r_d, T, \sigma, -1)$ XXX units, because S_t and $\frac{1}{S_t}$ have the same volatility. Of course two prices of the same option must be equal under the no-arbitrage principle. Therefore we get:

$$\frac{1}{S_t} O^{BS}(S_t, K, r_d, r_f, T, \sigma, \phi) = KO^{BS}(\frac{1}{S_t}, \frac{1}{K}, r_f, r_d, T, \sigma, -\phi). \qquad (A.30)$$

A similar relationship holds for a single barrier option (see Wystup, 2008):

$$\frac{1}{S_t} O^{BS}(S_t, K, B, r_d, r_f, T, \sigma, \phi, \eta) = KO^{BS}(\frac{1}{S_t}, \frac{1}{K}, \frac{1}{B}, r_f, r_d, T, \sigma, -\phi, -\eta). \qquad (A.31)$$

Bibliography

Bossens, F., Rayée, G., Skantzos, N.S. and Deelstra, G. (2010). Vanna-volga methods applied to FX derivatives. From theory to market practice, *Working Paper*.

Brigo, D. and Mercurio, F. (2007). *Interest Rates Models – Theory and Practice with Smile, Inflation and Credit*, Springer.

Carr, P., Hogan, A. and Verma, A. (2006). Vanna-volga method for 1st generation exotics in FX, *Bloomberg Research Paper*.

Castagna, A. (2009). *FX Options and Smile Risk*, Wiley Finance.

Castagna, A. and Mercurio, F. (2006). Consistent pricing of FX options, *Working Papers Series*, Banca IMI.

Castagna, A. and Mercurio, F. (2007). The vanna-volga method for implied volatilities, *Risk*, January, 106–111.

Cox, J. C., Ingersoll, J. E. and Ross, S. A. (1985). A Theory of the term structure of interest rates, *Econometrica* 53: 385–407.

Derman, E. and Kani, I. (1996). The ins and outs of barrier options: Part 1, *Derivatives Quaterly*, Winter 1996, 55–67.

Fisher, T. (2007). Variations on the vanna-volga adjustment, *Bloomberg Research Paper*

Garman, M. B. and Kohlhagen, S. W. (1983). Foreign currency option values, *Journal of International Monet & Finance*, 2: 231–237.

Hakala, J., Perissé, G. and Wystup, U. (2002). The pricing of first generation exotics, *Foreign Exchange Risk*, Risk Publications.

Haug, E. G. (2007). *The Complete Guide To Option Pricing Formulas (2nd ed)*, McGraw-Hill.

Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies*, 6: 327–343.

Janek, A., Kluge, T., Weron, R. and Wystup, U. (2011). FX Smile in the Heston Model, *in Statistical Tools for Finance and Insurance (2nd ed)*, eds. P. Čižek, W. Härdle, R. Weron, Springer-Verlag, Berlin, 133–162.

Kucharczyk, D. (2011). The Heston model for pricing path dependent options, *MSc Thesis*, Wroclaw University of Technology.

Lipton, A. and McGhee, W. (2002). Universal barriers, *Risk*, May, 81–85.

Reiswich, D. and Wystup, U. (2010). FX volatility smile construction, *Centre for Practical Quantitative Finance Working Paper Series #20*, Frankfurt School of Finance and Management.

Shkolnikov, Y. (2009). Generalized vanna-volga method and its applications, *NumeriX Research Paper*

Weber, A. and Wystup, U. (2009). Pricing Formulae for Foreign Exchange Options, MathFinance.

Weron, A. and Weron, R. (2005). *Inżynieria finansowa*, WNT (in Polish).

Wystup, U. (2002). Ensuring efficient hedging of barrier options, *RISK training course on pricing, hedging and trading exotic derivatives.*

Wystup, U. (2003). The market price of one-touch options in foreign exchange markets, *Derivatives week*, 12(13), 1–4.

Wystup, U. (2006). *FX Options and Structured Products*, Wiley Finance.

Wystup, U. (2008). Foreign exchange symmetries, *Centre for Practical Quantitative Finance Working Paper Series #9*, Frankfurt School of Finance and Management.

Wystup, U. (2008). Vanna-volga pricing, *Centre for Practical Quantitative Finance Working Paper Series #11*, Frankfurt School of Finance and Management.