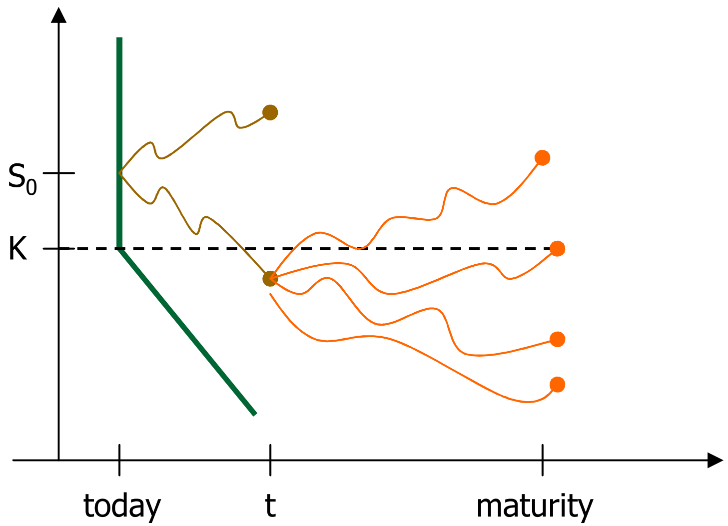
Various pricing methodologies for exotic options

Contents

- In this series we will cover
  - American exercise with Monte Carlo
  - Likelihood ratio method
  - Explicit / Implicit method in PDEs
  - Moment Matching
  - Vanna-Volga method

American exercise in Monte Carlo

- When is it optimal to exercise the option?



- Naïve approach. If at any time t:
  - Spot is out of-the-money, it is not optimal to exercise. Stop
  - Spot is in-the-money then
    - start new simulation from this spot
    - if (on average) final spot finishes more in-the-money, do not exercise now
    - if (on average) final spot finishes less in-the-money, exercise now

Least-squares Monte Carlo

- Since this has to be done for every time step t:
  Naïve Monte Carlo is clearly impractical
- Methodology for american exercise provided by
  - Longstaff & Schwartz (2001) *Rev Fin Studies* v.14 pp.113-147
- Method is not exact but quite accurate (versus e.g. PDE)
- Is not hard to implement
- But not as CPU-efficient as standard monte carlo
- Central idea
  - Work backwards starting from maturity
  - At each step compare immediate exercise value with expected cashflow from continuing
  - Exercise if immediate exercise is more valuable

Least-squares Monte Carlo (1)

- Generate spots for each path & for each time-step
- Make an N_{paths}xN_{steps} table of spot paths (according to some dynamics)
- Make an N_{paths}xN_{steps} empty table of cashflows (CF)

Spot Paths

$$N_{paths} \updownarrow \begin{pmatrix} \bullet & \bullet & & \bullet & \bullet \\ \bullet & \circ & & \circ & \bullet \\ \bullet & \bullet & & \bullet & \circ \\ \bullet & \bullet & \cdots & \circ & \bullet \\ \bullet & \bullet & & \bullet & \circ \\ \bullet & \circ & & \bullet & \circ \\ \bullet & \circ & & \circ & \bullet \end{pmatrix}$$

$$\xleftrightarrow{\quad N_{steps} \quad}$$

$\circ$ Out-of-the-money

$\bullet$ In-the-money

Cashflows

$$\begin{pmatrix} 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \end{pmatrix}$$

Least-squares Monte Carlo (2)

- If spot at maturity is
  - in-the-money: assign for this path CF=payoff value,
  - out-of-the-money: assign for this path CF=0,

Spot Paths

$$\begin{pmatrix} \bullet & \bullet & & \bullet & \bullet \\ \bullet & \circ & & \circ & \bullet \\ \bullet & \bullet & & \bullet & \circ \\ \bullet & \bullet & \cdots & \circ & \bullet \\ \bullet & \bullet & & \bullet & \circ \\ \bullet & \circ & & \bullet & \bullet \\ \bullet & \circ & & \circ & \bullet \end{pmatrix}$$

○ Out of-the-money

● In-the-money

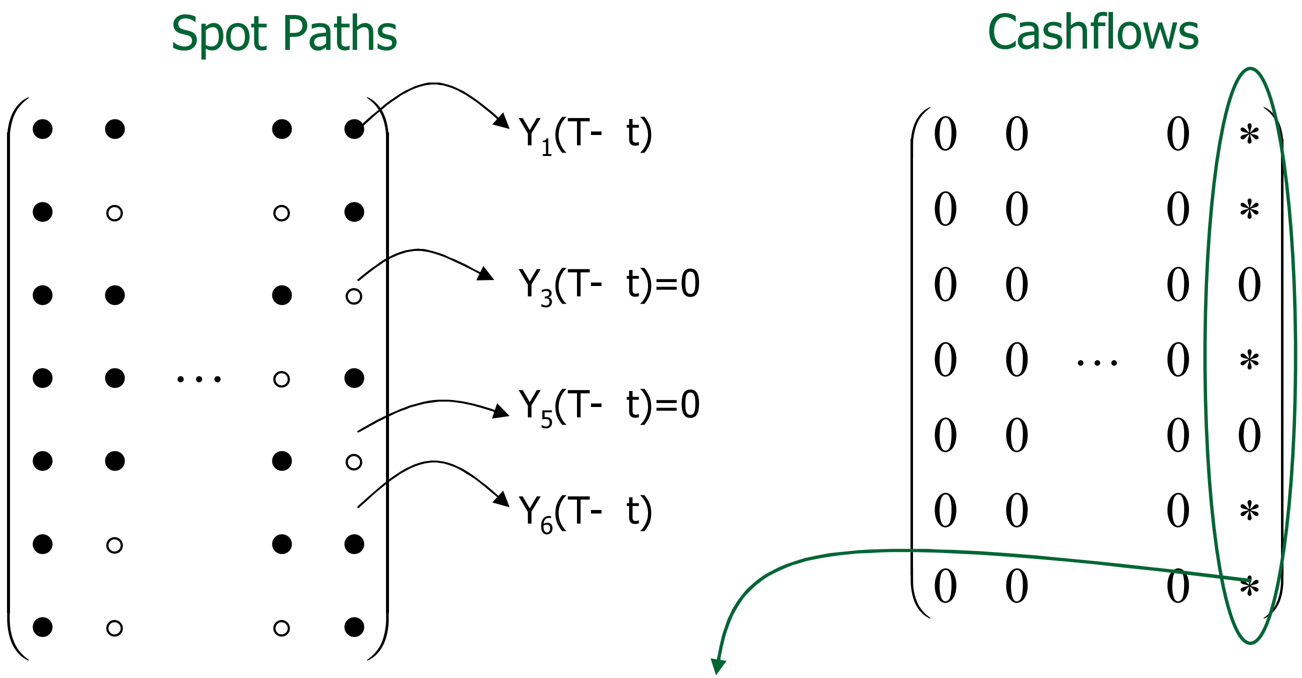
Cashflows

$$\begin{pmatrix} 0 & 0 & & 0 & * \\ 0 & 0 & & 0 & * \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & * \\ 0 & 0 & & 0 & * \end{pmatrix}$$

$*$ $CF=(S_{\text{this path}}(T)-K)^+$

Least-squares Monte Carlo (3)

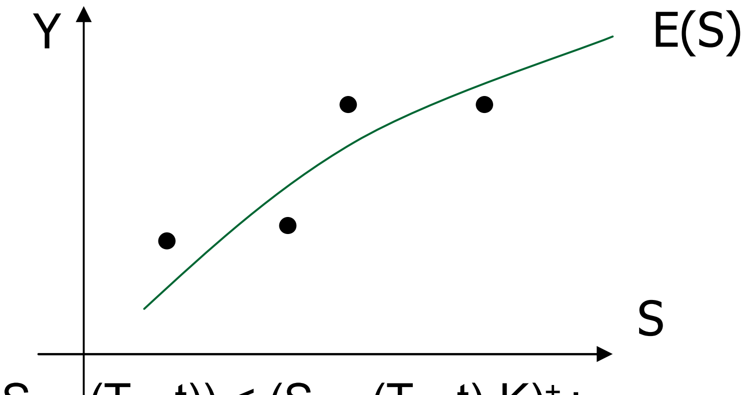
- Go one time-step backwards. If spot is
  - in-the-money: option holder must decide whether to exercise now or continue. Calculate Y=discounted cashflow at next step if option is not exercised now
  - out-of-the-money: assign for this path CF=0



$$Y_{path}(T- \ t) = DF(T- \ t,T) \cdot CF(T)$$

Least-squares Monte Carlo (4)

- On the pairs $\{S_{path\ i}, Y_{path\ i}\}$ pass a regression of the form
  $E(S) = a_0 + a_1 \cdot S + a_2 \cdot S^2$
- This function is an approximation to the expected payoff from continuing to hold the option from this time point on



- If $E(S_{path}(T-\ t)) < (S_{path}(T-\ t)-K)^+$:
  - exercise the option at this time step
  - Assign CF at this step = $(S_{path}(T-\ t)-K)^+$ and for all larger t set CF=0
- If $E(S_{path}(T-\ t)) > (S_{path}(T-\ t)-K)^+$:
  - Do not exercise the option at this time step
  - Maintain same value of cashflow at next steps

Least-squares Monte Carlo (5)

- Proceed similarly till the first time step and populate the matrix of cashflows

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- There should be one non-zero cashflow per path!

  (the option can be exercised only once)

$$\mathrm{Amer} = \frac{1}{N_{\mathrm{paths}}} \cdot \sum_{i=1}^{N_{\mathrm{paths}}} \mathrm{DF}\left(\mathrm{today}, \mathrm{t}_i^{\mathrm{exer}}\right) \cdot \mathrm{CF}\left(S_i\left(\mathrm{t}_i^{\mathrm{exer}}\right)\right)$$

- Callables are priced with the same idea

Greeks in Monte Carlo

- To calculate Greeks with Monte Carlo:
  - Bump sensitivity parameter (spot, vol, etc)
  - Recalculate market data with the bumped parameter (smile, curves, etc)
  - Re-run Monte Carlo (using same RNG)
  - Calculate Greeks as finite difference
  - For example,

$$\Delta = \frac{\text{Price}(S+\Delta S) - \text{Price}(S-\Delta S)}{2\cdot\Delta S} \qquad \Gamma = \frac{\text{Price}(S+\Delta S) - 2\cdot\text{Price}(S) + \text{Price}(S-\Delta S)}{(\Delta S)^2}$$

$$\text{Vega} = \frac{\text{Price}(\sigma+\Delta\sigma) - \text{Price}(\sigma)}{\Delta\sigma}$$

- This requires at least 12 Monte Carlo runs for all Greeks !
- Not ideal for impatient traders

Likelihood ratio method (1)

- This method allows us to calculate all Greeks within a single Monte Carlo

  Main idea:
- Express Greeks as payoffs
- Price the new “payoffs” with the same simulation

  Note:
- The analytics of the method simplify if spot is assumed to follow lognormal process (as in BS)
- The LR greeks will not be in general the same as the finite difference greeks !!
  - This is because of the modification of the market data when using the finite difference method

Likelihood ratio method (2)

- Consider an exotic option with a path-dependent payoff
- Its price will depend on all spots in the path

$$\text{Exotic} = \text{DF} \cdot \int dS_1 \cdots dS_m \cdot \text{PDF}(S_1, \cdots, S_m) \cdot \text{Prob}_{\text{surv}}(S_1, \cdots, S_m) \cdot \text{Payoff}$$

- PDF: probability density function of the spot

$$\text{PDF}(S_1, \ldots, S_m) = \prod_{i=1}^{m} \text{PDF}(S_i) = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma_i^2 \cdot \Delta t_i}} \frac{1}{S_i} e^{-\frac{1}{2} z_i^2}$$

  - z_i the Gaussian random number used to make the jump $S_{i-1} \longrightarrow S_i$

$$z_i = \frac{\log \frac{S_i}{S_{i-1}} - \left(r - \frac{1}{2}\sigma_i^2 \right) \cdot \Delta t_i}{\sigma_i \sqrt{\Delta t_i}}$$

- $\text{Prob}_{\text{surv}}$ the total survival probability for the spot path (given some barrier levels)

$$\text{Prob}_{\text{surv}} = \prod_{i=1}^{m} \text{Prob}_{\text{surv}}^{t_{i-1} \to t_i}$$

  - For explicit expressions for the surv.prob. of KO or DKO see previous slides

Likelihood ratio method (3)

- Derivative price

$$\text{Exotic} = \text{DF} \cdot \int dS_{1\cdots m} \cdot \text{Payoff} \cdot \text{PDF}\left(S_{1\cdots m}\right) \cdot \text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right)$$

- Sensitivity with respect to a parameter (=spot, vol, etc)

$$\frac{\partial \text{Exotic}}{\partial \alpha} = \text{DF} \cdot \int dS_{1\cdots m} \cdot \text{Payoff} \cdot \frac{\partial \left(\text{PDF}\left(S_{1\cdots m}\right) \cdot \text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right)\right)}{\partial \alpha}$$

$$= \text{DF} \cdot \int dS_{1\cdots m} \cdot \text{Payoff} \cdot \text{PDF}\left(S_{1\cdots m}\right) \cdot \text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right) \cdot \left[\frac{1}{\text{PDF}\left(S_{1\cdots m}\right)} \cdot \frac{\partial \text{PDF}\left(S_{1\cdots m}\right)}{\partial \alpha} + \frac{1}{\text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right)} \cdot \frac{\partial \text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right)}{\partial \alpha}\right]$$

- We read off the new payoff

$$\text{New Payoff} = \text{Payoff} \cdot \left[\frac{1}{\text{PDF}\left(S_{1\cdots m}\right)} \cdot \frac{\partial \text{PDF}\left(S_{1\cdots m}\right)}{\partial \alpha} + \frac{1}{\text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right)} \cdot \frac{\partial \text{Prob}_{\text{surv}}\left(S_{1\cdots m}\right)}{\partial \alpha}\right]$$

Likelihood ratio method (4)

- This is simple derivatives over analytic functions (see previous slide)!
- For example,
  - Delta becomes the new payoff

$$\Delta = \mathrm{DF} \cdot \mathrm{E}\left[\mathrm{Payoff} \cdot \left(\frac{1}{\mathrm{PDF}(S_{1\cdots m})} \frac{\partial \mathrm{PDF}(S_{1\cdots m})}{\partial S_0} + \frac{1}{\mathrm{Prob}_{\mathrm{surv}}^{t_0 \to t_1}} \frac{\partial \mathrm{Prob}_{\mathrm{surv}}^{t_0 \to t_1}}{\partial S_0} \right) \right]$$

  - To be priced with the same spot path as the Payoff itself
  - Similarly for other Greeks: more lengthy expressions but doable!

Finite Difference Methods

- <u>Explicit method</u>
- Spot derivatives are calculated at *t=(i+1)· t*

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2 \cdot \Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j+1} - 2 f_{i+1,j} + f_{i+1,j-1}}{(\Delta S)^2}$$

- <u>Implicit method</u>
- Spot derivatives are calculated at *t=i· t*

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2 \cdot \Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} - 2 f_{i,j} + f_{i,j-1}}{(\Delta S)^2}$$

Explicit method

- The difference equation becomes

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+\mu\cdot j\Delta S\cdot\frac{f_{i+1,j+1}-f_{i+1,j-1}}{2\cdot\Delta S}+\frac{1}{2}\sigma^2\cdot(j\Delta S)^2\frac{f_{i+1,j+1}-2\cdot f_{i+1,j}+f_{i+1,j-1}}{(\Delta S)^2}=r\cdot f_{i+1,j}$$

- and after some re-arrangement:

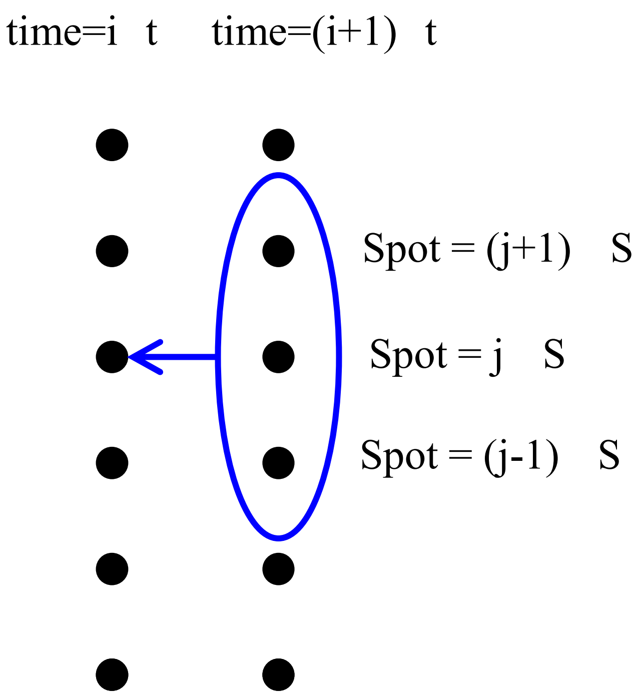
$$f_{i,j}=f_{i+1,j-1}\left(\frac{1}{2}\sigma^2 j^2\Delta t-\frac{1}{2}j\mu\Delta t\right)+f_{i+1,j}\left(1-\sigma^2 j^2\Delta t-r\Delta t\right)+f_{i+1,j+1}\left(\frac{1}{2}\sigma^2 j^2\Delta t+\frac{1}{2}j\mu\Delta t\right)$$

- more compactly:

$$f_{i,j}=f_{i+1,j-1}\cdot A_j+f_{i+1,j}\cdot B_j-f_{i+1,j+1}\cdot C_j$$

- For $i+1=T_{mat}$ the function $f_{i+1,j}$ is fully known
- Solve above equation iteratively for $f_{i,j}$ in every (i,j) until i=today

Explicit method schematically



- To calculate the option value at the boundary spots
  - S_{min} (with j=1)
  - S_{max} (with j=nbrSpots)

  we need extra equations, the boundary conditions
- We obtain these by requiring that at very low and very high spots the option has no convexity:

$$\frac{\partial^2 C}{\partial S^2} = 0 \Rightarrow C(j+1) - 2C(j) + C(j-1) = 0$$

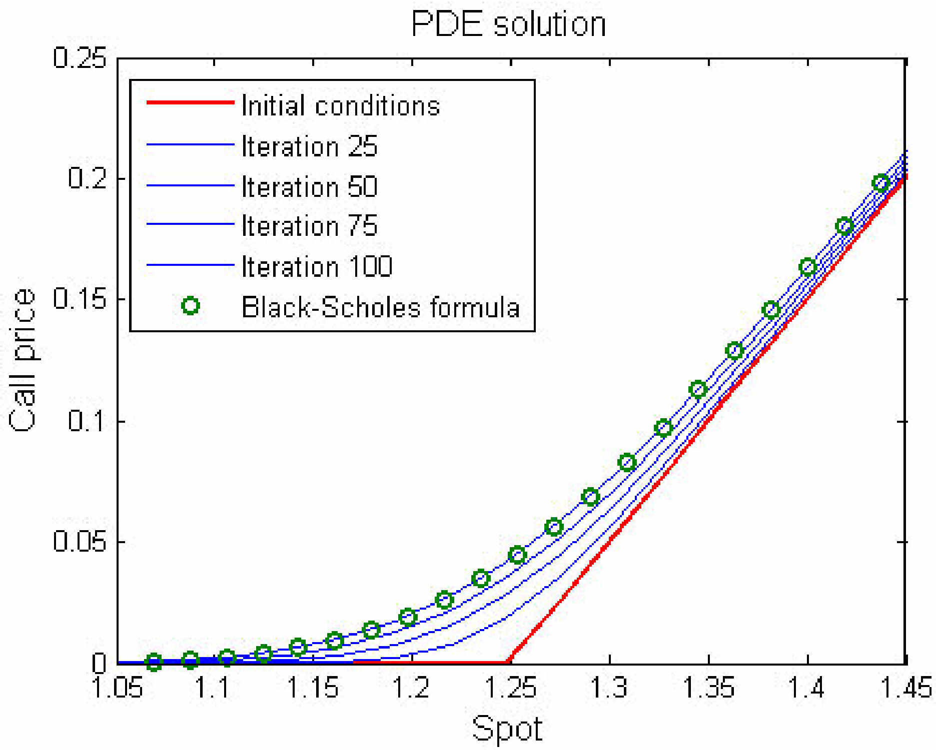
- This implies:

$$C(1) = 2C(2) - C(3)$$

$$C(N) = 2C(N-1) - C(N-2)$$

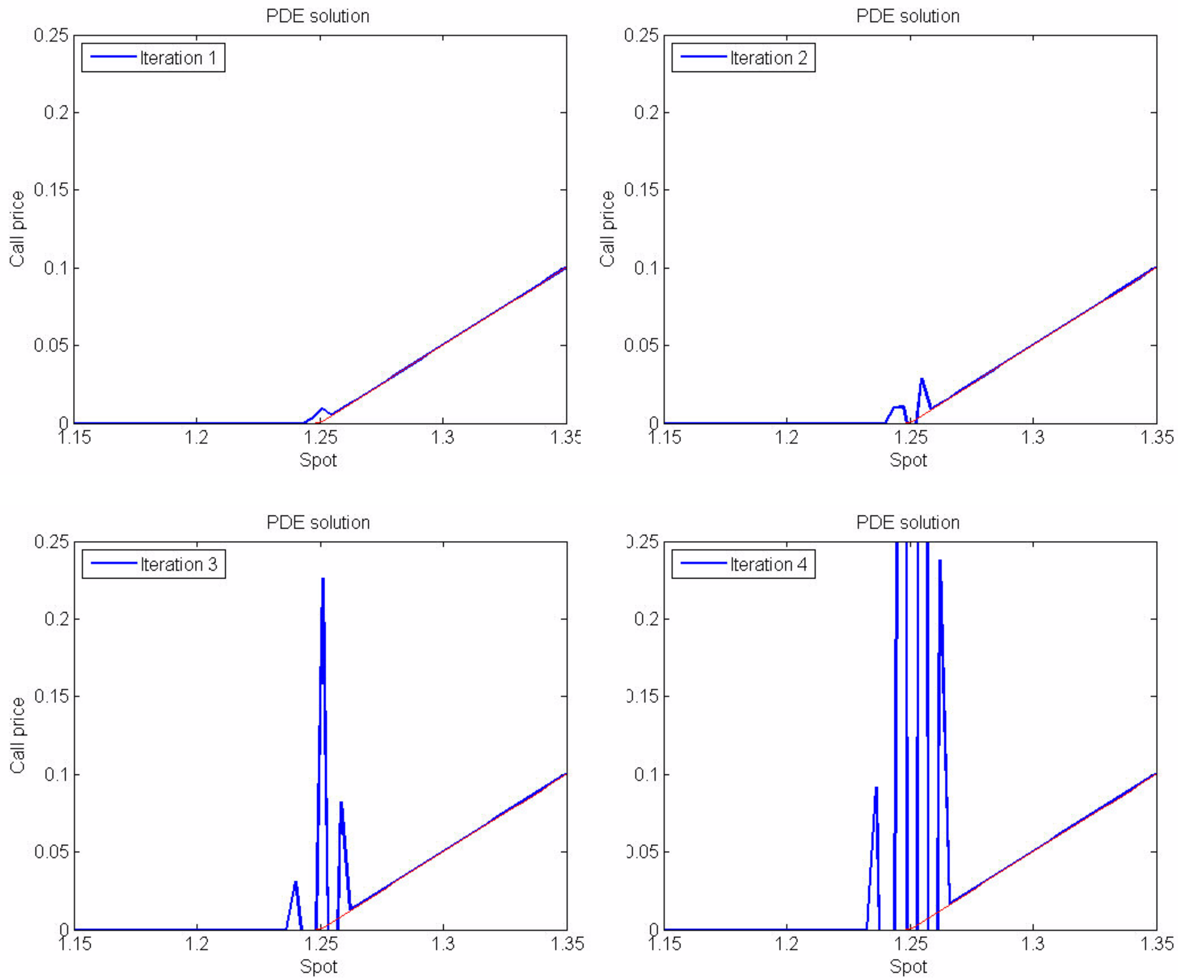
Explicit method at work

- PDE solution with
  - 100 time steps
  - 100 spots
  - t = 0.005
  - S = 0.025
- converges to the correct Black-Scholes solution



Explicit method (not) at work

- Unstable if number of time-steps is not big enough
- Oscillations are produced and propagate to all spots



Implicit method

- More complex but avoids instabilities of explicit method

- The difference equation becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + \mu \cdot j\Delta S \cdot \frac{f_{i,j+1} - f_{i,j-1}}{2 \cdot \Delta S} + \frac{1}{2}\sigma^2 \cdot (j\Delta S)^2 \frac{f_{i,j+1} - 2 \cdot f_{i,j} + f_{i,j-1}}{(\Delta S)^2} = r \cdot f_{i,j}$$

- and after some re-arrangement:

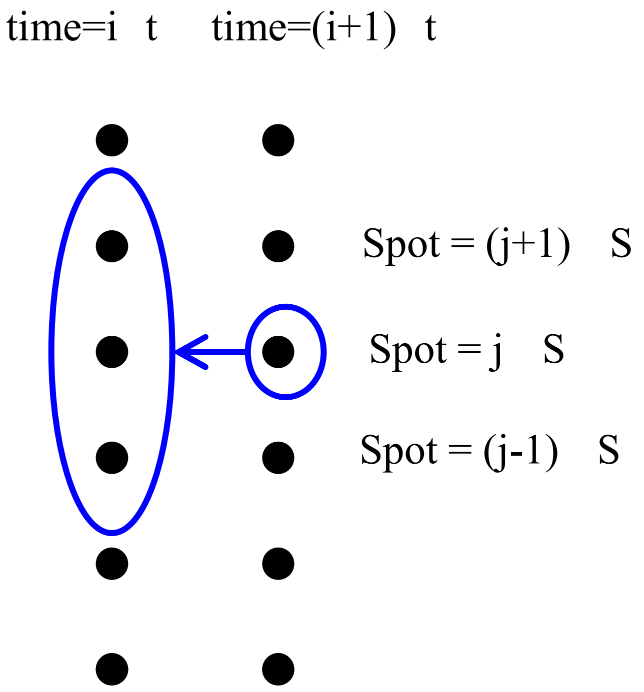
$$f_{i,j-1}\left(-\frac{1}{2}\sigma^2 j^2 \Delta t + \frac{1}{2} j\mu\Delta t\right) + f_{i,j}\left(1 + \sigma^2 j^2 \Delta t + r\Delta t\right) + f_{i,j+1}\left(-\frac{1}{2}\sigma^2 j^2 \Delta t - \frac{1}{2} j\mu\Delta t\right) = f_{i+1,j}$$

- more compactly:

$$f_{i,j-1} \cdot A_j + f_{i,j} \cdot B_j - f_{i,j+1} \cdot C_j = f_{i+1,j}$$

- For $i+1 = T_{mat}$ the function $f_{i+1,j}$ is fully known
- Solve above equation iteratively for $f_{i,j}$ in every (i,j) until i=today

Implicit method schematically



- 1 equation, 3 unknowns !
- We have to solve the entire system of equations for each time step
- Linear algebra methods
- LU decomposition
- Boundary conditions remain as before

Explicit vs Implicit methods

- In practise we use a combination of the two methods
- Crank-Nicolson method
- Combines efficiency and stability

Simple analytic methods: "moment matching"

- Average-rate option payoff with *N* fixing dates

$$\text{Asian} = \max\left(\frac{1}{N}\sum_{i=1}^{N} S_i - K, 0\right)$$

- Basket option with two underlyings

$$\text{Basket} = \max\left(a_1 \frac{S_1(T)}{S_1(t)} + a_2 \frac{S_2(T)}{S_2(t)} - K, 0\right)$$

- TV pricing can be achieved quickly via "moment matching"
- Mark-to-market requires correlated stochastic processes for spots/vols (more complex)

"Moment matching"

- To price Asian (average option) in TV we consider that
  - The spot process is lognormal
  - The *sum* of all spots is lognormal *also*

- Note: a sum of lognormal variables is *not* lognormal. Therefore this method is an approximation (but quite accurate for practical purposes)

- Central idea of moment matching
  - Find first and second moment of sum of lognormals:
    $E[\ {}_i S_i]$, $E[\ (\ {}_i S_i)^2\ ]$,
  - Assume sum of lognormals is lognormal (with known moments from previous step) and obtain a Black-Scholes formula with appropriate drift and vol

Asian options analytics (1)

- Prerequisites for the analysis: statistics of random increments
- Increments of spot process have 0 mean and variance T (time to maturity)

- $E[W_t]=0$, $E[W_t^2]=t$

- If $t_1<t_2$ then $E[W_{t1} \cdot W_{t2}] = E[W_{t1} \cdot (W_{t2}-W_{t1})] + E[W_{t1}^2] = t_1$
  (because W_{t1} is independent of $W_{t2}-W_{t1}$)

- More generally, $E[W_{t1} \cdot W_{t2}] = \min(t_1,t_2)$

- From this and with some algebra it follows that
  $E[S_{t1} \cdot S_{t2}] = S_0^2 \exp[r \cdot (t_1+t_2) + \;^2 \cdot \min(t_1,t_2)]$

Asian options analytics (2)

- Asian payoff contains sum of spots

$$X = \frac{1}{N}\sum_{i=1}^{N} S_i$$

- What are its mean (first moment) and variance?

$$\mathrm{E}[X] = \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N} S_i\right] = \frac{1}{N}\sum_{i=1}^{N}\mathrm{E}[S_i] = \frac{1}{N}\sum_{i=1}^{N}\mathrm{E}\left[S_0\cdot e^{\left(r-\frac{1}{2}\sigma^2\right)t_i+\sqrt{t_i}\cdot\sigma\cdot N(0,1)}\right] = \frac{1}{N}\sum_{i=1}^{N} S_0\cdot e^{r\cdot t_i}$$

$$\mathrm{E}[X^2] = \mathrm{E}\left[\frac{1}{N^2}\sum_{i=1}^{N} S_i\cdot\sum_{j=1}^{N} S_j\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}[S_i\cdot S_j] = \frac{1}{N^2}\sum_{i,j=1}^{N}\mathrm{E}\left[S_0^2\cdot e^{r\cdot\left(t_i+t_j\right)+\sigma^2\cdot\min(t_i,t_j)}\right]$$

- Looks complex but on the right-hand side all quantities are known and can be easily calculated !

- Therefore the first and second moment of the sum of spots can be calculated

Asian options analytics (3)

- Now assume that X follows lognormal process, with the (flat) vol, the drift

$$dX_t = \mu \cdot X_t \cdot dt + \lambda \cdot X_t \cdot dW_t$$

- Has solution (as in standard Black-Scholes)

$$X_T = S_0 \cdot e^{\left(\mu - \frac{1}{2}\lambda^2\right)T + \lambda \cdot W_T}$$

- Take averages in above and obtain first and second moment in terms of ,

$$\left.\begin{aligned} \mathrm{E}\left[X_T\right] &= S_0 \cdot e^{\mu \cdot T} \\ \mathrm{E}\left[X_T^2\right] &= S_0^2 \cdot e^{2\left(\mu - \frac{1}{2}\lambda^2\right)T} \cdot \mathrm{E}\left[e^{2\lambda W_T}\right] = \mathrm{E}^2\left[X_T\right] \cdot e^{\lambda^2 T} \end{aligned}\right\}$$

- Solving for drift and vol produces

$$\mu = \frac{1}{T} \cdot \log \frac{\mathrm{E}\left[X_T\right]}{S_0} \qquad \lambda = \sqrt{\frac{1}{T} \cdot \log \frac{\mathrm{E}\left[X_T^2\right]}{\mathrm{E}^2\left[X_T\right]}}$$

Asian options analytics (4)

- Since we wrote Asian payoff as max(X_T-K,0)
- We can quote the Black-Scholes formula

$$\text{Asian} = \text{DF} \cdot \left(e^{\mu T} \cdot S_0 \cdot N(d_1) - K \cdot N(d_2) \right)$$

- With

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(\mu + \frac{1}{2} \lambda^2 \right) \cdot T}{\lambda \cdot \sqrt{T}} \qquad d_2 = \frac{\ln \frac{S_0}{K} + \left(\mu - \frac{1}{2} \lambda^2 \right) \cdot T}{\lambda \cdot \sqrt{T}}$$

- And , are written in terms of E[X], E[X^2] which we have calculated as sums over all the fixing dates

- The “averaging” reduces volatility we expect lower price than vanilla

- Basket is based on similar ideas

Asian options analytics

- Exercise

- Assume that we approximate the Asian payoff by

$$\text{Approx} = \max\left(a \cdot S_T + b, 0\right)$$

- Find the constants *a* and *b* through moment matching

- Do a numerical comparison with the previous result.

Basket options analytics

- A basket option is defined by the payoff

$$A(t) = \sum_{m=1}^{n} a_i S^i(t) \qquad A(0) = A_0$$

- With a_i constants and $S^i(t)$ given by:

$$dS_t^i = (r - q_i) S_t^i dt + \sigma_i S_t^i dW_t^i$$

$$dW_t^i dW_t^j = \rho_{ij} dt$$

- Assume that the basket follows a lognormal process:

$$d\overline{A}(t) = (r - \overline{q}) \overline{A} dt + \overline{\sigma} \overline{A}(t) dW_t$$

- And show through moment-matching that

$$\overline{q} = -\tfrac{1}{T} \ln\left(\frac{\sum_{i=1..n} a_i S_0^i e^{-q_i T}}{\sum_{i=1..n} a_i S_0^i} \right) \qquad \overline{\sigma}^2 = \tfrac{1}{T} \ln\left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j e^{(-q_i - q_j + \rho_{ij}\sigma_i \sigma_j)T}}{e^{-2\overline{q}T} \left(\sum_{i=1..n} a_i S_0^i \right)^2} \right)$$

Vanna-Volga method

- Which model can reproduce market dynamics?
- Market psychology is not subject to rigorous math models…
- Brute force approach: Capture main features by a mixture model combining jumps, stochastic vols, local vols, etc
- But…
  - Difficult to implement
  - Hard to calibrate
  - Computationally inefficient

Vanna-Volga as an alternative pricer

- Vanna-Volga is an alternative pricing “recipie”
  - Easy to implement
  - No calibration needed
  - Computationally efficient

- But…
  - It is not a rigorous model
  - Has no dynamics

Vanna-Volga main idea

- The vol-sensitivities

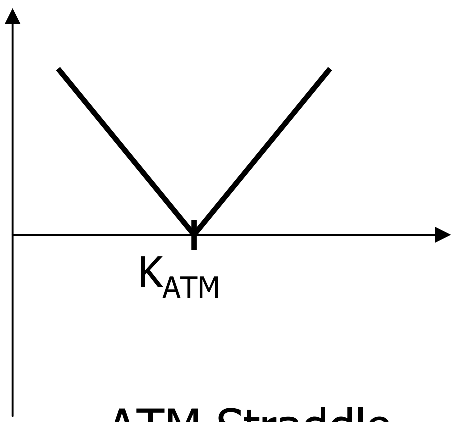
  Vega $\frac{\partial \text{Price}}{\partial \sigma}$ Vanna $\frac{\partial^2 \text{Price}}{\partial \sigma \partial S}$ Volga $\frac{\partial^2 \text{Price}}{\partial \sigma^2}$

  are responsible for the smile impact

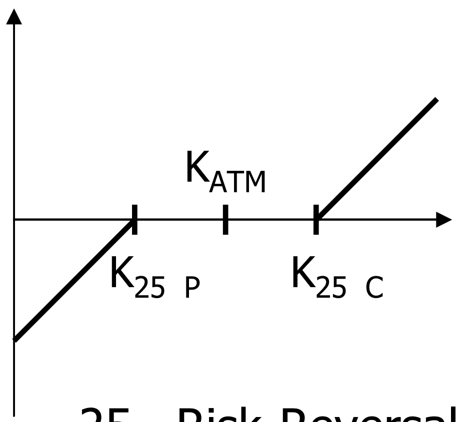
- Practical (trader's) recipie:
  - Construct portfolio of 3 vanilla-instruments which zero out the Vega,Vanna,Volga of exotic option at hand
  - Calculate the smile impact of this portfolio (easy BS computations from the market-quoted volatilities)
  - Market price of exotic = Black-Scholes price of exotic + Smile impact of portfolio of vanillas

Vanna-Volga hedging portfolio

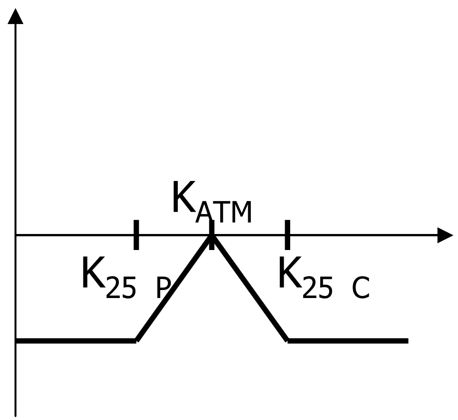
- Select three liquid instruments:
  - At-The-Money Straddle (ATM) =½ Call(K_{ATM}) + ½ Put(K_{ATM})
  - 25 -Risk-Reversal (RR) = Call(=¼) - Put(=-¼)
  - 25 -Butterfly (BF) = ½ Call(=¼) + ½ Put(=-¼) – ATM



ATM Straddle



25 Risk-Reversal



25 Butterfly

RR carries mainly Vanna　　　BF carries mainly Volga

Vanna-Volga weights

- Price of hedging portfolio P = $w_{ATM} \cdot ATM + w_{RR} \cdot RR + w_{BF} \cdot BF$
- What are the appropriate weights w_{ATM} , w_{RR}, w_{BF}?
  - Exotic option at hand X and portfolio of vanillas P are calculated using Black-Scholes
  - vol-sensitivities of portfolio P = vol-sensitivities of exotic X:

$$\begin{pmatrix} X_{vega} \\ X_{vanna} \\ X_{volga} \end{pmatrix} = \begin{pmatrix} ATM_{vega} & RR_{vega} & BF_{vega} \\ ATM_{vanna} & RR_{vanna} & BF_{vanna} \\ ATM_{volga} & RR_{volga} & BF_{volga} \end{pmatrix} \cdot \begin{pmatrix} w_{vega} \\ w_{vanna} \\ w_{volga} \end{pmatrix}$$

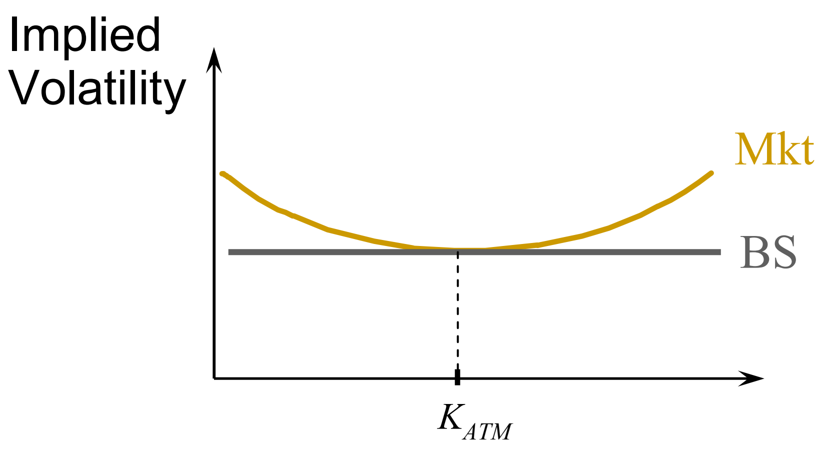
  - solve for the weights:

$$\vec{w} = A^{-1} \cdot \vec{X}$$

Vanna-Volga price

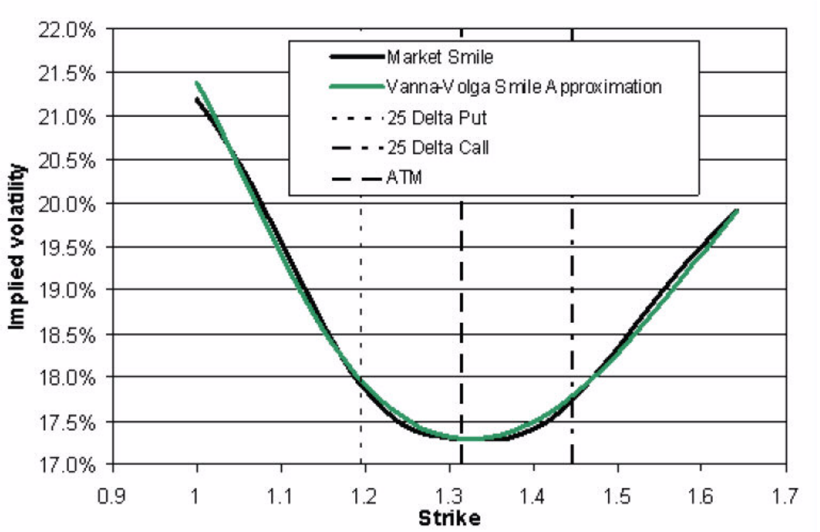
- Vanna-Volga market price is

$$X^{VV} = \underbrace{\boxed{X^{BS}}}_{\text{BS price}} + \underbrace{\boxed{w_{ATM} \cdot \cancel{(ATM^{mkt}-ATM^{BS})} + w_{RR} \cdot (RR^{mkt}-RR^{BS}) + w_{BF} \cdot (BF^{mkt}-BF^{BS})}}_{\text{Smile impact of vanilla hedging portfolio}}$$

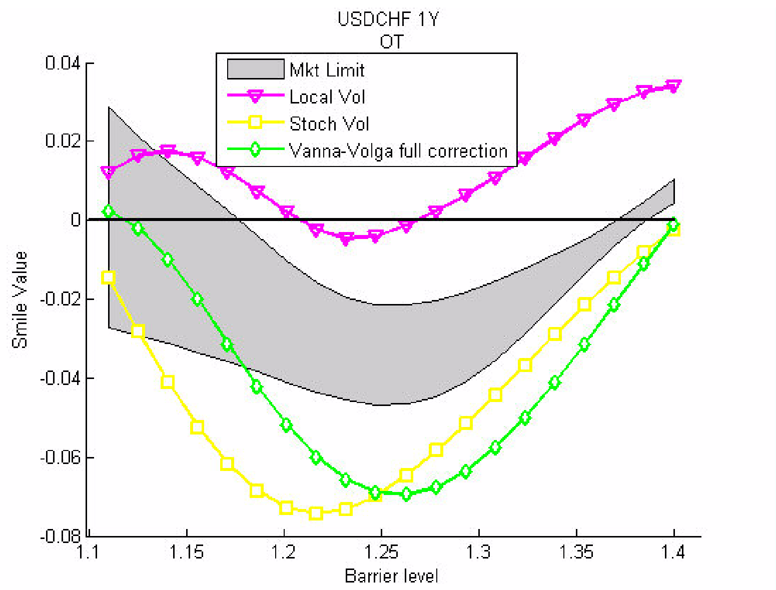


Vanna-Volga: full correction

- Applying the full VV correction to price a vanilla provides a reasonable approximation of the original smile:



- Applying the full VV correction to price an exotic option usually over-estimates the smile impact :



Vanna-Volga: attenuating the correction

- Starting from the original VV setting :

$$X^{VV} = X^{BS} + w_{RR} \cdot (RR^{Mkt}\text{-}RR^{BS}) + w_{BF} \cdot (BF^{Mkt}\text{-}BF^{BS})$$

- Apply some simple matrix calculus, drop vega contribution and introduce attenuation factors:

$$X^{VV} = X^{BS} + p_{VANNA}(\gamma)\, X_{VANNA}\, \Omega_{VANNA} + p_{VOLGA}(\gamma)\, X_{VOLGA}\, \Omega_{VOLGA}$$

$$\vec{I} = \begin{pmatrix} 0 \\ \mathrm{RR}^{\mathrm{mkt}} - \mathrm{RR}^{\mathrm{BS}} \\ \mathrm{BF}^{\mathrm{mkt}} - \mathrm{BF}^{\mathrm{BS}} \end{pmatrix} \qquad \begin{pmatrix} \Omega_{\mathrm{vega}} \\ \Omega_{\mathrm{vanna}} \\ \Omega_{\mathrm{volga}} \end{pmatrix} = (A^T)^{-1}\vec{I}$$

- Market practice : p_{VANNA} and p_{VOLGA} are functions of
  - γ = Survival probability
  - γ = Expected first exit time (FET) / T

$\Big\}$ obtained by solving BS PDE with appropriate boundary condition

Requirements on attenuation factors

$$0 < p_{VANNA}(\gamma),\ p_{VANNA}(\gamma) < 1$$

- As spot level approaches a "Knock-out" barrier, Vanna-Volga correction should fade-out
  - For volga term, always true as volga fades-out: $$\lim_{\gamma \to 0} X_{VOLGA} = 0$$
  - Not the case for Vanna, hence, we should impose:

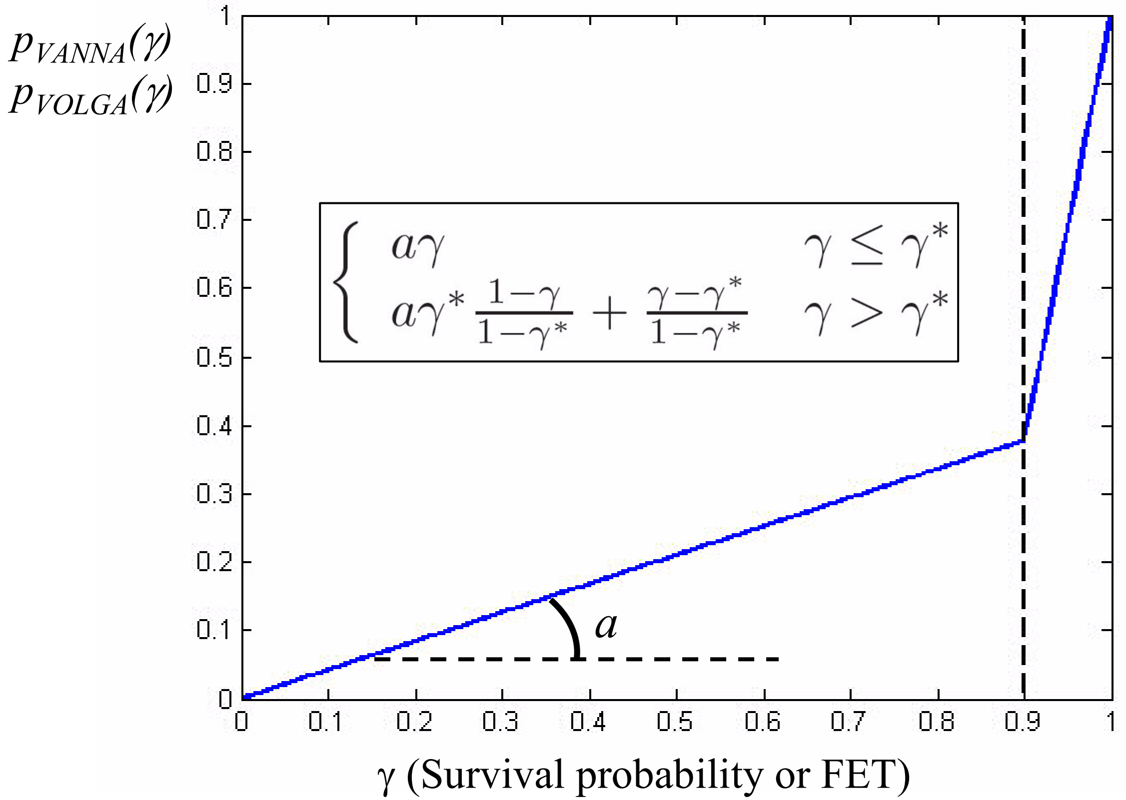
$$\lim_{\gamma \to 0} p_{VANNA}(\gamma) = 0$$

- Vanilla payouts (no barriers) are best priced with the full Vanna-Volga correction:

$$\lim_{\gamma \to 1} p_{VANNA}(\gamma) = \lim_{\gamma \to 1} p_{VOLGA}(\gamma) = 1$$

Example of attenuation factors choice

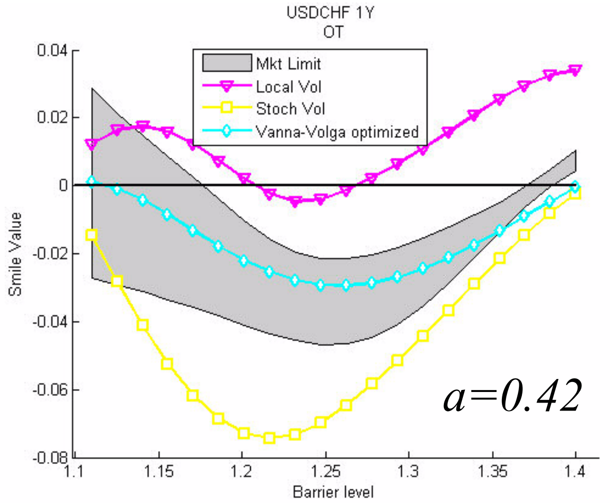
- Composite linear function



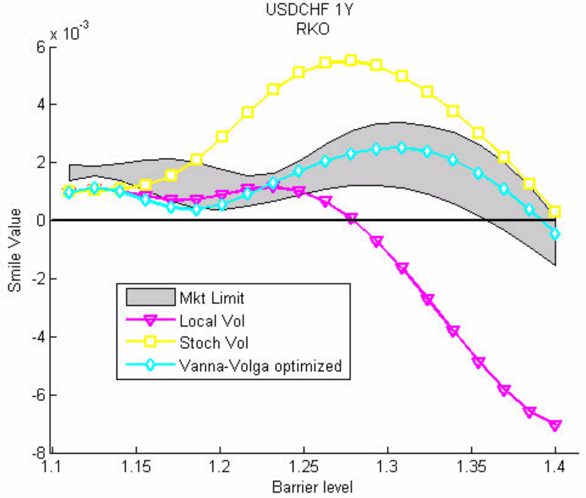
- *a* ? Calibration from market prices of selected exotic options (typically “touch” options)

Vanna-Volga: example of calibration

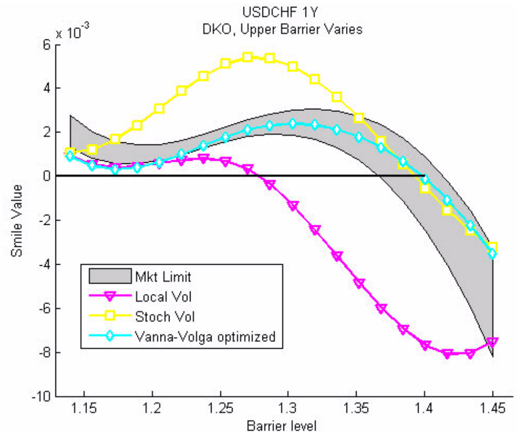
One-Touch



Call with barrier



Call with 2 barriers



- **Calibration on OT yields satisfactorily results for other exotic instruments**

- **Optimized Vanna-Volga approach yields better prices than Heston or Local-Vol**

Concluding remarks

- **Price of Exotic (Path-dependent) options:**
  - not uniquely determined by knowledge of vanilla prices
  - depends on smile dynamics (model)

- **In FX markets, Vanna-Volga is an appealing alternative to more rigorous stochastic models**
  - Easy implementation
  - Simple calibration
  - Computationally very efficient

- **Limitations**
  - Not a model, impossible to simulate dynamics
    - Limited to options for which the “survival probability” or “FET” can be computed
  - Correction based on smile at maturity date only, not on the entire volatility surface
  - Does not account for stochastic interest rates effects
    - Maturity <1 – 2Years
  - Does not account for jumps in the underlying process
    - Maturity >1 Month