Validation of Hull's Closed-Form Pricing Formula for Down & In European Barrier Call Options

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1 Executive Summary

This paper examines the validity of John Hull's analytical formula for pricing of down & in European barrier call options. We find that it works well as a quick-and-dirty estimate and potentially more than that for shorter maturities. However, great care must be taken to correct for a model assumption about continuous observation of the barrier, and hedging decisions should not be based solely on the analytical formula when the asset price is close to the barrier level.

In particular, Hull's formula gives results that are quite similar to the estimates provided by Bloomberg's Exotic Option Valuation tool, for both short and long maturities. This is only the case if one uses an adjustment suggested in the literature which artificially tweaks the barrier level, thereby mitigating the fact that real-world barriers are only monitored daily. Furthermore, the analytically derived price quotes can be replicated well by standard asset price simulation.

A modified simulation routine shows that Hull's price is robust to fluctuations in the risk-free interest rate and to the potential of large (non-normal) swings in asset returns. It is more difficult to verify the usefulness of the hedging parameters implied by the formula, but simulation and comparison with Bloomberg estimates at least do not show any glaring cause for concern.

In order to test the model against market prices an academic paper is quoted which finds that observed prices of similar barrier options can exceed theoretical and simulated values by more than 5%. Although this may partly be explained by hedging difficulties leading to higher risk premia, it is probably not the only cause.

The Hull formula is much more computationally efficient than the alternatives, but as a stand-alone model it cannot be used for more than rough estimates. In valuations where precision of price quotes and hedging parameters is important, the formula has too many weaknesses to be relied upon, especially for longer maturities.

2 Overview

In accordance with the OCC 2000-16 specifications, this paper lays out the model specifications and logic, tests it against other models and finally compares its predictions to observed market prices. First, the technical specifications are summarized in Section 3, then a discussion of the model assumptions follows in Section 4, Section 5 compares it to a benchmark model and seeks to replicate its price estimates and Greeks, a couple of robustness tests are carried out in Section 6, a study of observed market prices is quoted in Section 7 and finally Section 8 outlines the model strengths and weaknesses. The paper is rounded off in Section 9 by a list of relevant academic literature referenced in the text.

Note that the majority of the tests in this paper are only scratching the surface, in that they are mainly of a graphical and qualitative character rather than quantitative in a rigorous statistical sense. Furthermore, often only limited samples are considered. A complete validation would procure extra data and go more into details.

3 Technical Specification

Barrier options are path-dependent securities, the payoff of which depend on whether the underlying asset has breached a certain barrier level during the lifetime of the option. Eight types of barrier options exist but in this validation paper we focus on so-called *down & in barrier call options of the European kind*.[1] In addition to the specification of the underlying asset, the maturity and the strike price, a barrier level is agreed upon when buying the option. If the price of the underlying stays above this level until maturity, the option expires worthless. If the barrier is breached, however, the option turns into a plain vanilla European call option with the given tenor and strike.

Closed-form pricing formulas for general barrier options were originally derived by, among others, Merton (1973) and Rubinstein and Reiner (1991) – see these papers for precise documentation. Here we use the version quoted in Hull (2006) for down & in call options, namely

$$c_{\text{di}} = S_0 e^{-qT}(H/S_0)^{2\lambda}\Phi(y) - Ke^{-rT}(H/S_0)^{2\lambda-2}\Phi(y - \sigma\sqrt{T})$$

for $H \leq K$ and

$$\begin{aligned} c_{\text{di}} = c &- S_0 e^{-qT}\left[\Phi(x_1) - (H/S_0)^{2\lambda}\Phi(y_1)\right] \\ &+ Ke^{-rT}\left[\Phi(x_1 - \sigma\sqrt{T}) - (H/S_0)^{2\lambda-2}\Phi(y_1 - \sigma\sqrt{T})\right] \end{aligned}$$

for $H \geq K$. Here Φ is the cumulative distribution function of the standard normal distribution, S_0 is the time-0 price of the underlying, T is the maturity, K is the strike, H is the barrier level, r is the risk-free rate, q is the dividend

[1] Henceforth referred to as a "barrier option", unless otherwise stated.

rate, σ is the volatility, c is the price of the corresponding European call option according to the usual Black-Scholes formula and

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}, \quad y = \frac{\log[H^2/(S_0 K)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},$$

$$x_1 = \frac{\log(S_0/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \quad y_1 = \frac{\log(H/S_0)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}.$$

In the following a Monte Carlo C++ simulation routine is used. It is based on the Black, Scholes and Merton framework in which the evolution of the underlying satisfies (under the risk-neutral measure)

$$\Delta X = (r - \sigma^2/2)\Delta t + \sigma \Delta W \sqrt{\Delta t},$$

where $X = \log S$. The component ΔW is a random shock, assumed to be standard normally distributed. In the simulations, the step size Δt has been set to 1/252 years, i.e. approximately one business day. The C++ method employs the random number generator found in the freely available GSL library, and the RNG receives a new random number seed for each simulated asset. To ensure limited errors, 10,000 path simulations are done per pricing. The standard deviation of the Monte Carlo price tends to be about 2 cents on the dollar.

In the derivation of the above closed-form formula it is assumed that the barrier is observed continuously during the lifetime of the option. In reality, it is typically observed at close every business day. As noted by Hull, the literature suggests multiplying H by $\exp(-0.5826\sigma\sqrt{\Delta t_o})$ to correct for discrete barrier observation. Here Δt_o is the time interval between observations.

4 Review of Assumptions

Hull's closed-form formula for the pricing of down & in barrier options suffers from the same restricting assumptions as the usual Black-Scholes European option formulas. These are as follows:

- There are no arbitrage opportunities.
- The logarithm of the price level follows a diffusion process with normally distributed shocks, i.e. the price path is continuous (no jumps).
- The risk-free rate is constant.
- Hedging takes place continuously and without transaction costs.
- Volatility is exogenously given and constant.
- Specific to the barrier option model: The barrier level is observed continuously.

The no-arbitrage assumption is very prevalent in finance and is likely to hold within reasonable error in liquid markets. Hence, at least for exchange-traded barrier options it is a fairly safe restriction to impose. Log-normality of asset prices is not generally observed in the market, as returns tend to be leptokurtic. However, the model can fairly easily be extended to general distributions through simulation. That price paths are continuous is a more serious restriction and it requires sound consideration and scenario risk reporting by users of the model. One can circumvent the assumption of a constant risk-free rate by basing the formula on forward contracts on the asset. Of course, hedging is carried out discretely in the real world, but this turns out not to cause significant problems in the model framework. The effect of transaction costs is mitigated by having large trading desks focusing on net exposures.

The volatility assumptions are probably the most severe in the model. Because real-market volatility is not constant over time or over different asset price levels, and because implied volatility is not independent of asset price movements, it is important to develop comprehensive risk reporting of different volatility scenarios when using the closed-form formula.[2]

As mentioned in Section 3, discrete observation of the barrier level can be partly accommodated by adjusting the barrier level input in the model. Alternatively, one can use lattice models in order to directly impose a discreteness of time, but Easton and Gerlach (2007) claim that this still leads to inaccurate results. Monte Carlo simulation is a good, albeit computationally intensive, approach to the pricing of most exotic options, but in the case of barrier options it is necessary to run a very large number of asset path simulations, as the presence of the knock-in barrier can severely distort payoffs in limited samples.

5 Comparison to Other Models

As required by the OCC 2000-16 guidelines Hull's formula is now compared against a benchmark and sought replicated by a third model. The benchmark is chosen as the Bloomberg terminal's Exotic Option Valuation calculator (abbrev. OVX), and the replication model is simple Monte Carlo simulation. Figure 1 is a screenshot of the Bloomberg barrier option calculator. It features several more parameters than the Hull model but for consistency these are ignored in the following. According to the terminal help files, the calculator uses "analytical expressions" when the option is monitored continuously, while it employs a trinomial tree model (extending the work of Ritchken, 1995) in the case of discrete monitoring.

[2]The preceding discussion was based on lecture notes by Steve Allen.

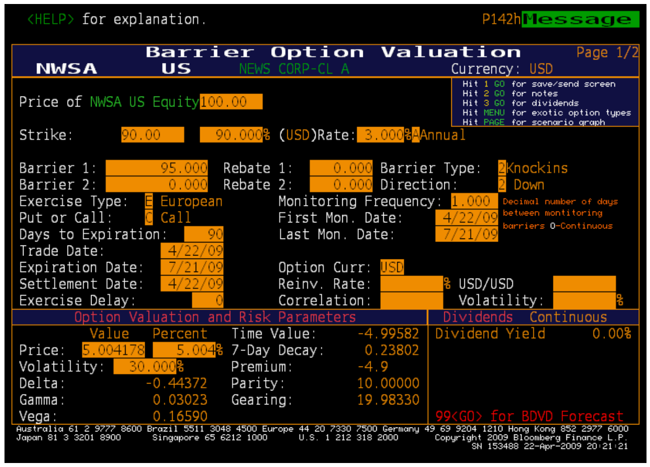

Figure 1: Barrier Option Valuation screen on the Bloomberg terminal.

5.1 Pricing

It turns out that the analytical expression favored by Bloomberg for continuous barrier monitoring must be very similar to Hull's. As a quick test, four barrier options were priced, all with initial asset price $S_0 = 100$, dividend rate $q = 0$ and maturity $T = 90$ days. The following table compares the two approaches:

	K	r	σ	H	Hull	Bloomberg	Rel. diff.
Barrier 1	105	0.03	0.3	95	1.3013	1.3010	−0.021%
Barrier 2	105	0.02	0.3	98	2.6856	2.6848	−0.028%
Barrier 3	110	0.03	0.4	95	1.8257	1.8251	−0.031%
Barrier 4	90	0.03	0.3	95	5.8709	5.8708	−0.002%

The difference is miniscule, although the Hull result is seemingly systematically smaller than the Bloomberg estimate. However, for all intents and purposes it may be supposed that

> Bloomberg's analytical formula is virtually identical to Hull's and any discrepancies are likely caused by rounding errors or slightly different treatments of the maturity day count.

We do not expand upon this issue since continuous monitoring is not of much practical relevance.

Moving on to discrete barrier monitoring, we consider two different barrier option types: Both have $S_0 = 100$ and $T = 90$ days, but Barrier 1 has $K = 90$ and $H = 95$, while Barrier 2 has $K = 105$ and $H = 98$. The risk-free rate is varied between $r = 0.01, 0.03, 0.07$, whereas the volatility is taken as $\sigma = 0.1, 0.3, 0.5, 0.7$. This creates a scenario price grid which lends to a comparison of the Hull and Bloomberg approaches. The barrier level in the Hull formula is adjusted as described above (this is crucial for precision, as argued later).

Furthermore, the monitoring interval in the OVX calculator is set to 1 business day. It turns out that for Barrier 1 the median relative difference in the 12 prices is -0.16%, while the average is 1.26%. For Barrier 2 the median is -5.62% and the average comes out as -6.98%. Hence, for Barrier 1 the differences between the models are negligible, whereas the discrepancy is more noticeable for Barrier 2. We suspect that this is due to the barrier level being very close to the initial asset price in the latter case.

All in all, however, it appears that

> The Hull model works well as a quick-and-dirty valuation method, if the Bloomberg calculator is to be trusted.

It is comforting to see that the vastly different modeling approaches undertaken by Hull and Bloomberg lead to fairly similar results.

The price quotes from the Hull formula are now sought replicated by Monte Carlo simulation. Since this is more easily done than retrieving Bloomberg calculations, it is possible to compare a larger sample of prices. Specifically a down & in option with $S_0 = 100$, $K = 105$, $H = 98$, $q = 0$ and $T = 90$ days is considered. The risk-free rate is varied between $0.01, 0.02, \ldots, 0.10$, while the volatility runs between $0.1, 0.2, \ldots, 1.0$. This leads to 100 scenario price quotes in total.

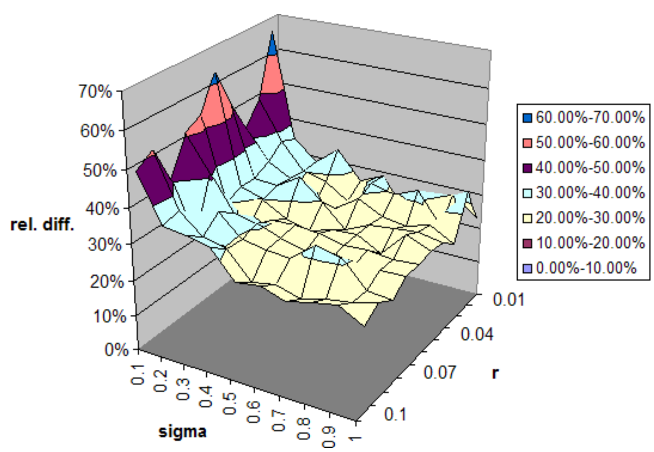

Figure 2: Graph of the relative difference between Hull and Monte Carlo, uncorrected, $T = 90$ days.

First the *uncorrected* Hull price is compared to the Monte Carlo estimate. As is evident in Figure 2, the discrepancies are huge. The median error is 28.73% with an interquartile distance of about 6%. This means that the uncorrected Hull formula tends to overshoot the Monte Carlo estimate consistently by approximately $1/4$. Furthermore, the graph shows that the relative error is decreasing in σ. These observations make intuitive sense, since a continuously observed barrier is more likely to be breached, especially if the volatility is small, leading to higher prices.

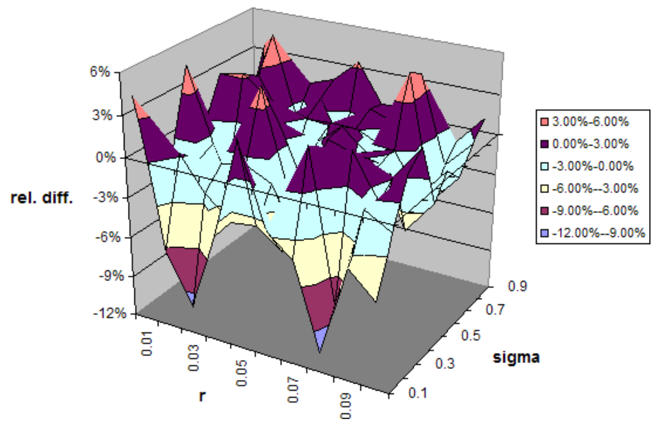

Figure 3: Graph of the relative difference between Hull and Monte Carlo, corrected, $T = 90$ days.

Figure 3 shows the same graph, but incorporating the correction of the barrier level to approximate discrete monitoring. Now the results are much more reasonable, and the errors seem unsystematic, except for very small volatilities. The median relative error is -0.56% with an interquartile distance of 3%.

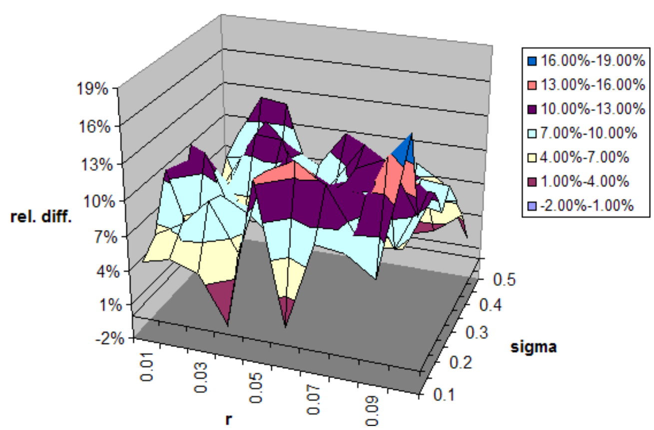

Figure 4: Graph of the relative difference between Hull and Monte Carlo, uncorrected, $T = 3$ years.

The exercise above is repeated for a barrier option with $S_0 = 100$, $K = 110$, $H = 90$, $q = 0$ and $T = 3$ years. The volatility is only varied up to 0.5 but in steps of 0.05, leading to 90 scenarios. Figures 4 and 5 show the relative differences in the uncorrected and the corrected cases, respectively. Because the

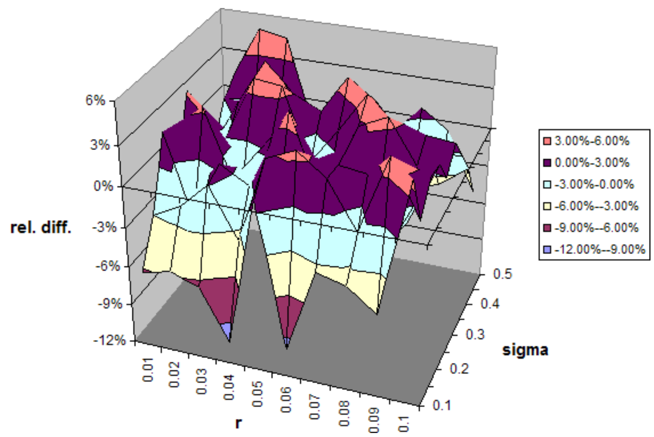

Figure 5: Graph of the relative difference between Hull and Monte Carlo, uncorrected, $T = 3$ years.

barrier level is no longer close to the starting asset price, the uncorrected relative differences are not unambiguously decreasing in σ, but the general observation holds that the uncorrected Hull formula overshoots the Monte Carlo results (by a median of 7.91%, interquartile range 5%), while the corrected prices are quite close (median -0.15%, interquartile range 5%). Thus, the error stemming from failing to adjust for discrete monitoring is not as substantial when the time horizon is large, but it is definitely still there.

We conclude that

> Monte Carlo simulation is perfectly capable of replicating the closed-form price within reasonable error. It is absolutely crucial to remember the barrier level adjustment for discrete monitoring in Hull's formula.

Thus, if one makes the necessary corrections, Hull's formula agrees well with both Bloomberg's Exotic Option Valuation calculator and simple Monte Carlo simulation.

5.2 Hedge parameters

We now briefly touch upon the Greeks in the model, focusing on delta, gamma and vega. The nature of a down & in barrier option calls for extra care when hedging, since for example high time-0 asset prices will require the underlying's price to fall significantly before the option kicks in, while at the same time this drop has an adverse effect on the final payoff given a breach of the barrier. Therefore, there are several counteracting forces to consider and, additionally, delta is very discountinuous right around the barrier level (in the extreme case $S_0 \leq H$ the option turns into a vanilla European call).

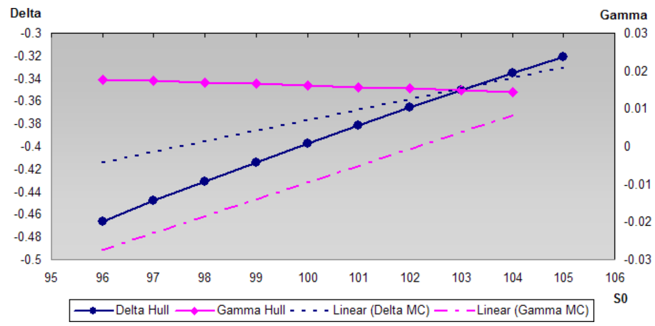

Figure 6: Delta and gamma as a function of time-0 asset price.

In the following a barrier option with $K = 90$, $H = 95$, $q = 0$, $T = 90$ days and $r = 0.03$ is considered. Figure 6 shows the variation of delta and gamma as a function of the time-0 asset price S_0. The volatility is set to $\sigma = 0.5$. The solid lines show the corrected Hull values (measured by finite differences, not by differentiating the analytical formula) while the dashed lines show linear regressions of the corresponding Monte Carlo values. It was necessary to carry out regressions since the innate variability of the simulations lead to quite unstable Greeks. Consequently, the standard deviation of the linear regression estimates are fairly high, but it is comforting to see that the regression lines are somewhat close to the Hull values, even if the gamma regression slopes the wrong way.

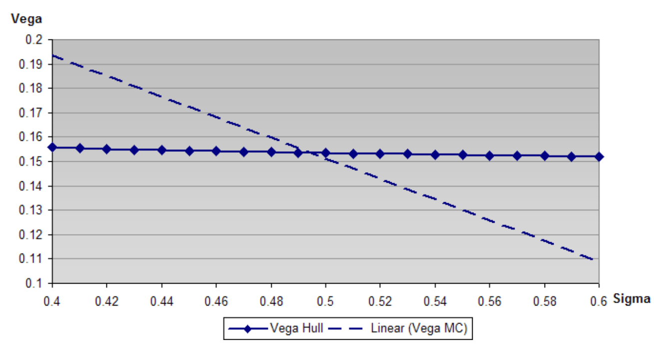

Figure 7: Vega as a function of volatility.

Figure 7 shows a similar graph for vega as a function of volatility. The time-0 asset price is set to $S_0 = 100$ in the computations. Again the regression on the Monte Carlo estimates is not so different from the analytical values that there is cause for alarm. However, the simulated vegas are much more unstable than the analytical ones.

It may be noted that the lattice-generated Greeks in the Bloomberg exotic option calculator for the above-mentioned barrier option (with $S_0 = 100$ and $\sigma = 0.5$) is a delta of -0.41466, a gamma of 0.01727 and a vega of 0.16047.

These are pretty close to the Hull values of -0.39740, 0.01613 and 0.15343, respectively.

Finally, the closed-form delta for $S_0 = 95$ and $\sigma = 0.5$ is -2.2636, much larger than the deltas for $S_0 = 96$ and up. This confirms the intuition about knock-in options and stresses the importance of collecting precise data when the asset price moves close to the barrier level.

The variability of the Monte Carlo prices makes it difficult to conclude much, but

> At least for short maturities, the Greeks implied by the corrected Hull formula can be very roughly confirmed by Monte Carlo simulation and they agree with Bloomberg's estimates in the limited sample considered here.

6 Sensitivity Analysis

In this section we examine the possible implications of two changes in the model, namely a stochastic risk-free rate and fat-tailed price shocks. Because we have established above that Monte Carlo simulation corresponds reasonably well with Hull's closed-form formula, the testing is carried out by simulation.

First we let the risk-free rate be normally distributed in the following sense: At every time step its mean is given by a specified $\overline{r}$ and its standard deviation is 10 basis points (BP). This captures the effect of fluctuations in the interest rate, but it would probably be more realistic to use an autoregressive process. The fixed parameters are set to $S_0 = 100$, $K = 105$, $q = 0$, $T = 90$ days, $\sigma = 0.5$ and $H = 98$. By running a simulation for $\overline{r} = 0.1, 0.2, \ldots, 1.0$ we get 10 price figures which can be compared with the corresponding corrected Hull quotes. The median relative error comes out at -0.36%. If we repeat the process with the standard deviation of the interest rate fluctuations set to 50 BP instead, the mean error is -0.91%. Comparing with the results from Section 5, we see

> For short maturities, stationary fluctuations of the risk-free rate do not significantly invalidate the use of Hull's formula, even if the daily fluctuations have a standard deviation of 50 basis points.

Of course, the error will be exacerbated by persistence in the interest rate shocks, as can be seen in the scenario analysis discussed above.

In order to gauge the importance of the normality assumption, we now let the exogenous shock ΔW in the evolution of the asset price be t-distributed with 7 degrees of freedom (this number was picked rather arbitrarily). This means that the shocks will have an excess kurtosis of 2 rather than 0. ΔW is scaled such that its variance is retained at 1. Then the simulations are run for the barrier option $S_0 = 100$, $K = 105$, $q = 0$, $H = 95$ and $T = 90$ days, with r varying between 0.01 and 0.07 in steps of 0.01 and σ varying between 0.3 and 1.0 in steps of 0.1.

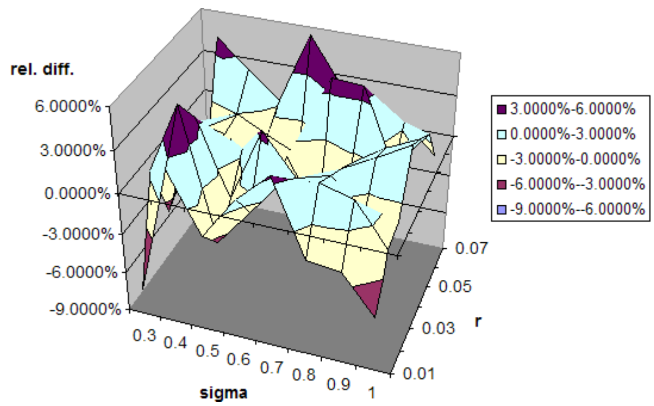

Figure 8: Graph of the relative difference between Hull and Monte Carlo with t-distributed shocks.

The relative differences between Hull's corrected prices and the Monte Carlo prices are shown graphically in Figure 8. The median error is 0.09% with an interquartile range of 4%. Because the graph does not reveal any particular pattern in the error distribution, we conclude that

> The validity of the Hull formula does not hinge upon the normality assumption, at least not for short maturities.

7 Comparison with Market Prices

Barrier options have been traded over the counter for at least two decades and they were first introduced as exchange-traded securities in 1991 on the CBOE and the American Exchange. However, it is not easy to obtain free-to-use historical quotes so instead we rely on research by Easton, Gerlach, Graham and Tuyl (2004), who observed and analyzed barrier option prices on the Australian Stock Exchange. Unfortunately, their research only covers knock-out options, but the conclusions should still apply – at least qualitatively – to knock-in options.

They use Monte Carlo simulation to determine a theoretical value for the barrier options and then compare this price (in a properly statistically adjusted manner) to the observed one in the market. Since it has been determined above that Hull's formula gives essentially the same results as simple Monte Carlo simulation, the conclusions are relevant for the purpose of this paper. As model inputs for volatility and risk-free rate they use historical implied volatilities and the Australian bank bill rate. The dividend yield was also backed out of market indexes.

After running their tests, Easton et al. conclude that the observed barrier option prices generally tend to be higher than the simulated prices, at a statistically significant level (although the median difference in all cases is less than

9 cents on the dollar). Because they also find that observed vanilla European option prices are *less* than the theoretical prices, the verdict is that something other than the model parameter inputs is causing the discrepancy. They theorize that the very high gammas of barrier options with underlyings near the barrier level make option writers demand a (perhaps overly) high premium. However, in a later paper (2007), two of the authors acknowledge that the instability of deltas near the barrier level does not hold for all barrier options.

This leads to the conclusion that

> To the extent that Australian knock-out barrier options can serve as a proxy for general down & in options, market prices of barrier options tend to exceed theoretical and simulated values, sometimes by more than 5%. This may partly, but not fully, be explained by larger premiums due to hedging difficulties.

8 Model Strengths and Weaknesses

According to the discussion above, we have established that *if (and only if) the correction for discrete monitoring of the barrier level is applied*, Hull's closed-form formula for down & in barrier options works well as a quick valuation device for options with short maturities. The formula is quite robust to changes in the model particulars, such as the constant interest rate and the normality assumptions. Furthermore, it corresponds well with the benchmark Bloomberg price calculator and it serves as more than a ballpark estimate of observed market prices, with a predictable sign of the price difference.

The formula's most obvious strength is its relative simplicity – if inputted into Excel it can value hundreds of barrier options in less than a second, which makes it vastly more efficient than lattice and especially Monte Carlo approaches. Because the framework is basically the same as for the well-known and widely used Black-Scholes formula, it is not conceptually context-dependent (although parameter estimates are, of course) and many of the model limitations can be addressed with the same routines that financial firms have employed for plain vanilla options since the 1970s. The Greeks implied by the formula are sharp unlike those derived from simulation.

However, there is no easy way to adapt Hull's formula to, for example, American-style options or baskets of barrier options with correlated underlying assets. It is only possible to approximate the effect of discrete barrier level monitoring, and the closer the barrier level is to the strike or the current asset price, the more dependent do the formula results become on the Black-Scholes assumptions. The Greeks are discontinuous near the barrier level so hedging, and therefore also modeling, is difficult in these cases.

In conclusion,

Strengths: Very efficient while being relatively precise; well-known framework; sharp Greeks.
Weaknesses: Almost the same as for Black-Scholes; not easily extendable; approximations necessary for discrete monitoring; fragile near barrier level; problematic when used for hedging decisions.

It is therefore reasonable to use Hull's formula for quick-and-dirty valuations, as well as for short-maturity estimates that do not require better than a 5–10% error margin. However, particular care must be taken if the strike or asset price is close to the barrier level, and in general it is a good idea to at least periodically verify the results using advanced lattice and/or Monte Carlo techniques.

9 Relevant Literature

- Easton, S., R. Gerlach, M. Graham, and F. Tuyl, 2004, 'An empirical examination of exchange-traded barrier options,' *Journal of Futures Markets* 24, 1049-1064.

- Easton, S., and R. Gerlach, 2007, 'Modelling exchange-traded barrier options traded in the Australian options market,' *Accounting and Finance* 47, 109-122.

- Hull, J., 2006, 'Options, futures, and other derivatives,' Pearson/ Prentice Hall, 6th ed., 533-535.

- Merton, R. C., 1973, 'Theory of rational option pricing,' *Bell Journal of Mathematics and Management Science* 4, 141-183.

- Ritchken, P., 1995, 'On pricing barrier options,' *Journal of Derivatives* winter, 19-28.

- Rubinstein, M., and E. Reiner, 1991, 'Breaking down the barriers,' *Risk* 4, 28-35.