Vanna-Volga methods applied to FX derivatives: a practitioner's approach

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Presentation layout

- **Problem statement:**
  - **Vanilla prices : the implied volatility surface**
  - **pricing exotic options : smile dynamics**
  - **Comparing prices from different models**

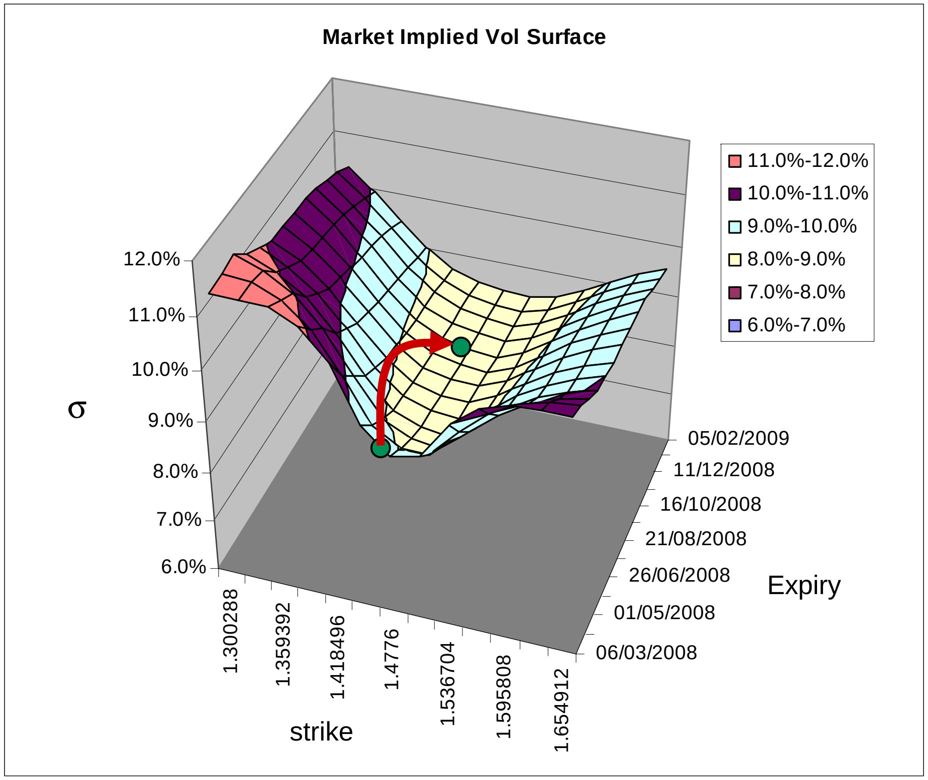
- **The Vanna-Volga Method**
  - **The general idea**
  - **Basic implementation for vanillas**
  - **Adaptations for the pricing of exotic options**
  - **A concrete calibration example**

- **Conclusions**

- **Bibliography**

Problem statement : the implied smile

- **Price of Vanillas (Call / Put) uniquely determined by the "Implied volatility" surface :**



Black-Scholes formula

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / K) + (r - q + \sigma(K,T)^2 / 2)T}{\sigma(K,T)\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - q - \sigma(K,T)^2 / 2)T}{\sigma(K,T)\sqrt{T}}$$

Terminal spot density:

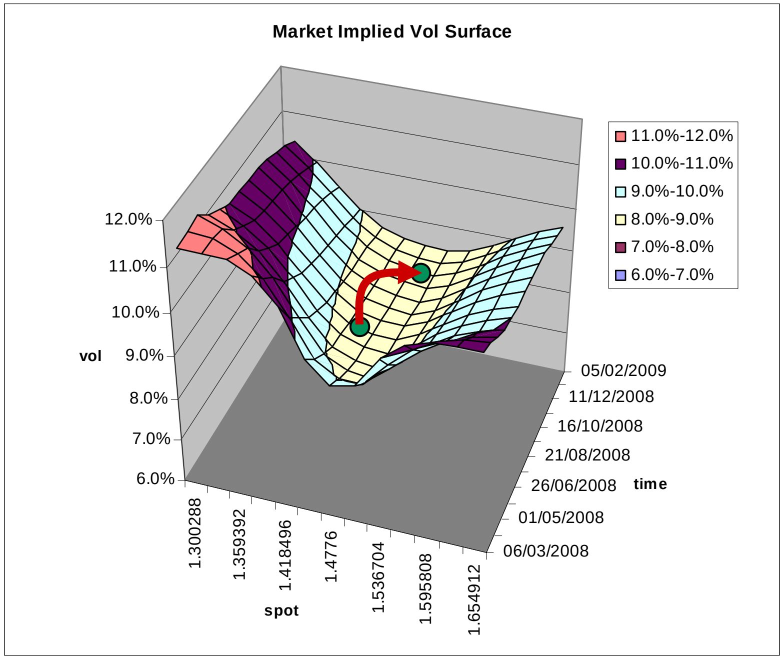
$$f(S,T) = e^{rT} \frac{d^2}{dK^2} \mathrm{Call}\,(S,T,K,\sigma(K))\Big|_{K=S}$$

High-amplitude movements more likely than predicted by a normal distribution

Problem statement : smile dynamics

- **Exotic options (e.g. calls/puts with barriers) :**
  - payout depends on entire spot path S(t), 0<t<T
  - Price not only function terminal density (vol surface), but also on **transition density** (=smile dynamics)

  $f(S_1,T_1,S_2,T_2)$ = density of spot at time T_2, conditional on spot starting from S_1 at time T_1.



Problem : transition densities are **not fully determined** even by the complete knowledge of the entire implied volatility surface

Problem statement : smile dynamics

- **Various stochastic models can be perfectly calibrated to the same implied smile (they all give the exact same price for vanilla options) and yet yield very different prices for exotic options.**

Schoutens, W., Simons, E. and Tistaert, J., "A Perfect calibration ! Now what ?" , Wilmott Magazine, March 2004

Problem statement : smile dynamics (cont'd)

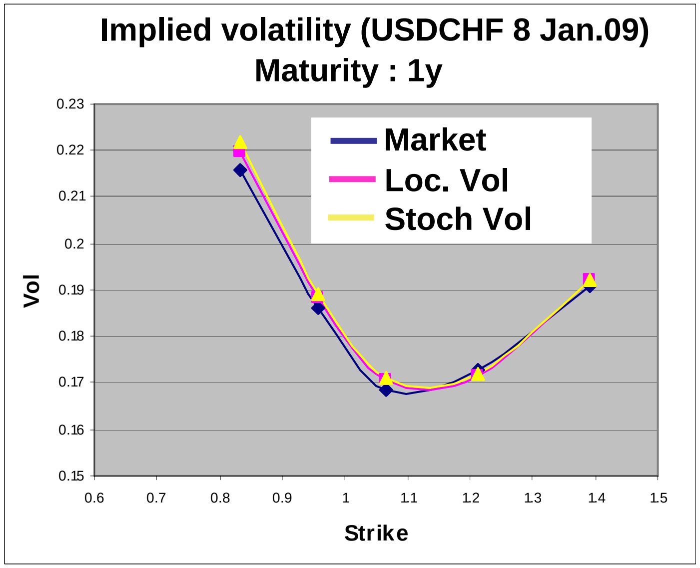
- **Example : 2 very different smile dynamics**

Pure local volatility (Dupire)

$$dS = \mu S\,dt + \sigma(S,t) S\,dW$$

Pure stoch. volatility (Heston)

$$\begin{cases} dS = \mu S\,dt + \sqrt{V} S\,dW_1 \\ dV = \kappa(V_\infty - V)dt + \varepsilon\sqrt{V}\,dW_2 \\ E\left[dW_1 dW_2\right] = \rho dt \end{cases}$$

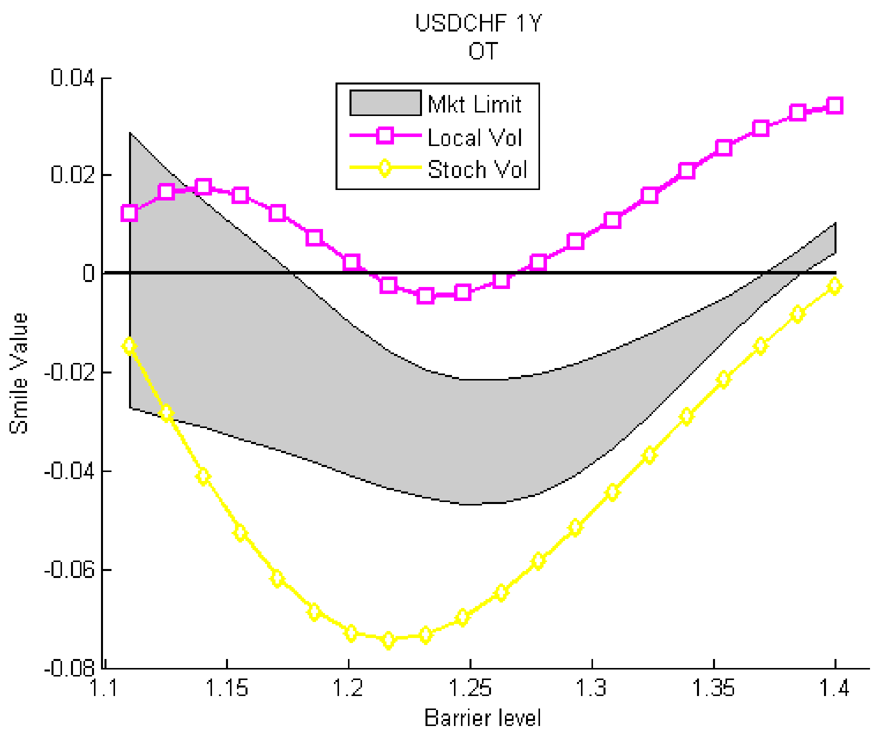


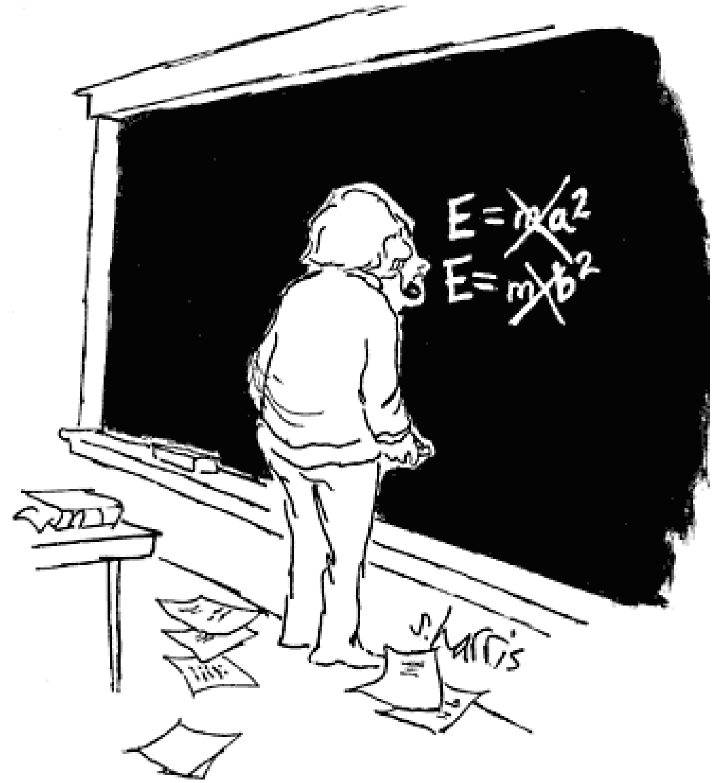
The 2 models are able to fit vanilla prices (after calibration of Heston)

But ….

Which model is better ?

Test on the pricing of a "One-Touch" option





Actual smile dynamics seems to follow a mixture between those 2 extreme regimes

A tradeoff

- **A rigorous mixture model – Stochastic vol / local vol / jumps – is certainly able to produce correct market prices, however…**
  - Complexity of implementation
  - Delicate calibration
  - Computationally demanding

- **Alternatively, the « Vanna-Volga » follows a more « cooking recipe » approach**
  - Strictly speaking, Vanna-Volga is not a model, rather a price adjustment method
  - Easy implementation
  - Simple / No calibration
  - Computationally very efficient

Vanna-Volga : main idea

- **Derives from the trader's idea that the smile adjustment to an option price is associated with costs incurred by hedging its volatility risk**
  - Build a portfolio of 3 vanilla strategies, which zeros out the vega ($\partial P/\partial\sigma$), Volga ($\partial^2 P/\partial\sigma^2$) and Vanna ($\partial^2 P/\partial\sigma\partial S$) of the option at hand.
  - Using the market-implied volatility surface, calculate the smile impact on the obtained portfolio of vanillas.

Market Price (Exotic Option)

=

Black-Scholes Price (Exotic Option)

+

Smile Impact (vanilla hedging portfolio)

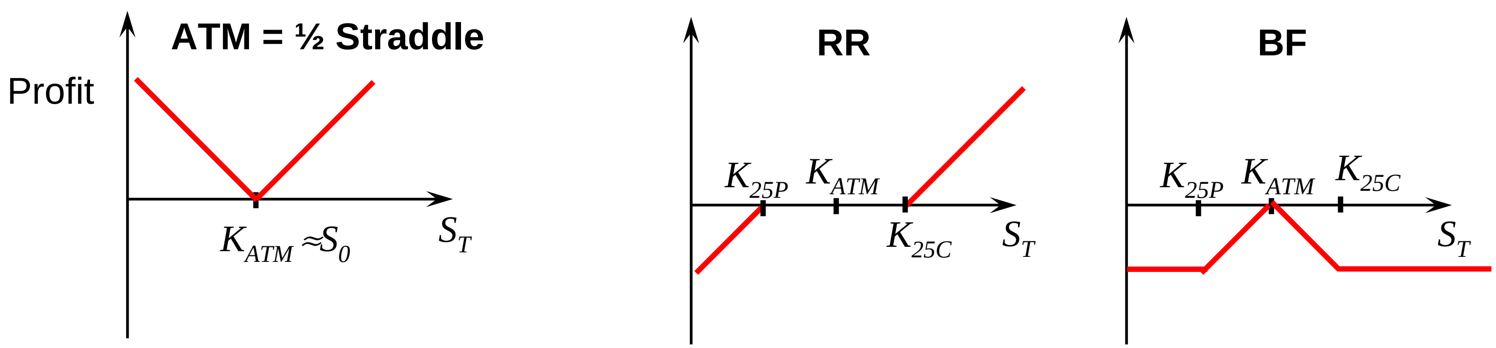
Vanna-Volga : the vanilla hedging portfolio

- **Composed of 3 liquid vanilla strategies:**

  - At-The-Money (ATM) = ½ Call($K_{ATM} \approx S_0$) + ½ Put ($K_{ATM} \approx S_0$)

  - 25Δ Risk-Reversal (RR) = Call(Δ = 0.25) - Put(Δ = -0.25)

  - 25Δ Butterfly (BF) = ½ Call(Δ = 0.25) + ½ Put(Δ = -0.25) - ATM



Vanna-Volga : the vanilla hedging portfolio

- **selecting the weights:**
  - Price of exotic option : X
  - Price of hedging portfolio : $P = w_{ATM} \cdot ATM + w_{RR} \cdot RR + w_{BF} \cdot BF$
  - X and P computed under Black-Scholes assumptions (no smile)
  - Zeroing out the sensitivity to volatility, up to 2nd order (Vega, Vanna and Volga) :

$$\begin{pmatrix} X_{\mathrm{vega}} \\ X_{\mathrm{vanna}} \\ X_{\mathrm{volga}} \end{pmatrix} = \begin{pmatrix} \mathrm{ATM}_{\mathrm{vega}} & \mathrm{RR}_{\mathrm{vega}} & \mathrm{BF}_{\mathrm{vega}} \\ \mathrm{ATM}_{\mathrm{vanna}} & \mathrm{RR}_{\mathrm{vanna}} & \mathrm{BF}_{\mathrm{vanna}} \\ \mathrm{ATM}_{\mathrm{volga}} & \mathrm{RR}_{\mathrm{volga}} & \mathrm{BF}_{\mathrm{volga}} \end{pmatrix} \begin{pmatrix} w_{\mathrm{ATM}} \\ w_{\mathrm{RR}} \\ w_{\mathrm{BF}} \end{pmatrix}$$

$$\vec{w} = A^{-1}\vec{x}$$

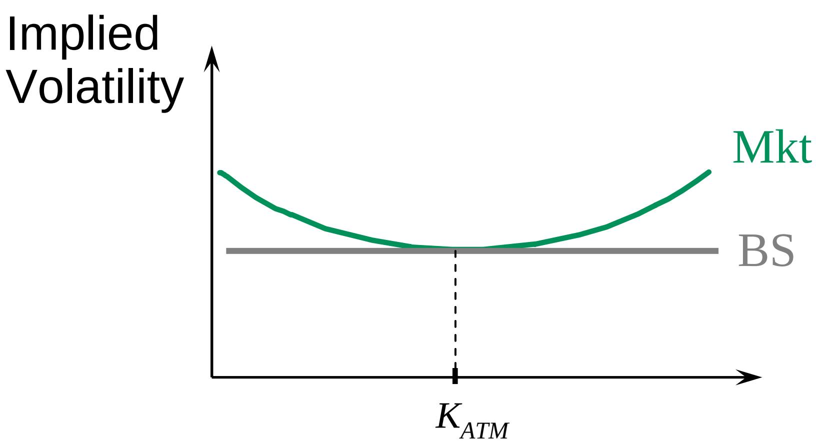
Vanna-Volga: full correction

- **Finally the « Vanna volga » price of an option is :**

$$X^{VV} = \boxed{X^{BS}} + \boxed{w_{ATM} \cdot \cancel{(ATM^{Mkt}\text{-}ATM^{BS})} + w_{RR} \cdot (RR^{Mkt}\text{-}RR^{BS}) + w_{BF} \cdot (BF^{Mkt}\text{-}BF^{BS})}$$

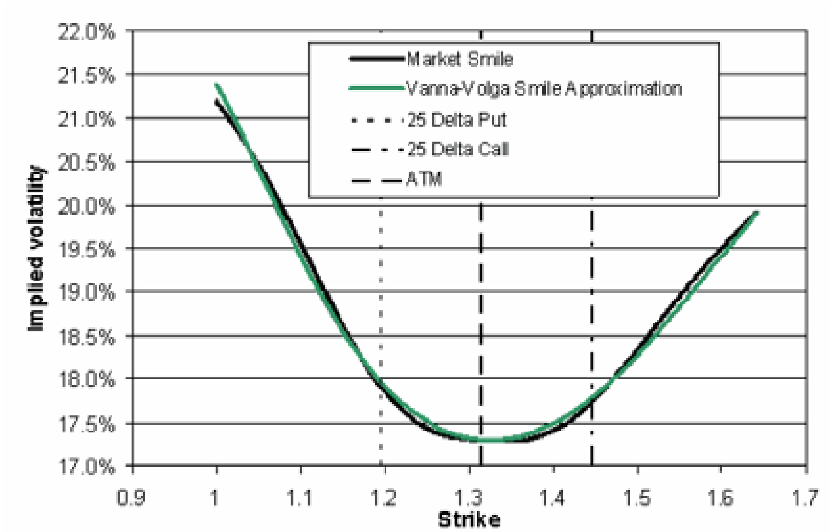
Black-Scholes Price

Smile Impact on vanilla hedging portfolio

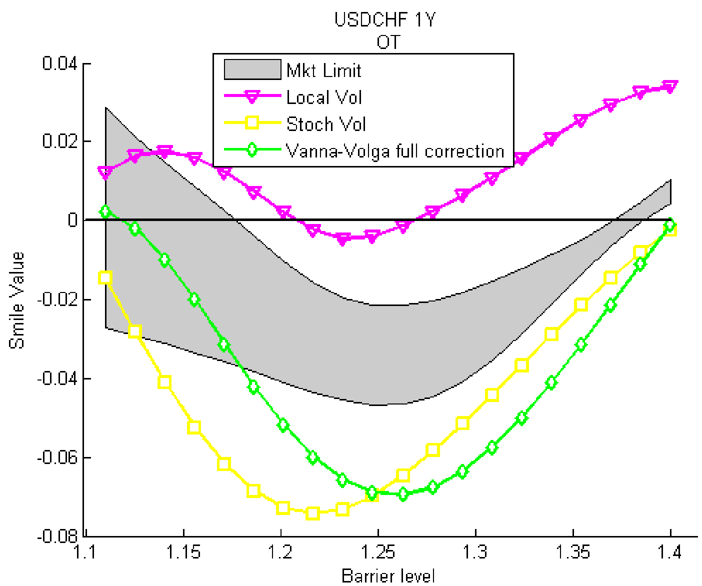


Vanna-Volga: full correction

- **Applying the full VV correction to price a vanilla provides a reasonable approximation of the original smile :**



- **Applying the full VV correction to price an exotic option usually over-estimates the smile impact :**



Vanna-Volga: attenuating the correction

- **Starting from the original VV setting :**

$X^{VV} = X^{BS} + w_{RR} \cdot (RR^{Mkt}-RR^{BS}) + w_{BF} \cdot (BF^{Mkt}-BF^{BS})$

- **Apply some simple matrix calculus, drop vega contribution and introduce attenuation factors:**

$X^{VV} = X^{BS} + p_{VANNA}(\gamma)\, X_{VANNA}\, \Omega_{VANNA} + p_{VOLGA}(\gamma)\, X_{VOLGA}\, \Omega_{VOLGA}$

$$\vec{I} = \begin{pmatrix} 0 \\ \mathrm{RR}^{\mathrm{mkt}} - \mathrm{RR}^{\mathrm{BS}} \\ \mathrm{BF}^{\mathrm{mkt}} - \mathrm{BF}^{\mathrm{BS}} \end{pmatrix} \qquad \begin{pmatrix} \Omega_{\mathrm{vega}} \\ \Omega_{\mathrm{vanna}} \\ \Omega_{\mathrm{volga}} \end{pmatrix} = (A^T)^{-1}\vec{I}$$

- **Market practice : p_{VANNA} and p_{VOLGA} are functions of**
  - γ = Survival probability
  - γ = Expected first exit time (FET) / T

  } obtained by solving BS PDE with appropriate boundary condition

Vanna-Volga: requirements on attenuation factors

$$0 < p_{VANNA}(\gamma), p_{VANNA}(\gamma) < 1$$

- **As spot level approaches a "Knock-out" barrier, Vanna-Volga correction should fade-out**
  - **For volga term, always true as volga fades-out:** $\lim_{\gamma \to 0} X_{VOLGA} = 0$
  - **Not the case for Vanna, hence, we should impose:**

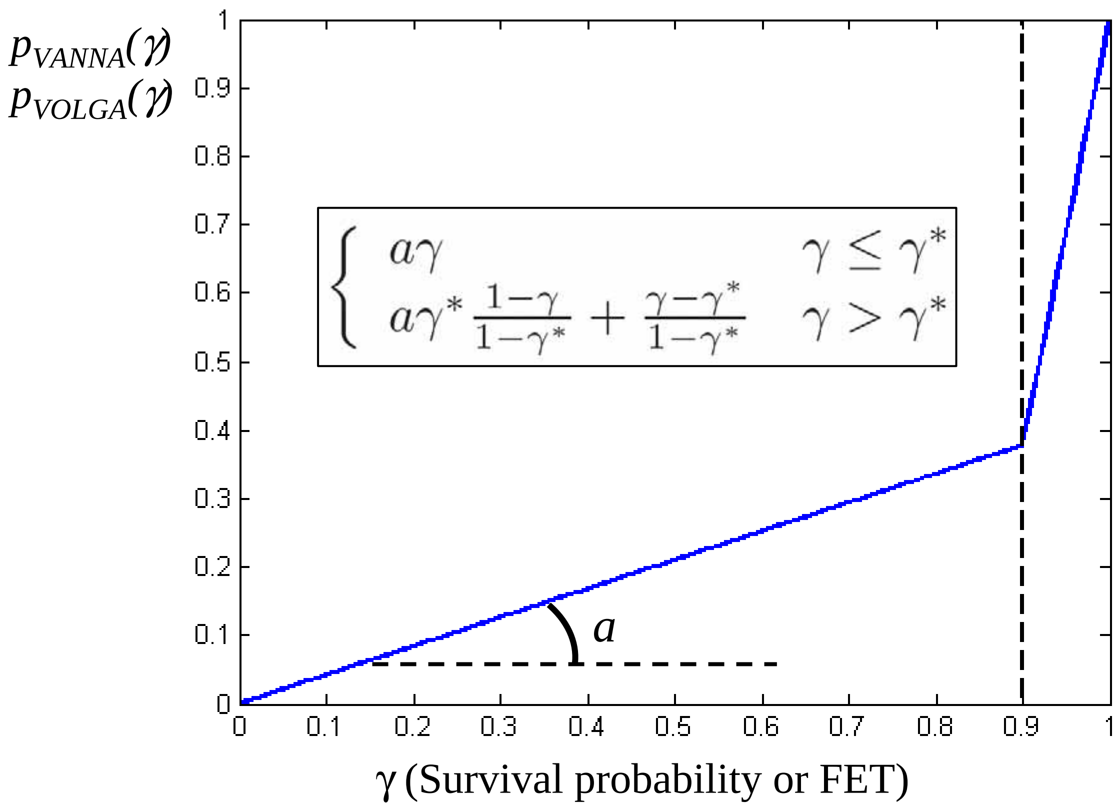
$$\lim_{\gamma \to 0} p_{VANNA}(\gamma) = 0$$

- **Vanilla payouts (no barriers) are best priced with the full Vanna-Volga correction:**

$$\lim_{\gamma \to 1} p_{VANNA}(\gamma) = \lim_{\gamma \to 1} p_{VOLGA}(\gamma) = 1$$

Vanna-Volga: example of attenuation factors choice

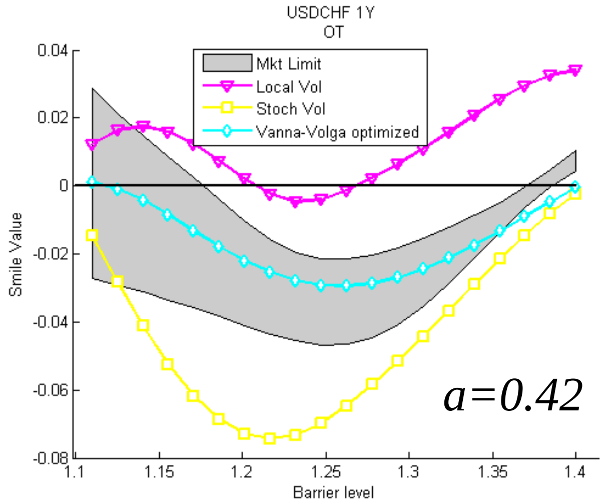
Composite linear function



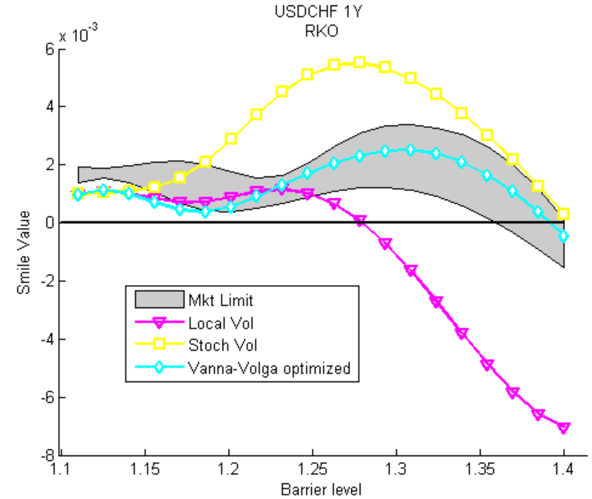
a **? Calibration from market prices of selected exotic options (typically "touch" options)**

Vanna-Volga: example of calibration

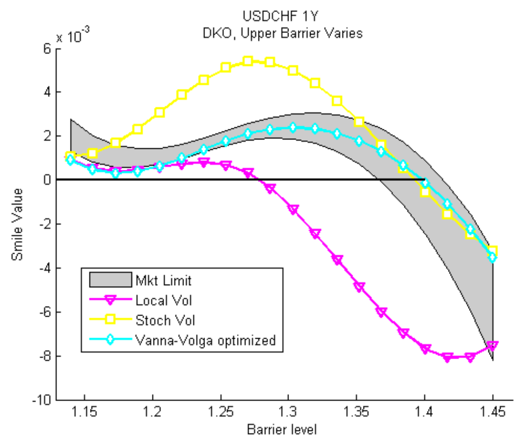
One-Touch



Call with barrier



Call with 2 barriers



- **Calibration on OT yields satisfactorily results for other exotic instruments**

- **Optimized Vanna-Volga approach yields better prices than Heston or Local-Vol**

Concluding remarks

- **Price of Exotic (Path-dependant) options:**
  - not uniquely determined by knowledge of vanilla prices
  - depends on smile dynamics (model)

- **In FX markets, Vanna-Volga is an appealing alternative to more rigorous stochastic models**
  - Easy implementation
  - Simple calibration
  - Computationally very efficient

- **Limitations**
  - Not a model, impossible to simulate dynamics
    - Limited to options for which the “survival probability” or “FET” can be computed
  - Correction based on smile at maturity date only, not on the entire volatility surface
  - Does not account for stochastic interest rates effects
    - Maturity <1 – 2Years
  - Does not account for jumps in the underlying process
    - Maturity >1 Month

Bibliography

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