

Inverse Nonlinear Fast Fourier Transform: Closing A Chapter in Quantum Signal Processing

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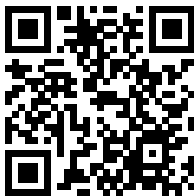
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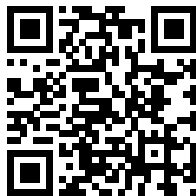
January 2026

QIP 2026, Riga

- Hongkang Ni, Rahul Sarkar, Lexing Ying, and Lin Lin. “Inverse nonlinear fast Fourier transform on SU (2) with applications to quantum signal processing.” arXiv preprint arXiv:2505.12615 (2025).
- <https://github.com/qsppack/QSPPACK>



Paper



Code



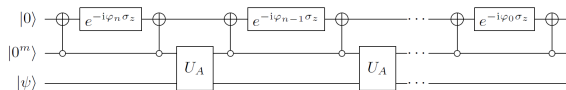
Slides

- 1 Introduction
- 2 QSP, GQSP, and NLFT
- 3 Layer Stripping: Stable or Not?
- 4 INLFFT: The Fast Algorithm
- 5 Conclusion

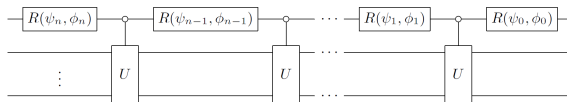
Matrix Function Computation: QSP² and GQSP³

Numerous applications:

- Hamiltonian simulation
- Linear system solver
- Eigenvalue problems
- Gibbs states preparation
- ...



QSP circuit: Implement block-encoding of $f(A)$ for Hermitian A .¹



GQSP circuit: Implement block-encoding of $f(U)$ for unitary U .

¹For non-Hermitian A , similar circuit implements singular value transformation (QSVT).

²(Low, Chuang, PRL 2017, QIP'17), (Gilyén, Su, Low, Wiebe, STOC 2019, QIP'19)

³(Motlagh, Wiebe, PRX Quantum 2024)

QSP Phase Factor Finding Problem

- Qubitization technique: only need to find phase factors for 2×2 matrix!
- Single qubit Pauli matrices

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- For $x = \cos \theta \in [-1, 1]$, rotation matrix

$$W(x) = e^{i\theta\sigma_X} = \begin{pmatrix} x & i\sqrt{1-x^2} \\ i\sqrt{1-x^2} & x \end{pmatrix}$$

Problem (QSP phase factor finding)

Given an even (or odd) real target polynomial $f(x)$ with $\|f\|_{L^\infty[-1,1]} \leq 1 - \eta$, how to find a sequence $\Phi = (\varphi_0, \dots, \varphi_n)$ such that $f(x) = \text{Im}[P(x)]$, where $P(x)$ is defined by

$$\begin{pmatrix} P(x) & * \\ * & * \end{pmatrix} = e^{i\varphi_0\sigma_Z} W(x) e^{i\varphi_1\sigma_Z} W(x) \dots e^{i\varphi_{n-1}\sigma_Z} W(x) e^{i\varphi_n\sigma_Z}?$$

GQSP Phase Factor Finding Problem

- Two-parameter single-qubit rotations

$$R(\psi, \phi) := \begin{pmatrix} \cos \psi & e^{i\phi} \sin \psi \\ -e^{-i\phi} \sin \psi & \cos \psi \end{pmatrix} \otimes I$$

- Lift the parity constraint on target polynomial

Problem (GQSP phase factor finding)

Given a target polynomial $b(\cdot)$ of degree n such that $\|b\|_{L^\infty(\mathbb{T})} \leq 1 - \eta$, how to find a sequence of angles

$$\{(\psi_k, \phi_k)\}_{k=0}^n$$

such that

$$\begin{pmatrix} * & b(U) \\ * & * \end{pmatrix} = R(\psi_0, \phi_0) \begin{pmatrix} U & \\ & I \end{pmatrix} \cdots \begin{pmatrix} U & \\ & I \end{pmatrix} R(\psi_n, \phi_n)?$$

In the early stages of QSP development..

Toward the first quantum simulation with quantum speedup

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Section H.3:

...However, this computation is difficult in practice, so we can only carry it out for very small instances. Specifically, we found the time required to calculate the angles to be **prohibitive for values of M greater than about 32**...It is a natural open problem to give a more practical method for computing the angles

Key problem 1: Efficiency

- Iterative methods: $\tilde{\mathcal{O}}(n) \sim \tilde{\mathcal{O}}(n^2)$. Fails in fully-coherent regime (small η).
- Direct methods:

Algorithm	Time complexity	Space complexity
Riemann-Hilbert ¹	$\tilde{\mathcal{O}}(n^4 + n\eta^{-1} \log^2(\epsilon^{-1}))$	$\mathcal{O}(n^2)$
Half-Cholesky ²	$\tilde{\mathcal{O}}(n^2 + n\eta^{-1} \log^2(\epsilon^{-1}))$	$\mathcal{O}(n)$
Layer stripping ^{3,4}	$\tilde{\mathcal{O}}(n^2 + n\eta^{-1} \log^2(\epsilon^{-1}))$	$\mathcal{O}(n)$
Inverse nonlinear FFT	$\tilde{\mathcal{O}}(n + n\eta^{-1} \log^2(\epsilon^{-1}))$	$\mathcal{O}(n)$

- Easily compute for $n \sim 10^6$

¹(Alexis, Lin, Mnatsakanyan, Thiele, Wang, CPAM 2026, QIP'25)

²(Ni, Ying, arXiv:2410.06409)

³(Y.-J. Tsai, Thesis, 2005)

⁴(Gilyén, Su, Low, Wiebe, STOC 2019, QIP'19)

Key problem 2: Numerical Stability

What is numerical stability?

- **Intuition:** Small errors should *not* grow uncontrollably as the algorithm proceeds.
- **Practical perspective:** Truncation errors negligible on common devices
e.g. 52 bits for IEEE double precision
- **Theoretical perspective:** Bit requirement grows only *poly-logarithmically* in problem parameters.
- **In our setting:** Bit requirement scales as

$$r = \text{polylog}(n, \eta^{-1}, \epsilon^{-1}),$$

Key problem 2: Numerical Stability

Iterative methods:

- Stable when the algorithm converges.

Direct methods:

- Step 1: Complementary polynomial finding:
 - Root finding approach¹: Unstable
 - Weiss algorithm²: Stable
- Step2: Layer stripping: ?

Problem

- *Is it possible to prove the layer stripping part of the direct methods is stable?*
Yes! (Under certain conditions)
- *Is it possible to develop a more efficient layer stripping type method that preserves stability?*
Yes!

¹(Gilyén, Su, Low, Wiebe, STOC 2019, QIP'19), (Haah, Quantum 2019)

²(Alexis, Lin, Mnatsakanyan, Thiele, Wang, CPAM 2026, QIP'25)

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Nonlinear Fourier Transform (NLFT)

- Given a compactly supported sequence $\gamma = (\gamma_k)_{0 \leq k \leq n}$ of complex numbers, we define its Nonlinear Fourier Transform¹ (NLFT) as

$$\widehat{\gamma} := \begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \prod_{k=0}^n \left[\frac{1}{\sqrt{1 + |\gamma_k|^2}} \begin{pmatrix} 1 & \gamma_k z^k \\ -\overline{\gamma_k} z^{-k} & 1 \end{pmatrix} \right]$$

Here, $a^*(z) := \overline{a(z^{-1})}$.

- The image of NLFT is

$$\mathcal{S} = \{(a, b) : aa^* + bb^* = 1, 0 < a(\infty) < \infty\}.$$

Problem (Inverse NLFT)

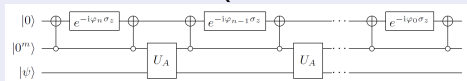
Given a Laurent polynomial $b(z)$ satisfying $\|b\|_{L^\infty(\mathbb{T})} \leq 1$, determine a compactly supported sequence γ such that

$$\widehat{\gamma} = \begin{pmatrix} * & b(z) \\ * & * \end{pmatrix}.$$

¹(Tao, Thiele, arXiv:1201.5129), (Y.-J. Tsai, Thesis, 2005), (Alexis, Mnatsakanyan, Thiele, 2024)

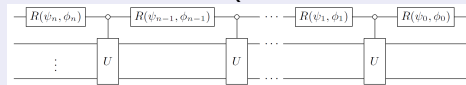
Converting (G)QSP to NLFT¹

QSP



$$e^{i\varphi_0\sigma_z} W(x) e^{i\varphi_1\sigma_z} W(x) \dots W(x) e^{i\varphi_n\sigma_z}.$$

GQSP



$$R(\psi_0, \phi_0) \cdot [U] \cdot \dots \cdot [U] \cdot R(\psi_n, \phi_n)$$

Nonlinear Fourier Transform

$$\begin{pmatrix} * & b(z) \\ * & * \end{pmatrix} = \prod_{k=0}^n \left[\frac{1}{\sqrt{1 + |\gamma_k|^2}} \begin{pmatrix} 1 & \gamma_k z^k \\ -\overline{\gamma_k} z^{-k} & 1 \end{pmatrix} \right]$$

¹See details in Section 3 of our paper. See also (Laneve, arXiv:2503.03026)

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The Layer Stripping Algorithm

QSP^{1,2}

- Given $\text{Im}[P(x)]$
- Find complementary polynomials $\text{Re}[P(x)]$ and $Q(x)$ such that

$$\begin{bmatrix} P(x) & iQ(x)\sqrt{1-x^2} \\ iQ^*(x)\sqrt{1-x^2} & P^*(x) \end{bmatrix}$$

is unitary for $x \in [-1, 1]$.

- The first phase factor ϕ_0 can be obtained from the leading coefficient of P and Q
- Undo the first rotation and retrieve the following ϕ_k 's sequentially using the same method

NLFT^{3,4}

- Given $b(z)$
- Find complementary Laurent polynomial $a(z)$ such that

$$\begin{bmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{bmatrix}$$

is unitary for $z \in \mathbb{T}$.

- The first coefficient γ_0 can be obtained from the leading coefficient of a and b
- Undo the first rotation and retrieve the following γ_k 's sequentially using the same method

¹(Gilyén, Su, Low, Wiebe, STOC 2019, QIP'19)

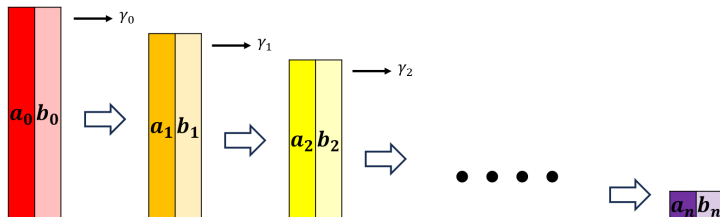
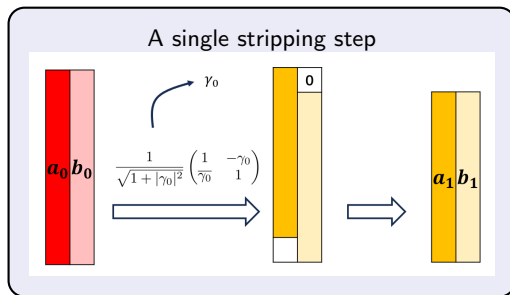
²(Haah, Quantum 2019)

³(Tao, Thiele, arXiv:1201.5129)

⁴(Alexis, Mnatsakanyan, Thiele, RMC 2024)

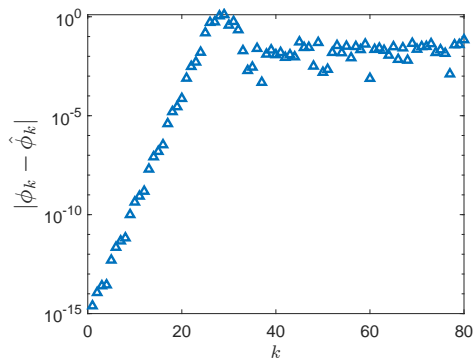
Layer Stripping (Matrix Form)

- $\mathbf{a}_0, \mathbf{b}_0$ are the coefficient vectors of $a(z), b(z)$



Instability of layer stripping

- The numerical error may accumulate exponentially during this sequential algorithm.
- Layer stripping works under $\tilde{\mathcal{O}}(n)$ bits precision floating point arithmetic.¹
- Instability justified by the following randomly generated example:



Error accumulation of the phase factors for an 80-degree polynomial pair $(a(z), b(z))$

¹(Haah, Quantum 2019)

Stable Layer Stripping Algorithm

- Matrix completion problem: Given $b(z)$, find $a(z)$ to complete $\begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix}$
- Combinatorially many choices¹ of $a(z)$
- Key insight²: let $a^*(z) := \overline{a(\overline{z^{-1}})}$ be an **outer** function: roughly, a^* has no zeros in the closed unit disk
- This outer a^* can be constructed efficiently and stably using the Weiss algorithm

Theorem

When **$a^*(z)$ is outer**, the layer stripping algorithm for the inverse NLFT problem is numerically stable.

¹(Gilyén, Su, Low, Wiebe, STOC 2019, QIP'19), (Haah, Quantum 2019), (Wang, Dong, Lin, Quantum 2022)

²(Alexis, Lin, Mnatsakanyan, Thiele, Wang, CPAM 2026, QIP'25)

- Use the displacement structured matrix theory¹
- Consider the unique matrix K satisfying

$$K - ZKZ^\dagger = \mathbf{a}_0\mathbf{a}_0^\dagger + \mathbf{b}_0\mathbf{b}_0^\dagger, \quad Z = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

- K has the triangular factorization

$$K = UU^\dagger, \quad U = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|} \hline \text{red} & \text{orange} & \text{yellow} & \text{light green} & \text{green} & \text{blue} & \text{purple} \\ \hline \end{array} \\ \begin{array}{l} a_0 \\ a_1 \\ a_2 \\ \\ \\ \\ a_n \end{array} \end{array}$$

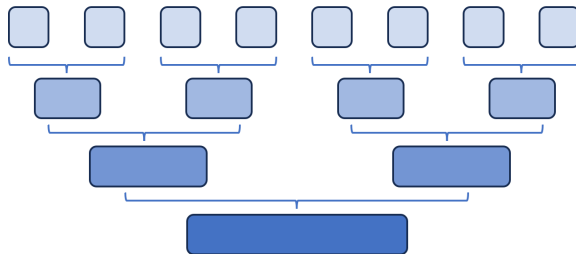
- Doing layer stripping \approx Factorizing matrix K (using Schur's algorithm)
- K is well-conditioned when $a^*(z)$ is outer.

¹(Kailath, Sayed, SIAM Review 1995)

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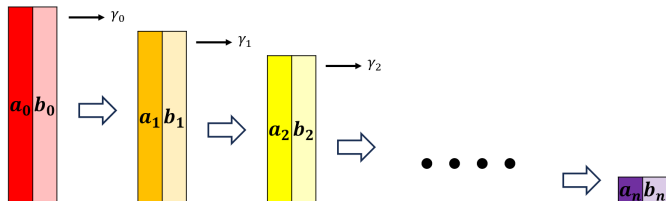
$$\begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \prod_{k=0}^n \left[\frac{1}{\sqrt{1 + |\gamma_k|^2}} \begin{pmatrix} 1 & \gamma_k z^k \\ -\overline{\gamma_k} z^{-k} & 1 \end{pmatrix} \right]$$

- Q: How to calculate the polynomial matrix product more efficiently?
- A: Divide and conquer + FFT!

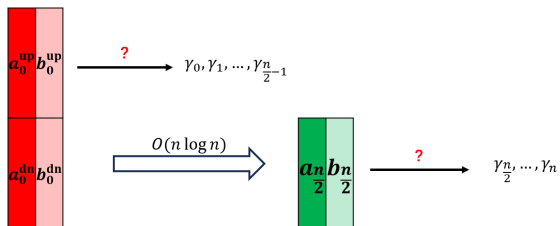


INLFFT

- Layer stripping: $\mathcal{O}(n^2)$



- INLFFT: $\mathcal{O}(n \log^2 n)$



$$\begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \prod_{k=0}^n \left[\frac{1}{\sqrt{1+|\gamma_k|^2}} \begin{pmatrix} 1 & \gamma_k z^k \\ -\gamma_k^* z^{-k} & 1 \end{pmatrix} \right]$$

Split the product and take inverse ($m = \lceil \frac{n}{2} \rceil$):

$$\prod_{k=m-1}^0 \left[\frac{1}{\sqrt{1+|\gamma_k|^2}} \begin{pmatrix} 1 & -\gamma_k z^k \\ \gamma_k^* z^{-k} & 1 \end{pmatrix} \right] \begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \begin{pmatrix} a_m(z) & z^m b_m(z) \\ -z^{-m} b_m^*(z) & a_m^*(z) \end{pmatrix}$$

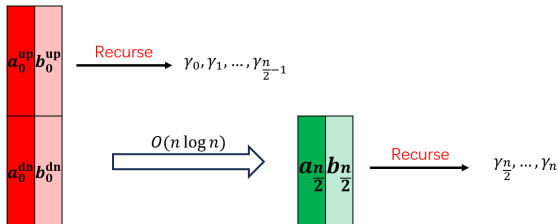
Algorithm Sketch

- ① Calculate the first half of the NLFT coefficients $\gamma_0, \dots, \gamma_{m-1}$.
- ② Use the divide-and-conquer type fast algorithm to calculate the product $\prod_{k=m-1}^0 \left[\frac{1}{\sqrt{1+|\gamma_k|^2}} \begin{pmatrix} 1 & -\gamma_k z^k \\ \gamma_k^* z^{-k} & 1 \end{pmatrix} \right]$.
- ③ Construct (a_m^*, b_m) by the above formula.
- ④ Retrieve the last half of the NLFT coefficients $\gamma_m, \dots, \gamma_n$ from (a_m^*, b_m) .

Algorithm Sketch

- ① Calculate the first half of the NLFT coefficients $\gamma_0, \dots, \gamma_{m-1}$.
- ② Use the divide-and-conquer type fast algorithm to calculate the product $\prod_{k=m-1}^0 \left[\frac{1}{\sqrt{1+|\gamma_k|^2}} \begin{pmatrix} 1 & -\gamma_k z^k \\ \overline{\gamma_k} z^{-k} & 1 \end{pmatrix} \right]$.
- ③ Construct (a_m^*, b_m) by the above formula.
- ④ Retrieve the last half of the NLFT coefficients $\gamma_m, \dots, \gamma_n$ from (a_m^*, b_m) .

- Key insight: $\gamma_0, \dots, \gamma_{m-1}$ only depends on the first half of \mathbf{a}_0 and \mathbf{b}_0
- Step 1 and 4 can be done recursively



Theorem

When $a^*(z)$ is *outer*, the INLFFT algorithm is numerically stable.

See Section 5.5 in our paper for proof.

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- **Algorithmic Contribution:**

- Developed INLFFT: A fast, divide-and-conquer implementation of the Inverse NLFT.
- Achieved near optimal complexity of $\tilde{O}(n)$, improving upon the previous $\tilde{O}(n^2)$ barrier.

- **Theoretical Contribution:**

- We resolved the long-standing open question regarding the numerical stability of layer stripping.
- We further established the stability result of INLFFT.

- **Closing a Chapter in (G)QSP Phase Factoring Finding**

- We unify QSP and GQSP within the NLFT framework
- This leads to phase factor finding algorithms that are both efficient and stable.

- **Further Directions:**

- Multi-variable QSP?
- NLFT in higher dimensions?

Thank you for your attention!