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ELLIPTIC PDE: FROM LAPLACE TO DE GIORGI.

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References

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1. Laplace and Heat EQs

1.1. **Problem 1. Inner regularity of** Δ. For convenience, setting $u^\epsilon = \chi_\epsilon * u$ and we will observe that the integral $\int_U u^\epsilon \Delta(\phi) \leq \sup_U \Delta u||u||_{L^1}$ is bounded as Δu will vanish outside of $supp(u)$ and that $supp(u) \subset\subset U$, so $supp(\Delta u)$ is bounded, and closed by definition, therefore compact, by which follows that $\sup_U \Delta u < \infty$.
Let us rephrase the integrals by integral by parts. In fact,

$$\int_U u^\epsilon \Delta(\phi) \mathrm{d}\mu(x) = [u^\epsilon \phi]|_{\partial U} - \int_U \phi \int_U u(t)\Delta_x \chi_\epsilon(x-t)\mathrm{d}\mu(t)\mathrm{d}\mu(x)$$

while noticing $[u^\epsilon \phi]|_{\partial U} = 0$ as ϕ has compact support inside U, and that by the first observation we see that an application of Fubini's theorem implies

$$-\int_U \int_U \phi(x)u(t)\Delta_x \chi_\epsilon(x-t)\mathrm{d}\mu(x)\mathrm{d}\mu(t) = -\int_U u(t)\int_U \phi(x)\Delta_x \chi_\epsilon(x-t)\mathrm{d}\mu(x)\mathrm{d}\mu(t) =$$

$$= -\int_U u(t)\int_U \phi(x)\Delta_x \chi_\epsilon(x-t)\mathrm{d}\mu(x)\mathrm{d}\mu(t) = \int_U u(t)\Delta_t \int_U \phi(x)\chi_\epsilon(x-t)\mathrm{d}\mu(x)\mathrm{d}\mu(t)$$

$$= \int_U u(t)\Delta(\phi * \chi_\epsilon)\mathrm{d}\mu(t)$$

If, say $u \in C(U)$, then a well-known fact is that enjoys the uniform convergence under convolution on a compact set, one can see, for example [1] or [2] : $u_\epsilon := u * \chi_\epsilon(x) \longrightarrow u(x)$ for $\epsilon \longrightarrow 0$. Apply a different convolution by χ_t to u_ϵ, and one sees that $u_{\epsilon t} \longrightarrow u_t$ as $\epsilon \longrightarrow 0$. We will use this observation to obtain harmonicity. Now consider u_ϵ, it is smooth and by above inequality and u is weakly harmonic, one can in fact obtain $\int_U u^\epsilon \Delta(\phi)\mathrm{d}\mu(x) = \int_U u(t)\Delta(\phi * \chi_\epsilon)\mathrm{d}\mu(t) = 0$ as $\phi * \chi_\epsilon$ satisfies the conditions prescribed above pretty obviously. An integration by parts gives $\int_U \phi\Delta(u^\epsilon)\mathrm{d}\mu(x)$, and as $\Delta(u^\epsilon)$ is the smooth function with support in U, we may just put $\phi = \Delta(u^\epsilon)$, and so above becomes $\int_U \Delta(u^\epsilon)^2\mathrm{d}\mu(x) = 0$, since u^ϵ is smooth, one has $u^\epsilon = 0$, i.e., harmonic.
Harmonictiy is a local property, we need no further inspection other than look at a fixed $x \in U$ locally. We can then choose s so that $x \in U_s := \{d(x, \partial U) > s\}$. Setting $t > 0$, and we again set u_{st}, u^s is harmonic so enjoys Mean Value Property, and by lecture, or

[2], $u_{st}(z) = u_s(z)$. By commutativity of $*$, $u_{st} = u_{ts}$, and then as $s \to 0$, $u_s(x) \to u(x)$ by previous discussions, and $u_{st} = u_{ts}$ and $u_{st}(x) \to u_t(x)$, therefore $u(x) = u_t(x)$, this says u is harmonic in neighborhood of x in U_t, and finishes the proof.

1.2. **Problem 2. Schwartz Principal**. A. Check if $v \in C^2(B_1(0))$. To do this we only have to check the points on $\{x_n = 0\} \cap B_1(0)$ if v is of C^2. We only need to approach from the upper sphere and lower sphere, i.e., $x \to 0^+$ from U^+, and $x \to 0^-$ from U^-. Then $\frac{\partial v}{\partial z_j} = -\frac{\partial u(x_1, x_2, \ldots, x_{n-1}, -x_n)}{\partial x_j} = -\frac{\partial u}{x_j}(x_1, \ldots, x_{n-1}, -x_n)$, and similar take x_i direction partial derivative, then $\frac{\partial^2 v}{\partial z_i z_j} = \frac{\partial^2 u}{x_j}(x_1, \ldots, x_{n-1}, -x_n)$, they are of C^1 as when v_i approaching from points in U^+ to $\{x_n = 0\} \cap B_1(0)$, we have a sign problem, namely from positive direction $u_i(x_1, x_2, \ldots, x_n)$, and from negative we have $u_i(x_1, x_2, \ldots, -x_n)$, but u being zero in $\{x_n = 0\} \cap B_1(0)$ we then solve the problem by the $x^+, x^- \to 0$. Then C^2 proceeds similarly.
B. Again, for the similar consideration, we know that harmonicity of v is local, therefore, we can have faith only looking at $\{x_n = 0\} \cap B_1(0)$. Let us note here, $v \in C^2$(in fact by Problem 1, we only need v to be continuous, for which case is obviously true), and we only need to check the mean value property. For $r > 0$, $B_r(x)$ for $x \in \{x_n = 0\} \cap B_1(0)$, and we want to show $0 = v(x) = \int_{\partial B_r(x)} v(y) \mathrm{d}\mu(y)$. This is quite a consequence of definition:

$$\int_{\partial B_r(x)} v \mathrm{d}\mu(y) = \int_{\partial B_r^+(x)} u(x_1, x_2, \ldots, x_n) \mathrm{d}\mu(y) + \int_{\partial B_r^-(x)} -u(x_1, x_2, \ldots, -x_n) \mathrm{d}\mu(y)$$

$$= \int_{\partial B_r^+(x)} u(x_1, x_2, \ldots, x_n) \mathrm{d}\mu(y) - \int_{\partial B_r^+(x)} u(x_1, x_2, \ldots, x_n) \mathrm{d}\mu(y) = 0$$

Therefore harmonicity follows.

1.3. **Problem 3. Some Uniqueness**. A. on the bounded solution. For the existence, the lecture gives the proof by means of Green's function $u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} \mathrm{d}y$.
The uniqueness asks if there is a harmonic function u, such that $u = 0$ on $\partial \mathbb{R}^n_+$. By means of above reflection principal, one can in fact reflect the solution u so that u is a harmonic function in $\mathbb{R}^n$. Then as u is bounded everywhere(actually the continuity implies we can put $u \in L^\infty$), an application of Louiville's theorem show that u reduces to a constant, while $u = 0$ at boudanry, this says $u = 0$. Therefore should the original question has 2 solutions w_1, w_2, $w_1 - w_2 = 0$ by above, i.e., the solution is unique.
B. Essentially with similar analysis as before, we ask for a nonzero solution with 0 in the boundary while harmonic on unbounded domain. In $\mathbb{R}^3$, we can find $u = 1 - \frac{1}{\sqrt{(x^2+y^2+z^2)}}$ for domain $\Omega := \mathbb{R}^3 - B_1(0)$. We see $u = 0$ on the boudanry sphere, and it is bounded by 1, it is easy to see that it is harmonic as it is away from singularity 0, and $\Delta u = \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}} = 0$. It is clearly nonzero, so the solution is not unique.

1.4. **Problem 4. Kernel estimates** K. In fact if we observe that $K(x,y) = \int_{\partial\mathbb{R}^n_+} \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n} = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{1}{|x-y|^n}$. After a change of variable, for example, by a translation, and moreover, in the second equality, putting $y \to \frac{y}{x_n}$ to cancel out the factor

$$K(x,y) = \frac{2x_n}{n\alpha(n)}\frac{1}{x_n^{n-1}}\int_{\partial\mathbb{R}^n_+}\frac{1}{|1+|\frac{y}{x_n}|^2||^{n/2}} = \frac{2}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{1}{|1-|y|^2|^{\frac{n}{2}}}$$

Now applying co-area formula we in fact are evaluating

$$\int_{\partial\mathbb{R}^n_+}\frac{1}{|1-|y|^2|^{\frac{n}{2}}} = \int_0^\infty\int_{\partial B_r(0)}\frac{1}{(1+r^2)^{n/2}}\mathrm{d}\mu(S)\mathrm{d}r = \frac{2(n-1)\alpha(n-1)}{n\alpha(n)}\int_0^\infty\frac{r^{n-2}}{(1+r^2)^{n/2}}\mathrm{d}r$$

We need to take care of last term, for which we will do iterations for this integral, named as I_n, for $n=2$, it is just $\pi/2$ as it is well known calculus result, and for $n=3$, $I_3=1$. And for general n, one proceeds to

$$\int_0^\infty\frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}}dr = \big(-\frac{1}{n-2}\frac{1}{(1+r^2)^{\frac{n-2}{2}}}r^{n-3}\big)_0^\infty + \frac{n-3}{n-2}\int_0^\infty\frac{r^{n-4}}{(1+r^2)^{\frac{n-2}{2}}}dr$$

or one has $I_n = \frac{n-3}{n-2}I_{n-2}$ as we have the data for $n=2,3$, we can do reccurence to have formula: $I_n = \frac{(n-3)!!}{(n-2)!!}\frac{\pi}{2}$ for n even and $I_n = \frac{(n-3)!!}{(n-2)!!}$ for n odd. For the coeffecients except I_n we have $2\alpha(n-1)/\alpha(n) = \frac{2\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\pi}}$ by $\alpha(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Since it is a wellknown combinatorics facts(can be done by generating function), $\Gamma(\frac{n}{2}) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}}$ for n odd and $\Gamma(n) = n!$ for n integer, we can then check $I_n = \frac{1}{2\alpha(n-1)/\alpha(n)}$ by n's parity, which is true. This concludes the problem.

1.5. **Problem 5**. Denote: $H := \frac{\partial}{\partial t} - \Delta$.
Consider $f(x,t) = u^2 - u$, and we can solve the nonhomogeneous problem $Hu = f$ on tube domain $B_R^T := \{(z,t)||z| < R, t \in \mathbb{R}, 0 \le t \le T\}$ with boundary condition $u(z,t)$. By the uniqueness of solution, we know that if we solve for this problem, the solution $\hat{u}$ in fact just $u|_{B_R^T}$.

Examining on the boundary data, on the parabolic boundary $\partial_0 B_R^T := (\partial B_R \times T) \cup (B_R \times \{t=0\})$, the data $u(x,0) \in (0,1)$, can be now be used "fiber-wise"(one would, certainly imagine $\pi : B_R^T \to B_R$ as a smooth fiber bundle projection.).

Fixing x, along the "fiber" line $l : (x,t)$ we have variation on x to be zero, to put into math , we have $\Delta u = 0$, and the PDE is just locally $\frac{\partial}{\partial t}u = u^2 - u$, which is an ODE, we can easily solve this, to be $u = \frac{1}{1-e^{-t}-C}$ for some constants C, by $t=0$, we have $u(x,0) = \frac{1}{-C} \in (0,1)$, so $1 - C - e^{-t} > 1$, i.e., $\frac{1}{1-C-e^{-t}} = u(x,t) < 1$.

Apply this observation to each points $z \in \partial B_R$, we have $u(x,t)$ on $\partial_0 B_R^T$ to be within $(0,1)$. Apply the maximum principal, we have

$$\max_{\partial_0 B_R^T} u = \max_{\bar{B}_R^T} u$$

that is, we have that in $\bar{B}_R^T$, $u \in (0, 1)$. As $\hat{u}$ to be the restriction of u on $\bar{B}_R^T$, we have $u \in (0, 1)$, we realize this applies to arbitrary R, T, also for every (x, t) we can find them in some B_R^T, therefore for arbitrary (x, t), the value $u(x, t) \in (0, 1)$.

2. Some Topics About Heat Equations

2.1. **Problem 1**. One simply imitate the method used in regularity of Heat equation.
A. To set up, we can first work on some fixed points, say $(0,T)$. Define a closed circular cylinder $C(x,t;r) : \{(y,s) : |x-y| < r, t-r^2 < s < t\}$. We first use r to denote a fixed length of radius. Define also $C' := C(x,t;3/4r), C'' := C(x,t;1/2r)$. We may construct a mollifier ζ such that vanishes outside the parabolic boundary of C, and 1 on C', so we have extended u the smooth solution to $v = u\zeta$. Therefore $v = 0$ on the bottom boundary, $\mathbb{R}^n \times \{0\}$. And $(\frac{\partial}{\partial t} + \Delta)v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta =: G$. Setting $w = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)G\mathrm{d}y\mathrm{d}s$. Therefore w satisfies $(\frac{\partial}{\partial t} + \Delta)w = 0$ on $\mathbb{R}^n \times (0,t_0)$ and vanishes on $\mathbb{R}^n \times \{t = 0\}$. That u is bounded means we have the uniqueness of unbounded domain solution of heat equation, that is, $v = w$. So we can exhibit $v = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)G\mathrm{d}y\mathrm{d}s$. Now we take a look at points in C'', as ζ vanishes outside C, the former integral is just $v = \int_C \Phi(x-y,t-s)G(y,s)\mathrm{d}y\mathrm{d}s$. As we have noted, G vanishes in some neighborhood of singularity of Φ, in other words, it is legal to integrate by parts:

$$v = \int_C u(y,s)(\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) + 2D_y\Phi(x-y,t-s)D\zeta(y,s))\mathrm{d}y\mathrm{d}s$$

and we may just take S to mean the expression $\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) + 2D_y\Phi(x-y,t-s)D\zeta(y,s)$.
First we may assume $r = 1$, for which case $|\nabla u| \leq \int_{C(x,t;1)} |\nabla S||u|\mathrm{d}\mu \leq KA$, as S is bounded by K on compact sets, and $|u| \leq A$.
For general r, we can put $w = u(rx, r^2t)$, it is clear that it satisfies the heat equation. By above, we have $|\nabla v| \leq KA$.
The homogeneity of heat equation just says $\nabla w = r\nabla u(rx, r^2t)$ and $|\nabla u| \leq \frac{KA}{r}$ in domains $C'' := C(x,t;r/2)$.
B. Global observation. In fact, we observe that to give a supremum of all $x \in U_\epsilon \times [\epsilon \times \infty)$ by estimate $\frac{C}{\epsilon}$ is shockingly same as for supremum of $x+y$(a translation by y), this is due to that whenever (x,t) is, we are repeating the above process and it is the same parabolic domain, provided ϵ is fixed. Therefore, the same parabolic domain, combine with u uniformaly bounded by A, then integral involve K stay under the same bound. So take $r = \epsilon$, we have $\sup_{U_\epsilon \times [\epsilon^2,\infty]} u \leq \frac{C}{\epsilon}$. The reason with the domain $U_\epsilon \times [\epsilon^2, \infty)$ is that we need the part we truncated away(i.e., the part for which $C - C''$ as in previous notation) to get away from the domain outside U_T.

Example 2.1. One can use a similar path as Example 2.2, where one uses the function $u = \sum_{n=0}^\infty \frac{\chi^{(n)}(t)}{(2n)!} x_2^{2n}$ with $\chi(t) = \begin{cases} e^{-\frac{1}{t^2}}, & t > 0 \\ 0, & t = 0 \end{cases}$. For the current form of heat equation: $\Delta u = \frac{\partial u}{\partial t}$. Therefore it is rather clear that $\Delta u = \sum_{n=0}^\infty \frac{\chi^{(n)}(t)}{(2n-1)!} x_2^{2n-1}$, after suitable replacement of χ, and the use of comparison of series expansion $\sum_{n=0}^\infty \frac{(-1)^k}{(2n-1)!} x_2^{2n-1} = \sin(x)$ and $\cos(x)$, we can find the case $|\Delta u| = \frac{C}{\epsilon}$, and it cannot be better since if so, then this will contradicts this case.

2.2. **Problem 2**. We can re-write $\frac{EE''-E'^2}{E^2} \geq 0$, or $(\frac{E'}{E})' \geq 0$ or $\log''(E) \geq 0$. Take $f = \log E$, then it is obvious that f is convex, which is by a well known criterion: f of class C^2 is convex iff $f'' \geq 0$.
By convexity definition, i.e., Jensen inequality, $\log(E) \leq \frac{T-t}{T}\log(E(0)) + \frac{t}{T}\log(E(T))$ Alternatively one can view this to be the fact chord $\frac{T-t}{T}\log E(0) + \frac{t}{T}\log E(1)$lies above $\log E$. Take exponential at the same side we have the desired form.

2.3. **Problem 3**. We would follow the idea in Problem. Let Σ be the annulus $1/2 < |z| < 1$, and that $\phi(x) = w(x_0, T) + \epsilon(|x|^{2-n} - 1)$, we wish to check $\nabla\phi \cdot x$ is strictly positive on $|z| = 1$, for which we compute $\nabla\phi \cdot x = (n-2)(\sum_{k=1}^n x_k^2)^{n/2-1}$, and this is clearly positive whence $n > 2$.

Also let us notice $w - \phi = w - w(x_0, T) - \epsilon(|x|^{2-n} - 1)$, it is 0 whenever it hits (x_0, T) We wish to show it is the maximal. So except for (x_0, T) we wish that

$$\frac{w - w(x_0, T)}{|x|^{2-n} - 1} \leq \epsilon$$

As $|x|^{2-n} - 1 \geq 0$, and $w - w(x_0, T) \leq 0$, we have that take any positive number ϵ will do.

As it is maximum, the directional derivatives at (x_0, T) pointing outwards, therefore becomes

$$\nabla(w - \phi) \cdot x = \frac{\partial w - \phi}{\partial \mathrm{n}} - \lim_{h \to 0} \frac{(w - \phi)(x_0 + \mathrm{n}h, T) - (w - \phi)(x_0, T)}{|h|} \geq 0$$

which follows naturally $\nabla(w) \cdot x_0 \geq \nabla(\phi) \cdot x_0 > 0$.

2.4. **Problem 4**.

2.4.1. *U is unbounded.* We probably need this domain to be bounded in this case, as one can see in following counterexample.

Example 2.2. Let $U := \{x \in \mathbb{R}^3 : x_1 > 0\}$, i.e., an upper half plane, and we then identity $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, with $(x_1, x_2) \times t$. We check the function $u = \sum_{n=0}^{\infty} \frac{\chi^{(n)}(t)}{(2n)!} x_2^{2n}$ with $\chi(t) = \begin{cases} e^{-\frac{1}{t^2}}, & t > 0 \\ 0, & t = 0 \end{cases}$. For the current form of heat equation $\Delta u = \frac{\partial u}{\partial t}$, it clearly satisfies the equation, and therefore a solution. Also, notice that the normal derivative on boundary $x_1 = 0$ is simply $\Delta u \cdot (-1, 0) = (u_{x_1}, u_{x_2}) \cdot (-1, 0) = -\frac{\partial u}{\partial x_1}$, however our u when fixed x_2, t is simply constant: there is no x_1 term, so the normal derivative vanishes as well. On the boundary $\bar{U} \times \{t = 0\}$, it is by definition we have $u = 0$ as g vanishes identically on $\bar{U} \times \{t = 0\}$. Therefore we conclude that u is a solution for no-flux boundary problem. Also $u = 0$ is also an no-flux solution, this combines with above provides a counterexample.

2.4.2. *U is bounded.* For this case, for any $x \in U$, we then notice that as the boundary is of bounded sectional curvature, we in fact have an inner ball enclosing x and the sphere will be tangent to the boundary. If $u(x) = 0$, then we are done. Now suppose it is not 0. Let us assume that the intersection of the sphere and ∂U to be $x_0 \times (0, T]$, and notice that they share the same outer-pointing normal. Then by an application of maximal principal of bounded domain, we deduce that u on $B_r(x) \times (0, T]$ must attain its maximum value on either the bottom boundary: $\bar{U} \times \{t = 0\}$ or on the tube-shaped boundary: $\partial U \times (0, T]$. We want to show that given the no-flux boundary, it is only possible to attain the maximum in the bottom boundary, i.e., $u \le 0$. Since if so, then clearly $-u$ satisfies the no-flux condition as well and at x is nonzero, apply our claim, we have $-u \le 0$ as well, so $u = 0$ identically on $B_r(x) \times (0, T]$.
We prove our claim by contradiction. If not, so we have a maximum on $\partial U \times (0, T]$, we know that the location of maximum must occur either on $x_0 \times (0, T]$ or inside U_T, however, the later cannot be possible by the maximum principle applied on U_T, as heat equation solution will not attain maximum inside U_T, so it has to be some (x_0, T_0), it is a strict maximum since if not, then it has to be reduce to be a constant, and $u = 0$ in the bottom boundary says u must be reduce to 0 as well. So apply Hopf lemma, we conclude that $\frac{\partial u}{\partial \mathrm{n}} > 0$, however our boundary condition requires the normal derivatives to vanish, it is a contradiction.

3. Homework 3.

3.1. **Problem 1.** The basic idea is to build a prototype function u that has a singularity that has a discontinuity x_0 so essential that, the difference in a small neighborhood of x_0 cannot be properly controlled and we cannot remedy this discontinuity by modifying on a small set of measure zero, and the same time derivatives are reasonable. The first functions of this sort seems to be $\log$. For bounded functions, we can in fact compose the former with e^{it}, i.e., $\sin x$, $\cos x$ to make use of the oscillatory.

For simplicity, we consider $U = B_1(0)$ sitting in $\mathbb{R}^2$. A first example is yet a unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$. This function is in the space $W^{1,2}(U)$. This function clearly blows up at $(0,0)$, however if we compute $\int_U |u|^2 = \alpha(2)\int_0^1 \left|\log\log(1+\frac{1}{r})\right|^2 r\mathrm{d}r$, and setting $\sigma = \frac{1}{r}$ we have

$$\alpha(2)\int_1^\infty \frac{1}{\sigma^3}\left|\log\log(1+\sigma)\right|^2 \mathrm{d}r$$

. By an application of L'Hospital's rule, we can see $\frac{\left|\log\log(1+\sigma)\right|^2}{\sigma} \to 0$ as σ tends to 0. So for $t > T$, we have the former expression bounded by 1. Therefore we have $\int_U |u|^2 = \alpha(2)\int_1^T \frac{1}{\sigma^3}\left|\log\log(1+\sigma)\right|^2 \mathrm{d}r + \alpha(2)\int_T^\infty \frac{1}{\sigma^2}\mathrm{d}\sigma < \infty$. Now for derivatives. We easily see that the weak derivative is $u_i = \frac{-x_i}{\log\left(1+\frac{1}{|x|}\right)(1+\frac{1}{|x|})|x|^3}$, and we will proceed similar as above.

$$\int_U |u_i|^2 \mathrm{d}x \le \alpha(2)\int_0^1 \left|\frac{-x_i}{\log\left(1+\frac{1}{|x|}\right)(1+\frac{1}{|x|})|x|^3}\right|^2 r\mathrm{d}r = \alpha(2)\int_1^\infty \frac{\sigma}{(1+\sigma)^2|\log(1+\sigma)|^2}\mathrm{d}\sigma$$

A change of variable $\tau \to \log(1+\sigma)$:

$$\le \frac{\alpha(2)}{2}\int_{\log 2}^\infty \frac{1}{\tau^2}\frac{1}{e^{2\tau}}e^{2\tau}\mathrm{d}\tau < \infty$$

So we see that $u \in W^{1,2}(U)$.

Now choose a cutoff function χ, such that $v := \chi u$ is compactly supported in U, and agree with u in $B_{1/2}(0)$.

As the compositions of $W^{1,2}$ functions are always of class $W^{1,2}$, we see that $f := \sin(v)$ is in fact $W^{1,2}$. It is compactly supported, bounded, and we show now f is not a continuous function up to a measure zero set.

Suppose not, we have a continuous function g remedies the singularity at $(0,0)$ up to a null set. Then take $\delta > 0$, such that $|x| < \delta$, we have that $|g(x) - g(0)| < 1/8$. Since f, g differs by a null set, in every t, on $B_t(0)$ f, g agree almost every point. Since B_t is in fact compact(taken as closure), g is in fact uniformly continuous, so we have $|g(x) - g(y)| < 1/8$ for every $x, y \in B_t(0)$. Therefore we take x, y outside the null set f, g differs, such that $v(x) = 2N\pi$ while $v(y) = \frac{\pi}{2} + 2M\pi$ for some M, N, this is due to v blows up at $(0,0)$, one can have any value near $(0,0)$, then $|g(x) - g(y)| = 1 > 1/8$, a contradiction.

3.2. **Problem 2**. In class we have proven that $f \in W_0^{1,p}(U)$ for $f \in W^{1,p}(U)$ if and only if $Ef = 0$(E for extension operator, as theorem 2 in p.275 of Evans.). So to find a counterexample it is enough to talk about a continuous function in $\bar{U}$ such that its boundary value on $S^1 = \partial\mathbb{D}$ is simply nonzero. One can find, for example, constant function 1.

It is not possible to approximate constant function, such as 1, by compactly smooth function in H^1 sense, since $\mathbb{R}^n$ is already noncompact, so the content for which $|\psi_n - 1| = 1$ will not be finite, therefore the integral will not converge in H^1 sense. However, it is possible to give a different topology to make it converge, namely, to give a topology for convergence on every compact set, since $\mathbb{R}^n$ can be exausted by compact sets, this topological vector space is in fact Frechet space, but this is already very weak notion of convergence.

3.3. **Problem 3**. We will use a well known theorem in text book *An Introduction to PDE* by Evans, that is, provided U is open bounded and boundary of type C^1, then $u : U \to \mathbb{R}$ is Lipschitz if and only if $u \in W^{1,\infty}(U)$. One can in fact extend this theorem with regard to boundary a little bit, with same argument(since, usually, arguments about Lipschitz will ultimately use absolute continuity by Reidemeister.) to exploit the local convex refinements of open covering.

To avoid long argument, we will only shortly present the proof of extension theorem, so that the proof of $p = \infty$ is then completed. The main idea is to replace the smooth functions u_m to be Lipschitz functions, and so that we do not need to check it is of C^1, rather than to see if it is Lipschitz. We will use this fact several times later.

We start with half plane case. Set the hemisphere in upper half plane and lower half plane, B^+, B^- with the definition $B_1 \cap \mathbb{R}^{n,+}, \mathbb{R}^{n,-}$ the half planes with different signs of x_n, respectively. Assume we have u defined in B^+, then we reflect by defining $v := u(x_1, x_2, \dots, |x_n|)$ notice the former is the composition of two Lipschitz mapping, therefore itself is Lipschitz as well, and so v is of type $W^{1,\infty}$.

Now we check $||v||_{W^{1,\infty}(B)} \le K||u||_{W^{1,\infty}(B^+)}$. To see this, for multi α equals to 0, i.e., v itself, the inequality is clear. We need to specify the weak derivative of v. This is purely computation: for $\phi \in C_c^\infty(B)$

$$\int_B |x|\phi'(x)\mathrm{d}x = \int_{B^+} x\phi'(x) - \int_{B^-} x\phi'(x)\mathrm{d}x = -\int_{B^+} \phi(x)\mathrm{d}x + \int_{B^-} \phi(x)\mathrm{d}x = \int_B \mathrm{sgn}x\phi(x)\mathrm{d}x$$

and combined with composition of weak derivatives of order 1, we can derive $D^\alpha v = (-\mathrm{sgn}x)D^\alpha u$, and this clearly leads to the inequality we want, with K independent of u we chose.

Now for the case where ∂U is a C^1 manifold, and the boundary is compact, this leads to finite number of charts (U_i, ψ_i) of ∂U. Notice that all C^1 mapping on compact sets are Lipschitz, so all charts ψ_i are in fact Lipschitz. Now for a local chart U_i, we set $w := u \circ \psi$, therefore w is still Lipschitz, so we may proceed as above to get $||w||_{W^{1,\infty}(B)} \le K||w||_{W^{1,\infty}(B^+)}$. At last throw back to the original coordinate, and we are done with local picture. Denote the extended function of u in this local setting as $\bar{u}$. To achieve a global one, we get a finite refinements of charts(for example, in Evans) as well as $U_0 \subset\subset U$ and ζ_i the partition of unity for these open sets. Setting $g = \sum_{k=0}^N \zeta_k \bar{u}_i$, where $u_0 = u$. The estimate of u_i passes to g, and we have $||g||_{W^{1,\infty}(B)} \leq K||u||_{W^{1,\infty}(B^+)}$. Further we may construct support or g to lie within V, $U \subset\subset V$. Now we can take $Eu := g$, and the map is clearly linear.

3.4. **Problem 4**. The natural candidate for this is simply distance function $\rho(x) :=$ **dist**$(x, \partial U)$. But to proceed, we will set $g_n := \max(0, 1 - n\rho(x))$. Clearly g_n is just continuous on closure of U, and $0 < g_n < 1$. Moreover, pointwise g_n descends to 0, if $x \in U$ not on the boundary. Moreover, we will have pointwise dominating relation: $g_1 > g_n$ for $n > 1$, and g_1 is integrable(since U is bounded.), an application to Dominated Convergence Theorem we conclude $\int_U |g_n|^p \to 0$ while $n \to \infty$. This integral, while passed to p-norm, we see that $||g_n||_{L^p(U)} \to 0$. If T is continuous between the Banach spaces, then the operator norm $\sup \frac{||Tg_n||_{L^p}}{||g_n||_{L^p}}$ is finite.

However $Tg_n = g_n|_{\partial U} = 1$, and then $||Tg_n||_{L^p} = K > 0$, however, this means the operator norm of T is infinite.

3.5. **Problem 5**. Let $u \in H^2(U) \cap H_0^1(U)$. Then there is sequence $u_k \to u$ in H^2 sense, and $w_k \to u$ in H^1, for $w_k \in C_c^\infty(U)$. Then

$$\int_{\partial U} u_k \frac{\partial u_k}{\partial n} \mathrm{d}S = \int_{\partial U} \operatorname{div}(u_k \nabla u_k)\mathrm{d}x = \int_{\partial U} \left|\nabla u_k\right|^2 \mathrm{d}x + \int_{\partial U} u_k \Delta u_k \mathrm{d}x$$

And the validity of which is assured by derivatives are of L^2. Now the difference of $\int_{\partial U} \left|\nabla u_k\right|^2 \mathrm{d}x + \int_{\partial U} u_k \Delta u_k \mathrm{d}x$ and $\int_{\partial U} \left|\nabla u\right|^2 \mathrm{d}x + \int_{\partial U} u \Delta u \mathrm{d}x$ are just

$$\int_{\partial U} (\left|\nabla u_k\right| + \left|\nabla u\right|)(\left|\nabla u_k\right| - \left|\nabla u\right|)\mathrm{d}x + \int_{\partial U} (u - u_k)\Delta(u + u_k)\mathrm{d}x - \int_{\partial U} u\Delta u_k - u_k \Delta u \mathrm{d}x$$

By Cauchy Shwartz, the first term is bounded by $2|||\nabla_k| + |\nabla u|||_{L^2}|||\nabla_k| - |\nabla u|||_{L^2}$, $||(u - u_k)||_{L^2}||\Delta(u + u_k)||_{L^2}$ and the last term will be tends to zero by comparing with $||u\Delta u||_{L^2}$. Above three terms will vanish as $k \to \infty$.
It is quite clear that as w_k vanishes on ∂U, then $\int_{\partial U} w_k \frac{\partial u_k}{\partial n} = \int_{\partial U} \operatorname{div}(w_k \nabla u_k)\mathrm{d}x = 0$, so

$$\int_{\partial U} u_k \frac{\partial u_k}{\partial n} \mathrm{d}S = \int_{\partial U} \operatorname{div}((u_k - w_k)\nabla u_k)\mathrm{d}x = \int_U \nabla(u_k - w_k)\nabla(u_k)\mathrm{d}\mu + \int_U (u_k - w_k)\Delta(u_k)\mathrm{d}\mu$$

Since $u_k - w_k$ converges in H^1, then by Hölder inequality, clearly these two terms tend to zero. Therefore $\int_{\partial U} u \frac{\partial u}{\partial n} \mathrm{d}S = 0$. This implies $\int_{\partial U} \left|\nabla u\right|^2 \mathrm{d}x + \int_{\partial U} u \Delta u \mathrm{d}x = 0$. Apply Hölder, we have $||u\Delta u||_{L^2} \leq ||u||_{L^2}||\Delta u||_{L^2}$, combined with equality we had, then $||\nabla u||_{L^2} \leq ||u||_{L^2}^{\frac{1}{2}}||\Delta u||_{L^2}^{\frac{1}{2}}$.

4. Homework 4.

4.1. **Problem 1. [Evans 11]**. Let $K \subset\subset \Omega$ and K be compact set, and in $\mathbb{R}^n$ by an application of Heine-Borel this is finite and bounded, and therefore of finite Lesbugue measure. $u \in L^1(K)$, upon mollifying, if we choose the mollifiers χ_ϵ so $D(u * \chi_\epsilon) = Du * \chi_\epsilon = 0$. Ω is connected, and $u * \chi_\epsilon$ is of C^∞, then $u * \chi_\epsilon = C_\epsilon$, a constant a.e. in $L^1(K)$, and since $L^1(K)$ is complete, upon taking $\epsilon = \frac{1}{n}$, then $\{G_n := C_{\frac{1}{n}}\}$ will converge to a limt as G_n is Cauchy. And resulting limit, in L^1 sense, is constant a.e. C so $u = C$ in K a.e..

We can exaust U, if not finite, by $\overline{B_n(0) \cap U}$, and upon an exaustion of compact subsets: $U \subset \bigcup_n \overline{B_n(0) \cap U}$, we may therefore conclude u to be constant a.e..

4.2. **Problem 2. [Evans 12]**. Set $w = \chi_{(0,1)}$ be the indicator function on the real line $\mathbb{R}$. It is clearly an example that is not in $W^{1,1}$(in near boundary, the derivatives will blow up). Take any $V \subset U$. If any of $\sup V$, or $\inf V$ reaches 1 or 0, then clearly $dist(V, \partial U) = 0$ and thus L^1 norm of w is simply $\int_V \chi_{(0,1)} \mathrm{d}\mu = \mu(V \cap (0,1)) < \infty$ by monotonicity of measure. If not, then $\sup V$, or $\inf V$ are in U, this tells us that any derivatives of w is simply 0, for which case we conclude $||D^h u||_{L^1(V)} = 0$, for which case we conclude that for $0 < |h| < \frac{1}{2} dist(V, \partial U)$, $||D^h u||_{L^1(V)} < \infty$.

4.3. **Problem 3. [Evans 15]**. An application of definition of norm: $||u||_{L^2} \leq ||\text{Avg}(u)||_{L^2} + ||u - \text{Avg}(u)||_{L^2}$ by the notation $\text{Avg}(u) := \frac{1}{\mu(U)} \int_U u \mathrm{d}\mu$. By the *je ne sais quoi* of controls of Du over $u - \text{Avg}(u)$, or Poincaré Inequality, we have on the right hand side, $||u||_{L^2} \leq ||\text{Avg}(u)||_{L^2} + ||u - \text{Avg}(u)||_{L^2} \leq C||Du||_{L^2} + ||\text{Avg}(u)||_{L^2} \leq \mu(U)^{\frac{1}{2}} |\text{Avg}(u)| + C||Du||_{L^2}$. We may consider on $||\text{Avg}(u)||_{L^2}$. Set $V := \{u \neq 0\}$. By assumption, $\mu(U - V) \geq \alpha$. The previous holds as V is borel, therefore measurable.

$$\text{Avg}(u) := \frac{1}{\mu(U)} \int_U u \mathrm{d}\mu \leq \frac{1}{\mu(U)} \Big(\int_V \chi_V \Big)^{\frac{1}{2}} \Big(\int_V u^2 \Big)^{\frac{1}{2}} \mathrm{d}\mu$$
$$= \frac{1}{\mu(U)} \big(\mu(V)\big)^{\frac{1}{2}} \Big(\int_V u^2 \Big)^{\frac{1}{2}} \mathrm{d}\mu \leq \frac{1}{\mu(U)} \big(\mu(U) - \alpha\big)^{\frac{1}{2}} \Big(\int_V u^2 \Big)^{\frac{1}{2}} \mathrm{d}\mu := \theta \Big(\int_V u^2 \Big)^{\frac{1}{2}} \mathrm{d}\mu$$

Then
$$||u||_{L^2} \leq \mu(U)^{\frac{1}{2}} |\text{Avg}(u)| + C||Du||_{L^2} \leq \theta\, \mu(U)^{\frac{1}{2}} ||u||_{L^2} + C||Du||_{L^2}$$
and so
$$(1 - \theta\, \mu(U)^{\frac{1}{2}}) ||u||_{L^2} \leq C||Du||_{L^2}$$
, taking square so that we have $\int_U u^2 \mathrm{d}\mu \leq C' \int_U |Du|^2 \mathrm{d}\mu$ for some constants C'.

4.4. **Problem 4. [Evans 17.]** First we notice $F(u) < \text{essup} F'|u| + F(0)$, so $F(u) \in L^1$. Let us choose $u_m \in W^{1,p} \cap \mathscr{C}^\infty(U)$ to approximate $u \in W^{1,p}(U)$. Taking difference: $\int_U \left|F(u_m) - F(u)\right|^p \mathrm{d}\mu \leq \sup \left|F'\right|^p \int_U \left|u_m - u\right|^p \to 0$ as $m \to \infty$. And as F' is bounded, more importantly is of class $\mathscr{C}^1$, F is so Lipshitz, and $\int_U \left|F'(u_m) Du_m - F'(u) Du\right|^p \mathrm{d}\mu \leq \sup \left|F'\right|^p \int_U \left|Du_m - Du\right|^p + \int_U \left|Du\right|^p \left|F'(u_m) - F'(u)\right|^p$

Now passing to pointwise series, $u_m \to u$ a.e. As F' is continuous, and thus $F'(u_m) \to F'(u)$ a.e. point-wisely by continuity. That F' is bounded by M implies that $|F'(u_m)| <$

M and the later is obviously in L^1 as U is bounded. An application of dominated convergence theorem implies that $\int_U |Du|^p |F'(u_m) - F'(u)|^p \to 0$. Then, combining with convergence in $W^{1,p}$ we see that $F'(u_m)Du_m, F(u_m) \to F'(u)Du, F(u)$ in L^p. This establish the existence part. The uniqueness part follows from the uniqueness of weak derivative: $\int F'(u)Du\psi \mathrm{d}\mu = \int F(u)\psi' \mathrm{d}\mu$. Now the derivatives of different directions are easily deduced.

4.5. **Problem 5**. We first approximate $u^+ := \max(u, 0)$ by a $\mathscr{C}^1$ function $\zeta_n := \left(z^2 + \epsilon_n^2\right)^{1/2} - \epsilon_n$ for $z \geq 0$ and simply 0 otherwise. We notice that $\epsilon_n \to 0$ means that, $\zeta_n \to u^+$.

ζ_n has derivatives $\zeta_n' = \frac{z}{\sqrt{z^2+\epsilon^2}} = \frac{1}{\sqrt{1+(\epsilon/z)^2}}$ which is bounded and of L^p clearly. Therefore by Problem 4, $\zeta_n(u)$ is in $W^{1,p}$, and that its weak solution is simply $\zeta_n'(u)Du$. First we see that $\zeta_n \to u^+$ in L^1. This follows from $|\sqrt{u^2 + \epsilon_n^2} - \epsilon_n| \leq u$ and the later is of L^1 and the point-wise convergence, we pass the difference $\sqrt{u^2 + \epsilon_n^2} - u$ to L^1 norm, and by dominated convergence theorem, this limit goes to 0 as the difference goes to 0 pointwise. We next prove that $\zeta_n'(u)Du \to Du^+$, clearly pointwisely this is true since when $n \to \infty$, $\epsilon \to 0$, we have ζ_n in the limit for 1 while $u \geq 0$ and vanishes elsewhere. We check for L^1 convergence similarly, one uses the fact that $|\zeta_n'(u)Du| \leq |Du^+|$ since $0 < \zeta_n' < 1$. Thus, an application of dominated covergence theorem will effect $\zeta_n'(u)Du \to Du^+$ in L^1 sense.

Notes. Here it is important that we do estimate for Du^+ as a whole, since while separating $\zeta_n'Du$ by Hölder will cause us having to consider bounded domain. I wasted on some time on carrying on the above estimates. This(using DCT instead) is in fact a sharp estimate, and one sees that Heaviside is not a counterexample here. There is a similar problem(17) in Evans, and it requires the domain to be bounded. Clearly the author had probably been fallen into that thoght(using Hölder), too.

We therefore record that $\zeta_n \to u^+$ and derivatives in L^1 sense accordingly. If we choose $\psi \in \mathscr{C}_c^\infty$: $\int \zeta_n'(u)Du\psi = \int \zeta_n(u)\psi'$ this is true since ζ_n is differentiable. Upon passing to the limit will allow us conclude that the formula is also true, that is we have $\int Du^+\psi = \int u^+\psi'$. This confirms that Du^+ is a weak solution for u^+, and to check that it is of L^p this is trivial because $|Du^+| \leq |Du|$.

That $|u| \in W^{1,p}(U)$ follows from the identity $|u| = u^+ + (-u)^+$, and we know that since $u \in W^{1,p}$ is a Banach space, and by our previous results: $|u| \in W^{1,p}$. By what follows, we can compute $\int |u|\psi' \mathrm{d}\mu = \int (u^+ + (-u)^+)\psi' \mathrm{d}\mu = \int (Du^+ + D(-u)^+)\psi \mathrm{d}\mu$ so we can define $D|u| = Du^+ + D(-u)^+$. To put into different situations, we have Du for $u > 0$ and $D(-u) = -Du$(we remind the fact that here, the weak derivative comes from u not u^+) for $u < 0$. While $u = 0$, the result is $D|u| = Du - Du = 0$.

Now for any $f, g \in W^{1,p}$, we have $\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$. It follows from above observation that this function lies in $W^{1,p}$.

4.6. **Problem 6. Rellich-Kondrachov**.

4.6.1. ***Modification to Fit the Original Proof.*** Fix q, for $1 \leq q \leq p^*$, and U is open, for any $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then Poincarë inequality $||u||_{L^q} \leq C||Du||_{L^p}$ establishes $||u||_{L^q} \leq C||Du||_{W^{1,p}(U)}$, and that $W^{1,p}(U) \subset L^q(U)$. Then for the sequential compactness is then all the same for the proof of Rellich-Konfrachov.

4.6.2. *Using Rellich-Kondrachov.* That the set inclusion $W^{1,p}(U) \subset L^q(U)$ still is the same as above. Fix q, for $1 \leq q \leq p^*$, and U is open, for any $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then Poincarë inequality $||u||_{L^q} \leq C||Du||_{L^p}$ establishes $||u||_{L^q} \leq C||Du||_{W^{1,p}(U)}$, and that $W^{1,p}(U) \subset L^q(U)$. Now for the sequence converging it is a bit different. We can enclose an openset V with $\mathscr{C}^\infty$ boundary containing U, and for every functions $u \in W_0^{1,p}(U)$, we may simply get an extension $u \in W^{1,p}(V)$ by setting $u = 0$ on $V - U$. Notice that in this case, it is very easy to see that Du is also extended by the same way: namely set to be 0 when in $V - U$. This due to the fact that we can split the integral to integrate over $U, V - U$, and on $V - U$, by above definition, it has zero contribution, so that $\int_V D_u\psi = \int_V u\psi'$ after adding $\int_U D_u\psi = \int_U u\psi'$ and $\int_{V-U} D_u\psi = \int_{V-U} u\psi'$(for which case $Du = 0 = u$).

By an application of Rellich-Kondrachov lemma, it is clear that $W^{1,p}(V) \subset\subset L^q(U)$ for $1 < q < p^*$ fixed. As the above identification every elements in $W_0^{1,p}(U)$ can be identified with an element in $W^{1,p}(V)$, so for every bounded sequence $u_n \in W^{1,p}(U)$, we may have $u_n \in W^{1,p}(V)$. And it is also bounded, because on $V - U$ u_n vanishes identically, so has zero contribution to the integral similarly, for weak derivatives. Thus by Rellich-Kondrachov, we have a convering subsequence u_{n_k}. We wish to show that while restricting back to U, this subsequence is still convergent, for which case our theorem is proved. Now the subsequence converges in $L^q(V)$ norm, for it to converge in $L^q(U)$ norm, if we can show that assume $u_{n_k} \to u^*$, then u^* vanishes outside U a.e. (This idea can be dated back from Hurwitz's theorem on Normal family of holomorphic functions), we can conclude the convergence is in L^q since $\int_V \left|u_{n_k} - u^*\right|^p = \int_U \left|u_{n_k} - u^*\right|^p + \int_{V-U} \left|u_{n_k} - u^*\right|^p$ and since u_n all vanishes outisde U, if also u^* is zero except for a null set, we conclude the convergence in $L^q(V)$. But this is clear: $\int_{V\setminus U} |u^*|^q \mathrm{d}\mu = \int_{V\setminus U} |u^* - u_{n_k}|^q \mathrm{d}\mu \leq \int_V |v - u_{n_k}|^q \mathrm{d}\mu$, and the last term goes to 0 means that the integral is 0. Therefore u^* is 0 a.e. outside U, and the above deduction concludes the problem.

4.7. **Problem 7. An Counter Example.** Notice that we are bounded away from $(0, 0)$, the singularity of $x^{\frac{1}{p}}$. Therefore the function is smooth and the weak derivative is just $D_x u = \frac{1}{p} x^{\frac{1-p}{p}}, D_y u = 0$. We check if $u \in W^{1,p}$. The $L^1(U)$ norm of u is just $\int_0^1 \int_{-x^r}^{x^r} x^{1/p} \mathrm{d}y\mathrm{d}x = 2\int_0^1 x^{r+\frac{1}{p}} \mathrm{d}x = \frac{2}{r+\frac{1}{p}+1} < \infty$. and for weak derivatives: $\int_0^1 \int_{-x^r}^{x^r} \frac{1}{p^p} x^{1-p} \mathrm{d}y\mathrm{d}x = \frac{2}{p^p} \int_0^1 x^{r+1-p} \mathrm{d}x = \frac{2}{p^p(r-p+2)} < \infty$. upon taking $\frac{1}{p}$ power, we have $\frac{\sqrt[p]{2}}{p\sqrt[p]{2-p+r}} < \infty$. So $u \in W^{1,p}$. Consider the essential sup of u. Take any ball of 0 of radius r. If r is very small, we know that $B_r(0) \cap U$ has nonzero measure, however, u is not bounded in this set, hence the essential sup of u is unbounded. In other words, $u \notin L^\infty$. It does not violate Sobolev embedding. Recall we have interpolation relation: for $u \in W^{k,p}$, $\frac{1}{\infty} = \frac{1}{p} - \frac{k=1}{n=2}$, i.e., $0 = \frac{1}{p} - \frac{1}{2}$, so $p = 2$. However, in this case we have $p > 2$.

5. On The Elliptic Estimates.

5.1. **Euler-Langrange Functionals**. We may define the energy $E(u) = \int \frac{1}{2}cu^2 + \frac{1}{2}A\nabla u \cdot \nabla u - fu$.

According to the definition, set $G(\epsilon) = E(u + \epsilon v) - E(u)$. Since $G(\epsilon) = o(\epsilon)$ if and only if the coeffecients of ϵ terms vanish, so we only care about ϵ terms in G. This is not hard to compute:

$$\epsilon \int cuv + \frac{1}{2}\nabla v \cdot \nabla u + \frac{1}{2}A\nabla u \cdot \nabla v - fv \mathrm{d}\mu$$

We recall that $A = A^t$, thus the term above can be re-written as

$$\begin{aligned}\epsilon \int cuv + \frac{1}{2}A\nabla v \cdot \nabla u + \frac{1}{2}\nabla u \cdot A^t\nabla v - fv \mathrm{d}\mu &= \epsilon \int cuv + \frac{1}{2}A\nabla v \cdot \nabla u + \frac{1}{2}\nabla u \cdot A\nabla v - fv \mathrm{d}\mu \\ &= \epsilon \int cuv + A\nabla v \cdot \nabla u - fv \mathrm{d}\mu\end{aligned}$$

and since they must vanish, so $\int cuv + A\nabla v \cdot \nabla u - fv \mathrm{d}\mu = 0$, but this is just the definition of weak solution of $Lu = f$. One can easily deduce the above by integration by parts for: $\int_U -\nabla \cdot (A\nabla u)v + cuv = \int fv$.

5.2. **Characterization of K^***. Now, we first observe that, for $Ku = \gamma f$ to hold, we have the equivalent condition that $B_\gamma[u, \psi] = (f, \psi)$ for test functions $\psi \in H_0^1$. We wish to prove $K^*v = v$, then $B^*[v, \varphi] = 0$ for $\varphi \in H_0^1$. For any $\psi \in H_0^1$, set $Kw = \gamma\psi$(in other words, for $L_\gamma^{-1}w = \psi$, fix ψ, and the linear form $B_\gamma[\psi, \phi] = (w, \phi)$ on H_0^1 can be represented by w because of Riesz representation theorem, and the fact that a closed subspace of a Hilbert space is still a Hilbert space). So, we may proceed on computing $B_\gamma^*[v, \psi] = B_\gamma[\psi, v]$, and by above observation, $B_\gamma[\psi, v] = (w, v) = (v, w) = (K^*v, w) = (v, Kw) = (\gamma\psi, v) = \gamma(v, \psi)$. And then it follows that $B[\psi, v] = 0$ or $B^*[v, \psi] = 0$. Notice that this is true for any $\psi \in H_0^1$, thus we conclude that v is a weak solution of L^* homogeneous problem.

5.3. **Some Notions of weak* Convergence**. For n_j, we have the following equality, $B[u_{n_j}, \psi] = (f_{n_j}, \psi)$, and the right hand side converges to 0 in L^2 sense because $|(f_{n_i}, \psi)| \le ||f_{n_j}||_{L^2}||\psi||_{L^2} \to 0$.

We wish to show that the left hand side converges to 0 as well. We observe that since coeffecients of L are of L^∞, therefore upon multiplication, the terms of sort: Lu are all of H_0^1. Thus, what is left for us, is that we wish to show the following convergence

$$\int_U \sum a_{ij} \cdot (u_k)_{x_i}v_{x_j} + \sum b_i \cdot (u_k)_{x_i}v + cu_kv\mathrm{d}x \to \int_U \sum a_{ij}u_{x_i}v_{x_j} + \sum b_iu_{x_i}v + cuv\mathrm{d}x \tag{5.1}$$

$$\lambda(u_k, v) \to \lambda(u, v) \tag{5.2}$$

Now, we notice that for T bounded(or continuous) in a Banach space, it preserves the weak convergence. Thus, we can view an multiplication of $a_{i,j}, b_i, c \in L^\infty$ as a linear operator from $H_0^1 \to L^2$, and to see that they are bounded, it suffices to notice the operator norm of each are bounded by their L^∞ norm, which is finite. So a_{ij}, b_i, c preserves weak convergence. Thus $u_{n_j} \rightharpoonup u$ in H_0^1, and notice that $\frac{\partial}{\partial x_k} : u_{n_j} \to \frac{\partial u_{n_j}}{\partial x_k}$ is a continuous operator, easily seen by $||T|| = \sup \frac{||\frac{\partial u_{n_j}}{\partial x_k}||_{L^2}}{||u_{n_j}||_{H_0^1}} \leq 1$. Therefore $\frac{\partial}{\partial x_k}$ sends $u_{n_j} \rightharpoonup u$ to $\frac{\partial u_{n_j}}{\partial x_k} \rightharpoonup \frac{\partial u}{\partial x_k}$. In conclusion, combining the weak convergence of partials and multiplication by a L^∞ function, we can conclude that $\int_U \sum a_{ij} \cdot (u_k)_{x_i} + \sum b_i \cdot (u_k)_{x_i} + cu_k \mathrm{d}x \rightharpoonup \int_U \sum a_{ij} u_{x_i} + \sum b_i u_{x_i} + cu \mathrm{d}x$ in L^2. Therefore take $v \in H_0^1$, we have the following individual convergence:

$$\int_U \sum a_{ij} \cdot (u_k)_{x_i} v_{x_j} \to \int_U \sum a^{ij} u_{x_i} v_{x_j} \tag{5.3}$$

$$\int_U \sum b_i \cdot (u_k)_{x_i} v \to \int_U \sum b_i u_{x_i} v \tag{5.4}$$

$$\int_U c u_k v \to \int_U cuv \tag{5.5}$$

$$\lambda(u_k, v) = \int_U \lambda u_k v \longrightarrow \int_U \lambda uv = \lambda(u, v) \tag{5.6}$$

Adding up (5.3) to (5.6) we have the desired (5.1) and (5.2). And thus we conclude that u is indeed a weak solution.

5.4. **Rearrangement Argument**. We may define the "energy functional"

$$B[u, v] = \int \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial x_j x_k} \frac{\partial^2 v}{\partial x_j x_k} \mathrm{d}\mu \tag{5.7}$$

for the test functions $u, v \in H_0^2$. To get an unique weak solution, we can apply Lax-Milgram. It suffices to show that $||u||_{H_0^2}^2 \lesssim B[u, u]$ for all $u \in H_0^2$.

Let u_i be the i-th partial. Now if we take

$$\operatorname{div}(u_i \nabla u_i) = \sum_{i,j} u_{i,j} u_{i,j} + \sum_{i,j} u_i u_{ijj} \tag{5.8}$$

. It is then problematic to deal with the third partial, since we only assume $u \in H_0^2$. To avoid this problem, we notice that after an integration by parts, and an application of change of differentiation of order we may have

$$\int_U u_i u_{ijj} \mathrm{d}\mu = \int_U u_i u_{jji} \mathrm{d}\mu = -\int_U u_{ii} u_{jj} \tag{5.9}$$

since the partials vanish on boundary. The reason we can exchange the order of differentiation is that since $\int_U u_i u_{ijj} \mathrm{d}\mu = -\int_U u_{ii} u_{jj} \mathrm{d}\mu < \infty$, and we may approximate u with

$u_n \in \mathscr{C}_0^\infty$. Thus, we apply this to (5.8) and conclude that

$$\int_U \operatorname{div}(u_i \nabla u_i) = \int_U \sum_{i,j} u_{i,j} u_{i,j} + \sum_{i,j} u_i u_{ijj} = \int_U \sum_{i,j} u_{i,j} u_{i,j} - \int_U \sum_{i,j} u_{ii} u_{jj} \tag{5.10}$$

$$\int_U \operatorname{div}(u_i \nabla u_i) = \int_{\partial U} u_i \nabla(u_i) \cdot \mathrm{n} \mathrm{d}\mu(\partial U) = 0 \tag{5.11}$$

(5.11) follows from Divergence theorem. Then we have $\int_U |D^2 u|^2 \mathrm{d}\mu = \int_U \sum_{i,j} u_{i,j} u_{i,j} = \int_U \sum_{i,j} u_{ii} u_{jj} = \int_U |\Delta u|^2 \mathrm{d}\mu$.

Compute the L^2 norm of partial of first order, first observe u vanishes on boundary, $||Du||_{L^2}^2 = -\int u \Delta u \mathrm{d}\mu \le ||u||_{L^2} ||\Delta u||_{L^2}$, and AM-GM gives for positive t, $||u||_{L^2} ||\Delta u||_{L^2} \le t||u||_{L^2}^2 + \frac{1}{4t}||\Delta u||_{L^2}^2$, further, recall that we can control the original function with its first order partial, by Poincaré, we have $||Du||_{L^2}^2 \le tK||Du||_{L^2}^2 + \frac{1}{4t}||\Delta u||_{L^2}^2$ for some K. Choose, say $t = \frac{1}{2K}$, so $||Du||_{L^2}^2 \lesssim ||\Delta u||_{L^2}^2$. To sum up, we have

$$||u||_{L^2} \le \sqrt{K} ||Du||_{L^2} \tag{5.12}$$

$$||Du||_{L^2}^2 \le \sqrt{K} ||\Delta u||_{L^2}^2 \tag{5.13}$$

Combine above we see that $||u||_{H_0^2}^2 \lesssim ||\Delta u||_{L^2}^2$, and observe by an application from the indentity we showed in the begining, we have $B[u, u] = ||\Delta u||_{L^2}^2$, and we conclude that $||u||_{H_0^2}^2 \lesssim B[u, u]$.

Notes. The skill in controlling $||Du||_{L^2}^2$ by AM-GM seems to be known as rearrangement arguments. This operates on the boundary of norms of functions, and (as the nature of AM-GM) is pretty sharp in terms of estimates. So it seems to me that, in terms of regularization effects, Δ^2 seems to be very limited, as opposed to Δ(which is elliptic!).

5.5. **Poisson Equation Revisited**. We first look at v smooth, where calculus is available, and observe that in fact H^1 can be approximated by C^∞, so here we relaxed on the compactly support for smooth functions, but rather will apply trace theorem to specify the boundary data. Let continuous trace operator be T. By integration by parts, $\int_U f v \mathrm{d}\mu = \int_U -\Delta u v = \int_U \nabla u \nabla v - \int_{\partial U} v \frac{\partial u}{\partial \nu} = \int_{\partial U} vu + \int_U \nabla u \nabla v$. The last term holds by $u + \frac{\partial u}{\partial \nu} = 0$ on boundary. We may thus extend densely for this operator B, to be

$$B[u, v] = \int_{\partial U} TuTv + \int_U \nabla u \nabla v \tag{5.14}$$

Thus, if u is ever a weak solution of Robin boundary condition, $B[u, v] = (f, v)$ for any $v \in H^1$.

To show the existence of solution, we appeal to Lax-Milgram. For instance, it is rather clear that $B[u, v] \lesssim ||u||_{H^1} ||v||_{H^1}$, since T is continuous and terms of $\nabla u \nabla v$ can be dealt with partials of first order in the later terms.

We concentrate on $||u||_{H^1}^2 \lesssim B[u, u]$. Suppose not, say for each k, we have $\frac{1}{k}||u_k||_{H^1}^2 >$

$B[u_k, u_k]$. WLOG, we can assume u_k of unit length in H_0^1. Then

$$\frac{1}{k} > B[u_k, u_k] \tag{5.15}$$

From (5.14), we have then $\frac{1}{k} > ||Du_k||_{L^2}^2$, and $||u_k||_{H^1} < 1 + \frac{1}{k}$, so the sequence u_k is uniformly bounded. By Rellich-Kondrachov Lemma, one immediately sees that we can have a convergent subsequence in L^2, say u_k without too much subscripts. $u_k \to u^*$. We hope that u^* has a weak derivative, and yet this is not hard to see, for instance, for $\psi \in \mathscr{C}_0^\infty$, we have $\int u^* D\psi \, \mathrm{d}\mu = \lim \int u_k D\psi = \lim \int -Du_k \psi \leq ||Du_k||_{L^2} ||\psi||_{L^2}$ now as L^2 norm of Du_k is bounded by $\frac{1}{k}$, we have then $\int u^* D\psi \, \mathrm{d}\mu = 0$. Now we may take $Du^* = 0$. And $u_k \to u^*$ in H^1 implies $||u^*||_{L^2} = 1$ as well. Also, weak derivative vanishes meaning that u^* is constant on U by last homework. Now we proceed to see its trace. By combining (5.15) and T is continuous: $||Tu^*||_{L^2} \leq ||T(u^* - u_k)||_{L^2} + ||Tu_k||_{L^2} \lesssim ||u^* - u_k||_{H^1} + \sqrt{\frac{1}{k}}$ and the last term will tend to 0.
Thus $Tu^* = 0$. Now as u^* can be approximated by smooth functions, one has those functions vanishes on the boundary, but have to stay constant inside U, so they must vanish on U, resulting L^2 norm to vanish, but this contradicts the assumption that $||u^*||_{L^2} = 1$.

5.6. **Discrete Derivatives**. By definition of weak derivatives, one has for $v \in H_0^1$ test functions.

$$\int \nabla u \nabla v + c(u)v = \int fv \tag{5.16}$$

Let $v = -D_k^{-h}(D_k^h u)$ as in Evans for h very small but nonzero. By the formula $\int_U vD_k^{-h} w \, \mathrm{d}\mu = \int_U wD_k^h v \, \mathrm{d}\mu$, applied to

$$-c(u)D_k^{-h}(D_k^h u) = D_k^h c(u) D_k^h u \tag{5.17}$$

and

$$-\nabla u \nabla D_k^{-h}(D_k^h u) = -\nabla u D_k^{-h}(D_k^h \nabla u) = |D_k^h Du|^2 \tag{5.18}$$

In view of above, then (5.16) becomes

$$\int |D_k^h Du|^2 + D_k^h c(u) D_k^h u = -\int f D_k^{-h}(D_k^h u) \tag{5.19}$$

An application of mean value theorem says

$$D_k^h c(u) D_k^h u = \frac{c(u(x + he_k) - u(x))}{h} D_k^h u \geq c'(u(x_0))|D_k^h u|^2 \geq 0 \tag{5.20}$$

Combining (5.17) to (5.19), one has

$$\int_U |D_k^h Du|^2 \leq \Big| \int_U f D_k^{-h}(D_k^h u) \Big| \leq ||f||_{L^2} ||D_k^{-h}(D_k^h u)||_{L^2} \tag{5.21}$$

By an application of Cauchy Schwartz:

(5.22)
$$||f||_{L^2}||D_k^{-h}(D_k^h u)||_{L^2} \leq \frac{1}{2}\Big(\int_U |f|^2 + \int |D_k^{-h}(D_k^h u)|^2\Big) \leq \frac{1}{2}\Big(\int_U |f|^2 + \int |D(D_k^h u)|^2\Big)$$

Therefore one has $\int_U |D_k^h Du|^2 \leq \int_U |f|^2$ for arbitrary h. By Theorem 3. of Evans 5.8.2. and estimate the vector length of $D^h Du$ by $D_k^h Du$, we conclude that $||D^2 u||_{L^2} \lesssim \int_U |f|^2$.

5.7. **Weak Maximal Principle**. Compute, for instance, the partials $|Du|_{ij}^2 = 2\big(Du \cdot \frac{\partial}{\partial x_i x_j} Du + \frac{\partial Du}{\partial x_i} \cdot \frac{\partial Du}{\partial x_j}\big)$. We find $\frac{\partial}{\partial x_i x_j} Du$ to be present in $D(Lu) = 0$. Therefore we can expand it, say $Da^{ij}u_{ij} + a^{ij}\frac{\partial}{\partial x_i x_j} Du = 0$; Therefore one has $2Du \cdot Da^{ij}u_{ij} + 2a^{ij}Du \cdot \frac{\partial}{\partial x_i x_j} Du = 0$. Adding back $-2a^{ij}\frac{\partial Du}{\partial x_i} \cdot \frac{\partial Du}{\partial x_j}$ we have

$$2Du \cdot Da^{ij}u_{ij} - 2a^{ij}\frac{\partial Du}{\partial x_i} \cdot \frac{\partial Du}{\partial x_j} = -a^{ij}|Du|_{ij}^2 \tag{5.23}$$

Notice that $\frac{\partial^2 u^2}{\partial x_i x_j}$ is nothing but $2u$ multiply by a sum of partials $u_i u_j + u_{ij}$. Following this, we look at $\sum a^{ij}u_{ij} = 0$, and if we multiply by $2u$, we have $0 = \sum a^{ij}2uu_{ij}$, adding back the crossing term $\sum a^{ij}2u_i u_j$ we have

$$\sum a^{ij}2u_i u_j = \sum a^{ij}\frac{\partial^2 u^2}{\partial x_i x_j} \tag{5.24}$$

Combining (5.23), (5.24) one has

$$\sum -a^{ij}\frac{\partial^2\big(\lambda u^2 + |Du|^2\big)}{\partial x_i x_j} = \sum 2Du \cdot Da^{ij}u_{ij} - 2a^{ij}\frac{\partial Du}{\partial x_i} \cdot \frac{\partial Du}{\partial x_j} - a^{ij}2\lambda u_i u_j \tag{5.25}$$

Since a^{ij} has bounded derivatives, and that by uniform ellipticity, we have

$$\sum -a^{ij}\frac{\partial^2\big(\lambda u^2 + |Du|^2\big)}{\partial x_i x_j} \leq 2M\sum\big(\sum u_k\big)u_{ij} - 2\theta\sum\frac{\partial Du}{\partial x_i} \cdot \frac{\partial Du}{\partial x_j} + \lambda u_i u_j \tag{5.26}$$

We are done if we can find for individual pairing: $\lambda u_k u_k + \frac{1}{n^3}u_{ij}u_{ij} \geq \frac{M}{\theta}u_k u_{ij}$. One can always multiply by $\frac{\theta}{M}$ to reduce to following. We invoke the Young inequality: $ab \leq sa^p + \frac{(sp)^{-\frac{q}{p}}}{q}b^q$. Let $p = q = 2$, and $a = u_{ij}$, $b = u_k$, $s = \frac{1}{n^3}\frac{\theta}{M}$, we can then solve for λ. So take such λ, we see that the left hand side of (5.26) is always non-positive. This is the desired λ.

Now, since in L, we do not have c terms, also u is smooth; this allows us to apply weak maximum principal. By $Lv \leq 0$ in U, by weak maximum principal, one has $\max_{\overline{U}} v = \max_{\partial U} v$. So $\sup_U |Du|^2 + \lambda u^2 \leq \sup_{\partial U} |Du|^2 + \lambda u^2$, and therefore $\sup_U |Du|^2 \leq \sup_{\partial U} |Du|^2 + \lambda u^2 \leq (\sup_{\partial U} |Du| + |u|)^2$ so $\sup_U |Du| \leq (\sup_{\partial U} |Du| + |u|)$, i.e.

$$||Du||_{L^\infty(U)} \lesssim ||Du||_{L^\infty(\partial U)} + ||u||_{L^\infty(\partial U)} \tag{5.27}$$

6. On de Giorgi's Elliptic Estimates

We would refer to this note of de Giorgi's elliptic estimates as given in class.
We wish to solve following PDE and ellipticity:

$$\nabla(A\nabla u) = 0 \tag{6.1}$$

$$\frac{1}{\Lambda} \le A \le \Lambda \tag{6.2}$$

Here we work out some details. The first gives an affirmation for a nonlinear estimate, the second asserts the use of oscillation estimate to help with the regularity(here, we are dealing with $C^{2,\alpha}$, for instance). The fourth is addressed by an estimate by continuous representatives of a H^1 function.

6.1. **Problem 1**. We replace U_k by its inductive relation and conclude that

$$U_k \le C^{p_{n,k}} U_0^{\beta^{k+1}} \tag{6.3}$$

where $p_{n,k} = \sum_{j=1}^{k}(k-j)\beta^j$. To facilitate estimate of asymptotic data of $p_{n,k}$, we would then use Cauchy-Schwartz to reduce $p_{n,k} \le \sqrt{\sum_{j=0}^{k} j^2 \sum_{j=0}^{k} \beta^{2j}} = \sqrt{\frac{k(k+1)(2k+1)(\beta^{2k}-1)}{6(\beta^2-1)}} \lesssim \sqrt{k^3\beta^{2k}} \lesssim k^2\beta^k$. Thus $C^{p_{n,k}} \lesssim C^{k^2\beta^k}$. If we put, $U_0 = \frac{1}{C^2}$ say, then we have $C^{p_{n,k}} U_0^{\beta^{k+1}} \lesssim \frac{C^{k^2\beta^k}}{C^{2\beta^{k+1}}} = C^{k^2-\beta^{k+1}}$ now, since $k^2 - \beta^{k+1}$ will tend to $-\infty$ as $k \to \infty$ as $\beta > 1$, we see that left hand side of (6.3) will tend to 0.

6.2. **Oscillation Implies Hölder Continuity**. Take a region precompact in U where we are considering the PDE, say $U^\epsilon := \{x : d(x,U) \ge \epsilon\}$. In following we will prove that $u \in C^\alpha(U^\epsilon)$. Freeze $z \in U^\epsilon$, and we put $w_n(x) = u(z + \epsilon\frac{x}{4^n})$, and $w_0 = u$. We wish w_n is a solution to our equation and that we do not alter Λ in (6.2) at all, thus, we may re-adjust a bit, by taking $A_n(x) = A(z + \frac{x}{4^n})$. This is good, since we can apply oscillation estimate on all w_n, by viewing the ball $x \in B_{1/2}, B_2$ around z but with reduced radius of $z + \epsilon\frac{x}{4^n}$. This yields $\operatorname{osc}_{|x-z|\le\frac{1}{4}} u(x) \le 2\lambda||u||_{L^\infty(B_1)}$, and inductively $\frac{1}{\lambda}\operatorname{osc}_{|x-z|\le\frac{1}{4^n}} u(x) \le \operatorname{osc}_{|x-z|\le\frac{1}{4^{n-1}}} u(x)$. This results in $\sup_{|x-z|\le\frac{1}{4^n}} |u(x) - u(z)| \le 2\lambda^n||u||_{L^\infty(B_1)}$. If we take, say $\alpha = -\log_4\lambda$, then $\sup_{|x-z|\le\frac{1}{4^n}} |u(x)-u(z)| \le 2\lambda^n||u||_{L^\infty(B_1)} = 2||u||_{L^\infty(B_1)}(\frac{1}{4^n})^{-\log_4\lambda}$ for any n. In particular, we have $\sup_{|x-z|=\frac{1}{4^n}} |u(x)-u(z)| \le 2\lambda^n||u||_{L^\infty(B_1)} = 2||u||_{L^\infty(B_1)}(\frac{1}{4^n})^{-\log_4\lambda} = 2||u||_{L^\infty(B_1)}||x-z||^\alpha$. The choice of z is arbitrary, and we have proven this holds true for any y, z, $||y-z|| = \frac{1}{4^n}$ for any n, while the bound, $2||u||_{L^\infty(B_1)}$ is universal as long as $y, z \in B_1$. Thus this gives the Hölder continuity in B_1.

6.3. **Maximal Energy**. Let us set $\mathcal{M}$ to be the measurable sets of measure 1. Let $A \in \mathcal{M}$, and B be the ball. We will show that $F(B) = \int_B |f|^2 \mathrm{d}\mu$ is the maximal of all $F(A), A \in \mathcal{M}$. That is, we will show that $D := F(B) - F(A) \ge 0$. Recall that $\mathcal{M}$ is an σ algebra. Thus $A-B, B-A, A\cap B \in \mathcal{M}$, and therefore $D = \int_{B-A}|f|^2\mathrm{d}\mu - \int_{A-B}|f|^2\mathrm{d}\mu$. Since $\mu(A) = \mu(B) = 1$, $\mu(A-B) = \mu(B-A)$. Also let us notice that on $B-A$, $f(x) \ge f(\frac{1}{\sqrt{\pi}})$ since f is non-increasing, while on $A-B$, $f(x) \le f(\frac{1}{\sqrt{\pi}})$. Thus $D =$

$\int_{B-A}|f|^2\mathrm{d}\mu - \int_{A-B}|f|^2\mathrm{d}\mu \geq \mu(B-A)f(\frac{1}{\sqrt{\pi}}) - \mu(A-B)f(\frac{1}{\sqrt{\pi}}) = 0$. This concludes the proof.

6.4. **Approximate By Continuous Function**. We need to have a a.e. alternative for H^1 function so that we can have validity of fundamental theorem of calculus, we can relax up to a.e. because in our proof, we only are required to do integration.

Theorem 6.1. *Let $u \in H^1(I)$, $I = [a, b]$, then there is a continuous function w, such that $w = u$ a.e., and $w(a) - w(b) = \int_b^a u'(x)\mathrm{d}\mu(x)$.*

before the proof, we need a technical lemma.

Lemma 6.2. $w(x) = \int_{x_0}^x v(t)\mathrm{d}t$ *is continuous with weak derivative* $-v$.

Proof. For test functions, $\int_I w\psi'\mathrm{d}\mu = \int_I \int_{x_0}^x v(\zeta)\mathrm{d}\zeta\psi'\mathrm{d}\mu$, by Fubini and reversing the sign we may have $-\int_a^{x_0} v\mathrm{d}t \int_a^t \psi'\mathrm{d}x + \int_{x_0}^b v\mathrm{d}t \int_t^b \psi'\mathrm{d}x = -\int_I v\psi\,\mathrm{d}t$. □

Next we prove the theorem.

Proof. Set $w(x) = \int_b^x u'(\zeta)\mathrm{d}\zeta$. By Lemma 6.2, we have a weak derivative $-u'$. That is $\int_I w\psi' = \int_I -u'\psi$, and $\int_I u\psi' = \int_I -u'\psi$ implies that $\int_I (w-u)\psi' = 0$ for every ψ. Thus we conclude by homework(weak derivative 0 implies constant a.e.) that $u = w + K$ for some constant K, and $K = u(b)$, thus $w(b) = u(b)$. □

The proof will be similar to Lemma 10, as presented in the document, but to do integration, we might be careful. To choose $x, y \in A, C$, we can first fix $x \in A$, and then we have a ray from x to any $y \in C$, we can write, for instance the line $x + ty$ for some $y \in C$, since we know that we can find a continuous representative for u a.e. on this line, so that for a.e. y on the line $\frac{1}{2}(\bar{w}(y) - \bar{w}(x)) = \int_0^1 (y-x)\nabla\bar{w}(x + t(y-x))\mathrm{d}t$ holds. The choice of x is arbitrary, as we have seen in the theorem, the continuous representative $\bar{w}_{cts}$ always have the end point value x to be $\bar{w}(x)$.

Thus in this way, we can integrate by co-area theorem, or Fubini, that $\frac{1}{2}\int\int_C 1\mathrm{d}r\mathrm{d}S \leq \int\int_C \int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\mathrm{d}s\mathrm{d}r\mathrm{d}S \leq \int_{B^1}\int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\mathrm{d}s\mathrm{d}y$ This is valid since on each r direction, we have fixed x, the set where difference of $\bar{w}$, and $\bar{w}_{cts}$ is not 0 is a nullset, hence the integral exhibits no difference.

When we integrate over A, the method is the same, while we move the integral around justified by Fubini. And these steps after are all the same.

Thus we may conclude we can extend the same proof to H^1, however with very tight estimate.

7. Appendix.

7.1. **On the Equivalence of Hilbert's 19th Problem**.

7.2. **Weak Maximum Principal**. This is the theorem for elliptic operator homogeneous L.

6.4.1. Weak maximum principle.

First, we identify circumstances under which a function must attain its maximum (or minimum) on the boundary. We always assume $U \subset \mathbb{R}^n$ is open, bounded.

THEOREM 1 (Weak maximum principle). *Assume* $u \in C^2(U) \cap C(\bar{U})$ *and*

$$c \equiv 0 \quad \text{in } U.$$

(i) *If*

$$Lu \leq 0 \quad \text{in } U, \tag{3}$$

then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(ii) *If*

$$Lu \geq 0 \quad \text{in } U, \tag{4}$$

then

$$\min_{\bar{U}} u = \min_{\partial U} u.$$

Remark. A function satisfying (3) is called a *subsolution*. We are thus asserting that *a subsolution attains its maximum on* ∂U. Similarly, if (4) holds, u is a *supersolution* and attains its minimum on ∂U. □

Proof. 1. Let us first suppose we have the strict inequality

$$Lu < 0 \quad \text{in } U, \tag{5}$$

and yet there exists a point $x_0 \in U$ with

$$u(x_0) = \max_{\bar{U}} u. \tag{6}$$

Now at this maximum point x_0, we have

$$Du(x_0) = 0 \tag{7}$$

and

$$D^2u(x_0) \leq 0. \tag{8}$$

2. Since the matrix $A = ((a^{ij}(x_0)))$ is symmetric and positive definite, there exists an orthogonal matrix $O = ((o_{ij}))$ so that

$$OAO^T = \operatorname{diag}(d_1, \ldots, d_n), \quad OO^T = I, \tag{9}$$

with $d_k > 0$ $(k = 1, \ldots, n)$. Write $y = x_0 + O(x - x_0)$. Then $x - x_0 = O^T(y - x_0)$, and so

$$u_{x_i} = \sum_{k=1}^n u_{y_k} o_{ki}, \; u_{x_i x_j} = \sum_{k,l=1}^n u_{y_k y_l} o_{ki} o_{lj} \quad (i, j = 1, \ldots, n).$$

Hence at the point x_0,

$$\begin{aligned} \sum_{i,j=1}^n a^{ij} u_{x_i x_j} &= \sum_{k,l=1}^n \sum_{i,j=1}^n a^{ij} u_{y_k y_l} o_{ki} o_{lj} \\ &= \sum_{k=1}^n d_k u_{y_k y_k} \quad \text{by (9)} \\ &\leq 0, \end{aligned} \tag{10}$$

since $d_k > 0$ and $u_{y_k y_k}(x_0) \le 0$ $(k = 1, \dots, n)$, according to (8).

3. Thus at x_0

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} \ge 0,$$

in light of (7) and (10). So (5) and (6) are incompatible, and we have a contradiction.

4. In the general case that (3) holds, write

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1} \quad (x \in U),$$

where $\lambda > 0$ will be selected below and $\epsilon > 0$. Recall (as in the proof of Theorem 4 in §6.3.2) that the uniform ellipticity condition implies $a^{ii}(x) \ge \theta$ $(i = 1, \dots, n,\ x \in U)$. Therefore

$$\begin{aligned} Lu^\epsilon &= Lu + \epsilon L(e^{\lambda x_1}) \\ &\le \epsilon e^{\lambda x_1} \left[-\lambda^2 a^{11} + \lambda b^1\right] \\ &\le \epsilon e^{\lambda x_1} [-\lambda^2 \theta + \|\mathbf{b}\|_{L^\infty} \lambda] \\ &< 0 \quad \text{ in } U, \end{aligned}$$

provided we choose $\lambda > 0$ sufficiently large. Then according to steps 1 and 2 above $\max_{\bar{U}} u^\epsilon = \max_{\partial U} u^\epsilon$. Let $\epsilon \to 0$ to find $\max_{\bar{U}} u = \max_{\partial U} u$. This proves (i).

5. Since $-u$ is a subsolution whenever u is a supersolution, assertion (ii) follows. □

We next modify the maximum principle to allow for a *nonnegative* zeroth-order coefficient c. Remember from §A.3 that $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$.

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