L^2 ESTIMATES OF $\bar{\partial}$ PROBLEM IN PSEUDOCONVEX DOMAIN, I

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Abstract. Survey and examination on the phenomenon of regularity of $\bar{\partial}$ Problem in $\mathbb{C}^n$ via the technique of L^2 estimates. Some applications to Cousin's Problem.

References

1. Lars Hörmander *An Introduction to Complex Analysis in Several Variables.* Addison-Wesley, Reading, Massachusetts, 1993.
2. Yum-Tong Siu *Introduction to several complex variables.* River Edge, NJ : World Scientific, 2005.
3. Volker Scheidemann *Introduction to Complex Analysis in Several Variables* Basel : Birkhäser Verlag

1. Holomorphicity in $\mathbb{C}^n$

Set $z_k = x_k + \sqrt{-1}y_k$, the coordinate of $\mathbf{z} \in \mathbb{C}^n$, and the domain of consideration Ω is open in $\mathbb{C}^n$. Let $\mathrm{d}z_j, \mathrm{d}\bar{z}_j$ be duals of $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ living in $T_{\mathbf{z}}(\Omega), \bar{T}_{\mathbf{z}}(\Omega)$. Write $u \in C^1(\Omega)$ after upgrading in its DeRham cochain,

$$\mathrm{d}u = \sum_{k=1}^{n} \frac{\partial u}{\partial z_k}\mathrm{d}z_k + \frac{\partial u}{\partial \bar{z}_k}\mathrm{d}\bar{z}_k(*)$$

where $\frac{\partial u}{\partial z_k} = \frac{1}{2}(\frac{\partial u}{\partial z_{2k-1}} - \sqrt{-1}\frac{\partial u}{\partial z_{2k}})$, and $\frac{\partial u}{\partial \bar{z}_k} = \frac{1}{2}(\frac{\partial u}{\partial z_{2k-1}} + \sqrt{-1}\frac{\partial u}{\partial z_{2k}})$.

We will first consider polydisc $D := \prod D_k$, where D_k is an disk of variable z_k, and the boundary(in manifold sense) ∂D, which is just $\prod \partial D_k$ we call it distinguished boundary.

We say $u \in C^1(\Omega)$ is analytic, or in $A(\Omega)$ the set of all analytic functions if $\bar{\partial}u = 0$. In the language of differential forms, we observe that from $(*)$, $\mathrm{d}u = \partial u + \bar{\partial}u$, and if $u \in A(\Omega)$, then we only have ∂u left. In terms of its differentials, it is the linear combination of $\mathrm{d}z_k, \mathrm{d}\bar{z}_k$, we call the differential forms in the span of $\mathrm{d}z_k$ to be of type (1, 0), and in the span of $\bar{\mathrm{d}}z_k$ to be of type (1, 0).

The next fact gives the representation of function values by integral in current setting.

Theorem 1.1. *Let u be a continuous function in the closure of a polydisc $\bar{D}$, and analytic in z_j when other variables kept fixed in its interior. Then*

$$u = \frac{1}{2\pi\sqrt{-1}} \int_{\partial_0 D} \frac{u}{\prod_{k=1}^{n}(\zeta_k - z_k)} \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n$$

This fact immediately gives out that $u \in C^\infty(D)$ and u is analytic. Further, if we set Ω to be arbitrary open set in $\mathbb{C}^n$, and $u \in A(\Omega)$, then $u \in C^\infty(\Omega)$ and all derivatives are analytic.

Under what regularity condition will a function $u : W \subset \mathbb{C}^n \to \mathbb{C}$ be analytic? Unlike the real case, we only have some partial results such as f differentiable a.e. with respect to Lesbegue measure when f is Lipschitz continuous, in our case, we only require variable-wise analyticity, that is,

Theorem 1.2. *u is analytic if for particular z_k, z_j for $j \neq k$ is fixed, $u(z_k)$ is analytic for every k.*

Like in $\mathbb{C}$, we have notions of analytic continuation, that is we can extend the domain of definition of f to the maximal set D_f , and in $\mathbb{C}$, If U is a domain, and $A(U)$ be the family of analytic functions on U, then $\cap_{f \in A(U)} D_f = U$, this is true since we can always define $f = \frac{1}{z-a}$, for a not in U, so a is not in D_f , therefore not in the intersection.

Yet for $n > 1$, this fails to be true.

Example 1.3. Set $S = \{(z,w) \in \mathbb{C}^2 ||z| < 1, \beta < 1, \beta < |w| < 1\} \cup \{(z,w) \in \mathbb{C}^2 ||z| < \alpha < 1, |w| < 1\}$. Then any analytic function can be extended on $\hat{S} = \{(z,w) \in \mathbb{C}^2 ||z| < 1, |w| < 1\}$.

To characterize domain of holomorphy, we have several definitions.

Definition 1.4. A domain U is a domain of holomorphy, if there is no open set V such that UV and every analytic function on U can be extended on V.

Definition 1.5. An $A(\Omega)$-hull of a compact set $K \subset \Omega$ is the set $\hat{K}_\Omega := \{z; |f(z)| \leq \sup_{f \in A(\Omega), \zeta \in K} |f(\zeta)|\}$.

We present following theorems as for references of characterization of domain of holomorphy and pseudoconvex domain, one can find proofs of these theorems in usual contemporary texts, for example, one could consult chapter 2 of [1].

Theorem 1.6. *(Cartan-Thullen) Let Ω be a domain. Then TFAE:*
(1) Ω is a domain of holomorphy
(2) For any compact subset K, $\hat{K}_\Omega$ is relatively compact in Ω
(3) There is a analytic function $f \in A(\Omega)$, Ω is its domain of holomorphy

Definition 1.7. A plurisubharmonic function u is a semicontinuous real-valued function while being subharmonic while restricing on every complex line. The set of plurisubharmonic functions on Ω is $P(\Omega)$.

Definition 1.8. Distance function d on $\Omega \subset \mathbb{C}^n$ is $d(x) = dist(x, \mathbb{C}^n - \Omega)$ for $x \in \Omega$.

Theorem 1.9. *If Ω is a domain of holomorphy, $-\log d$ is plurisubharmonic.*

Definition 1.10. We define $P(\Omega)$-hall to be the set $\hat{K}^P_\Omega := \{z \in \Omega; |f(z)| \leq \sup_{f \in P(\Omega), \zeta \in K} |f(\zeta)|\}$.

Theorem 1.11. *The followings are equivalent, and one satisfies one of (i), (ii) and (iii) below is called pseudoconvex domain.*
(i) $-\log d$ is plurisubharmonic in Ω;
(ii) Ω supports a continuous strong exaustion function u which is also plurisubharmonic;
(iii) Ω is plurisubharmonically convex, that is, $\hat{K}^P_\Omega := \{z; |f(z)| \leq \sup_{f \in P(\Omega), \zeta \in K} |f(\zeta)|\} \subset\subset \Omega$ for any compact $K \subset \Omega$.

Suppose Ω being a pseudoconvex domain, the origin of such domain, if $\partial\Omega$ is a C^2 manifold(In fact, since every C^k manifold can be viewed as a C^∞ manifold, by taking the maximum Atlas, we C^2 can be relaxed for C^1, though this does not help much.), is equivalent to Ω is a convex open domain in Euclidean sense. For general case, we define the pseudoconvex domain in terms of plurisubharmonic funtions living on them, this can be found to be an high dimension analog when we treat a Riemann Surface M by the subharmonic functions living on it. The above theorems imply that a domain of holomorphy is a pseudoconvex domain.

2. L^2 Estimates of $\bar{\partial}$ Problem

Since last section asserts that all domain of Holomorphy is a pseudoconvex domain, we will show converse is true.

Before this proving the equivalence, we will first study the solution of

$$\bar{\partial}u = f$$

where $f \in \Omega_{p,q}^{C^\infty}$ with the property that $\bar{\partial}f = 0$.

the existence theorem of $\bar{\partial}$ problem, and regularity concerning the solution of which, by means established by L. Hörmander, if among the other, in 1960s.

We will generalize the tools for function spaces which $\bar{\partial}$ acts on, as even C_c^∞ behaves wildly in terms of taking limit, we will extend this function to be an element in larger space with adequate tools to move things around. Let the complex of differential forms of coeffecients of $L^2(W,\phi)$ functions be

$$\Omega_{(p,q)}^{L^2(W,\phi)}, \Omega_{(p+1,q)}^{L^2(W,\phi)}$$

, where $W \subset \mathbb{C}^n$ for some domain, clearly both of which are Hilbert space. Heuristically we wish to solve nonhomogeneous $\bar{\partial}u = f$, for f holomorphic function satisying $\bar{\partial}f = 0$. We simply regard it as an operator on Hilbert space H_1, H_2, then we may, after noticing that as $\Omega_{(p,q)}^{\mathscr{C}^\infty(W)}, \Omega_{(p+1,q)}^{\mathscr{C}^\infty(W)}$ are dense in $\Omega_{(p,q)}^{L^2(W,\phi)}, \Omega_{(p+1,q)}^{L^2(W,\phi)}$, define T on the previous ones, and then extend to $\Omega_{(p,q)}^{L^2(W,\phi)}, \Omega_{(p+1,q)}^{L^2(W,\phi)}$.

In fact, we can form a sequence

$$\Omega_{(p,q)}^{L^2(W,\phi)} \underset{\bar{\partial}}{\rightarrow} \Omega_{(p,q+1)}^{L^2(W,\phi)} \underset{\bar{\partial}}{\rightarrow} \Omega_{(p,q+2)}^{L^2(W,\phi)}$$

The problem of deciding (p,q) forms is just when, under some nice densities, that we have the kernel of the latter operator between 2 Hilbert spaces is the image of the former. This, then reduces the existence to problems of estimates of operators in Hilbert spaces. We will prove following lemmas with regard to any Hilbert spaces H.

Theorem 2.1. *Let $T : H_1 \to H_2$ be a densely defined, closed operator between Hilbert spaces. If a closed set Z contains image of T. Then, for shorthand let the domain of T^* be D_{T^*}, we have $Z = R_T$ if and only if $||f||_{H_2} \leq C||T^*f||_{H_1}$, for f runs through $Z \cap D_{T^*}$. Here, we define D_T to be the domain of T, and R_T for the image of T, for convenience.*

Proof. We first estimate the size of set

$$B = \{f : f \in Z \cap D_{T^*}, ||T^* f||_{H_1} \leq 1\}$$

Our goal is to show it is bounded by some constants C. It suffices to show that for fixed $g \in Z$, $|\langle f, g\rangle_{H_2}|$ is bounded aways from some constants; suppose we have $Tu = \psi$ for $u \in D_T$, we see that $|\langle f, g\rangle_{H_2}| \leq |\langle T^* f, u\rangle_{H_1}| \leq ||u||_{H_1}$which proves one direction.

Let us assume that the inequality holds, and $\psi \in Z$. H_i are Hilbert spaces, so naturally we can identify $T^{**} = T$, therefore $Tu = \psi$ for $u \in H_1$ is then translated to $\langle u, T^* f\rangle = \langle \psi, f\rangle$ as $\langle T^{**}u, f\rangle = \langle f, T^{**}u\rangle = \langle T^* f, u\rangle = \langle f, Tu\rangle = \langle \psi, f\rangle = \langle u, T^* f\rangle$, therefore to speak with onto ψ it is enough to speak of the identity above.

To find u satisfying above, let us denote by β the functional send $T^* f$ to $\langle f, \psi\rangle$, it is easy to check that β is linear, and by Hahn-Banach, one could see that if β is dominated by semi-norm $C||\psi||_{H_2}||T^* f||_{H_1}$ for some constants C, we can extend β to be defined on H_1, as it is a Hilbert space, by Riesz representation, one deduce that there is a u, such that $\langle T^* f, u\rangle_{H_1} = \beta(T^* f) = \langle f, \psi\rangle_{H_2}$, so we have a solution u. Now we only need to show the inequality $C||\psi||_{H_2}||T^* f||_{H_1} \geq |\langle \psi, f\rangle|_{H_2}$.

If f is in $Z \cap D_{T^*}$, then clearly the desired estimate follows from the assumption, yet if not, then since Z is closed, therefore $Z = (Z^{\perp})^{\perp}$, and $f \in D_{T^*}$, we only need to consider when f is orthogonal to Z, in which case $\langle \psi, f\rangle_{H_2} = 0$ and $\langle T^* f, T^* f\rangle = \langle f, TT^* f\rangle = 0$as we assume $R_T \subset Z$. This concludes $R_T = Z$.

□

Next, we provide informations on T^*.

Theorem 2.2. *With same notations as in above, for every $g \in H_1$, such that g is orthogonal to* $\ker(T)$*, there is $f \in D_{T^*}$ such that $T^* f = g$, and $||g||_{H_2} \leq C||f||_{H_1}$ for some constants C.*

Proof. Observe that $R_{T^*} \perp \ker T$ and $R_T \perp \ker T^*$, so $g \in \overline{R_{T^*}}$, and $R_T = Z \subset \ker T^*$, which means we can reduce T^* from D_{T^*} to $D_{T^*} \cap Z$ with same image. Yet the inequality we gave at above theorem only says the range $R_{T^*_{red}}$ is in fact closed, therefore we can find $f \in D_{T^*}$, such that $T^* f = g$, the norm bounds follow from the last theorem. □

Set this Z to be the kernel of the latter operator, in

$$\Omega^{L^2(W,\phi)}_{(p,q)} \underset{\bar{\partial}}{\rightarrow} \Omega^{L^2(W,\phi)}_{(p,q+1)} \underset{\bar{\partial}}{\rightarrow} \Omega^{L^2(W,\phi)}_{(p,q+2)}$$

which is just all the elements in $\Omega^{L^2(W,\phi)}_{(p,q+1)}$ for which f satisfying $\bar{\partial} f = 0$ in distribution sense, and if we may define E be the latter operator, or $\bar{\partial} : \Omega^{L^2(W,\phi)}_{(p,q+1)} \rightarrow \Omega^{L^2(W,\phi)}_{(p,q+2)}$. If, we take another function ϕ_3, $Z = \ker E$, if we want to make use of previous two lemmas, then we need to show the estimates in (2.1), then we prove a stronger estimate that

$$||f||^2_{L^2(W,\phi_2)} \leq C(||T^* f||^2_{L^2(W,\phi_3)} + ||Wf||^2_{L^2(W,\phi_1)})$$

for $f \in D_{T^*} \cap D_E$, notice that when $f \in Z, E(f) = 0$. We cannot prove the estimates when the set we are choosing behaves so wild, to find a more appropriate set of density, namely C^∞. In fact we can prove the estimates under such, of which the details are entailed below as a series of lemmas.

Theorem 2.3. *Setting η_λ, for $\lambda \in \Lambda$ a net, to be a family of C^∞ functions which vanishes outside a compact set of Ω for λ is large enough, this can be done by mollifiers by Friedrichs. If $\phi_2 \in C^1(\Omega)$ and*

$$e^{-\phi_{j+1}} \sum_{k=1}^{n} |\frac{\partial \eta_\lambda}{\partial \bar{z}_k}| \leq e^{-\phi_j}$$

for $j = 1,2; \lambda = 1,2,\ldots.$ Then $\Omega^{\mathscr{C}^\infty(W)}_{(p,q+1)}$ is dense in $D_{T^*} \cap D_E$ under the norm

$$f \to ||f||^2_{L^2(W,\phi_2)} + ||T^* f||^2_{L^2(W,\phi_3)} + ||Ef||^2_{L^2(W,\phi_1)}$$

Proof. From computation we see that $E(\eta_\lambda f) - \eta_\lambda Ef = \bar{\partial}\eta_\lambda \wedge f,\ f \in D_E$, and then it follows from the estimates in assumption that $|E(\eta_\lambda f) - \eta_\lambda Ef|^2 e^{-\phi_3} \leq |f|^2 e^{-\phi_2}$, as the righthand side is L^1, by Dominated Convergence, one sees that the left hand side must converges to the point-wise limit as well, which is just 0, since $\bar{\partial}\eta_\lambda$ vanishes very quickly, so in fact $||E(\eta_\lambda f) - \eta_\lambda Ef||^2_{L^2(W,\phi_3)} \to 0$.

We would like to imitate this estimate, for example, $T^*(\eta f)$. In fact, if $f \in D_{T^*}, \eta \in C_0^\infty(\Omega)$, clearly that $\eta f \in D_{T^*}$, and we test it by $u \in D_T$ via

$$\begin{aligned}\langle \eta f, Tu\rangle_{L^2(W,\phi_2)} &= \langle \eta f, T(\bar{\eta}u)\rangle_{L^2(W,\phi_2)} + \langle \eta f, \bar{\eta}Tu - T(\bar{\eta}u)\rangle_{L^2(W,\phi_2)} \\ &= \langle \eta T^* f, u\rangle_{L^2(W,\phi_1)} + \langle f, \bar{\eta}Tu - T(\bar{\eta}u)\rangle_{L^2(W,\phi_2)}\end{aligned}$$

So the sum is continuous with $||u||_{L^2(W,\phi_1)}$, since no derivative appears in the last term, by Riesz representation theorem, we can again find an element $v \in L^2_{p,q}(\Omega,\phi_1)$ with $\langle v, u\rangle_{L^2(W,\phi_1)} = \langle \eta f, Tu\rangle_{L^2(W,\phi_2)}, u \in D_T$. This in turn gets us the desried $T^*(\eta f) = v$, now repeat above estimate, we obtain

$$\langle T^*(\eta_\lambda f) - \eta_\lambda T^* f, u\rangle_{L^2(W,\phi_1)} \leq \int |f||u| e^{-\frac{\phi_1+\phi_2}{2}} \mathrm{d}\mu$$

which says

$$|T^*(\eta_\lambda f) - \eta_\lambda T^* f|^2 e^{-\phi_1} \leq |f|^2 e^{-\phi_2}$$

with right hand side is L^1, so repeat above we conclude $||T^*(\eta_\lambda f) - \eta_\lambda T^* f||^2_{L^2(W,\phi_1)} \to 0$ as $\lambda \to \infty$. Therefore $\eta_\lambda f \to f$ as $\lambda \to \infty$ and $f \in D_{T^*} \cap D_E$.

Now we will apply one particular theorem about mollifiers. Let χ be mollifiers, that is $C_0^\infty(\mathbb{R}^n)$ and $\int \chi = 1$. Setting $\chi_s(x) = \frac{\chi(x/s)}{s^n},\ x \in \mathbb{R}^n$, if g is L^2 function, then we know that $g * \chi_s$ is C^∞ with support has no point at distance $> s$ from the support of g , if we normalize χ so that $supp(\chi) \subset B^n$. While $s \to 0$, $g * \chi_s$ will converge to g in L^2 sense.

Take $f \in D_{T^*} \cap D_E$, and similarly define $f_s = f * \chi_s$, but take n above to be $2n$ for this case. While *$s \to 0$, $supp(f * \chi_s) \subset K \subset\subset \Omega$, and $||f - f * \chi_s||_{L^2(W,\phi_2)}$. For E, as $E(f * \chi_s) = E(f) * \chi_s$, $||Ef - E(f * \chi_s)||_{L^2(W,\phi_3)}$.

To express T^*, we write $e^{\phi_2 - \phi_1} T^* = \xi + \mu$, ξ is a constant coeffecients operator, and μ degree 0, then $(\xi + \mu)(f * \chi_s) = ((\xi + \mu)f) * \chi_s + \mu(f * \chi_s) - (\mu f) * \chi_s$convergent to $(\xi + \mu)f + \mu f - \mu f$, so $||T^*(f * \chi_s) - T^* f||_{L^2(W,\phi_1)} \to 0$ as well.

This completes the proof.

□

Next to do estimate, it is necessary to carry out specific forms of T^*, as we just avoided.

Definition 2.4. 1. We use summation sign $\sum_{|I|=n}^{<}$ to denote the summation in partial order given by index set I.
2. Therefore a (p,q) form is then $u = \sum_{|I|=p}^{<}\sum_{|J|=q}^{<} u_{I,K}\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K$.
3. We sometimes use D in short hand to denote C^∞, and spaces of (p,q) forms on W with L^p coeffecients by $\Omega_{p,q}^{L^p}(W)$.

Since our case seems to be quite feasible to start with a specific construction, by actually starting computing T^*, we can choose

$$u = \sum_{|I|=p}^{<}\sum_{|J|=q}^{<} u_{I,K}\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \in D_{(p,q)}(\Omega)$$

and

$$f = \sum_{|I|=p}^{<}\sum_{|J|=q+1}^{<} u_{I,K}\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \in L^2_{(p,q+1)}(\Omega,\phi_2)$$

Since $f_{I,J}$ is defined for all J as an antisymmetric function of indices in J, and

$$\bar{\partial}u = \sum_{|I|=p}^{<}\sum_{|J|=q}^{<}\sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k} u_{I,K}\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \wedge \mathrm{d}\bar{z}_k$$

Thus, we see that if $f \in D_{T^*}$, using the fact that we want the solution of $Tu = f$, we estimate the norm of solution u by

$$\int \sum_{I,K}^{<} (T^*f)_{I,K}\overline{u_{I,K}}e^{-\phi_1}\mathrm{d}\mu = \langle T^*f, u\rangle_{\phi_1} = \langle f, Tu\rangle_{\phi_2} = (-1)^{p-1}\int \sum_{I,K}^{<}\sum_{k=1}^{n}\frac{\partial}{\partial \bar{z}_k} f_{I,jK}e^{-\phi_2}\mathrm{d}\mu$$

So by property of inner product, we see that

$$T^*f = (-1)^{p-1}\sum_{I,K}^{<}\sum_{k=1}^{n} e^{\phi_1}\frac{\partial f_{I,jK}e^{-\phi_2}}{\partial \bar{z}_k}\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \quad (2.4)$$

The job right now is to create a way using the convergence theorems we had in hand to make a solution.

Recall that η_λ, for $\lambda \in \Lambda$ a net, to be a family of C^∞ functions which vanishes outside a compact set of Ω for λ is large enough. Now choose any function ψ, for which

$$\sum_{k=1}^{n} |\frac{\partial \eta_j}{\partial \bar{z}_k}|^2 \le e^{\psi}, j = 1,2,3,4,...$$

For convenience, setting $\phi_1 = \phi - 2\psi, \phi_1 = \phi - \psi, \phi_3 = \phi$. Notice that this means whatever ψ is, the condition in

$$e^{-\phi_{j+1}}\sum_{k=1}^{n} |\frac{\partial \eta_\lambda}{\partial \bar{z}_k}| \le e^{-\phi_j}$$

. Since by the theorems we have just recorded concerning the existence of a solution by some operator, we only need to look at the estimates in norms, namely

$||T^*f||^2_{L^2(W,\phi_3)}, ||Wf||^2_{L^2(W,\phi_1)}$, this could be done if we keep ψ fixed(as any ψ does not impact), and making some estimates for ϕ, so we can apply the condition for a pseudoconvex domain(namely there is a strongly exaustion function which is also plurisubhamonic). For the convenience, ϕ is supposed to be C^2. As

$$\bar{\partial} f = \sum_{|I|=p}^{<} \sum_{|J|=q}^{<} \sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k} f_{I,K} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \wedge \mathrm{d}\bar{z}_k$$

the norm is

$$\sum_{I,J,L}^{<} \sum_{j=1,l=1}^{n} \frac{\partial}{\partial \bar{z}_k} f_{I,K} \overline{\frac{\partial}{\partial \bar{z}_k} f_{I,K}} \varepsilon^{jJ}_{lL}$$

The essence of this estimate is to play with this sum.

For ε^{jJ}_{lL}, we see that it is 0 whenever $j \in J, l \in L$, and moreover, whenever $\{j\} \cup J = \{l\} \cup L$. For the rest nontrivial case, it is $\mathrm{sgn}(\frac{jJ}{lL})$.

If $j = l$, by above $J = L$ and j,l not in J,L. Therefore it is $\sum_{I,J}^{<} \sum_{j \notin J} |\frac{\partial}{\partial \bar{z}_j} f_{I,K}|^2$, as the permutation is just identity. Now if $j \neq l$, then $K = J - \{l\} = L - \{j\}$. We can form $\mathrm{sgn}(\frac{jJ}{lL}) = \mathrm{sgn}(\frac{jJ}{jlK})\mathrm{sgn}(\frac{jlK}{ljK})\mathrm{sgn}(\frac{ljK}{lL}) = -\mathrm{sgn}(\frac{J}{lK})\mathrm{sgn}(\frac{jK}{L})$, so the terms of summation become

$$-\sum_{I,K}^{<} \sum_{j \neq l} \frac{\partial}{\partial \bar{z}_j} f_{I,lK} \overline{\frac{\partial}{\partial \bar{z}_l} f_{I,jK}}$$

, and combine them we have

$$|\bar{\partial} f|^2 = \sum_{I,J}^{<} \sum_{j} |\frac{\partial}{\partial \bar{z}_j} f_{I,K}|^2 - \sum_{I,K}^{<} \sum_{j,k} \frac{\partial}{\partial \bar{z}_k} f_{I,jK} \overline{\frac{\partial}{\partial \bar{z}_l} f_{I,kK}}$$

Define $\delta_j w = e^{\phi} \frac{\partial w e^{-\phi}}{\partial z_j} = \frac{\partial w}{\partial z_j} - w \frac{\partial \phi}{\partial z_j}$, we see that $-\delta_j$ is in fact adjoint with $\frac{\partial}{\partial z_j}$, in sense of following inner proudct $\langle w_1, \frac{\partial w_2}{\partial z_j} \rangle_{L^2(\Omega,\phi)} = -\langle \delta_k w_1, w_2 \rangle_{L^2(\Omega,\phi)}$. From the computation,

$$T^* f = (-1)^{p-1} \sum_{I,K}^{<} \sum_{k=1}^{n} e^{\phi_1} \frac{\partial (f_{I,jK} e^{-\phi_2})}{\partial \bar{z}_k} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \ (\spadesuit)$$

,

$$e^{\psi} T^* f = (-1)^{p-1} \sum_{I,K}^{<} \sum_{j=1}^{n} \delta_j f_{I,jK} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K + (-1)^{p-1} \sum_{I,K}^{<} \sum_{j=1}^{n} f_{I,jK} \frac{\partial \psi}{\partial z_j} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K$$

By HÃűlder applied on the norm, and view $e^{-\phi}\mathrm{d}\mu$ a measure, we see that

$$\int \sum_{I,K}^{<} \sum_{j,k=1}^{n} \delta_j f_{I,jK} \overline{\delta_j f_{I,jK}} e^{-\phi} \mathrm{d}\mu \leq 2||T^* f||^2_{\phi_1} + 2 \int |f|^2 |\psi|^2 e^{-\phi} \mathrm{d}\mu$$

and now combining the result of $|\bar{\partial} f|^2$, we now obtain

$$\int \sum_{I,K}^{<} \sum_{j,k=1}^{n} (\delta_j f_{I,jK} \overline{\delta_j f_{I,jK}} - \frac{\partial f_{I,jK}}{\partial \bar{z}_k} \overline{\frac{\partial f_{I,jK}}{\partial \bar{z}_k}}) e^{-\phi} \mathrm{d}\mu + \int \sum_{I,K}^{<} \sum_{j,k=1}^{n} |\frac{\partial f_{I,jK}}{\partial \bar{z}_k}|^2 e^{-\phi} \mathrm{d}\mu$$

$$\leq 2||T^*f||^2_{\phi_1} + ||Wf||^2_{\phi_3} + 2\int |f|^2|\psi|^2 e^{-\phi}\mathrm{d}\mu$$

Recall that we have the adjoint property for $\delta_j, \frac{\partial}{\partial z_j}$, now we also notice $\delta_j \frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j}\delta_j = \frac{\partial^2 \phi}{\partial \bar{z}_k \partial z_j}$, then we can shift the differentiation in the left side of above inequality, or

$$\sum_{I,K}^{<} \int \sum_{j,k=1}^{n} f_{I,jK}\overline{f_{I,kK}} \frac{\partial^2 \phi}{\partial \bar{z}_k \partial z_j} e^{-\phi}\mathrm{d}\mu + \sum_{I,J}^{<} \sum_{k=1}^{n} \int |\frac{\partial f_{I,J}}{\partial \bar{z}_k}|^2 e^{-\phi}\mathrm{d}\mu$$

$$\leq 2||T^*f||^2_{\phi_1} + ||Ef||^2_{\phi_3} + 2\int |f|^2|\psi|^2 e^{-\phi}\mathrm{d}\mu,\ f \in D_{(p,q+1)}(\Omega)\ (\textbf{Norm estimate})$$

So we can assume ϕ is strically plurisubharmonic, then it satisfies tangential relation

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial \bar{z}_k \partial z_j} w_j \overline{w_k} \geq c \sum_{k=1}^{n} |w_k|^2,\ w \in \mathbb{C}^n$$

for $c > 0$, so the above inequality is just

$$\int (c - 2|\partial \psi|^2)|f|^2 e^{-\phi}\mathrm{d}\mu \leq 2||T^*f||^2_{\phi_1} + ||Ef||^2_{\phi_3}$$

By the convergence theorem mentioned above, we now have proven: ϕ_j for $j = 1,2,3$ as defined above, and $\psi, \phi \in C^2$, and suppose

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial \bar{z}_k \partial z_j} w_j \overline{w_k} \geq 2(|\partial \psi|^2 + e^{\psi}) \sum_{k=1}^{n} |w_k|^2,\ w \in \mathbb{C}^n\ (\star)$$

we have $||T^*f||^2_{L^2(W,\phi_3)} + ||Ef||^2_{L^2(W,\phi_1)} \geq ||f||^2_{L^2(W,\phi_2)}$.

Then we have a solution for $\bar{\partial} u = f$ in distribution sense.

Theorem 2.5. *Let Ω be an pseudoconvex domain in $\mathbb{C}^n$, then $\bar{\partial} u = f$ always has a solution $u \in L^2_{(p,q)}(\Omega, loc)$ for every $f \in L^2_{(p,q)}(\Omega, loc)$ such that $\bar{\partial} f = 0$.*

Proof. By the familiar equivalence of pseudoconvex domain, we can choose a plurisubharmonic function $\rho \in C^\infty(\Omega)$, which is also a strongly exaustion function, that is $\Omega^c = \{z|\rho(z) \leq c\} \subset\subset \Omega$ for arbitrary c.

As it is plurisubharmonic, choose ξ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \bar{z}_k} w_j \bar{w}_k \geq \xi \sum_{k=1}^{n} |w_k|^2$$

We can choose ξ to be continuous and positive. Let ν be a smooth convex increasing function and $\eta = \nu(\rho)$, the above equation is

$$\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \bar{z}_k} w_j \bar{w}_k \geq \nu'(\rho)\xi \sum_{k=1}^{n} |w_k|^2$$

To satisfy $(\star)$, we need $\nu'(\rho)\xi \geq 2(|\partial \psi|^2 + e^{\psi})$. Setting Ω_t as begining, we need the above equation to satisfy $\nu'(t) \geq \sup_{\Omega_t} \frac{2(|\partial \psi|^2 + e^{\psi})}{\xi}$ This is a finite increasing function, therefore we can construct ν , also making every $f \in \Omega^{L^2(loc)}_{p,q+1}(W)$, we have $f \in$

$\Omega^{L^2(W,\phi-\psi)}_{p,q+1}(W)$, then by the theorem in the beginning, we conclude that there is a solution $u\in\Omega^{L^2(W,\phi-2\psi)}_{p,q+1}(W)$.

□

Using the theory of Sobolev space, we can formulate the regularity of solution u.

Let H^s be the space of functions whose t-th derivatives are in L^2, for all $t\le s$, and $H^s(W,loc)$ for those functions satisfying above in some compact sets $K\subset\subset W$. We similarly define the space, complex of differential forms of coefficients of $H^s(W,loc)$ functions be $\Omega^{H^s(W,loc)}_{(p,q)}$, where $W\subset\mathbb{C}^n$ for some domain.

Definition 2.6. Define operator ♣ of summation of derivatives along z_j direction from putting $\phi_i=0$ in (♠) by following

$$♣f=\sum_{I,K}^{<}\sum_{k=1}^{n}\frac{\partial f_{I,jK}}{\partial\bar z_k}\mathrm{d}z^I\wedge\mathrm{d}\bar z^K$$

With the hope that when estimates of sum of partials are continuous with the norm of operator, we can, in fact obtain the information with regard to partials in z_j's direction.

Thus, we need the estimate for this operator. In fact, suppose $f\in\mathscr{C}^\infty_{p,q+1}$ are smooth, with compact support, and by Norm Estimate, setting $\phi=\psi=0$ we have simpler forms:

$$\sum_{I,J}^{<}\sum_{k=1}^{n}\int|\frac{\partial f_{I,J}}{\partial\bar z_k}|^2\mathrm{d}\mu\le 2||♣f||^2_{L^2}+||\bar\partial f||^2_{L^2}\mathrm{d}\mu,\ f\in D_{(p,q+1)}(\Omega)\ (\textbf{Simpler Norm Estimate})$$

Yet we only have L^2 solutions, this means we have to replace f by $f*\chi_s$ as we just mentioned in the proof of existence theorem, also we requre, as in the Simpler Norm Estimate has implied, $\bar\partial f, ♣f$ are all in $\Omega^{L^2}_{p,q+2},\Omega^{L^2}_{p,q}$. We know what ♣ is, explicitly a combination of partials, so applying the Simpler Norm Estimate on ♣ applied to $f*\chi_s-f*\chi_t$, we only need to take care of 2 norm from the Simpler Norm Estimate. One for ♣ is quite obvious, since $♣(f*\chi_s)=(♣f)*\chi_s$, when $s\to 0$, this is just $♣f$, similar for $\bar\partial(f*\chi_s)$. Now the convergence of norms imply the individual convergence of $\chi_s*\frac{\partial f_{I,J}}{\partial\bar z_j}$ in L^2, as L^2 is complete, this says the partials of f in any z_j direction is L^2.

Next, we show for this case, that is, $f\in L^2$, with compact support, and $\frac{\partial f}{\partial\bar z_k}\in L^2$ as well, then $f\in H^1$, in other words, we need to show $\frac{\partial f}{\partial z_j}$ to be L^2. We first notice that if f is smooth and allows one to do integration by parts:

$$\int|\frac{\partial f}{\partial z_j}|^2\mathrm{d}\mu=\int\frac{\partial f}{\partial z_j}\overline{\frac{\partial f}{\partial z_j}}\mathrm{d}\mu=-\int\frac{\partial^2 f}{\partial\bar z_j z_j}\bar f\mathrm{d}\mu=\int|\frac{\partial f}{\partial\bar z_j}|^2\mathrm{d}\mu$$

Even if f is not smooth, we can approximate it by $f*\chi_s$, by the same arguments as above, we conclude that in fact $\frac{\partial f}{\partial z_j}$ is L^2.

Combining those facts, for $f\in\Omega^{L^2}_{p,q}$, and if $\bar\partial f, ♣f$ are all in $\Omega^{L^2}_{p,q+2},\Omega^{L^2}_{p,q}$., we find that $f\in\Omega^{H^1}_{p,q+1}$. With those estimates in mind, we then prove that we can "lift" the regularity grading of solution of $\bar\partial u=f$ by means of Sobolev space.

Theorem 2.7. *Let W be an pseudoconvex domain in $\mathbb{C}^n$, $0 \leq s \leq \infty$, then $\bar{\partial}u = f$ always has a solution $u \in \Omega_{(p,q)}^{H^{s+1}(W,loc)}$ for every $f \in \Omega_{(p,q+1)}^{H^s(W,loc)}$ such that $\bar{\partial} f = 0$ for $q > 0$; while for $q = 0$, this property is automatically satisfied.*

Proof. We first treat the case for $q = 0$. As a cosequence of Existence of Solution, the differential relation is at the disposal:

$$\bar{\partial}u = f; \frac{\partial u_I}{\partial \bar{z}_j} = f_{I,j} \in H^s(W, loc)$$

Suppose $u \in H^t(W, loc)$for some t, with $0 \leq t \leq s$, as we have seen, $t = 0$ is the worst case, so this inequality holds unless we are done. By Leibniz rule, for $\frac{\partial \rho u_I}{\partial \bar{z}_j} = \rho f_{I,j} + \frac{\partial \rho}{\partial \bar{z}_j} u_I$ is of $H^t(W, loc)$. This means that if $\eta := \frac{\partial^t}{\partial \bar{z}_j{}^t}$, then $\frac{\partial \eta}{\partial \bar{z}_j} \in L^2$ in every $\bar{z}_j$ direction. By the observations we made before, $\eta \in H^1$, so all derivatives of ρu_I of order $t+1$ are in L^2, i.e., $u_I \in H^{t+1}(W, loc)$, and repeat above, we see that it stops at $t = s$, i.e., $u_I \in H^{s+1}(W, loc)$.

Before moving to case for which $q > 0$, we make following observation: $Tu = f$ meaning that we can in fact choose $u \in (\ker T)^{\perp}$, as we can always throw out parts in $\ker T$, and as in Hilbert spaces one obtains closure by taking orthogonal complements, we see that $u \in \overline{R_{T^*}}$. $(\star\star)$

Now notice that we are working in L^2, so $\phi - \psi = 0$, this means

$$T^* f = (-1)^{p-1} \sum_{I,K}^{<} \sum_{k=1}^{n} e^{\phi_1} \frac{\partial (f_{I,jK})}{\partial \bar{z}_k} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K \ (\spadesuit')$$

as $u \in \bar{R}_{T^*}$, one sees that

$$ue^{-\phi_1} = (-1)^{p-1} \sum_{I,K}^{<} \sum_{k=1}^{n} \frac{\partial (f_{I,jK})}{\partial \bar{z}_k} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K = (-1)^{p-1} \clubsuit f$$

so $\clubsuit^2 = 0$ implies $\clubsuit(ue^{-\phi_1}) = 0$.

As $\clubsuit$ is an operator being the combination of partials, then, to simplify the notation, we can just put

$$\bar{\partial}u = f; \clubsuit u = \lambda$$

as an alternative forms, with λ a differential operator of order 0 with $\mathscr{C}^\infty$ functions as coeffecients.

Now we will use the above simplification of $\clubsuit$ to explicitly estimate the size of derivatives along $z_j, \bar{z}_j$ direction. Specifically, assume that, similar for $q = 0$, we set $u \in \Omega_{p,q}^{H^t}(W)$ for $0 \leq t \leq s$, say ρ is a smooth function with compact support, so smilar to above case $\bar{\partial}(\rho u) \in \Omega_{p,q+1}^{H^t}(W)$, and $\clubsuit(\rho u) \in \Omega_{p,q-1}^{H^t}(W)$. Setting D^t to be the differentiation of t times, then $D^t(\rho u) \in H^1$ as we can apply the result obtained above, with $\bar{\partial} f, \clubsuit f$ are in $\Omega_{p,q+2}^{L^2}, \Omega_{p,q}^{L^2}$. Repeat the argument we obtain the similar result.

□

The Sobolev lemma tells us that

$$H^{s+2n}_{(p,q)} \subset \mathscr{C}^s_{(p,q)}(\Omega)$$

and then the above theorem concludes following.

Theorem 2.8. *If W is an pseudoconvex domain in $\mathbb{C}^n$, then $\bar{\partial}u = f$ always has a solution $u \in \Omega^{C^\infty(W)}_{(p,q)}$ for every $f \in \Omega^{C^\infty(W)}_{(p,q+1)}$ so long as $\bar{\partial}f = 0$.*

Here we construct equivalence of the solvability and regularity of $\bar{\partial}u = f$ on W and W being a domain of holomorphy.

Theorem 2.9. *Suppose that W is a domain in $\mathbb{C}^n$ that $\bar{\partial}u = f$ always has a solution $u \in \Omega^{C^\infty(W)}_{(p,q)}$ for every $f \in \Omega^{C^\infty(W)}_{(p,q+1)}$ such that $\bar{\partial}f = 0$. Then W is a domain of holomorphy.*

Proof. When $n = 1$ the theorem is true for all open sets, one can prove by choosing a sequence of countable discrete subset $S \subset W$, when taken the closure, the closure includes ∂W. By Weiersterass Theorem, one can always find a function $f \in A(W)$, with polynomial of $z - s_j$ as principal parts, yet it is not identically 0. Assume we intersect with W any open set U, we have for some $s \in U \cap W$ being the limit point of S, so if we can analytically continue along U, since $s_j \to s$ f are all 0, this says $f = 0$ on $W \cup U$, which contradicts our assumption.

So for general n, it suffices to consider open convex subsets U sharing a common boundary point with W, there is $f \in A(W)$ cannot be continued to that point. Let's assume the common point is just the origin, and that the hyperplane meets U: $H := \{z_n = 0\}, H \cap U = U' \neq \emptyset$. Since U is convex, and that $0 \in \partial U$, then $0 \in \partial U'$ as well, or the boudary of set $H \cap W$, with its topology being open in $\mathbb{C}^{n-1}$, with $i : H \cap W \hookrightarrow W$, and $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$. So now we have seen the case in $n = 1$ is true, our strategy is to lift f. That is, one shows that for every $f \in \Omega^{C^\infty}_{p,q}(H \cap W)$, $\bar{\partial}f = 0$, one can find $F \in \Omega^{C^\infty}_{p,q}(W)$, with $\bar{\partial}F = 0$ and $f = i^*F$.

Set $E := \{z | \pi(z) \in W - W \cap H\}$, as E and $H \cap W$ are disjoint and we can find a smooth function ψ being 1 in neighborhood of $H \cap W$ and 0 in neighborhood of E. Then form $\psi\pi^* f \in \Omega^{C^\infty}_{0,q}(W)$ is 0 near E , and $i^*\psi\pi^* f = f$.

We then set $F = \psi\pi^* f - z_n G$ with G undetermined in $\Omega^{C^\infty}_{0,q}(W)$ form so that $\bar{\partial}F = 0$, this translates to solving

$$\bar{\partial}G = \frac{\partial\bar{\psi} \wedge \pi^* f}{z_n}$$

Now $\bar{\partial}\frac{\partial\bar{\psi} \wedge \pi^* f}{z_n} = 0$, so we can solve, by assumption, for G, so $i^*F = i^*\pi^* f = f$, F has the prescribed property.

We now prove by induction on dimension. Given $f \in \Omega^{C^\infty}_{0,q}(W \cap H)$, with $\bar{\partial}f = 0$, we see that there is F such that $\bar{\partial}F = 0, i^*F = f$, so we can solve $\bar{\partial}u_n = F$, for $u_n \in \Omega^{C^\infty}_{0,q}(W)$, and $u_{n-1} = i^*u_n$, we have $\bar{\partial}u_{n-1} = f$. By inductive hypothesis we see that $H \cap W$ is a domain of holomorphy, and by definition there is a function $\rho \in A(H \cap W)$ cannot be continued out of U'. We can choose $\Psi \in A(W)$, and $i^*\Psi = \rho$, then it cannot be continued in a neighborhood of U. This finish the proof.

$\square$

Combine above theorems, we see that pseudoconvex domain is a domain of holomorphy, as the converse to this is proven in last section, we may conclude:

Theorem 2.10. *Pseudoconvex domain is equivalent to domain of holomorphy.*

3. Application to Cousin Problem

Given $\Omega \subset \mathbb{C}^n$, setting U_i to be a family of open covers for which the family of meromorphic functions $f_j \in \mathscr{M}(U_j)$, and $f_j - f_i \in A(U_i \cap U_j)$. If Ω is pseudoconvex, then Cousin Problem asks, do we have $f \in \mathscr{M}(\Omega)$, such that $f - f_i \in A(U_i)$?

Theorem 3.1. *Ω is pseudoconvex, then Cousin Problem is solvable.*

Proof. If we can find $u_i \in A(U_i)$ such that on $U_i \cap U_j$, $f_j - f_i = u_j - u_i$, then put $f|_{U_i} = f_i - u_i$, then on $U_i \cap U_j$ one has

$$f = f_j - u_j = f_i - u_i$$

this gives the explicit construction.

Now set $f_{ij} = f_j - f_i \in A(U_i \cap U_j)$, from the definition

$$f_{ij} + f_{ji} = 0; \ (f_{ij} + f_{jk} + f_{ki})_{U_i \cap U_j \cap U_k} = 0$$

Set ρ_i to be a partition of unity subcoordinate to the open cover U_i, we can define $g_j = \sum_i \rho_i f_{ij}$, and then $g_j - g_i = \sum_k \rho_k f_{kj} - \sum_k \rho_k f_{ki} = \sum_k \rho_k f_{ij} = f_{ij}$, so on U we have $0 = \bar{\partial} f_{ij} = \bar{}_j - \bar{\partial} g_i$, and take

$$w = \bar{\partial} g_j = \bar{\partial} g_i$$

w is in fact $\bar{\partial}$ closed, and it is $(0,1)$ form. As U is pseudoconvex, we have solvable $\bar{\partial} u = w$. Define $u_i = g_i - u$, u_i is analytic as $\bar{\partial} u_i = \bar{\partial} g_i - \bar{\partial} u = w - w = 0$, and $u_j - u_i = g_j - g_i = f_{ij} = f_j - f_i$. $\square$

However, if we ask, instead of meromorphic functions(blows up at some points), not the points they blow up, rather the points where they vanish, could we still have the same result?

In $\mathbb{C}$, the answer is confirmative, as Weiersterass product can do the job as the zeroes of holomorphic functions are discrete in $\mathbb{C}$. When the dimension goes up, the closed set for which holomorphic function vanishes are not discrete points, but hyperplane.

Therefore Cousin Problem (II) asks, given $\Omega \subset \mathbb{C}^n$, setting U_i to be a family of open covers for which the family of meromorphic functions $f_j \in A(U_j)$, such that $\frac{f_i}{f_j}$ does not vanish on $U_i \cap U_j$, is there $f \in A(\Omega)$ such that $\frac{f_i}{f}$ does not vanish on U_i?

Essentially we need Ω to be pseudoconvex and some extra condition.

Theorem 3.2. *Ω is pseudoconvex, and there is $g \in C^\infty(\Omega)$, such that $\frac{f_i}{g}$ does not vanish in U_i, then Cousin (II) is solvable.*

Proof. Suppose U_i is Euclidean convex, for example, we can take refinements of original U_i. Set $f_{ij} = \frac{f_j}{f_i} = \frac{f_j/g}{f_i/g}$, then by the assumption, we can define $\log f_{ij} = \log f_j/g - \log f_i/g$, with appropriate choice of $\arg z$, on $U_i \cap U_j$, one see that

$$0 = \bar{\partial} \log f_{ij} = \bar{\partial} \log f_j/g - \bar{\partial} \log f_i/g$$

Define on U_i, $w = \bar{\partial} \log f_i/g$, above says w is $(0,1)$ form globally and $\bar{\partial} w = 0$.

By Ω is pseudoconvex, there is u such that $\bar{\partial} u = w$, if we set $v_i = \log \frac{f_i}{g} - u$, or $\bar{\partial v_i} = w - \bar{\partial u} = 0$, so v_i holomorphic on U_i. As $0 = \bar{\partial} \log f_{ij} = \bar{\partial} \log f_j/g - \bar{\partial} \log f_i/g$, so $f_{ij} = \frac{e^{v_j}}{e^{v_i}} = \frac{f_j}{f_i}$. On $U_i \cap U_j$, $\frac{f_i}{e^{v_i}} = \frac{f_j}{e^{v_j}}$. Take $f = \frac{f_i}{e^{v_i}}$, then we can define this globally by the identity before. Also $f - f_i$ is nonzero and holomorphic, this concludes the proof. $\square$

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