ST552, Homework 4

Due Wednesday, Oct 9, 2013

1. Fill out the missing steps in our derivation of the best (minimum variance) affine unbiased estimator (MVAUE) for the Aiken model: $\mathrm{E}(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{b}$, $\mathrm{Cov}(\boldsymbol{y}) = \sigma^2\boldsymbol{V}$, assuming $\boldsymbol{V}$ is positive definite.

   (a) Show that if $\boldsymbol{V}$ is positive definite, then $\boldsymbol{V}^{-1}$ is positive definite.

   (b) If $\boldsymbol{V}$ is positive definite, show that $\mathcal{C}(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X}) = \mathcal{C}(\boldsymbol{X})$.

   (c) Prove that the minimum trace affine unbiased estimator (MTAUE) is unique by showing that

   $$\boldsymbol{A} = \boldsymbol{\Lambda}(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^T\boldsymbol{V}^{-1}$$

   is invariant to the choice of the generalized inverse $(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X})^{-}$. (Hint: you may use the fact there exists a positive definite matrix $\boldsymbol{V}^{1/2}$ such that $\boldsymbol{V} = \boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2}$.)

2. Fill out the missing steps in our derivation of the best (minimum variance) affine unbiased estimator (MVAUE) for the Aiken model with linear constraints $\boldsymbol{R}\boldsymbol{b} = \boldsymbol{r}$. We assume $\boldsymbol{V} \succ \boldsymbol{0}_{n\times n}$ and let $\boldsymbol{G} = \boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X} + \boldsymbol{R}^T\boldsymbol{R}$.

   (a) Show that $\boldsymbol{G}$ is positive semidefinite and $\mathcal{C}(\boldsymbol{G}) \supset \mathcal{C}(\boldsymbol{X}^T) \cup \mathcal{C}(\boldsymbol{R}^T)$.

   (b) Show that $\mathcal{C}(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T) \supset \mathcal{C}(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{X}^T)$ and thus $\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T)^{-}\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda} = \boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}$.

   (c) Check that the variance of the MTAUE is

   $$\mathrm{Cov}(\widehat{\boldsymbol{\Lambda}\boldsymbol{b}}) = \sigma^2\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^T - \sigma^2\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T)^{-}\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^T.$$

   (d) Show that the MTAUE is also the MVAUE.

3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices of same number of rows. Show that the following statements are equivalent. (Hint: I followed the route $(b) \to (a) \to (h), (i) \to (d), (f) \to (c), (e) \to (g) \to (b)$.)

   (a) $\mathcal{C}(\boldsymbol{A}) \cap \mathcal{C}(\boldsymbol{B}) = \{\boldsymbol{0}\}$

   (b) $\mathrm{rank}(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T) = \mathrm{rank}(\boldsymbol{A}) + \mathrm{rank}(\boldsymbol{B})$

   (c) $\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{A}$ is idempotent

   (d) $\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{A} = \boldsymbol{A}^{-}\boldsymbol{A}$

   (e) $\boldsymbol{B}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{B}$ is idempotent

   (f) $\boldsymbol{B}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{B} = \boldsymbol{B}^{-}\boldsymbol{B}$

   (g) $\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{B} = \boldsymbol{0}$

   (h) $(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}$ is a generalized inverse of $\boldsymbol{A}\boldsymbol{A}^T$

   (i) $(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}$ is a generalized inverse of $\boldsymbol{B}\boldsymbol{B}^T$

4. In class we showed that the MVAUE for Aitken model (non-singular $\boldsymbol{V}$) with linear constraints $\boldsymbol{R}\boldsymbol{b} = \boldsymbol{r}$ is

$$\widehat{\boldsymbol{\Lambda}\boldsymbol{b}} = \boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{X}^{T}\boldsymbol{V}^{-1}\boldsymbol{y} + \boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{R}^{T}(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^{T})^{-}(\boldsymbol{r} - \boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{X}^{T}\boldsymbol{V}^{-1}\boldsymbol{y})$$

where $\boldsymbol{G} = \boldsymbol{X}^{T}\boldsymbol{V}^{-1}\boldsymbol{X} + \boldsymbol{R}^{T}\boldsymbol{R}$, and has variance

$$\mathrm{Cov}(\widehat{\boldsymbol{\Lambda}\boldsymbol{b}}) = \sigma^{2}\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^{T} - \sigma^{2}\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{R}^{T}(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^{T})^{-}\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^{T}.$$

Show that in the special case $\mathcal{C}(\boldsymbol{X}^{T}) \cap \mathcal{C}(\boldsymbol{R}^{T}) = \{\boldsymbol{0}\}$, the MVAUE simplifies to

$$\widehat{\boldsymbol{\Lambda}\boldsymbol{b}} = \boldsymbol{\Lambda}\boldsymbol{G}^{-}(\boldsymbol{X}^{T}\boldsymbol{V}^{-1}\boldsymbol{y} + \boldsymbol{R}^{T}\boldsymbol{r}),$$

with variance matrix

$$\mathrm{Cov}(\widehat{\boldsymbol{\Lambda}\boldsymbol{b}}) = \sigma^{2}\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^{T} - \sigma^{2}\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{R}^{T}\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^{T}.$$

(Hint: Use Q3 with $\boldsymbol{A} = \boldsymbol{X}^{T}\boldsymbol{V}^{-1/2}$ and $\boldsymbol{B} = \boldsymbol{R}^{T}$.)

5. (Generalized inverse of bordered Gramian matrix) Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a positive semidefinite matrix and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$. Let $\boldsymbol{N} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{B}^{T}$ and $\boldsymbol{C} = \boldsymbol{B}^{T}\boldsymbol{N}^{-}\boldsymbol{B}$. Show the following facts.

   (a) $\mathcal{C}(\boldsymbol{A}) \subset \mathcal{C}(\boldsymbol{N})$, $\mathcal{C}(\boldsymbol{B}) \subset \mathcal{C}(\boldsymbol{N})$, $\mathcal{C}(\boldsymbol{N}) = \mathcal{C}((\boldsymbol{A}, \boldsymbol{B}))$

   (b) $\boldsymbol{N}\boldsymbol{N}^{-}\boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{N}\boldsymbol{N}^{-}\boldsymbol{B} = \boldsymbol{B}$

   (c) $\mathcal{C}(\boldsymbol{B}^{T}) = \mathcal{C}(\boldsymbol{C})$, $\mathrm{rank}(\boldsymbol{B}) = \mathrm{rank}(\boldsymbol{C})$

   (d) A generalized inverse of the *bordered Gramian matrix*

   $$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^{T} & \boldsymbol{0} \end{pmatrix}$$

   is

   $$\boldsymbol{Z}^{-} = \begin{pmatrix} \boldsymbol{N}^{-} - \boldsymbol{N}^{-}\boldsymbol{B}\boldsymbol{C}^{-}\boldsymbol{B}^{T}\boldsymbol{N}^{-} & \boldsymbol{N}^{-}\boldsymbol{B}\boldsymbol{C}^{-} \\ \boldsymbol{C}^{-}\boldsymbol{B}^{T}\boldsymbol{N}^{-} & -\boldsymbol{C}^{-} + \boldsymbol{C}\boldsymbol{C}^{-} \end{pmatrix},$$

   where $\boldsymbol{N} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{B}^{T}$ and $\boldsymbol{C} = \boldsymbol{B}^{T}\boldsymbol{N}^{-}\boldsymbol{B}$, and

   $$\boldsymbol{Z}\boldsymbol{Z}^{-} = \begin{pmatrix} \boldsymbol{N}\boldsymbol{N}^{-} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}\boldsymbol{C}^{-} \end{pmatrix}.$$

   (e) In the special case, $\mathcal{C}(\boldsymbol{B}) \subset \mathcal{C}(\boldsymbol{A})$, the generalized inverse in (d) takes a simpler form

   $$\boldsymbol{Z}^{-} = \begin{pmatrix} \boldsymbol{A}^{-} - \boldsymbol{A}^{-}\boldsymbol{B}\boldsymbol{\Lambda}^{-}\boldsymbol{B}^{T}\boldsymbol{A}^{-} & \boldsymbol{A}^{-}\boldsymbol{B}\boldsymbol{\Lambda}^{-} \\ \boldsymbol{\Lambda}^{-}\boldsymbol{B}^{T}\boldsymbol{A}^{-} & -\boldsymbol{\Lambda}^{-} \end{pmatrix},$$

   where $\boldsymbol{\Lambda} = \boldsymbol{B}^{T}\boldsymbol{A}^{-}\boldsymbol{B}$, and

   $$\boldsymbol{Z}\boldsymbol{Z}^{-} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{A}^{-} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}\boldsymbol{B}^{-} \end{pmatrix}.$$

   (Hint: it's easier to verify directly instead of deriving from (d).)