

## ST552, Homework 4

Due Wednesday, Oct 9, 2013

1. Fill out the missing steps in our derivation of the best (minimum variance) affine unbiased estimator (MVAUE) for the Aiken model:  $E(\mathbf{y}) = \mathbf{X}\mathbf{b}$ ,  $\text{Cov}(\mathbf{y}) = \sigma^2\mathbf{V}$ , assuming  $\mathbf{V}$  is positive definite.

- (a) Show that if  $\mathbf{V}$  is positive definite, then  $\mathbf{V}^{-1}$  is positive definite.
- (b) If  $\mathbf{V}$  is positive definite, show that  $\mathcal{C}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X}) = \mathcal{C}(\mathbf{X})$ .
- (c) Prove that the minimum trace affine unbiased estimator (MTAUE) is unique by showing that

$$\mathbf{A} = \mathbf{\Lambda}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}^T\mathbf{V}^{-1}$$

is invariant to the choice of the generalized inverse  $(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-}$ . (Hint: you may use the fact there exists a positive definite matrix  $\mathbf{V}^{1/2}$  such that  $\mathbf{V} = \mathbf{V}^{1/2}\mathbf{V}^{1/2}$ .)

2. Fill out the missing steps in our derivation of the best (minimum variance) affine unbiased estimator (MVAUE) for the Aiken model with linear constraints  $\mathbf{R}\mathbf{b} = \mathbf{r}$ . We assume  $\mathbf{V} \succ \mathbf{0}_{n \times n}$  and let  $\mathbf{G} = \mathbf{X}^T\mathbf{V}^{-1}\mathbf{X} + \mathbf{R}^T\mathbf{R}$ .

- (a) Show that  $\mathbf{G}$  is positive semidefinite and  $\mathcal{C}(\mathbf{G}) \supset \mathcal{C}(\mathbf{X}^T) \cup \mathcal{C}(\mathbf{R}^T)$ .
- (b) Show that  $\mathcal{C}(\mathbf{R}\mathbf{G}^{-}\mathbf{R}^T) \supset \mathcal{C}(\mathbf{R}\mathbf{G}^{-}\mathbf{X}^T)$  and thus  $\mathbf{R}\mathbf{G}^{-}\mathbf{R}^T(\mathbf{R}\mathbf{G}^{-}\mathbf{R}^T)^{-}\mathbf{R}\mathbf{G}^{-}\mathbf{\Lambda} = \mathbf{R}\mathbf{G}^{-}\mathbf{\Lambda}$ .
- (c) Check that the variance of the MTAUE is

$$\text{Cov}(\widehat{\mathbf{\Lambda}\mathbf{b}}) = \sigma^2\mathbf{\Lambda}\mathbf{G}^{-}\mathbf{\Lambda}^T - \sigma^2\mathbf{\Lambda}\mathbf{G}^{-}\mathbf{R}^T(\mathbf{R}\mathbf{G}^{-}\mathbf{R}^T)^{-}\mathbf{R}\mathbf{G}^{-}\mathbf{\Lambda}^T.$$

- (d) Show that the MTAUE is also the MVAUE.

3. Let  $\mathbf{A}$  and  $\mathbf{B}$  be two matrices of same number of rows. Show that the following statements are equivalent. (Hint: I followed the route  $(b) \rightarrow (a) \rightarrow (h), (i) \rightarrow (d), (f) \rightarrow (c), (e) \rightarrow (g) \rightarrow (b)$ .)

- (a)  $\mathcal{C}(\mathbf{A}) \cap \mathcal{C}(\mathbf{B}) = \{\mathbf{0}\}$
- (b)  $\text{rank}(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T) = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$
- (c)  $\mathbf{A}^T(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}\mathbf{A}$  is idempotent
- (d)  $\mathbf{A}^T(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}\mathbf{A} = \mathbf{A}^{-}\mathbf{A}$
- (e)  $\mathbf{B}^T(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}\mathbf{B}$  is idempotent
- (f)  $\mathbf{B}^T(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}\mathbf{B} = \mathbf{B}^{-}\mathbf{B}$
- (g)  $\mathbf{A}^T(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}\mathbf{B} = \mathbf{0}$
- (h)  $(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}$  is a generalized inverse of  $\mathbf{A}\mathbf{A}^T$
- (i)  $(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)^{-}$  is a generalized inverse of  $\mathbf{B}\mathbf{B}^T$

4. In class we showed that the MVAUE for Aitken model (non-singular  $\mathbf{V}$ ) with linear constraints  $\mathbf{R}\mathbf{b} = \mathbf{r}$  is

$$\widehat{\mathbf{b}} = \mathbf{\Lambda}\mathbf{G}^{-}\mathbf{X}^T\mathbf{V}^{-1}\mathbf{y} + \mathbf{\Lambda}\mathbf{G}^{-}\mathbf{R}^T(\mathbf{R}\mathbf{G}^{-}\mathbf{R}^T)^{-}(\mathbf{r} - \mathbf{R}\mathbf{G}^{-}\mathbf{X}^T\mathbf{V}^{-1}\mathbf{y})$$

where  $\mathbf{G} = \mathbf{X}^T\mathbf{V}^{-1}\mathbf{X} + \mathbf{R}^T\mathbf{R}$ , and has variance

$$\text{Cov}(\widehat{\mathbf{b}}) = \sigma^2\mathbf{\Lambda}\mathbf{G}^{-}\mathbf{\Lambda}^T - \sigma^2\mathbf{\Lambda}\mathbf{G}^{-}\mathbf{R}^T(\mathbf{R}\mathbf{G}^{-}\mathbf{R}^T)^{-}\mathbf{R}\mathbf{G}^{-}\mathbf{\Lambda}^T.$$

Show that in the special case  $\mathcal{C}(\mathbf{X}^T) \cap \mathcal{C}(\mathbf{R}^T) = \{\mathbf{0}\}$ , the MVAUE simplifies to

$$\widehat{\mathbf{b}} = \mathbf{\Lambda}\mathbf{G}^{-}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{y} + \mathbf{R}^T\mathbf{r}),$$

with variance matrix

$$\text{Cov}(\widehat{\mathbf{b}}) = \sigma^2\mathbf{\Lambda}\mathbf{G}^{-}\mathbf{\Lambda}^T - \sigma^2\mathbf{\Lambda}\mathbf{G}^{-}\mathbf{R}^T\mathbf{R}\mathbf{G}^{-}\mathbf{\Lambda}^T.$$

(Hint: Use Q3 with  $\mathbf{A} = \mathbf{X}^T\mathbf{V}^{-1/2}$  and  $\mathbf{B} = \mathbf{R}^T$ .)

5. (Generalized inverse of bordered Gramian matrix) Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be a positive semidefinite matrix and  $\mathbf{B} \in \mathbb{R}^{n \times m}$ . Let  $\mathbf{N} = \mathbf{A} + \mathbf{B}\mathbf{B}^T$  and  $\mathbf{C} = \mathbf{B}^T\mathbf{N}^{-}\mathbf{B}$ . Show the following facts.

- (a)  $\mathcal{C}(\mathbf{A}) \subset \mathcal{C}(\mathbf{N})$ ,  $\mathcal{C}(\mathbf{B}) \subset \mathcal{C}(\mathbf{N})$ ,  $\mathcal{C}(\mathbf{N}) = \mathcal{C}((\mathbf{A}, \mathbf{B}))$
- (b)  $\mathbf{N}\mathbf{N}^{-}\mathbf{A} = \mathbf{A}$ ,  $\mathbf{N}\mathbf{N}^{-}\mathbf{B} = \mathbf{B}$
- (c)  $\mathcal{C}(\mathbf{B}^T) = \mathcal{C}(\mathbf{C})$ ,  $\text{rank}(\mathbf{B}) = \text{rank}(\mathbf{C})$
- (d) A generalized inverse of the *bordered Gramian matrix*

$$\mathbf{Z} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{pmatrix}$$

is

$$\mathbf{Z}^{-} = \begin{pmatrix} \mathbf{N}^{-} - \mathbf{N}^{-}\mathbf{B}\mathbf{C}^{-}\mathbf{B}^T\mathbf{N}^{-} & \mathbf{N}^{-}\mathbf{B}\mathbf{C}^{-} \\ \mathbf{C}^{-}\mathbf{B}^T\mathbf{N}^{-} & -\mathbf{C}^{-} + \mathbf{C}\mathbf{C}^{-} \end{pmatrix},$$

where  $\mathbf{N} = \mathbf{A} + \mathbf{B}\mathbf{B}^T$  and  $\mathbf{C} = \mathbf{B}^T\mathbf{N}^{-}\mathbf{B}$ , and

$$\mathbf{Z}\mathbf{Z}^{-} = \begin{pmatrix} \mathbf{N}\mathbf{N}^{-} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\mathbf{C}^{-} \end{pmatrix}.$$

- (e) In the special case,  $\mathcal{C}(\mathbf{B}) \subset \mathcal{C}(\mathbf{A})$ , the generalized inverse in (d) takes a simpler form

$$\mathbf{Z}^{-} = \begin{pmatrix} \mathbf{A}^{-} - \mathbf{A}^{-}\mathbf{B}\mathbf{\Lambda}^{-}\mathbf{B}^T\mathbf{A}^{-} & \mathbf{A}^{-}\mathbf{B}\mathbf{\Lambda}^{-} \\ \mathbf{\Lambda}^{-}\mathbf{B}\mathbf{A}^{-} & -\mathbf{\Lambda}^{-} \end{pmatrix},$$

where  $\mathbf{\Lambda} = \mathbf{B}^T\mathbf{A}^{-}\mathbf{B}$ , and

$$\mathbf{Z}\mathbf{Z}^{-} = \begin{pmatrix} \mathbf{A}\mathbf{A}^{-} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}\mathbf{B}^{-} \end{pmatrix}.$$

(Hint: it's easier to verify directly instead of deriving from (d).)