ST758, Homework 1

Due Thursday, Sep 12, 2013

1. Show the following facts about the *Gramian* matrix $\boldsymbol{A}^\intercal \boldsymbol{A}$.

   (a) $\boldsymbol{A}^\intercal \boldsymbol{A}$ is symmetric and positive semidefinite.

   (b) $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^\intercal) = \text{rank}(\boldsymbol{A}^\intercal \boldsymbol{A}) = \text{rank}(\boldsymbol{A}\boldsymbol{A}^\intercal)$.

   (c) $\boldsymbol{A}^\intercal \boldsymbol{A} = \boldsymbol{0}$ if and only if $\boldsymbol{A} = \boldsymbol{0}$.

   (d) $\boldsymbol{B}\boldsymbol{A}^\intercal \boldsymbol{A} = \boldsymbol{C}\boldsymbol{A}^\intercal \boldsymbol{A}$ if and only if $\boldsymbol{B}\boldsymbol{A}^\intercal = \boldsymbol{C}\boldsymbol{A}^\intercal$.

   (e) $\boldsymbol{A}^\intercal \boldsymbol{A}\boldsymbol{B} = \boldsymbol{A}^\intercal \boldsymbol{A}\boldsymbol{C}$ if and only if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{C}$.

   (f) For any generalized inverse $(\boldsymbol{A}^\intercal \boldsymbol{A})^-$, $[(\boldsymbol{A}^\intercal \boldsymbol{A})^-]^\intercal$ is also a generalized inverse of $\boldsymbol{A}^\intercal \boldsymbol{A}$. Note $(\boldsymbol{A}^\intercal \boldsymbol{A})^-$ is not necessarily symmetric.

   (g) $(\boldsymbol{A}^\intercal \boldsymbol{A})^- \boldsymbol{A}^\intercal$ is a generalized inverse of $\boldsymbol{A}$.

   (h) $\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}(\boldsymbol{A}^\intercal \boldsymbol{A})^- \boldsymbol{A}^\intercal$, where $\boldsymbol{A}^+$ is the Moore-Penrose inverse of $\boldsymbol{A}$.

   (i) Let $\boldsymbol{P}_{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{A}^\intercal \boldsymbol{A})^- \boldsymbol{A}^\intercal$. Show that $\boldsymbol{P}_{\boldsymbol{A}}$ is symmetric, idempotent, invariant to the choice of generalized inverse $(\boldsymbol{A}^\intercal \boldsymbol{A})^-$, and projects onto $\mathcal{C}(\boldsymbol{A})$.

2. (a) Show the Sherman-Morrison formula

$$(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{u}^\intercal)^{-1} = \boldsymbol{A}^{-1} - \frac{1}{1 + \boldsymbol{u}^\intercal \boldsymbol{A}^{-1}\boldsymbol{u}} \boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{u}^\intercal \boldsymbol{A}^{-1},$$

   where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix and $\boldsymbol{u} \in \mathbb{R}^n$. This formula supplies the inverse of the symmetric, rank-one perturbation of $\boldsymbol{A}$.

   (b) Show the Woodbury formula

$$(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^\intercal)^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{U}(\boldsymbol{I}_m + \boldsymbol{V}^\intercal \boldsymbol{A}^{-1}\boldsymbol{U})^{-1}\boldsymbol{V}^\intercal \boldsymbol{A}^{-1},$$

   where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is nonsingular, $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{n \times m}$, and $\boldsymbol{I}_m$ is the $m \times m$ identity matrix. In many applications m is much smaller than n. Woodbury formula generalizes Sherman-Morrison and is valuable because the smaller matrix $\boldsymbol{I}_m + \boldsymbol{V}^\intercal \boldsymbol{A}^{-1}\boldsymbol{U}$ is typically much easier to invert than the larger matrix $\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^\intercal$.

   (c) Show the binomial inversion formula

$$(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{B}\boldsymbol{V}^\intercal)^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{B}^{-1}(\boldsymbol{B}^{-1} + \boldsymbol{V}^\intercal \boldsymbol{A}^{-1}\boldsymbol{U})^{-1}\boldsymbol{B}\boldsymbol{V}^\intercal \boldsymbol{A}^{-1},$$

   where $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular.

   (d) Show the identity

$$\det(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^\intercal) = \det(\boldsymbol{A})\det(\boldsymbol{I}_m + \boldsymbol{V}^\intercal \boldsymbol{A}^{-1}\boldsymbol{U}).$$

   This formula is useful for evaluating the density of a multivariate normal with covariance matrix $\boldsymbol{A} + \boldsymbol{U}\boldsymbol{U}^\intercal$.

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3. Let $\boldsymbol{u} = (1, 2, 3, 3, 2, 1)^\intercal$.

   (a) Compute $\boldsymbol{U} = \boldsymbol{I} - (2/d)\boldsymbol{u}\boldsymbol{u}^\intercal$ where $d = \boldsymbol{u}^\intercal\boldsymbol{u}$. (This type of matrix is known as an 'elementary reflector' or a 'Householder transformation.')

   (b) Let $\boldsymbol{C} = \boldsymbol{U}\boldsymbol{U}$, the matrix product of $\boldsymbol{U}$ and itself. Find the largest and smallest off-diagonal elements of $\boldsymbol{C}$.

   (c) Find the largest and smallest diagonal elements of $\boldsymbol{C}$.

   (d) Compute $\boldsymbol{U}\boldsymbol{u}$. (matrix times vector)

   (e) Compute the scalar $\max_i \sum_j |U(i,j)|$

   (f) Print the second row of $\boldsymbol{U}$.

   (g) Print the elements of the third column below the diagonal.

   (h) Let $\boldsymbol{A}$ be the first three columns of $\boldsymbol{U}$. Compute $\boldsymbol{P} = \boldsymbol{A}\boldsymbol{A}^\intercal$.

   (i) Show that $\boldsymbol{P}$ is idempotent by recomputing (e) with $\boldsymbol{P}\boldsymbol{P} - \boldsymbol{P}$.

   (j) Let $\boldsymbol{B}$ be the last three columns of $\boldsymbol{U}$. Compute $\boldsymbol{Q} = \boldsymbol{B}\boldsymbol{B}^\intercal$.

   (k) Show that $\boldsymbol{Q}$ is idempotent by recomputing (e) with $\boldsymbol{Q}\boldsymbol{Q} - \boldsymbol{Q}$.

   (l) Compute $\boldsymbol{P} + \boldsymbol{Q}$.

4. Read in the matrix in the file 'hoium.dat' on the verification rate of NWS warnings. The columns are county, number of warnings, number verified, fraction verified, population density, county area, and per capita income. Compute the correlation between number of warnings and the area, and the correlation between fraction verified and per capita income.

5. Let the $n \times n$ matrix $\boldsymbol{A}$ have elements $A(i,j) = r^{|i-j|}$; use $r = 0.90$.

   (a) Compute and print $\boldsymbol{A}$ for $n = 7$.

   (b) Compute and print the Cholesky factorization for $\boldsymbol{A}$ for $n = 7$.

   (c) Compute the Cholesky factorization for $n = 20$. Does it fail? If not, find the determinant.