ST758, Homework 2

Due Tuesday, Sep 23, 2014

1. Show the following facts about triangular matrices. A unit triangular matrix is a triangular matrix with all diagonal entries being 1.

   (a) The product of two upper (lower) triangular matrices is upper (lower) triangular.

   (b) The inverse of an upper (lower) triangular matrix is upper (lower) triangular.

   (c) The product of two unit upper (lower) triangular matrices is unit upper (lower) triangular.

   (d) The inverse of a unit upper (lower) triangular matrix is unit upper (lower) triangular.

   (e) An orthogonal upper (lower) triangular matrix is diagonal.

2. Suppose symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has entries $a_{ij} = i(n - j + 1)$ and $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ has entries $b_{ij} = \sum_{k=1}^{i} \sigma_k^2$ for $j \geq i$ and $\sigma_k^2 \geq 0$. Show that $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite.

3. (a) Show the Sherman-Morrison formula

$$(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}})^{-1} = \boldsymbol{A}^{-1} - \frac{1}{1 + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}^{-1}\boldsymbol{u}} \boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}^{-1},$$

   where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix and $\boldsymbol{u} \in \mathbb{R}^n$. This formula supplies the inverse of the symmetric, rank-one perturbation of $\boldsymbol{A}$.

   (b) Show the Woodbury formula

$$(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^{\mathsf{T}})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{U}(\boldsymbol{I}_m + \boldsymbol{V}^{\mathsf{T}}\boldsymbol{A}^{-1}\boldsymbol{U})^{-1}\boldsymbol{V}^{\mathsf{T}}\boldsymbol{A}^{-1},$$

   where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is nonsingular, $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{n \times m}$, and $\boldsymbol{I}_m$ is the $m \times m$ identity matrix. In many applications m is much smaller than n. Woodbury formula generalizes Sherman-Morrison and is valuable because the smaller matrix $\boldsymbol{I}_m + \boldsymbol{V}^{\mathsf{T}}\boldsymbol{A}^{-1}\boldsymbol{U}$ is typically much easier to invert than the larger matrix $\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^{\mathsf{T}}$.

   (c) Show the binomial inversion formula

$$(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{B}\boldsymbol{V}^{\mathsf{T}})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{U}(\boldsymbol{B}^{-1} + \boldsymbol{V}^{\mathsf{T}}\boldsymbol{A}^{-1}\boldsymbol{U})^{-1}\boldsymbol{V}^{\mathsf{T}}\boldsymbol{A}^{-1},$$

   where $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular.

   (d) Show the identity

$$\det(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^{\mathsf{T}}) = \det(\boldsymbol{A})\det(\boldsymbol{I}_m + \boldsymbol{V}^{\mathsf{T}}\boldsymbol{A}^{-1}\boldsymbol{U}).$$

   This formula is useful for evaluating the density of a multivariate normal with covariance matrix $\boldsymbol{A} + \boldsymbol{U}\boldsymbol{U}^{\mathsf{T}}$.

   (e) Consider the $n \times n$ matrix

$$\boldsymbol{M} = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \vdots & \vdots \\ b & b & \cdots & a \end{pmatrix}.$$

Show that $\boldsymbol{M}$ has inverse and determinant

$$
\begin{aligned}
\boldsymbol{M}^{-1} &= \frac{1}{a-b}\left[\boldsymbol{I}_n - \frac{b}{a+(n-1)b}\mathbf{1}_n\mathbf{1}_n^T\right] \\
\det(\boldsymbol{M}) &= (a-b)^{n-1}[a+(n-1)b].
\end{aligned}
$$

4. Write an R function, with interface `solve.power(A, k, b)`, to solve linear equation $\boldsymbol{A}^k\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is non-singular and $\boldsymbol{b} \in \mathbb{R}^n$. Test your function in simulated data.

5. Consider a mixed effects model

$$
y_i = \boldsymbol{x}_i^t\boldsymbol{\beta} + \boldsymbol{z}_i^t\boldsymbol{\gamma} + \epsilon_i, \quad i = 1, \ldots, n,
$$

where ϵ_i are independent normal errors $N(0, \sigma_0^2)$, $\boldsymbol{\beta} \in \mathbb{R}^p$ are fixed effects, and $\boldsymbol{\gamma} \in \mathbb{R}^q$ are random effects assumed to be $N(\mathbf{0}_q, \sigma_1^2\boldsymbol{I}_q)$ independent of ϵ_i. Show that $\boldsymbol{y} \sim N(\boldsymbol{X\beta}, \sigma_0^2\boldsymbol{I}_n + \sigma_1^2\boldsymbol{ZZ}^t)$, where $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{X} \in \mathbb{R}^{n\times p}$, and $\boldsymbol{Z} \in \mathbb{R}^{n\times q}$. Write an R function, with interface $\texttt{dmvnorm.lowrank}(\texttt{y}, \texttt{mu}, \texttt{Z}, \texttt{sigma0}, \texttt{sigma1}, \texttt{log} = \texttt{FALSE})$, that evaluates the (log)-density of a multivariate normal with mean $\boldsymbol{\mu}$ and covariance $\sigma_0^2\boldsymbol{I} + \sigma_1^2\boldsymbol{ZZ}^t$ at $\boldsymbol{y}$. Test your function on simulated data.

6. In class we learnt about the BLAS and how it has become a de facto standard for basic linear algebra operations. R uses the BLAS and LAPACK libraries extensively to accelerate certain types of operations. Apart from `*`, `%*%`, `eigen`, and `qr`, what other common functions use the BLAS and LAPACK routines to speed up calculations?

   Find out if R uses an optimized BLAS on your system. If yes, what version is used? If not, find out if any optimized BLAS is available for your system.