exercise6

8.

Consider the following distribution

$$f(x|k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$$

a.

Knowing that this distribution gives the density of the sum of k i.i.d. exponential random variables with rate λ, use the moment generating function method to derive analytical expressions for its mean and variance

we know this is a gamma distribution from the question

the mgf can be calculated with

$$\begin{aligned}
M_x(t) &= \int_{-\infty}^{\infty} e^{tx} f(x|k,\lambda)dx \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx \\
&= \int_{0}^{\infty} e^{tx} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx && f(x|k,\lambda) \text{is only defined for} x \geq 0 \\
&= \int_{0}^{\infty} \frac{\lambda^k x^{k-1} e^{(t-\lambda)x}}{(k-1)!} dx \\
&= \frac{\lambda^k}{(k-1)!} \int_{0}^{\infty} x^{k-1} e^{(t-\lambda)x} dx
\end{aligned}$$

at this point, for clarity, we focus on the integral part

$$\begin{aligned}
\int_{0}^{\infty} x^{k-1} e^{(t-\lambda)x} dx &= \frac{1}{(t-\lambda)} \int_{0}^{\infty} x^{k-1} de^{(t-\lambda)x} \\
&= \frac{1}{(t-\lambda)} \left(x^{k-1} e^{(t-\lambda)x} \Big|_0^{\infty} - \int_{0}^{\infty} e^{(t-\lambda)x} dx^{k-1} \right) \\
&= \frac{k-1}{(t-\lambda)} \left(x^{k-1} e^{(t-\lambda)x} \Big|_0^{\infty} - \int_{0}^{\infty} e^{(t-\lambda)x} x^{k-2} dx \right)
\end{aligned}$$

to evaluate $x^{k-1} e^{(t-\lambda)x} \Big|_0^{\infty}$

we need to discuss in cases,

if $t-\lambda \geq 0$, $x^{k-1}e^{(t-\lambda)x}\Big|_0^\infty$ is obviously ∞, which make the whole mgf ∞ and meaningless, therefore we assume this case would never happen, in other words, we enforce $\lambda > t$

we rewrite the formula

$$x^{k-1}e^{(t-\lambda)x} = \frac{x^{k-1}}{e^{(\lambda-t)x}}$$

notice that the numerator and the denominator have similar characteristics in the limit, or

if $x \to 0, x^{k-1} \to 0 \quad (k \geq 1), e^{(\lambda-t)x} \to 0$

and if $x \to \infty, x^{k-1} \to \infty \quad (k \geq 1), e^{(\lambda-t)x} \to \infty \quad (\lambda > t)$

also notice that in the general case, this also applies for all $x^a, x > 0$ and also apply when x^a is multiplied by any constant, since $C * 0 = 0, C * \infty = \infty$

therefore, by recursively applying L'hopital rule, we get

$\lim_{x\to\infty} \frac{x^{k-1}}{e^{(k-t)x}} = \lim_{x\to\infty} \frac{(k-1)x^{k-2}}{(\lambda-t)e^{(\lambda-t)x}} = \cdots = \lim_{x\to\infty} C * \frac{1}{e^{(\lambda-t)x}} = 0$
and $\lim_{x\to 0} \frac{x^{k-1}}{e^{(\lambda-t)x}} = \frac{0}{1} = 0$

so $x^{k-1}e^{(t-\lambda)x}\Big|_0^\infty = 0$

back to the integral, this gives

$$\begin{aligned}\int_0^\infty x^{k-1}e^{(t-\lambda)x}dx &= \frac{k-1}{(t-\lambda)}\left(x^{k-1}e^{(t-\lambda)x}\Big|_0^\infty - \int_0^\infty e^{(t-\lambda)x}x^{k-2}dx\right) \\ &= \frac{k-1}{\lambda-t}\int_0^\infty e^{(t-\lambda)x}x^{k-2}dx \\ &= \text{this iterates}\end{aligned}$$

if we denote $I_{k-1} = \int_0^\infty x^{k-1}e^{(t-\lambda)x}dx$ and so forth, then $I_{k-1} = \frac{k-1}{\lambda-t}I_{k-2}$

and actually $I_n = \frac{n}{\lambda-t}I_{n-1}$

the final term $I_0 = \int_0^\infty e^{(t-\lambda)x}dx = \frac{1}{t-\lambda}e^{(t-\lambda)x}\Big|_0^\infty = \frac{1}{t-\lambda}e^u\Big|_{u=0}^{-\infty} = \frac{1}{\lambda-t}e^u\Big|_{u=-\infty}^{0} = \frac{1}{\lambda-t}$

so the integral can be evaluated as

$$\int_0^\infty x^{k-1}e^{(t-\lambda)x}dx = \frac{1}{(t-\lambda)}\int_0^\infty x^{k-1}de^{(t-\lambda)x} = \frac{1}{\lambda-t}\prod_{n=1}^{k-1}(\frac{n}{\lambda-t}) = \frac{(k-1)!}{(\lambda-t)^k}$$

therefore, finally,

$$M_x(t) = \frac{\lambda^k}{(k-1)!}\int_0^\infty x^{k-1}e^{(t-\lambda)x}dx = \frac{\lambda^k}{(k-1)!}\frac{(k-1)!}{(\lambda-t)^k} = \left(\frac{\lambda}{\lambda-t}\right)^k$$

therefore

$$\mu = \frac{dM_x(t)}{dt}\Big|_{t=0} = \frac{d}{dt}\Big(\frac{\lambda}{\lambda - t}\Big)^k\Big|_{t=0}$$
$$= k\lambda^k(\lambda - t)^{-(k+1)}\Big|_{t=0}$$
$$= \frac{k}{\lambda}$$

and

$$E[X^2] = \frac{d^2M_x(t)}{dt^2}\Big|_{t=0}$$
$$= \frac{d(k\lambda^k(\lambda - t)^{-(k+1)})}{dt}\Big|_{t=0}$$
$$= k(k+1)\lambda^k(\lambda - t)^{-(k+2)}\Big|_{t=0}$$
$$= \frac{k(k+1)}{\lambda^2}$$

therefore

$$Var(X) = E[X^2] - E[X]^2 = \frac{k(k+1)}{\lambda^2} - (\frac{k}{\lambda})^2 = \frac{k}{\lambda^2}$$

b

Assume in this part that you collected the following n = 10 samples from the distribution

(2.846,3.445,4.376,0.402,2.893,2.522,0.458,3.742,3.156,1.030)

Give unbiased estimates for the mean and variance of this distribution. Using the results of part a., obtain estimates for k and λ, ensuring that the obtained values are admissible for this distribution.

the mean value of this sample is

$$\frac{1}{n}\sum_{n=1}^{10} x_n$$
$$= \frac{2.846 + 3.445 + 4.376 + 0.402 + 2.893 + 2.522 + 0.458 + 3.742 + 3.156 + 1.030}{10}$$
$$= \frac{24.87}{10} = 2.487$$

the variance of this sample is

$$\sigma^2 = \frac{1}{n-1}\sum_{n=1}^{10}(x_n-\mu)^2$$

$$\begin{aligned}
=&\frac{1}{9}\Big((2.846-2.487)^2+(3.445-2.487)^2+\\
&(4.376-2.487)^2+(0.402-2.487)^2+\\
&(2.893-2.487)^2+(2.522-2.487)^2+\\
&(0.458-2.487)^2+(3.742-2.487)^2+\\
&(3.156-2.487)^2+(1.030-2.487)^2\Big)\\
=&\frac{1}{9}\Big(0.359^2+0.958^2+1.889^2+(-2.085)^2\\
&0.406^2+0.035^2+(-2.029)^2+1.255^2\\
&0.669^2+(-1.457)^2\Big)\\
=&\frac{17.3905}{9}\approx 1.9323
\end{aligned}$$

therefore

$\mu = \frac{k}{\lambda} = 2.487$

$\sigma^2 = \frac{k}{\lambda^2} = 1.9323$

so $\lambda = \frac{\mu}{\sigma^2} \approx \frac{2.487}{1.9323} = 1.287$

$k = \mu\lambda = 2.487 * 1.287 \approx 3.201$

and since $\lambda > 0, k \geq 1$, this set of parameters is admissable

(c)

Assume in this part that k is known. Suppose now that you have collected n independent samples $(x_1, x_2, \ldots, x_n)$ from this distribution. Derive an analytical expression for the maximum likelihood estimate for λ based on these samples, expressing it as a function of the sample mean X and the known value of k

(to be consistent with the notation in the lectures, the log in the following proof is actually ln)

to get the MLE of this distribution, we first try to obtain the log-likelihood function of the distribution

$$
\begin{aligned}
\ell(\lambda) = \log L(\lambda) &= \log \prod_{i=1}^{n} f(x_i|\lambda, k) && \text{k is known} \\
&= \log \prod_{i=1}^{n} \frac{\lambda^k x_i^{k-1} e^{-\lambda x_i}}{(k-1)!} \\
&= nk \log \lambda + (k-1) \sum_{i=1}^{n} \log x_i + -\lambda \sum_{i=0}^{n} x_i \log e - n \log(k-1)!
\end{aligned}
$$

we would want to maximize, or calculate the value of λ that make the derivative 0

$\frac{d\ell(\lambda)}{\lambda} = \frac{nk}{\lambda} - \sum_{i=0}^{n} x_i$

therefore $\lambda = \frac{nk}{\sum_{i=0}^{n} x_i} = \frac{k}{\bar{X}}$