exercise4

5

(assessed)

Let X and Y be continuous random variables

(a)

Show that $P(X < Y + 1) = \int_{-\infty}^{\infty} P(X < y + 1|Y = y) f_Y(y) dy$

since y is a continuous random variable

following the total probability law

$$P(E) = \sum_i P(E|F_i)P(F_i)$$

therefore,

$$\begin{aligned} P(X < Y + 1) &= \sum P(X < y + 1|Y = y)P(Y = y) \qquad\qquad (\forall y \in (-\infty, \infty)) \\ &= \int_{-\infty}^{\infty} P(X < y + 1|Y = y)P(Y = y)dy \\ &= \int_{-\infty}^{\infty} P(X < y + 1|Y = y) f_Y(y) dy \end{aligned}$$

(b)

Hence or otherwise, if X and Y are $Exp(\lambda)$ and $N(0, 1)$ respectively and independent, give an expression for the probability that $X < Y + 1$. Your answer need to hold for any value of λ and show the details of the derivations

since $X \sim Exp(\lambda)$, $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

and since $Y \sim N(0, 1)$ $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

so (and since $X \sim Exp(\lambda)$ is only meaningful for $x \geq 0$, given that $Y = y$)

$$\begin{aligned}
P(X < y+1|Y = y) &= \int_{-\infty}^{y+1} f_X(x)dx \qquad \text{(as mentioned } Exp(\lambda)\text{is only nonzero when } x \geq 0) \\
&= \int_{0}^{y+1} f_X(x)dx \\
&= \int_{0}^{y+1} \lambda e^{-\lambda x}dx \\
&= -\int_{-\lambda x=0}^{-\lambda(y+1)} e^{\lambda x}d(-\lambda x) \\
&= -e^{-\lambda x}\Big|_{-\lambda x=0}^{-\lambda(y+1)} \\
&= 1 - e^{-\lambda(y+1)}
\end{aligned}$$

therefore, using the lemma we proved in (a)

and also for x to be greater than 0, the lower bound of y should be -1

$$
\begin{aligned}
P(X<Y+1) &= \int_{-\infty}^{\infty} P(X<y+1|Y=y) f_Y(y) dy \\
&= \int_{-1}^{\infty} P(X<y+1|Y=y) f_Y(y) dy \\
&= \int_{-1}^{\infty} (1-e^{-\lambda(y+1)})(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}) dy \\
&= \int_{-1}^{\infty} (\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}-\lambda(y+1)}) dy \\
&= \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}-\lambda(y+1)} dy
\end{aligned}
$$

$$
\begin{aligned}
&\text{(since the first term is in the form of a normal distribution pdf)} \\
&= \Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} - \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\Big) - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}-\lambda(y+1)} dy \\
&= \Big(1-\Phi(-1)\Big) - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}-\lambda(y+1)} dy \\
&\Big(\text{the integral from } -\infty \text{ to } \infty \text{ of a normal distribution pdf is 1} \\
&\quad \text{and} \\
&\int_{-\infty}^{-1} \phi(x) dx = \Phi(-1)\Big) \\
&= \Phi(1) - e^{-\lambda+\frac{\lambda^2}{2}} \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}-\lambda y-\frac{\lambda^2}{2}} dy \\
&\Big(\Phi(x) = 1-\Phi(-x) \text{ as in the lectures}\Big) \\
&= \Phi(1) - e^{-\lambda+\frac{\lambda^2}{2}} \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+\lambda)^2}{2}} dy \\
&= \Phi(1) - e^{-\lambda+\frac{\lambda^2}{2}} \int_{-1+\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&\text{(substituting } y+\lambda \text{with } z) \\
&= \Phi(1) - e^{-\lambda+\frac{\lambda^2}{2}} (\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - \int_{\infty}^{-1+\lambda} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz) \\
&= \Phi(1) - e^{-\lambda+\frac{\lambda^2}{2}} (1-\Phi(-1+\lambda)) \\
&\Big(\text{the integral from } -\infty \text{ to } \infty \text{ of a normal distribution pdf is 1} \\
&\quad \text{and} \\
&\int_{-\infty}^{-1+\lambda} \phi(x) dx = \Phi(-1+\lambda)\Big) \\
&= \Phi(1) - e^{-\lambda+\frac{\lambda^2}{2}} \Phi(1-\lambda) \\
&\Big(\Phi(x) = 1-\Phi(-x) \text{ as in the lectures}\Big)
\end{aligned}
$$

I don't think we should decompose any further, $\Phi(1)$ is a constant and $\Phi(1-\lambda)$ depends on λ, unrolling them gives another nasty expression with the normal distributions.

and also, the domain of Φ is $\mathbb{R}$ or $(-\infty, \infty)$, therefore, λ can take any values in $\mathbb{R}$

(c)

Assume in this part that $\lambda = 1$, i.e. X and Y+1 have the same mean. Show that this implies for the result obtained in part (b) that $P(X < Y + 1) = \Phi(1) - \frac{1}{2\sqrt{e}}$

since $\lambda = 1$

substituting this in the expression we obtained in (b) gives

$$\begin{aligned} P(X < Y + 1) &= \Phi(1) - e^{-1+\frac{1}{2}}\Phi(0) \\ &= \Phi(1) - \frac{1}{\sqrt{e}}\frac{1}{2} \\ &= \Phi(1) - \frac{1}{2\sqrt{e}} \end{aligned}$$