coursework1

Problem1:

(i)

Show that for any pair of vector $v, w \in \mathbb{R}^m$,we have $|v \bullet w| \leq ||v||_1||w||_\infty$

assume $v = [v_1, v_2, \dots, v_m]^T$ and $w = [w_1, w_2, \dots, w_m]^T$

by definition

$||v||_1 = \sum_{i=0}^{m} |v_i|$

$||w||_\infty = \max\{w_1, w_2, \dots, w_m\}$

$$\begin{aligned}
|v \bullet w| &= |v^T w| \\
&= |\sum_{i=0}^{m} v_i w_i| && \text{(by definition of dot product)} \\
&= \sum_{i=0}^{m} |v_i w_i| && \text{(since } \left|\sum a\right| = \sum |a|\text{)} \\
&\leq \sum_{i=0}^{m} |v_i||w_i| && \text{(by triangular inequality)} \\
&\leq \sum_{i=0}^{m} |v_i|\Big|||w||_\infty\Big| && \text{(by definition of } \mathrm{l}_\infty\text{)} \\
&= ||w||_\infty \sum_{i=0}^{m} |v_i| && \text{(since } ||w||_\infty \text{ doesnt involve i)} \\
&= ||w||_\infty ||v||_1 && \text{(by definition of } ||v||_i\text{)} \\
&= ||v||_1 ||w||_\infty && \text{(since both terms are scalar values)}
\end{aligned}$$

(ii)

As in the lecture notes, define the l_∞ matrix norm for $A \in \mathbb{R}^{m \times n}$ by

$$||A||_\infty := \max_{1 \leq i \leq m} ||a^i||_1$$

where a^i is the ith row of A. Show carefully using part (i) that

$$||A||_\infty = \max\{||Ax||_\infty : ||x||_\infty \leq 1\} = \max\{||Ax||_\infty : ||x||_\infty = 1\}$$

by the definition of $||A||_\infty$

we have

for any $A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n$

assume $A = [a^1, a^2, \dots, a^m], x = [x_1, x_2, \dots, x_n]^T$ where $\forall i \leq m, a_i \in \mathbb{R}^n$

$$||A||_\infty = \max_{1\leq i\leq m} ||a^i||_1$$

and

$$\begin{aligned} ||Ax||_\infty &= \max_{1\leq i\leq m} ||a^i x||_1 && \text{(by definition of the norm)} \\ &\leq \max_{1\leq i\leq m} ||a^i||_1 ||x||_\infty && \text{by part (i)} \end{aligned}$$

if at this point, we add the constraint of $||x||_\infty$, or $\Big|\Big|[x_1, x_2, \dots, x_n]\Big|\Big| \leq 1$ as required in $\max\{||Ax||_\infty : ||x||_\infty \leq 1\}$

then

$$\begin{aligned} ||Ax||_\infty &\leq \max_{1\leq i\leq m} ||a^i||_1 ||x||_\infty && \text{by part (i)} \\ &= \max_{1\leq i\leq m} ||a^i||_1 && \max ||x||_\infty = 1 \\ &= ||A||_\infty \end{aligned}$$

and also, since the max can only be obtained when $||x||_\infty = 1$

we proved that

$$\max\{||Ax||_\infty : ||x||_\infty \leq 1\} = \max\{||Ax||_\infty : ||x||_\infty = 1\}$$

in conclusion

$$||A||_\infty = \max\{||Ax||_\infty : ||x||_\infty \leq 1\} = \max\{||Ax||_\infty : ||x||_\infty = 1\}$$

Problem 2:

Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

A is in dimension 2×3 with $3 > 2$

therefore, we first calculate

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

then we find the eigenvectors of A^TA

$det(A^TA - \lambda) = 0$

$$det(\begin{bmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{bmatrix}) = 0$$

so $(13-\lambda)((13-\lambda)(8-\lambda)-4) - 12(12(8-\lambda)-(-4)) + 2(-24-2(13-\lambda)) = 0$
$(13-\lambda)(\lambda^2 - 21\lambda + 100) - 12(100 - 12\lambda) + 2(-50 + 2\lambda) = 0$
$(13\lambda^2 - 273\lambda + 1300) + (-\lambda^3 + 21\lambda^2 - 100\lambda) - 1200 + 144\lambda + (-100 + 4\lambda) = 0$
$-\lambda^3 + 34\lambda^2 - 255\lambda = 0$

$\lambda(\lambda - 9)(\lambda - 25) = 0$

therefore the eigenvalues are 25, 9, 0

(skipping eigenvector calculation)

and the eigenvectors are $v_1 = [1,1,0]^T, v_2 = [1,-1,4]^T, v_3 = [-2,2,1]^T$

after normalising

$v_1 = \frac{1}{\sqrt{2}}[1,1,0]^T, v_2 = \frac{1}{3\sqrt{2}}[1,-1,4]^T, v_3 = \frac{1}{3}[-2,2,1]$

therefore in the SVD expression of $A = USV^T$

$$V = [v_1, v_2, v_3] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

and each $u_i = \frac{1}{\sigma_i}Av_i$

therefore

- $u_1 = \frac{1}{5}Av_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$
- $u_2 = \frac{1}{3}Av_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$
- u_3 does not exist since the corresponding eigenvalue is 0

therefore $U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

and $S = diag(\sigma_1, \sigma_2) = diag(5,3) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

in conclusion the SVD representation is

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = A = USV^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Problem 3:

Show that for any matrix $A \in \mathbb{R}^{mn}$ the two matrices A^TA and A have the same nullspace, Deduce carefully that the three matrices A, A^TA, AA^T have the same rank

first we prove that A^TA and A have the same nullspace

1. $null(A^TA) \subseteq null(A)$
   for arbitrary vector $x \in \mathbb{R}^m$

   if $x \in null(A)$, or $Ax = 0$

   then $A^TAx = A^T(Ax) = A^T0 = 0$

   so $x \in null A^TA$

   therefore $null(A^TA) \subseteq null(A)$
2. $null(A) \subseteq null(A^TA)$
   for arbitrary vector $x \in \mathbb{R}^m$

   if $x \in null(A)$, or $A^TAx = 0$

   then $x^TA^TAx = 0$

   therefore $(Ax)^T(Ax) = 0$

   thus $Ax = 0$, or $x \in null(A)$

   so $null(A) \subseteq null(A^TA)$
   in conclusion $null(A) = null(A^TA)$

the in the exact similar way we can also prove $null(A) = null(A^TA)$

and since $null(M) + rank(M) = \text{cols of M}$ for any matrix M

$A \in \mathbb{R}^{m\times n}. A^TA \in \mathbb{R}^{n\times n}$

therefore, they have the same number of columns and the same rank

$rank(A) = rank(A^TA) = n - null(A)$

in addition using singular value decomposition

assume

$A = USV^T$

then $AA^T = USV^T(USV^T)^T = USV^TVS^TU^T$

and $A^TA = (USV^T)^TUSV^T = VS^TU^TUSV^T$

since V and U in singular value decomposition are orthonormal

therefore $UU^T = 0, VV^T = 0$

$AA^T = USS^TU^T, A^TA = VS^TSV^T$

in SVD, S is a diagonal matrix with the singular values on the diagonals, we assume S is shape $n \times m$, with $m \leq n$ (the other case is just the reverse)

then SS^T would give a $m \times m$ square matrix with the n eigenvalues on the diagonal and the rest 0

S^TS would give a $n \times n$square matrix with the n eigenvalues the fill the whole diagonal

or to be more specific $rank(SS^T) = rank(S^TS)$

also notice that $A^TA = (A^TA)^T$ therefore A^TA
is symmetric, the same for AA^T

therefore $AA^T = USS^TU^T$ and $A^TA = VS^TSV$ posses the form of the spectral theorem, so the eigenvalues of AA^T is the non-zero elements on the diagonal of SS^T and the eigenvalues of A^TA is the non-zero elements on the diagonal of S^TS

as mentioned before, the non-zero elements of SS^T and S^TS are the same, so AA^T and A^TA possess the same set of eigenvalues,

therefore $rank(AA^T) = rank(A^TA)$

in conclusion $rank(A) = rank(AA^T) = rank(A^TA)$

Problem 4:

which one of the following two matrices A and B does not have a Cholesky decomposition? Find the Cholesky decomposition of the other matrix

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}, B = \begin{bmatrix} 5 & 7 & 1 \\ 7 & 6 & 4 \\ 1 & 4 & 13 \end{bmatrix}$$

for a matrix to have a Cholesky decomposition, the matrix has to be semi-definite

therefore we test we a random vector x say $x = [1, -1, 0]$

$$\begin{aligned} x^TAx &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & -3 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 13 > 0 \end{aligned}$$

$$\begin{aligned} x^TAx &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 7 & 1 \\ 7 & 6 & 4 \\ 1 & 4 & 13 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -3 < 0 \end{aligned}$$

We cannot say A is semi-positive definite, but we are certain that B is not semi-positive definite

so B is the one that does not have a Cholesky decomposition

therefore, assume $A = LL^T$ where L is an upper triangular matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$
\begin{aligned}
\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} &= A \\
=LL^T &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \\
&= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}
\end{aligned}
$$

therefore

$$
\left\{
\begin{array}{ccc}
l_{11}^2 = 25 & \Rightarrow & l_{11} = 5 \\
l_{11}l_{21} = 15 & \Rightarrow & l_{21} = 3 \\
l_{11}l_{31} = -5 & \Rightarrow & l_{31} = -1 \\
\hline
l_{21}^2 + l_{22}^2 = 18 & \Rightarrow & l_{22} = 3 \\
l_{21}l_{31} + l_{22}l_{32} & \Rightarrow & l_{32} = 1 \\
\hline
l_{31}^2 + l_{32}^2 + l_{33}^2 = 11 & \Rightarrow & l_{33} = 3
\end{array}
\right.
$$

in conclusion, the Cholesky decomposition is

$$
A = LL^T = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}
$$