Modeling θ_T as a Double Mean-Reverting Process in SVI Parameterization

In the context of the **Stochastic Volatility Inspired (SVI)** parameterization, θ_T represents the total variance up to maturity T. When modeling θ_T as a **double mean-reverting process**, we're considering a stochastic volatility model where the variance itself mean-reverts to a stochastic long-term mean, which also mean-reverts to a constant level. This adds an extra layer of dynamics to capture more complex market behaviors.

Double Mean-Reverting Process

First Mean Reversion (Variance to Stochastic Mean Level):
The instantaneous variance v_t mean-reverts to a stochastic mean level m_t:

$$dv_t = \kappa_v(m_t - v_t)dt + \sigma_v \sqrt{v_t}\, dW_t^v$$

- v_t: Instantaneous variance at time t.
- κ_v: Speed of mean reversion of v_t toward m_t.
- σ_v: Volatility of the variance process.
- dW_t^v: Brownian motion driving v_t.

Second Mean Reversion (Mean Level to Constant Level):
The mean level m_t mean-reverts to a constant long-term mean θ:

$$dm_t = \kappa_m(\theta - m_t)dt + \sigma_m\, dW_t^m$$

- m_t: Stochastic mean level at time t.
- κ_m: Speed of mean reversion of m_t toward θ.
- σ_m: Volatility of the mean level process.
- dW_t^m: Brownian motion driving m_t.
- θ: Constant long-term mean level.

Computing θ_T Under Double Mean Reversion

The total variance θ_T up to time T is given by the expected integrated variance:

$$\theta_T = \mathbb{E}\left[\int_0^T v_t\, dt\right]$$

To derive θ_T under the double mean-reverting process, follow these steps:

1. Compute the Expected Mean Level $\mathbb{E}[m_t]$

Solve the stochastic differential equation (SDE) for m_t:

$$\mathbb{E}[m_t] = m_0 e^{-\kappa_m t} + \theta(1 - e^{-\kappa_m t})$$

- m_0: Initial mean level at $t = 0$.

2. Compute the Expected Variance $\mathbb{E}[v_t]$

Given the dependency of v_t on m_t, the expected variance is:

$$\mathbb{E}[v_t] = v_0 e^{-\kappa_v t} + \kappa_v \int_0^t e^{-\kappa_v (t-s)} \mathbb{E}[m_s] \, ds$$

- v_0: Initial variance at $t = 0$.

Substitute $\mathbb{E}[m_s]$ into the integral:

$$\mathbb{E}[v_t] = v_0 e^{-\kappa_v t} + \kappa_v \int_0^t e^{-\kappa_v (t-s)} \left[m_0 e^{-\kappa_m s} + \theta(1 - e^{-\kappa_m s}) \right] ds$$

3. Integrate $\mathbb{E}[v_t]$ Over Time to Find θ_T

Compute θ_T by integrating $\mathbb{E}[v_t]$ from 0 to T:

$$\theta_T = \int_0^T \mathbb{E}[v_t] \, dt$$

This results in:

$$\theta_T = \int_0^T \left[v_0 e^{-\kappa_v t} + \kappa_v \int_0^t e^{-\kappa_v (t-s)} \left(m_0 e^{-\kappa_m s} + \theta[1 - e^{-\kappa_m s}] \right) ds \right] dt$$

4. Simplify the Expression

While the integral may seem complex, it can be simplified using standard integration techniques and properties of exponential functions. The key steps involve:

- Evaluating the double integral.
- Applying the convolution of exponential functions.
- Simplifying the resulting expressions.

Note: Due to the complexity, the final expression may not be closed-form but can be computed numerically.

Final Formula for θ_T

An approximate expression for θ_T is:

$$\theta_T = \left(\frac{v_0}{\kappa_v}\right)\left(1 - e^{-\kappa_v T}\right) + \left(\frac{\kappa_v m_0}{\kappa_v - \kappa_m}\right)\left(\frac{1 - e^{-\kappa_m T}}{\kappa_m} - \frac{1 - e^{-\kappa_v T}}{\kappa_v}\right) + \theta\left(T - \frac{1 - e^{-\kappa_m T}}{\kappa_m}\right)$$

This formula assumes $\kappa_v \neq \kappa_m$. If $\kappa_v = \kappa_m$, the expression needs adjustment to avoid division by zero.

Considerations

- **Correlation Between Brownian Motions:** If dW_t^v and dW_t^m are correlated, this affects the variance of v_t but not the expectation $\mathbb{E}[v_t]$.
- **Parameter Estimation:** Estimating the parameters κ_v, κ_m, σ_v, σ_m, v_0, and m_0 requires calibration to market data.
- **Numerical Methods:** In practice, numerical integration or simulation may be necessary to compute θ_T accurately.

Conclusion

By modeling θ_T as a double mean-reverting process, you capture richer dynamics in the implied volatility surface. The formula involves integrating the expected variance over time, accounting for both layers of mean reversion.

Key Formula:

$$\theta_T = \int_0^T \mathbb{E}[v_t]\, dt$$

with $\mathbb{E}[v_t]$ computed based on the mean-reverting processes of v_t and m_t.

References

- **Gatheral, J. (2006).** *The Volatility Surface: A Practitioner's Guide.* Wiley.
- **Christoffersen, P., Heston, S., & Jacobs, K. (2009).** *The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work So Well.* Management Science.

These references provide deeper insights into stochastic volatility models and the implementation of double mean-reverting processes in financial modeling.