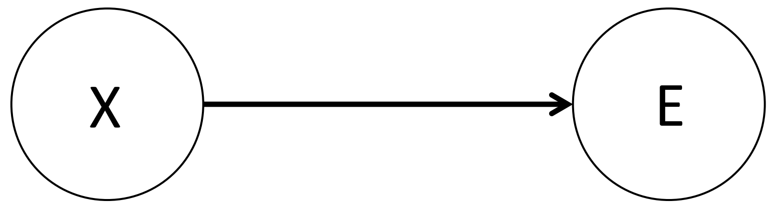
CS221 Section 7
Bayesian Networks

Nov 9th 2018

Review: Bayes Rule

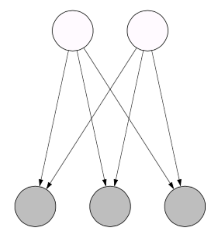


- Variables X, E,
- Prior P(X), likelihood P(E | X)
- Given evidence E = e, what is P(X | E)?
- Bayes rule: P(X | E) = (P(X) P(E | X)) / P(E)
- Posterior probability

Roadmap

- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence
- Gibbs Sampling

Bayesian Networks



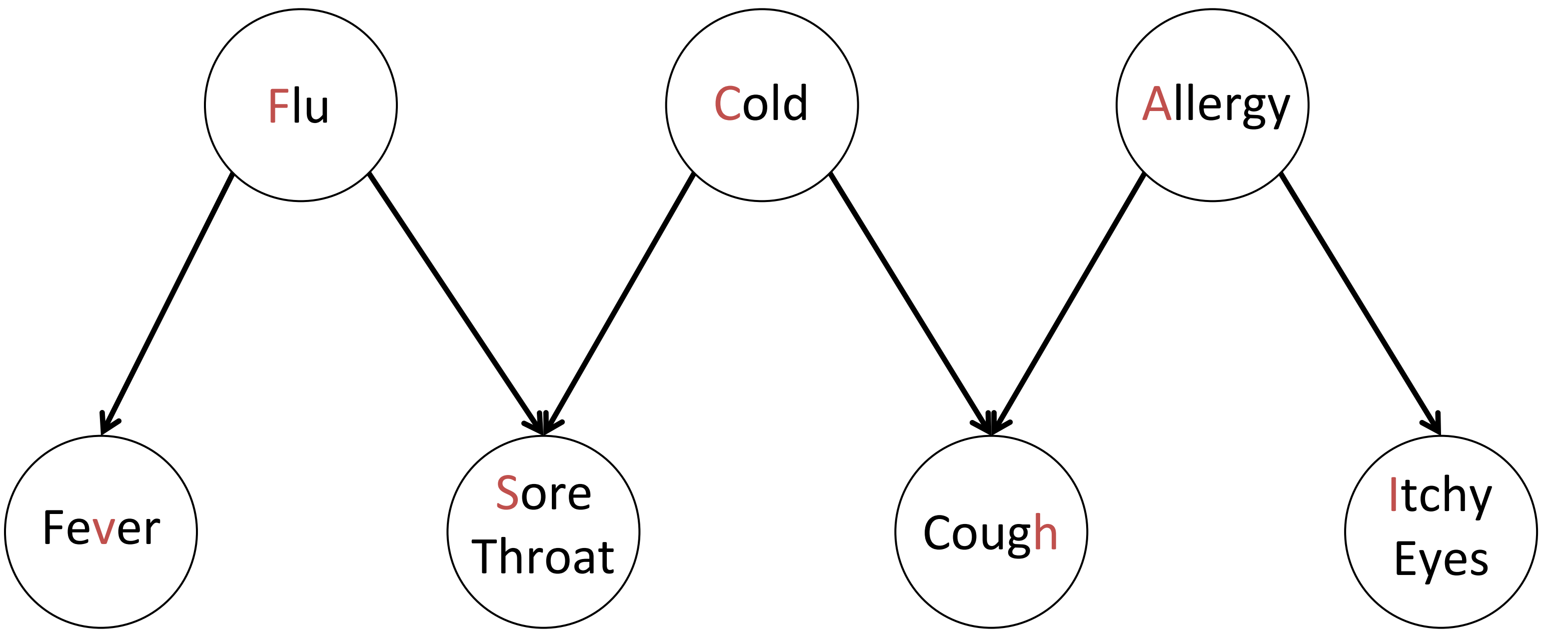
Definition: Bayesian network

Let $X = (X_1, \dots, X_n)$ be random variables.

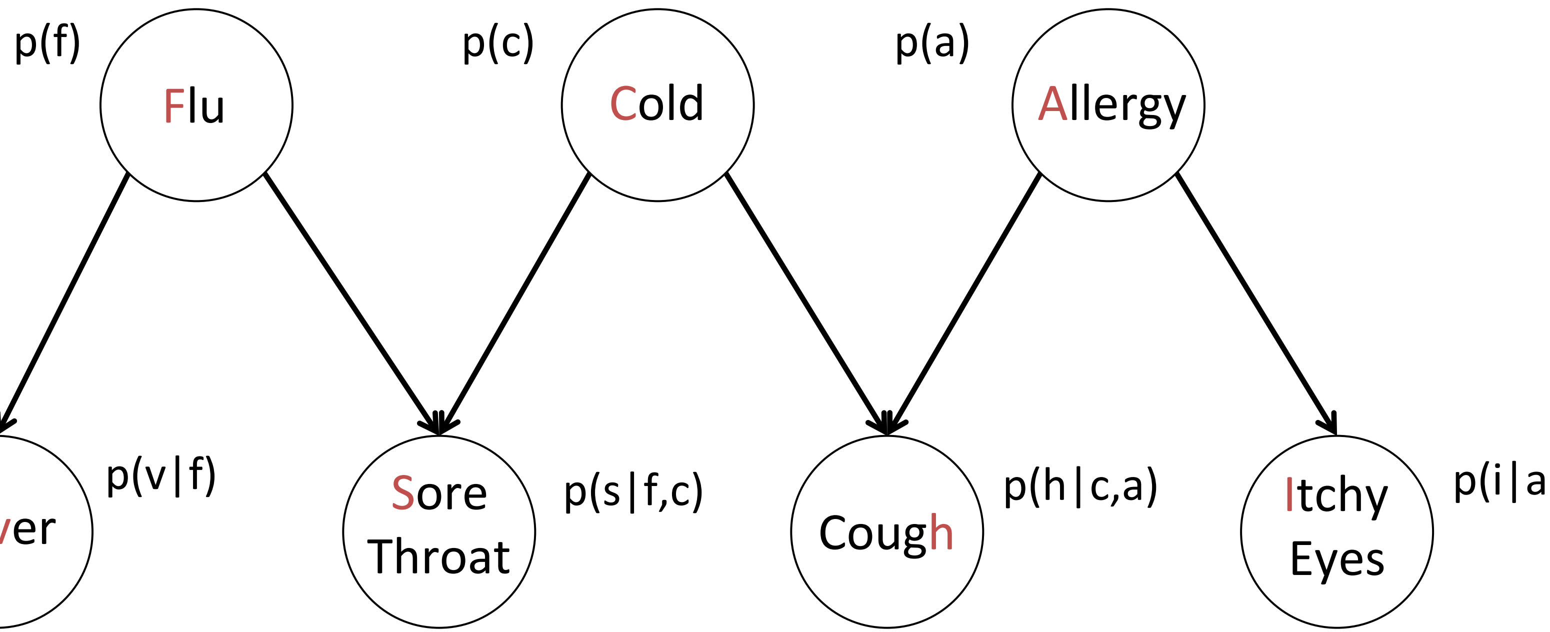
A **Bayesian network** is a directed acyclic graph (DAG) that specifies a joint distribution over X as a product of local conditional distributions, one for each node:

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} p(x_i \mid x_{\text{Parents}(i)})$$

Bayesian Networks



A Bayesian network represents a joint probability distribution.



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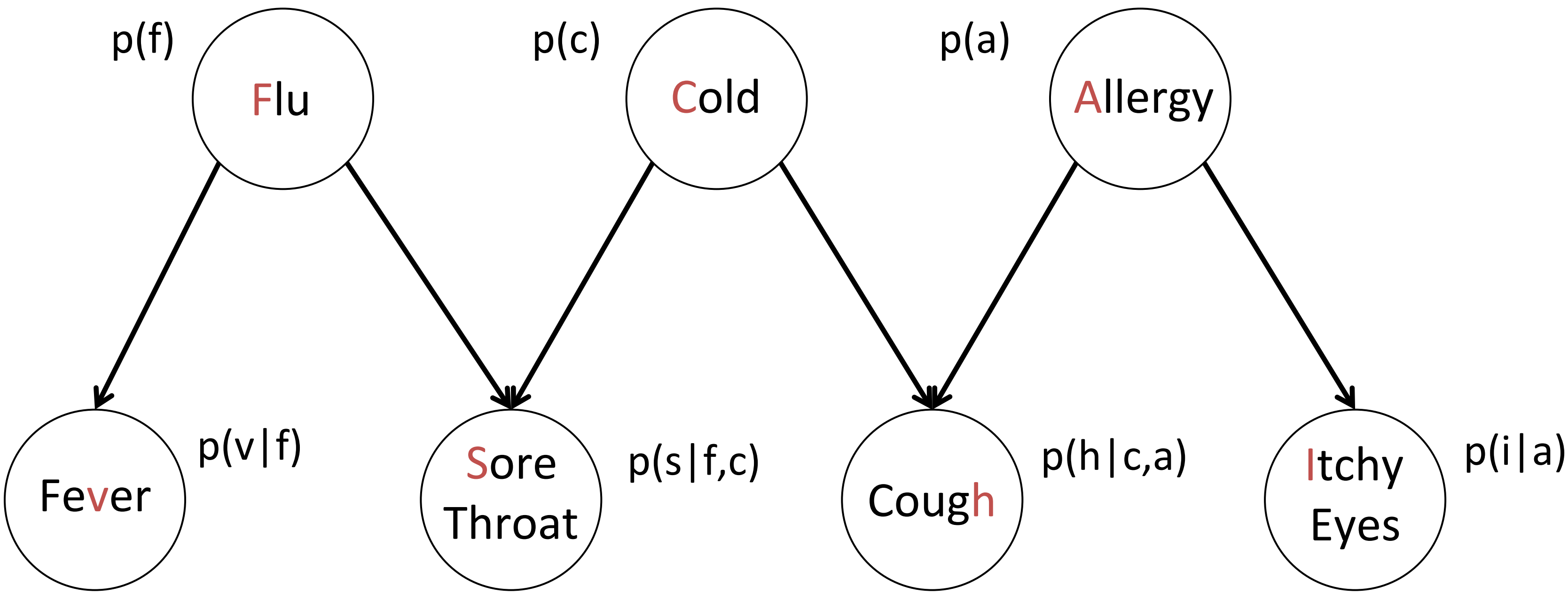


$P(F=f, C=c, A=a, V=v, S=s, C=c, I=i) = p(f)p(c)p(a)p(v|a)p(s|f, c)p(h|c,a)p(i|a)$

Roadmap

- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence
- Gibbs Sampling

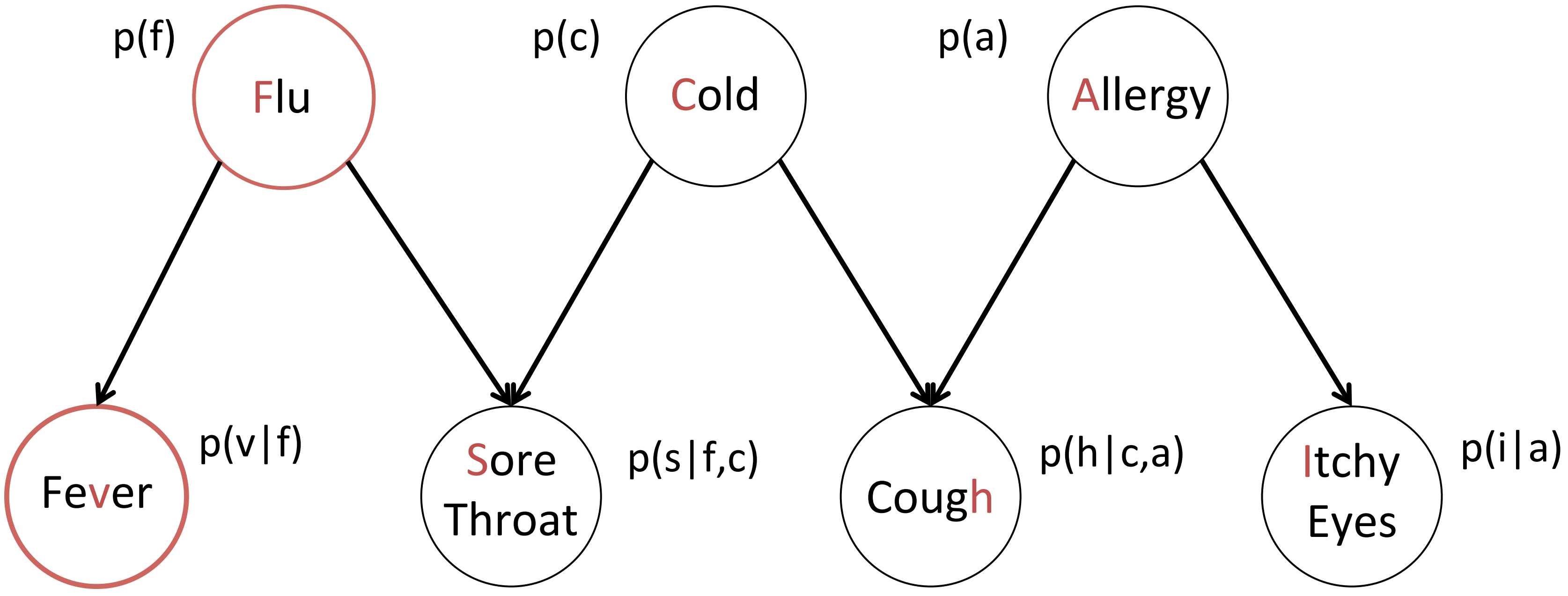
Probabilistic Queries – Cookbook

Given a query $P(Q|E=e)$

1. Remove (marginalize) variables not ancestors of Q or E.
2. Convert Bayesian network to factor graph.
3. Condition (shade nodes / disconnect) on E = e.
4. Remove (marginalize) nodes disconnected from Q.
5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

Probabilistic Queries - Examples

$P(F=1|V=1) = ?$



Probabilistic Queries - Examples

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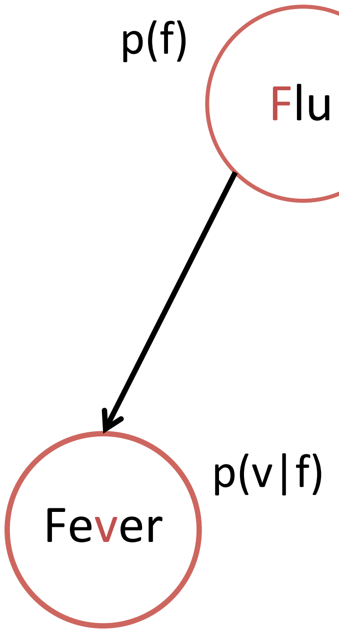

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Probabilistic Queries - Examples

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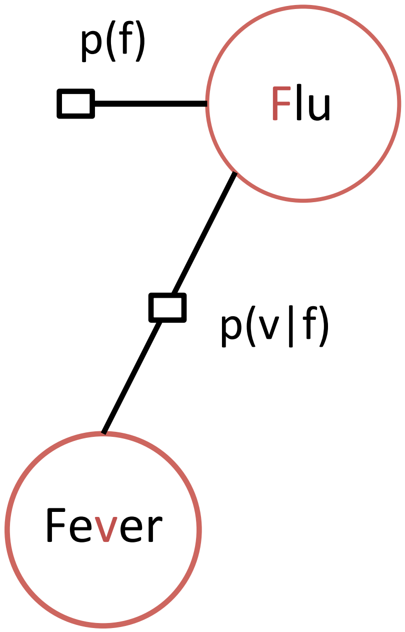

2. Convert Bayesian network to factor graph.

Probabilistic Queries - Examples

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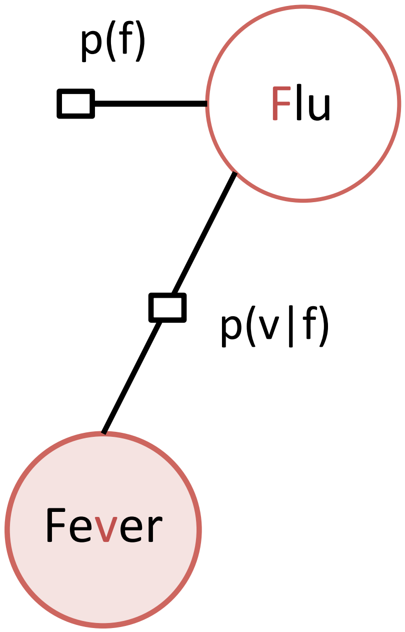
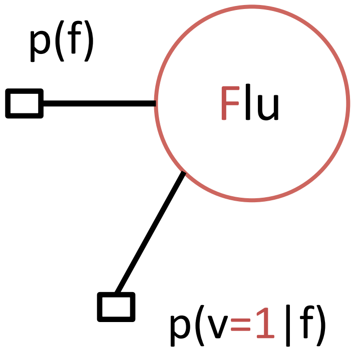


3. Condition on E = e.
   3.1 shade nodes

Probabilistic Queries - Examples

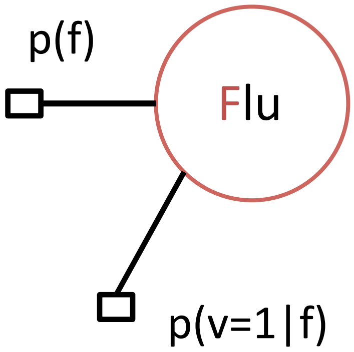
$P(F=1|V=1) = ?$



3. Condition on E = e.
   3.2 disconnect

Probabilistic Queries - Examples

$P(F{=}1 \mid V{=}1) = ?$



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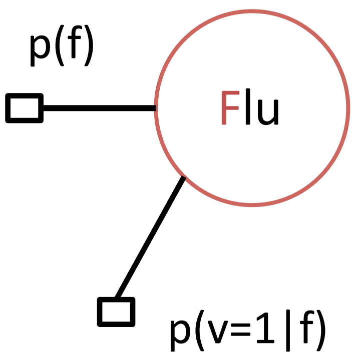
Probabilistic Queries - Examples

$P(F=1|V=1) = ?$



5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

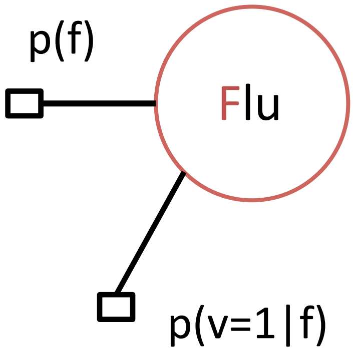
f	p(f)
0	1-α
1	α

f	v	p(v|f)
0	0	0.70
0	1	0.30
1	0	0.20
1	1	0.80

$$P(F=f|V=1) \propto p(f)\, p(v=1|f)$$

Probabilistic Queries - Examples

P(F=1|V=1) = ?



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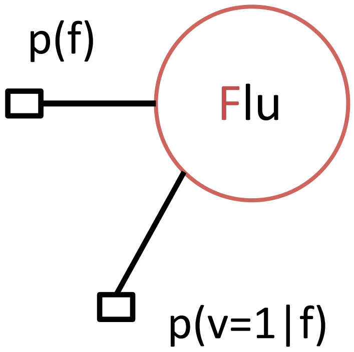
$$P(F=f|V=1) \propto p(f)\, p(v=1|f) = \begin{cases} (1-\alpha)*0.30, & f=0 \\ & \end{cases}$$

Probabilistic Queries - Examples

$P(F=1|V=1) = ?$



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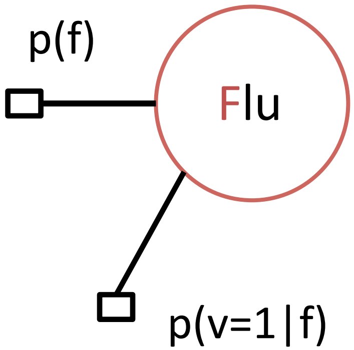
$$P(F=f|V=1) \propto p(f)\, p(v=1|f) = \begin{cases} (1-\alpha) * 0.30, & f = 0 \\ \alpha * 0.80, & f = 1 \end{cases}$$

Probabilistic Queries - Examples

P(F=1|V=1) = ?



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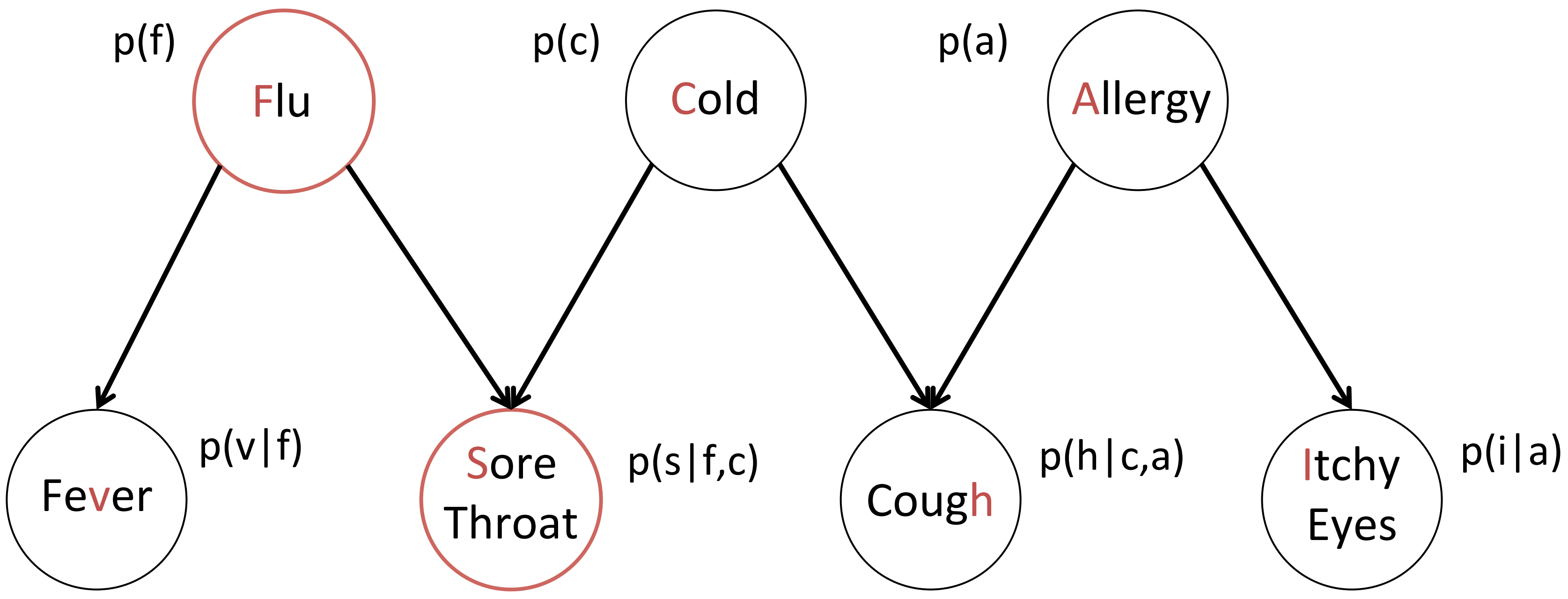
$$P(F=f|V=1) \propto p(f)\, p(v=1|f) = \begin{cases} (1-\alpha)*0.30, & f=0 \\ \alpha*0.80, & f=1 \end{cases}$$

$$P(F=1|V=1) = \frac{\alpha*0.80}{\alpha*0.80+(1-\alpha)*0.30}$$

Probabilistic Queries - Examples

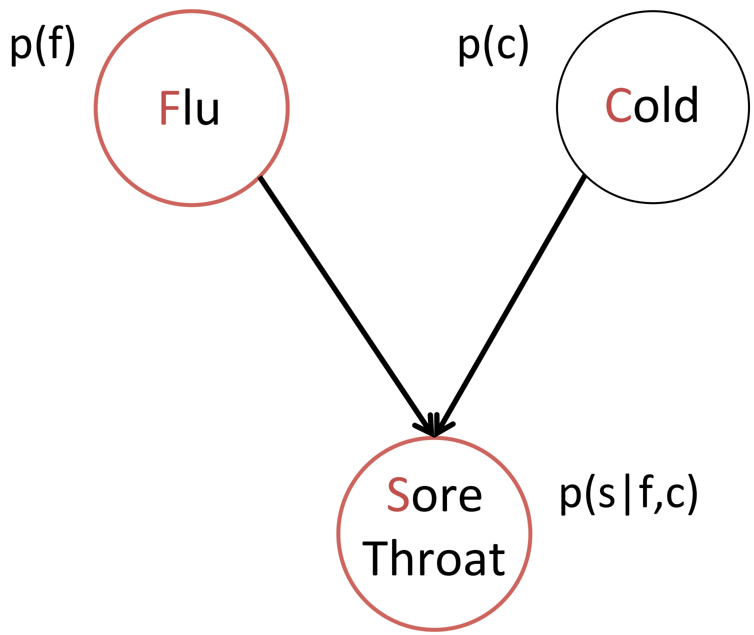
$P(F=1|S=1) = ?$



Probabilistic Queries - Examples

P(F=1|S=1) = ?

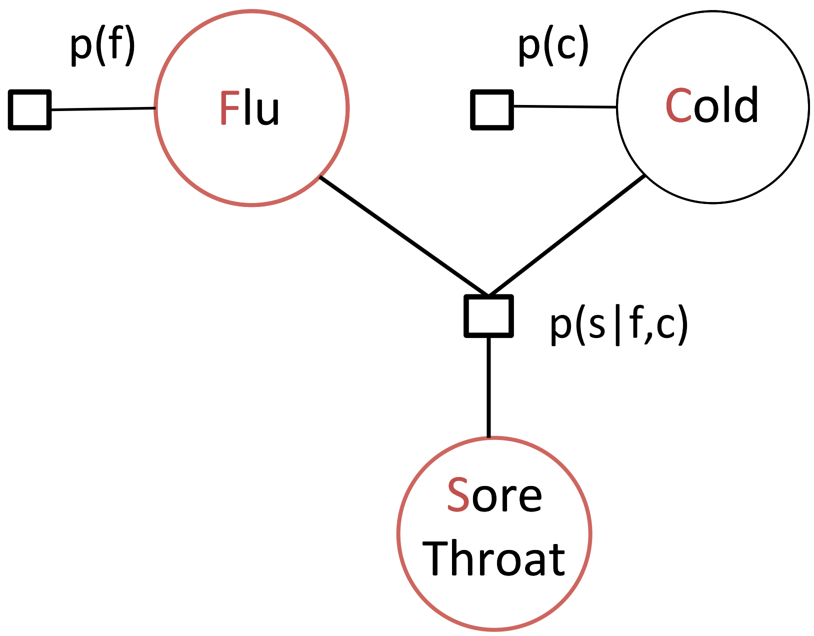
1. Remove (marginalize) variables not ancestors of Q or E.



Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

2. Convert Bayesian network to factor graph.

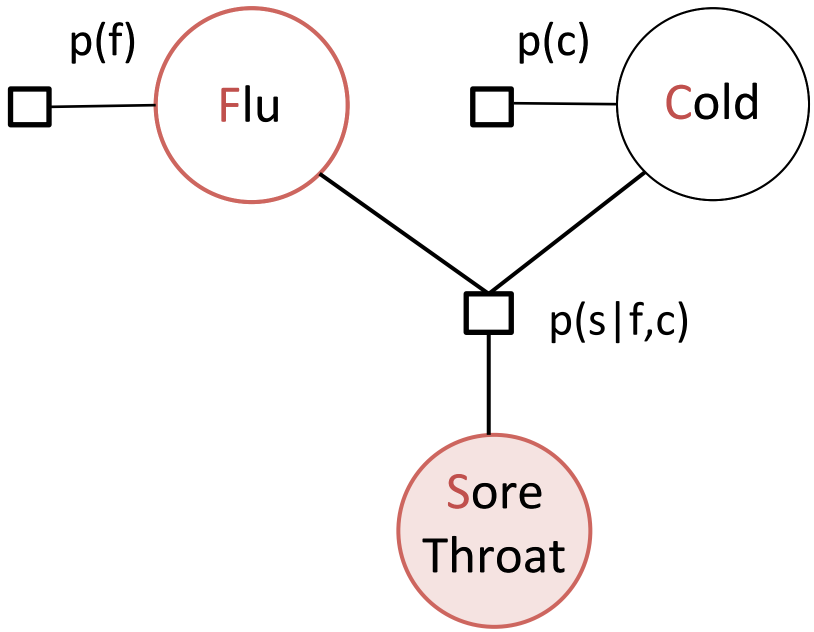


★ One factor per variable!

Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

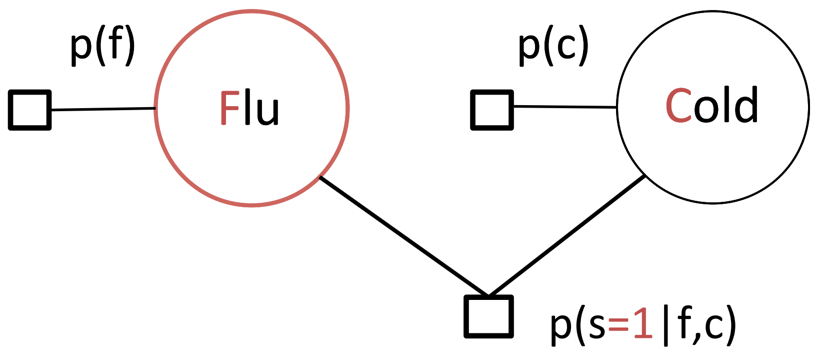
3. Condition on E = e.
   3.1 shade nodes



Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

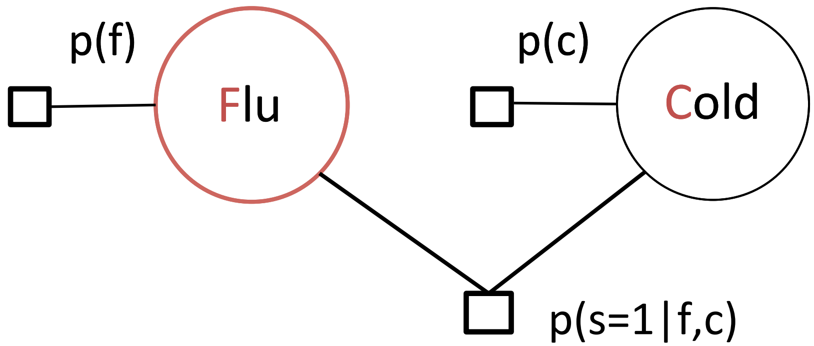
3. Condition on E = e.
   3.2 disconnect



Probabilistic Queries - Examples

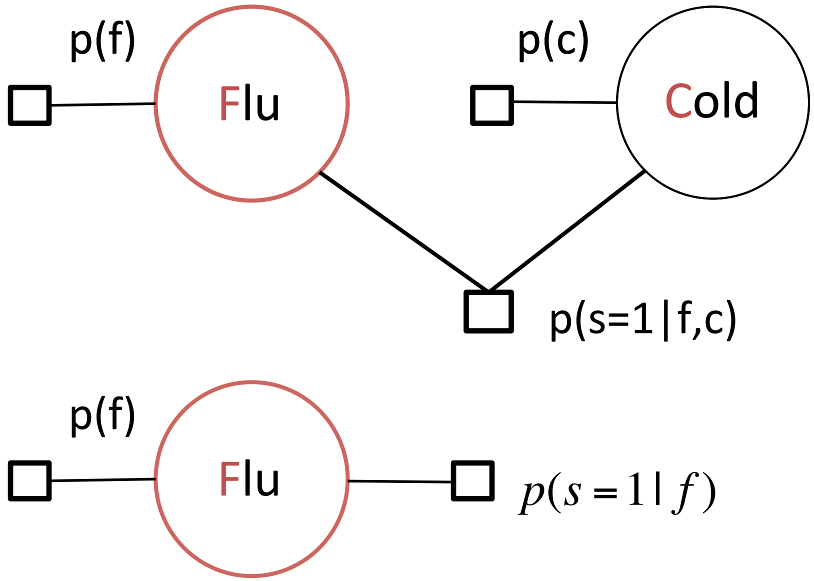
P(F=1|S=1) = ?

4. Remove (marginalize) nodes disconnected from Q.



Probabilistic Queries - Examples

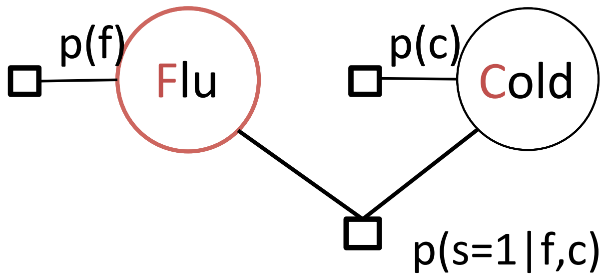
$P(F=1|S=1) = ?$

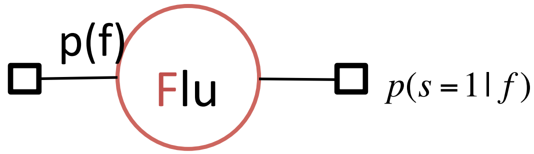


5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

Probabilistic Queries - Examples

P(F=1|S=1) = ?





5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

$$p(s=1|f)$$

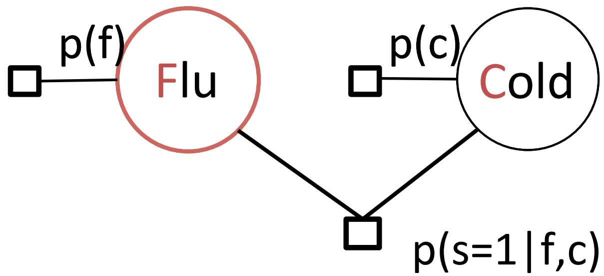
$$=\sum_c p(c)p(s=1|f,c)$$

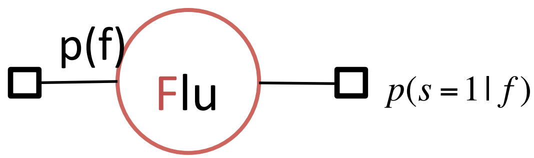
$$=p(c=0)p(s=1|f,c=0)+p(c=1)p(s=1|f,c=1)$$

f	p(s=1,f)
0	?
1	?

Probabilistic Queries - Examples

P(F=1|S=1) = ?





$$p(s=1|f)$$
$$=\sum_c p(c)p(s=1|f,c)$$
$$=p(c=0)p(s=1|f,c=0)+p(c=1)p(s=1|f,c=1)$$
$$=\begin{cases}(1-\beta)*0+\beta*0.75, & f=0\\ & \end{cases}$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

f	p(f)
0	1-α
1	α

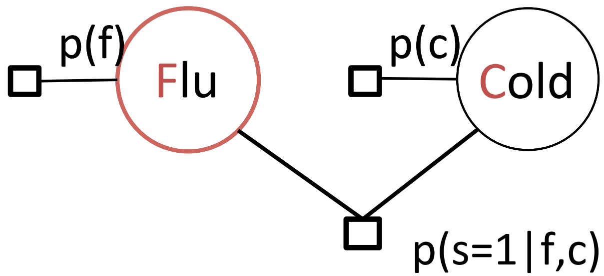
c	p(c)
0	1-β
1	β

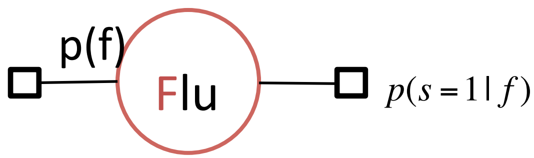
s	f	c	p(s|f,c)
0	0	0	1.00
1	0	0	0
0	1	0	0.30
1	1	0	0.70
0	0	1	0.25
1	0	1	0.75
0	1	1	0.10
1	1	1	0.90

f	p(s=1,f)
0	β*0.75
1	?

Probabilistic Queries - Examples

P(F=1|S=1) = ?





$$p(s=1|f)$$
$$=\sum_c p(c)p(s=1|f,c)$$
$$=p(c=0)p(s=1|f,c=0)+p(c=1)p(s=1|f,c=1)$$
$$=\begin{cases}(1-\beta)*0+\beta*0.75, & f=0\\ (1-\beta)*0.70+\beta*0.9, & f=1\end{cases}$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

f	p(f)
0	1-α
1	α

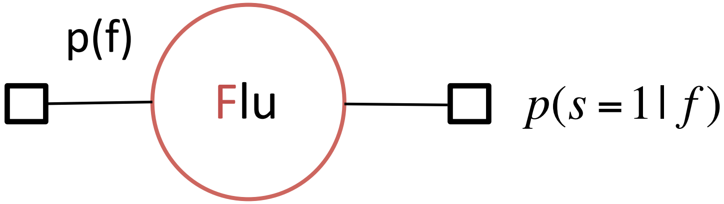
c	p(c)
0	1-β
1	β

s	f	c	p(s|f,c)
0	0	0	1.00
1	0	0	0
0	1	0	0.30
1	1	0	0.70
0	0	1	0.25
1	0	1	0.75
0	1	1	0.10
1	1	1	0.90

f	p(s=1,f)
0	β*0.75
1	((1-β)*0.7+β*0.9)

Probabilistic Queries - Examples

P(F=1|S=1) = ?



$$P(F=f\mid S=1)$$
$$\propto p(f)p(s=1\mid f)$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

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f	p(s=1|f)
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Probabilistic Queries - Examples

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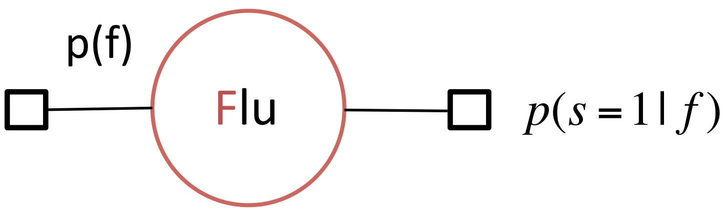

$$P(F=f \mid S=1)$$

$$\propto p(f)p(s=1\mid f)$$

$$=\begin{cases} (1-\alpha)\beta * 0.75, & f=0 \\ \end{cases}$$

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f	p(f)
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f	p(s=1|f)
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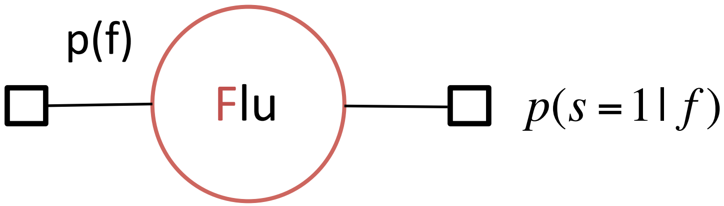
Probabilistic Queries - Examples

P(F=1|S=1) = ?



$$P(F=f\,|\,S=1)$$

$$\propto p(f)p(s=1\,|\,f)$$

$$=\begin{cases} (1-\alpha)\beta * 0.75, & f=0 \\ \alpha((1-\beta)*0.70+\beta*0.9), & f=1 \end{cases}$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

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0	1-α
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f	p(s=1|f)
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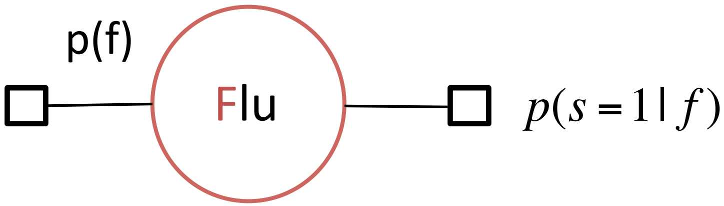
1	(1-β)*0.7+β*0.9

Probabilistic Queries - Examples

P(F=1|S=1) = ?

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).



f	p(f)
0	1-α
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f	p(s=1|f)
0	β*0.75
1	(1-β)*0.7+β*0.9

$$P(F=f \mid S=1)$$

$$\propto p(f)p(s=1 \mid f)$$

$$= \begin{cases} (1-\alpha)\beta * 0.75, & f=0 \\ \alpha((1-\beta)*0.70+\beta*0.9), & f=1 \end{cases}$$

$$P(F=1 \mid S=1) = \frac{p(f=1)p(s=1 \mid f=1)}{p(f=1)p(s=1 \mid f=1)+p(f=1)p(s=1 \mid f=1)}$$

$$= \frac{\alpha((1-\beta)*0.70+\beta*0.9)}{(1-\alpha)\beta*0.75+\alpha((1-\beta)*0.70+\beta*0.9),}$$

Probabilistic Queries – Cookbook

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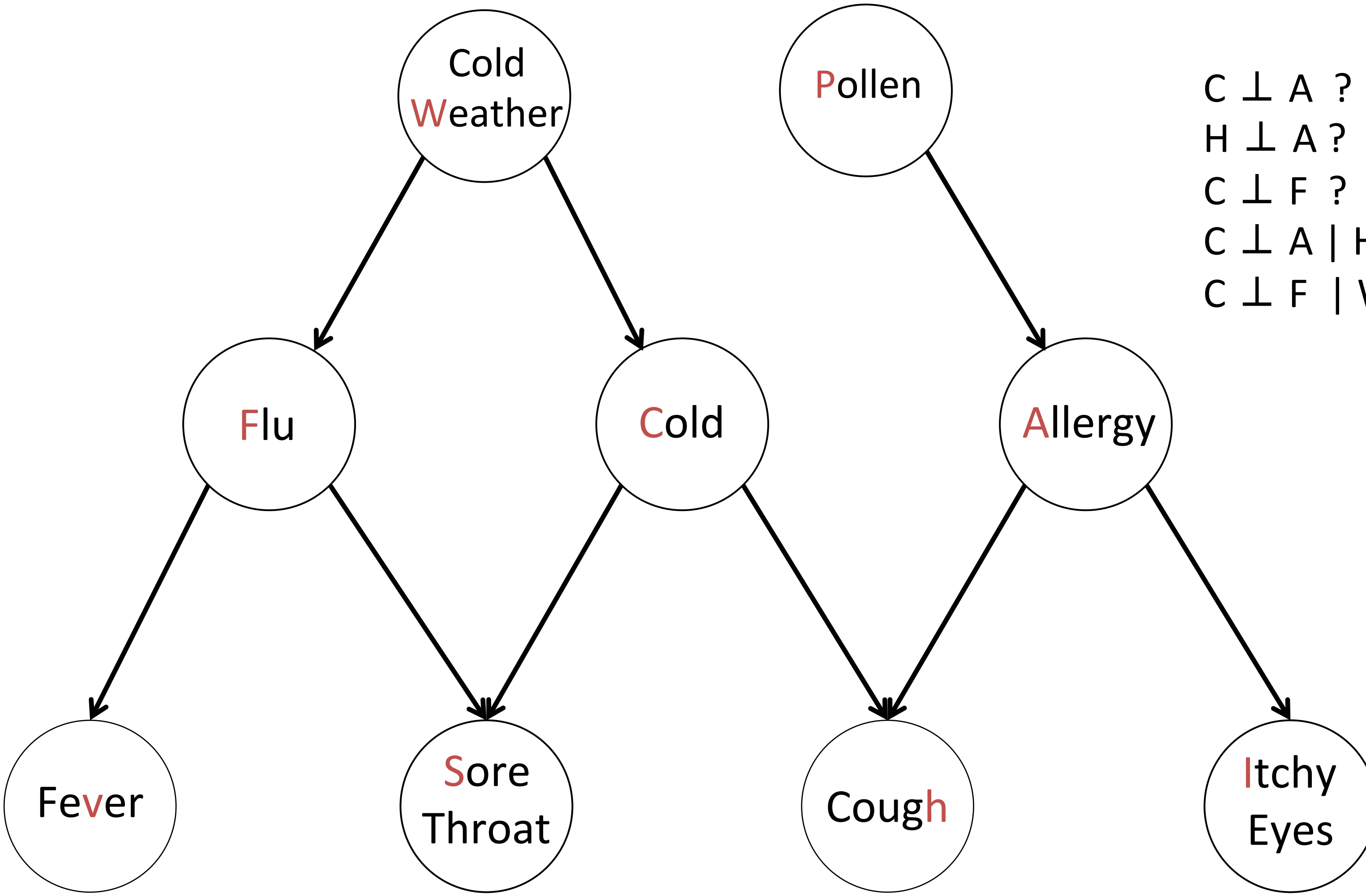
Roadmap

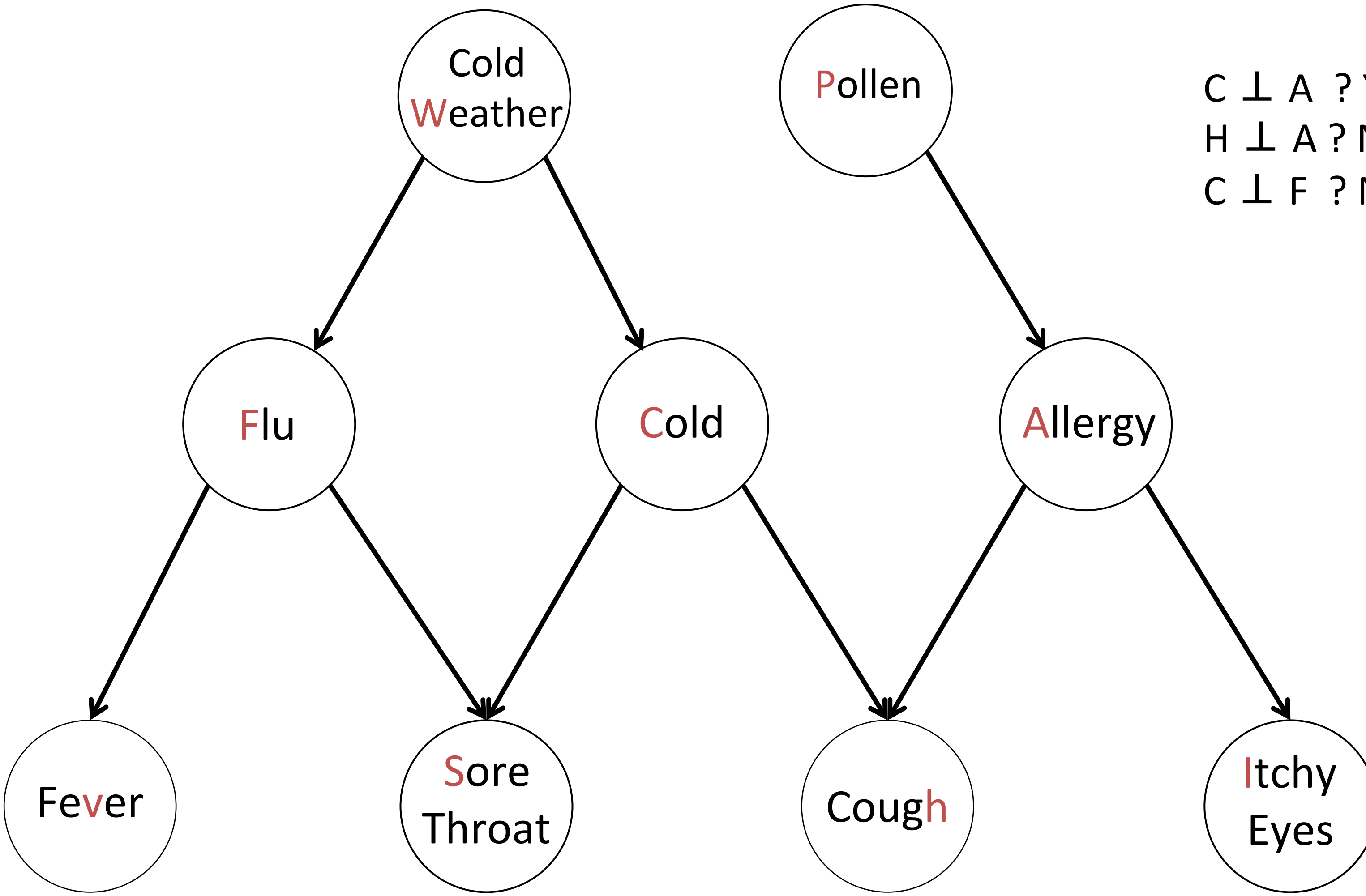
- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence
- Gibbs Sampling

Conditional Independence



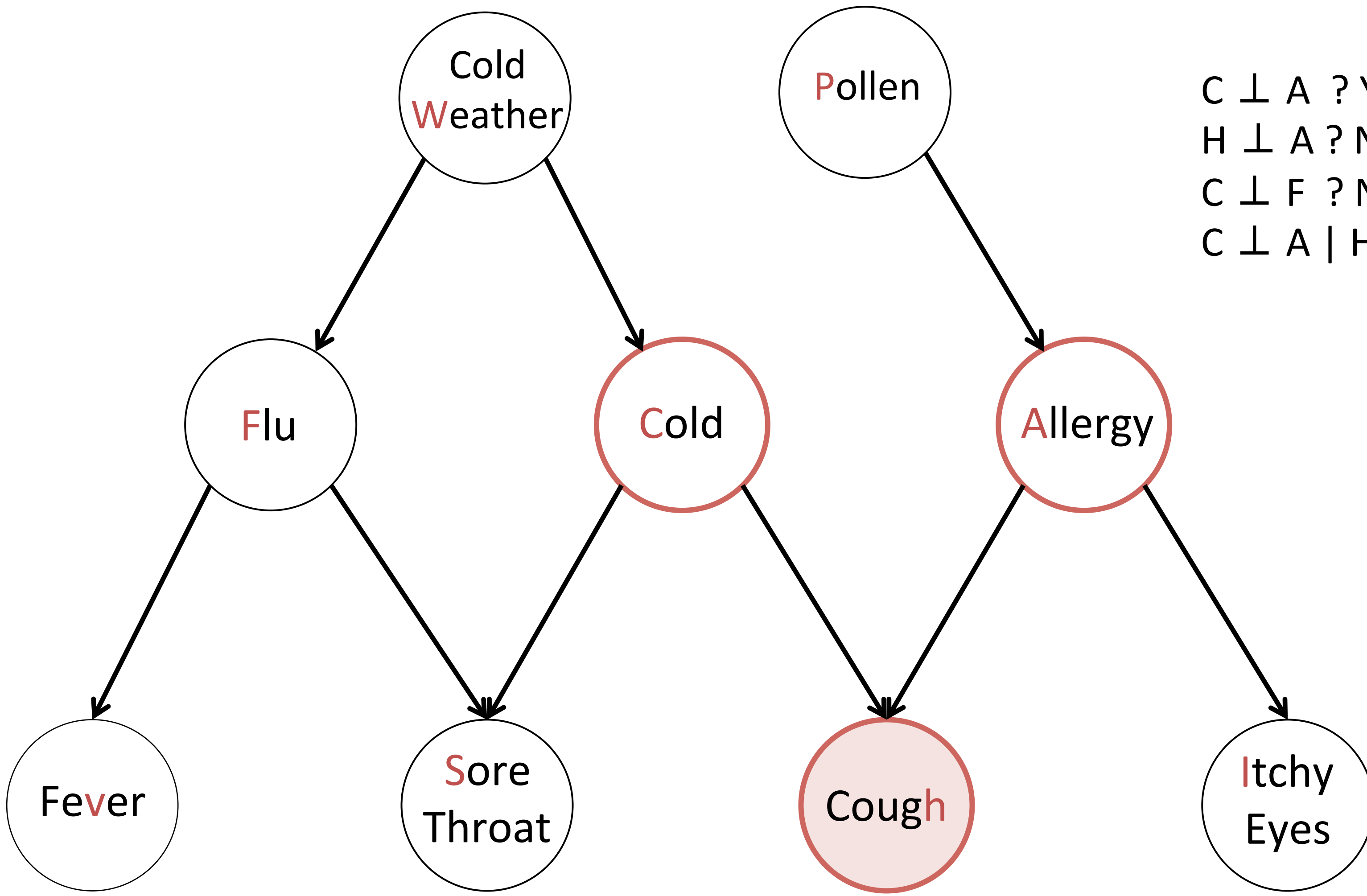
$C \perp A$?
$H \perp A$?
$C \perp F$?
$C \perp A \mid H$?
$C \perp F \mid W$?

Conditional Independence



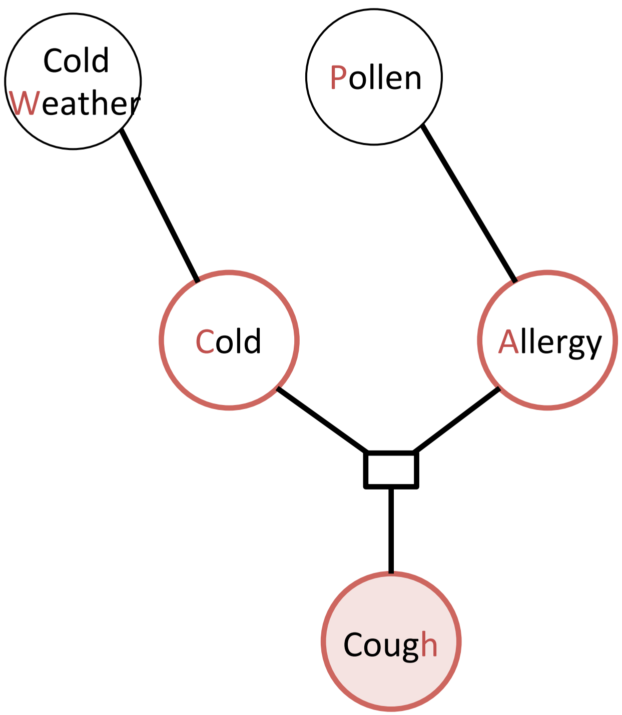
C ⊥ A ? Yes
H ⊥ A ? No
C ⊥ F ? No

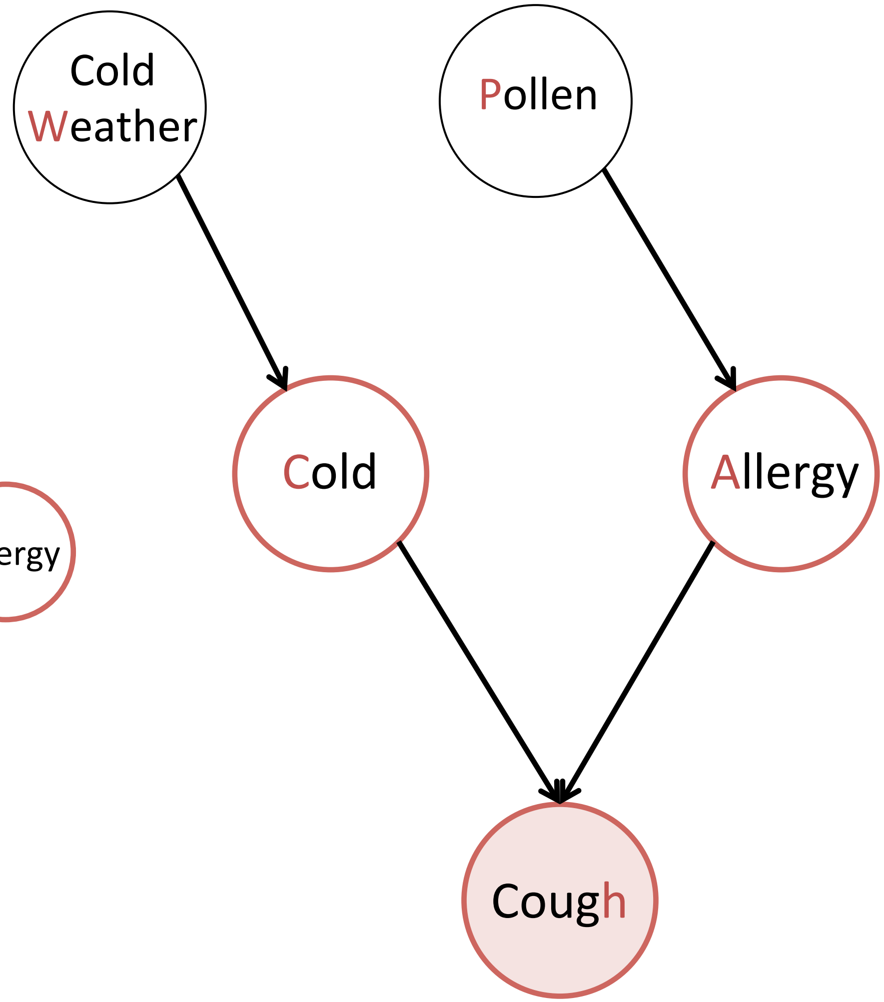
Conditional Independence



$C \perp A$? Yes
$H \perp A$? No
$C \perp F$? No
$C \perp A \mid H$?

Conditional Independence

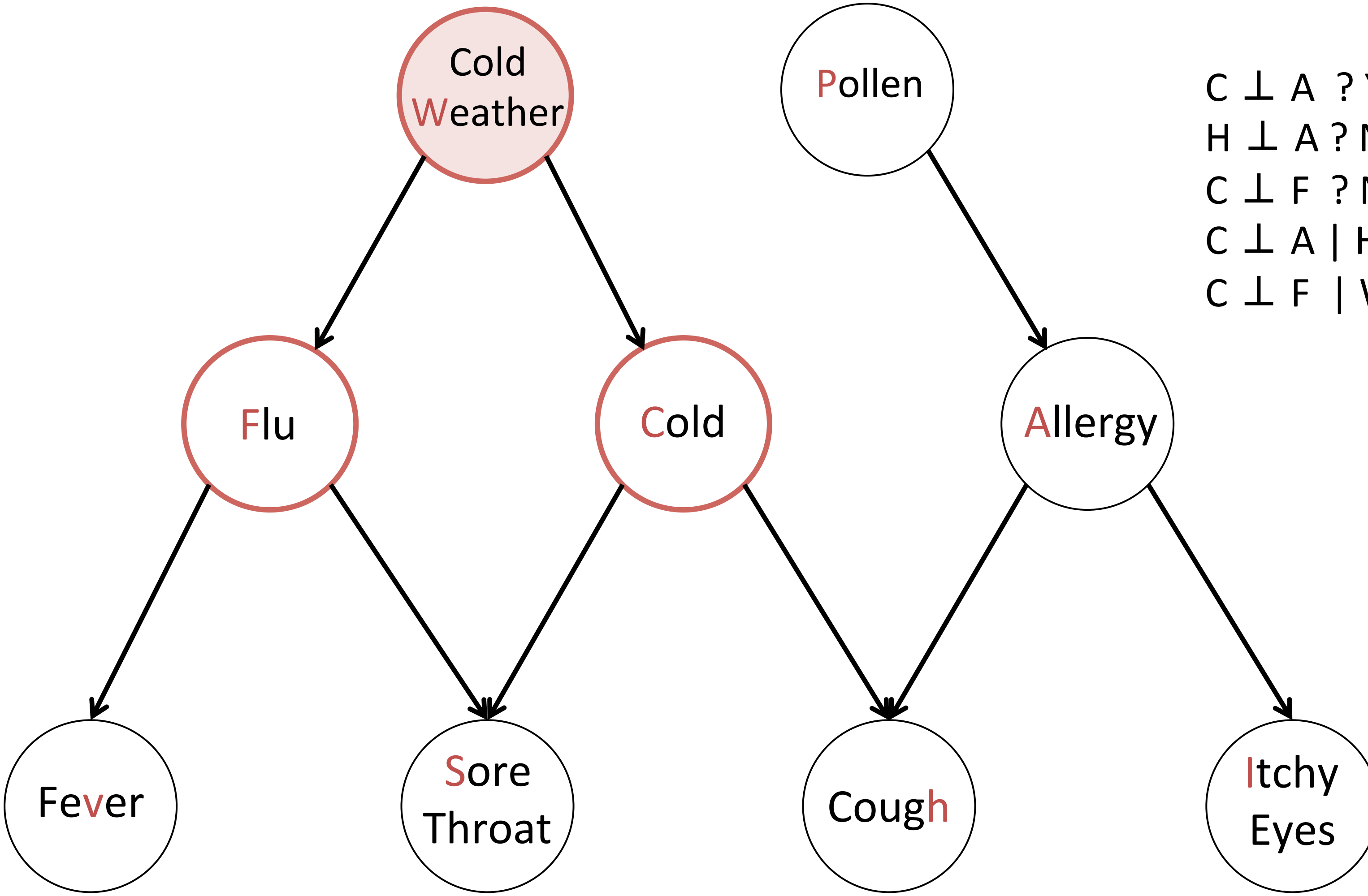




C $\perp$ A ? Yes
H $\perp$ A ? No
C $\perp$ F ? No
C $\perp$ A | H?

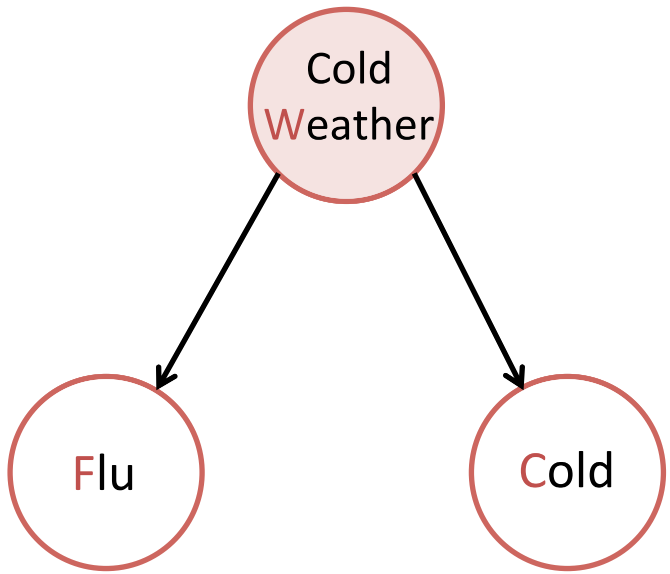
Explaining Away!

Conditional Independence

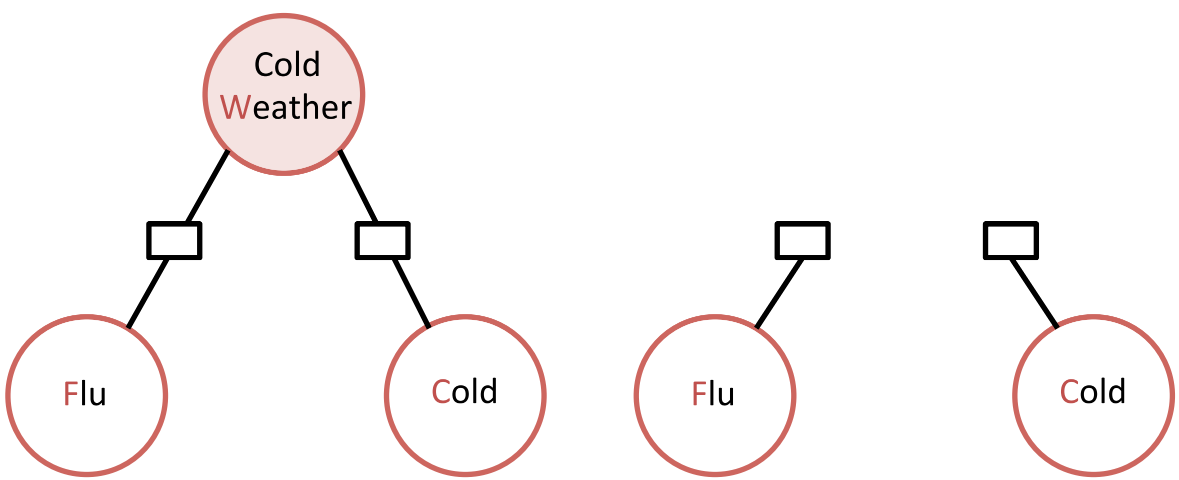


$C \perp A$? Yes
$H \perp A$? No
$C \perp F$? No
$C \perp A \mid H$? No
$C \perp F \mid W$?

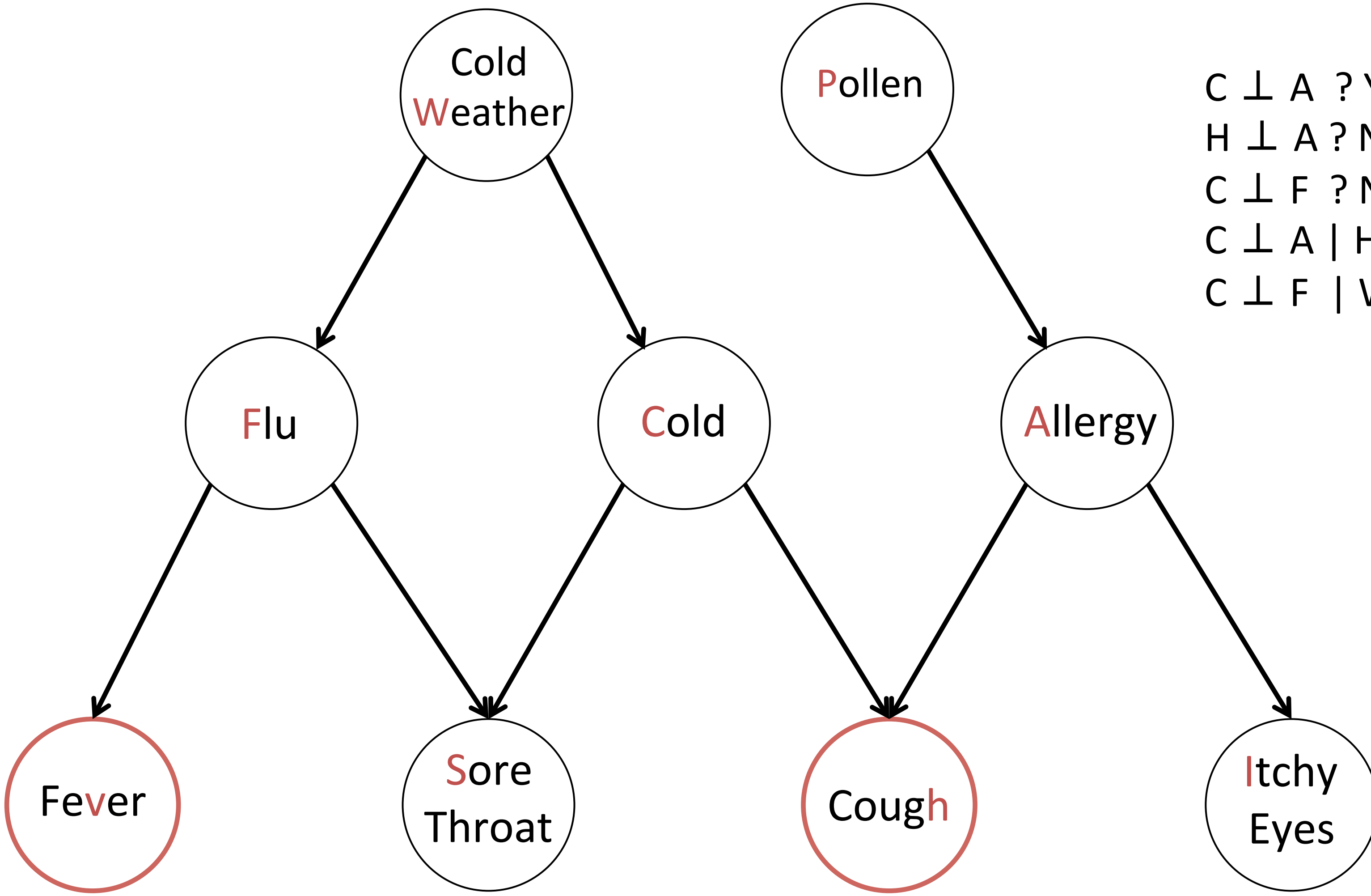
Conditional Independence



C ⊥ A ? Yes
H ⊥ A ? No
C ⊥ F ? No
C ⊥ A | H? No
C ⊥ F | W?

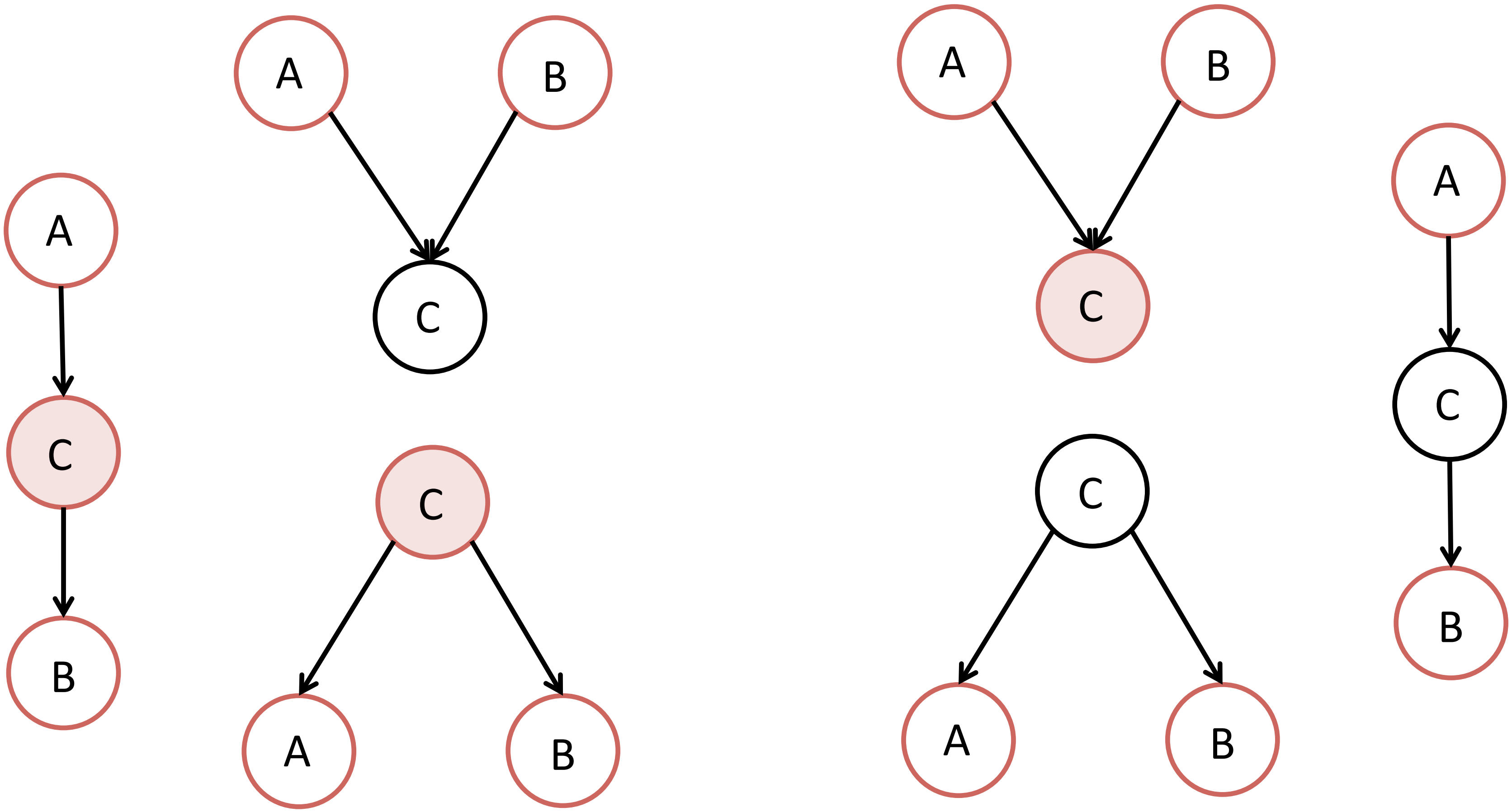


Conditional Independence



$C \perp A$? Yes
$H \perp A$? No
$C \perp F$? No
$C \perp A \mid H$? No
$C \perp F \mid W$? Yes

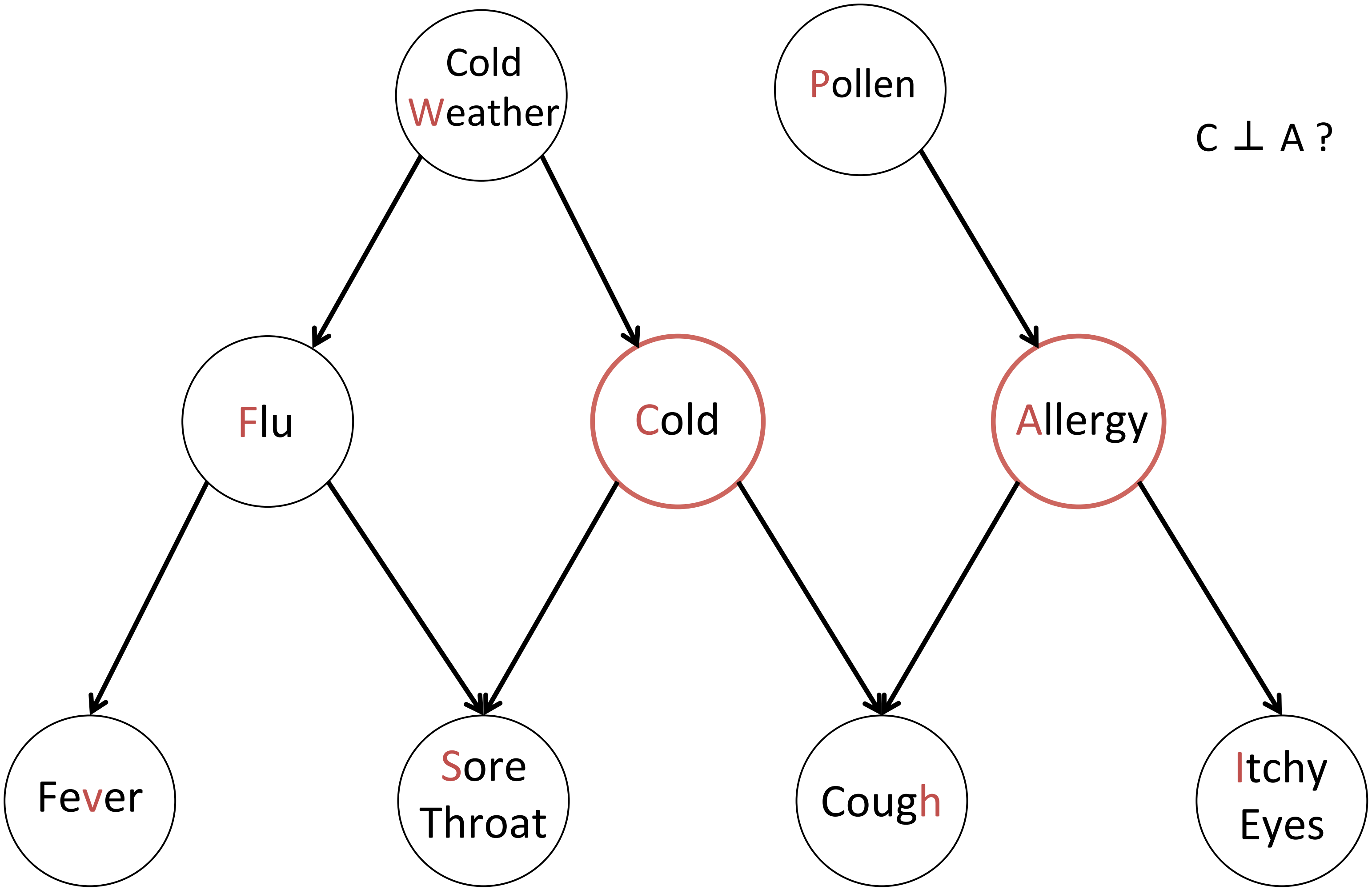
Patterns



Independent

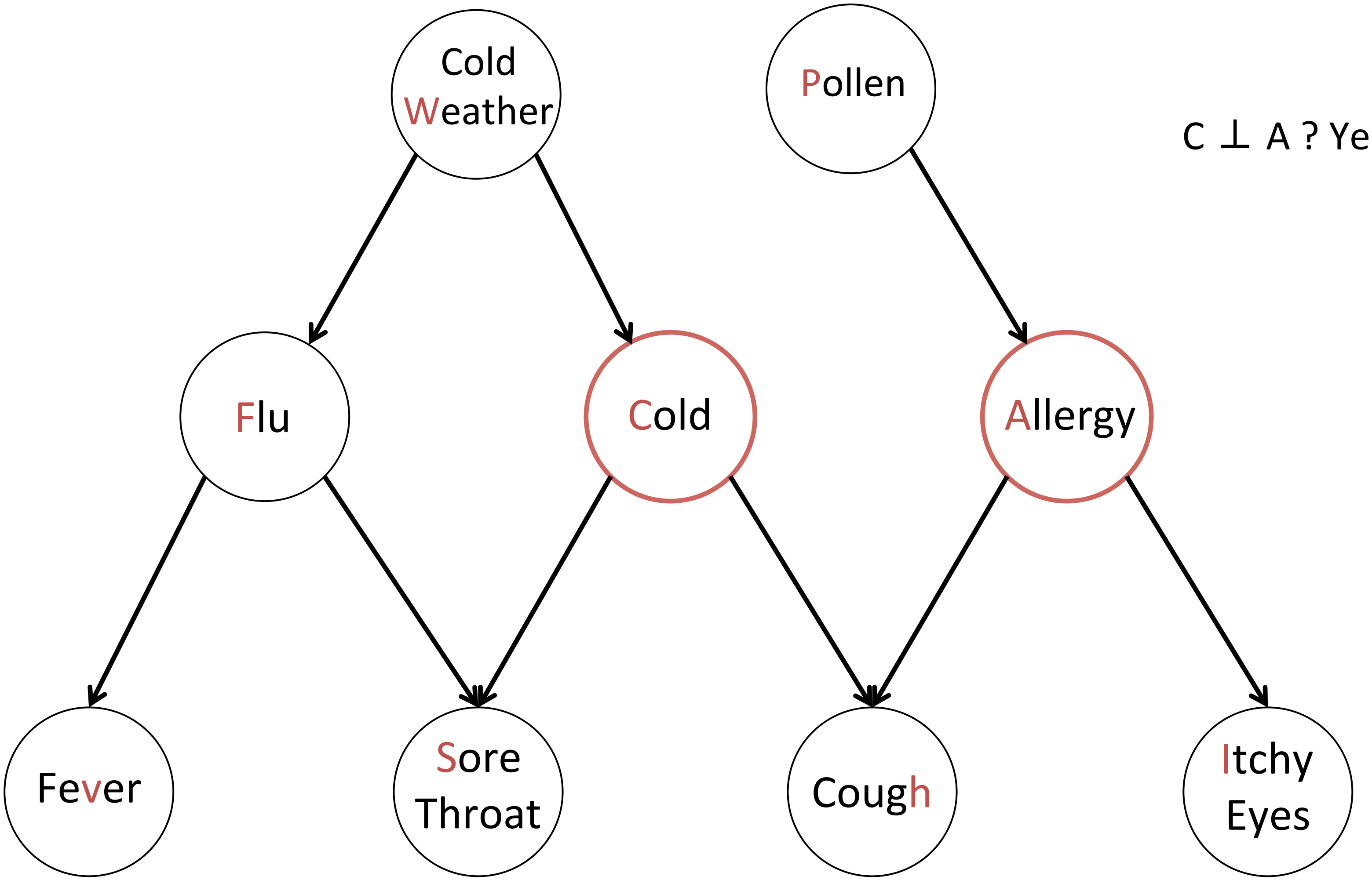
Dependent

Conditional Independence



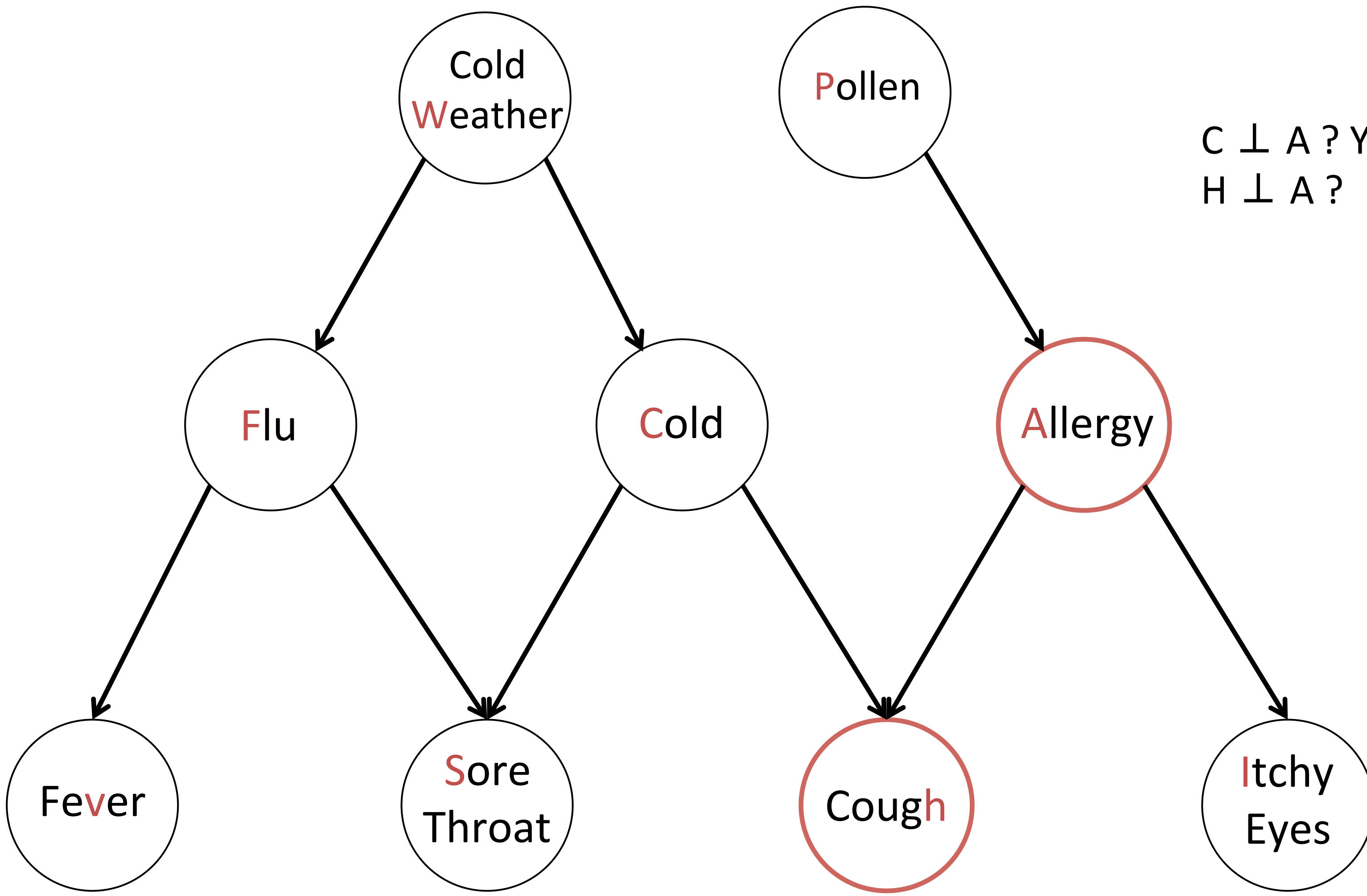
C ⊥ A ?

Conditional Independence



C ⊥ A ? Yes

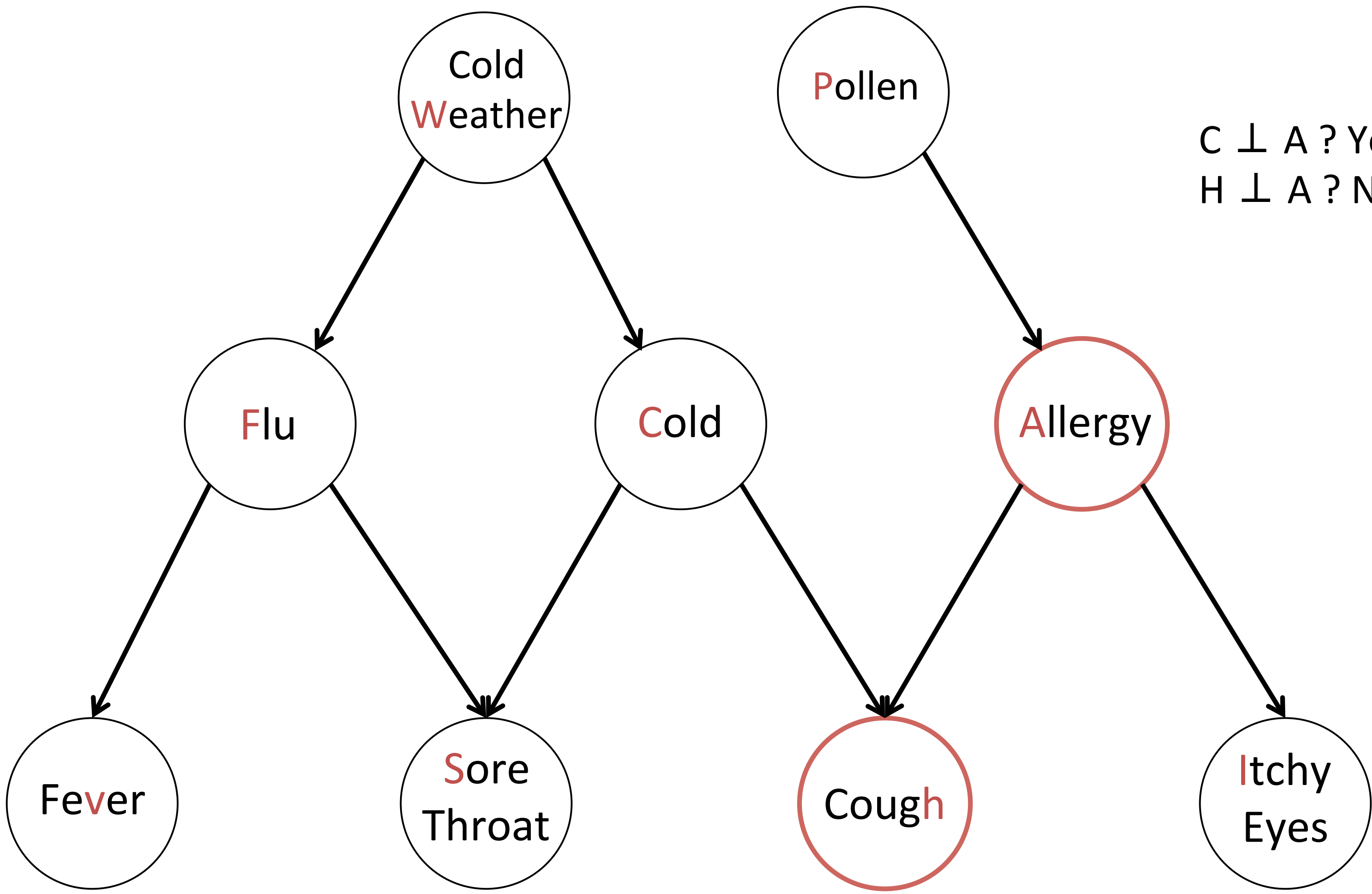
Conditional Independence



C ⊥ A ? Yes

H ⊥ A ?

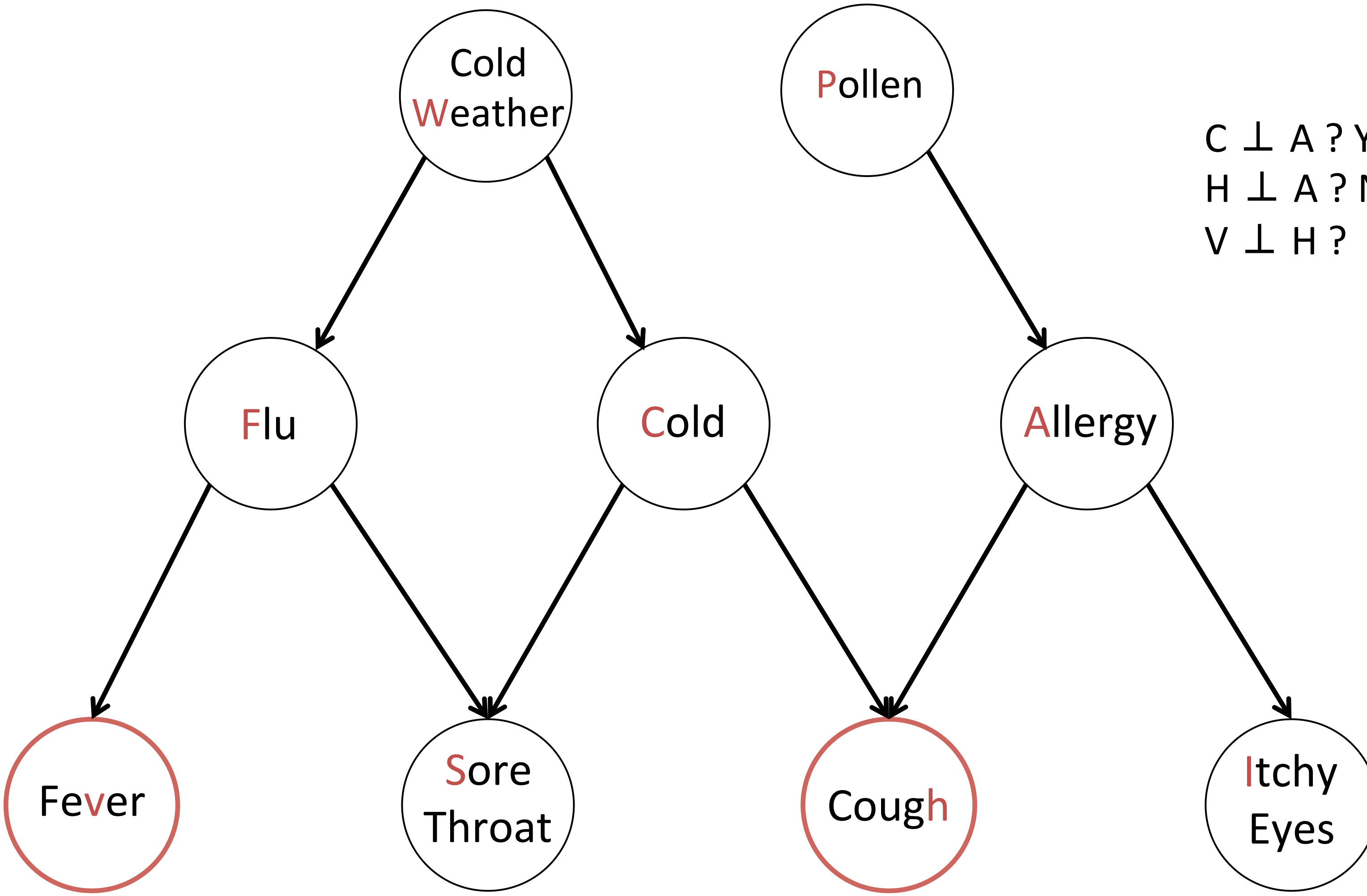
Conditional Independence



C ⊥ A ? Yes

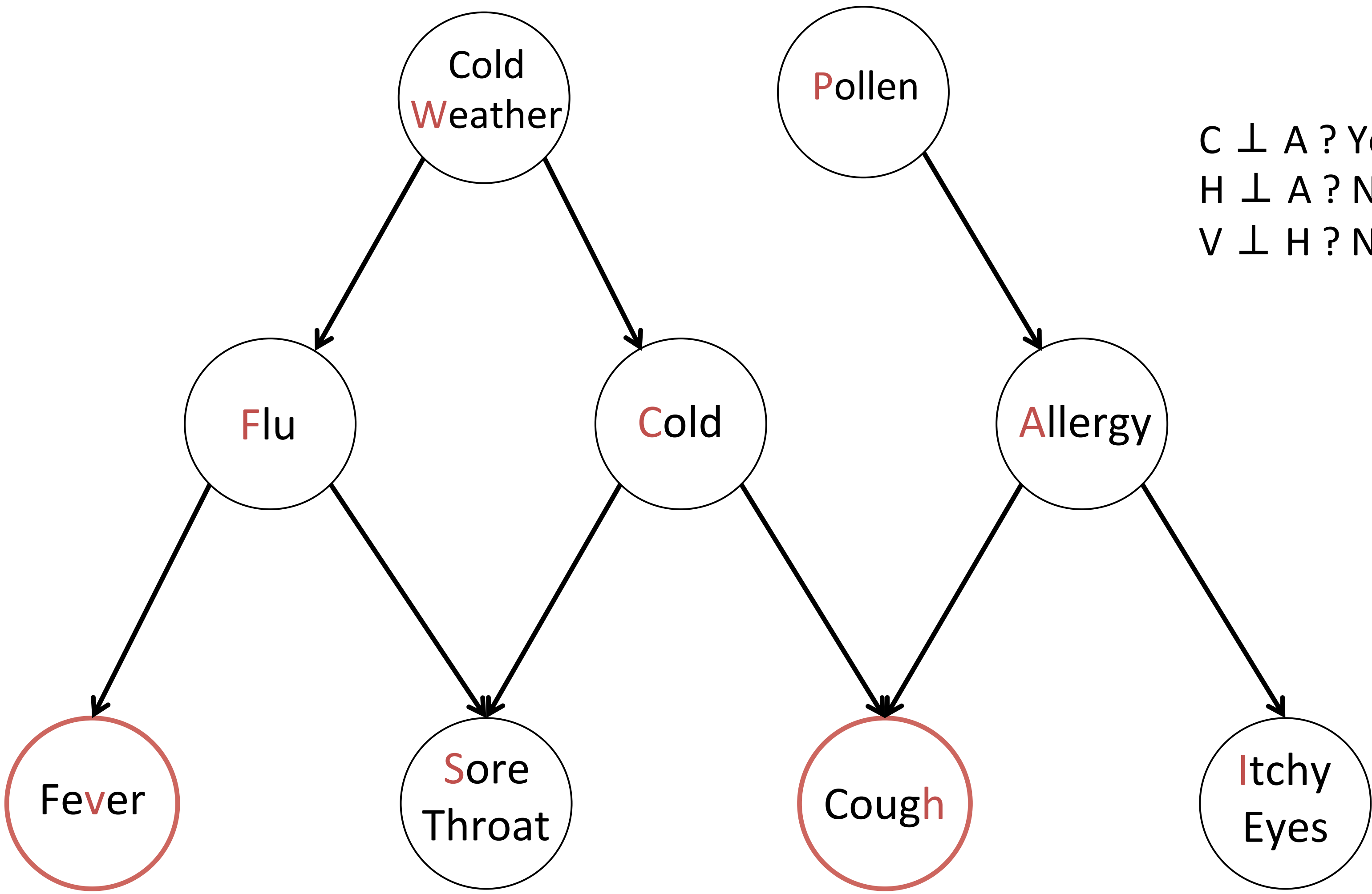
H ⊥ A ? No

Conditional Independence



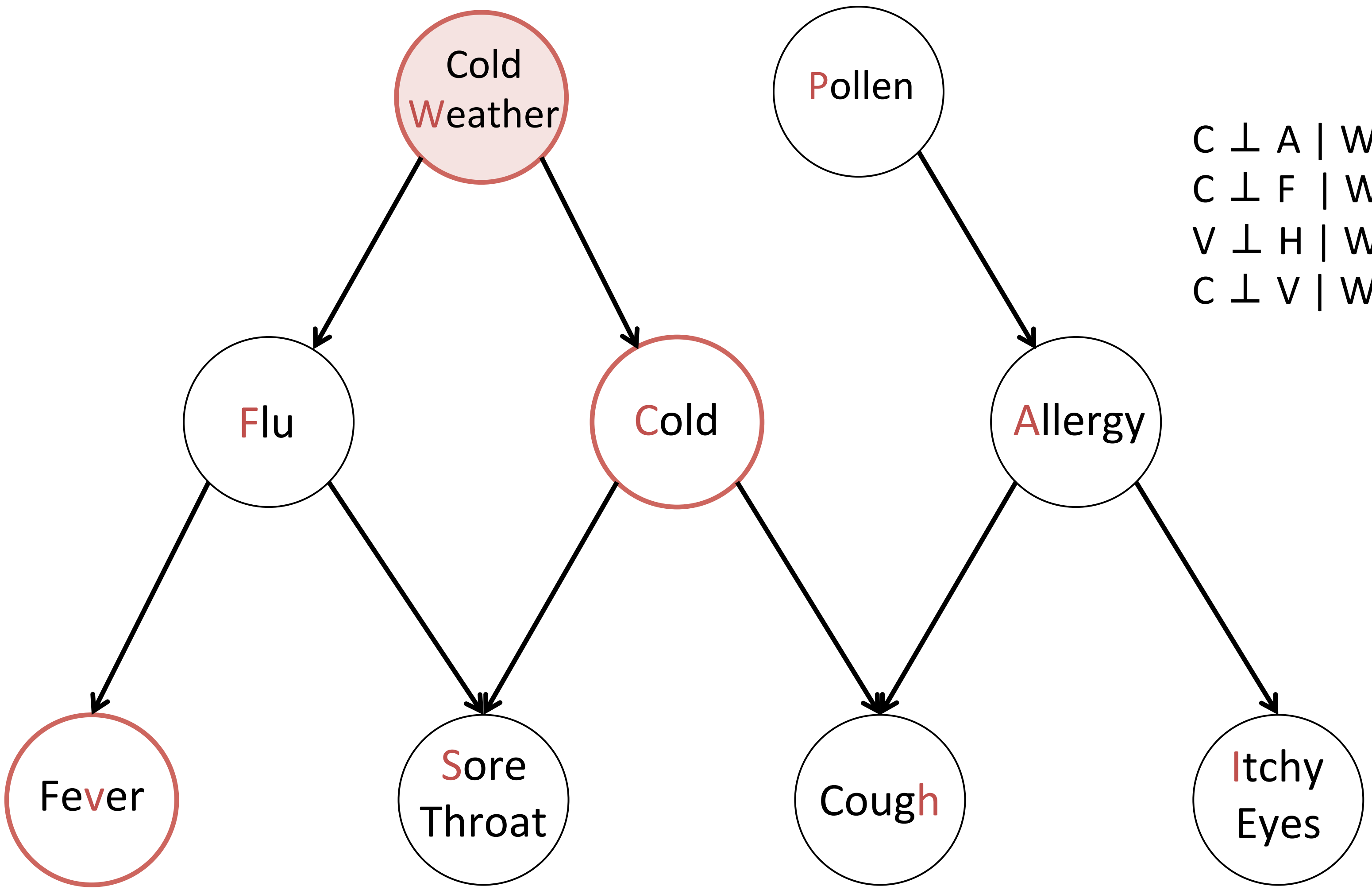
C ⊥ A ? Yes
H ⊥ A ? No
V ⊥ H ?

Conditional Independence



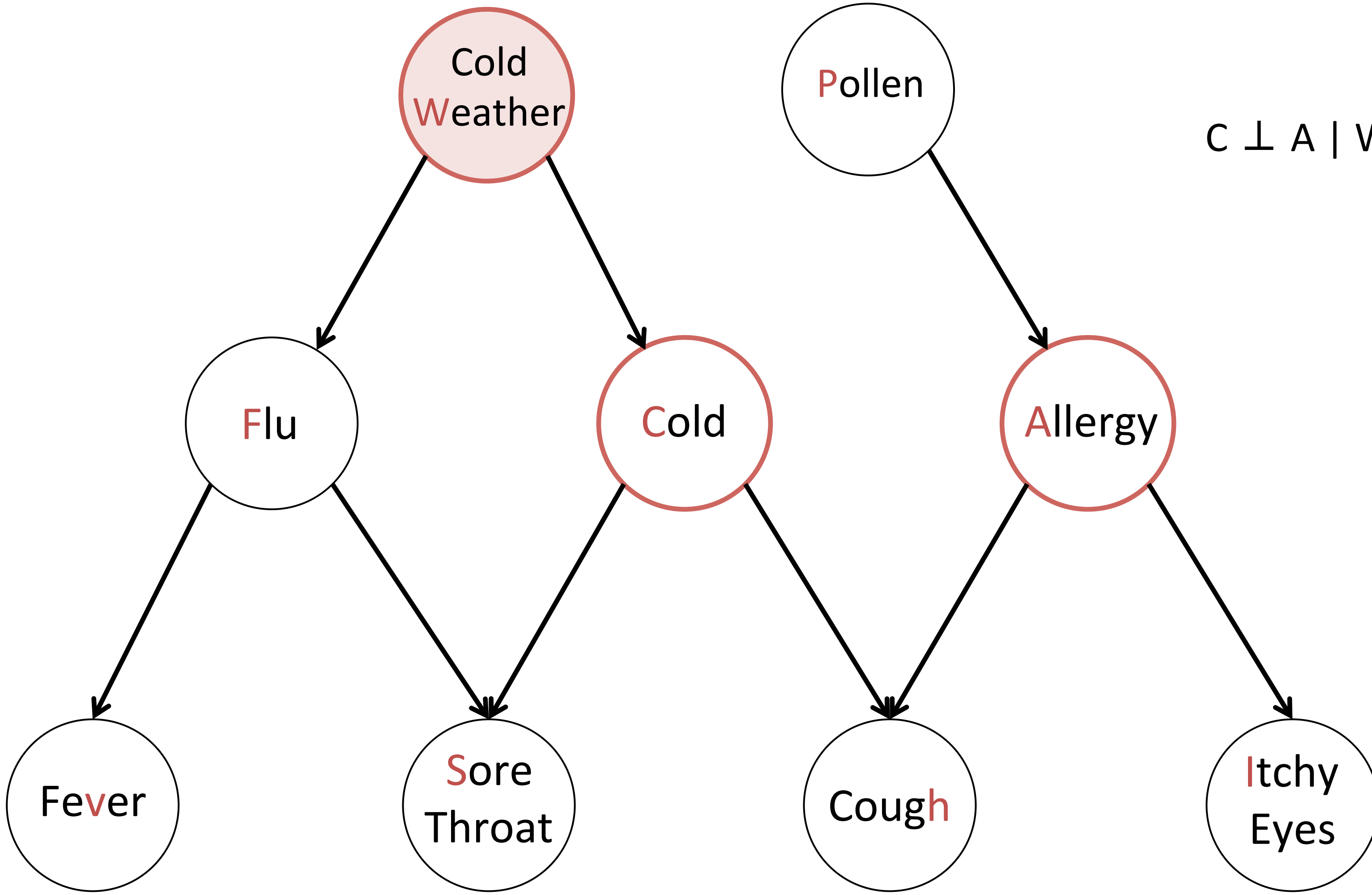
C ⊥ A ? Yes
H ⊥ A ? No
V ⊥ H ? No

Conditional Independence



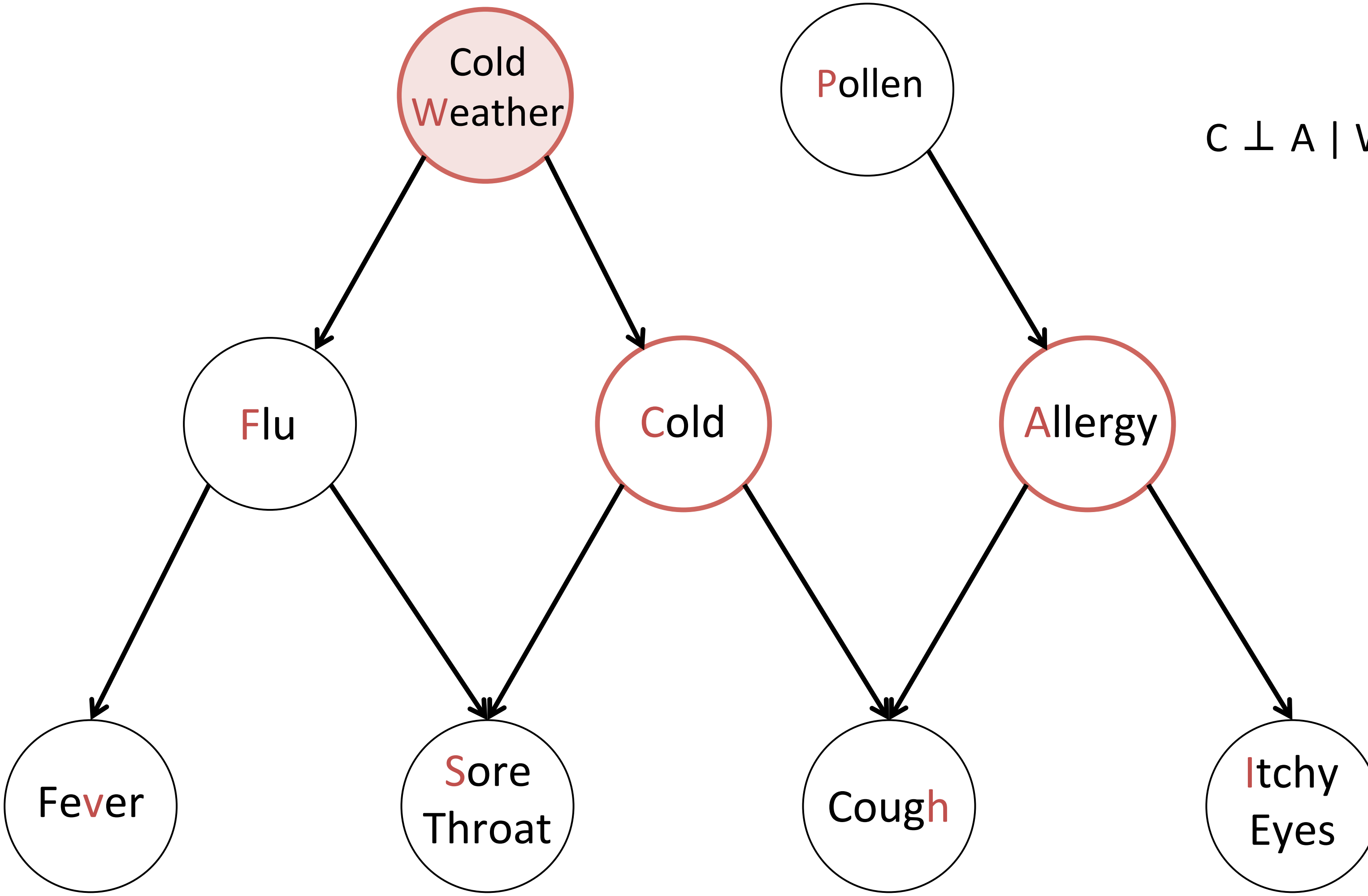
$C \perp A \mid W?$
$C \perp F \mid W?$
$V \perp H \mid W?$
$C \perp V \mid W?$

Conditional Independence



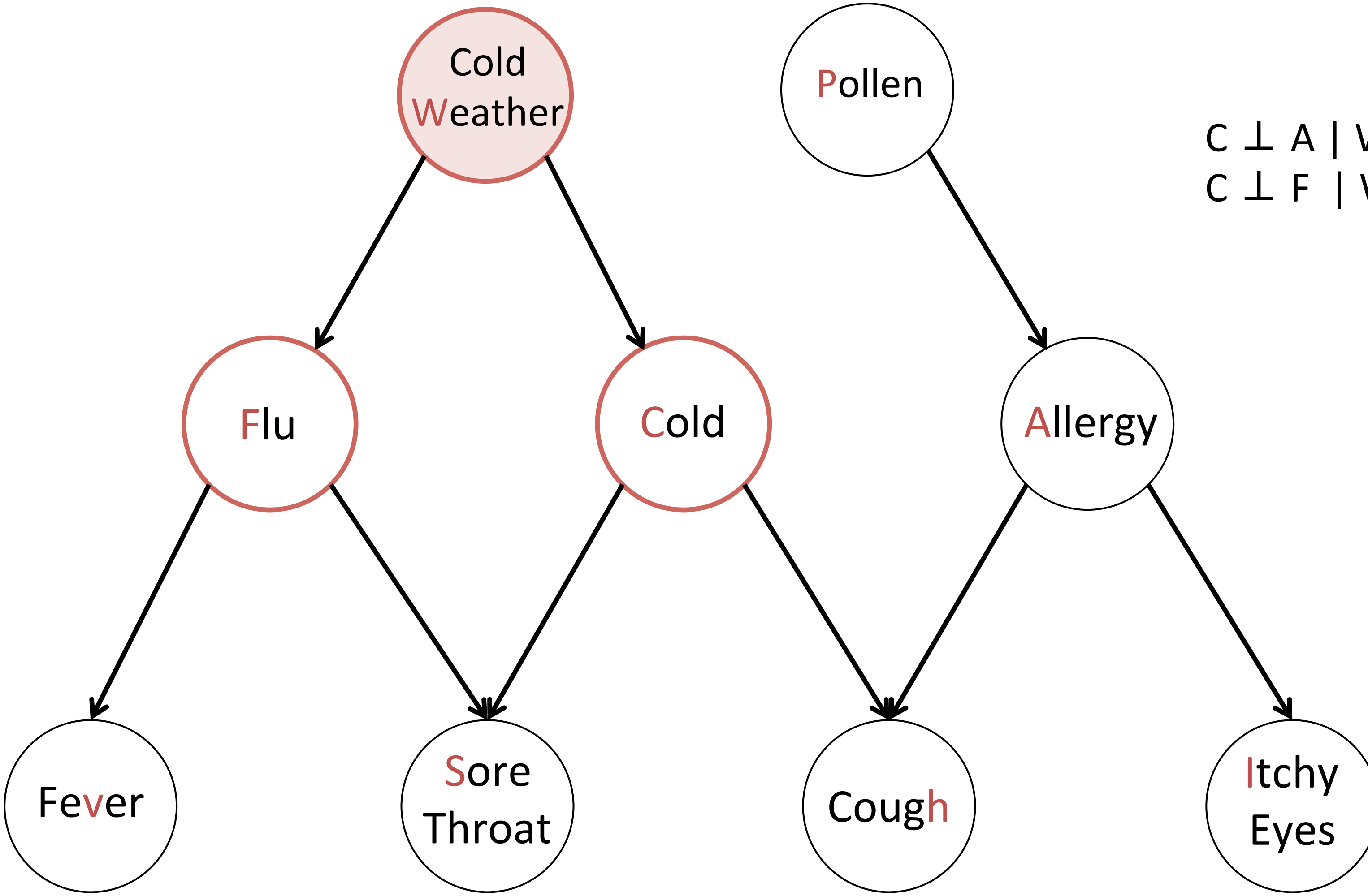
$C \perp A \mid W?$

Conditional Independence



C ⊥ A | W? Yes

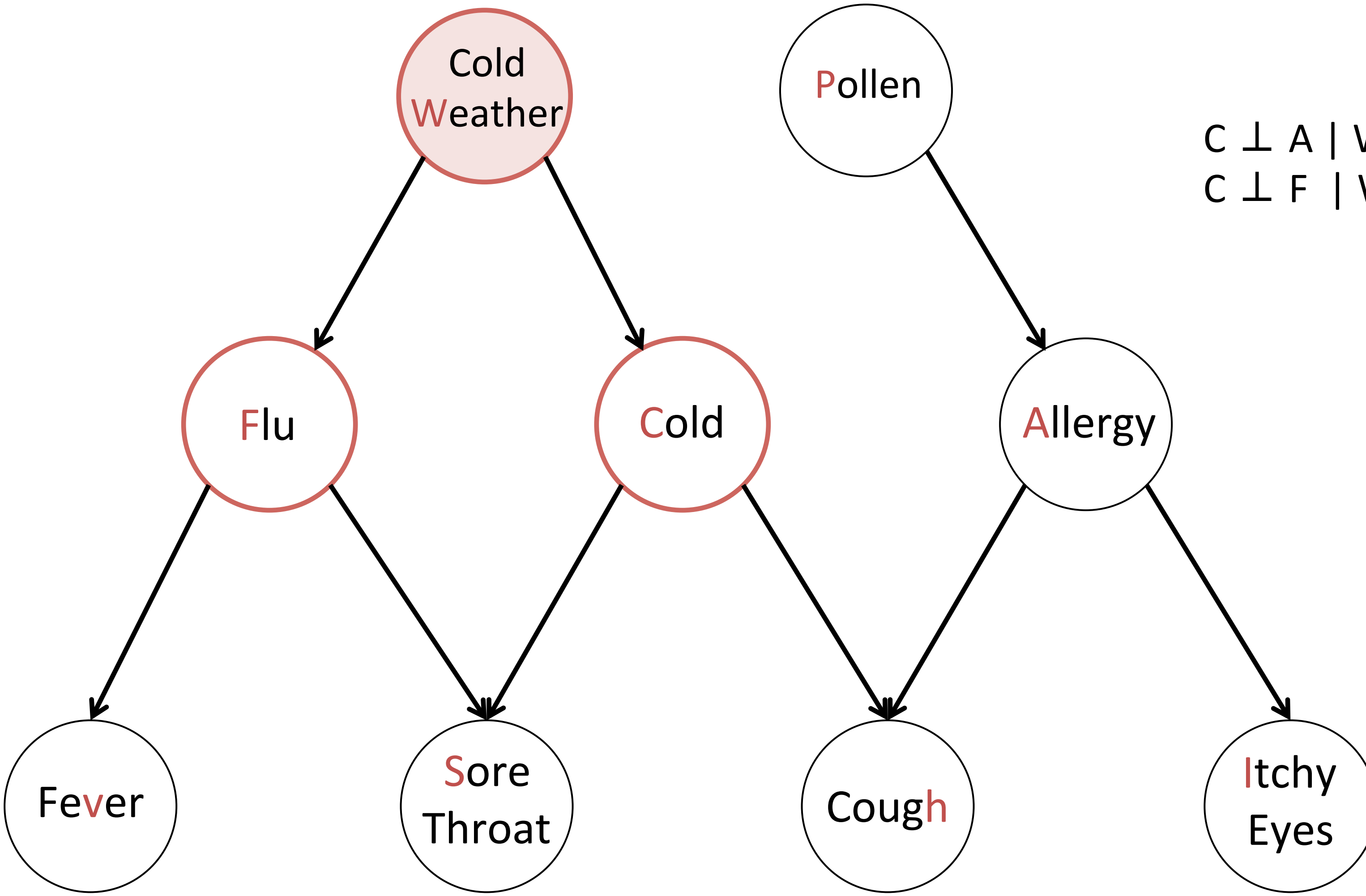
Conditional Independence



$C \perp A \mid W$? Yes

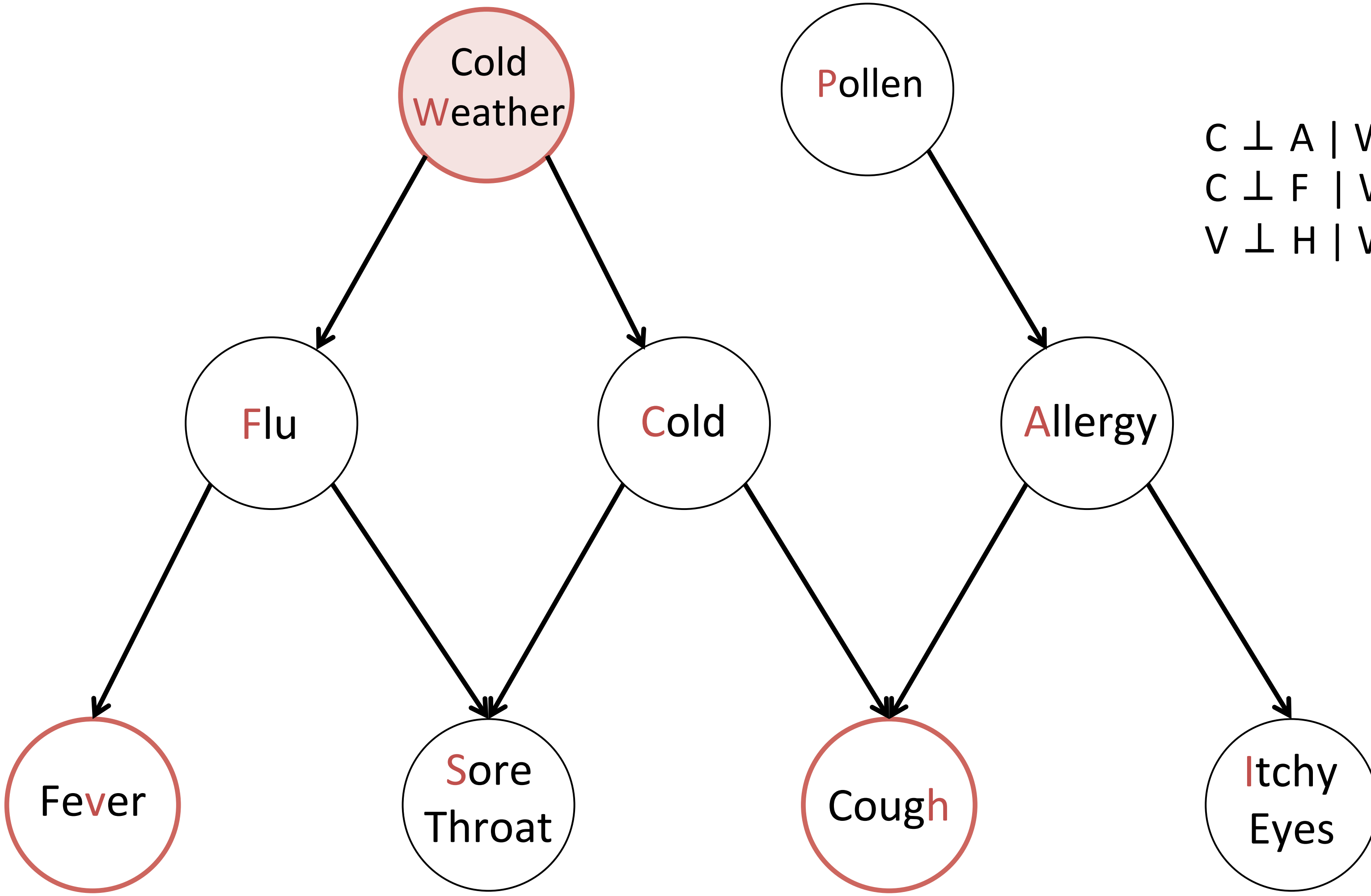
$C \perp F \mid W$?

Conditional Independence



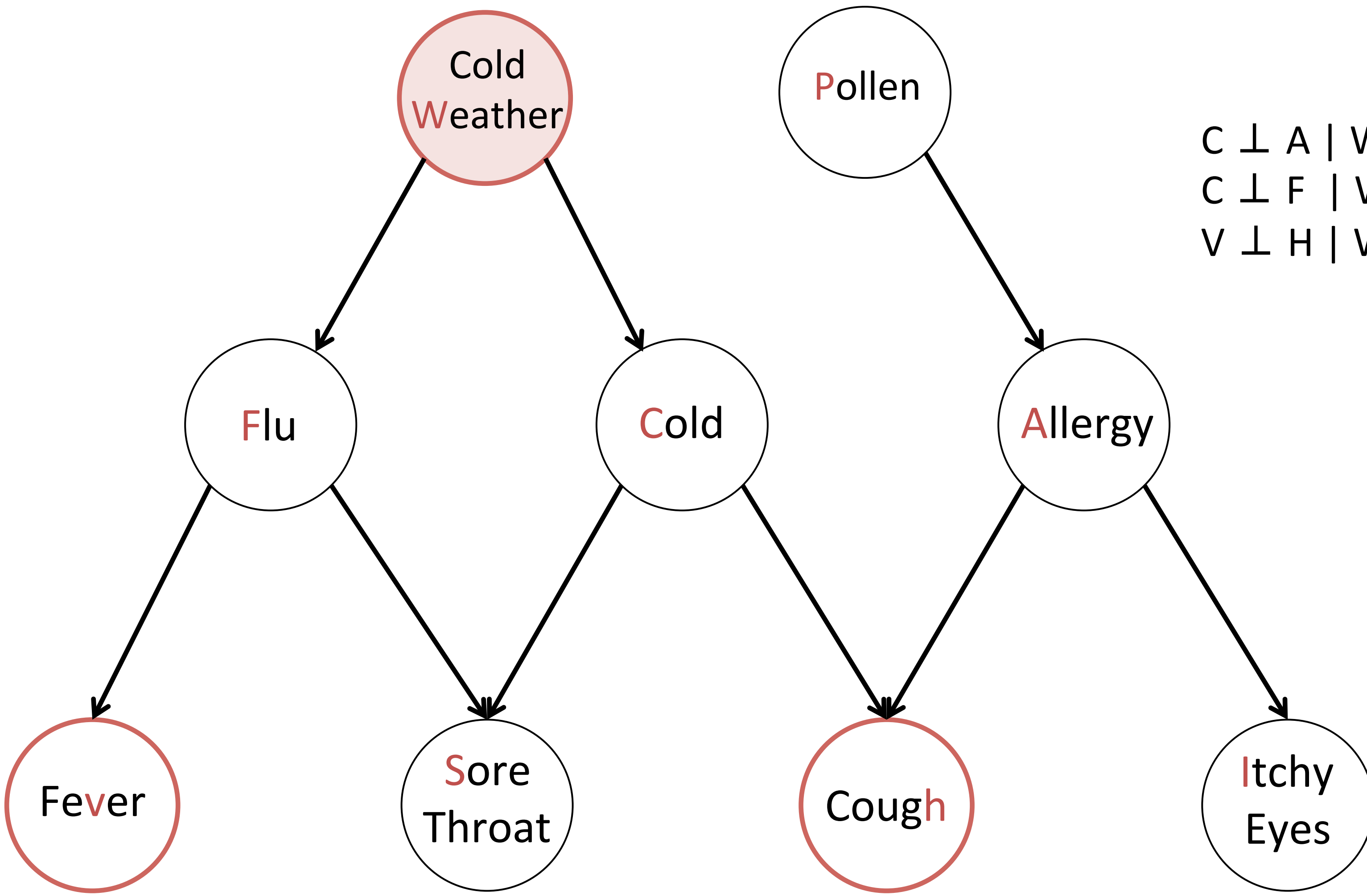
C ⊥ A | W? Yes
C ⊥ F | W? Yes

Conditional Independence



C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W?

Conditional Independence

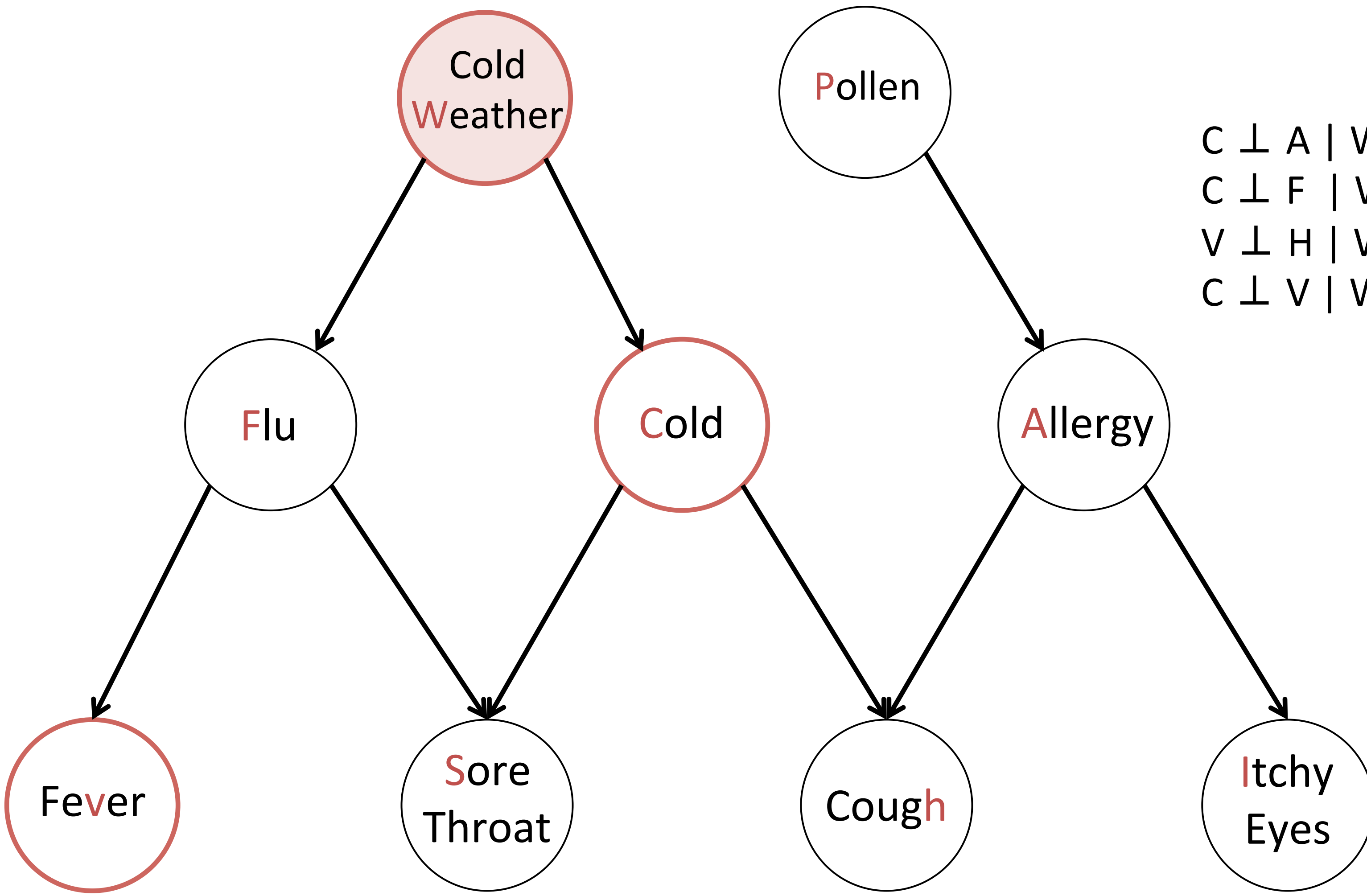


C ⊥ A | W? Yes

C ⊥ F | W? Yes

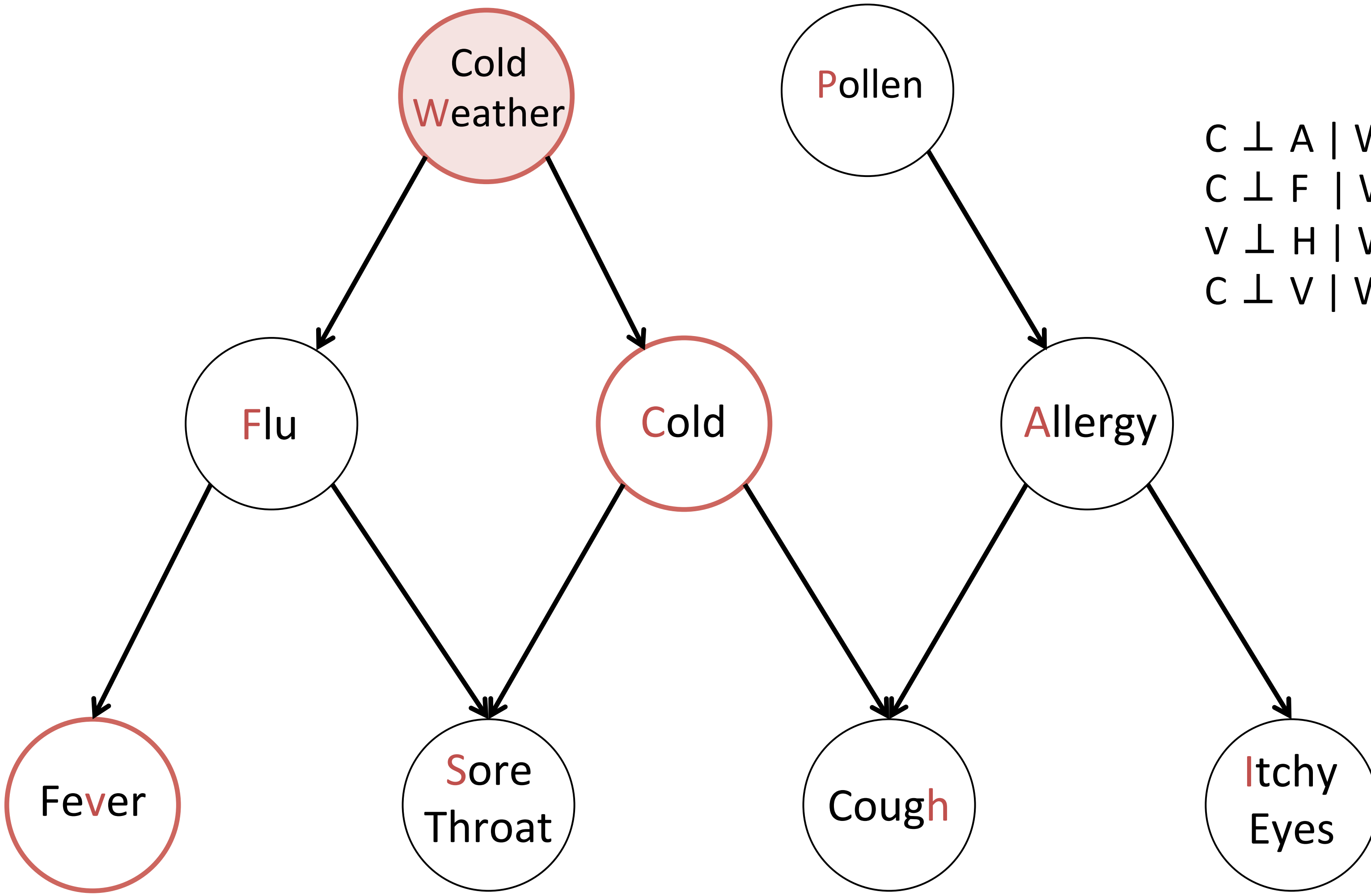
V ⊥ H | W? Yes

Conditional Independence



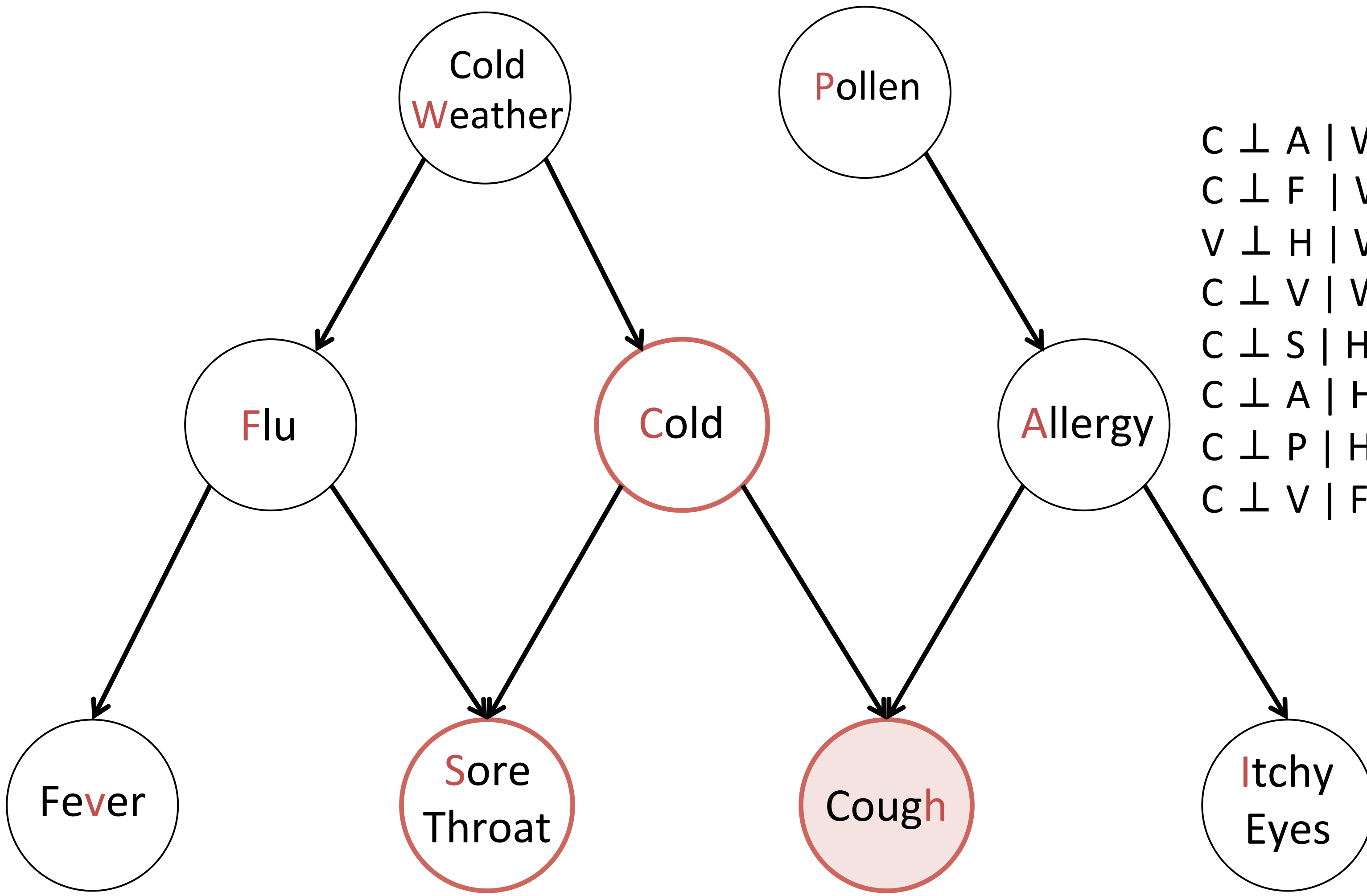
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W?

Conditional Independence



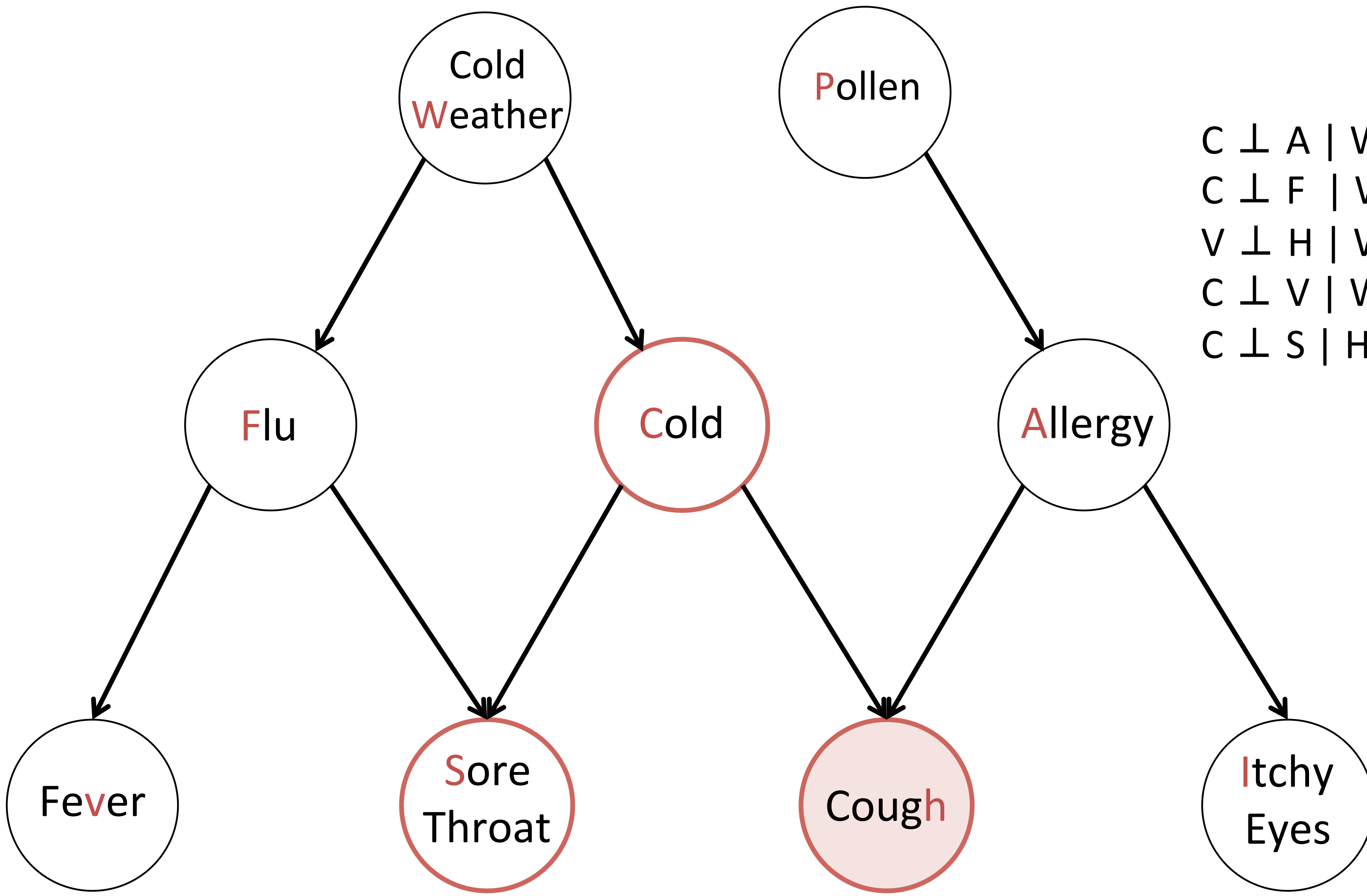
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes

Conditional Independence



$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$?
$C \perp A \mid H$?
$C \perp P \mid H$?
$C \perp V \mid F$?

Conditional Independence



$C \perp A \mid W$? Yes

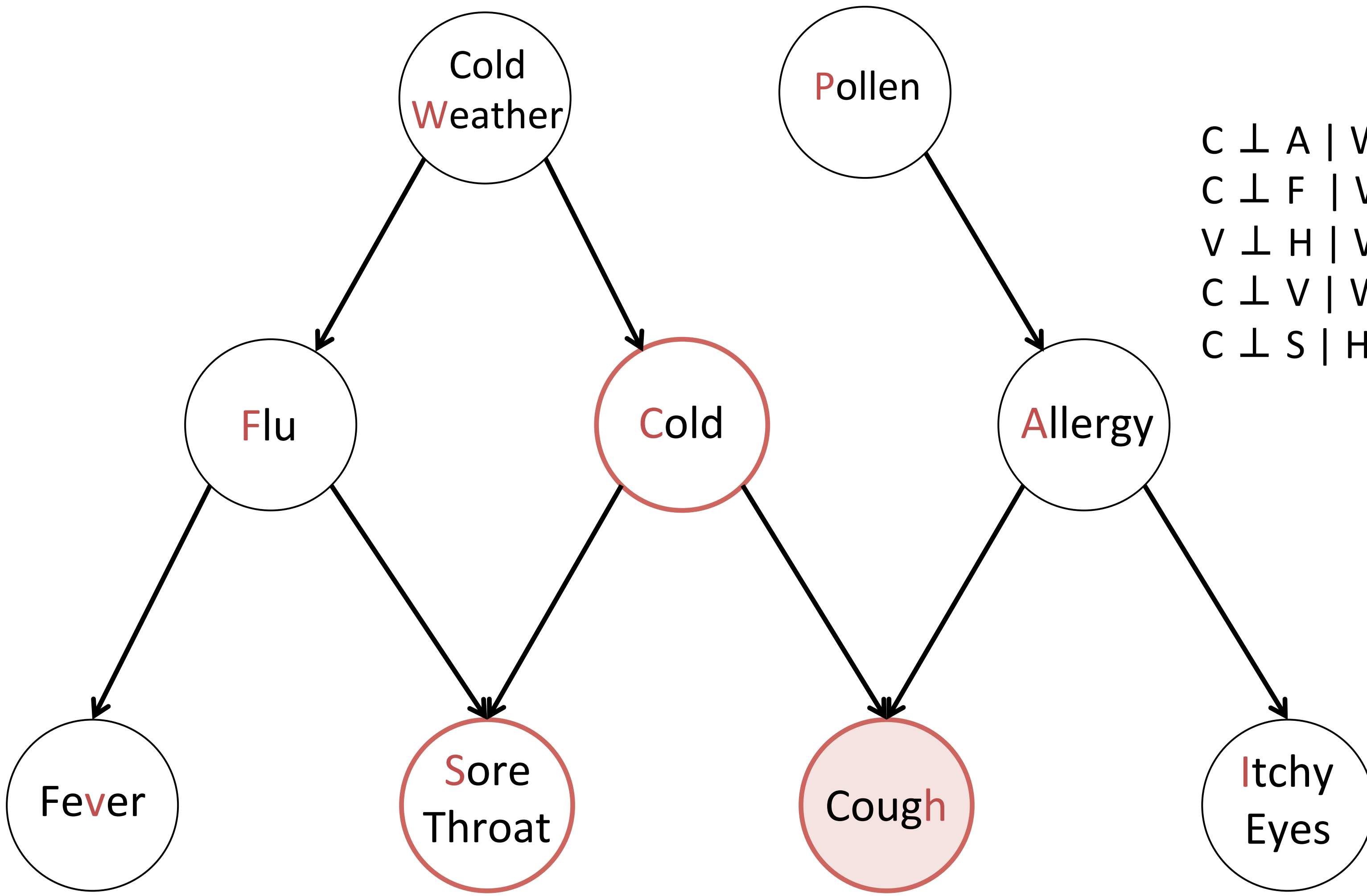
$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

$C \perp V \mid W$? Yes

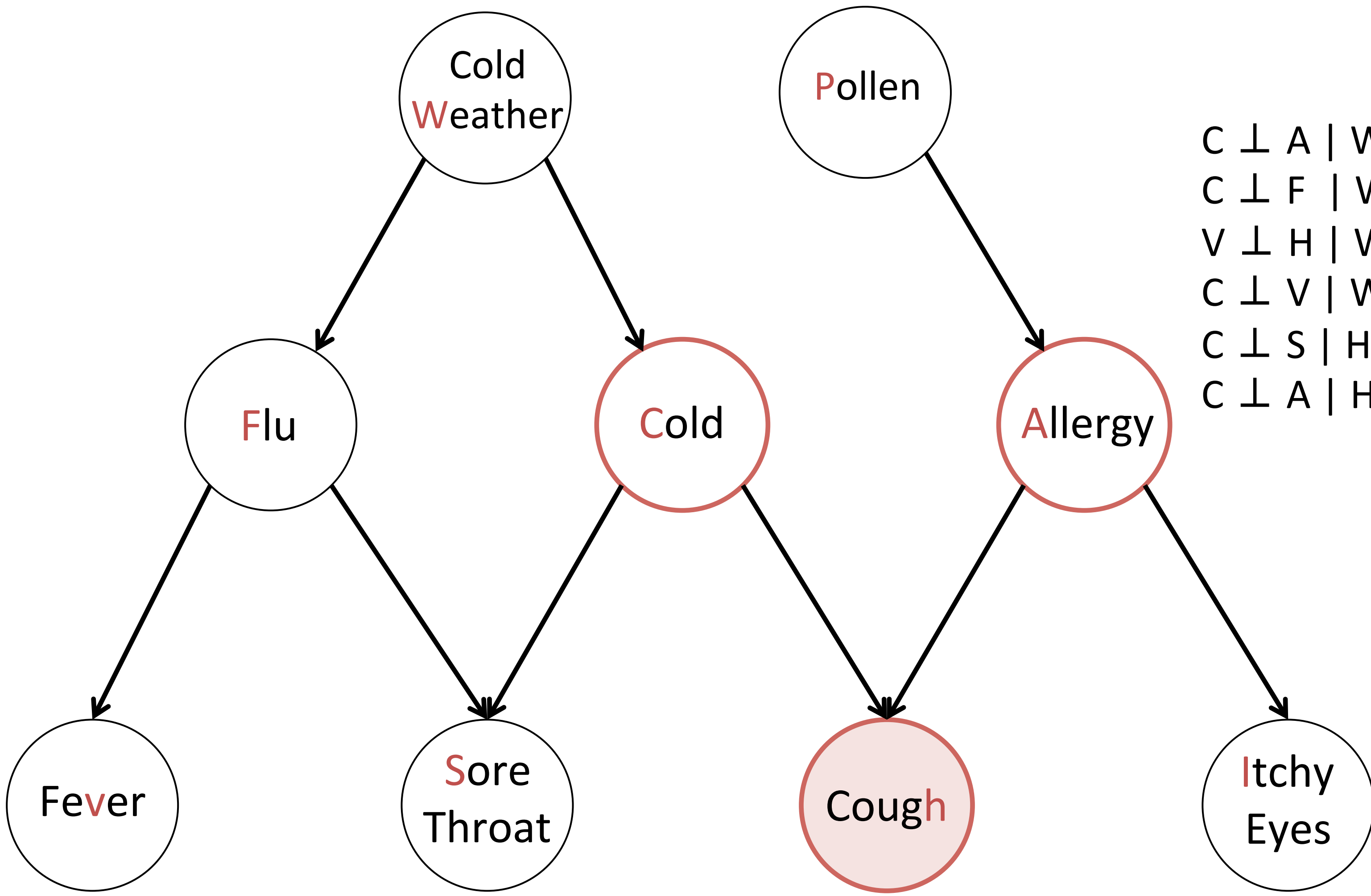
$C \perp S \mid H$?

Conditional Independence



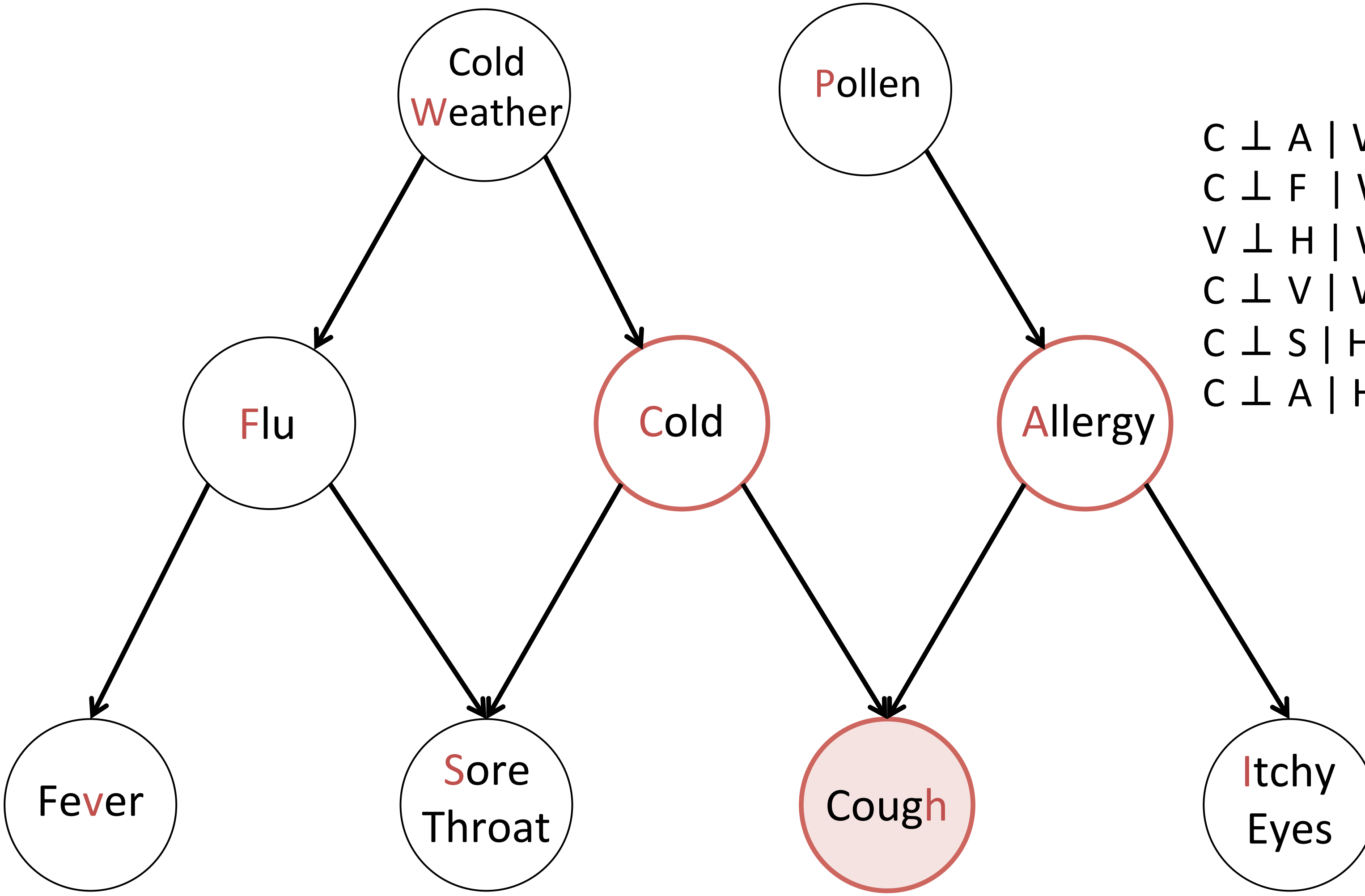
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No

Conditional Independence



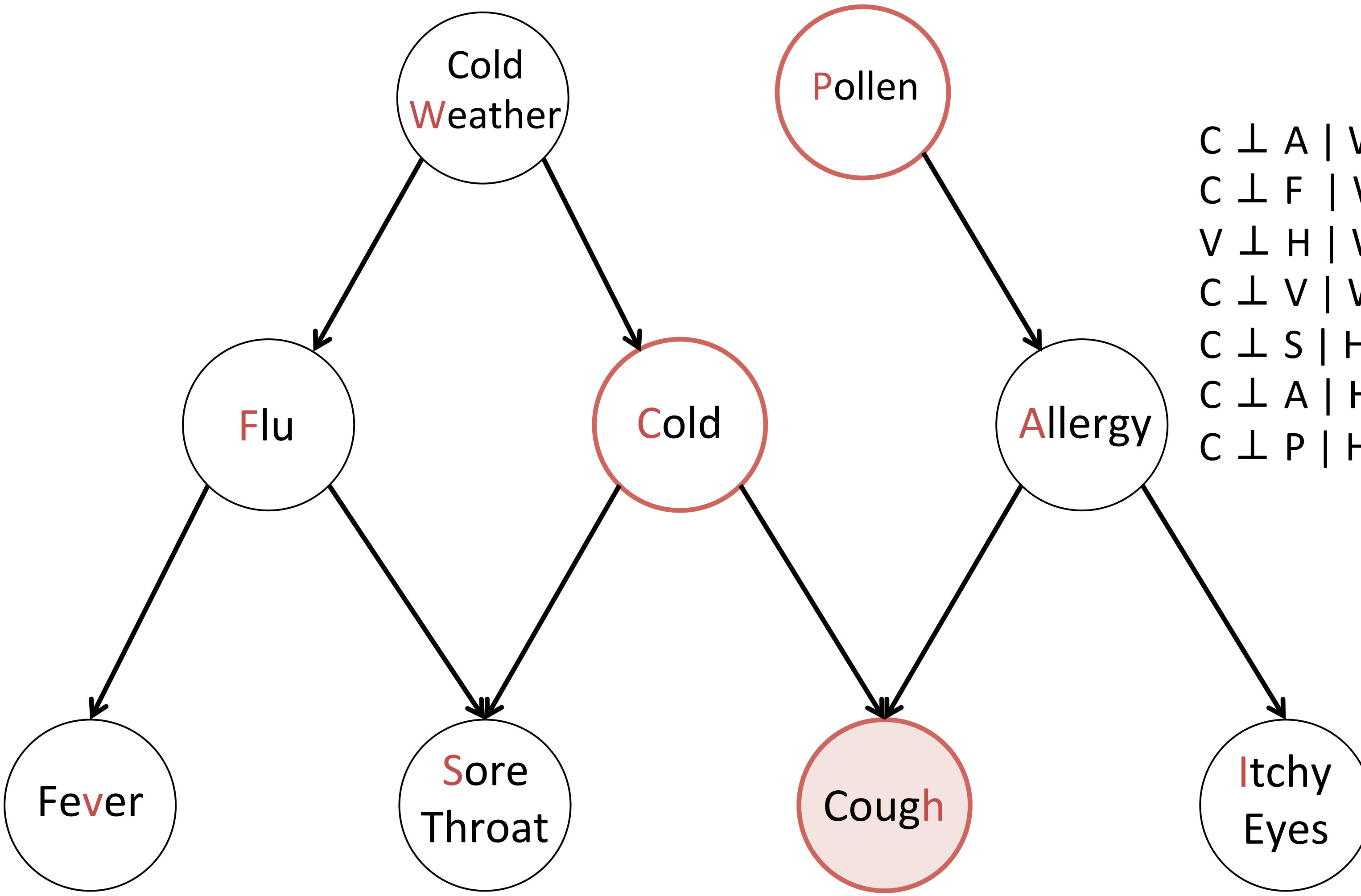
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes
C ⊥ S | H? No
C ⊥ A | H?

Conditional Independence



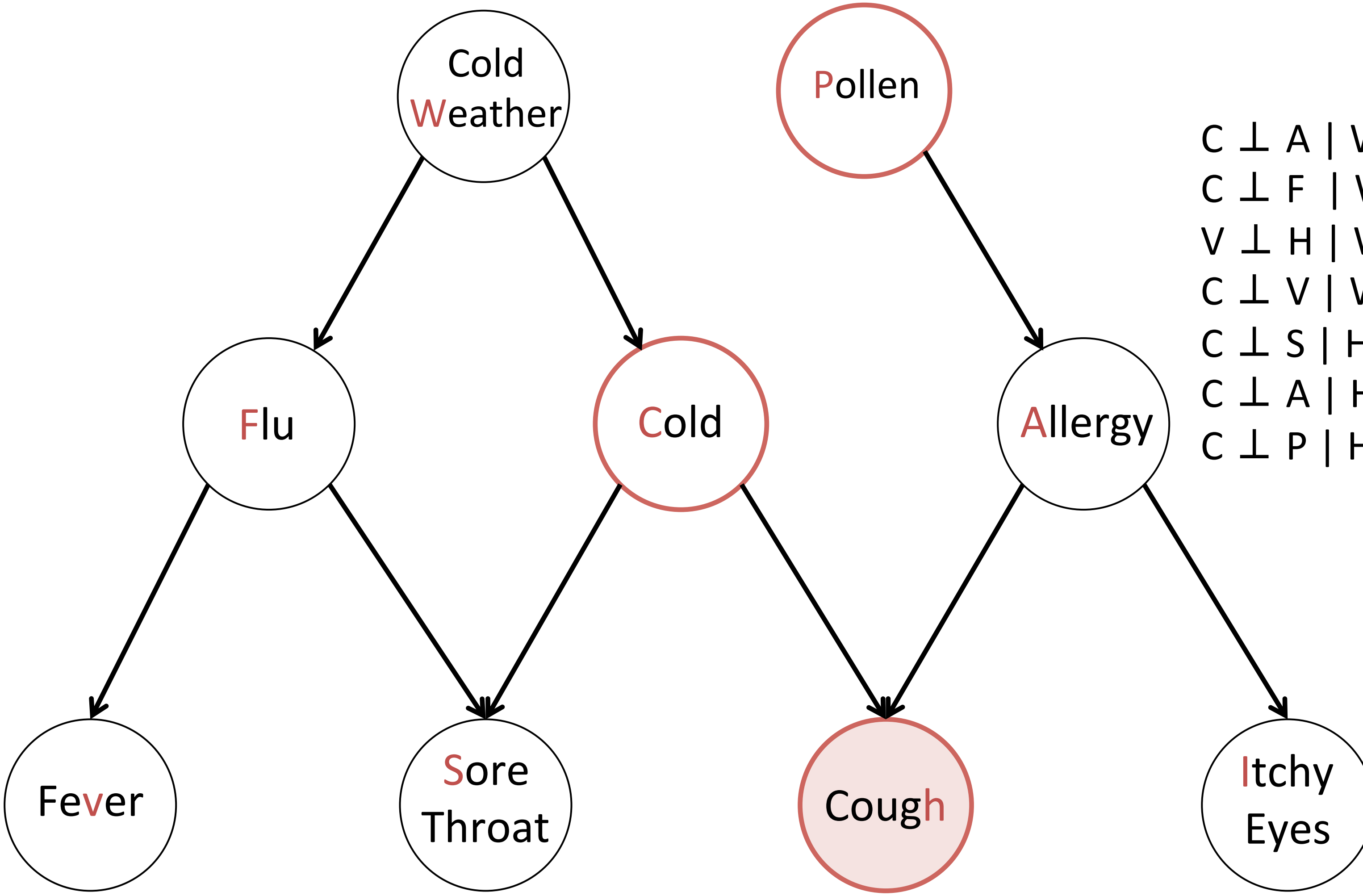
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No

Conditional Independence



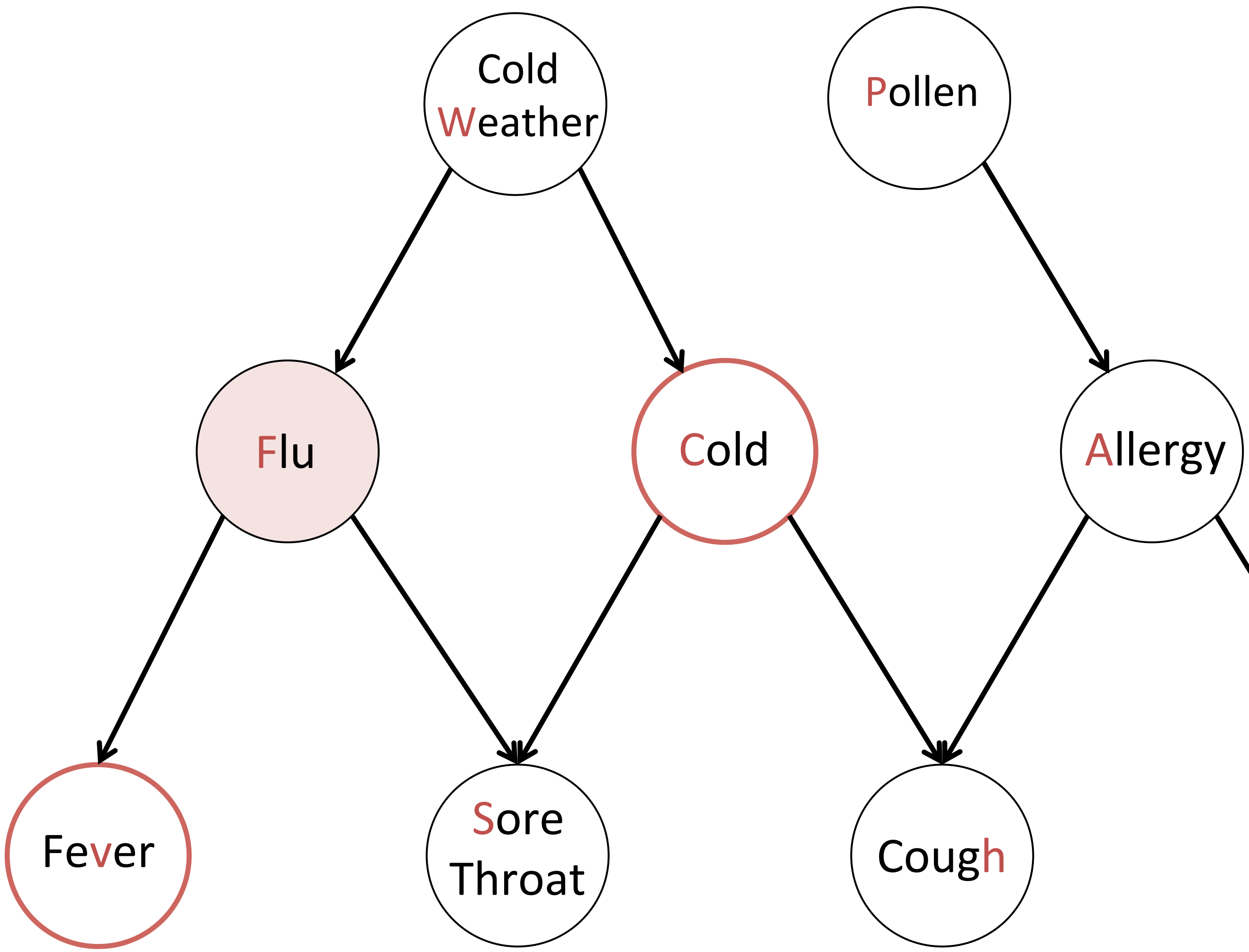
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$?

Conditional Independence



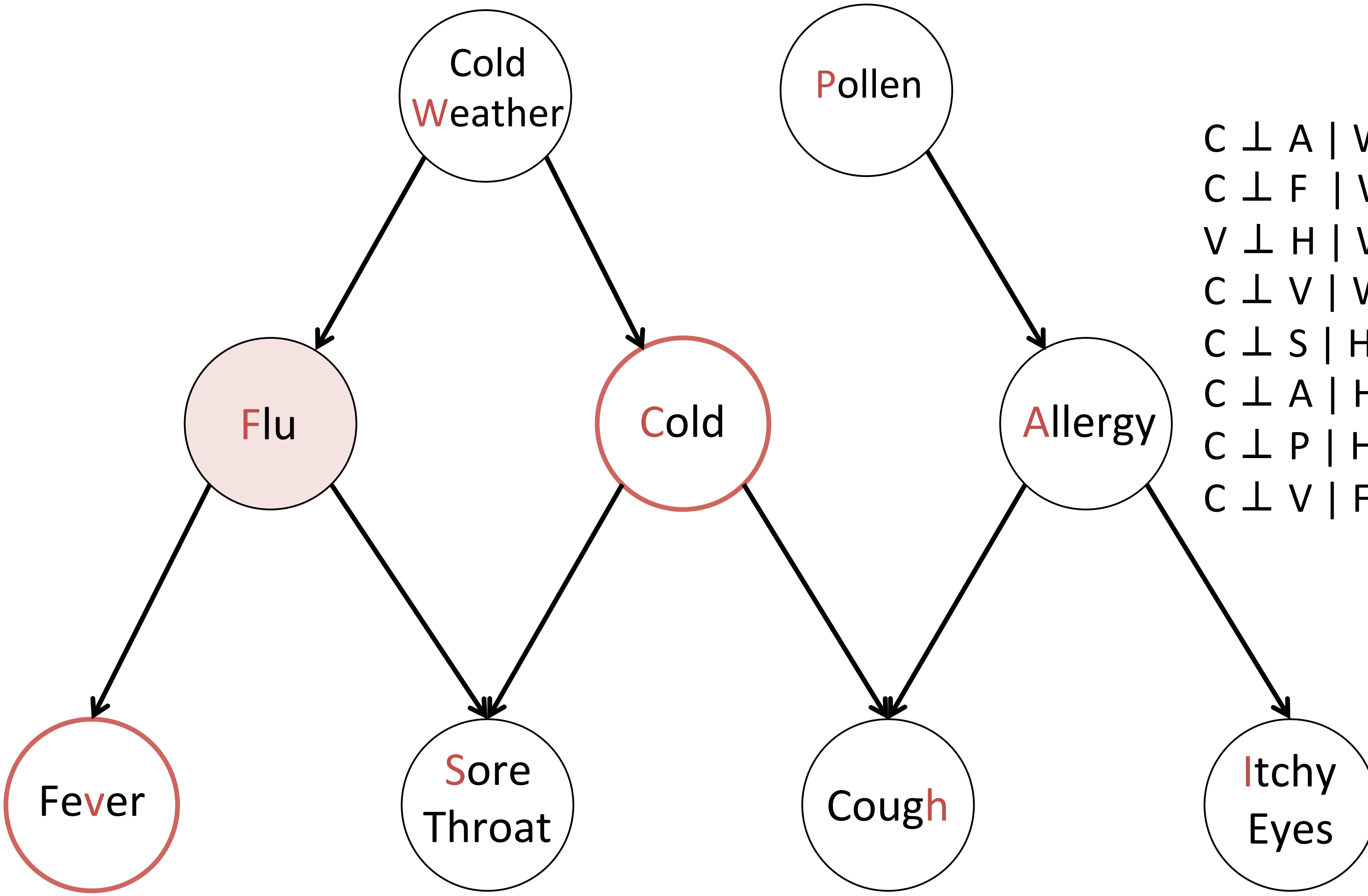
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$? No

Conditional Independence



$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$? No
$C \perp V \mid F$?

Conditional Independence



$C \perp A \mid W$? Yes

$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

$C \perp V \mid W$? Yes

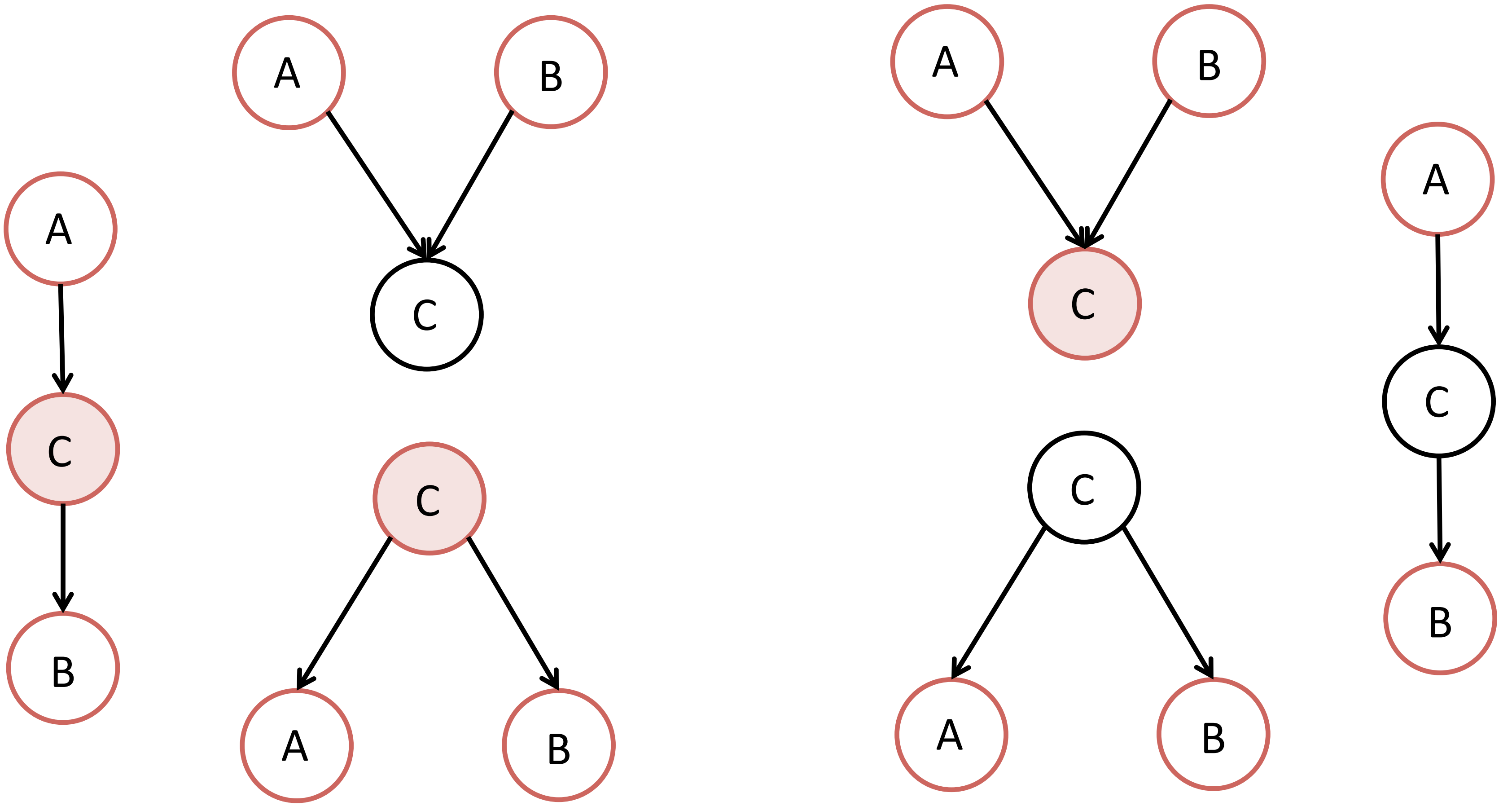
$C \perp S \mid H$? No

$C \perp A \mid H$? No

$C \perp P \mid H$? No

$C \perp V \mid F$? Yes

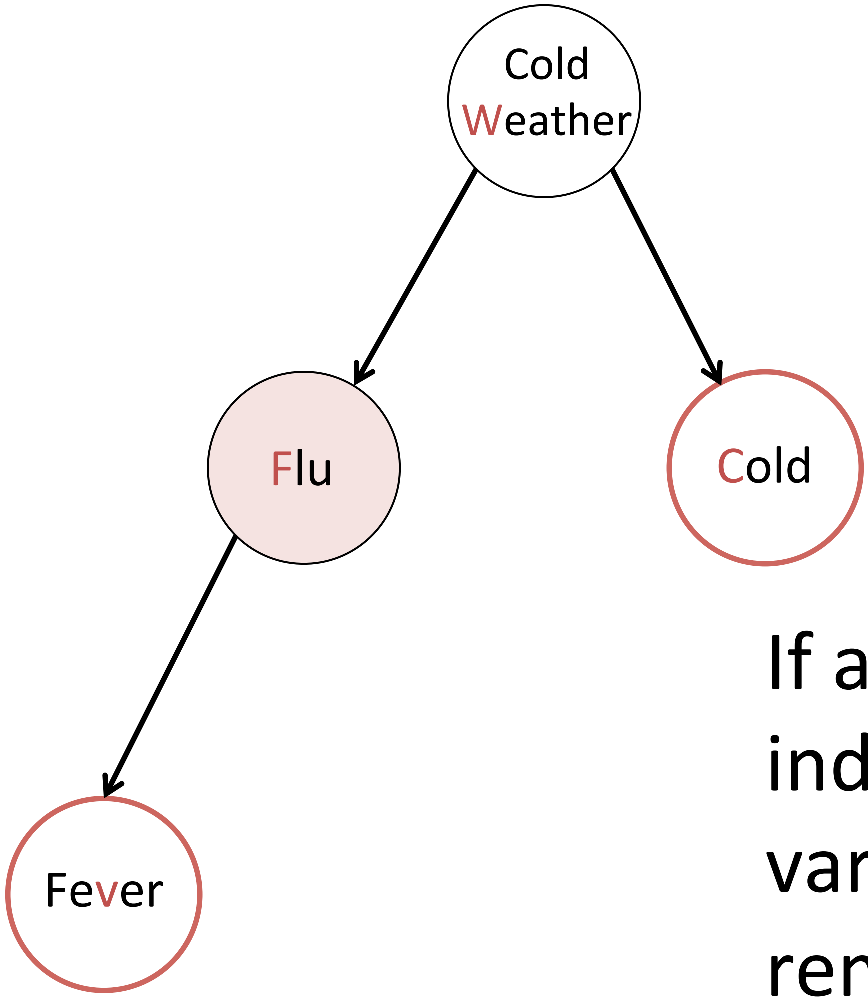
Patterns



Independent

Dependent

Conditional Independence



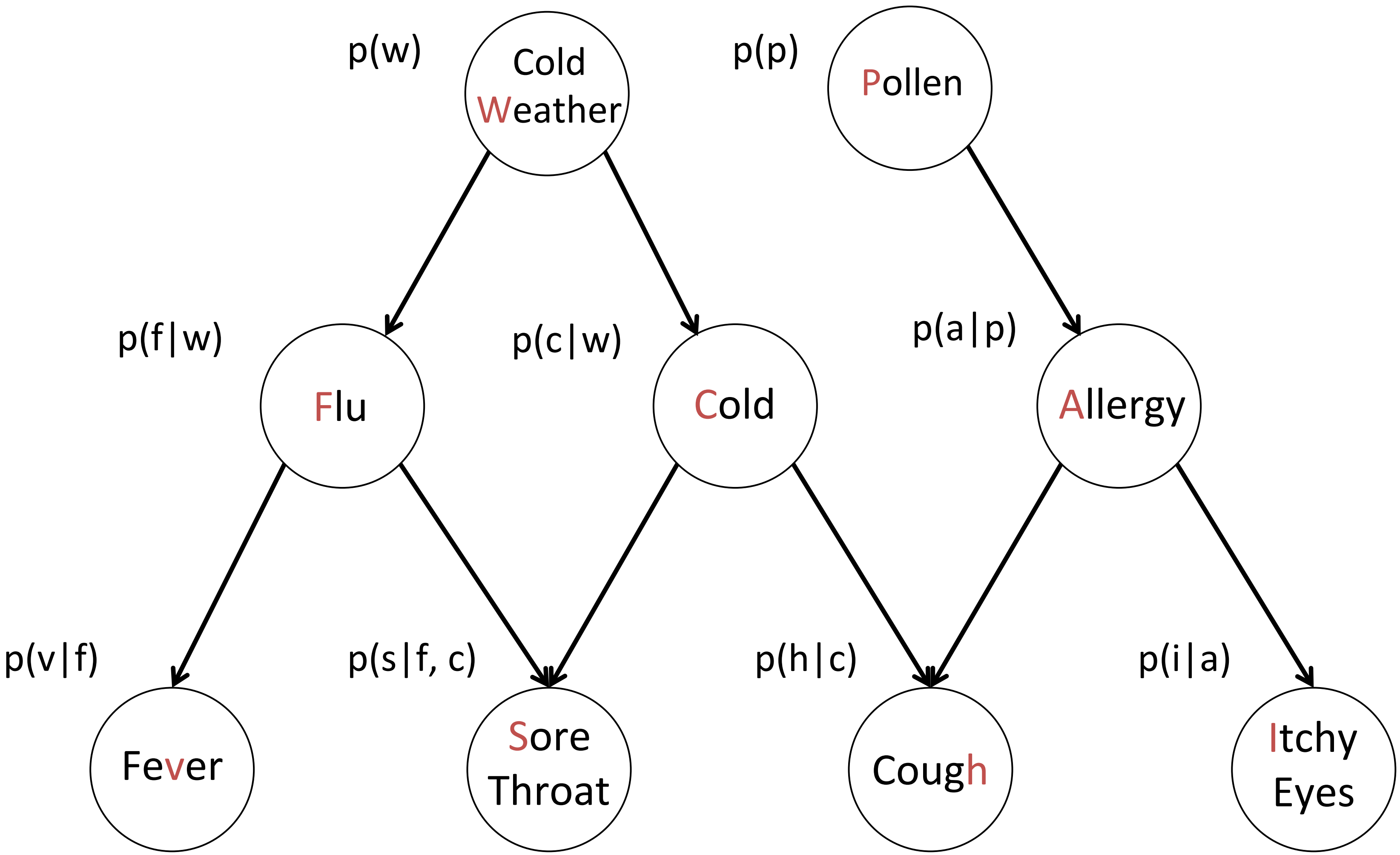
P(C=c|F=f) = ?

If a variable (Fever) is independent of the Query variable Q (Cold), we can remove(marginalize) it.

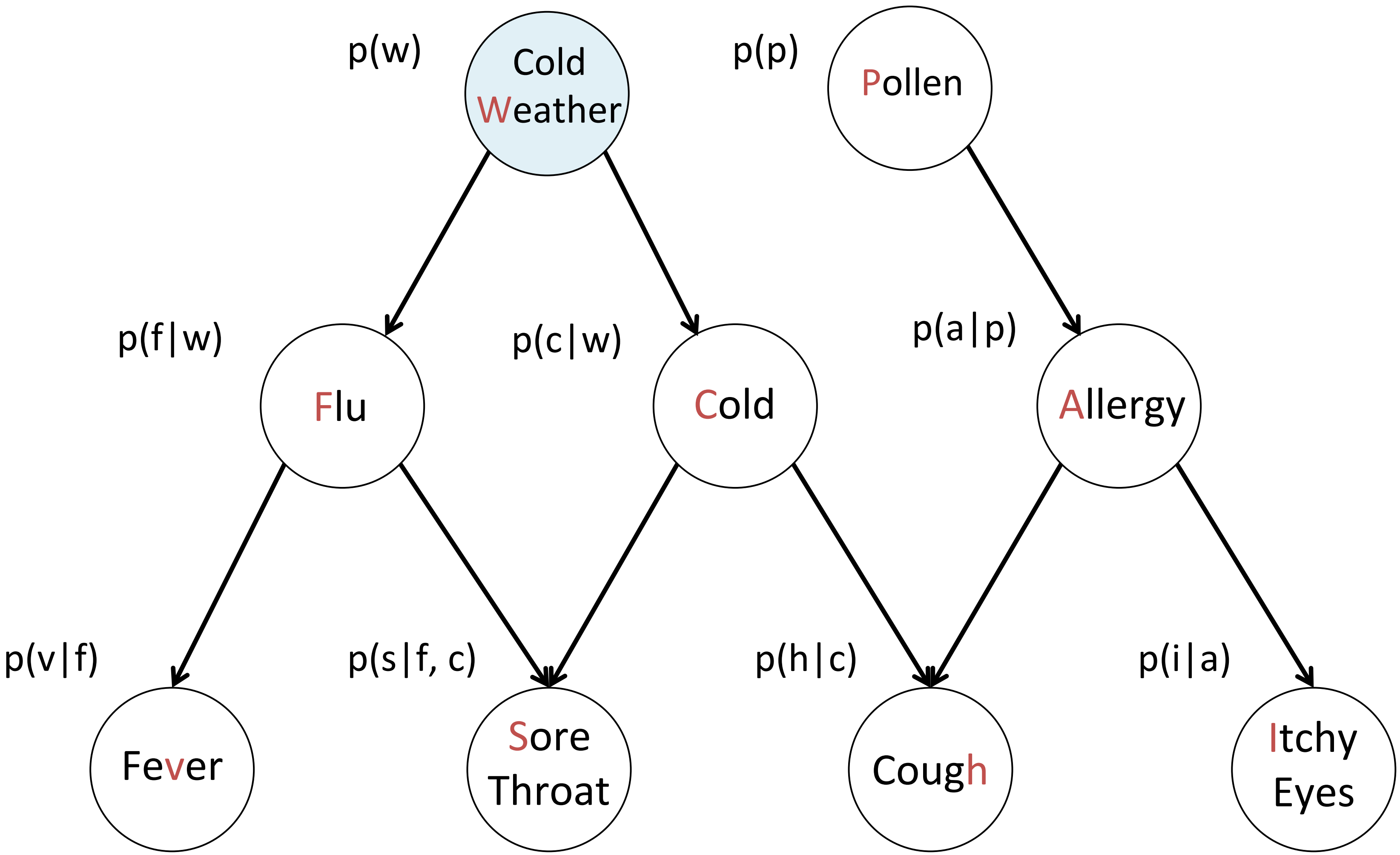
Roadmap

- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence
- Gibbs Sampling

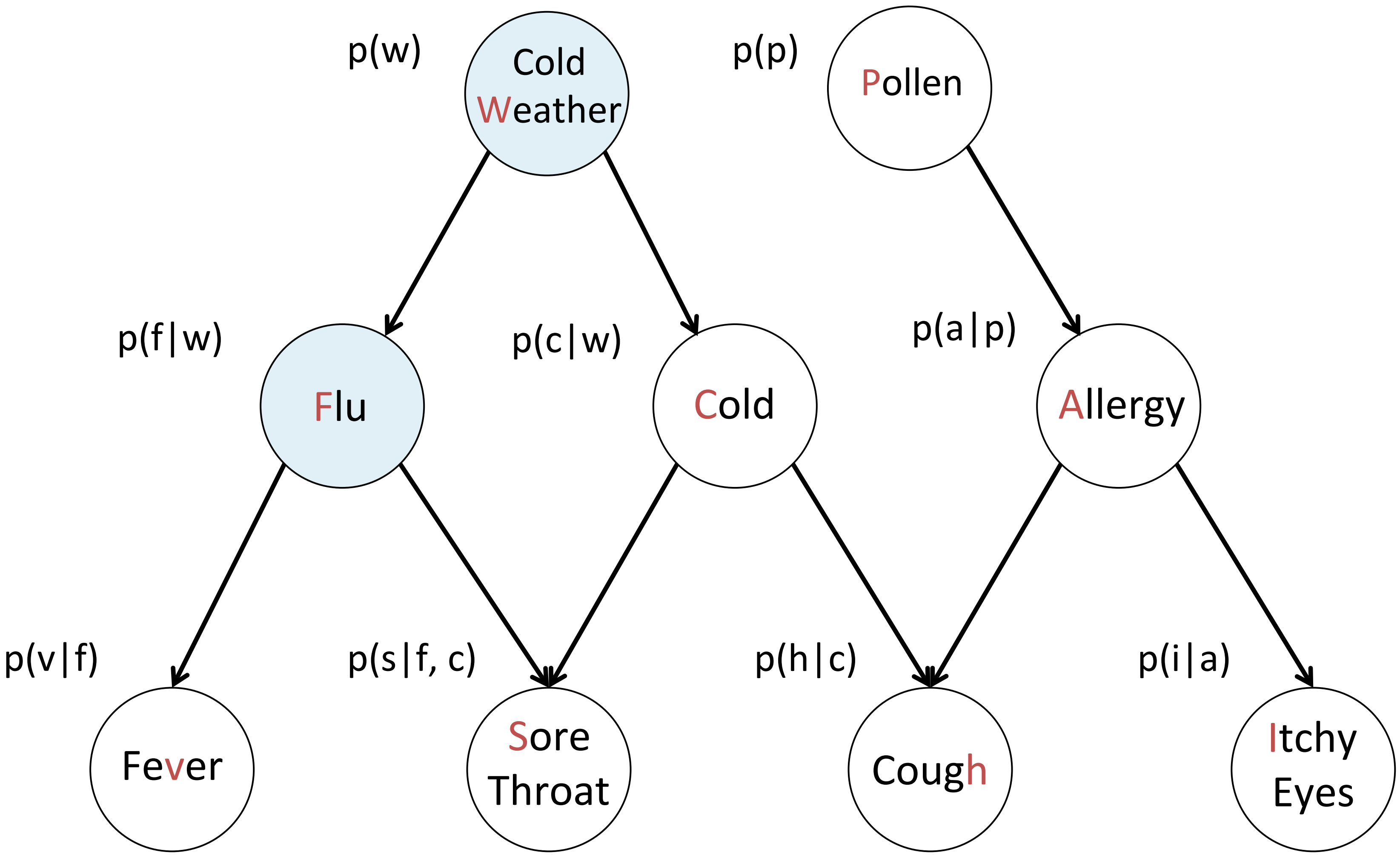
Sample 1M samples from joint distribution



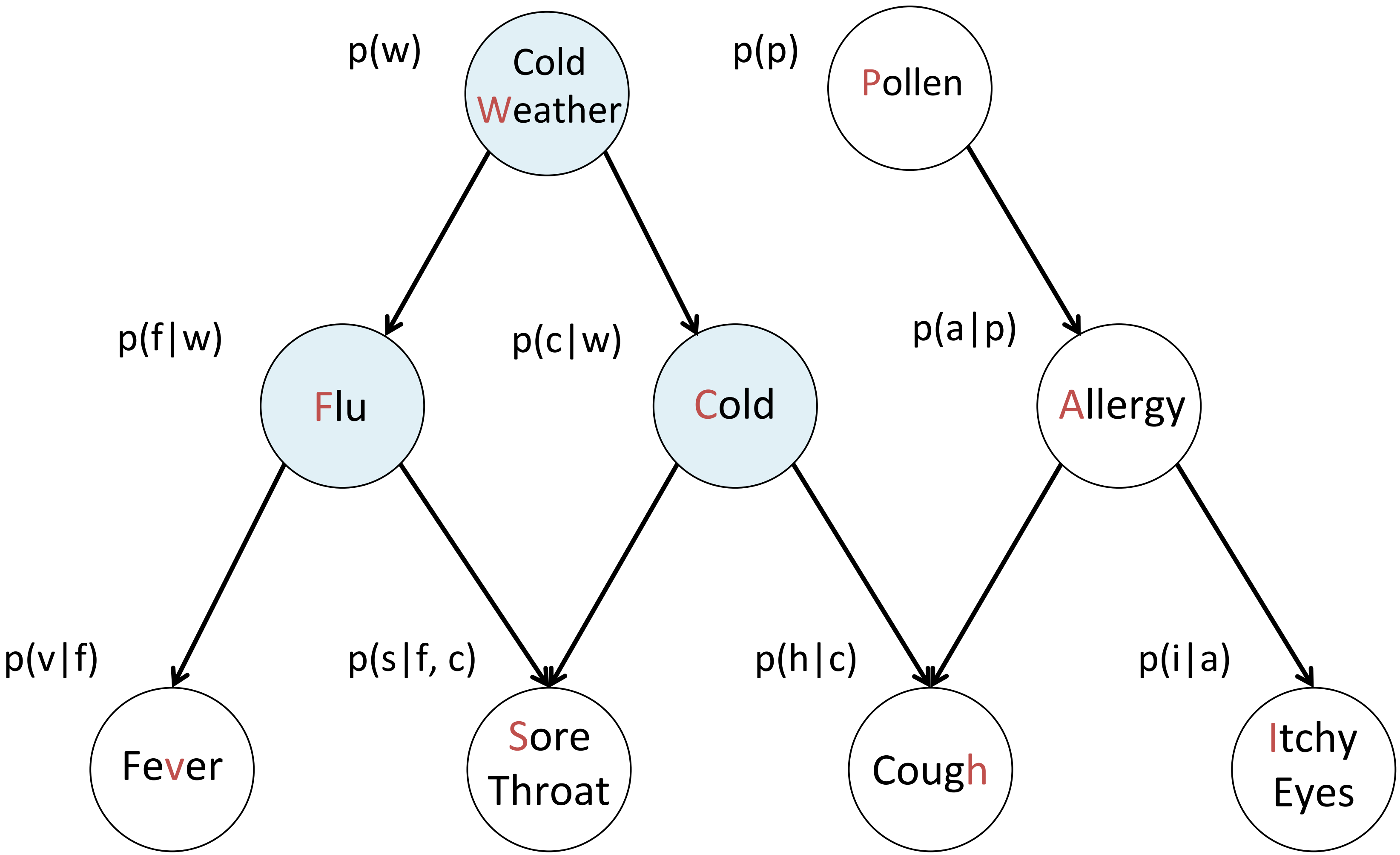
Forward Sampling



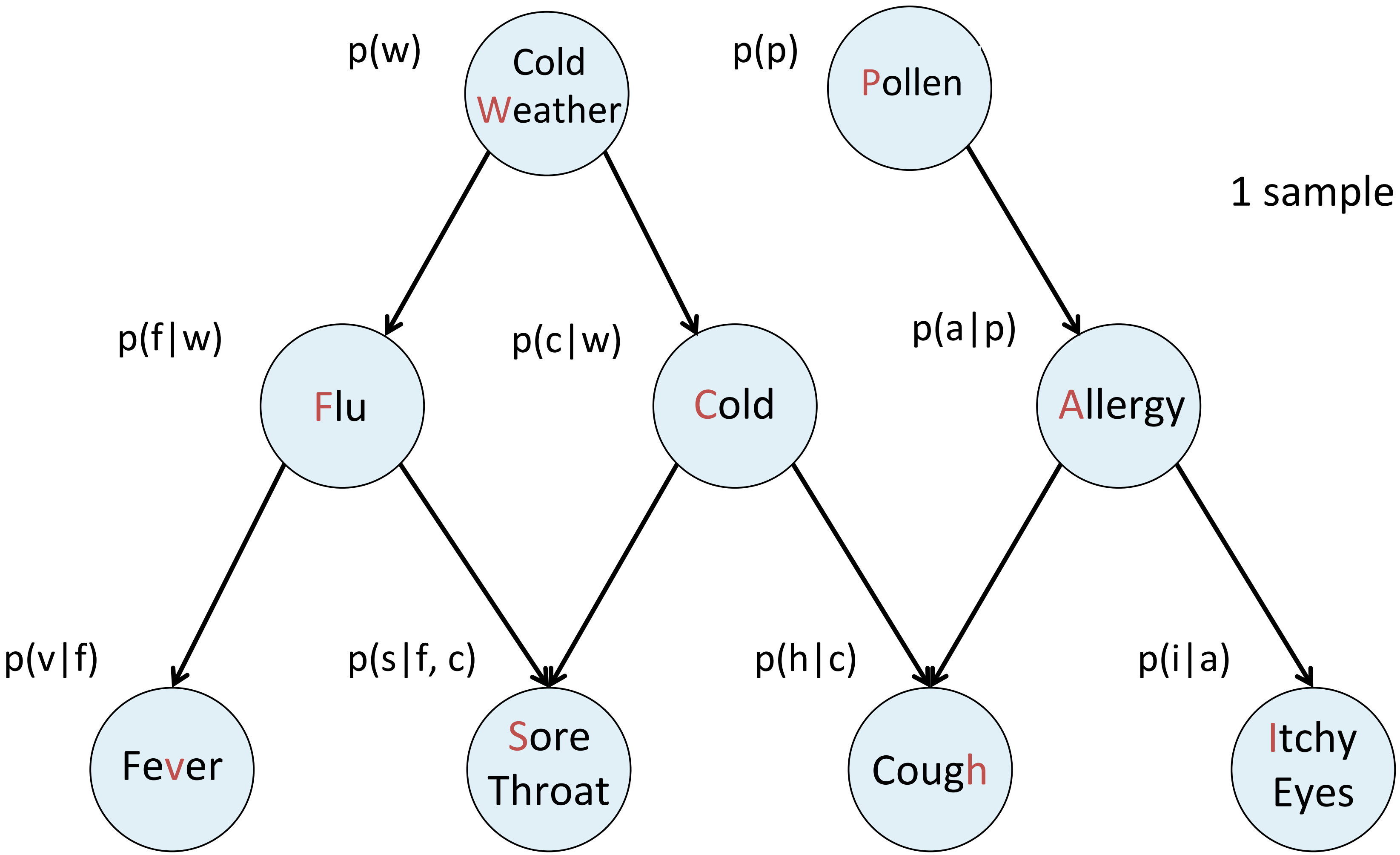
Forward Sampling



Forward Sampling



Forward Sampling



Gibbs Sampling

Algorithm: Gibbs sampling

Initialize x to a random complete assignment

Loop through $i = 1, \ldots, n$ until convergence:

for each v, compute weight of $\{X_i : v\} \cup x \setminus \{x_i\}$

Choose $\{X_i : v\} \cup x \setminus \{x_i\}$ with prob prop. to weight

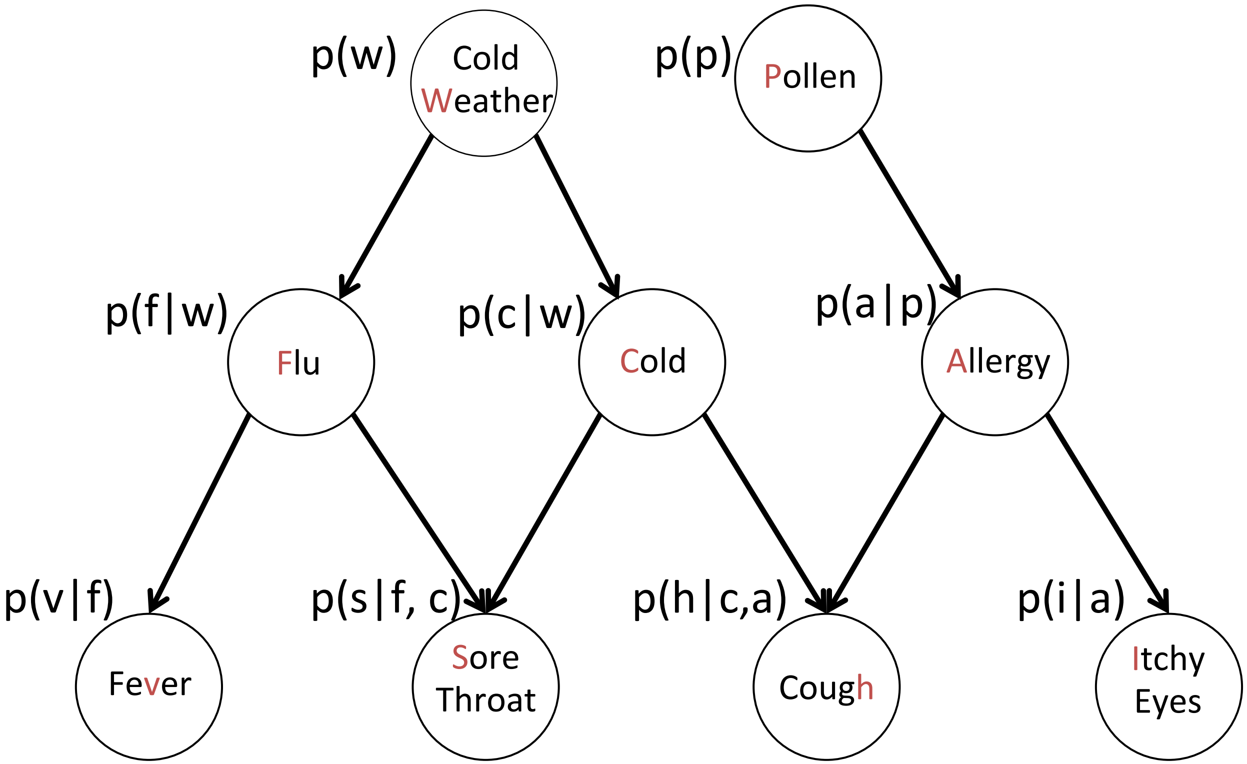
Gibbs sampling (probabilistic interpretation)

Loop through $i = 1, \ldots, n$ until convergence:

Set $X_i = v$ with prob. $\mathbb{P}(X_i = v \mid X_{-i} = x_{-i})$

(notation: $X_{-i} = X \setminus \{X_i\}$)

Gibbs Sampling

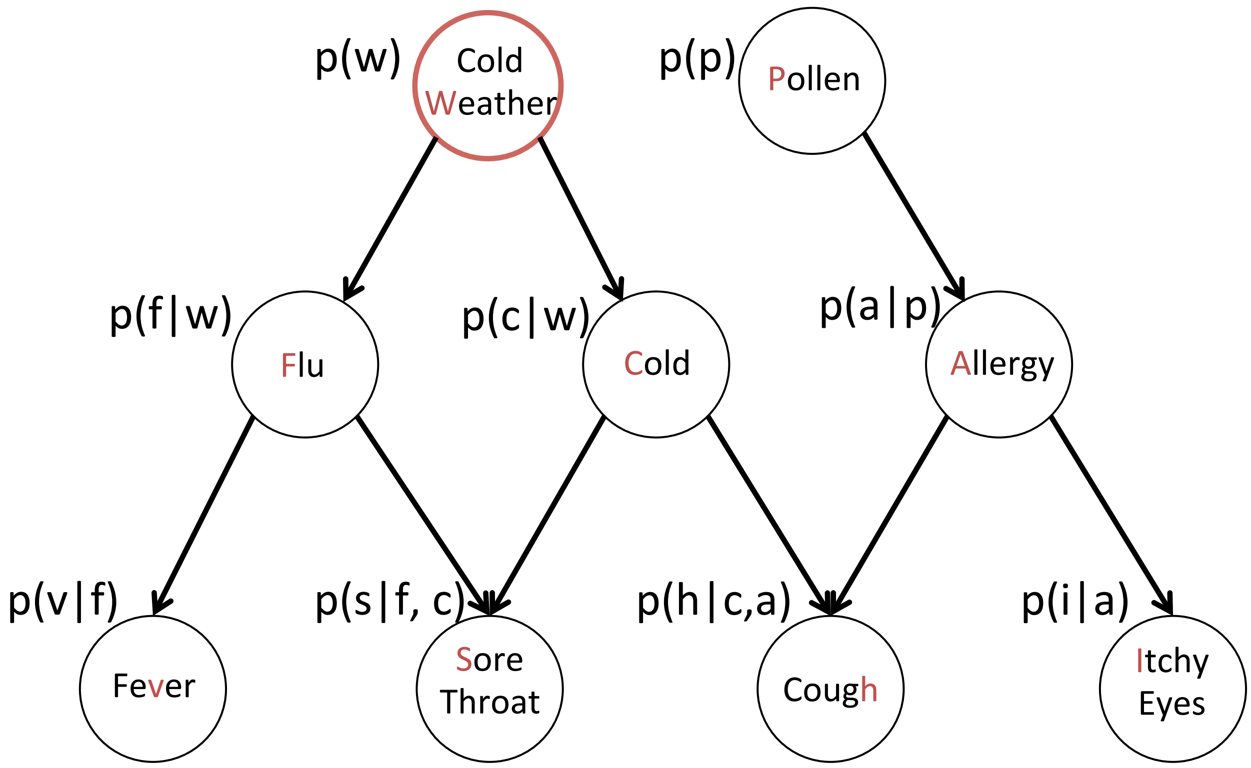


$$P(W{=}w|F{=}1, P{=}1, C{=}0, \ldots, I{=}0)$$

$$= P(W{=}w|F{=}1, C{=}0)$$

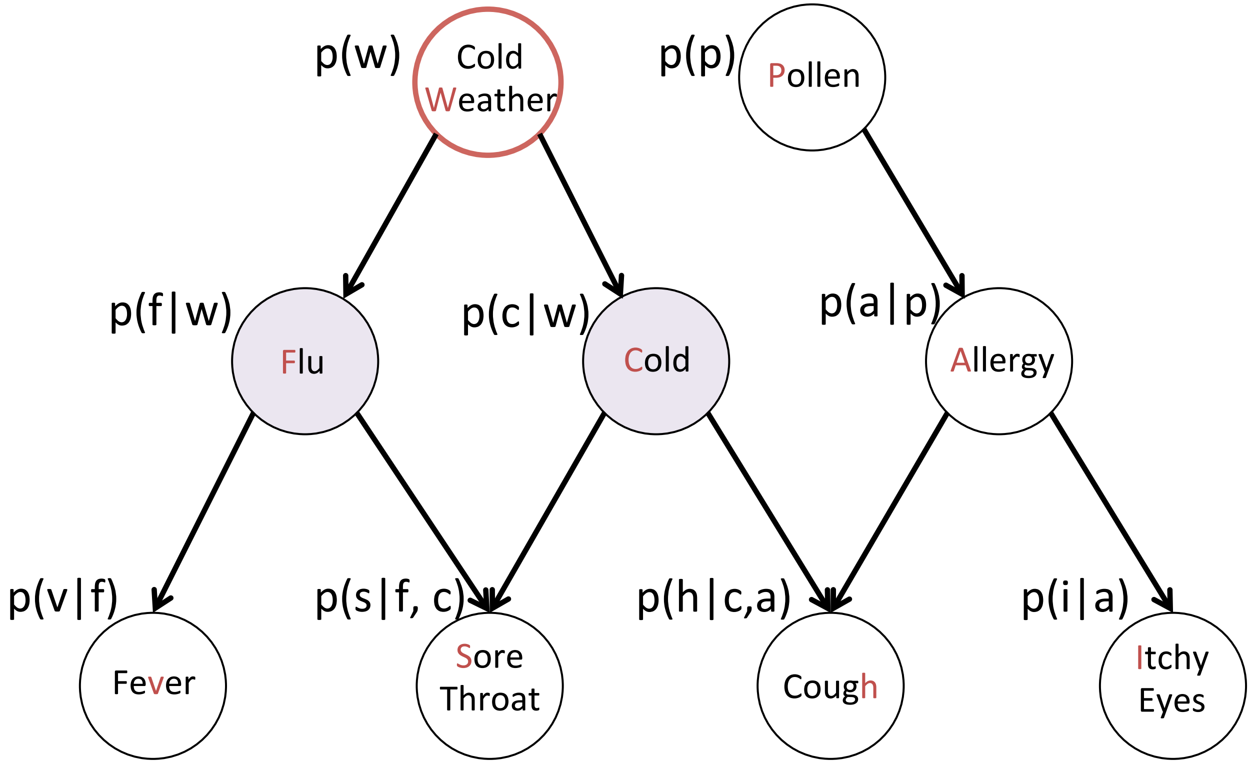
Gibbs Sampling

How do we sample a new value for W?



Gibbs Sampling

How do we sample a new value for W?

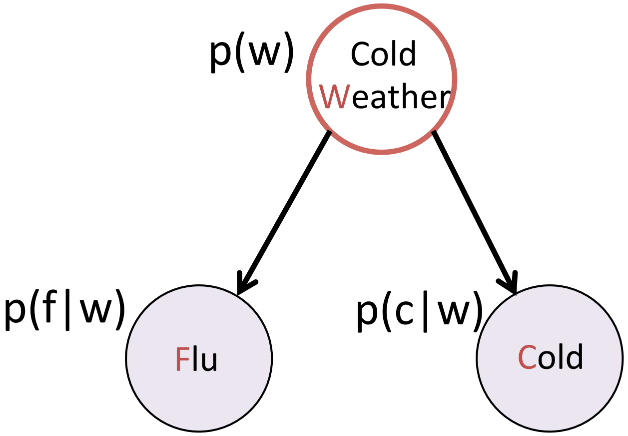


$$P(W=w|F=1, P=1, C=0, \ldots, I=0)$$

$$= P(W=w|F=1, C=0)$$

Markov Blanket!

Gibbs Sampling

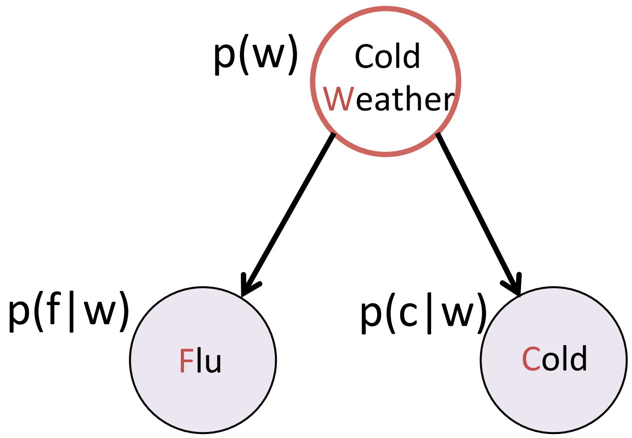


w	p(w)
0	0.4
1	0.6

w	f	p(f|w)
0	0	0.95
0	1	0.05
1	0	0.80
1	1	0.20

w	c	p(f|w)
0	0	0.88
0	1	0.12
1	0	0.70
1	1	0.30

Gibbs Sampling



w	p(w)
0	0.4
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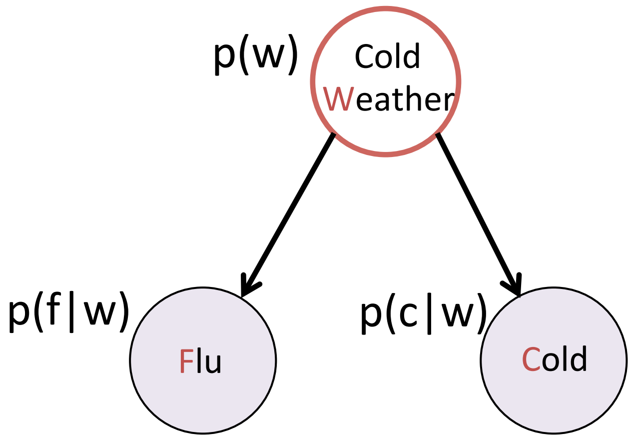
w	f	p(f|w)
0	0	0.95
0	1	0.05
1	0	0.80
1	1	0.20

w	c	p(f|w)
0	0	0.88
0	1	0.12
1	0	0.70
1	1	0.30

$$P(W=w|F=1, P=1, C=0, \ldots, I=0)$$
$$= P(W=w|F=1, C=0)$$
$$\propto P(F=1|W=w)*P(C=0|W=w)*P(W=w)$$

Gibbs Sampling



w	p(w)
0	0.40
1	0.60

w	f	p(f|w)
0	0	0.95
0	1	0.05
1	0	0.80
1	1	0.20

w	c	p(f|w)
0	0	0.88
0	1	0.12
1	0	0.70
1	1	0.30

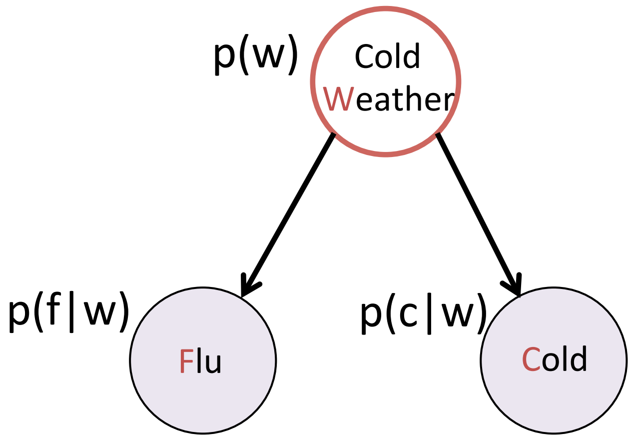
$$P(W{=}w|F{=}1, P{=}1, C{=}0, \ldots, I{=}0)$$

$$= P(W{=}w|F{=}1, C{=}0)$$

$$\propto P(F{=}1|W{=}w)*P(C{=}0|W{=}w)*P(W{=}w)$$

$$= \begin{cases} 0.05*0.88*0.40, & W = 0 \\ & \end{cases}$$

Gibbs Sampling



w	p(w)
0	0.40
1	0.60

w	f	p(f|w)
0	0	0.95
0	1	0.05
1	0	0.80
1	1	0.20

w	c	p(f|w)
0	0	0.88
0	1	0.12
1	0	0.70
1	1	0.30

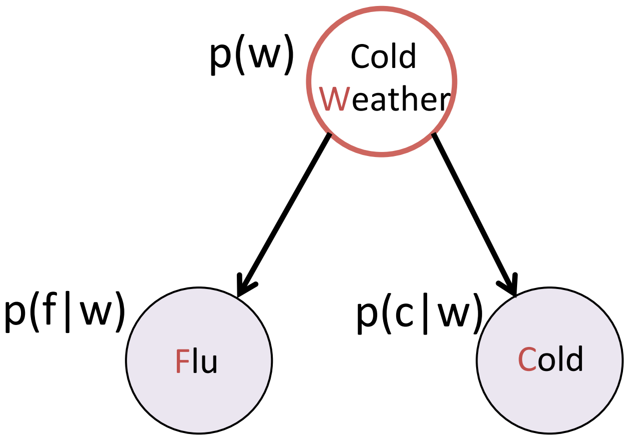
$$P(W{=}w|F{=}1, P{=}1, C{=}0, \ldots, I{=}0)$$

$$= P(W{=}w|F{=}1, C{=}0)$$

$$\propto P(F{=}1|W{=}w)*P(C{=}0|W{=}w)*P(W{=}w)$$

$$= \begin{cases} 0.05*0.88*0.40, & W=0 \\ 0.20*0.70*0.60, & W=1 \end{cases}$$

Gibbs Sampling



w	p(w)
0	0.40
1	0.60

w	f	p(f|w)
0	0	0.95
0	1	0.05
1	0	0.80
1	1	0.20

w	c	p(f|w)
0	0	0.88
0	1	0.12
1	0	0.70
1	1	0.30

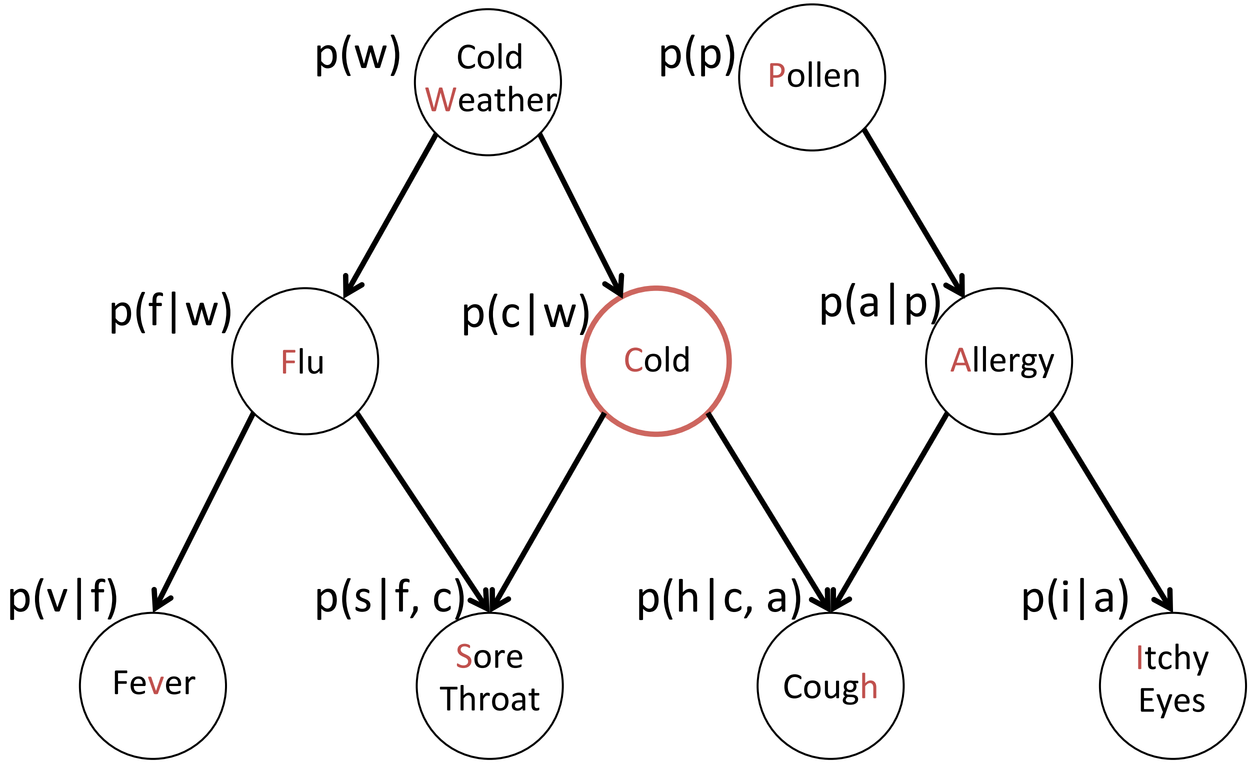
$$
\begin{aligned}
& P(W{=}w|F{=}1, P{=}1, C{=}0, \ldots, I{=}0) \\
= \; & P(W{=}w|F{=}1, C{=}0) \\
\propto \; & P(F{=}1|W{=}w)*P(C{=}0|W{=}w)*P(W{=}w) \\
= \; & \begin{cases} 0.05*0.88*0.40, & W=0 \\ 0.20*0.70*0.60, & W=1 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
& P(W=w \mid F=1, C=0) \\
= \; & \begin{cases} 0.0176/(0.0176+0.084), & w=0 \\ 0.084/(0.0176+0.084), & w=1 \end{cases} \\
= \; & \begin{cases} 0.173, & w=0 \\ 0.827, & w=1 \end{cases}
\end{aligned}
$$

Sample a new w!

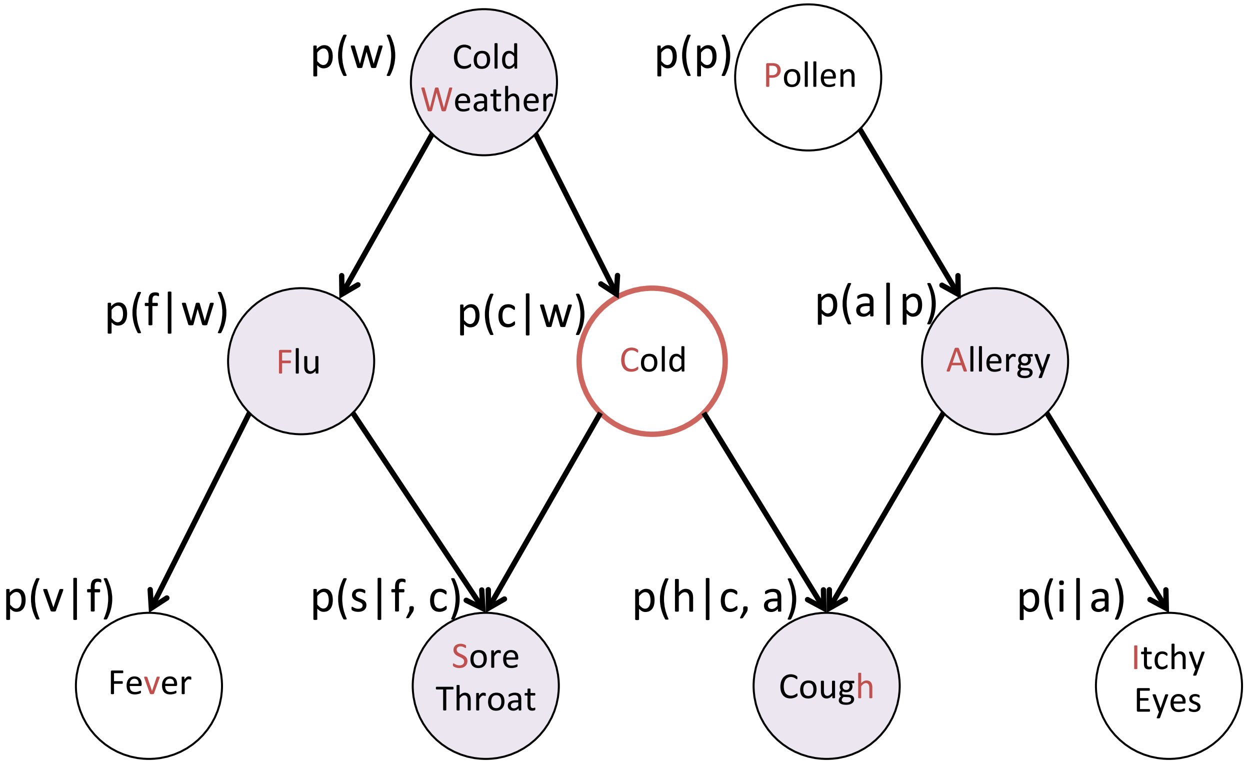
Gibbs Sampling

How do we sample a new value for C?



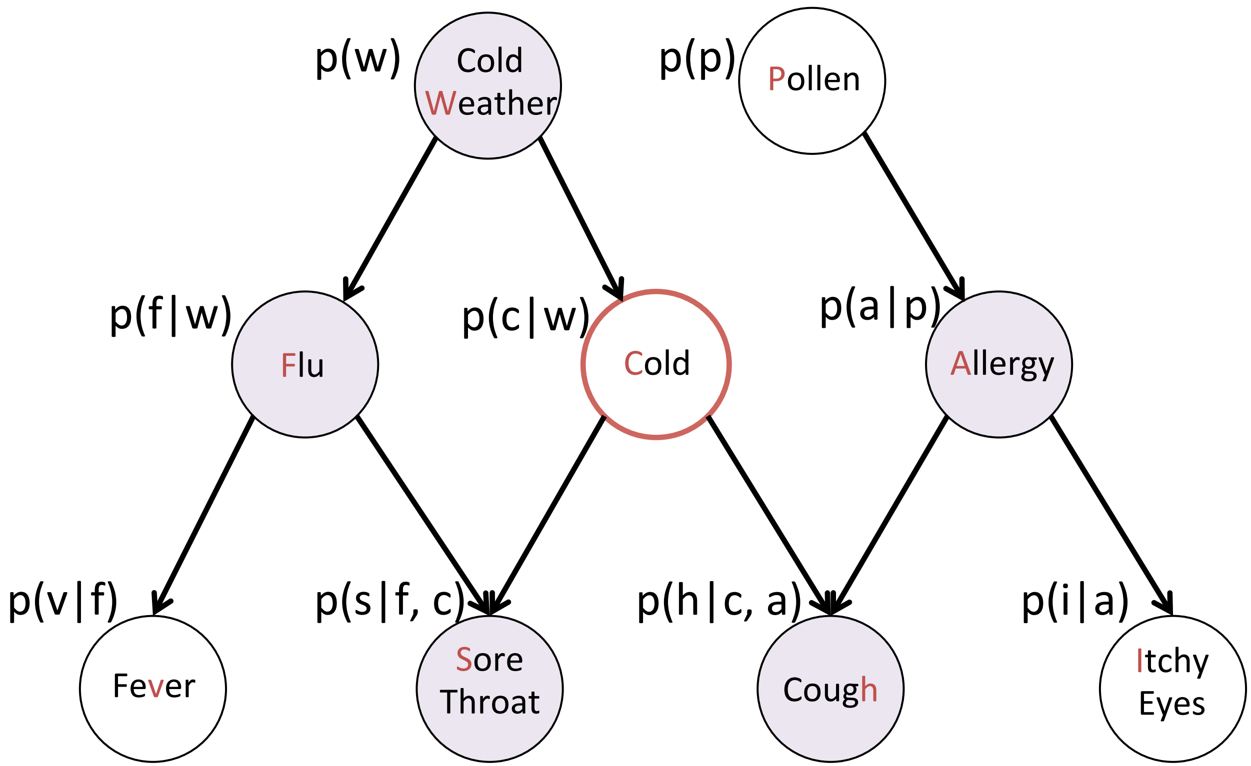
Gibbs Sampling

How do we sample a new value for C?



Gibbs Sampling

How do we sample a new value for C?

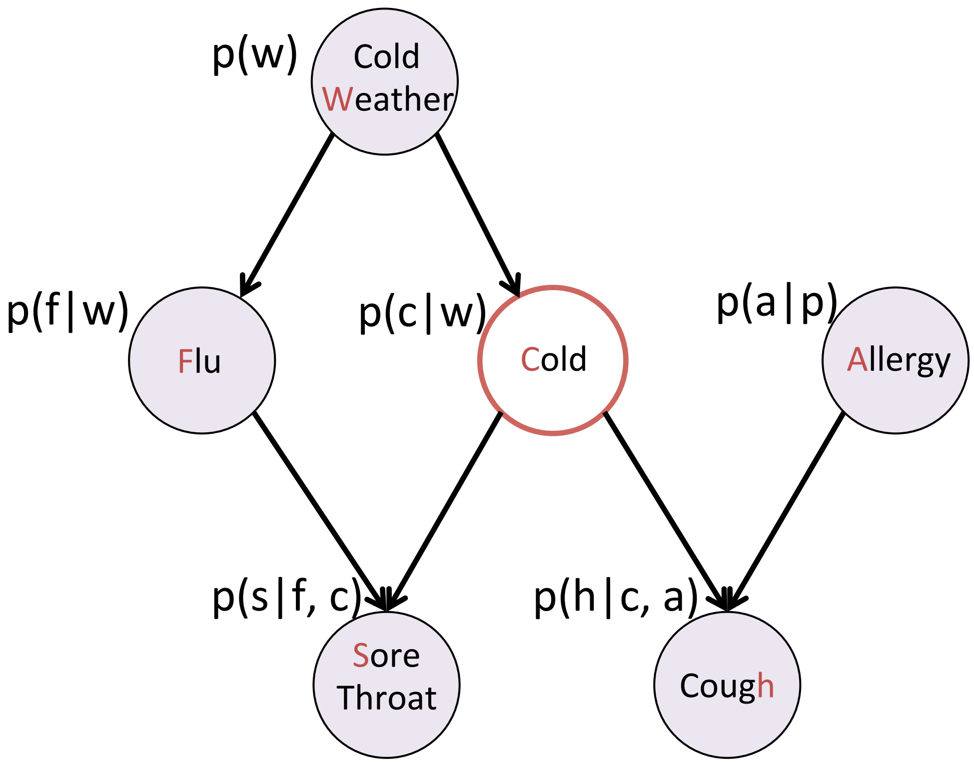


$$P(C=c \mid W=1, F=1, P=1, \ldots, I=0)$$

$$= P(C=c \mid W=1, F=1, S=0, H=1, A=1)$$

Markov Blanket!

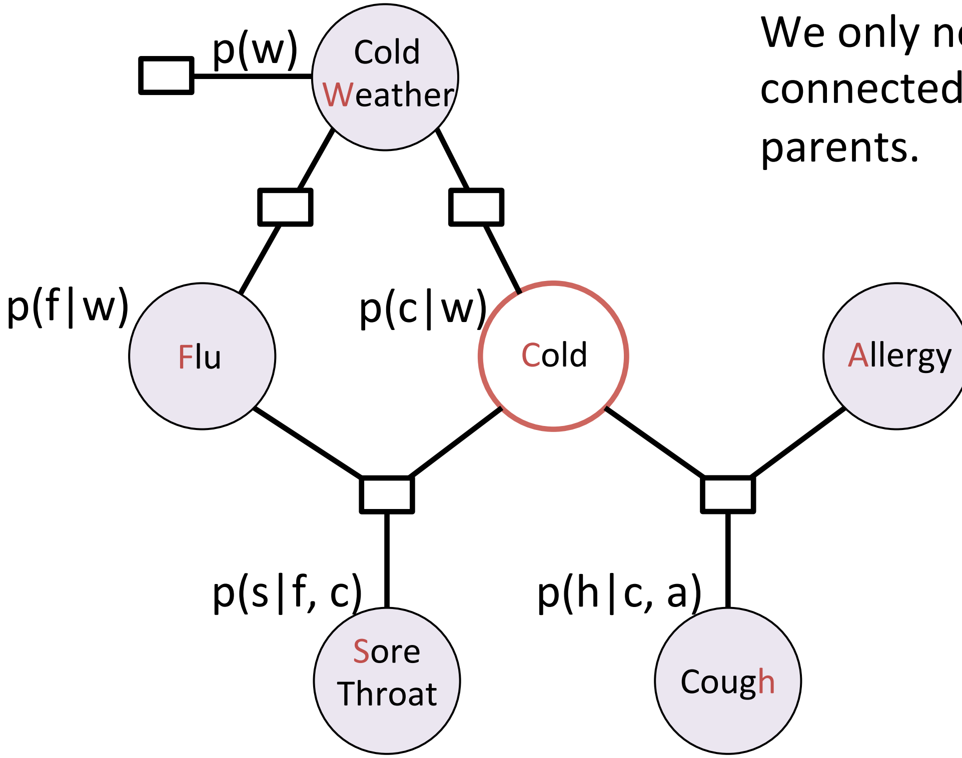
Gibbs Sampling



$$P(C=c \mid W=1, F=1, P=1, \ldots, I=0$$

$$= P(C=c \mid W=1, F=1, S=0, H=1, A=1)$$

Gibbs Sampling From a Factor Graph Perspective



We only need look at the factors connected to the children and the parents.

P(C=c | W=1, F=1, P=1, ..., I=0

= P(C=c | W=1, F=1, S=0, H=1, A=1)

= p(w) p(f | w) p(c | w) p(s | f, c) p(h | c, a)

Questions?