# Solving the SIR-Macro Model in "The Macroeconomics of Epidemics"

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April 6, 2020

This note describes how to compute transition dynamics in the SIR-Macro model developed by Eichenbaum, Rebelo, and Trabandt (2020). An implementation in Python is available on GitHub https://github.com/bbardoczy/sir-macro.

#### 1 General model

Let's start by summarizing the equilibrium conditions of the SIR-Macro model in its most general form. For a detailed description, please consult the paper.

- Vaccine that protects susceptible people from getting infected is discovered with probability  $\nu_t$ .
- **Treatment** that cures infected people is discovered with probability  $\xi_t$ .
- Susceptible people:

The Bellman equation is

$$U_t^s = \max_{c_t^s, n_t^s, \tau_t} \left\{ \log c_t^s - \frac{\theta}{2} (n_t^s)^2 + (1 - \nu_t) \beta \left[ (1 - \tau_t) U_{t+1}^s + \tau_t U_{t+1}^i \right] + \nu_t \beta U_{t+1}^r \right\}$$
s.t.  $(1 + \mu_{ct}) c_t^s = A n_t^s + \Gamma_t$  (\lambda\_t^s)

$$\tau_t = \pi_1 c_t^s (I_t C_t^I) + \pi_2 n_t^s (I_t N_t^I) + \pi_3 I_t \tag{\mu_t^s}$$

The FOCs are

$$0 = \frac{1}{c_t^s} - \lambda_t^s (1 + \mu_{ct}) - \mu_t^s \pi_1(I_t C_t^I)$$
(1.1)

$$0 = -\theta n_t^s + \lambda_t^s A - \mu_t^s \pi_2(I_t N_t^I)$$

$$\tag{1.2}$$

$$0 = \beta(1 - \nu_t) \left( U_{t+1}^s - U_{t+1}^i \right) - \mu_t^s \tag{1.3}$$

#### • Infected people:

The Bellman equation is

$$U_t^i = \max_{c_t^i, n_t^i} \left\{ \log c_t^i - \frac{\theta}{2} (n_t^i)^2 + (1 - \xi_t) \beta \left[ (1 - \pi_r - \pi_{dt}) U_{t+1}^i + \pi_r U_{t+1}^r \right] + \xi_t \beta U_{t+1}^r \right\}$$
s.t.  $(1 + \mu_{ct}) c_t^i = A \phi n_t^i + \Gamma_t$   $(\lambda_t^i)$ 

The FOCs are

$$0 = \frac{1}{c_t^i} - \lambda_t^i (1 + \mu_{ct}) \tag{1.4}$$

$$0 = -\theta n_t^i + \lambda_t^i \phi A \tag{1.5}$$

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#### • Recovered people:

The Bellman equation is

$$\begin{split} U_t^r &= \max_{c_t^r, n_t^r} \left\{ \log c_t^r - \frac{\theta}{2} (n_t^r)^2 + \beta U_{t+1}^r \right\} \\ \text{s.t. } (1 + \mu_{ct}) c_t^r &= A n_t^r + \Gamma_t \end{split}$$
  $(\lambda_t^r)$ 

The FOCs are

$$0 = \frac{1}{c_t^r} - \lambda_t^r (1 + \mu_{ct}) \tag{1.6}$$

$$0 = -\theta n_t^r + \lambda_t^r A \tag{1.7}$$

#### • Epidemic laws of motion:

$$\pi_{dt-1} = \pi_d + \kappa I_{t-1}^2 \tag{1.8}$$

$$T_{t-1} = \pi_1(S_{t-1}c_{t-1}^s)(I_{t-1}c_{t-1}^i) + \pi_2(S_{t-1}n_{t-1}^s)(I_{t-1}n_{t-1}^i) + \pi_3S_{t-1}I_{t-1}$$
(1.9)

$$S_t = S_{t-1} - T_{t-1} (1.10)$$

$$I_t = I_{t-1} + T_{t-1} - (\pi_r + \pi_{dt-1})I_{t-1}$$
(1.11)

$$R_t = R_{t-1} + \pi_r I_{t-1} \tag{1.12}$$

$$D_t = D_{t-1} + \pi_{dt-1} I_{t-1} \tag{1.13}$$

starting from  $S_0 = 1 - \varepsilon$ ,  $I_0 = \varepsilon$ ,  $R_0 = 0$ ,  $D_0 = 0$ .

#### • Medical preparedness:

$$\pi_{dt} = \pi_d + \kappa I_t^2 \tag{1.14}$$

• Government budget:

$$\mu_{ct}C_t = \Gamma_t(S_t + I_t + R_t) \tag{1.15}$$

• Aggregation:

$$C_t = S_t c_t^s + I_t c_t^i + R_t c_t^r \tag{1.16}$$

$$N_t = S_t n_t^s + I_t n_t^i + R_t n_t^r (1.17)$$

• Production:

$$C_t = A\left(S_t n_t^s + \phi I_t n_t^i + R_t n_t^r\right) \tag{1.18}$$

### 2 Algorithm

We're looking for H-period long sequences of endogenous variables. H has to be long enough so that the model gets close to their post-epidemic steady state. We found that H=250 suffices for all cases in the paper. The problem boils down to finding the root of a function. We'll use a quasi-Newton method to do so.

- 1. Specify containment policies and discovery probabilities  $\{\mu_{ct}, \xi_t, \nu_t\}_{t=0}^{H-1}$
- 2. Guess employment  $\{n_t^s, n_t^i, n_t^r\}_{t=0}^{H-1}$ . In practice, we start from the steady state.
- 3. Solve for consumption:

$$c_t^r = \frac{A}{(1 + \mu_{ct})\theta n_t^r} \tag{2.1}$$

$$\Gamma_t = (1 + \mu_{ct})c_t^r - An_t^r \tag{2.2}$$

$$c_t^i = \frac{A\phi n_t^i + \Gamma_t}{1 + \mu_{ct}} \tag{2.3}$$

$$c_t^s = \frac{An_t^s + \Gamma_t}{1 + \mu_{ct}} \tag{2.4}$$

4. Iterate forward the epidemic laws of motion.

Initial conditions:

$$S_0 = 1 - \varepsilon \tag{2.5}$$

$$I_0 = \varepsilon \tag{2.6}$$

$$R_0 = 0 (2.7)$$

$$D_0 = 0 (2.8)$$

Iteration for  $t = 1, \ldots, H - 1$ :

$$\pi_{dt-1} = \pi_d + \kappa I_{t-1}^2 \tag{2.9}$$

$$T_{t-1} = \pi_1(S_{t-1}c_{t-1}^s)(I_{t-1}c_{t-1}^i) + \pi_2(S_{t-1}n_{t-1}^s)(I_{t-1}n_{t-1}^i) + \pi_3S_{t-1}I_{t-1}$$
(2.10)

$$S_t = S_{t-1} - T_{t-1} (2.11)$$

$$I_t = I_{t-1} + T_{t-1} - (\pi_r + \pi_{dt-1})I_{t-1}$$
(2.12)

$$R_t = R_{t-1} + \pi_r I_{t-1} \tag{2.13}$$

$$D_t = D_{t-1} + \pi_{dt-1} I_{t-1} \tag{2.14}$$

5. Solve for value functions by backward iteration.

Transmission probability:

$$\tau_t = \pi_1 c_t^s (I_t C_t^I) + \pi_2 n_t^s (I_t N_t^I) + \pi_3 I_t \tag{2.15}$$

Terminal conditions:

$$U_H^r = U_{ss} (2.16)$$

$$U_H^i = \frac{\log(\phi c_{ss}^i) - \frac{\theta}{2}(n_{ss}^i)^2 + \beta(1 - \xi_{H-1})\pi_r U_H^r + \beta \xi_{H-1} U_H^r}{1 - \beta(1 - \pi_r - \pi_{dH-1})(1 - \xi_{H-1})}$$
(2.17)

$$U_H^s = U_H^r \tag{2.18}$$

Iterate backwards:

$$U_t^r = \log c_H^r - \frac{\theta}{2} (n_H^r)^2 + \beta U_{t+1}^r$$
(2.19)

$$U_t^i = \log c_H^i - \frac{\theta}{2} (n_H^i)^2 + (1 - \xi_t) \beta \left[ (1 - \pi_r - \pi_{dt}) U_{t+1}^i + \pi_r U_{t+1}^r \right] + \xi_t \beta U_{t+1}^r$$
 (2.20)

$$U_t^s = \log c_t^s - \frac{\theta}{2} (n_t^s)^2 + (1 - \nu_t)\beta \left[ (1 - \tau_t) U_{t+1}^s + \tau_t U_{t+1}^i \right] + \nu_t \beta U_{t+1}^r$$
(2.21)

6. Multipliers and aggregates:

$$\mu_t^s = \beta(1 - \nu_t)(U_{t+1}^s - U_{t+1}^i) \tag{2.22}$$

$$\lambda_t^s = \frac{\theta n_t^s + \mu_t^s \pi_2 I_t n_t^i}{A} \tag{2.23}$$

$$\lambda_t^i = \frac{\theta n_t^i}{\phi A} \tag{2.24}$$

$$C_t = S_t c_t^s + I_t c_t^i + R_t c_t^r \tag{2.25}$$

$$N_t = S_t n_t^s + I_t n_t^i + R_t n_t^r$$
 (these are raw hours!) (2.26)

7. Residuals:

$$0 = \mu_{ct}C_t - \Gamma_t(S_t + I_t + R_t)$$
 (2.27)

$$0 = \lambda_t^i (1 + \mu_{ct}) - \frac{1}{c_t^i} \tag{2.28}$$

$$0 = \lambda_t^s (1 + \mu_{ct}) + \mu_t^s \pi_1(I_t C_t^I) - \frac{1}{c_t^s}$$
(2.29)

8. Update guesses using a Newton step.

Let  $x^j$  be the current guess and f be the function that maps guesses into residuals. The next guess should be

$$x^{j+1} = x^j - [f'(x_{ss})]^{-1} f(x^j)$$
(2.30)

Note that there's no need to recompute the Jacobian f' at every step. The Jacobian at the initial guess is a sufficiently good updating rule to make the algorithm converge in 8-10 steps to a tolerance level of  $10^{-8}$ . This trick provides large efficiency gains whenever it's costly to compute the Jacobian.

## References

EICHENBAUM, M. S., S. REBELO, AND M. TRABANDT (2020): "The Macroeconomics of Epidemics," NBER Working Paper 26882.