Things to think about

- Character of a representation
- Irreducible representation
- Decomposation into irreducible representation

1. Representation Scheme of G-action on Coherent Sheaves

1.1. **Geometric point of view of locally free sheaves and their endomorphisms.** Let X be a Noetherian scheme over an algebraically closed field k. A vector bundle $\pi : V \to X$ of rank r is a scheme over X that is locally trivial, *i.e.* there is an affine cover $U_i = \text{Spec } A_i$ of X and isomorphisms

$$\psi_i : \pi^{-1}(U_i) \to U_i \times_k \mathbb{A}_k^r$$

The transition functions between local frames on V need to satisfy cocycle condition.
The category of vector bundles over X is equivalent to the category of locally free O_X-modules. Given a vector bundle $\pi : V \to X$ of rank r, its sections (on open sets) form a locally free sheaf $\mathscr{E}$ of rank r. To recover V from $\mathscr{E}$, one takes Spec sym$\mathscr{E}^\vee$.
An endomorphism of vector bundle $\pi : V \to X$ is an X-morphism $V \to V$, *i.e.* a k-morphism that preserves the fibers.

Lemma 1. *There is a vector bundle $E(\pi) : E(V) \to X$ whose global sections are in one-to-one correspondence with endomorphisms of $\pi : V \to X$.*

Proof. Let $\mathscr{E}$ be locally free sheaf of sections of $\pi : V \to X$. Then,

$$E(V) = \text{Spec sym } \mathscr{E}^\vee \otimes \mathscr{E}$$

The morphism $E(\pi) : E(V) \to X$ is given by the O_X-algebra structure of sym $\mathscr{E}^\vee \otimes \mathscr{E}$. □

Lemma 2. *$E(V)$ has an open subscheme $A(\pi) : A(V) \to X$ whose global sections are in one-to-one correspondence with automorphisms of $\pi : V \to X$.*

Proof. The complement of zero section of $E(V)$ is $A(V)$. □

1.2. **Representation Scheme of G-action on a vector bundle.** Let G be a group. An G-action on $\pi : V \to X$ is a group homomorphism

$$T : G \to \text{Aut}_X(V)$$

where the group structure on $\text{Aut}_(V)$ is given by composition.
The main theorem of this section is the following

Theorem 1. *Suppose G is finitely presented. Then, for any vector bundle $\pi : V \to X$, here is a scheme of finite type*

$$ES(\pi) : ES(V) \to X$$

whose global sections are in one-to-one correspondence with the G-action on $\pi : V \to X$.

Lemma 3. *Suppose G is finitely presented, and $\pi : V \to X$ is trivial. Then, there is a scheme of finite type over X*

$$ES(\pi) : ES(V) \to X$$

whose global sections are in one-to-one correspondence with the G-actions on V.

Proof. Suppose $G =< g_1, \cdots, g_n >$ is a free group on n letters. For any G-action on V

$$T : G \to \mathrm{Aut}_X(V)$$

$T(g_i)$'s do not need to satisfy any relations. Therefore, giving an G-action on V is equivalent to specifying a global section of

$$\prod_n (A(\pi)) : \prod_n A(V) \to X$$

Now, suppose G has relations $r_1, \cdots, r_s$ (monomials in g_i). Let $< e_1, \cdots, e_r >$ be a global frame of the vector bundle

$$\pi : V \to X$$

and let $< {e_1}^\vee, \cdots, {e_r}^\vee >$ be the dual frame for the dual bundle

$$\pi^\vee : V^\vee \to X$$

Then, $< x_{ij} = e_i \otimes {e_j}^\vee >$ is a global frame for the bundle

$$\mathscr{E}(\pi) : \mathscr{E}(V) \to X$$

and $\mathscr{E}(V) = \mathrm{Spec} \quad _X[x_{ij}]$, where $O_X[x_{ij}]$ is the symmetric algebra generated by x_{ij} over O_X.
Let $\tau := (s_1, \cdots, s_n) : X \to \prod_n A(V)$ be the section corresponding to $(T(g_1), \cdots, T(g_n))$. Then, as $\mathscr{E}(\pi) : \mathscr{E}(V) \to X$ has global frame.
τ is given by an O_X-algebra homomorphism:

$$\bigotimes_{k=1}^{n} O_X[x_{ij}^k] \to O_X$$

where $i, j = 1, \cdots, r$. For each k, $O_X[x_{ij}^k]$ is the sheaf of regular functions of $\mathscr{E}(V)$.

Let X^k be the matrix $[x_{ij}^k]$, and let J be the ideal sheaf of $\bigotimes_{k=1}^{n} O_X[x_{ij}^k]$ defined by the relations

$$r_i(X^1, \cdots, X^n) = I, i = 1, \cdots, s$$

Then, as an O_X-algebra map, τ factors through

$$\bigotimes O_X[x_{ij}^k] \to \bigotimes O_X[x_{ij}^k]/J$$

Conversely, let $\alpha : \bigotimes O_X[x_{ij}^k]/J \to O_X$ be any O_X-algebra map. Let M^k be the matrix over O_X correpsonding to $\alpha(X^k)$, then

$$r_i(M^1, \cdots, M^n) = I, i = 1, \cdots, s$$

i.e. $T(g_i) = M_i$ defines an G-action of $\mathscr{E}(\pi) : \mathscr{E}(V) \to X$.
So $ES(V)$ is the closed subscheme of $\prod_n A(V)$ defined by the ideal sheaf J. $\square$

To prove the main theorem, we need an elementary linear algebra fact.

Lemma 4. *Let M_1 and M_2 be free modules over a ring A of rank n. Let M_1 and M_2 be generated by the basis $< e_1, \cdots, e_n >$ and $< s_1, \cdots, s_n >$, respectively. Suppose $\phi : M_1 \to M_2$ is an isomorphism given by the matrix T. Then, the image of ${e_i}^\vee \otimes e_j$ under $\phi^{\vee -1} \otimes \phi$ is given by the ij-entry of*

$$T({s_i}^\vee \otimes s_j)T^{-1}$$

where $({s_i}^\vee \otimes s_j)$ is the matrix whose ij-entry is ${s_i}^\vee \otimes s_j$

Proof. The proof is done by elementary computation. Suppose

$$\phi(e_i) = a_{i1}s_1 + \cdots a_{in}s_n$$

Then, $T = (a_{ij})$. Write $T^{-1} = (b_{ij})$. Then, the map $\phi^{\vee -1}$ is given by

$$\phi^{\vee -1}({e_i}^\vee) = b_{1i}s_1 + b_{2i}s_2 + \cdots + b_{ni}s_n$$

(the inverse transpose of T).
Then, the conclusion of the lemma follows directly. $\square$

Proof. (Theorem 1) Let U_i = Spec A_i be an affine cover of X that trivilizes $\pi : V \to X$. Write V_i for the restriction of V on U_i and V_{ij} as the restriction on U_{ij}. As proved in Lemma 2, G-actions on $\pi_i : V_i \to U_i$ and $\pi_{ij} : V_{ij} \to U_{ij}$ can be represented by ES(π_i); ES$(V_i) \to U_i$ and ES(π_{ij}) : ES$(V_{ij}) \to U_{ij}$.
To make the notation clean. Assume G is generated by one element g with the relation r. The general case can be proved similarly. In that

case, $\mathrm{ES}(\pi_{ij}) : \mathrm{ES}(V_{ij}) \to U_{ij}$ is a subscheme of $\mathrm{ES}(\pi_{ij}) : \mathrm{ES}(V_{ij}) \to U_{ij}$ is a subscheme of $\mathrm{E}(\pi_{ij}) : \mathrm{E}(V_{ij}) \to U_{ij}$.
To prove the theorem, it is enough to show that the gluing isomorphism $\phi_{ij} : E(V_{ij}) \to E(V_{ji})$ descends to an isomorphism $\bar{\phi_{ij}} : \mathrm{ES}(V_{ij}) \to \mathrm{ES}(V_{ij})$.
Let $V_i = \text{Spec } A_i[x_1^i, \cdots, x_n^i]$, and let

$$\phi_{ij} A_{ij}[x_1^j, \cdots, x_n^j] \to A_{ij}[x_1^i, \cdots, x_n^i]$$

be the ring map that glues V_{ij} onto V_{ji}. Let T be the matrix representing this map.
Set $x_{rs}^i := {x^i_r}^\vee \otimes x_s^i$. The ring map that glues $\mathrm{E}V_{ij}$ on $\mathrm{E}V_{ji}$ is therefore

$${\phi_{ij}}^{\vee -1} \otimes \phi_{ij} A_{ij}[x_{rs}^j] \to A_{ij}[x_{rs}^i]$$

By Lemma 4, the map ${\phi_{ij}}^{\vee -1}$ is represented by

$$(x_{rs}^j) \mapsto T(x_{rs}^i)T^{-1}$$

Let J^i be the ideal of $A_i[x_{rs}^i]$ defining $\mathrm{ES}(V_i) \to U_i$. Use the description of the map ${\phi_{ij}}^{\vee -1}$ above, one can see that the image of J^j and J^i are the same ideal. Hence, ${\phi_{ij}}^{\vee -1}$ descends to a map

$$\bar{\phi_{ij}}^{\vee -1} : A_{ij}[x_{rs}^j]/J^j \to A_{ij}[x_{rs}^i]/J^i$$

□

Remark 1. *Let $x \in X$ be a closed point. The fiber of ES(V) over x is the representation variety Hom$_k(G, GL(V_x))$.*

1.3. **A(V)-action on ES(V) by conjugation.** Suppose $X = \text{Spec } A$. Let $\pi : V \to X$ be a trivial vector bundle. of rank n.

Lemma 5. *$A(\pi) : A(V) \to X$ is a group scheme.*

Proof. Clear. □

Write $A[x_{11}, \cdots, x_{nn}]$ for the ring of regular functions on $\mathrm{E}(V)$, $R := A[x_{11}, \cdots, x_{nn}, d^{-1}]$ for the ring of regular functions on $\mathrm{A}(V)$ where d is the determinant of the matrix (x_{ij}).
$\mathrm{A}(V)$ acts on $\mathrm{ES}(V)$ via conjugation. Ring-theoretically, the action is given by descending the morphism

$$\phi : S \to S \otimes R \tag{1.3.1}$$

$$X_i \mapsto XX_iY \tag{1.3.2}$$

to

$$\bar{\phi} : S/J \to S/J \otimes R$$

Lemma 6. *The morphism $\bar{\phi}$ is well-defined.*

Proof. As R is flat over A, $J \otimes R \hookrightarrow S \otimes R$. So it suffices to show that $\phi(J) \subset J \otimes R$.
Let $M_i, i = 1, \cdots, s$ be the matrices such that comp$(M_i - I_r)$ generates the ideal J. Each M_i can be written as $X_1^{i_1} \cdots X_n^{i_n}$ for some $i_1, \cdots, i_n$. Therefore,

$$\phi(M_i - I_r) = X(M_i - I_r)Y$$

So each component of $X(M_i - I_r)Y$ can be written as an element in $J \otimes R$. □

Definition 1. *An element $f \in S$ is said to be invariant under ϕ if*

$$\phi(f) = f \otimes 1$$

Similarly, an element $\bar{f} \in S/J$ is said to be invariant under $\bar{\phi}$ if

$$\bar{\phi}(\bar{f}) = \bar{f} \otimes R$$

Denote the subring of invariant elements of S by S^ϕ. For each matrix X_i, write char(X_i) as

$$t^r + \Gamma_i^{r-1} t^{r-1} + \cdots \Gamma^0$$

and let $\bigwedge(X_i)$ be the set

$$\{\Gamma^{r-1}, \cdots, \Gamma^0\}$$

Proposition 1. *S^ϕ is finitely generated by the union of*

$$\bigwedge(X_i), i = 1, \cdots, n$$

Proof. Induction on the rank of the vector bundle. Bootstrape the matrix □

For each element $g \in G$, let char$(T(g))$ be the characteristic polynomial of $T(g)$. It can be written as

$$t^r + a_{r_1} t^{r-1} + \cdots + a_0$$

where $a_i \in H^0(X, O_X)$.

Definition 2. *Two actions T_1 and T_2 are said to* have the same charateristic polynomials *if for each element of $g \in G$, char$(T_1(g)) =$ char$(T_2(g))$.*

Lemma 7. *If two actions are conjugate, then the have the same charateristic polynomial.*

Remark 2. *The converse is not true. Theorem of MacDuffee.*

Let $Y :=$ Spec $(S/J)^{\bar{\phi}}$, and let $\beta : Y \to X$ be its structure morphism, and let $\rho :$ ES$(V) \to Y$ be the map corresponding to the inclusion of the rings $(S/J)^{\bar{\phi}} \to S/J$ Let $\rho :$ ES$(V) \to Y$ be the natural map.

Proposition 2. *Each section $T : X \to ES(V)$, induces a section $\bar{T} : X \to Y$, such that the following diagram commutes Insert a diagram here If T_1 and T_2 have the same charateristic ploynomials, then $\bar{T_1} = \bar{T_2}$; Moreover, any $A(V)$-equivariant morphism $f : ES(V) \to Z$ over X with the above properties factors uniquely through $\rho : ES(V) \to Y$.*

Proof. Y is the categorical quotient. Show for any $f : \mathrm{ES}(V) \to Z$ with the above property, the following diagram commutes

$$\begin{array}{ccc}
\mathrm{A}(V) \times_X \mathrm{ES}(V) & \longrightarrow & \mathrm{ES}(V) \\
\downarrow & & \downarrow \\
\mathrm{ES}(V) & \longrightarrow & Z
\end{array}$$

How does $(S/J)^{\bar{\phi}}$ look like? Consider the simply case when Z is affine ∎

Now, let $\pi : V \to X$ be any vector bundle. Let $U_i = \text{Spec } A_i$ be any open cover of X, and let $U_{ij} = \text{Spec } A_{ij}$ denote the intersection. Let $\pi_i : V_i \to U_i$ be the restriction of V on U_i. On each V_i, denote the chosen frame as i-frame. Use V_{ij} to denote V_i(with i-frame) restricts to U_{ij}. Use $\psi_{ij} : \mathrm{ES}(V_{ij}) \to \mathrm{ES}(V_{ji})$ and $\tau_{ij} : \mathrm{A}(V_{ij}) \to \mathrm{A}(V_{ji})$ to denote the respective transition function.

Proposition 3. *$A(\pi) : A(V) \to X$ is a group scheme.*

Proof. 1. The existence of a multiplication map

$$\mu : \mathrm{A}(V) \times_X \mathrm{A}(V) \to \mathrm{A}(V)$$

Let $\mu_i : \mathrm{A}(V_i) \times_{U_i} \mathrm{A}(V_i) \to \mathrm{A}(V_i)$ denote the local multiplication map. It is easy to check (by Lemma 4) that the following diagram commutes

$$\begin{array}{ccc}
A_{ij}[y^i_{rs}] & \xrightarrow{\mu_i{}^\#} & A_{ij}[y^i_{rs}] \otimes A_{ij}[y^i_{rs}] \\
\downarrow{\scriptstyle \tau_{ij}{}^\#} & & \downarrow{\scriptstyle \tau_{ij}{}^\# \otimes \tau_{ij}{}^\#} \\
A_{ij}[y^j_{rs}] & \xrightarrow{\mu_j} & A_{ij}[y^j_{rs}] \otimes A_{ij}[y^j_{rs}]
\end{array}$$

Therefore, μ_i's glue to a morphism μ. μ is associative, because μ_i's are.
2. The existence of an inverse map

$$\eta : \mathrm{A}(V) \to X$$

Same as above. ∎

Over U_i, let $\phi_i : \mathrm{A}(V_i) \times_{U_i} \mathrm{ES}(V_i) \to \mathrm{ES}(V_i)$ denote the action of $\mathrm{A}(V_i)$ on $\mathrm{ES}(V_i)$ by conjugation.

Proposition 4. *The maps ϕ_i glue to a morphism*

$$\phi : A(V) \times ES(V) \to ES(V)$$

Proof. The following diagram commutes

$$\begin{array}{ccc} \frac{S_{ij}}{J_{ij}} & \xrightarrow{\phi_i{}^{\#}} & \frac{S_{ij}}{J_{ij}} \otimes R_{ij} \\ \Big\downarrow{\scriptstyle \tau_{ij}{}^{\#}} & & \Big\downarrow{\scriptstyle \psi_{ij}{}^{\#} \otimes \tau_{ij}{}^{\#}} \\ \frac{S_{ji}}{J_{ji}} & \xrightarrow{\phi_j{}^{\#}} & \frac{S_{ji}}{J_{ji}} \otimes R_{ji} \end{array}$$

□

Over each U_i, let $\rho_i : \mathrm{ES}(V_i) \to Y_i$ be the quotient map where Y_i is the spectrum of the invariant functions on $\mathrm{ES}(V_i)$ under the action of $\mathrm{A}(V_i)$.

Proposition 5. *The transition functions $\psi_{ij} : ES(V_{ij}) \to ES(V_{ji})$ descend to Y_{ij},* i.e. *there are morphisms $\bar{\psi_{ij}} : Y_{ij} \to Y_{ji}$ so that the diagram*

$$\begin{array}{ccc} ES(V_{ij}) & \xrightarrow{\rho_{ij}} & Y_{ij} \\ \Big\downarrow{\scriptstyle \bar{\psi_{ij}}} & & \Big\downarrow{\scriptstyle \bar{\psi_{ij}}} \\ ES(V_{ji}) & \xrightarrow{\rho_{ji}} & Y_{ji} \end{array}$$

Proof. $(\frac{S_{ij}}{J_{ij}})^{\phi_{ij}{}^{\#}}$ is the kernel of the map

$$\frac{S_{ij}}{J_{ij}} \xrightarrow{\phi_{ij}{}^{\#} - \mathrm{id}} \frac{S_{ij}}{J_{ij}} \otimes R_{ij}$$

By the proof of Proposition 4, we conclude that $\tau_{ij}{}^{\#}$ maps $(\frac{S_{ij}}{J_{ij}})^{\phi_{ij}{}^{\#}}$ isomorphically onto $(\frac{S_{ji}}{J_{ji}})^{\phi_{ji}{}^{\#}}$

□

Let Y be the schemed obtained by gluing Y_i and Y_j along Y_{ij}.

Theorem 2. *The morphisms $\rho_i : ES(V_i) \to Y_i$ glue and define a morphism $\rho : ES(V) \to Y$. For each section $T : X \to ES(V)$, let $\bar{T}$ be the induced map $X \to Y$. If T_1 and T_2 are sections corresponding to conjugate G-actions on V, then $\bar{T_1} = \bar{T_2}$.*

Proof. □

Before moving on, think about the A-moudle strunture of $(S/J)^{\bar{\phi}}$

Theorem 3. *Suppose X is integral and G is a finite group. Then, $\beta : Y \to X$ is a finite morphism.*

Proof. Without loss of generality, one can assume X is affine. Let $X = \text{Spec } A$. Let L be the field of fraction of A, and let $\bar{L}$ be the algebraic closure of L. The extension $A \to \bar{L}$ is flat This does not work, think about the the example $k[x] \to k[x, \frac{1}{x}]$ □

1.4. **Irreducible representations.** Throughout this section, G is assumed to be a finite group. Let $T : G \to \text{End}(V)$ be a representation. For each element $g \in G$, one can define the subscheme V^g of V over X fixed by $T(g)$ as the fiber product of the following two maps

$$\begin{aligned} \text{id} :& V \to V \\ T(g) :& V \to V \end{aligned}$$

Definition 3. *The G-invariant subscheme of V is the fiber product of*

$$T(g) : V \to V$$

for all elements $g \in G$. It is denoted by V^G.

As G is a finite group, V^G is a closed subscheme of V.

Definition 4. *A representation $T : G \to End(V)$ is said to be irreducible, if V^G is the zero section of $\pi : V \to X$.*

The goal of this section is to develop a character theory similar to that of the representation of a group into a finite dimensional vector space.

Analogue of Schur's lemma

Proposition 6. *Let $\pi_1 : V_1 \to X$ and $\pi_2 : V_2 \to X$ be two vector bundles over X, and let $T_1 : G \to End(V_1)$ and $T_2 : G \to End(V_2)$ be two irreducible representations. Let $f : V_1 \to V_2$ be a morphism of vector bundles, such that for all $g \in G$, $f \circ T_1(g) = T_2(g) \circ f$. Then,*

(1) *If T_1 and T_2 are not isomorphic, then f maps V_1 onto the zero-section of V_2.*
(2) *If X is projective, $V_1 = V_2$, and $T_1 = T_2$, then f is a multiplication by scalar.*

Proof. The proof is a direct generalization in Serre's book.
In the first part, to prove f is an isomorphism, one shows that both f and its dual are injective. □

1.5. **When G acts non-trivially on X.** *Suppose G acts nontrivially on X, is there any scheme over X whose X-valued points correspond to G-action on $\pi : V \to X$?*
Let M be an A-module. Suppose G act nontrivially on A. For each $g \in G$, let $\mu(g)$ be the automorphism of A induced from the G-action.

Definition 5. *A G-equivariant struncture on M is a collection of set map*

$$\lambda_g : M \to M$$

indexed by elements of G such that

- *For every element $a \in A$,*

$$\lambda_g(am) = \mu(a)\lambda_g(m)$$

- *For $g_1, g_2 \in G$,*

$$\lambda_{g_1 g_2}(m) = \lambda_{g_1}(\lambda_{g_2}(m))$$

View M as A^G-module, then each λ_g becomes a map of A^G-module.

Lemma 8. *Two different equivariant structures on M become two different equivariant structures on M viewed as A^G-module.*

Proof. Clear. □

Let G be a finite group, and let $f : X \to Z := X/G$ be the quotient map. Then, in many good cases f_*V is a vector bundle on Z. A G-equivariant structure on V can be viewed as an G-action on f_*V in the sense of the previous chapters. And by the above lemma, two distinct G-equivairant structure on V become to two distinct G-actions on f_*V. One can ask, is there a subscheme W of ES(f_*V) such that the Z-valued points of W correspond to equivariant structures on V?