VANISHING THEOREMS FOR PARABOLIC HIGGS BUNDLES

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Abstract. The main result is a Kodaira vanishing theorem for semistable parabolic Higgs bundles with trivial parabolic Chern classes. This implies a Kodaira-Saito vanishing theorem for complex variations of Hodge structure.

Introduction

Let X be a complex smooth projective variety and $D \subset X$ a divisor with simple normal crossings. In an earlier paper [A], the first author reproved (and slightly extended) Saito's Kodaira vanishing theorem for (complex) variations of Hodge structures on $X - D$ with unipotent monodromy around D, by deducing it from a more general vanishing theorem for Higgs bundles. In this paper, our goal is to extend this further to complex polarized variations of Hodge structure without any unipotency condition. By work of Mochizuki and Simpson, such a variation determines a parabolic Higgs bundle, which consists of a vector bundle E on X, a map $\theta : E \to \Omega^1_X(\log D) \otimes E$ such that $\theta \wedge \theta = 0$ and an $\mathbb{R}$-indexed filtration on $E^* \subset E(*D)$ satisfying appropriate conditions. The first condition tells us that we can form a "de Rham" complex

$$\mathrm{DR}(E, \theta) := E \xrightarrow{\theta} \Omega^1_X(\log D) \otimes E \xrightarrow{\theta} \Omega^2_X(\log D) \otimes E \xrightarrow{\theta} \dots$$

For parabolic Higgs bundles coming from complex variations, this is just the Kodaira-Spencer complex. Also in this case, the sections of $E|_{X-D}$ lie in E^α if their norms, with respect to the Hodge metric, are $O(|f(x)|^{\alpha-\epsilon})$, where f is a local equation for D. The filtration is also related to the monodromy about D; in particular, it would be trivial in the unipotent case, but not otherwise. One reason for keeping track of this filtration is that enters into natural modifications of Chern classes et cetera, where the usual formulas need to be corrected along D. These notions will be reviewed in the paper.

Our main result is as follows: Given a slope semistable parabolic Higgs bundle (E, E^*, θ) with trivial parabolic Chern classes, and an ample line bundle L,

$$\mathbb{H}^i(X, \mathrm{DR}(E, \theta) \otimes L) = 0$$

for $i > \dim X$. Moreover if there is a decomposition $E = E_+ \oplus E_-$ such that $\theta(E) \subseteq \Omega^1_X(\log D) \otimes E_-$, we can deduce from the vanishing theorem that E_+ is nef. The assumptions of these results, and therefore its conclusion, hold for Higgs bundles arising from complex variations of Hodge structure. In particular, we recover the well known semipositivity results of Fujita, Kawamata and many others.

A special case of the main theorem where the filtration is trivial and θ is nilpotent was proved in [A]. (This was proved using characteristic p methods, but a trick used here leads in princple to a characteristic 0 proof. See remark 7.5.) The proof of the more general theorem is by reducing it to this special case. The reduction is done in stages, and the sketch is probably clearer if we describe the process in reverse from general to special. Using an approximation argument, we reduce to the case where the weights, which are the numbers where the filtration E^* jumps, are rational. Yokogawa [Y] shows that there is a moduli space of parabolic Higgs bundles with fixed rational weights and vanishing parabolic Chern classes which are semistable in the appropriate sense. Then using the natural $\mathbb{C}^*$-action on this space and upper semicontinuity of cohomology, we reduce to the case where θ is nilpotent. Again using the rationality of the weights, a result of Biswas [B1] shows there exists a branched covering $\pi : Y \to X$ and a nilpotent semistable Higgs bundle $(\mathcal{E}, \vartheta)$ on Y such that $\mathbb{H}^i(X, \mathrm{DR}(E, \theta) \otimes L)$ is a summand of $\mathbb{H}^i(Y, \mathrm{DR}(\mathcal{E}, \vartheta) \otimes \pi^* L)$. This is zero by the special case.

1. Parabolic bundles

Let X be a complex smooth projective variety with a reduced simple normal crossing divisor $D = \sum_{i=1}^n D_i$. Let $j : U = X - D \to X$ denote the inclusion of the complement. We fix this notation throughout the paper. We use the following definition, which is equivalent to the one given by Maruyama and Yokogawa [MY], although different notationally.

Definition 1.1. *A parabolic sheaf on (X, D) is a torsion free $\mathcal{O}_X$-module E, together with an decreasing $\mathbb{R}$-indexed filtration by coherent subsheaves such that*

- P1. $E^0 = E$.
- P2. $E^{\alpha+1} = E^\alpha(-D)$.
- P3. $E^{\alpha-c} = E^\alpha$ *for any* $0 < c \ll 1$.
- P4. *The subset of α such that $Gr^\alpha E \neq 0$ is discrete in $\mathbb{R}$. Here $Gr^\alpha E := E^\alpha / E^{\alpha+\epsilon}$ for $0 < \epsilon \ll 1$.*

We refer to the filtration as a *parabolic structure*. The numbers α such that $Gr^\alpha E \neq 0$ are called *weights*. A weight is normalized if it lies in $[0, 1)$. The axioms imply that the ordered set of positive normalized weights $0 < \alpha_1 < \alpha_2 < ... < \alpha_l < 1$ together with the reindexed filtration

$$E = F^0(E) \supsetneq F^1(E) = E^{\alpha_1} \supsetneq F^2(E) = E^{\alpha_2} \ldots \supsetneq F^{\ell+1}(E) = E(-D)$$

determines the whole parabolic structure. We refer to the last filtration as a *quasi-parabolic structure*. Thus a parabolic structure consists of a quasi-parabolic structure together with a choice of normalized weights. In certain situations, we will need to perturb the weights.

Definition 1.2. *Given $\epsilon > 0$, we say that two parabolic sheaves E^* and E'^* are ϵ-close if the underlying sheaves with quasi-parabolic structures are the isomorphic, and the normalized weights satisfy $|\alpha_i - \alpha_i'| < \epsilon$.*

Definition 1.3. *A parabolic bundle on (X, D) consists of a vector bundle E on X with a parabolic structure, such that as a filtered bundle, E is Zariski locally a sum of rank one bundles. (See the discussion after example 1.5 for further explanation).*

Many authors use a weaker definition. However, we have followed Iyer and Simpson [IS] in adopting what they call a locally abelian parabolic bundle as our definition. Certain notions and constructions given later (weight vectors, Biswas' correspondence) become more straightforward with this definition.

We describe a few basic examples.

Example 1.4. *Any vector bundle E can be given a parabolic structure with a single weight 0 and $E^i = E(-iD)$. We refer to this as the trivial parabolic structure.*

Example 1.5. *Given any line bundle L and coefficients $\beta_i \in [0,1)$ for each component D_i of D, we have the following parabolic line bundle*

$$L^\alpha := L(\sum -\lfloor 1+\alpha-\beta_i \rfloor D_i) \tag{1}$$

where $\lfloor\cdot\rfloor$ is the floor function.

We will see shortly that the weights are exactly the β_i. We can assemble these into a vector $(\beta_1,\ldots,\beta_n) \in \mathbb{R}^n$ that we call the *normalized weight vector* for L^*. We can make this independent of the labeling by viewing it as an element of $Hom_{Set}(\mathrm{Comp}(D),\mathbb{R})$, where $\mathrm{Comp}(D)$ is the set of irreducible components of D. We can recover the normalized weight vector from the parabolic structure alone: the ith component of the weight vector is β_i if and only if $Gr^{\beta_i}L$ is nonzero at the generic point of D_i. It follows easily that any parabolic line bundle is isomorphic to the one above for some unique normalized weight vector. Our definition says that Zariski locally a parabolic bundle is a direct sum of parabolic line bundles. It would equivalent to formulate this in the analytic topology. The proof is implicit in the argument given the first paragraph of [IS, p 361].

We will determine the weights and quasi-parabolic structure for the above example. To simplify the notation, reindex the D_i, so that $0 \le \beta_1 \le \ldots \le \beta_n < 1$. The following is straightforward.

Lemma 1.6. *The set of normalized weights is exactly the set $\{\beta_i\}$. If we list the weights union 0 in increasing order $0 = \beta_{r_0} < \beta_{r_1} \ldots < \beta_{r_\ell}$, then*

$$F^i(L) = L(-D_1 - \ldots - D_{r_i})$$

We want extend the notion of normalized weight vectors to parabolic bundles. Given a Zariski open $U \subseteq X$, we have a restriction $Hom(\mathrm{Comp}(D),\mathbb{R}) \to Hom(\mathrm{Comp}(U\cap D),\mathbb{R})$. Suppose that we are given a Zariski open cover $\{U_i\}$ of X such that each $E|_{U_i}$ is a sum of parabolic line bundles. We say that β is a *normalized weight vector* of E if for each i, $\beta|_{U_i}$ is a normalized weight vector of a line bundle summand of $E|_{U_i}$. This notion is easily seen to be independent of the cover.

Example 1.7. *Suppose that (V_o,∇_o) is a vector bundle with an integrable connection with regular singularities over U. By Deligne [D], for each v α there exists a unique extension*

$$\nabla^\alpha : V^\alpha \to \Omega^1_X(\log D)\otimes V^\alpha$$

with the eigenvalues of the residue $Res_{D_i}(\nabla^\alpha) \in \mathrm{End}(V^\alpha \otimes \mathcal{O}_{D_i})$ having real parts in $[\alpha, 1+\alpha)$, for each irreducible component D_i of D. This again forms a parabolic bundle, that we refer to as the Deligne parabolic bundle. If the monodromy of ∇_o around each component of D is unipotent, then V_* is trivial.

Definition 1.8. *A parabolic Higgs sheaf or bundle on (X,D) is a parabolic sheaf or bundle E^* together with holomorphic map*

$$\theta : E \to \Omega^1_X(\log D)\otimes E$$

such that

$$\theta\wedge\theta = 0$$

and

$$\theta(E^\alpha) \subseteq \Omega^1_X(\log D) \otimes E^\alpha$$

Natural examples come from variations of Hodge structure. These will be discussed in more detail in section 5.

2. Biswas' correspondence

We will assume in this section that the weights are rational with denominator dividing a fixed positive integer N. Recall that Kawamata [K, Theorem 17] has constructed a smooth projective variety Y, and a finite Galois map $\pi : Y \to X$, such that $\tilde{D} := (\pi^* D)_{red}$ is a simple normal crossing divisor. Let $\pi^* D_i = k_i N(\tilde{D}_i)$ for some integer $k_i > 0$, where $\tilde{D}_i = (\pi^* D_i)_{red}$. Let G denote the Galois group of π. A *G-equivariant vector bundle* on Y, is a bundle $p : V \to Y$ (viewed geometrically rather than as a sheaf) on which G acts compatibly with p.

We list some basic classes of examples.

Example 2.1. *For the above Galois covering $\pi : Y \to X$ with Galois group G and any vector bundle V over Y, $\pi^* V$ can be made into a G-equivariant bundle, so that the projections p*

$$\begin{tikzcd} \pi^* V \arrow[r] \arrow[d, "p"] & V \arrow[d, "p"] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

are compatible with the G-action. Fix any point $y \in Y$, then the action of isotropy subgroup of the point y on the fiber $(\pi^* V)_y$ is trivial.

Example 2.2. *With the same notation as the above, the line bundle $\mathcal{O}_Y(\tilde{D}_i)$ has an equivariant structure compatible with the one on $\pi^* \mathcal{O}_X(D_i)$ under the isomorphism $\mathcal{O}_Y(\tilde{D}_i)^{\otimes k_i N} \cong \pi^* \mathcal{O}_X(D_i)$.*

There is a Higgs version of G-equivariant bundle given by Biswas [B2]. A *G-equivariant Higgs bundle* is a pair (V, θ), with a G-equivariant bundle V on Y and an equivariant morphism $\theta : V \to \Omega^1_Y(\log D) \otimes V$, such that $\theta \wedge \theta = 0$. Then we have the following results given by Biswas.

Theorem 2.3 (Biswas [B1] [B2, Theorem 5.5])**.** *With the notation as above, we have the following two equivalences of categories:*

1. *An equivalence $E^* \mapsto \mathcal{E}$ between the category of parabolic bundles on X with weights in $\frac{1}{N}\mathbb{Z}^n$ and G-equivariant bundles on Y.*
2. *An equivalence $(E^*, \theta) \mapsto (\mathcal{E}, \vartheta)$ between the category of parabolic Higgs bundles on X with weights in $\frac{1}{N}\mathbb{Z}^n$ and G-equivariant Higgs bundles on Y.*

We recall the construction in one direction for (1). Given an G-equivariant bundle $\mathcal{E}$ on Y, we obtain a parabolic bundle

$$E^\alpha = (\pi_*(\mathcal{E} \otimes \mathcal{O}_Y(\lfloor -\alpha \cdot \pi^* D \rfloor))^G \tag{2}$$

where $\lfloor -\alpha \cdot \pi^* D \rfloor = \sum_i \lfloor -\alpha_i k_i N \rfloor \tilde{D}_i$.

Suppose that (V_o, ∇_o) is a vector bundle with connection satisfying the assumptions of Example 1.7. In addition suppose that the eigenvalues of the monodromy around D are Nth roots of unity. Then the weights of the Deligne parabolic bundle lie in $\frac{1}{N}\mathbb{Z}$. Furthermore $(\tilde{V}_o, \Box_o) := (\pi^* V_o, \pi^* \nabla_o)$ has unipotent local monodromies. Let (V, ∇) and $(\tilde{V}, \Box)$ denote Deligne's extensions of V_o and $\tilde{V}_o$. The functoriallity of this construction [D, prop 5.4], shows that $\tilde{V}$ is equivariant.

Lemma 2.4. *Biswas' construction applied to $\tilde{V}$ yields V^*.*

Proof. Since we will not need this result, we will merely outline the proof when $\dim X = 1$. Working locally, we may assume that V^* is a line bundle on X. Suppose the residue of ∇ at D is given by β. Let

$$\pi : Y \to X$$

by the cyclic cover of degree N branched over D such that $\beta \in \frac{1}{N}\mathbb{Z}$. Then, π is locally given by $y^N = x$, where x and y are local coordinates defined on coordinate neighborhoods $U \subset X$ and $W \subset Y$. We assume that D and $\tilde{D} = (f^* D)_{\mathrm{red}}$ are defined by $x = 0$ and $y = 0$. Let $G \cong \mathbb{Z}/N\mathbb{Z}$ be the Galois group of π, and let μ denote a generator.

Let e be a local frame of V^0 such that ∇^0 is given by the connection matrix

$$\beta \frac{dx}{x}$$

Let $j \in \mathbb{N}$ be an integer such that

$$\frac{j}{N} = \beta$$

Then, the connection matrix of $\pi^* V^0$ locally on W will be given by

$$j \frac{dy}{y}$$

with respect to the frame $s = f^* e$.

Let $W^* = W - \tilde{D}$. We have an inclusion of bundles on $Y - \tilde{D}$

$$\phi : \pi^* V_o \hookrightarrow \tilde{V}$$

which extends to an isomorphism

$$\phi : \pi^* V^0(j\tilde{D}) \to \tilde{V}$$

Locally on W, it is given by

$$\phi : O_W \cdot s \to O_W \cdot s$$
$$y^{-j} s \mapsto s$$

Now, $f^* V^*(j\tilde{D})$ has a natural G-action, and the above isomorphism respect this action. Locally on W, the action of G on $\tilde{V}$ can be described as

$$\mu \cdot s = \mu^{-j} \times s$$

We claim that

$$(\pi_* \tilde{V} \otimes O_Y(\lfloor -\alpha N \rfloor \tilde{D}))^G$$

is the extension of V^o whose residue lies in $[\alpha, 1+\alpha)$.

Since $(\pi_* \pi^* V_o)^G = V^o$, We have the natural inclusion

$$V^o \hookrightarrow (\pi_* \tilde{V})^G$$

on $X - D$.

Next, let ∇^α be the connection on $(\pi_*\tilde{V} \otimes O_Y(\lfloor -\alpha N \rfloor \tilde{D}))^G$, we compute

$$\mathrm{Res}^D \nabla^\alpha$$

Locally on U, $y^{j-\lfloor -\alpha N \rfloor} s$ is a basis for $(f_*\tilde{V})^G$. So

$$\mathrm{Res}_D \nabla^\alpha = \frac{j - \lfloor -\alpha N \rfloor}{N}$$

which lies in $[\alpha, 1{+}\alpha)$, as claimed. This proves the lemma.

□

3. Parabolic Chern classes

We give a quick definiton of parabolic Chern classes. Given the polynomial

$$\prod_1^r (1 + t(x_i + y_i))$$

we may write the coefficient of t^k as a polynomial $P_k(s_1, \dots, s_r, y_1 \dots, y_r)$ in the elementary symmetric polynomials $s_i = s_i(x_1, \dots, x_r)$ and the remaining variables y_j. Given a rank r parabolic bundle E^* with normalized weight vectors $\alpha^{(1)} = (\alpha_i^{(1)}), \dots, \alpha^{(r)} = (\alpha_i^{(r)})$, we define the parabolic Chern class by

$$\text{par-c}_k(E^*) = P_k(c_1(E), \dots c_r(E), \sum \alpha_i^{(1)}[D_i], \dots, \sum \alpha_i^{(r)}[D_i])$$

We can unpack this formula with the help of the splitting principle. For a parabolic line bundle L^* with notation as in Example 1.5, the weights of L^* in the interval $[0, 1)$ is (β_i). Then we see that the parabolic first Chern class of L^* is

$$\text{par-c}_1(L^*) = c_1(L) + \sum_i \beta_i [D_i]. \tag{3}$$

Given a parabolic bundle E^* of rank r, let $p : Fl(E) \to X$ denote the full flag bundle of E. The pullback p^*E carries a filtration $F^i \subset E$ by subbundles such that associated graded $G_i = F^i/F^{i+1}$ are line bundles. The parabolic structure on E can be pulled back to a parabolic structure on p^*E along p^*D, and each G_i carries the induced parabolic structure. One sees from above that:

Lemma 3.1. *The parabolic Chern classes satisfy*

$$1 + \sum p^* \text{par-c}_i(E^*) = \prod_i (1 + \text{par-c}_1(G_i^*))$$

Lemma 3.2 (Biswas)**.** *Given any parabolic vector bundle E^* with weights in $\frac{1}{N}\mathbb{Z}$, let $\pi : Y \to X$ and $\mathcal{E}$ be the G-equivariant bundle corresponding to E^* as in theorem 2.3. Then*

$$\pi^* \text{par-c}_i(E^*) = c_i(\mathcal{E})$$

Proof. We can use lemma 3.1 and the injectivity of the map $H^*(X, \mathbb{R}) \to H^*(Fl(E), \mathbb{R})$ to reduce this to the case where $i = 1$ and $E^* = L^*$ is a line bundle. Let us use $\mathcal{L}$ instead of $\mathcal{E}$. Then

$$\pi^* \text{par-c}_1(L^*) = c_1(\pi^* L) + \sum_i \beta_i [\pi^* D_i] = c_1(\pi^* L) + \sum_i k_i \beta_i N [\tilde{D}_i].$$

By Biswas [B1, (3.11)],

$$c_1(\mathcal{L}) = c_1(\pi^* L) + \sum_i \beta_i k_i N [\tilde{D}_i] = \pi^* \text{par-}c_1(L^*).$$

□

Lemma 3.3. *Given $\epsilon > 0$ and a parabolic bundle E^* with trivial parabolic Chern classes, there exists a parabolic bundle E'^* with trivial parabolic Chern classes and rational weights which is ϵ-close to E^*.*

Proof. We first treat the case where E^* is a line bundle with normalized weight vector $\alpha = (\alpha_i)$. By the assumption, we have

$$\text{par-}c_1(E^*) = c_1(E) + \sum_{j=1}^{n} \alpha_j \cdot [D_j] = 0 \ \in H^2(X, \mathbb{R})$$

Since $c_1(E)$ and $[D_j]$ are in $H^2(X, \mathbb{Q})$, the above equation defines a rational affine subspace of $\mathbb{R}^n$. The rational vectors are dense in this subspace. Therefore we can choose a rational vector $\alpha' = (\alpha'_i)$ with $|\alpha'{}_j - \alpha_j| < \epsilon$ for all i and

$$c_1(E) + \sum_{j=1}^{n} \alpha'_j \cdot [D_j] = 0 \ \in H^2(X, \mathbb{R}).$$

Now we do the general case. By the assumption that $\text{par-c}_i(E^*) = 0$ in $H^{2i}(X, \mathbb{R})$, we have $p^* \, \text{par-c}_i(E^*) = 0$ in $H^{2i}(Fl(E), \mathbb{R})$. By Lemma 3.1, we know that $\text{par-}c_1(G_k^*) = 0$ in $H^2(Fl(E), \mathbb{R})$ for all k. After identifying $\text{Comp}(D) = \text{Comp}(p^* D)$, we may identify weight vectors of E^* and $p^* E^*$. It is easy to see that the normalized weight vectors of E^* are precisely the weight vectors of the various G_k^*. By the first paragraph, we can find $G_k'^*$, ϵ-close to G_k^*, having rational normalized weights, and $\text{par-}c_1(G_k'^*) = 0$. Let E'^* be E^* as a quasi-parabolic bundle, but with the normalized weight vectors of $G_k'^*$.

□

4. Stability and Semistability

In this section, we will recall the definitions of (semi)stability for parabolic and equivariant Higgs sheaves. There are in fact two different notions: μ-, or slope, (semi)stability and p- , or Hilbert polynomial, (semi)stability. The μ-(semi)stability condition behaves well with respect to Biswas' correspondence. While p-(semi)stability is more convenient for the construction of moduli space.

We fix a very ample line bundle H on X. For a parabolic sheaf E^*, we have the following numerical invariants defined by Maruyama and Yokogawa [MY]. The parabolic Hilbert polynomial of E^* is

$$\text{par-}P_{E^*}(m) := \int_0^1 P_{E^t}(m) dt \tag{4}$$

where $P_{E^t}(m)$ is the Hilbert polynomial of E^t with respect to H. The normalized parabolic Hilbert polynomial of E^* is $\text{par-}p_{E^*}(m) := \text{par-}P_{E^*}(m)/\operatorname{rank}(E)$. The parabolic degree of E^* is defined to be

$$\text{par-deg}(E^*) := \int_0^1 deg(E^t) dt + \operatorname{rank}(E) \cdot \deg(D) \tag{5}$$

where $deg(E^t)$ is the usual degree of E^t, with respect to H. The parabolic H-slope of E^* is $\text{par-}\mu_H(E^*) := \text{par-deg}(E^*)/\operatorname{rank}(E)$.

For a parabolic Higgs sheaf (E^*, θ), the above invariants are defined to be that of its underlying parabolic sheaf E^*. We have the following proposition.

Proposition 4.1. *For any parabolic bundle E^*, we have*

$$\text{par-deg}(E^*) = \text{par-}c_1(E^*) \cdot H^{d-1}$$

(We recall that $d = \dim X$.)

Proof. By taking top exterior powers, we may reduce to the case where $E^* = L^*$ is a line bundle. We may assume that L^* is as described in example 1.5 with $0 \le \beta_1 \le \ldots \le \beta_n < \beta_{n+1} = 1$. In fact, we may assume that the β_i form a strictly increasing sequence, because both sides of the expected formula depend continuously on these parameters. Then by lemma 1.6, the normalized weights are given by $\beta_1, \ldots, \beta_n$ and the filtration by $F^i(L) = L(-D_1 - \ldots - D_i)$.

By (5), the left hand side of the purported equation is

$$\begin{aligned}\text{par-deg}(L^*) &= \sum_{i=0}^{n} \deg(F^i(L)) \cdot (\beta_{i+1} - \beta_i) + \deg(D)\\ &= \sum_{i=0}^{n} (c_1(L) - \sum_{j=1}^{i} c_1(D_j)) \cdot H^{d-1} \cdot (\beta_{i+1} - \beta_i) + \deg(D)\\ &= \deg(L) + \sum_{i=0}^{n} \beta_i \deg(D_i)\end{aligned}$$

By (3), the right hand side is

$$\begin{aligned}\text{par-}c_1(L^*) \cdot H^{d-1} &= (c_1(L) + \sum_{j=1}^{n} \beta_j \cdot c_1(D_j)) \cdot H^{d-1}\\ &= \deg(L) + \sum_{i=0}^{n} \beta_i \deg(D_i)\end{aligned}$$

□

We can define (semi)stability of parabolic and G-equivariant Higgs bundles using the numerical invariants defined above.

Definition 4.2 ([B2][MY])**.** *1) A parabolic Higgs sheaf (E^*, θ) on X is called μ_H-semistable (resp. μ_H-stable) if for any coherent saturated subsheaf V of E, with $0 < \text{rank}V < \text{rank}E$ and $\theta(V) \subseteq V \otimes \Omega^1_X(\log D)$, the condition*

$$\text{par-}\mu_H(V^*) \le \text{par-}\mu_H(E^*) \ (resp.\ \text{par-}\mu_H(V^*) < \text{par-}\mu_H(E^*))$$

is satisfied, where V^* carries the induced the parabolic structure from E^*, i.e. $V_\alpha := V \cap E^\alpha$. Stability or semistability for a G-equivariant Higgs sheaf $(\mathcal{E}, \vartheta)$ is defined similarly, where in addition $\mathcal{V}$ is required to be an equivariant subsheaf.

2) A parabolic Higgs sheaf (E^*, θ) on X is called p-semistable (resp. p-stable) if for any coherent saturated subsheaf V of E, with $0 < \text{rank}V < \text{rank}E$, and $\theta(V) \subseteq V \otimes \Omega^1_X(\log D)$, the condition

$$\text{par-}p_{V^*}(m) \le \text{par-}p_{E^*}(m) \ (resp.\ \text{par-}p_{V^*}(m) < \text{par-}p_{E^*}(m))$$

is satisfied for all sufficiently large integers m, where V^* carries the induced parabolic structure.

Lemma 4.3. *A μ_H-stable parabolic Higgs bundle (E^*, θ) on (X, D) is p-stable.*

Proof. Denote H by $\mathcal{O}_X(1)$. For any torsion free sheaf E, $P_E(m) = \dim H^0(X, E \otimes \mathcal{O}_X(m))$, for $m \gg 0$. In particular, P_E can be uniquely written in the form

$$P_E(m) = \sum_{i=0}^{d} a_i(E) \frac{m^i}{i!}.$$

The rank of E is $\mathrm{rank}(E) = \frac{a_d(E)}{a_d(\mathcal{O}_X)}$. By Hirzebruch-Riemann-Roch formula, we have $\deg(E) = a_{d-1}(E) - \mathrm{rank}(E) \cdot a_{d-1}(\mathcal{O}_X)$.

Now we consider the parabolic Higgs bundle (E^*, θ). For the parabolic Hilbert polynomial, we have

$$\text{par-}P_{E^*}(m) = \sum_{i=0}^{d} (\sum_{j=0}^{l} a_i(F^j(E))(\alpha_{j+1} - \alpha_j)) \frac{m^i}{i!},$$

where $\alpha_0 = 0$ and $\alpha_{l+1} = 1$. Then the normalized parabolic Hilbert polynomial of (E^*, θ) is

$$\text{par-}p_{E^*}(m) = \frac{a_d(\mathcal{O}_X)}{a_d(E)} \sum_{i=0}^{d} (\sum_{j=0}^{l} a_i(F^j(E))(\alpha_{j+1} - \alpha_j)) \frac{m^i}{i!}. \tag{6}$$

For the parabolic degree, we have

$$\text{par-deg}(E^*) = \sum_{j=1}^{l} a_{d-1}(F^j(E))(\alpha_{j+1} - \alpha_j) + \mathrm{rank}(E) \cdot (\deg(D) - a_{d-1}(\mathcal{O}_X)).$$

Then the parabolic H-slope is

$$\text{par-}\mu_H(E^*) = \frac{a_d(\mathcal{O}_X)}{a_d(E)} \sum_{j=1}^{l} a_{d-1}(F^j(E))(\alpha_{j+1} - \alpha_j) + \deg(D) - a_{d-1}(\mathcal{O}_X). \tag{7}$$

Since the parabolic Higgs bundle (E^*, θ) is μ_H-stable, for any coherent subsheaf V of E, satisfying the conditions of definition 4.2, we have $\text{par-}\mu_H(V^*) < \text{par-}\mu_H(E^*)$, i.e.,

$$\frac{a_d(\mathcal{O}_X)}{a_d(V)} \sum_{j=1}^{l} a_{d-1}(F^j(V))(\alpha_{j+1} - \alpha_j) < \frac{a_d(\mathcal{O}_X)}{a_d(E)} \sum_{j=1}^{l} a_{d-1}(F^j(E))(\alpha_{j+1} - \alpha_j). \tag{8}$$

In general, for any parabolic Higgs sheaf (G^*, θ), by the above equation (6), the leading term of $\text{par-}p_{G^*}(m)$ is always $\frac{m^d}{d!}$, and the $d-1$ degree term is

$$\frac{a_d(\mathcal{O}_X)}{a_d(G)} \sum_{j=1}^{l} a_{d-1}(F^j(G))(\alpha_{j+1} - \alpha_j) \frac{m^{d-1}}{(d-1)!}.$$

Hence we get

$$\text{par-}p_{V^*}(m) < \text{par-}p_{E^*}(m),$$

for all sufficiently large integers m, by the previous inequality (8). $\square$

The μ-(semi)stability condition behaves well in Biswas's correspondence. In fact, we have the following result by Biswas.

Lemma 4.4 (Biswas [B2, Theorem 5.5])**.** *Under Biswas's correspondence in Theorem 2.3, μ_H-semistable (Higgs) bundles with weights in $\frac{1}{N}\mathbb{Z}$ correspond to $\mu_{\pi^* H}$-semistable G-equivariant (Higgs) bundles.*

5. Review of Nonabelian Hodge theory

Natural examples of parabolic Higgs bundles come from variations of Hodge structures. Suppose that (V_o, ∇_o) is a flat bundle underlying a polarized variation of Hodge structure on $X-D$ [G, SW] with unipotent monodromy around components of D. Then we can form the Deligne canonical extension V to V_o. The bundle V_o also carries a Hodge filtration $F_o^\bullet$ satisfying Griffiths' transversality. By a theorem of Schmid [SW], the Hodge filtration extends to a filtration $F^\bullet$ of V. Let $E = \mathrm{Gr}_F V$, and $\theta = \mathrm{Gr}_F \nabla$. Then, as observed already in [A], (E, θ) is a Higgs bundle with trivial parabolic structure with trivial Chern classes. If the monodromies are quasi-unipotent, as in geometric examples, then we may use a Galois G-cover $\pi : Y \to X$, as in section 2, such that $\pi^*\nabla^o$ is unipotent. The Higgs bundle associated to $\pi^*\nabla^o$ is naturally G-equivariant, and thus via theorem 2.3, we get a parabolic Higgs bundle on X with rational weights and trivial parabolic Chern classes. To more general complex variations of Hodge structures, we can also associate a parabolic Higgs bundles with vanishing parabolic Chern classes (but real weights), but this relies on nonabelian Hodge theory. We review these ideas now, since they will be needed later. Let us start with a definition of a complex polarized variation of Hodge structures or a $\mathbb{C}$-PVHS from the C^∞ point of view. Let $A^{i,j}(H)$ denote the space of C^∞ (i,j)-forms with values in a bundle H.

Definition 5.1. *A complex polarized variation of Hodge structures over $U = X-D$ is a C^∞-vector bundle H with a decomposition $H = \bigoplus_p H^p$, a flat connection $\mathcal{D}$ and a horizontal indefinite Hermitian form k_H. These are required to satisfy Griffiths' transversality*

$$\mathcal{D} : H^p \longrightarrow A^{0,1}(H^{p+1}) \oplus A^{1,0}(H^p) \oplus A^{0,1}(H^p) \oplus A^{1,0}(H^{p-1}),$$

the decomposition $\bigoplus_p H^p$ is orthogonal with respect to k_H, and k_H is positive (negative) definite on H^p with p is even (odd).

To relate this to the more traditional perspective, decompose $\mathcal{D}$ into operators of types $(1,0)$ and $(0,1)$

$$\mathcal{D} = \mathcal{D}^{1,0} + \mathcal{D}^{0,1}$$

The operator $\mathcal{D}^{0,1}$ defines a complex structure on H, and let V_o denote the corresponding holomorphic bundle. The operator $\nabla = \mathcal{D}^{1,0}$ induces a holomorphic connection on V_o, and $F^pV_o = H^p \oplus H^{p+1} \oplus \dots$ forms a holomorphic subbundle such that $\nabla(F^pV_0) \subset \Omega^1_U \otimes F^{p-1}V_o$. The graded holomorphic bundle $E_o = \mathrm{Gr}_F V_o$ carries a Higgs field $\theta = \mathrm{Gr}_F \nabla$.

The Higgs bundle (E_o, θ) can be constructed from a different point of view which is more general. First observe that after changing signs of k_H on odd H^p, we get a positive definite Hermitian form K_H. Suppose more generally that we are given a C^∞ flat bundle $(H, \mathcal{D})$ with a Hermitian metric K over U, we can decompose $\mathcal{D} = \mathcal{D}^{1,0} + \mathcal{D}^{0,1}$ as above. Let δ' and δ'' be operators of type $(1,0)$ and $(0,1)$ such that $\mathcal{D}^{1,0} + \delta''$ and $\delta' + \mathcal{D}^{0,1}$ are metric connections with respect to the metric K, i.e.,

$$(\mathcal{D}^{0,1}u, v)_K + (u, \delta' v)_K = \mathcal{D}^{0,1}(u,v)_K,$$

$$(\delta'' u, v)_K + (u, \mathcal{D}^{1,0} v)_K = \delta''(u,v)_K,$$

for all local sections u, v of H. Define

$$\bar{\partial} := \frac{1}{2}(\mathcal{D}^{0,1} + \delta'')$$
$$\theta := \frac{1}{2}(\mathcal{D}^{1,0} - \delta')$$

Definition 5.2. *A triple $(H, \mathcal{D}, K)$ on U is called a harmonic bundle if the pseudo-curvature $G_K := \bar{\partial}\theta = 0$. A harmonic bundle $(H, \mathcal{D}, K)$ is tame if the eigenvalues of the associated Higgs field θ (which are multivalued 1-forms) have order at most 1, near the divisor D.*

Given a harmonic bundle $(H, \mathcal{D}, K)$, H equipped with $\mathcal{D}^{0,1}$ becomes a holomorphic bundle V_o over U with a holomorphic connection ∇ induced from $\mathcal{D}^{1,0}$; H equipped with $\bar{\partial}$ becomes a holomorphic bundle E_o over U and θ becomes a holomorphic Higgs field $\theta : E_o \to \Omega^1_U \otimes E_o$. If $(H, \mathcal{D}, K)$ is tame, then both V_o and E_o extend to parabolic bundles over X making the latter into a parabolic Higgs bundle. Roughly speaking, $E^\alpha \subset j_* E_o$ is generated by sections s, with $|s(x)|_{K_H} = O(|f(x)|^{\alpha - \epsilon})$, where f is a local equation for D, and similarly for V^α.

We have the following correspondence given by Simpson [S3, main theorem] for curves and Mochizuki [M1, thm 1.4] for higher dimensional quasi-projective varieties.

Theorem 5.3 (Kobayashi-Hitchin Correspondence)**.** *For the quasi-projective variety $X - D$ and ample line bundle H over X, we have a one to one correspondence between tame harmonic bundles $(H, \mathcal{D}, K)$ and μ_H-polystable parabolic Higgs bundles (E^*, θ) with vanishing parabolic Chern classes, where "μ_H-polystable" bundle means a direct sum of μ_H-stable bundles.*

Mochizuki [M2, thm 1.1] gives a stronger statement, which will not need. A key example of a tame harmonic is given by $\mathbb{C}$-PVHS. This has certain extra features as well.

Proposition 5.4. *Any $\mathbb{C}$-PVHS $(H, \mathcal{D}, k_H)$ with metric K_H over U is a tame harmonic bundle. The resulting parabolic structure on V agree Deligne parabolic structure. The filtration*

$$F^p V^\alpha = V^\alpha \cap j_* F^p V$$

gives a filtration by subbundles, and the associated graded $Gr_F V^\alpha$ can be identified with E^α.

A proof can be found in [Br, section 7], although the result is not stated explicitly in this form. The proposition gives a grading on E by $Gr^p_F V^0$. Let $F^{\max} V = Gr^p_F V^0 = F^p V^0$, where p is the largest integer for which this is nonzero. We will refer to this as the *smallest Hodge bundle* associated to the variation.

Corollary 5.5. *A $\mathbb{C}$-PVHS gives rise to a μ_H-polystable parabolic Higgs bundle (E^*, θ) with vanishing parabolic Chern classes. Furthermore θ is nilpotent in the sense that it has zero eigenvalues.*

Proof. The last statement follows from the fact that θ shifts the grading by -1. $\square$

***

Next, we recall some facts about the moduli space of parabolic Higgs sheaves and the Hitchin fibration from Yokogawa [Y]. Let Γ denote the following data: a positive integer m, system of rational weights $0 < \alpha_1 < \alpha_2 < ... < \alpha_l < 1$ and polynomials $P, P_1, ..., P_l$. Consider the following contravariant functor:

$$\overline{\mathfrak{M}}(X, D, \Gamma) : Sch/\mathbb{C} \longrightarrow \mathcal{S}et$$

which assigns to any scheme the set of isomorphism classes of flat families of rank m parabolic Higgs sheaves (E^*, θ) over $(X \times S, D \times S)$ with the following properties

1. For each closed point s of S, $({E_s}^*, \theta_s) := (E^*, \theta)_s$ has weights α with quasi-parabolic structure $E_s \supsetneq F^1(E_s) \supsetneq \ldots \supsetneq F^l(E_s) \supsetneq E_s(-D \times \{s\})$.
2. $({E_s}^*, \theta_s)$ is p-semistable.
3. The Hilbert polynomial of E_s with respect to polarization H is P. The Hilbert polynomials of $E_s/F^i(E_s)$ are P_i.
4. The parabolic Chern classes of $({E_s}^*, \theta_s)$ vanish.

Define an equivalence relation on $\overline{\mathfrak{M}}(X, D, \Gamma)(S)$ by $(E^*, \theta) \sim (E'^*, \theta')$ if and only if there exists a line bundle $\mathcal{L}$ over S, such that $Gr^W_*(E^*, \theta) \cong Gr^{W'}_*(E'^*, \theta') \otimes \mathcal{L}$, where W, W' are Jordan-Hölder filtrations (defined on [Y, p 457]). Define

$$\overline{\mathcal{M}}(X, D, \Gamma) = \overline{\mathfrak{M}}(X, D, \Gamma)(S)/\sim$$

We denote by $\mathcal{M}(X, D, \Gamma)$ the subfunctor of $\overline{\mathcal{M}}(X, D, \Gamma)$ consisting of all flat families of p-stable parabolic Higgs bundles. Then we have the following theorem given by Yokogawa [Y].

Theorem 5.6 (Yokogawa)**.** *There exist quasiprojective moduli spaces $M(X, D, \Gamma) \subset \overline{M}(X, D, \Gamma)$ coarsely representing the functors $\mathcal{M}(X, D, \Gamma)$ and $\overline{\mathcal{M}}(X, D, \Gamma)$ respectively. The closed points of $M(X, D, \Gamma)$ are in one to one correspondence to the isomorphic classes of p-stable parabolic Higgs bundles (E^*, θ) of rank m over (X, D) with weights α, Hilbert polynomials $\mathcal{P}$, and vanishing Chern classes.*

Remark 5.7. *Points of $\overline{M}(X, D, \Gamma)$ are $\sim$-equivalence classes of p-semistable sheaves on X.*

We will need to recall a few details of the construction. Yokogawa [Y, §2] shows there is a scheme R^{ss} on which a special linear group G acts, such that there are inclusions

$$M(X, D, \Gamma) \subset \overline{M}(X, D, \Gamma) \subset R^{ss} /\!\!/ G$$

where the last space is the GIT quotient. Let $R^0 = R^0(X, D, \Gamma)$ denote the preimage of $\overline{M}(X, D, \Gamma)$ in R^{ss}. Then also by construction, $X \times R^0$ comes with a family of parabolic sheaves inducing the quotient map $R^0 \to \overline{M}(X, D, \Gamma)$. We will refer to this as the semi-universal sheaf.

Yokogawa has also generalized the construction and properties of the Hitchin map of Simpson [S1, S2] in the non-log case.

Theorem 5.8 (Yokogawa)**.** *There is a Hitchin map*

$$\mathfrak{h} : \overline{M}(X, D, \Gamma) \longrightarrow \mathfrak{V}(X, m) := \bigoplus_{i=0}^{m-1} H^0(X, Sym^i \Omega^1_X(\log D)).$$

given by sending (E^*, θ) to its characteristic polynomial. This map is projective.

Remark 5.9. *Note that $\mathfrak{h}(E^*, \theta) = 0$ if and only if θ is nilpotent.*

6. Vanishing Theorem: nilpotent case

In this section, we prove a vanishing theorem for de Rham complex of any μ_H-semistable parabolic Higgs bundle (E^*, θ), with vanishing parabolic Chern classes and nilpotent Higgs field θ.

Lemma 6.1. *Let (E^*, θ) be a μ_H-semistable parabolic Higgs bundle. Then there exist $\epsilon > 0$ such that any parabolic Higgs bundle ϵ-close to (E^*, θ) is μ_H-semistable.*

Proof. Suppose (E^*, θ) is stable. Let us denote the normalized weights by $\{\alpha_1, ..., \alpha_r\}$, and the quasiparabolic structure by $E = F^0(E) \supsetneq F^1(E) \supsetneq \ldots \supsetneq F^r(E) \supsetneq E(-D)$. Denote the degree of each bundle $F^i(E)$ by $d_i(E)$ and $d_i(V) = \deg V \cap F^i(E)$ for any subsheaf $V \subseteq E$. Suppose V satisfies the conditions in definition 4.2, then

$$\frac{\sum_{i=0}^r d_i(V)(\alpha_{i+1} - \alpha_i)}{\operatorname{rank} V} < \frac{\sum_{i=0}^r d_i(E)(\alpha_{i+1} - \alpha_i)}{\operatorname{rank} E}$$

or equivalently

$$\sum_{i=0}^r (\operatorname{rank} V \cdot d_i(E) - \operatorname{rank} E \cdot d_i(V))(\alpha_{i+1} - \alpha_i) > 0.$$

Since for any V, $\operatorname{rank} V \cdot d_i(E) - \operatorname{rank} E \cdot d_i(V)$ are integers, we can find $\epsilon > 0$ such that

$$\sum_{i=0}^r (\operatorname{rank} V \cdot d_i(E) - \operatorname{rank} E \cdot d_i(V))(\alpha'_{i+1} - \alpha'_i) > 0$$

for $|\alpha'_i - \alpha_i| < \epsilon$.

If (E^*, θ) is semistable, we have the Jordan-Hölder filtration of (E^*, θ)

$$0 \subset W_1 \subset W_2 \subset ... \subset (E^*, \theta)$$

such that the quotients W_i/W_{i-1} are stable. We apply the above argument to these subquotients. □

Proposition 6.2. *Let (E^*, θ) be a μ_H-semistable parabolic Higgs bundle with zero parabolic Chern classes. There exists a μ_H-semistable parabolic Higgs bundle $({E'}^*, \theta')$ with the same properties and rational weights such that $(E, \theta) = (E', \theta')$.*

Proof. This follows from lemma 3.3 and 6.1.

□

We discuss the penultimate form of the main result. Given any parabolic Higgs bundle (E^*, θ), we have the associated de Rham complex

$$\mathrm{DR}(E, \theta) = E \xrightarrow{\theta} \Omega_X(\log D) \otimes E \to \ldots.$$

Theorem 6.3. *Let (E^*, θ) be a μ_H-semistable parabolic Higgs bundle on (X, D) with vanishing Chern classes and with θ nilpotent. Let L be an ample line bundle on X. Then*

$$H^i(X, \mathrm{DR}(E, \theta) \otimes L) = 0$$

for $i > d$, where $d = \dim X$.

Proof. For the μ_H-semistable parabolic Higgs bundle (E^*, θ) on X, with trivial parabolic Chern classes, and θ nilpotent, by proposition 6.2, we can find a new semistable parabolic Higgs bundle (E'^*, θ) on X, with trivial parabolic Chern classes, θ nilpotent, and rational weights, such that $(E', \theta) = (E, \theta)$. Thus we can assume that the weights of (E^*, θ) are in $\frac{1}{N}\mathbb{Z}^n$, for some integer N.

Consider the Galois covering $\pi : Y \to X$ as described in Section 2. By Theorem 2.3, Lemma 3.2, and Lemma 4.4, we can find a $\mu_{\pi^* H}$-semistable G-equivariant Higgs bundle $(\mathcal{E}, \vartheta)$ on Y, such that $c_i(\mathcal{E}) = 0$ and ϑ is nilpotent. The nilpotency of ϑ is coming from Biswas's construction of $\mathcal{E}$. Actually by Biswas [B1, (3.3)], $\mathcal{E} \subset \pi^*(E \otimes \mathcal{O}_X(D))$, over which θ acts nilpotently. Now we can apply the first main theorem of [A] to conclude

$$\mathbb{H}^i(Y, \mathrm{DR}(\mathcal{E}, \vartheta) \otimes \pi^* L) = 0$$

By the projection formula, the pushforward of the de Rham complex $\mathrm{DR}(\mathcal{E}, \vartheta) \otimes \pi^* L$ through π is the complex $\pi_* \mathrm{DR}(\mathcal{E}, \vartheta) \otimes L$. Since $E = (\pi_* \mathcal{E})^G$ and the differential forms on the quotient space X are coming from the G-invariant differential forms on Y, after taking the G-invariant subcomplex, we get the de Rham complex $\mathrm{DR}(E, \theta) \otimes L$.

The next lemma shows that the natural map

$$\mathbb{H}^i(X, \mathrm{DR}(E, \theta) \otimes L) \to \mathbb{H}^i(Y, \mathrm{DR}(\mathcal{E}, \vartheta) \otimes \pi^* L)^G$$

is an isomorphism. □

Lemma 6.4. *Given any cochain complex of complex vector spaces $(C^\bullet, d)$ with a finite group G action, there is a natural isomorphism $H^i((C^\bullet)^G) \cong H^i(C^\bullet)^G$.*

Proof. This follows from the exactness of the functor $(-)^G$ (Maschke's theorem). □

Corollary 6.5. *For a Higgs bundle (E^*, θ) coming from a $\mathbb{C}$-PVHS we have*

$$\mathbb{H}^i(\mathrm{DR}(E, \theta) \otimes L) = 0$$

for $i > d$.

7. Vanishing theorem: general case

In this section, we will use the results in previous sections to prove the vanishing theorem for semistable parabolic Higgs bundles with vanishing parabolic Chern classes.

Lemma 7.1. *Let L be a line bundle on X. Let $(E_1^*, \theta_1) \sim (E_2^*, \theta_2)$ be equivalent p-semistable bundles with (E_1, θ_1) p-polystable (a direct sum of p-stable bundles). If*

$$\mathbb{H}^i(\mathrm{DR}(E_1, \theta_1) \otimes L) = 0$$

then

$$\mathbb{H}^i(\mathrm{DR}(E_2, \theta_2) \otimes L) = 0$$

Proof. The assumptions say that $(E_1, \theta_1) \cong Gr_*^J(E_2, \theta_2)$ J is a Jordan-Hölder filtration on (E_2, θ_2), and that

$$\mathbb{H}^i(\mathrm{DR}(Gr_*^J(E_2, \theta_2)) \otimes L) = 0$$

The conclusion follows easily from the exact sequences

$$0 \to \mathrm{DR}(J_{i-1}(E_2, \theta_2)) \to \mathrm{DR}(J_i(E_2, \theta_2), \theta) \to \mathrm{DR}(Gr_i^J(E_2, \theta_2)) \to 0$$

and induction.

□

The following is the main theorem.

Theorem 7.2. *Let (E_*, θ) be a μ_H-semistable parabolic Higgs bundle with vanishing parabolic Chern classes. Let L be an ample line bundle over X. Then*

$$\mathbb{H}^i(X, \mathrm{DR}(E, \theta) \otimes L) = 0$$

for $i > d$, where $d = \dim X$.

Proof. By lemmas 7.1 and 4.3, it suffices assume that (E^*, θ) is μ_H-stable. By proposition 6.2, there is no loss of generality in assuming that (E^*, θ) has rational weights $0 < \alpha_1 < \alpha_2 < ... < \alpha_l < 1$ Let $m = \operatorname{rank} E$, and $E \supsetneq F^1(E) \supsetneq ... \supsetneq F^l(E) \supsetneq E(-D)$ denote the quasi-parabolic structure. Denote the Hilbert polynomials of E with respect to H by P, and the Hilbert polynomials of $E/F^i(E)$ by P_i, respectively, and let $\Gamma = (m, \alpha_i, P, P_i)$. Then by theorem 5.6, the isomorphism class (E^*, θ) can be regarded as a closed point, which is denoted by p, in the moduli space $M(X, D, \Gamma) \subset \overline{M}(X, D, \Gamma)$ described earlier. Now we consider the Hitchin map $\mathfrak{h} : \overline{M}(X, D, \Gamma) \to \mathfrak{V}(X, m)$. We can assume that $\mathfrak{h}(p) \neq 0$, otherwise θ is nilpotent and we are done by theorem 6.3. Let $\mathbb{C} \cdot \mathfrak{h}(p)$ be the complex affine line passing through 0 and $\mathfrak{h}(p)$ in $\mathfrak{V}(X, m)$. Also, considering the $\mathbb{C}^*$-action

$$t : \overline{M}(X, D, \Gamma) \to \overline{M}(X, D, \Gamma)$$

$$(E^*, \theta) \mapsto (E^*, t\theta)$$

we will get a $\mathbb{C}^*$-orbit C_p of the point p as a curve in $\overline{M}(X, D, \Gamma)$. By properness of the Hitchin map, i.e., theorem 5.8, we can extend the curve C_p to $\overline{C}_p$ by adding a point p_0 in the fiber $\overline{M}(X, D, \Gamma)_0$ over 0 of the Hitchin fibration.

Now by earlier remarks, we get the following commutative diagram

$$\begin{tikzcd}
\widetilde{C}_p \arrow[r] \arrow[d] & R^0(X, D, \Gamma) \arrow[d, "pr"] & \tilde{p} \arrow[d, mapsto] \\
\overline{C}_p \arrow[r] \arrow[d] & \overline{M}(X, D, \Gamma) \arrow[d, "\mathfrak{h}"] & p \arrow[d, mapsto] \\
\mathbb{C} \cdot \mathfrak{h}(p) \arrow[r, hook] & \mathfrak{V}(X, m) & \mathfrak{h}(p)
\end{tikzcd}$$

where $\widetilde{C}_p$ is some curve in $R^0(X, D, \Gamma)$ mapping finitely to $\overline{C}_p$. Let $\tilde{p}, \tilde{p}_0 \in \widetilde{C}_p$ lie over p and p_0 respectively $R^0(X, D, \Gamma)_0$ by $\tilde{p}_0$. Now pullback the semi-universal parabolic bundle to the curve $\widetilde{C}_p$ to obtain a parabolic bundle $(\mathcal{E}^*, \vartheta)$ over $X \times \widetilde{C}_p$ which is flat over $\widetilde{C}_p$. Note that for any point $q \in \widetilde{C}_p$ not lying over p_0, the parabolic Higgs bundle $(\mathcal{E}^*, \vartheta)_q$ is equivalent (under $\sim$) to $(E^*, t\theta)$, for some $t \in \mathbb{C}^*$, and is therefore stable. Consequently, $(\mathcal{E}^*, \vartheta)_q \cong (E^*, t\theta)$.

Now consider the parabolic Higgs bundle $({E_0}^*, \theta_0)$ over X corresponding to p_0. This is a fixed point for the $\mathbb{C}^*$-action, so by Mochizuki [M1, proposition 1.9] it must, in fact, come from a $\mathbb{C}$-PVHS. Hence by corollary 6.5, we have

$$\mathbb{H}^i(\mathrm{DR}((E_0, \theta_0)) \otimes L) = 0, \text{ for } i > d.$$

By proposition 5.4 and theorem 5.3, we have that $({E_0}^*, \theta_0)$ is μ_H-polystable. Hence it is parabolic p-polystable by lemma 4.3. Since $(\mathcal{E}^*, \vartheta)_{\tilde{p}_0} \sim ({E_0}^*, \theta_0)$, lemma 7.1

implies

$$\mathbb{H}^i(\mathrm{DR}((\mathcal{E},\vartheta)_{\tilde{p}_0})\otimes L)=0,\ \text{for } i>d.$$

Since

$$q\mapsto \dim \mathbb{H}^i(\mathrm{DR}((\mathcal{E},\vartheta)_q)\otimes L)$$

is upper semi-continuous,

$$\mathbb{H}^i(\mathrm{DR}((\mathcal{E},\vartheta)_q)\otimes L)=0$$

for $i>d$, and q in a small open nighborhood of $\tilde{p}_0$ in $\widetilde{C}_p$. Thus we get

$$\mathbb{H}^i(\mathrm{DR}(E,t\theta)\otimes L)=0$$

for $i>d$ and t small enough. This implies

$$\mathbb{H}^i(\mathrm{DR}(E,\theta)\otimes L)=0$$

for $i>d$.

□

As an application, we can generalize the semipositivity theorem of Fujita, Kawamata and many others. See Brunebarbe [Br] for a fairly recent result along these lines, together with some history.

Corollary 7.3. *Suppose that for (E,θ) is a μ_H-semistable parabolic Higgs bundle with vanishing parabolic Chern classes, and that there is a decomposition $E=E_+\oplus E_-$ such that $\theta(E)\subseteq \Omega^1_X(\log D)\otimes E_-$. Then*

$$H^i(X,\omega_X\otimes E_+\otimes L)=0$$

for $i>0$ and E_+ is nef.

Proof. The assumptions imply that

$$\omega_X\otimes E_+\subseteq \mathrm{DR}(E,\theta)$$

is a direct summand. Therefore, there is an injection

$$H^i(X,\omega_X\otimes E_+\otimes L)\to \mathbb{H}^i(X,\mathrm{DR}(E,\theta)\otimes L)$$

This proves the vanishing statement.

For any $n>0$, the nth symmetric power S^nE satisfies the same assumptions as in the corollary with $(S^nE)_+=S^n(E_+)$. Therefore

$$H^i(X,\omega_X\otimes S^nE_+\otimes L)=0$$

for all $n>0$. Now apply [A, lemma 3.1] to conclude that E_+ is nef. □

Corollary 7.4. *Let $F^{\max}V$ be the smallest Hodge bundle associated to a $\mathbb{C}$-PVHS, then*

$$H^i(X,\omega_X F^{\max}V\otimes L)=0$$

for $i>0$, and $F^{\max}V$ is nef.

Remark 7.5. *As explained in the introduction, the proof of theorem ultimately hinges on* [A, thm 1] *which was proved by characteristic p methods. It is possible to give an entirely characteristic* 0 *proof with above trick as follows. The deformation argument used above shows that it suffices to prove*

$$\mathbb{H}^i(\mathrm{DR}(E,\theta)\otimes L)=0,\quad i>d$$

when (E,θ) comes from a $\mathbb{C}$-PVHS H. Since $H\oplus\bar{H}$ is an $\mathbb{R}$-PVHS, we are reduced to proving vanishing in this case. This can be done in principle *by adapting Schnell's* *proof of Saito's vanishing* [SC] *to the category of pure* $\mathbb{R}$*-Hodge modules introduced in* [SM].

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