Statement of the algorithm and problem

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1 Mathematical framework

Denote $\mathbb{R}$ as the real line. With the equivalence relation $T_x\mathbb{R}^3 \backsimeq \mathbb{R}^3$, tensor $\boldsymbol{A}$ in Euclidian space could be defined as $\boldsymbol{A} \in \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \cdots \otimes \mathbb{R}^3 \backsimeq \mathbb{R}^{3^{n_A}}$, in which n_A is the order of this tensor.

Define rotation action $R \in SO(3)$ on tensors of different order as follow:

$$R(\boldsymbol{u}) \in T_x(\mathbb{R}^3), (R(\boldsymbol{u}))_i = \sum_{j=0}^{3} R_{ij} u_j \quad \text{for} \quad \boldsymbol{u} \in T_x(\mathbb{R}^3)$$

$$R(\boldsymbol{A}) \in T_x(\mathbb{R}^3) \otimes T_x(\mathbb{R}^3), R(\boldsymbol{A})_{ij} = \sum_{m,n}^{(3,3)} R_{im} R_{jn} A_{mn} \quad \text{for} \quad \boldsymbol{A} \in T_x(\mathbb{R}^3) \otimes T_x(\mathbb{R}^3)$$

$$R(\boldsymbol{B}) \in T_x(\mathbb{R}^3) \otimes T_x(\mathbb{R}^3) \otimes T_x(\mathbb{R}^3), R(\boldsymbol{B})_{ijk} = \sum_{l,m,n}^{(3,3,3)} R_{il} R_{jm} R_{kn} A_{lmn},$$
$$\text{for} \quad \boldsymbol{B} \in T_x(\mathbb{R}^3) \otimes T_x(\mathbb{R}^3) \otimes T_x(\mathbb{R}^3)$$

Rotation action on tensors of arbitrary orders can be defined analogously. Consider the tensor tuple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \in \mathbb{R}^{3^{n_A}+3^{n_B}+3^{n_C}}$. Rotation of such tensor tuple can be defined as follow:

$$R(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) = (R(\boldsymbol{A}), R(\boldsymbol{B}), R(\boldsymbol{C}))$$

Define equivalence class of tensor tuple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ under rotation as follow:

$$[(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})] := \{(R(\boldsymbol{A}), R(\boldsymbol{B}), R(\boldsymbol{C})) | R \in SO(3)\}$$
$$[(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})] \in \mathbb{R}^{3^{n_A}+3^{n_B}+3^{n_C}}/SO(3)$$

We hope to establish the following mapping:

$$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \xrightarrow{\pi(\cdot)} [(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})] \xrightarrow{l(\cdot)} (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})^s$$

in which $\pi((\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}))$ is the canonical projection of quotient relation. $l([(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})])$ is (horizontal) lift of quotient manifold. $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})^s \in \mathbb{R}^{3^{n_A}+3^{n_B}+3^{n_C}}$ is invariant (tensor tuple in standard framework).

2 Algorithm

For tensor tuple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \in \mathbb{R}^{3^{n_A}+3^{n_B}+3^{n_C}}$, conduct singular value decomposition on each tensor:

$$\boldsymbol{A} = \boldsymbol{U}_1^A \boldsymbol{U}_2^A \dots \boldsymbol{U}_{n_A}^A \Sigma_A (\boldsymbol{V}_1^A)^T (\boldsymbol{V}_2^A)^T \dots (\boldsymbol{V}_{n_A}^A)^T$$

$$\boldsymbol{B} = \boldsymbol{U}_1^B \boldsymbol{U}_2^B \dots \boldsymbol{U}_{n_B}^B \Sigma_B (\boldsymbol{V}_1^B)^T (\boldsymbol{V}_2^B)^T \dots (\boldsymbol{V}_{n_B}^B)^T$$

$$\boldsymbol{C} = \boldsymbol{U}_1^C \boldsymbol{U}_2^C \dots \boldsymbol{U}_{n_C}^C \Sigma_C (\boldsymbol{V}_1^C)^T (\boldsymbol{V}_2^C)^T \dots (\boldsymbol{V}_{n_C}^C)^T$$

From all matrices $\boldsymbol{U}_1^A, \boldsymbol{U}_1^B, \boldsymbol{U}_1^C, \dots, \boldsymbol{U}_{n_A}^A, \boldsymbol{U}_{n_B}^B, \boldsymbol{U}_{n_C}^C$, arbitrarily choose three independent column vectors and form a position matrix $[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$. Compute the standard position matrix $[\boldsymbol{e}_1^s, \boldsymbol{e}_2^s, \boldsymbol{e}_3^s]$ as follow:

$$\boldsymbol{e}_1^s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{e}_2^s = \begin{pmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 \\ 1 - \sqrt{1 - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2} \\ 0 \end{pmatrix}$$

$$\boldsymbol{e}_3^s = \begin{pmatrix} e_{3,1}^s \\ e_{3,2}^s \\ e_{3,3}^s \end{pmatrix}$$

$$e_{3,1}^s = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3$$

$$e_{3,2}^s = \frac{\boldsymbol{e}_2 \cdot \boldsymbol{e}_3 - \boldsymbol{e}_{3,1}^s (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)}{\sqrt{1 - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2}}$$

$$e_{3,3}^s = \sqrt{1 - (\boldsymbol{e}_{3,1}^s)^2 - (\boldsymbol{e}_{3,2}^s)^2}$$

Compute the rotation matrix $\boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})}$ as follow:

$$\boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})} [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = [\boldsymbol{e}_1^s, \boldsymbol{e}_2^s, \boldsymbol{e}_3^s]$$

$$\boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})} = [\boldsymbol{e}_1^s, \boldsymbol{e}_2^s, \boldsymbol{e}_3^s][\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]^{-1}$$

A set of invariant $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})^s$ then can be computed as follow:

$$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})^s = \boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})}((\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})) = (\boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})}(\boldsymbol{A}), \boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})}(\boldsymbol{B}), \boldsymbol{R}_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})}(\boldsymbol{C}))$$

3 Proposition that need to be proven

Problem Prove that the map $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \to (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})^s$ constructed in the Algorithm part is a realization of $\pi(l(\cdot))$ in the first part.

(Personally I think at least two points need to be proven. First, for $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_1$ and $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_2$ equivalent under $SO(3)$ action, the algorithm should give the same result. Second, for $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_1$ and $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_2$ not in the same equivalence class defined above, the algorithm should give different results.)