§2. Presentations of Groups and Monodromy Relations

Let us discuss first some elementary group-theoretic notions. For any set S, we denote by $F(S)$ the *free group* generated by S, which is unique up to isomorphism, see [La], p. 34. When we say that a group G has a *presentation*

$$G = \langle (x_i) \rangle_{i \in I} : (r_j)_{j \in J} \rangle$$

with generators x_i *and relations* r_j, we mean the following.

Let X be the set consisting of the elements x_i, $i \in I$. The relations r_j are elements in the free group $F(X)$, i.e., they are "words" in the letters x_i and x_i^{-1}. Let R be the normal subgroup in $F(X)$ generated by all these elements $r_j, j \in J$. Then the quotient group $F(X)/R$ is isomorphic to the given group G, see [La], pp. 36–37. Usually the relations r_j are written in the more intuitive form

$$r_j = 1, \qquad j \in J,$$

since these equalities hold in $F(X)/R = G$.

(2.1) **Examples.** (i) In the notation above, the *free group* $F(X)$ is generated by the elements x_i, $i \in I$, with no relations, i.e., $J = \varnothing$.

(ii) The *finite cyclic group* $\mathbb{Z}/n\mathbb{Z}$ of order n has the following presentation

$$\mathbb{Z}/n\mathbb{Z} = \langle x \colon x^n = 1 \rangle.$$

(iii) The binary k-dihedral group $\tilde{D}_k$ is an important type of finite subgroup in $\mathrm{SL}(2, \mathbb{C})$ and has the following presentation, see [L3], p. 51,

$$\tilde{D}_k = \langle a, b \colon a^2 = b^k = (ab)^2 \rangle.$$

$\tilde{D}_k$ is a finite group of order $4k$ and satisfies

$$[\tilde{D}_k, \tilde{D}_k] = \mathbb{Z}/k\mathbb{Z},$$

$$\tilde{D}_k/[\tilde{D}_k, \tilde{D}_k] = \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{for } k \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{for } k \text{ even}, \end{cases}$$

see [L3], pp. 53 and 64.

(iv) Suppose, given two groups,

$$G^k = \langle (x_i^k)_{i \in I^k}; (r_j^k)_{j \in J^k} \rangle \qquad \text{where} \quad k = 1, 2.$$

Then the *free product* group $G^1 * G^2$ is the product of these groups in the category of groups (and not the product in the category of abelian groups, which is denoted by $G^1 \times G^2$ and exists only when both G^1 and G^2 are abelian).

In terms of presentation, we have

$$G^1 * G^2 = \langle (x_i^1)_{i \in I^1}, (x_i^2)_{i \in I^2} : (r_j^1)_{j \in J^1}, (r_j^2)_{j \in J^2} \rangle,$$

i.e., we put together all the generators and all the relations.

For instance, the group

$$(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) = \langle a, b \colon a^2 = b^3 = 1\rangle$$

is known to be isomorphic to the group

$$\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/(\pm I),$$

e.g., we can identify a with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and b with $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, see [CM], p. 85. In particular, it is clear from this isomorphism that the group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ is an infinite noncommutative group.

We present now a *"topological" construction* for the group

$$(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z}) \qquad \text{for} \quad (p, q) = 1.$$

First, some general constructions. Let S be a connected, locally contractible topological space and let $N \colon S \to S$ be a homeomorphism of finite order d. Let $p \colon \tilde{S} \to S$ be the universal covering space of S. Then the group $G = \pi_1(S)$ acts on $\tilde{S}$ as the group of covering transformations. Let $\tilde{N} \colon \tilde{S} \to \tilde{S}$ be a lifting of the homeomorphism N and let $\tilde{G}$ be the subgroup in the group of all homeomorphisms of $\tilde{S}$ generated by G and $\tilde{N}$. Since two liftings of N differ by an element in G, it is clear that this group $\tilde{G}$ is well defined, i.e., depends only on S and N.

For this reason, we use the notation $\tilde{G}(S, N)$ when we want to be more accurate. It is also clear that G is a normal subgroup in $\tilde{G}$ and that $\tilde{G}/G \simeq \mathbb{Z}/d\mathbb{Z}$. In other words, $\tilde{G}$ is an extension of the group $G = \pi_1(S)$ by the group $\mathbb{Z}/d\mathbb{Z}$, i.e,

$$0 \to \pi_1(S) \to \tilde{G} \to \mathbb{Z}/d\mathbb{Z} \to 0. \tag{2.2}$$

Some simple properties of the action of the group $\tilde{G}$ on the space $\tilde{S}$ are contained in the following.

(2.3) **Exercise.** Show that:

(i) $\tilde{S}/\tilde{G} = S/\langle N\rangle$ where $\langle N\rangle$ is the finite cyclic group of order d generated by the homeomorphism N.

(ii) Let $\tilde{s} \in \tilde{S}$ and $s = p(\tilde{s}) \in S$ and let $\tilde{G}_{\tilde{s}}$ and $\langle N\rangle_s$ be the corresponding isotropy groups. Then

$$\tilde{G}_{\tilde{s}} = \langle N\rangle_s.$$

(iii) If the group $\langle N\rangle$ acts freely on S, then the group $\tilde{G}$ acts freely on $\tilde{S}$ and

$$\pi_1(S/\langle N\rangle) = \tilde{G}.$$

(2.4) **Example.** Let $S = F$ be the Milnor fiber associated to the hypersurface V in $\mathbb{P}^n$ and let $N = h$, the monodromy homeomorphism. Using (2.3(iii)) we deduce

$$\pi_1(U) = \tilde{G}(F, h).$$

In fact, the extension (2.2) in this case is nothing other than the extension (1.9). The next example is quite different, since the actions involved are no longer free.

(2.5) **Example.** Let $F_a = \{\lambda \in \mathbb{C}; \lambda^a = 1\}$ and $h_a: F_a \to F_a$

$$h_a(x) = \exp\left(\frac{2\pi i}{a}\right)\cdot x$$

be the Milnor fiber and the monodromy homeomorphism of the singularity x^a for some integer $a > 1$. Take two positive integers p, q such that $(p, q) = 1$ and consider the join space

$$S = F_p * F_q \quad \text{and the join map} \quad N = h_p * h_q.$$

It is clear that the quotient $S/\langle N\rangle$ can be identified with a segment $[\alpha, \beta]$, with $\alpha \in F_p$, $\beta \in F_q$, i.e., the segment $[\alpha, \beta]$ is part of the join S.

The universal covering space $\tilde{S}$ in this case can be regarded as a tree, since it is a 1-complex which is simply-connected. Let $[\tilde{\alpha}, \tilde{\beta}]$ be a segment in this tree $\tilde{S}$, which is a lift of the segment $[\alpha, \beta]$. Using (2.3) (i) and (ii), we obtain:

(i) The segment $[\tilde{\alpha}, \tilde{\beta}]$ is a fundamental domain of $\tilde{S}$ mod $\tilde{G}$, see [Se], p. 48.

(ii) $\tilde{G}_{\tilde{\alpha}} = \langle N\rangle_\alpha = \mathbb{Z}/q\mathbb{Z}$, $\tilde{G}_{\tilde{\beta}} = \langle N\rangle_\beta = \mathbb{Z}/p\mathbb{Z}$.

(iii) The segment $[\tilde{\alpha}, \tilde{\beta}]$ is invariant only by the identity element in $\tilde{G}$.

Using now a basic result on groups acting on graphs, see [Se], p. 48, we get an isomorphism

$$\tilde{G}(F_p * F_q, h_p * h_q) = (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z}).$$

We now discuss briefly two important examples of *braid groups*. Let M be a connected manifold and for $n \geq 2$ consider the configuration space

$$M^{(n)} = \{(x_1, \ldots, x_n) \in M^n; x_i \neq x_j \text{ for } i \neq j\}.$$

The full symmetric group Σ_n on n letters acts on $M^{(n)}$ in an obvious way and let

$$\tilde{M}^{(n)} = M^{(n)}/\Sigma_n$$

be the corresponding quotient.

(2.6) **Definition.** The fundamental group $\pi_1(M^{(n)})$ (resp. $\pi_1(\tilde{M}^{(n)})$) is called the *pure braid group* (resp. the *full braid group*) on n strings of the manifold M.

(2.7) **Example** ($M = \mathbb{C}$). Consider the canonical projection

$$\sigma = (\sigma_1, \ldots, \sigma_n): \mathbb{C}^n \to \mathbb{C}^n/\Sigma_n = \mathbb{C}^n,$$

where σ_k is the kth symmetric function in $x_1, \ldots, x_n$. We can identify the points in the base space $\mathbb{C}^n/\Sigma_n$ with monic polynomials

$$p = x^n + a_1 x^{n-1} + \cdots + a_n.$$

Then $\tilde{\mathbb{C}}^{(n)}$ can be identified with the space of all such polynomials p having no multiple roots, i.e.,

$$\tilde{\mathbb{C}}^{(n)} = \mathbb{C}^n \setminus \Delta,$$

where Δ is the *discriminant hypersurface.* The group

$$B_n(\mathbb{C}) = \pi_1(\tilde{\mathbb{C}}^{(n)})$$

is called the *classical braid group of Artin* on n strings. Our discussion above implies

$$B_n(\mathbb{C}) = \pi_1(\mathbb{C}^n \setminus \Delta). \tag{2.8}$$

From a purely algebraic point of view, we have the following presentation for the group $B_n(\mathbb{C})$, see [Bi], p. 18.

(2.9) **Theorem** (Artin). *The group $B_n(\mathbb{C})$ admits a presentation with generators $g_1, \ldots, g_{n-1}$ and defining relations*

$$g_i g_j = g_j g_i \quad \text{for} \quad |i-j| \geq 2, \quad 1 \leq i, j \leq n-1,$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2.$$

(2.10) **Example** ($M = \mathbb{P}^1$). The set of unordered n-points in $\mathbb{P}^1$ corresponds to the set of homogeneous polynomials

$$\tilde{p} = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

of degree n in two variables x, y (modulo multiplicative nonzero constants). These polynomials $\tilde{p}$ form a projective space $\mathbb{P}^n$ and we have a *projective discriminant hypersurface* $\tilde{\Delta} \subset \mathbb{P}^n$, consisting of those polynomials $\tilde{p}$ with multiple roots. Hence the full braid group $B_n(\mathbb{P}^1)$ of the projective line $\mathbb{P}^1 \simeq S^2$ is given by

$$B_n(\mathbb{P}^1) = \pi_1(\mathbb{P}^n \setminus \tilde{\Delta}). \tag{2.11}$$

We have the following analog of (2.9).

(2.12) **Theorem.** *The braid group $B_n(\mathbb{P}^1)$ admits a presentation with generators $g_1, \ldots, g_{n-1}$ and defining relations*

$$g_i g_j = g_j g_i \quad \text{for} \quad |i-j| \geq 2, \quad 1 \leq i, j \leq n-1$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2,$$

$$g_1 \cdots g_{n-2} g_{n-1}^2 g_{n-2} \cdots g_1 = 1.$$

As a trivial case, consider the case $n = 2$. Then the discriminant $\tilde{\Delta}$ has the equation

$$a_1^2 - 4a_0 a_2 = 0,$$

i.e., $\tilde{\Delta}$ is a smooth conic on $\mathbb{P}^2$. By (1.3) and (1.13) we get

$$\pi_1(\mathbb{P}^2 \setminus \tilde{\Delta}) = \mathbb{Z}/2\mathbb{Z}.$$

This clearly agrees with the above presentation for $B_n(\mathbb{P}^1)$.

(2.13) **Remark.** The complement $\mathbb{C}^n \setminus \Delta$ offers a nice example of a $K(\pi, 1)$-space, see [Sp], p. 424, and [B4]. Indeed, we have

$$\pi_i(\mathbb{C}^n \setminus \Delta) = \pi_i \tilde{\mathbb{C}}^{(n)}) = \pi_i(\mathbb{C}^{(n)})$$

for $i > 1$.

The natural projection

$$\mathbb{C}^{(n)} \to \mathbb{C}^{(n-1)}, \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}),$$

is a fibration with typical fiber

$$F = \mathbb{C} \setminus \{(n-1)\text{-points}\}.$$

Since $\pi_i(F) = 0$ for $i > 1$, it follows by induction on n that

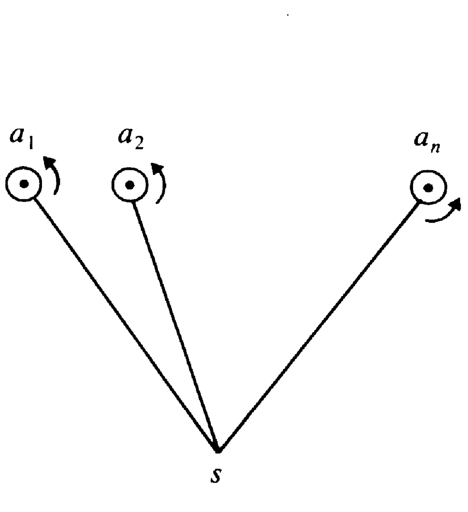
$$\pi_i(\mathbb{C}^{(n)}) = 0 \qquad \text{for} \quad i > 1,$$

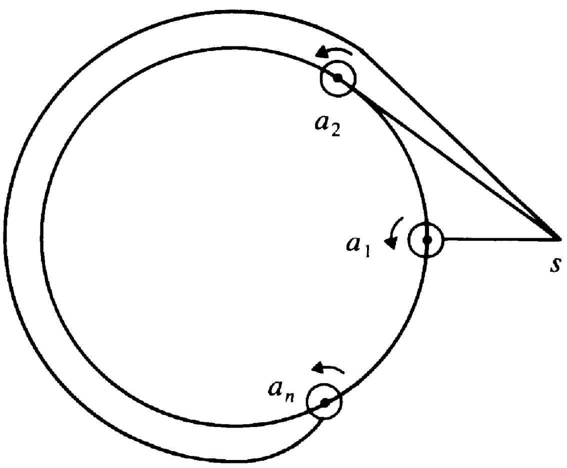
i.e., $\mathbb{C}^n \setminus \Delta$ is indeed a $K(\pi, 1)$-space.

(2.14) **Exercise.** Is $\mathbb{P}^n \setminus \tilde{\Delta}$ a $K(\pi, 1)$-space? *Hint.* Use the fact that $\pi_2(\mathbb{P}^1) = \pi_2(S^2) = \mathbb{Z}$.

(2.15) **Exercise.** Let $A = \{a_1, \ldots, a_n\}$ be a set consisting of n distinct points in $\mathbb{C}$. Show that $\pi_1(\mathbb{C} \setminus A)$ is a free group on n generators. *Hint.* Consider a system of n loops $\gamma_1, \ldots, \gamma_n$ (as defined, for instance, in (3.3.8)) going once around each of the points a_i.

Two possible illustrations are the following, in which the loops are numbered using the convention introduced after (3.3.3):





In the first illustration the points a_i are all situated on a line, while in the second picture they are situated on a circle. Note that in both cases the composition

$$\gamma_n\gamma_{n-1}\cdots\gamma_1$$

is a loop γ going once anticlockwise around all the points a_i. Show that

$$\pi_1(\mathbb{C}\setminus A) = F(\gamma_1, \ldots, \gamma_n).$$

We call such a set $\gamma_1, \ldots, \gamma_n$ of loops in $\pi_1(\mathbb{C}\setminus A)$ a *set of generating loops*.

(2.16) **Exercise.** Let $A = \{a_1, \ldots, a_n\}$ be a set consisting of n distinct points in $\mathbb{P}^1$. Show that $\pi_1(\mathbb{P}^1\setminus A)$ is a free group on $(n-1)$ generators. *Hint.* There are two possible ways to solve this exercise:

(i) Note that $\mathbb{P}^1\setminus\{a_n\} = \mathbb{C}$.

(ii) Suppose that all the points a_i are in the "finite part" $\mathbb{C}$ of the projective line $\mathbb{P}^1$. Let $\gamma_1, \ldots, \gamma_n$ be a set of generating loops in $\pi_1(\mathbb{C}\setminus A)$ as in (2.15). Then the loop $\gamma = \gamma_n\gamma_{n-1}\cdots\gamma_1$ is trivial in $\pi_1(\mathbb{P}\setminus A)$. Hence $\pi_1(\mathbb{P}\setminus A)$ has a presentation

$$\langle \gamma_1, \ldots, \gamma_n : \gamma_n\gamma_{n-1}\cdots\gamma_1 = 1\rangle$$

from which it is clear that $\pi_1(\mathbb{P}\setminus A) = F(\gamma_1, \ldots, \gamma_{n-1})$.

We recall now a very useful tool for doing computations of fundamental groups (for a proof, see, for instance, [CF]).

(2.17) **Theorem** (van Kampen). *Let X be a topological space with an open covering $X = X_1 \cup X_2$ such that $X_0 = X_1 \cap X_2$, X_1, and X_2 are all nonempty path-connected spaces. Suppose there are given presentations*

$$\pi_1(X_1) = \langle (x_i)_{i\in I} : (r_j)_{j\in J}\rangle,$$

$$\pi_1(X_2) = \langle (y_k)_{k\in K} : (s_l)_{l\in L}\rangle,$$

$$\pi_1(X_0) = \langle (z_m)_{m\in M} : (t_n)_{n\in N}\rangle,$$

and let

$$(i_a)_\# : \pi_1(X_0) \to \pi_1(X_a)$$

be the morphisms induced by the inclusions for $a = 1, 2$. Then the fundamental group of X has the following presentation:

$$\pi_1(X) = \langle (x_i)_{i\in I}, (y_k)_{k\in K} : (r_j)_{j\in J}, (s_l)_{l\in L}, (u_m)_{m\in M}\rangle,$$

where

$$u_m\colon (i_1)_\#(z_m) = (i_2)_\#(z_m) \qquad \textit{for} \quad m \in M.$$

(2.18) **Remark.** This theorem also holds when X_1 and X_2 are closed sets and satisfy a long list of additional conditions, see [CF], p. 65, and [Om]. Unfortunately, it is precisely this sophisticated version of the van Kampen theorem which we need in the sequel.

Before going further, we introduce an important class of groups, following Oka [O5]. Let p, q be two positive integers and consider the group

(2.19) $$G(p, q) = \langle \beta, (a_i)_{i \in \mathbb{Z}}: \beta = a_{p-1} \cdots a_0, R_1, R_2 \rangle$$

where

$$R_1: a_i = a_{i+q} \quad \text{for any} \quad i \in \mathbb{Z},$$

$$R_2: a_{i+p} = \beta a_i \beta^{-1} \quad \text{for any} \quad i \in \mathbb{Z}.$$

Note that this presentation has infinitely many generators and relations, but it is clear that the group $G(p, q)$ is finitely generated (e.g., by the elements $a_0, \dots, a_{p-1}$).

(2.20) **Exercise.** Show that when $(p, q) = 1$, the group $G(p, q)$ has the following simpler presentation:

$$G(p, q) = \langle \alpha, \beta: \alpha^p = \beta^q \rangle.$$

Hint. Prove the following claims:

(a) $\beta = a_j \cdots a_{j-p+1}$ for any $j \in \mathbb{Z}$;
(b) $\beta^q = a_{pq-1} a_{pq-2} \cdots a_1 a_0 = \alpha^p$ where $\alpha = a_{q-1} \cdots a_0$;
(c) $a_r = \beta^m a_0 \beta^{-m}$ if $r = mp + nq$;
(d) $a_0 = \beta^l \alpha^k$ if $1 = lp + kq$.

The topological significance of the abstract group $G(p, q)$ is explained by the following result.

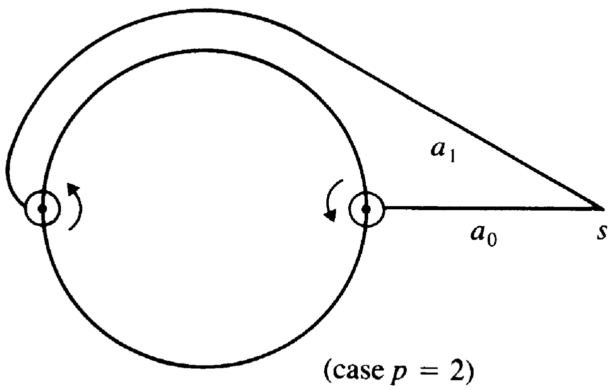
(2.21) **Proposition** (Oka [O5]). *Consider the affine plane curve*

$$C: x^p - y^q = 0$$

for some positive integers $p, q \geq 1$. Then

$$\pi_1(\mathbb{C}^2 \setminus C) = G(p, q).$$

Proof. Let $\varphi: \mathbb{C}^2 \setminus C \to \mathbb{C}$, $(x, y) \mapsto y$ be the second projection. Then φ is a locally trivial fibration over $\mathbb{C}^*$ with fiber $F = \varphi^{-1}(1)$. We consider the fibers $\varphi^{-1}(t)$ as subsets in $\mathbb{C}$ by projection onto the x-coordinate. Take generators $a_0, \dots, a_{p-1}$ of $\pi_1(F)$ as shown in the figure below.



(case $p = 2$)

If D is the closed disc $\{y \in \mathbb{C};\ |y| \leq 1\}$, then $\varphi^{-1}(D)$ is a deformation retract of the complement $\mathbb{C}^2 \setminus C$. Let D^+ and D^- be the upper and lower closed half-discs in D, respectively. Note that the map $y \mapsto |y|$ induces deformation retracts of the half-discs D^+ and D^- onto the segment $[0, 1]$. These deformation retracts can be lifted to produce deformation retracts r^+ and r^- of the sets $\varphi^{-1}(D^+)$ and $\varphi^{-1}(D^-)$, respectively, onto $\varphi^{-1}([0, 1])$. Note also that $\varphi^{-1}([0, 1])$ can be deformed onto $F = \varphi^{-1}(1)$. We apply van Kampen's theorem (2.17) twice (in fact, the version of it involving a closed covering as in (2.18)).

First consider the closed covering

$$\varphi^{-1}([-1, 1]) = \varphi^{-1}([-1, 0]) \cup \varphi^{-1}([0, 1]).$$

Let $F^- = \varphi^{-1}(-1)$ and note that $\varphi^{-1}(0) = \mathbb{C}^*$. This covering leads to the following commutative diagram

$$\begin{tikzcd}
\pi_1(F) \arrow[r, "\sim"'] & \pi_1(\varphi^{-1}([0, 1])) \arrow[r] & \pi_1(\varphi^{-1}([-1, 1])) & \\
\pi_1(\mathbb{C}^*) \arrow[ur, "j'_{\#}"] \arrow[r, "j''_{\#}"] & \pi_1(\varphi^{-1}([-1, 0])) \arrow[ur] & \pi_1(F^-). \arrow[l, "\sim"']
\end{tikzcd}$$

Here $\pi_1(F) = F(a_0, \ldots, a_{p-1})$ as in (2.15) and $\pi_1(F^-) = F(b_0, \ldots, b_{p-1})$ for a set of generating loops $b_0, \ldots, b_{p-1}$ in F^- chosen similarly to the loops $a_0, \ldots, a_{p-1}$ in F.

The generator σ of $\pi_1(\mathbb{C}^*) = \mathbb{Z}$ can be taken to be a large circle in $\mathbb{C}^*$ going anticlockwise and hence

$$j'_{\#}(\sigma) = a_{p-1} \cdots a_0,$$
$$j''_{\#}(\sigma) = b_{p-1} \cdots b_0.$$

Hence, by van Kampen's theorem (2.17) we have

$$\pi_1(\varphi^{-1}([-1, 1])) = \langle a_0, \ldots, a_{p-1}, b_0, \ldots, b_{p-1} : a_{p-1} \cdots a_0 = b_{p-1} \cdots b_0 \rangle.$$

Now apply van Kampen's theorem to the closed covering

$$\varphi^{-1}(D) = \varphi^{-1}(D^+) \cup \varphi^{-1}(D^-).$$

To do this, we consider the following commutative diagram:

$$\begin{tikzcd}
\pi_1(F) \arrow[r, "\sim"] & \pi_1(\varphi^{-1}(D^+)) \arrow[r] & \pi_1(\varphi^{-1}(D)) & \\
\pi_1(\varphi^{-1}([-1, 1])) \arrow[ur, "k'_{\#}"] \arrow[r, "k''_{\#}"] & \pi_1(\varphi^{-1}(D^-)) \arrow[ur] & \pi_1(F) \arrow[l, "\sim"']
\end{tikzcd}$$

Using the notation introduced above, we have

$$k'_{\#}(a_i) = a_i, \qquad k'_{\#}(b_i) = (r^+)_{\#}(b_i) \quad \text{and} \quad k''_{\#}(a_i) = a_i, \qquad k''_{\#}(b_i) = (r^-)^{\#}(b_i).$$

On the other hand, the monodromy homeomorphism $h\colon F \to F$ of the fibration induced by φ over $\mathbb{C}^*$ corresponds clearly to the composition

$$F \xrightarrow{(r^+)^{-1}} F_- \xrightarrow{r^-} F.$$

Since $r^+: F_- \to F$ is a homeomorphism, it follows that the elements $(r^+)_\#(b_i)$ for $i = 0, \ldots, p-1$ generate the group $\pi_1(F)$. Using van Kampen's theorem (2.17) it follows that

$$\pi_1(\varphi^{-1}(D)) = \langle a_0, \ldots, a_{p-1}: h_\#(a_i) = a_i \rangle.$$

The relations $h_\#(a_i) = a_i$ are called the *monodromy relations* associated to the projection φ of the pair $(\mathbb{C}^2, C)$ at the origin on $\mathbb{C}$. In our case, it is easy to see that

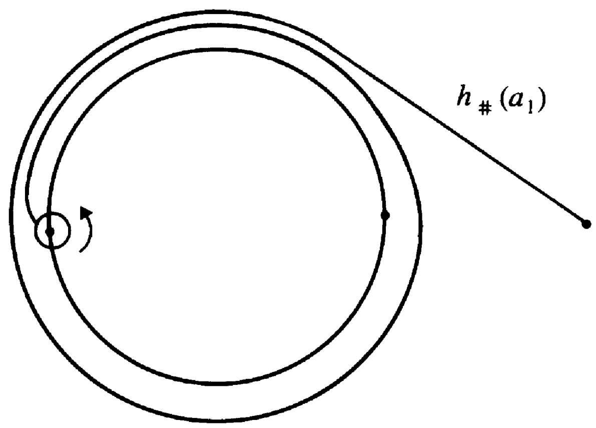
$$h_\#(a_j) = \begin{cases} \beta^m a_{r+j} \beta^{-m} & \text{for } j = 0, \ldots, p-r-1, \\ \beta^{m+1} a_{p-r+j} \beta^{-m-1} & \text{for } j = p-r, \ldots, p-1, \end{cases}$$

where the integers m and r are defined by the equation $q = mp + r$, $0 \le r < p$, and

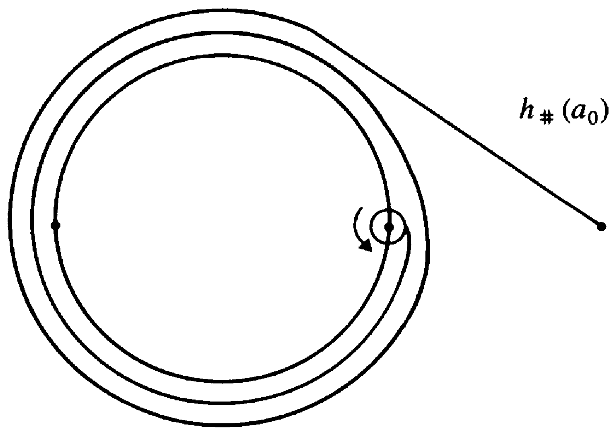
$$\beta = a_{p-1} \cdots a_0.$$

As an example we draw the loops $h_\#(a_j)$ in the case $p = 2$, $q = 3$.

(Note that the monodromy homeomorphism $h: F \to F$ is the exactly rotation with angle $2\pi q/p$.)



Hence $h_\#(a_0) = a_1 a_0 a_1 (a_1 a_0)^{-1}$.



Hence $h_\#(a_1) = (a_1 a_0)^2 a_0 (a_1 a_0)^{-2}$.

To end the proof of (2.21), we can add new generators and new relations to the above presentation of the group

$$\pi_1(\mathbb{C}^2, C) = \pi_1(\varphi^{-1}(D)),$$

namely,

$$a_{kp+j} = \beta^k a_j \beta^{-k} \qquad \text{for} \quad k \in \mathbb{Z}$$

and $0 \leq j < p$. Then the monodromy relation $h_{\#}(a_j) = a_j$ becomes

$$a_j = a_{j+q}$$

and this clearly ends the proof. □

(2.22) **Special Cases.**

(i) $(C, 0)$ is a node A_1, i.e., $p = q = 2$. Then

$$\pi_1(\mathbb{C}^2 \backslash C) = \langle a_0, a_1 \colon a_0 a_1 = a_1 a_0 \rangle = \mathbb{Z}^2.$$

(ii) $(C, 0)$ is a cusp A_2, i.e., $p = 2$, $q = 3$ (recall the illustration in the proof above). Then

$$\pi_1(\mathbb{C}^2 \backslash C) = \langle a_0, a_1 \colon a_0 a_1 a_0 = a_1 a_0 a_1 \rangle.$$

This group is thus isomorphic to the braid group $B_2(\mathbb{C})$ or to the group

$$\pi_{2,3} = \langle \alpha, \beta \colon \alpha^2 = \beta^3 \rangle$$

of the trefoil knot.

In fact, for any pair (p, q) such that $(p, q) = 1$, using (2.20) we get

$$\pi_1(\mathbb{C}^2 \backslash C) = \langle \alpha, \beta \colon \alpha^p = \beta^q \rangle,$$

which is the same as the fundamental group of the torus knot of type (p, q) discussed in (2.1.6).

(iii) $(C, 0)$ is smooth and the line $y = 0$ is an inflectional tangent of order p, i.e., $q = 1$. Then

$$\pi_1(\mathbb{C}^2 \backslash C) = \langle a_0, \ldots, a_{p-1} \colon a_0 = a_1 = \cdots = a_{p-1} \rangle \simeq \mathbb{Z}.$$

In all the above examples, the relations have been obtained by writing explicitly the corresponding monodromy relations $h_{\#}(a_j) = a_j$.

(2.23) **Remark.** Exactly the same computations as in (2.21) and (2.22) work in the local case, i.e., for a plane curve singularity $(C, 0)$ such that the line $\varphi^{-1}(0)$ is not contained in C. Of course, the monodromy relations in such a case might be much more difficult to work out. (The germ of the projection $\varphi \colon (\mathbb{C}^2 \backslash C, 0) \to (\mathbb{C}, 0)$ induces fibration over a smaller punctured disc at the origin of $\mathbb{C}$ and it is the monodromy of this local fibration which is meant here.)

§3. The van Kampen–Zariski Theorem

In this section we discuss a general method for finding a presentation of the fundamental group $G = \pi_1(\mathbb{P}^2 \setminus C)$ of a given (reduced) plane curve C.

First we consider the easiest part, namely, finding a set of generators for this group G. A special case of the Zariski theorem of Lefschetz type (1.6.5) is the following.

(3.1) **Proposition.** *For any hypersurface $V \subset \mathbb{P}^n$ and any line L in $\mathbb{P}^n$ intersecting V transversally and avoiding the singular part $S(V)$, there is an epimorphism*

$$\pi_1(L \setminus (V \cap L)) \to \pi_1(\mathbb{P}^n \setminus V)$$

induced by the inclusion.

Note that for such a line L, the intersection $V \cap L$ consists exactly of $d = \deg(V)$ points and hence by (2.16) the group $\pi_1(L \setminus (V \cap L))$ is a free group on $(d-1)$-generators.

To following result describes the *behavior of the fundamental group with respect to degenerations* of curves.

(3.2) **Corollary.** *Let C_t, $t \in [0, \varepsilon]$, be a smooth family of plane curves in $\mathbb{P}^2$ such that:*

(i) *the family C_t for $t \in (0, \varepsilon]$ is equisingular;*
(ii) *the limit curve $C_0 = \lim_{t \to 0} C_t$ is a reduced curve.*

Then there is an epimorphism

$$\pi_1(\mathbb{P}^2 \setminus C_0) \to \pi_1(\mathbb{P}^2 \setminus C_\varepsilon).$$

Proof. We recall that a familly C_t, $t \in (0, \varepsilon]$, of plane curves is *equisingular* if the singular points of the curve C_t can be indexed as $a_1(t), \ldots, a_p(t)$ in such a way that all the families of singularities $(C_t, a_i(t))$ are μ-constant. Then, by our discussion in Chapter 1, §3, it follows that the topological type of the pair $(\mathbb{P}^2, C_t)$ is independent of t for $t \in (0, \varepsilon]$.

Consider now a "tubular neighborhood" T of the curve C_0 in $\mathbb{P}^2$. In other words, T is a small open neighborhood of the curve C_0, which retracts to C_0, see [Df6] and our discussion in Chapter 5, §2, below.

Choose a t such that $\varepsilon \gg t > 0$ and such that the curve C_t is contained in the neighborhood T. Let L be a generic line with respect to both curves C_0 and C_t. Then the intersection $L \cap C_0$ (resp. $L \cap C_t$) consists of d points $p_1^0, \ldots, p_d^0$ (resp. $p_1^t, \ldots, p_p^t$) where $d = \deg C_t = \deg C_0$. We can arrange that the intersection $L \cap T$ consists of d disjoint small discs $D_1, \ldots, D_d$, each of them containing a pair of points p_i^0 and p_i^t for some $i = 1, \ldots, d$.