VANISHING THEOREMS FOR THE DE RHAM COMPLEX OF UNITARY LOCAL SYSTEM

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ABSTRACT. We will prove a Kodaira-Nakano type of vanishing theorem for the logarithmic de Rham complex of unitary local system. We will then study the weight filtration on the logarithmic de Rham complex, and prove a stronger statement for the associated graded complex.

INTRODUCTION

Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and $D \subset X$ a simple normal crossing divisor. In [2], Deligne constructed the canonical extension E for any local system $\mathcal{L}$ defined on U (over complex topology). E is equipped with a flat connection,

$$\nabla : E \to E \otimes \Omega_X(\log D)$$

and it is characterized by the following two properties

(1) The flat sections of ∇ coincide with $\mathcal{L}$ on U.
(2) The eigenvalues of the residue of ∇ lie in the strip

$$\{z \in \mathbb{C} | 0 \leq \text{Re}(z) < 1\}$$

Let V be a unitary local system on $U := X - D$ (over complex topology). Let (E, ∇) be the canonical extension of V. Write $\text{DR}_X(D, E)$ for the de Rham complex

$$E \xrightarrow{\nabla} E \otimes \Omega_X(\log D) \to \cdots \xrightarrow{\nabla} E \otimes \Omega_X^n(\log D)$$

First, we will prove a Kodaira-Nakano type of vanishing theorem

Theorem 1. *Let L be an ample line bundle on X, then*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$.

The de Rham complex $\text{DR}_X(D, E)$ comes with an increasing filtration $F^\cdot$ (Hodge filtration or "naive" filtration) and a decreasing filtration $W_\cdot$ (weight filtration). The weight filtration $W_\cdot$ will be defined in Section 2. $F^\cdot$ and $W_\cdot$ togethe will define a mixed Hodge structure on $\text{DR}_X(D, E)$.
Then, we will prove a more refined version of the above theorem

Theorem 2. *Let L be an ample line bundle on X, then*

$$H^q(X, Gr_\cdot^W(\Omega_X^p \otimes E) \otimes L) = 0$$

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1. Residue map

In this section we will define a residue map Res(E) on the complex DR(D, E, ∇). Similar to the usual residue map on the holomorphic de Rham complex, $\Omega_X^{\cdot}$, Res(E) will define a weight filtration on DR(D, E, ∇). Res(E) has been defined and studied in [8].

For $m = 1, \cdots n$, let D_m be the union of m-fold intersection of components of D; Let $\tilde{D}_m$ be the disjoint union of components of D_m; Let $v_m : \tilde{D}_m \to X$ be the composition of the projection map onto D_m and the inclusion map. $\tilde{C}_m := v_m^* D_{m+1}$ is either empty or a normal

Theorem 3. [8]

(1) *$V_m := j_* V|_{D_m - D_{m+1}}$ is a unitary local system on $D_m - D_{m+1}$.*
(2) *There exist a unique subvectorbundle E_m of $v_m^* E$ and a unique holomorphic integrable connection ∇_m on E_m with logarithmic poles along C_m such that*
$$\ker \nabla_m|_{\tilde{D}_m - \tilde{C}_m} = v_m^{-1} V_m$$
(3) *There exists a unique subvectorbundle E_m^* of $v_m^* E$ with*
$$E_m \oplus E_m^* = v_m^* E$$

Proof. All of the statements above are local. Therefore, we can assume X is a polydisk. Write $X = \Delta_1 \times \cdots \times \Delta_n$, and let z_i be the coordinate on Δ_i. Suppose D is defined by
$$z_1 \times \cdots \times z_s = 0$$

1. The local system V on U is equivalent to an unitary representation
$$T : \pi_1(U) \to \mathrm{GL}(r, \mathbb{C})$$
As $\pi_1(U)$ is abelian and T is unitary, we can simultaneously diagonalize all $T(\gamma_i$, where γ_i's form a generating set of $\pi_1(U)$ (see Appendix 1). Therefore, we can assume V is a direct sum of rank 1 unitary local systems. Write
$$V = V^1 \oplus \cdots \oplus V^r$$
For each V^i, let $\lambda_{i,j}$ be its monodromy around D^j. So V^i extends to D^j if and only if $\lambda_{i,j} = 1$.

Now let $D^{j1} \cap \cdots \cap D^{jm}$ be one component of D_m, and let $x \in D^{j1} \cap \cdots \cap D^{jm}$. Then, near x V_m is
$$\bigoplus_{\lambda_{i,j1} = \cdots = \lambda_{i,jm} = 1} V^i$$
This shows that V_m is a unitary local system.

2. The uniqueness of the subvectorbundle E_m follows from the uniqueness of canonical connection. Therefore, we only need to show the existence part. Use the notation from part 1, and assume V decomposes as direct sum of rank 1 unitary local system V^i. Let E^i be the canonical connection of V^i. Then, it is clear that
$$E_m = \bigoplus_{\lambda_{i,j1} = \cdots = \lambda_{i,jm} = 1} v_m^* E^i$$

3. E inheits a flat Hermitian form from V. Define E_m^* as the complement of E_m with respect to this metric. On Δ, E_m^* is the direct sum of $v_m^* E^i$ not appearing in the definition of E_m. □

Remark 1. *E_m could have different ranks on different component of $\tilde{D}_m$.*

For each $m \leq p \leq \dim D_m$, there exists a residue map crossing divisor in $\tilde{D}_m$.

$$\mathrm{Res}_m : \Omega_X^p(\log D) \to v_{m*}(\Omega_{\tilde{D}_m}^{p-m})$$

Res_m commutes with exterior derivative d, making it a homomorphism of complexes

$$\mathrm{Res}_m : \Omega_X^{\cdot}(\log D) \to v_{m*}\Omega_{\tilde{D}_m}^{\cdot}(\log \tilde{C}_m)[-m]$$

Consider the following variation of the residue map Res_m

$$\begin{aligned} \mathrm{Res}_m(E) : \Omega_X^p(\log D) \otimes E &\xrightarrow{\mathrm{Res}_m \otimes \mathrm{id}} v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m)) \otimes E \\ &= v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes v_m^* E) \\ &\xrightarrow{\mathrm{id} \otimes p_m} v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes E_m) \end{aligned}$$

where $p_m : v_m^* E \to \tilde{E}_m$ is the projection onto the E_m component.

Lemma 1. [8] *$Res_m(E) \circ \nabla = \nabla_m \circ Res_m(E)$,* i.e. *$Res_m(E)$ is homomorphism of complexs*

$$DR(D, E, \nabla) \to v_{m*}DR(\tilde{C}_m, E_m, \nabla_m)[-m]$$

2. Weight filtration on the de Rham Complex

The residue map

$$\mathrm{Res}_m(E) : \mathrm{DR}(D, E, \nabla) \to v_{m*}\mathrm{DR}(\tilde{C}_m, E_m, \nabla_m)[-m]$$

can be used to define a weight filtration $W_.$ on $\mathrm{DR}(D, E, \nabla)$ [8]

$$\begin{aligned} &W_m(\mathrm{DR}(D, E, \nabla)) = \ker \mathrm{Res}_{m+1}(E) && \text{if } m \geq 0 \\ &W_m(\mathrm{DR}(D, E, \nabla)) = 0 && \text{if } m < 0 \end{aligned}$$

Local descriptions of $W_m(\mathrm{DR}(D, E, \nabla))$ have been given in [8]. We will review them here:
Let $\Delta = \Delta_1 \times \cdots \times \Delta_n$ be a polydisk of X with coordinate $z_1, \cdots, z_n$. Suppose D is defined as

$$z_1 \times \cdots \times z_s = 0$$

As in part 1 of Theorem 3, we assume V is the direct sum of rank 1 unitary local systems on Δ, and write

$$V = V^1 \oplus \cdots \oplus V^r$$

Definition 1. *We say $\frac{dz_j}{z_j}$ acts on V^i by identity if $\lambda_{i,j} = 1$, i.e. the monodromy of V^i by a small circle around D_j is the identity.*

Let E^i be be canonical extension of V^i on Δ;
Let μ_i be a generator of E^i, then

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p} \otimes \mu_i$$

is in $W_m(\mathrm{DR}(D, E, \nabla))$ if and only if there are at most m log forms acting on V^i by identity.

Proposition 1. [8]

(1) *$W_.(DR(D, E, \nabla))$ is an increasing filtration.*
(2) *$Res_m(E)$ induces an isomorphism*

$$Gr_m^W(DR(D, \nabla, E)) \to v_{m*}(W_0(DR(\tilde{C}_m, \tilde{E}_m, \tilde{\nabla}_m))[-m])$$

Proof. The statements are local. We can assume X is a polydisk and V is a unitary local system of rank 1.

1. From the local description of $W_m(\mathrm{DR}(D, E, \nabla))$, it is clear that $W_.$ is an increasing filtration.

2. Let s be a section $W_m(\mathrm{DR}(D, E, \nabla))$. Use the local description above, s is of the form

$$\omega \otimes \mu$$

where

$$\omega = \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p}$$

and ω has at most m log 1-forms acting on V by identity. μ is a generating section of E.
First, we show $\mathrm{Res}_m(E)(s) \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m$.

$$\mathrm{Res}_m(E)(s) = \mathrm{Res}_m(\omega) \otimes \mu_m$$

By the construction of ω, $\mathrm{Res}_m(\omega)$ does not have log form $\frac{dz_j}{z_j}$ acting on V by identity. This shows that

$$\mathrm{Res}_m(E)(s) \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m)$$

If $\omega_0 \otimes \mu_m \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m)$, to get a preimage in $W_m(\Omega^p_X(\log D) \otimes E)$, simply take

$$\omega_m \wedge \omega_0 \otimes \mu$$

where ω_m is any m-form. And $\omega_m \wedge \omega_0 \otimes \mu \in W_m(\Omega^p_X(\log D) \otimes E)$ by the construction of ω_0. This shows that

$$\mathrm{Res}_m(E) : W_m(\mathrm{DR}(D, E, \nabla)) \to W_0(\mathrm{DR}(\tilde{C}_m, \tilde{E}_m, \tilde{\nabla}_m))$$

is surjective.
If $\mathrm{Res}_m(E)(s) = 0$, that means in ω, there are at most $m-1$ log forms acting on V by identity. This is precisely the local description of $W_{m-1}(\Omega^p_X(\log D) \otimes E)$. $\square$

3. Mixed Hodge Structure on the de Rham Complex

The framework for studying the mixed Hodge structure on $\mathrm{DR}_X(D,E)$ has been worked out by Deligne in [3] and [4]. The analysis of the mixed Hodge structure on $\mathrm{DR}_X(D,E)$ was given by Timmerscheidt in [8]. We will give an overview about the results from both authors. The vanishing theorem in the following section is a consequence of the mixed Hodge structure on $\mathrm{DR}_X(D,E)$.
Let A denote $\mathbb{Z},\mathbb{Q}$ or $\mathbb{R}$ and $A\otimes\mathbb{Q}$ the field $\mathbb{Q}$ or $\mathbb{R}$.
Assume V has a A-lattice throughout this section, i.e. there exists a unitary local system V_A defined over A such that

$$V = V_A \otimes_A \mathbb{C}$$

Let $D^+(A)$(resp. $D^+(\mathbb{C})$) denote the derived category of A-modules (resp. $\mathbb{C}$-vector spaces)
The main result of this section is

Theorem 4. [8]*(Proposition 6.4)*

$$(\mathbb{R}j_*V_A, (\mathbb{R}j_*V_{A\otimes\mathbb{Q}}, \tau), (DR_X(D,E), F, W))$$

is an A-cohomological mixed Hodge complex.

For readers' sake, we included all relvant definitions involved in the above theorem here. They can be found in [4] or [5](Section 5)
Define Hodge structure of weight n here

Definition 2. *(Hodge Complex (HC)) A Hodge A-complex K of weight n consists of*

1. *A complex K_A of A-modules, such that $H^k(K_A)$ is an A-module of finite type for all k;*
2. *A filtered complex $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces ;*
3. *Anisomorphism*
$$\alpha : K_A \otimes \mathbb{C} \to K_{\mathbb{C}}$$
in $D^+(\mathbb{C})$;

The following axioms must be satisfied

1. *The spectral sequence defined by $(K_{\mathbb{C}}, F)$ degenerates at E_1;*
2. *for all k, the filtration F on $H^k(K_{\mathbb{C}}) \cong H^k(K_A)\otimes\mathbb{C}$ defines a A-Hodge strucutre of weight $n+k$ on $H^k(K_A)$*

Definition 3. *Let X be a topological space. An A-Cohomological Hodge Complex (CHC) K of weight n on X, consists of:*

1. *A complex of sheaves K_A of A-modules on X;*
2. *A filtered complex of sheaves $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces on X;*
3. *an isomorphism*
$$\alpha : K_A \otimes \mathbb{C} \to K_{\mathbb{C}}$$
in $D^+(X,\mathbb{C})$

Moreover, the triple $(R\Gamma(K_A), R\Gamma(K_{\mathbb{C}}, F), R\Gamma(\alpha))$ is a Hodge Complex of weight n

Definition 4. *(Mixed Hodge Complex) An A-Mixed Hodge Complex (MHC) K consists of:*

1. *A complex K_A of A-modules such that $H^k(K_A)$ is an A-module of finite type for all k;*

(2) *A filtered complex* $(K_{A\otimes\mathbb{Q}}, W)$ *of* $A\otimes\mathbb{Q}$*- vector spaces with an increasing filtration* W*;*
(3) *An isomorphism* $K_A\otimes\mathbb{Q}\cong K_{A\otimes\mathbb{Q}}$ *in* $D^+(A\otimes\mathbb{Q})$*;*
(4) *A bi-filtered complex* $(K_{\mathbb{C}}, W, F)$ *of* $\mathbb{C}$*-vector spaces with an increasing (resp. descreasing) filtration* W *(resp.* F*) and an isomorphism:*

$$\alpha : (K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C}\cong(K_{\mathbb{C}}, W)$$

in $D^+F(\mathbb{C})$.

Moreover, the following axiom needs to be satisfied: For all n, the system consisting of

- *the complex* $Gr_n^W(K_{A\otimes\mathbb{Q}})$ *of* $A\otimes\mathbb{Q}$*- vector spaces,*
- *the complex* $Gr_n^W(K_{\mathbb{C}}, F)$ *of* $\mathbb{C}$*-vector spaces with induced* F *filtration,*
- *the isomorphism*

$$Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}})\otimes\mathbb{C}\to Gr_n^W(K_{\mathbb{C}})$$

is an $A\otimes\mathbb{Q}$*-Hodge Complex of weight* n*.*

Definition 5. *(Cohomological Mixed Hodge Complex (CMHC)) An* A*-Cohomological Mixed Hodge Complex* K *(CMHC) on a topological space* X *consists of:*

(1) *A complex of sheaves* K_A *of sheaves of* A*-modules on* X *such that* $H^k(X, K_A)$ *are* A*-modules of finite type;*
(2) *A filtered complex* $(K_{A\otimes\mathbb{Q}}, W)$ *of sheaves of* $A\otimes\mathbb{Q}$*-vector spaces on* X *with an increasing filtration* W *and an isomorphism*

$$K_A\otimes\mathbb{Q}\cong K_{A\otimes\mathbb{Q}}$$

in $D^+(X, A\otimes\mathbb{Q})$*;*
(3) *A bi-filtered complex of sheaves* $(K_{\mathbb{C}}, W, F)$ *of* $\mathbb{C}$*-vector spaces on* X *with an increasing (resp. descreasing) filtration* W *(resp.* F*) and an isomorphism:*

$$\alpha : (K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C}\to(K_{\mathbb{C}}, W)$$

in $D^+F(X,\mathbb{C})$.

Moreover, the following axiom needs to be satisfied: For all n*, the system consisting of:*

- *the complex* $Gr_n^W(K_{A\otimes\mathbb{Q}})$ *of sheaves of* $A\otimes\mathbb{Q}$*-vector spaces on* X*,*
- *the complex* $Gr_n^W(K_{\mathbb{C}}, F)$ *of sheaves of* $\mathbb{C}$*-vector spaces with induced* F *filtration,*
- *the isomorphism*

$$Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}})\otimes\mathbb{C}\to Gr_n^W(K_{\mathbb{C}})$$

is an $A\otimes\mathbb{Q}$*-Cohomological Hodge Complex of weight* n*.*

The following example of Cohomological Mixed Hodge Complex can be found in [4] and [5]

Example 1. *Let* X *be a smooth projective variety over* $\mathbb{C}$*,* $D\subset X$ *a simple normal crossing divisor. Let* $U:=X-D$ *and let*

$$j : U\to X$$

be the inclusion map.
Let $\mathbb{Q}_U$ *be the constant sheaf with* $\mathbb{Q}$*-coefficient on* U*.* $\mathbb{R}j_*\mathbb{Q}_U\otimes\mathbb{C}=\mathbb{R}\mathbb{C}_{U*}$ *is quasi-isomorphic to the logarithmic de Rham complex*

$$O_X\xrightarrow{d}\Omega_X(\log D)\xrightarrow{d}\cdots\xrightarrow{d}\Omega_X^n(\log D)$$

For any complex $K^\cdot$ of sheaves on X, let τ be the canonical increasing filtration

$$\tau_m K^q = \begin{cases} K^q & \text{if } q < m \\ \ker d^q \subset K^q & \text{if } q = m \\ 0 & \text{if } q > m \end{cases}$$

See [5]*(Corollary 6.4) for the following result:*
The system consisting of

(1) $(\mathbb{R}j_*\mathbb{Q}_U, \tau)$;
(2) $(\Omega_X^\cdot(\log D), W, F)$ *with usual weight and Hodge filtration W and F;*
(3) *The quasi-isomorphism*

$$(\mathbb{R}j_*\mathbb{Q}_U, \tau) \otimes \mathbb{C} \cong (\Omega_X(\log D)^\cdot, W)$$

is a Cohomological Mixed Hodge Complex on X.

4. Vanishing Theorem for the de Rham Complex

we have seen in the previous section that if V has a real lattice, then

$$(\mathbb{R}j_*V_A, (\mathbb{R}j_*V_{A\otimes\mathbb{Q}}, \tau), (\mathrm{DR}_X(D, E), F, W))$$

is an A-cohomological mixed Hodge complex. As a result of the general theory developed in [4], we have

Theorem 5. *Assume there is a real-valued unitary locayl system $V_\mathbb{R}$ defined on U such that*

$$V = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$$

Let V and $DR_X(D, E)$ be as above. The spectral sequence associated to the Hodge filtration on $DR_X(D, E)$.

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes E) => \mathbb{H}^{p+q}(X, DR_X(D, E))$$

degenerates at E_1

If V does not have an A-lattice with $A \subset \mathbb{R}$, then we cannot expect $\mathrm{DR}_X(D, E)$ to carry a mixed Hodge structure. However, the degeneration of Hodge spectral sequence still holds true:
Let $\bar{V}$ denote the conjugate of V, i.e. the monodromy representation of $\bar{V}$ is the complex conjugate of the monodromy representation of V

Lemma 2. *There exists a real unitary local system $W_\mathbb{R}$ of rank $2r$ such that*

$$V \oplus \bar{V} \cong W_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$$

Proof. We will construct $W_\mathbb{R}$ locally, and show it is canonically determined by V. Over a polydisk, we can assume V is diagonal, and we write

$$V = \bigoplus_{j=1}^r V^j$$

where V^i is a unitary local system of rank 1 with monodromy

$$\lambda_j = \cos\theta_j + i \sin\theta_j$$

We will construct $W_\mathbb{R}^j$ for each j. The monodromy of $\bar{V}^j$ is $\bar{\lambda}_j$ and the monodromy of $V^j \oplus \bar{V}^j$ is

$$\begin{bmatrix} \cos\theta_j + i\sin\theta_j & 0 \\ 0 & \cos\theta_j - i\sin\theta_j \end{bmatrix}$$

Since

$$\begin{bmatrix} \cos\theta_j + i\sin\theta_j & 0 \\ 0 & \cos\theta_j - i\sin\theta_j \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$

have the same characteristic polynomial over $\mathbb{C}$, they must be conjugate over $\mathbb{C}$. Therefore, we can take W^j to be

$$\begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$

Then,

$$W_{\mathbb{R}} = \bigoplus_{j=1}^{r} W^j$$

□

Now, let V be any unitary local system on $X - D$

Corollary 1. *The Hodge spectral sequence*

$$E_1^{p,q} := H^q(X, \Omega_X^p(\log D) \otimes E) => \mathbb{H}^{p+q}(X, DR(D, E)) = H^{p+q}(X, \mathbb{R}j_*V)$$

degenerates at E_1.

Proof. Direct sum and taking cohomology commutes □

Theorem 6. [1]*(Corollary 3.5) Suppose U is an affine variety of complex dimension n. Then, for any constructible sheaf $\mathcal{L}$ on U*

$$H^k(U, \mathcal{L}) = 0$$

for $k > n$

Corollary 2. *Let V and $DR_X(D, E)$ be as above. Suppose U is affine, then*

$$H^q(X, \Omega_X^p(\log D) \otimes E) = 0$$

for $p + q > \dim X$

Lemma 3. *Suppose B is a smooth divisor transversal to D. Then, there is short exact sequence*

$$0 \to \Omega_X^p(\log D + B) \otimes O_X(-B) \xrightarrow{i} \Omega_X^p(\log D) \xrightarrow{r} \Omega_B^p(\log D \cap B) \to 0$$

where i is the inclusion map, and r is the restriction map.

Proof. For simplicity, we prove the case for $p = 1$. We may also assume X is affine. Let $X = \text{Spec}A$, and let $f_1, \cdots, f_s$ be the regular sequence corresponding to D, and let b be the defining equation of B.
The basis of $\Omega_X^1(\log D + B) \otimes O_X(-B)$ as an A-module is

$$\frac{df_1}{f_1} \otimes b, \cdots, \frac{df_s}{f_s} \otimes b, \frac{db}{b} \otimes b$$

The basis of $\Omega_X(\log D)$ as an A-module is

$$\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}$$

The basis of $\Omega_B(\log D \cap B)$ as an $\frac{A}{b}$-module is

$$\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}$$

where by abuse of notation f_i are regarded as their image in $\frac{A}{b}$.
Then, it is clear how to define i and r show that the above sequence is exact □

Lemma 4. *Suppose B is a smooth divisor transversal to D. Then, $E_B := E \otimes O_B$ is the canonical extension of $V_B := V|_{B-B\cap D}$.*

Proof. The statement is local, therefore we may assume X is a polydisk

$$\Delta_1 \times \cdots \times \Delta_n$$

such that the analytic coordinate of Δ_i, for $i = 1, \cdots, s$, are defining equation of D_i, and the analytic coordinate of Δ_n is the defining equation of B.
First, we study V_B by computing its monodromy representation:
Let $T : \pi_1(X - D, x) \to \mathrm{GL}(r, \mathbb{C})$ be the monodromy representation of V. For each generator γ_i of $\pi_1(X - D, x)$, let $\Gamma_i = T(\gamma_i)$. As Γ_i are commuting and unitary, we can use one matrix to diagolize all of them. Therefore, we can assume all Γ_i are diagonal matrices. Moreover, as V is undefined only on D, so for each i, $\Gamma_i^{jj} = 1$, for $j = s + 1, \cdots, n$.
Now, $B = \Delta_1 \times \cdots \times \Delta_{n-1}$, and the monodromy reprentation of $V|_{B-B\cap D}$ is given by

$$\pi_1(B - B \cap D) \xrightarrow{i} \pi_1(X - D) \xrightarrow{T} \mathrm{GL}(r, \mathbb{C})$$

where i is the natural inclusion map. It is clear that one can choose the basis of $\pi_1(B - B \cap D)$ and $\pi_1(X - D)$ such that i can be realized as the identity map. Therefore, the monodromy representations of $V_{B-B\cap D}$ are also Γ_i, for $i = 1, \cdots, s$.
To show $E|_B$ is the canonical extension of $V_{B-B\cap D}$, we compute the connection matrix of $E|_B$ and relate it to the monodromy representations of $V|_{B-B\cap D}$.
One can assume E is trivial over X. Choose a local frame of V on X, and use it as a trivialization of E. With respect to this trivialization, the connection ∇ can be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalues in the stripe

$$\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.
Now, restrict E to B, we see that the connection $\nabla|_B$ can still be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

As monodromy representations of $V_{B-B\cap D}$ are Γ_i, it follows that $E|_B$ is the canonical extension of $V_{B-B\cap D}$. □

Theorem 7. *Suppose L is very ample on X. Then*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof. Let B be a smooth divisor transversal to D such that $L \cong O_X(B)$. By Lemma 3 we have the following exact sequence

$$0 \to \Omega_X^p(\log D + B) \xrightarrow{i} \Omega_X^p(\log D) \otimes O_X(B) \xrightarrow{r} \Omega_B^p(\log D \cap B) \otimes O_X(B) \to 0$$

Tensor it by E and take the cohomology sequence, we get:

$$\cdots H^q(X, \Omega_X^p(\log D + B) \otimes E) \to H^q(X, \Omega_X^p(\log D) \otimes O_X(B) \otimes E)$$
$$\to H^q(X, \Omega_B^p(\log B \cap D) \otimes O_X(B) \otimes E) \cdots$$

Therefore, to prove the theorem, it is enough to show
Claim 1: $H^q(X, \Omega_X^p(\log D + B) \otimes E) = 0$
Claim 2: $H^q(X, \Omega_B^p(\log B \cap D) \otimes O_X(B) \otimes E) = 0$ for $p + q > \dim X$.
Proof of claim 1: Consider the maps

$$X - (B + D) \xrightarrow{f} X - B \xrightarrow{h} X$$

Let V^o be the restriction of V on $X - (B + D)$. The complex $\mathrm{DR}(D + B, E, \nabla)$ is quasi-isomorphic to $\mathbb{R}(h \circ f)_* V^o$. Therefore,

$$H^k(X - (B + D), V^o) = \mathbb{H}^k(X, \mathrm{DR}(D + B, E, \nabla)$$

The claim then follows from Corollary 2.
End of Proof
Claim 2 follows from induction on the dimension of the variety.

Now to finish the proof, it remains to show the base case of Claim 2. One may assume now that X is a smooth projective curve over $\mathbb{C}$,
We need to show that

$$H^1(X, \Omega_X(\log D) \otimes E \otimes L) = 0$$

But for the curve case, $\Omega_X(\log D) \otimes O_X(B) = \Omega_X(\log D + B)$. So the result follows again from Theorem 2 □

Now suppose L is any ample line bundle. Let m be an integer such that $L^{\otimes m}$ is very ample. Take a smooth divisor B transversal to D such that $L^{\otimes m} \cong O_X(B)$. Let φ be the local equation of B on some affine open set, and let $\pi : X' \to X$ be the normalization of X in $\mathbb{C}(X)(\varphi^{\frac{1}{m}})$.

Proposition 2. *Let $\pi : X' \to X$, B and L be as above*

(1) *X' is smooth.*
(2) *$\pi^* B = m\tilde{B}$, where $\tilde{B} = (\pi^* B)_{red}$.*
(3) *$D' := \pi^* D$ is a normal crossing divisor on X'.*
(4) *$\tilde{B}$ is transversal to $\pi^* D$.*
(5) *$\pi^* \Omega_X^p(\log D) = \Omega_{X'}^p(\log D')$.*
(6) *$\pi^* E$ is the canonical extension of $\pi^{-1} V$.*

Proof. 1. We will construct X' by constructing its affine cover and specefiying the gluing morphisms. Let $U_i = \mathrm{Spec} A_i$ be an affine cover of X, and let f_i be the defining equation of D in A_i.
For each A_i, $\frac{A_i[Y]}{(Y^m - f_i)}$ is integrally closed in $\mathbb{C}(X)(f_i^{1/m})$. Therefore,

$$U_i' := \mathrm{Spec} \frac{A_i[Y]}{(Y^m - f_i)}$$

is the normalization of U_i in $\mathbb{C}(X)(f_i^{1/m})$
The same morphisms used to glue U_i into X can be used to glue U_i' into X'. Therefore, to show X' is smooth, it is enough to show $\frac{A_i[Y]}{(Y^m - f_i)}$ is a regular ring.

2. The local defining equation of $\tilde{B}$ is Y, and $\pi^*(f_i) = Y^m$

3. To see this, we describe $\pi^* D$ in $\pi^* U$ for any polydisk $U = \Delta_1 \times \cdots \times \Delta_n$. If $B \cap U \neq \varnothing$, then construct Δ_i such that defining equation of D_i, for $i = 1, \cdots, s$, are coordinates of D_i, for $i = 1, \cdots, s$; and the defining equation of B is the coordinate of D_n. Then,

$$\pi^* U = \Delta_1 \times \Delta_1 \cdots \Delta_{n-1} \times \Sigma^m$$

where Σ^m is the m-sheeted cover over a complex disk branched over the origin. In this case, $\pi^* D$ is still defined by $z_1 \times z_2 \times \cdots z_s$.
If $B \cap U = \varnothing$, then $\pi^* U$ is etale over U. Therefore, $\pi^* D$ is etale over D. So $\pi^* D$ is again a simple normal crossing divisor.

4.This is clear from the case 1 of part 3.

5. Straighforward computation. 6. We compute the monodromy representation of

$\pi^{-1} V$ first:
let $T : \pi_1(U - D, x) \to \mathrm{GL}(r, \mathbb{C})$ be the representation corresponding to the local system V.
Case 1: Suppose $x \notin B$, then $\pi^{-1}(U)$ is etale over U. Let U' be an component of $\pi^{-1}(U)$, and let $x' \in U'$ be a preimage of x. Then,

$$T' : \pi_1(U' - D', x') \xrightarrow{\pi_*} \pi_1(U - D, x) \xrightarrow{T} \mathrm{GL}(r, \mathbb{C})$$

is the representation corresponding to $\pi^{-1} V$.
Case 2: Suppose $x \in B$, then use the description from part 3, we know that

$$\pi^{-1} U = \Delta_1 \times \Delta_2 \times \cdots \times \Sigma^m$$

In both cases, $\pi^{-1} U - D'$ is homotopic to $S_1 \times S_2 \times \cdots \times S_s$ So we can define generators of $\pi_1(U' - D', x')$ and $\pi_1(U - D, x)$ such that π_* is the identity map.
To show $\pi^* E$ is the canonical extension of $\pi^{-1} V$, we only need to compute the connection matrix of $\pi^* E$ and relate it to the monodromies of $\pi^{-1} V$:
Let γ_i be a small circle around D_i, and let Γ_i be the monodromy $T(\gamma_i)$. As

$$\pi_* : \pi_1(U' - D', x') \to \pi_1(U - D, x)$$

is the identity map, Γ_i are also the monodromy representations of $\pi^{-1} V$. Next, we compute the connection matrix of E. Let U be small enough so that E is trivial over it. Choose a local frame of V, and use it as a trivialization of E. With respect to this trivialization, the connection ∇ can be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalue in the stripe

$$\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.
As $\pi^* z_i = z_i$, for $i = 1, \cdots, s$, we see that the $\pi^* \nabla$ over $\pi^{-1} U$ can be realized as:

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

This shows that $\pi^* E$ is the canonical extension of $\pi^{-1} V$.

$\square$

Corollary 3. *For any ample line bundle L on X,*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof. Let m, B and $\pi : X' \to X$ be as above. By Theorem 7

$$H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) = 0$$

for $p + q > \dim X' = \dim X$.
$\pi : X' \to X$ is a finite morphism, so for $i > 0$, $R^i\pi_*\mathscr{F} = 0$ for any coherent sheaf $\mathscr{F}$ on X'. This implies

$$\begin{aligned} H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) &= H^q(X, \pi_*(\pi^*(E \otimes \Omega_X^p(\log D) \otimes L))) \\ &= H^q(X, \pi_*(O_Y) \otimes E \otimes \Omega_X^p(\log D) \otimes L) \\ &= 0 \end{aligned}$$

for $p + q > \dim X$. The second equality follows from the projection formula.
As $\pi_*(O_Y) \cong \bigoplus_{i=0}^{m-1} O_X(-L^{\otimes i})$, the result follows. ☐

5. Partial weight filtration

In the previous section, we proved the vanishing theorem for the complex

$$\mathrm{DR}_X(D, E) \otimes O_X(B)$$

where B is a smooth very ample divisor transversal to D. The intermediate step for the proof is the vanishing theorem for the complex

$$\mathrm{DR}(D + B, E)$$

In this section, we define a partial weight filtration on the The complex

$$\mathrm{DR}_X(D + B, E)$$

It is a more refined weight filtration than the one defined in Section 2, and it will be used to prove the vanishing theorem for the graded complex

$$\mathrm{Gr}_{\cdot}^W \mathrm{DR}_X(D, E)$$

For simplicity, suppose V is a rank 1 unitary local system. We will define partial weight filtration by giving local description of forms. Then, we will show it is a well-defined global notion after Theorem 8. Let μ be a section of E. Recall that $W_m\Omega_X^p(\log D) \otimes E$ consists of sections of the form

$$\omega \otimes \mu$$

where ω can be written as

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \frac{dz_{j_2}}{z_{j_2}} \wedge \cdots \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \wedge \cdots \wedge dz_{j_p}$$

Moreover, ω has at most m log forms acting on V by identity.
Now let $W_m^{D^1}\mathrm{DR}(D, E)$ be the set of form that can be written as $\omega \otimes \mu$, where ω can be written as

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \frac{dz_{j_2}}{z_{j_2}} \wedge \cdots \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \wedge \cdots \wedge dz_{j_p}$$

Moreover, let g be the cardinality of the following set

$$\{\text{log forms in } \omega \text{ acting on } V \text{ by identity}\} \cap \{\frac{dz_2}{z_2}, \cdots, \frac{dz_s}{z_s}\}$$

Then $g \leq m$.
To generalize, $W_m^{D^{i_1}+\cdots D^{i_l}}\mathrm{DR}(D,E)$ is the set of forms that can be written as $\omega \otimes \mu$, where ω can be written as

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \frac{dz_{j_2}}{z_{j_2}} \wedge \cdots \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \wedge \cdots \wedge dz_{j_p}$$

Moreover, let g be the cardinality of the following set

$$\{\text{log forms in } \omega \text{ acting on } V \text{ by identity}\} \cap (\{\frac{dz_1}{z_1}, \cdots, \frac{dz_s}{z_s}\} - \{\frac{dz_{i_1}}{z_{i_1}}, \cdots \frac{dz_{i_l}}{z_{i_l}}\})$$

Write T for $D+B$. Let T_2 be the union of 2-fold intersections of components of T. Let $v_1 : \tilde{D}_1 \to D$ be the normalization map, i.e. $\tilde{D}_1$ is the disjoint union of components of D. Let F_1 be $v_1{}^*T_2$. Then, F_1 is a normal crossing divisor in D_1.
We have seen in Section 1 that the restiction of j_*V on $D_1 - T_2$ is a unitary local system, denote it by V_1; and let E_1 be the subbundle of v_1^*E which is the canonical extension of V_1.

Proposition 3. *There is an exact sequence*

$$0 \to W_0^B DR(D+B,E,\nabla) \xrightarrow{i} DR(D+B,E,\nabla) \xrightarrow{res} v_{1*}DR(F_1,E_1,\nabla_1) \to 0$$

where i is the inclusion map, and res is the residue map.

Proof. Suppose for simplicity D is smooth, i.e. D has only one component. Also, suppose V is a unitary local system of rank 1. Let μ be a local section of E.
Let z_1 be the local equation of D. Suppose $\frac{dz_1}{z_1}$ acts on V by identity, then V extends to a unitary local system on $D - D \cap B$. In this case, $D_1 = D$, and $F_1 = D \cap B$. Let z_n be the local equation for B. Then, locally over a polydisk

1. $W_0^B\Omega_X^p(\log D+B) \otimes E$ is generated by sections of the form

$$\frac{dz_n}{z_n} \wedge \omega \otimes \mu$$

where $\omega \in \Omega_X^{p-1}$.

2. $\Omega_X^p(\log D+B)$ is generated by sections of the form

$$\frac{dz_n}{z_n} \wedge \omega \otimes \mu$$

where $\omega \in \Omega_X^{p-1}(\log D)$.

3. $\Omega_{D_1}^{p-1}(\log F_1) \otimes E_1$ is generated by sections of the form

$$\frac{dz_n}{z_n} \wedge \omega \otimes \mu_1$$

where $\omega \in \Omega_{D_1}^{p-2}(\log F_1)$.

Use the local description, it is clear that the sequence is exact.

∎

Theorem 8. *Let (E_B, ∇_B) be the restriction of (E, ∇) on B, and let $DR(B \cap D, E_B, \nabla_B)$ be the complex*

$$0 \to E_B \to \Omega^1_B(B \cap D) \otimes E_B \cdots$$

then there is an exact sequence of complexes

$$\begin{aligned} 0 \to W^B_m DR_X(D+B, E) &\xrightarrow{i} W_m DR_X(D, E) \otimes O_X(B) \\ &\xrightarrow{r} W_m DR_B(D \cap B, E_B) \otimes O_X(B) \to 0 \end{aligned}$$

i is the inclusion map, and r is the restriction map.

Proof. For simplicity, we assume E has rank 1. The statement is local, so we work on a polydisk, and we use the notation from above. Let μ be a generating section of E, then

1. $W^B_m \mathrm{DR}_X(D+B, E) \otimes O_X(-B)$ is generated by

$$\omega \otimes \mu \otimes z_n$$

where $\omega \in \Omega^p_X(\log D + B)$ is a p-form that has at most m log forms coming from

$$\{\frac{dz_1}{z_1}, \cdots, \frac{dz_s}{z_s}\}$$

acting on E by identity.

2. $W_m \mathrm{DR}_X(D, E)$ is generated by

$$\omega \otimes \mu$$

where $\omega \in \Omega^p_X(\log D)$ is a p-form that has at most m log forms acting on E by identity.

3. $W_m \mathrm{DR}_B(D \cap B, E_B)$ is generated by

$$\omega \otimes \mu$$

where $\omega \in \Omega^p_B(\log B \cap D)$ is a p-form that has at most m log forms acting on μ_B by identity.

The map i is the natural inclusion map, i.e. $\frac{dz_n}{z_n} \otimes z_n \mapsto dz_n$; The map r is the restriction on B.

□

The above theorem also gives a description of

$$W^B_m \mathrm{DR}_X(D+B, E)$$

as the kernel of the restriction map

$$r : \mathrm{DR}_X(D+B, E) \otimes O_X(B) \to W_m \mathrm{DR}_B(B \cap D, E_B) \otimes O_X(B)$$

It means that $W_m \mathrm{DR}_X(D+B, E)$ is indeed globally well-defined.

6. Mixed Hodge Structure on the Complex $W_0^B\mathrm{DR}(D+B,E,\nabla)$

Throughout this section, we assume the unitary local system V has a real lattice $V_{\mathbb{R}}$ such that

$$V = V_{\mathbb{R}} \otimes \mathbb{C}$$

We will study the mixed Hodge structure on the complex

$$W_0^B\mathrm{DR}_X(D+B,E)$$

Consider the maps

$$X-(D+B) \xrightarrow{f} X-B \xrightarrow{h} X$$

Write V^o (resp. $V^o_{\mathbb{R}}$) for the restriction of V (resp. $V_{\mathbb{R}}$) on $X-(D+B)$.
Let τ be the canonical filtration on $\mathbb{R}h_*f_*V^o_{\mathbb{R}}$; let W be the increasing filtration on $W_0^B\mathrm{DR}_X(D+B,E)$ defined as

$$W_mW_0^B\mathrm{DR}_X(D+B,E) = \begin{cases} 0 & \text{if } m<0 \\ W_0\mathrm{DR}_X(D+B,E) & \text{if } m=0 \\ W_0^B\mathrm{DR}_X(D+B,E) & \text{if } m>0 \end{cases}$$

The main result of this section is

Theorem 9.

$$(\mathbb{R}h_*f_*V^o_{\mathbb{R}}, (\mathbb{R}h_*f_*V^o_{\mathbb{R}},\tau), (W_0^B DR_X(D+B,E), F^{\cdot}, W_{\cdot}))$$

is a $\mathbb{R}$-cohomological mixed Hodge complex.

Proposition 4. $\mathbb{R}h_*(f_*V^o)$ *is quasi-isomoprhic to*

$$W_0^B DR_X(D+B,E)$$

Proof. The statement is local, so we can assume X is a polydisk. For the basic case, one can assume V is of rank 1, D has two components D^1 and D^2 such that the monodromy of V around D^1 is trivial, and the monodromy of V around D^2 is nontrivial.
Let $Y=X-B$. Then, $\Omega^{\cdot}_Y(\log D^2)\otimes h^*E$ is a resolution of f_*V^o(see [8]).
Let $g: Y-D^2 \to Y$ be the inclusion map. By a theorem of Griffith[**?**] and Deligne[3], the inclusion map

$$i:\Omega^{\cdot}_Y(\log D^2) \to g_*\mathscr{A}^{\cdot}_{Y-D^2}$$

is a quasi-isomorphism. Therefore, f_*V is quasi-isomorphic to

$$g_*\mathscr{A}^{\cdot}_{Y-D^2}$$

As $g_*\mathscr{A}^{\cdot}_{Y-D^2}$ is a complex of flasque sheaves, $\mathbb{R}h_*f_*V$ is quasi-isomorphic to

$$h_*g_*\mathscr{A}^{\cdot}_{Y-D^2}$$

Now,

$$W_0^B\Omega^{\cdot}_X(\log D+B)\otimes E = \Omega^{\cdot}_X(\log D^2+B)\otimes E$$

But as we have seen the complex $\Omega^{\cdot}_X(\log D^2+B)$ is quasi-isomorphic to

$$(h\circ g)_*\mathscr{A}^{\cdot}_{Y-D^2}$$

So the result for the basic case follows.
Now, let V be of rank r. For each $i=1,2$, let Γ_i be the monodromy of V around D^i. As V is unitary, we can simultaneously diagonalize all Γ_1 and Γ_2. Therefore,

we can assume V is the direct sum of two rank 1 unitary local systems. As $\mathbb{R}h_*$ and f_* commutes with direct sum. The result follows.
Now, let V be of rank 1 and let $D^1, \cdots, D^s$ be components of D. Now let D_1 be the subdivisor of D over which V has identity monodromy; and let D_2 be the subdivisor of D over which V has nontrivial monodromy. Then, the result follows after the same steps in the basic case. $\square$

Proposition 5. *The inclusion map*

$$i : (W_0^B \mathit{DR}_X(D+B,E), \tau_\cdot) \to (W_0^B \mathit{DR}_X(D+B,E), W)$$

is a quasi-isomorphism of filtered complexes.

Proof. This is again a local statement, so we can assume X is a polydisk and V is of rank 1. We need to show that the induced maps of i

$$H^k(i) : H^k(\mathrm{Gr}_m^\tau W_0^B \mathrm{DR}_X(D+B,E)) \to H^k(\mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E))$$

are isomorphisms.

$$H^k(\mathrm{Gr}_m^\tau W_0^B \mathrm{DR}_X(D+B,E)) = \begin{cases} H^m(W_0^B \mathrm{DR}_X(D+B,E)) & \text{if } i = m \\ 0 & \text{otherwise} \end{cases}$$

Claim 1 If $m > 1$, then $H^m(W_0^B \mathrm{DR}_X(D+B,E)) = 0$.
Proof of Claim 1 We have a short exact sequence of complexes

$$0 \to W_0 \mathrm{DR}_X(D+B,E) \to W_0^B \mathrm{DR}_X(D+B,E) \xrightarrow{res} W_0 \mathrm{DR}_B(B \cap D, E_B)[-1] \to 0$$

where the $\mathrm{DR}(D \cap B, E_B, \nabla_B)$ is the complex

$$\cdots \to \Omega_B^m(\log B \cap D) \otimes E_B \xrightarrow{\nabla_B} \Omega_B^{m+1}(\log B \cap D) \otimes E_B \to \cdots$$

and the map res is the residue map.
Taking cohomology, we get

$$\begin{aligned} \cdots \to H^k(W_0 \mathrm{DR}_X(D+B,E)) \to H^k(W_0^B \mathrm{DR}_X(D+B,E)) \\ \to H^{k-1}(W_0 \mathrm{DR}_B(B \cap D, E_B)) \to \cdots \end{aligned}$$

Timmerscheidt proved in the Appendix D of [7] that $W_0 \mathrm{DR}_X(D+B,E)$ is a resolution of $(h \circ f)_* V$. Therefore, $W_0^B \mathrm{DR}_X(D+B,E)$ is exact. Likewise,

$$W_0 \mathrm{DR}_B(B \cap D, E_B)$$

is also exact.
So the conlusion follows.
End of Proof
The above proof also shows that

$$H^k(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = \begin{cases} H^1(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

$$H^k(\mathrm{Gr}_0^W W_0^B \mathrm{DR}_X(D+B,E)) = \begin{cases} H^0(W_0 \mathrm{DR}(D+B,E)) & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

Therefore, to prove

$$i : (W_0^B \mathrm{DR}_X(D+B,E),\tau) \to (W_0^B \mathrm{DR}_X(D+B,E),W)$$

is a quasi-isomorphism of filtered complexes, it remains to prove that both

$$H^0(i) : H^0(\mathrm{Gr}_0^\tau W_0^B \mathrm{DR}_X(D+B,E)) \to H^0(\mathrm{Gr}_0^W W_0^B \mathrm{DR}_X(D+B,E))$$

and

$$H^1(i) : H^1(\mathrm{Gr}_1^\tau W_0^B \mathrm{DR}_X(D+B,E)) \to H^1(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E))$$

are isomorphisms.
Now,

$$H^0(\mathrm{Gr}_0^\tau W_0^B \mathrm{DR}_X(D+B,E)) = \ker(E \xrightarrow{\nabla} W_0^B(\Omega_X^1(\log D+B)\otimes E)$$

and

$$H^0(\mathrm{Gr}_0^W W_0^B \mathrm{DR}_X(D+B,E)) = \ker(E \xrightarrow{\nabla} W_0(\Omega_X^1(\log D+B)\otimes E)$$

It is clear that the map $H^0(i)$ is an isomorphism.
To simplify notations, write $K^\cdot$ for $W_0^B \mathrm{DR}_X(D+B,E)$, from the proof of Claim 1, we have a commutative diagram

$$\begin{tikzcd}
H^1(\mathrm{Gr}_1^\tau K^\cdot) \arrow[r, "H^1(i)"] \arrow[d] & H^1(\mathrm{Gr}_1^W K^\cdot) \arrow[r, "res"] & H^1(W_0\mathrm{DR}_B(B\cap D,E_B)[-1]) \arrow[d] \\
H^1(K^\cdot) \arrow[rr, "res"] & & H^1(W_0\mathrm{DR}_B(B\cap D,E_B)[-1])
\end{tikzcd}$$

and the residue map on the second row is an isomorphism. As the residue map on the first row is an isomorphism (even on the complex level), we see that the map $H^1(i)$ is an isomorphism.

□

For reader's sake, we restate the main theorem of this Section:

Theorem 10.

$$(\mathbb{R}h_* f_* V_{\mathbb{R}}^o, (\mathbb{R}h_* f_* V_{\mathbb{R}}^o, \tau), (W_0^B DR_X(D+B,E), F^\cdot, W_.))$$

is a cohomological mixed $\mathbb{R}$-Hodge complex

Proof. The quasi-isomorphism

$$(\mathbb{R}h_* f_* V_{\mathbb{R}}^o, \tau)\otimes\mathbb{C} \to (W_0^B \mathrm{DR}_X(D+B,E), W_.)$$

was proved in the previous proposition.
It remains to show

$$(\mathrm{Gr}_m^\tau \mathbb{R}h_* f_* V_{\mathbb{R}}^o, (\mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E), F))$$

is a cohomological $\mathbb{R}$-complex of weight m, i.e. the Hodge spectral sequence of $(\mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E), F)$ degenerates at E_1, and the induced filtration on

$$\mathbb{H}^k(X, \mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E)) = \mathbb{H}^k(X, \mathrm{Gr}_m^\tau \mathbb{R}h_* f_* V_{\mathbb{R}}^o)\otimes\mathbb{C}$$

defines a pure $\mathbb{R}$-Hodge structure of weight $k+m$ on

$$\mathbb{H}^k(X, \mathrm{Gr}_m^\tau \mathbb{R}h_* f_* V_{\mathbb{R}}^o)$$

i.e. the induced filtration F on $\mathbb{H}^k(X, \mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E))$ is $m+k$ opposed to its conjugate.
For $m > 1$, all $\mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E)$ are 0, so we only need to show the case for $m = 0, 1$.
For $m = 0$,

$$(\mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E), F) = (W_0 \mathrm{DR}_X(D+B,E), F)$$

Timmerscheidt showed that it is a cohomological $\mathbb{R}$-complex of weight 0 in [7](Appendix D).
For $m = 1$, we have seen that

$$\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E) \cong W_0 \mathrm{DR}(B \cap D, E_B, \nabla_B)[-1]$$

Let F be the induced Hodge filtration on $\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)$, and let F_B be the usual Hodge filtration on $W_0 \mathrm{DR}(B \cap D, E_B, \nabla_B)$. let $\bar{F}$ and $\bar{F}_B$ be their conjugates. To show F and $\bar{F}$ are $k+1$ opposed on $\mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E))$, we show that

$$\mathrm{Gr}_q^{\bar{F}} \mathrm{Gr}_p^F \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = 0 \text{ if } p+q \neq k+1$$

As $\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E) \cong W_0 \mathrm{DR}_B(B \cap D, E_B)[-1]$,

$$\mathrm{Gr}_p^F \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = \mathrm{Gr}_{p-1}^{F_B} \mathbb{H}^{k-1}(B, W_0 \mathrm{DR}_B(B \cap D, E_B))$$
$$\mathrm{Gr}_q^{\bar{F}} \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = \mathrm{Gr}_{q-1}^{\bar{F}} \mathbb{H}^{k-1}(B, W_0 \mathrm{DR}_B(B \cap D, E_B))$$

Therefore, $\mathrm{Gr}_q^{\bar{F}} \mathrm{Gr}_p^F \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = 0$ if $p-1+q-1 \neq k-1$.
The E_1-degeneration of $(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E), F)$ follows from the E_1-degneration of $(W_0 \mathrm{DR}(B \cap D, E_B, \nabla_B), F_B)$.

$\square$

So far, we have shown that if V has a real lattice $V_{\mathbb{R}}$, i.e.

$$V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

Then, the Hodge spectral sequence

$$E_1^{p,q} = H^q(X, W_0^B \Omega_X^p(\log D + B) \otimes E) => \mathbb{H}^{p+q}(X, W_0^B \mathrm{DR}_X(D+B,E))$$

degenerates at E_1.
Now, consider the case when V does not have a real-lattice.

7. Vanishing Theorem for the complex $\mathrm{Gr}_{\cdot}^W \mathrm{DR}_X(D,E)$

Now, let V be any unitary local system over $\mathbb{C}$. We have seen in Section 4 that even if V does not have a real lattice, the spectral sequence of $(\mathrm{DR}_X(D,E), F)$ still have E_1-degeneration. Similarly, we have

Lemma 5. *Let B be a smooth divisor transversal to D, then The spectral sequence of $(W_0^B DR(D+B,E), F)$:*

$$E_1^{p,q} = H^q(X, W_0^B(\Omega_X^p(\log D + B))) => \mathbb{H}^{p+q}(X, W_0^B(DR_X(D+B,E)))$$

degenerates at E_1

Theorem 11. *Let B be a smooth very ample divisor transveral to D, then for $m = 0, \cdots, n-1$*

$$H^q(X, Gr_m^W DR^p(D, E, \nabla) \otimes O_X(B)) = 0$$

for $p+q > n+1$.

Proof. We show first that

$$H^q(X, W_0\mathrm{DR}^p(D, E, \nabla) \otimes O_X(B)) = 0$$

for $p+q > n+1$.
By Theorem 8, we have the exact sequence

$$0 \to W_0^B\mathrm{DR}_X(D{+}B, E) \to W_0\mathrm{DR}_X(D, E){\otimes}O_X(B) \to W_0\mathrm{DR}_B(B{\cap}D, E_B){\otimes}O_X(B) \to 0$$

Take cohomology sequence, we get

$$\cdots \to H^q(X, W_0^B(\Omega_X^p(\log D+B) \otimes E)) \to H^q(X, W_0(\Omega_X^p(\log D) \otimes E) \otimes O_X(B)) \to$$
$$\to H^q(B, W_0(\Omega_B^p(\log B \cap D) \otimes E) \otimes O_X(B)) \to \cdots$$

Therefore, it is enough to show
Claim 1: $H^q(X, W_0^B(\Omega_X^p(\log D+B) \otimes E)) = 0$ for $p+q > n$.
Claim 2: $H^q(B, W_0(\Omega_B^p(\log B \cap D) \otimes E_B)) \otimes O_X(B)) = 0$ for $p+q > n$.
Proof of Claim 1: Consider the maps

$$X - (B+D) \xrightarrow{f} X - B \xrightarrow{h} X$$

Write V^o for the restriction of V on $X - B$. We have seen in Lemma 5 that the spectral sequence

$$\begin{aligned} E_1^{p,q} = H^q(X, W_0^B(\Omega_X^p(\log D+B) \otimes E)) &=> \mathbb{H}^{p+q}(X, W_0^B\mathrm{DR}_X(D+B, E)) \\ &= H^{p+q}(X, \mathbb{R}h_* f_* V^o) \\ &= H^{p+q}(X - B, f_* V^o) \end{aligned}$$

As $X - B$ is affine, it follows from Theorem 6 that

$$H^q(X, W_0^B(\Omega_X^p(\log D+B) \otimes E)) = 0$$

for $p+q > n$.
End
Proof of Claim 2: Induct on dimension of X **End**
Therefore, it remains to show that if X is a smooth projective curve, then

$$H^1(X, W_0(\Omega_X(\log D) \otimes E) \otimes O_X(B)) = 0$$

But for the curve case,

$$W_0(\Omega_X(\log D) \otimes E) \otimes O_X(B) = W_0^B(\Omega_X(\log D+B) \otimes E)$$

Therefore the result follows again from Theorem 6
To finish the rest of the proof, we use the identification from proposition 1

$$(W_m/W_{m-1})\mathrm{DR}_X(D, E) \cong W_0\mathrm{DR}(\tilde{C}_m, E_m, \nabla_m)[-m]$$

and then apply the above argument to $\tilde{D}_m$. □

Corollary 4. *For any $m \in \mathbb{Z}$,*

$$H^q(X, W_m(\Omega_X^p(\log D) \otimes E) \otimes O_X(B)) = 0$$

for $p+q > \dim X$

Corollary 5. *Let L be an ample line bundle on X, then*

$$H^q(X, Gr^W_\cdot(\Omega^p_X(\log D)\otimes E)\otimes L) = 0$$

for $p+q>n$.

Proof. Like in Theorem 11, it is enough to show

$$H^q(X, W_0\mathrm{DR}^p(D,E,\nabla)\otimes L) = 0$$

for $p+q>n$.
Let m be a large enough integer such that $L^{\otimes m}$ is very ample. Let B be a smooth hyperplane divisor transversal to D so that

$$L \cong O_X(B)$$

Use the same idea from Corollary 3, we construct a cyclic cover of degree m branched over B

$$\pi : X' \to X$$

To finish the proof, it remains to show

$$\pi^* W_0\mathrm{DR}_X(D,E) = W_0\mathrm{DR}(\tilde{D},\tilde{E},\tilde{\nabla})$$

But this is clear from the local description of W_0. □

8. Appendix

8.1. **Linear algebra.**

Theorem 12. *Let U be an unitary matrix over $\mathbb{C}$, then U is diagonalizable.*

Theorem 13. *Let A and B be commuting diagonalizable $n\times n$ matrices over any field k, then A and B can be simultaneously diagolized.*

Proof. Let V be the vector space k^n. It is enough to show that A and B share the same eigenvectors.
Claim 1: A and B share at least one eigenvector.
Proof of Claim 1: Let v be an eigenvector of A with eigenvalue λ, then

$$ABv = BAv = B\lambda v = \lambda Bv$$

i.e. Bv is also an eigenvector of A with eigenvalue λ.
Let W be the subspace spanned by

$$v, Bv, \cdots, B^n v$$

Then, W is invariant under B. As V has a basis by eigenvectors of B, one can choose a vector $w\in W$ which is an eigenvector of B. Then, from the construction of W, w is also an eigenvector of A. **End**
Let w be as above, with $Bw=\mu w$; Let $e_1,\cdots,e_n$ be the standard basis of V; Let V' be the subspace spanned by $e_1,\cdots,e_{n-1}$; Let $\phi: V\to V$ be the linear map such that $\phi(e_n)=w$.

$$\phi^{-1}\circ A\circ\phi = A'\oplus \mathrm{Diag}(\lambda)$$
$$\phi^{-1}\circ B\circ\phi = B'\oplus \mathrm{Diag}(\mu)$$

where A' and B' are $n-1\times n-1$ submatrices of A and B, representing the restriction of A and B on V'.

Now, A' and B' are diagonlizable, and they commute, therefore, by inducting on the size of the matrix, we are done. $\square$

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