Let V be a unitary local syetem on $X - D$, and let E be its canonical extension. A weight filtration on the complex $DR_D(E, \nabla)$ has been defined and studied in [**?**]. We give a review here:

For simplicity, assume each component of D is smooth. For $m > 0$, let D_m be the union of m-fold intersection of components of D. Let $\tilde{D_m}$ be the disjoint union of components of D_m, and let $v_m : \tilde{D_m} \to X$ be the composition of projection onto D_m and the inclusion map. $\tilde{C_m} := v_m(D_{m+1})$ is either empty or a normal crossing divisor in $\tilde{D_m}$.

Residue map For each $m \leq p \leq \dim D_m$, there exists a residue map

$$\mathrm{Res}_m : \Omega_X^p(\log D) \to v_{m*}(\Omega_{\tilde{D_m}}^{q-m})$$

This map is defined as the following:

Let D_{m1} be one component of D_m, as the intersection of $D_{i1}, \cdots, D_{im}$. Then, the map Res_m sends dz_i/z_i to 1 if i appears in $i_1, \cdots, i_m$, and Res_m sends all other 1-form to 0. It can be checked that this map is well-defined independent of the chosen coordinate. Futhermore, this map commutes with exterior derivative d, making it a homomorphism of complexes

$$\mathrm{Res}_m : \Omega_X^{\cdot}(\log D) \to v_{m*}(\Omega_{\tilde{D_m}}^{\cdot}(\log \tilde{C_m})[-m]$$

Restriction of j_*V on D_m
D_{m+1}

Theorem 1. *[**?**]*

1. *$V_m := j_*V|_{D_m D_{m+1}}$ is a unitary local system on D_m*
   D_{m+1}.
2. *There exist a unique subvectorbundle E_m of ${v_m}^*E$ and a unique holomorphic integrable connection ∇_m on E_m with logarithmic poles along $\tilde{C_m}$ such that*
   $$\ker \nabla_m|_{\tilde{D_m}\tilde{C_m}} = v_m^{-1}V_m$$
3. *(E_m, ∇_m) is the canonical extension of $v_m^{-1}V_m$.*
4. *There exists a unique subvectorbundle M_m^* of ${v_m}^*M$ with*
   $$M_m \oplus M_m^* = {v_m}^*M$$
   such that for $\tilde{x} \in \tilde{D_m}$
   $\tilde{C_m}$
   $$M_{m,\tilde{x}}^* = {v_m}^*(\{\sigma \in M_{v_m(\tilde{x})} | (\sigma, \tau) = 0 \textit{ for all } \tau \in j_*(V)_{v_m(\tilde{x})}\})$$

The residue map and the decomposition of ${v_m}^*E$ into $E_m \oplus E_m^*$ enables us to define the following variation of residue map

$$\mathrm{Res}_m(E) : \Omega_x^p(\log D) \otimes E \xrightarrow{\mathrm{Res}_m \otimes \mathrm{id}} v_{m*}(\Omega_{\tilde{D_m}}^{p-m}(\log \tilde{C_m})) \otimes E \quad = v_{m*}(\Omega_{\tilde{D_m}}^{p-m}(\log \tilde{C_m}) \otimes {v_m}^*E)$$
$$\xrightarrow{\mathrm{id} \otimes p_m} v_{m*}(\Omega_{\tilde{D_m}}^{p-m}(\log \tilde{C_m}) \otimes E_m)$$

where p_m is the projection onto the M_m component.

Lemma 1. *[?] $Res_m(E) \circ \nabla = \nabla_m \circ Res_m(E)$,* i.e. *$Res_m(E)$ is homomorphism of complexs*

$$DR_D(E,\nabla) \to v_{m*}DR_{\tilde{C_m}}(E_m,\nabla_m)[-m]$$

Weight Filtration The weight filtration W_m on $DR_D(E,\nabla)$ is defined as

$$W_m(DR_D(E,\nabla)) = \ker \text{Res}_{m+1}(E) \text{ if } m \geq 0$$
$$W_m(DR_D(E,\nabla)) = 0 \text{if} m < 0$$

Proposition 1. *[?]*

- *$W_{\cdot}(DR_D(E,\nabla))$ is an increasing filtration.*
- *$Res_m(E)$ induces an isomorphism*

$$Gr_m^W(DR_D(E,\nabla)) \to v_{m*}(W_0(DR_{\tilde{C_m}}(E_m,\nabla_m)))$$

$W_0(DR_D(E,\nabla))$ was studied by Timmerscheidt in the Appendix D of [**?**]. The interesting thing about $W_0(DR_D(E,\nabla))$ is that it defines a Hodge structure on j_*V.

Theorem 2. *[?]*

1. *$W_0(DR_D(E,\nabla))$ is a resolution of j_*V.*
2. *The spectral sequence*

$$E_1^{p,q} := H^q(X, W_0^p(DR_D(E,\nabla))) => \mathbb{H}^{p+q}(X, j_*V)$$

degenerates at E_1.

3. *There exists a conjugate linear isomorphism*

$$H^q(X, W_0^p(DR_D(E,\nabla))) \cong H^p(X, W_0^q(DR_D(N,\nabla)))$$

where N is the canonical extension of $V^\vee$.

Proposition 2. *[?](1.5) Suppose U is an affine manifold. Let $\mathcal{L}$ be any local system on U. Then*

$$H^k(U,\mathcal{L}) = 0$$

for $k > \dim X$.

Therefore, back in our situation

Proposition 3. *Suppose $U := X - D$ is affine. Then*

$$H^q(X, W_0^p(DR_D(E,\nabla))) = 0$$

for $p + q > \dim X$.

Theorem 3. *Suppose $D_m - D_{m+1}$ is affine for each m. Then*

$$H^q(X, DR^p_D(E, \nabla)) = 0$$

for $p + q > \dim X$.

Proof. Consider the weight filtration $W_.$ on $DR_D(E, \nabla)$, we show that for all $m \geq 0$,

$$H^q(X, W^p_m) = 0$$

for $p + q > \dim X$.
For $m = 1$, consider the exact sequence

$$0 \to W_0 \to W_1 \to W_1/W_0 \to 0$$

by **??**,

$$W_1/W_0 \cong W_0(DR_{\tilde{C}_1}(E_1, \nabla_1))[-1]$$

Take cohomology sequence of the above exact sequence, one gets

$$\cdots \to H^q(X, W^p_0) \to H^q(X, W^p_1) \to H^q(\tilde{D}_1, W_0^{p-1}(DR_{\tilde{C}_1}(E_1, \nabla_1))) \to \cdots$$

Therefore, by the result for W_0, one has

$$H^q(X, W^p_1) = 0$$

for $p + q > \dim X$.
Repeat this process for all $m = 2, 3, \cdots$ □

Suppose $Y = \mathrm{Spec}A$ is a smooth affine variety, and Z is a smooth Weil divisor then Z is defined by one equation $f \in H^0(Y, O_Y)$. Therefore, $Y - Z = \mathrm{Spec}A_f$ is an affine variety.

Corollary 1. *Let $D = D_1 + \cdots + D_s$ be a simple normal crossing divisor on X such that D_s is very ample, and $D_s \cap D_i \neq \varnothing$. Let V be a unitary local system on $X - D$, and let E be its canonical extension. Then,*

$$H^q(X, DR^p_D(E, \nabla)) = 0$$

for $p + q > \dim X$

1 Vanishing Theorem

Lemma 2. *Suppose B is a smooth divisor transversal to D. Then, there is short exact sequence*

$$0 \to \Omega^p_X(\log D + B) \xrightarrow{i} \Omega^p_X(\log D) \otimes O_X(B) \xrightarrow{r} \Omega_B(\log D \cap B) \otimes O_X(B) \to 0$$

where i is the inclusion map, and r is the restriction map.

Lemma 3. *Suppose B is a smooth divisor transversal to D. Then, $E_B := E \otimes O_B$ is the canonical extension of $V_B := V|_{B-B\cap D}$.*

Theorem 4. *Suppose L is very ample on X. Then*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof. Tensor the exact sequence in **??** by E, and take cohomology long exact sequence. By the topological vanishing proved in the previous section, it remains to show

$$H^q(B, E \otimes \Omega_B^p(\log D \cap B) \otimes O_X(B)) = 0$$

for $p + q > \dim X$.
Then, by inducting on the dimension of X and **??**, one may assume that X is a curve.
In the curve case, one only needs to show

$$H^1(X, E \otimes \Omega_X^1(\log D) \otimes O_X(B)) = 0$$

But if X is curve, then $\Omega_X^1(\log D) \otimes O_X(B) = \Omega_X^1(\log D + B)$. Therefore, the above follows from the topological vanishing proved in the previous section. □

Now suppose L is any ample line bundle. Let m be an integer such that $L^{\otimes m}$ is very ample. Take a smooth divisor B transversal to D such that $L^{\otimes m} \cong O_X(B)$. Let φ be the local equation of B on some affine open set, and let $\pi : X^{prime} \to X$ be the normalization of X in $\mathbb{C}(X)(\varphi^{\frac{1}{m}})$.

Proposition 4. *1. X^{prime} is smooth.*

2. *$\pi^* B = m\tilde{B}$. $\tilde{B} = \pi^* B)_{red}$.*
3. *$D' := \pi^* D$ is a normal crossing divisor on X^{prime}.*
4. *$\tilde{B}$ is transversal to $\pi^* D$.*
5. *$\pi^* \Omega_X(\log D) = \Omega_X^{prime}(\log D')$.*
6. *$\pi^* E$ is the canonical extension of $\pi^{-1} V$.*

Proof. Choose $x \in X - D$. If $x \in B$, then let U be a polycylinder around x with coordinate $(z_1, \cdots, z_r, \cdots, z_n)$, such that if $D \cap U \neq \varnothing$, then $D \cap U$ is defined by $\prod_{i=1}^{r} z_i$ and B is defined by $z_n = 0$. Then, $\pi^{-1}(U)$ is a branched cover over U. Write $\pi^{-1}(U) = \cup_{i=1}^r U_i'$, where U_i' is a polycylinder with coordinate $(w_1, \cdots, w_r, \epsilon_i w_n)$, where $w_i = z_i$ for $i < n$, ϵ_i is a primitive m-th root of unity, and w_n is a fixed branch of $z_n^{1/m}$.

Alternatively, one can think of $\pi^{-1}(U)$ as the product

$$\Delta_1 \times \cdots \times \Delta_{n-1} \times \Sigma^m$$

where each Δ_i is complex disk, and Σ^m is the m-sheeted cover over Δ branched over the origin.
If $x \notin B$, then $\pi^*(U)$ is etale over U. Then, 1, 2, 3 and 4 are clear.
5. Let $T : \pi_1(U - D, x) \to \mathrm{GL}(r, \mathbb{C})$ be the representation corresponding to the local system V.
Case 1: Suppose $x \notin B$, then $\pi^{-1}(U)$ is etale over U. Let U' be an component of $\pi^{-1}(U)$, and let $x' \in U'$ be a preimage of x. Then, $T' : \pi_1(U' - D', x') \xrightarrow{\pi_*} \pi_1(U - D, x) \xrightarrow{T} \mathrm{GL}(r, \mathbb{C})$ is the representation corresponding to $\pi^{-1}V$.
Let γ_i be a small circle around D_i, and let Γ_i be the monodromy $T(\gamma_i)$. Let $N_1, \cdots, N_s$ be commuting matrices with eigenvalue in the stripe $\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$ such that $e^{-2\pi i N_i} = \Gamma_i$. Let U be small enough, so that $E_U \cong O_U^r$, then, ∇_E over U can be written as

$$d + \sum_{i=1}^{s} N_i \frac{dz_i}{z_i}$$

Let γ'_i be a small circle around D'_i. Then, $\pi_*(\gamma'_i) = \gamma_i$.
$\pi^* E$ is trivial over U', and the lifted connection $\pi^* \nabla_E$ over U' can be written as

$$d + \sum_{i=1}^{s} N_i \frac{dw_i}{w_i}$$

Therefore, $\pi^* E$ is the canonical connection of $\pi^{-1} V$.
Case 2: Suppposе $x \in B$, then $U' := \pi^{-1}(U)$ is a branched cover over U. Let x' be the preimage of x. Then, $T' : \pi_1(U' - D', x') \xrightarrow{\pi_*} \pi_1(U - D, x) \xrightarrow{T} \mathrm{GL}(r, \mathbb{C})$ is the representation corresponding to $\pi^{-1} V$.
The fundamental group $\pi_1(U' - D', x')$ in this case is also generated by small circles around D', because we can simply contract every polydisk or branched cover of a ploydisk after s to a point and ignore their contribution to the fundamental group.
The rest of the argument goes though like in Case 1. $\square$

Corollary 2. *For any ample line bundle L on X,*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof. Let m, B and $\pi : X' \to X$ be as above. By **??**

$$H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) = 0$$

for $p + q > \dim X' = \dim X$.

$\pi : X' \to X$ is a finite morphism, so for $i > 0$, $R^i{=}_*0$ for any coherent sheaf. This implies

$$\begin{aligned}
&H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) \\
&= H^q(X, \pi_*(\pi^*(E \otimes \Omega_X^p(\log D) \otimes L))) \\
&= H^q(X, \pi_*(O_Y) \otimes E \otimes \Omega_X^p(\log D) \otimes L) = 0
\end{aligned}$$

for $p + q > \dim X$. The second equality follows from the projection formula.
As $\pi_*(O_Y) \cong \bigoplus_{i=0}^{m-1} O_X(-L^{\otimes i})$, the result follows. $\square$

[?] **[?]**