To end the proof of (2.21), we can add new generators and new relations to the above presentation of the group

$$\pi_1(\mathbb{C}^2, C) = \pi_1(\varphi^{-1}(D)),$$

namely,

$$a_{kp+j} = \beta^k a_j \beta^{-k} \qquad \text{for} \quad k \in \mathbb{Z}$$

and $0 \leq j < p$. Then the monodromy relation $h_\#(a_j) = a_j$ becomes

$$a_j = a_{j+q}$$

and this clearly ends the proof. □

(2.22) **Special Cases.**

(i) $(C, 0)$ is a node A_1, i.e., $p = q = 2$. Then

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle a_0, a_1 \colon a_0 a_1 = a_1 a_0 \rangle = \mathbb{Z}^2.$$

(ii) $(C, 0)$ is a cusp A_2, i.e., $p = 2$, $q = 3$ (recall the illustration in the proof above). Then

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle a_0, a_1 \colon a_0 a_1 a_0 = a_1 a_0 a_1 \rangle.$$

This group is thus isomorphic to the braid group $B_2(\mathbb{C})$ or to the group

$$\pi_{2,3} = \langle \alpha, \beta \colon \alpha^2 = \beta^3 \rangle$$

of the trefoil knot.

In fact, for any pair (p, q) such that $(p, q) = 1$, using (2.20) we get

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle \alpha, \beta \colon \alpha^p = \beta^q \rangle,$$

which is the same as the fundamental group of the torus knot of type (p, q) discussed in (2.1.6).

(iii) $(C, 0)$ is smooth and the line $y = 0$ is an inflectional tangent of order p, i.e., $q = 1$. Then

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle a_0, \ldots, a_{p-1} \colon a_0 = a_1 = \cdots = a_{p-1} \rangle \simeq \mathbb{Z}.$$

In all the above examples, the relations have been obtained by writing explicitly the corresponding monodromy relations $h_\#(a_j) = a_j$.

(2.23) **Remark.** Exactly the same computations as in (2.21) and (2.22) work in the local case, i.e., for a plane curve singularity $(C, 0)$ such that the line $\varphi^{-1}(0)$ is not contained in C. Of course, the monodromy relations in such a case might be much more difficult to work out. (The germ of the projection $\varphi \colon (\mathbb{C}^2 \setminus C, 0) \to (\mathbb{C}, 0)$ induces fibration over a smaller punctured disc at the origin of $\mathbb{C}$ and it is the monodromy of this local fibration which is meant here.)

§3. The van Kampen–Zariski Theorem

In this section we discuss a general method for finding a presentation of the fundamental group $G = \pi_1(\mathbb{P}^2 \setminus C)$ of a given (reduced) plane curve C.

First we consider the easiest part, namely, finding a set of generators for this group G. A special case of the Zariski theorem of Lefschetz type (1.6.5) is the following.

(3.1) **Proposition.** *For any hypersurface $V \subset \mathbb{P}^n$ and any line L in $\mathbb{P}^n$ intersecting V transversally and avoiding the singular part $S(V)$, there is an epimorphism*

$$\pi_1(L \setminus (V \cap L)) \to \pi_1(\mathbb{P}^n \setminus V)$$

induced by the inclusion.

Note that for such a line L, the intersection $V \cap L$ consists exactly of $d = \deg(V)$ points and hence by (2.16) the group $\pi_1(L \setminus (V \cap L))$ is a free group on $(d - 1)$-generators.

To following result describes the *behavior of the fundamental group with respect to degenerations* of curves.

(3.2) **Corollary.** *Let C_t, $t \in [0, \varepsilon]$, be a smooth family of plane curves in $\mathbb{P}^2$ such that:*

(i) *the family C_t for $t \in (0, \varepsilon]$ is equisingular;*
(ii) *the limit curve $C_0 = \lim_{t \to 0} C_t$ is a reduced curve.*

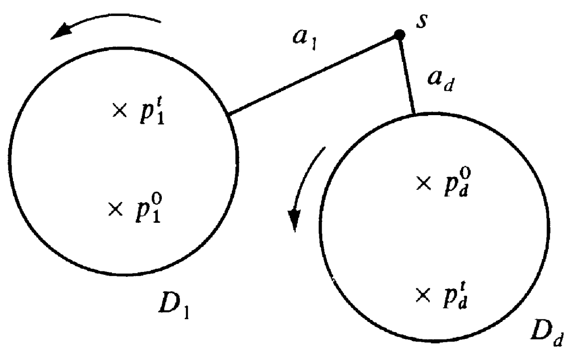
Then there is an epimorphism

$$\pi_1(\mathbb{P}^2 \setminus C_0) \to \pi_1(\mathbb{P}^2 \setminus C_\varepsilon).$$

Proof. We recall that a familly C_t, $t \in (0, \varepsilon]$, of plane curves is *equisingular* if the singular points of the curve C_t can be indexed as $a_1(t), \ldots, a_p(t)$ in such a way that all the families of singularities $(C_t, a_i(t))$ are μ-constant. Then, by our discussion in Chapter 1, §3, it follows that the topological type of the pair $(\mathbb{P}^2, C_t)$ is independent of t for $t \in (0, \varepsilon]$.

Consider now a "tubular neighborhood" T of the curve C_0 in $\mathbb{P}^2$. In other words, T is a small open neighborhood of the curve C_0, which retracts to C_0, see [Df6] and our discussion in Chapter 5, §2, below.

Choose a t such that $\varepsilon \gg t > 0$ and such that the curve C_t is contained in the neighborhood T. Let L be a generic line with respect to both curves C_0 and C_t. Then the intersection $L \cap C_0$ (resp. $L \cap C_t$) consists of d points $p_1^0, \ldots, p_d^0$ (resp. $p_1^t, \ldots, p_p^t$) where $d = \deg C_t = \deg C_0$. We can arrange that the intersection $L \cap T$ consists of d disjoint small discs $D_1, \ldots, D_d$, each of them containing a pair of points p_i^0 and p_i^t for some $i = 1, \ldots, d$.



Consider a system of paths $a_1, \ldots, a_d$ as drawn in the figure above, i.e., each of these paths is a path in the line L, starting from a base point $s \in \mathbb{P}^1 \setminus (C_0 \cup C_t)$ and going around one of the discs D_i for each $i = 1, \ldots, d$. It is clear that the classes of these loops generate both groups $\pi_1(\mathbb{P}^2 \setminus C_0)$ and $\pi_1(\mathbb{P}^2 \setminus C_t)$.

Consider the morphisms induced by the corresponding inclusions

$$\pi_1(\mathbb{P}^2 \setminus C_0) \xleftarrow[j_\#]{\sim} \pi_1(\mathbb{P}^2 \setminus T) \xrightarrow[k_\#]{} \pi_1(\mathbb{P}^2 \setminus C_t)$$

and note that $j_\#$ is an isomorphism since $\mathbb{P}^2 \setminus T$ is a deformation retract of $\mathbb{P}^2 \setminus C_0$. Since $\pi_1(\mathbb{P}^2 \setminus C_t) = \pi_1(\mathbb{P}^2 \setminus C_\varepsilon)$, it follows that $k_\# \circ j_\#^{-1}$ is the epimorphism that we are looking for. □

(3.3) **Exercise.** Show that the assumption C_0 reduced in (3.2) is crucial. *Hint.* Consider the family of conics in $\mathbb{P}^2$

$$Q_t\colon x(x + ty) = 0, \qquad t \in [0, 1).$$

For $t \neq 0$, we have $\pi_1(\mathbb{P}^2 \setminus Q_t) = \mathbb{Z}$, while $\pi_1(\mathbb{P}^2 \setminus Q_0) = 0$.

We also remark that the epimorphism in (3.1) does *not* exist for some special lines.

(3.4) **Example.** Consider the smooth conic

$$C\colon x^2 + yz = 0 \quad \text{in } \mathbb{P}^2$$

and the line $L\colon y = 0$, which is a tangent to C at the point $a = (0:0:1)$.

Since $C \cap L = \{a\}$, it follows that the group $\pi_1(L \setminus C \cap L) = \pi_1(\mathbb{C}) = 0$ has no epimorphism onto the group

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/2\mathbb{Z}.$$

There is, however, a *larger class* of lines L for which (3.1) holds than stated there. Let $p \in \mathbb{P}^2$ be a point and let $\mathbb{P}^1 \subset \mathbb{P}^2$ be a line which does not contain the point p. Consider the projection

$$\varphi\colon \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$$

with center p. Namely, for each point $x \in \mathbb{P}^2 \setminus \{p\}$, $\varphi(x)$ is the unique intersection point of the line $\overline{px}$ with the fixed line $\mathbb{P}^1$. We can use the points $a \in \mathbb{P}^1$ to parametrize the set of lines in $\mathbb{P}^2$ passing through the point p: the line $\varphi^{-1}(a) \cup \{p\}$ is denoted in the sequel by L_a.

We say that a line L_a through p is *exceptional* with respect to a curve C if it satisfies at least one of the following conditions:

(i) L_a is tangent to the curve C;
(ii) L_a passes through a singular point q of the curve $C, q \neq p$.

Here we say that a line L is tangent to the curve C at a point a if we have

$$(L, C)_a > \mathrm{mult}_a(C).$$

In particular, note that in (i) above the tangent point of the line L_a with the curve C may be the fixed point p. Let B be the union of all these exceptional lines through p with respect to the curve C and let $a_1, \ldots, a_s$ be the corresponding points in $\mathbb{P}^1$.

(3.5) **Lemma.** *With the above notations, the restriction*

$$\varphi\colon \mathbb{P}^2 \setminus (C \cup B) \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_s\}$$

is a locally trivial fibration.

Proof. Let $Z = B_p(\mathbb{P}^2)$ be the blow-up of the projective plane $\mathbb{P}^2$ at the point p, see, for instance, [D4], p. 170. Let $\pi\colon Z \to \mathbb{P}^2$ be the canonical projection and let $E = \pi^{-1}(p)$ be the exceptional divisor. Let $\overline{\varphi} = \varphi \circ \pi\colon Z \setminus E \to \mathbb{P}^1$ and note that $\overline{\varphi}$ extends in an obvious way to a map

$$\tilde{\varphi}\colon Z \to \mathbb{P}^1.$$

Let $\tilde{C}, \tilde{B} \subset Z$ be the proper transforms of the curves C and B, respectively, and set

$$Z_0 = Z \setminus \tilde{B}, \qquad Z_\infty = Z_0 \cap E, \qquad \tilde{C}_0 = \tilde{C} \cap Z_0.$$

Then the restriction

$$\tilde{\varphi}\colon Z_0 \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_s\}$$

is a proper submersion and induces a locally trivial fibration of the pair $(Z_0, \tilde{C}_0 \cup Z_\infty)$ over $\mathbb{P}^1 \setminus \{a_1, \ldots, a_s\}$ by (1.3.1). Since

$$\mathbb{P}^2 \setminus (B \cup C) = Z_0 \setminus (\tilde{C}_0 \cup Z_\infty),$$

this ends the proof of (3.5). □

(3.6) **Corollary.** *Asssume that no exceptional line in B is a component of the curve C. The for any line L_a through p which is not exceptional, there is an epimorphism*

$$\pi_1(L_a \setminus C) \to \pi_1(\mathbb{P}^2 \setminus C)$$

induced by the inclusion.

In other words, given a curve C and a point p in the projective plane $\mathbb{P}^2$, the result (3.1) holds for any line through the point p which is generic with respect to the curve C (even when $p \in C$).

Proof. The space $F = L_a \setminus C$ is exactly the fiber of the fibration φ considered in (3.5). The exact homotopy sequence of a fibration implies

$$\pi_1(F) \xrightarrow{j_\#} \pi_1(\mathbb{P}^2 \setminus (C \cup B)) \xrightarrow{\varphi_\#} \pi_1(\mathbb{P}^1 \setminus A) \to 1,$$

where $A = \{a_1, \ldots, a_s\}$ and j is the corresponding inclusion.

Hence a system of generators for the group $\pi_1(\mathbb{P}^2 \setminus (C \cup B)$ can be obtained as follows:

(i) take any generators $g_1, \ldots, g_m$ for $\pi_1(F)$ and consider the elements $\bar{g}_i = j_\#(g_i)$ for $i = 1, \ldots, m$;
(ii) consider a system of generating loops $\alpha_1, \ldots, \alpha_s$ for $\pi_1(\mathbb{P}^1 \setminus A)$ as in (2.16) and lift then to some loops $\bar{\alpha}_1, \ldots, \bar{\alpha}_s$ in $\pi_1(\mathbb{P}^2 \setminus (C \cup B))$.

The morphism

$$\pi_1(\mathbb{P}^2 \setminus (C \cup B)) \xrightarrow{k_\#} \pi_1(\mathbb{P}^2 \setminus C)$$

induced by the inclusion is an epimorphism, by a transversality argument similar to the one in the proof of (1.1). Moreover, $k_\#(\bar{\alpha}_j) = 0$ for any loop $\bar{\alpha}_j$, since this loop goes once around the line L_{a_j} and becomes trivial when this line is added back to the space $\mathbb{P}^2 \setminus (C \cup B)$. It follows that the elements

$$k_\#(\bar{g}_i) \qquad \text{for} \quad i = 1, \ldots, m,$$

generate the group $\pi_1(\mathbb{P}^2 \setminus C)$, i.e., the morphism

$$k_\# \circ j_\#$$

is an epimorphism. □

(3.7) **Exercise.** Let the curve C consist of d lines, all passing through a point p. Compute the group $\pi_1(\mathbb{P}^2 \setminus C)$ and compare the result to (3.6).

(3.8) **Corollary.** *Let C be an irreducible curve of degree d such that there is a point $p \in C$ with*

$$\mathrm{mult}_p(C) = d - 1.$$

Then the group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.

Proof. Use (3.6) and note in this case that $L_a \setminus C \simeq \mathbb{C}^*$. The result (3.8) can be informally stated as follows: if an irreducible curve C has a "very bad" singularity (i.e., a singularity having the highest possible multiplicity), then the corresponding group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. □

The next result shows that the same conclusion holds when the irreducible curve C has just one singular point which is not very degenerate, in the following precise sense. First we give a local definition.

(3.9) **Definition.** Let $X\colon g = 0$ be an IHS at the origin of $\mathbb{C}^{n+1}$. Let m denote the maximal ideal in the ring $\mathcal{O}_{n+1}$ of germs of analytic functions at the origin. The *μ-constant determinacy order* of the singularity $(X, 0)$, denoted by $\mu\text{-det}(X, 0)$, is the smallest positive integer s such that the family

$$g_t = g + th, \qquad t \in [0, \varepsilon),$$

is μ-constant for any germ $h \in m^s$ and $\varepsilon > 0$ small enough (here ε may depend on h).

(3.10) **Example.** Let $X\colon g = 0$ be a plane curve singularity which is an *ordinary r-fold* point, see [Hn], p. 38, for a definition. Using the formula for the Milnor number of a semiweighted homogeneous singularity, recall (3.1.19), we deduce that $\mu\text{-det}(X, 0) = r$.

(3.11) **Proposition.** *Assume that the irreducible curve C of degree d has just one singular point p and that*

$$\mu\text{-det}(C, p) < d.$$

Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.

Proof. Let $m = \operatorname{mult}_p(C)$ and let $L_1, \ldots, L_s$ be all the exceptional lines through p. Let $\mathbb{P}^1$ be a line in $\mathbb{P}^2$ such that:

(i) $\#(\mathbb{P}^1 \cap C) = d$, i.e., $\mathbb{P}^1$ is transverse to C;
(ii) $\mathbb{P}^1$ does not pass through any of the intersection points in $L_i \cap C$, $i = 1, \ldots, s$. Choose a linear system of coordinates $(x : y : z)$ on $\mathbb{P}^2$ such that:

(α) $p = (0 : 0 : 1)$;
(β) $z = 0$ is an equation for $\mathbb{P}^1$.

In the affine coordinate chart $\mathbb{C}^2$ given by $z = 1$ the curve C is given by an equation

$$f = f_m(x, y) + \cdots + f_{d-1}(x, y) + f_d(x, y) = 0,$$

where f_k denotes the homogeneous component of degree k in this equation.

Moreover, we have the following:

$$(3.12) \qquad \begin{cases} \text{(i) } f_m \neq 0; \\ \text{(ii) } f_d \text{ has no multiple factors;} \\ \text{(iii) } (f_d, f_{d-1}) = 1. \text{ In particular, } f_{d-1} \neq 0. \end{cases}$$

Indeed, (i) and (ii) are obvious and to prove (iii) we can proceed as follows. Consider the line

$$L = \{(tx, ty), t \in \mathbb{C}\} \qquad \text{for some} \quad (x, y) \in \mathbb{C}^2 \setminus \{0\}.$$

If the equations $f_{d-1} = 0$ and $f_d = 0$ had (x, y) as a common root, then the corresponding line L would have a multiple intersection point with C on the line at infinity $z = 0$. But this is a contradiction to our choice of this line at infinity $\mathbb{P}^1$.

Consider now the family of curves $\bar{C}_t$ in $\mathbb{P}^2$ having the following affine equation

$$\bar{C}_t\colon f_t = t(f_m + \cdots + f_{d-2}) + f_{d-1} + f_d = 0$$

for $t \in \mathbb{C}$. Using (3.12(iii)), we find out that there is a finite set $B \subset \mathbb{C}$ such that:

(i) $\bar{C}_t$ is an irreducible curve for $t \in \mathbb{C}\setminus B$;
(ii) $\bar{C}_t$ has exactly one singular point, namely, p for $t \in \mathbb{C}\setminus B$; and
(iii) $\{0, 1\} \subset \mathbb{C}\setminus B$.

(*Hint*. In fact, we first prove (iii) and then use standard semicontinuity arguments from algebraic geometry.)

Consider now the function

$$\mu\colon \mathbb{C}\setminus B \to \mathbb{Z}, \qquad \mu(t) = \mu(\bar{C}_t, p),$$

the Milnor number of the singularity $(\bar{C}_t, p)$. Since μ is an upper semicontinuous function in Zariski topology, there is a finite set $B^1 \subset \mathbb{C}\setminus B$ such that the function μ takes its minimal value on the Zariski open set $\mathbb{C}\setminus(B \cup B^1)$. Using the condition $\mu\text{-det}(C, p) < d$, it follows that for all t close enough to 1, the value $\mu(\bar{C}_t, p)$ is equal to $\mu(C, p)$. In other words, $1 \in \mathbb{C}\setminus(B \cup B^1)$. Choose now a path $\gamma\colon [0, 1] \to \mathbb{C}$ such that $\gamma(0) = 0$, $\gamma(1) = 1$, and $\gamma((0, 1]) \subset \mathbb{C}\setminus(B \cup B^1)$. If we define $C_t = \bar{C}_{\gamma(t)}$ for $t \in [0, 1]$, then the family of plane curves C_t has all the properties required to apply (3.2), with $\varepsilon = 1$. To end the proof, we use (3.8). □

(3.13) **Remark.** It is not true that *any* irreducible curve C having just one singularity of multiplicity $m < d - 1$ can be deformed into an irreducible curve C_0 having a singularity of multiplicity $d - 1$. Consider, for instance, the quartic curve

$$C\colon x^2z^2 + 2xy^2z + y^4 - x^3y = 0.$$

Then C has just one singular point, namely $p = (0:0:1)$, which is a singularity of type A_6.

A list of all the isolated singularities of the form

$$C_0\colon f_3(x, y) + f_4(x, y) = 0,$$

and such that the curve C_0 is irreducible, gives the possibilities D_4, D_5, and E_6, see [BG], p. 273. But the singularity A_6 cannot be deformed into any of these three singularities, for obvious reasons.

(3.14) **Remark.** An example of an irreducible curve $C \subset \mathbb{P}^2$, having just one singular point and such that the group $\pi_1(\mathbb{P}^2\setminus C)$ is not abelian, is given below in (4.21).

Consider again the fibration φ from (3.5), but this time with the point p *not* situated on the curve C. The fundamental group $\pi_1(\mathbb{P}^1\setminus\{a_1, \ldots, a_s\})$ is free with $s - 1$ generators $\gamma_1, \ldots, \gamma_{s-1}$, where γ_i is a loop based at a base point $a \in \mathbb{P}^1 \setminus \{a_1, \ldots, a_s\}$ and goes once around the point a_i for $i = 1, \ldots, s - 1$ as in (2.16). Each loop γ_i induces a monodromy automorphism

$$T_i\colon \pi_1(F) \to \pi_1(F) \qquad \text{where} \quad F = L_a \setminus C.$$

Since $L_a \cap C$ consists exactly of $d = \deg(C)$ points, it follows that $\pi_1(F)$ is a free group on $d - 1$ generators $g_1, \ldots, g_{d-1}$, which may be chosen as in (2.16).

It follows that the automorphism T_i is completely determined by giving the elements

$$T_i(g_j) \qquad \text{for} \quad j = 1, \ldots, d - 1.$$

If $[\beta]$ denotes the image in $G = \pi_1(\mathbb{P}^2 \setminus C)$ of an element $\beta \in \pi_1(F)$ under the morphism induced by the inclusion, it follows easily that we have the *monodromy relations*

$$[T_i(g_j)] = [g_j] \qquad \text{for all} \quad j = 1, \ldots, d - 1 \quad \text{and} \quad i = 1, \ldots, s - 1.$$

The proof of (2.21) suggests that these monodromy relations are the only ones (i.e., any other relations can be deduced from them). This is exactly the content of the following famous result, conjectured by O. Zariski [Z1] and proved by van Kampen [vK]. For a modern proof we refer to Cheniot [Ch1] and [Ch2].

(3.15) **Theorem** (van Kampen–Zariski). *The group $G = \pi_1(\mathbb{P}^2 \setminus C)$ admits the following presentation*:

$$G = \langle g_j, j = 1, \ldots, d - 1\colon T_i(g_j) = g_j, i = 1, \ldots, s - 1, j = 1, \ldots, d - 1 \rangle.$$

(3.16) **Remark.** Assume that the exceptional line L_{a_i} intersects the curve C *nontransversally* only at a point q and let $m = (C, L_{a_i})_q > 1$. As a generic line L_a through p moves toward L_{a_i}, m point in the intersection $L_a \cap C$ will collapse to the point q.

It is clear that among the monodromy relations $T_i(g_j) = g_j$, only those associated with loops g_j around one of these m points may be nontrivial. Indeed, outside a small ball centered at q, the fibration φ is trivial over D, where D is a small disc centered at a_i in $\mathbb{P}^1$. In other words, the interesting relations coming from the monodromy around the exceptional line L_{a_i} can be *localized* at the point q, recall (2.23). It follows that in general the number of nontrivial relations is much less than $(s - 1)(d - 1)$.

Lastly, we present two consequences of Theorem (3.15), both of them already discussed in Zariski's classical paper [Z1].

(3.17) **Corollary.** *Let C be a curve which has an inflectional tangent of order $d = \deg(C)$. Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. In particular, if C is smooth curve, then*

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/d\mathbb{Z}.$$

Proof. The assumption means that there is a line L_0 in $\mathbb{P}^2$ such that $L_0 \cap C = \{a\}$, and the point a is a smooth point on the curve C. Let p be a point in $L_0 \setminus C$ and apply (3.15). As remarked in (3.16), the monodromy relations obtained by going around the special line L_0 are the same as in (2.22(iii)). Hence all the generators of $\pi_1(\mathbb{P}^2 \setminus C)$ are equal, which implies that this group is abelian.

To prove the statement about smooth curves, first we recall that for any two smooth curves C and C^1—both of degree d—their complements $\mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus C^1$ are topologically equivalent, see (1.3.4). Then note that the curve

$$C^1: x^d + xy^{d-1} + z^d = 0$$

is smooth and the line $L_1: x = 0$ is an inflectional tangent of order d for it. □

(3.18) **Corollary** (Compare to (1.13)). *Let $C \subset \mathbb{P}^2$ be a nodal curve (i.e., all the singularities of C are nodes A_1). Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.*

Proof (Along the lines proposed by Zariski [Z1]). Assume first that C consists of d lines, namely, $L_1, \ldots, L_d$, meeting only two at a time. Then the generator g_j of $G = \pi_1(\mathbb{P}^2 \setminus C)$ corresponds to a loop in L_a around the intersection point $L_a \cap L_j$ for $j = 1, \ldots, d-1$. Let L_{ij} be the exceptional line joining the point p with the intersection point $L_i \cap L_j$ for $1 \leq i < j \leq d-1$. Then the monodromy relations obtained by going around the line L_{ij} imply $g_i g_j = g_j g_i$, use (3.16) and (2.22(i)). Hence all the generators commute with each other and G is an abelian group.

To treat the general case of a nodal curve C, we use the Severi remark (for a complete proof, see Harris [Hs]) that any nodal curve can be deformed into a union of lines, meeting only two at a time. Then the result follows from the first part of this proof via (3.2). □

(3.19) **Remark.** One may try to obtain a "degenerate van Kampen–Zariski theorem" similar to (3.15) in the situation where the center of projection p is a point *on* the curve C. If $m = \operatorname{mult}_p(C)$, then we get a system of generators for the group G having $d - m - 1$ elements from (3.6) (we have to take p not on a linear component of the curve C, if any exists). Note also that the *monodromy relations* obtained by going round the exceptional lines through p are still relations among these $(d - m - 1)$ generators.

The major drawback of this approach is that the monodromy relations do *not* form a complete set of relations necessary for a presentation of the group G, see (3.20) for a simple example. However, in many cases, we can use this approach to derive valuable information on the group G, see (3.21) for an example.

(3.20) **Example.** Consider the cuspidal cubic $C: xy^2 - z^3 = 0$ and let $p = (1:0:0)$. Then there is just one exceptional line. It follows that the fibration φ from (3.5) is trivial in this case (the base space $\mathbb{P}^1 \setminus \{a_1\}$ is contractible). Hence the "degenerate van Kampen–Zariski theorem" gives us in this case one generator g and no relation. On the other hand, we have

$$G = \langle g \colon g^3 = 1 \rangle$$

from (3.8).

(3.21) **Exercise.** Let $C \subset \mathbb{P}^2$ be a curve of degree d and let $p \in C$ be a point with $\mathrm{mult}_p(C) = d - 2$. Assume that one of the following two conditions is fulfilled:

(i) The curve C has a node q, $q \neq p$, and no line through p is an irreducible component for C;
(ii) The curve C is irreducible, $C \setminus \{p\}$ is smooth, and the tangent cone to the singularity (C, p) consists of exactly one line.

Then the fundamental group $G = \pi_1(\mathbb{P}^2 \setminus C)$ is abelian. *Hint.* Apply the "degenerate van Kampen–Zariski theorem" to the curve C and the center of projection p. From (2.6) we get that G can be generated by two elements g_1 and g_2. In case (i), use (3.16) and (2.22(i)) to get $g_1 g_2 = g_2 g_1$. In case (ii), consider the restriction

$$\psi \colon C \setminus \{p\} \to \mathbb{P}^1$$

of the projection with center p. Since im $\psi = \mathbb{C}$ is simply-connected, ψ cannot be a covering map. This implies the existence of a tangent to the curve C at a point q, passing through p. Use (3.16) and (2.22(iii)) to deduce $g_1 = g_2$. In both cases (i) and (ii), the monodromy relations obtained are sufficient to imply that G is abelian.

§4. Two Classical Examples

In this section we discuss two examples due to Zariski [Z1]. They present the simplest curves $C \subset \mathbb{P}^2$ whose fundamental groups $\pi_1(\mathbb{P}^2 \setminus C)$ are *not* abelian.

(4.1) **Exercise.** (i) List all the plane curves of degree 1, 2, and 3 up to linear equivalence. For the degree 3 case, we can have a look at the list in [D4], p. 51.

(ii) Show that all the fundamental groups $G = \pi_1(\mathbb{P}^2 \setminus C)$ are abelian for $\deg(C) \leq 3$, except in the case where C consists of three concurrent lines.

(iii) Compute all the groups G for $\deg(C) \leq 3$.

Passing to the plane curves of degree 4, let us recall that there are irreducible quartic curves having as singularities three cusps A_2, e.g., the following "normal form":

(4.2) $$C \colon f = x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x + y + z) = 0,$$

see [BG] for details and (4.7).

The following result says that all the other irreducible quartics are not interesting for our discussion.

(4.3) **Proposition.** *If $C' \subset \mathbb{P}^2$ is an irreducible quartic curve, not a three cuspidal quartic, then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C')$ is abelian.*

Proof. When C' is smooth, we can use (3.17). When C' has a point with multiplicity 3, then use (3.8). Assume now that all the singular points of the curve C' have multiplicity 2, i.e., they are A_k-singularities, for some integers $k \geq 1$. If there is an A_1-singularity, the result follows from (3.11) (if this is the only singularity of C') or from (3.21(i)) (if there are some other singularities). If C' has just one singular point, we can use (3.21(ii)). Hence we can assume that C' has at least two singularities and each of them is of type A_k for some $k \geq 2$.

Let C be an irreducible curve of degree d having m singular points $a_1, \dots, a_m$. The following simple formula for the Euler characteristic of the curve C is a special case of formula (5.4.4), which is proved in Chapter 5.

$$\chi(C) = 2 - (d-1)(d-2) + \sum_{i=1,m} \mu(C, a_i). \tag{4.4}$$

Since C is irreducible, it follows that $b_0(C) = b_2(C) = 1$, and hence $\chi(C) \leq 2$. This implies the following *upper bound* for the sum of the Milnor numbers of the singularities which may occur on an irreducible curve of degree d in $\mathbb{P}^2$

$$\sum_{i=1,m} \mu(C, a_i) \leq (d-1)(d-2). \tag{4.5}$$

In our case, we have $(d-1)(d-2) = 6$ since $d = 4$. From our assumption on C', it follows that either C' is a three cuspidal quartic or C' has exactly two singularities, say a_1 and a_2.

In the second case, we may assume

$$\mu(C', a_1) \leq \mu(C', a_2). \tag{4.6}$$

We have to show that in this latter case the group $\pi_1(\mathbb{P}^2 \setminus C')$ is abelian. Choose coordinates $(x:y:z)$ on $\mathbb{P}^2$ such that $a_1 = (0:0:1)$ and C' has an equation

$$Az^2 + 2Bz + C = 0,$$

where A, B, and C are binary forms in x and y of degrees 2, 3, and 4, respectively, and $A = x^2$. As in [BG], we consider the discriminant

$$\Delta = B^2 - AC.$$

The map $\psi\colon C^1 \setminus \{a_1\} \to \mathbb{P}^1, (x:y:z) \mapsto (y:z)$ is a ramified covering. Its ramification locus

$$R = \{(x:y) \in \mathbb{P}^1;\ \#\psi^{-1}(x, y) < 2\}$$

is given by $x = 0$ or $\Delta(x, y) = 0$. It is easy to see that a root $(\alpha : \beta)$ with $\alpha \neq 0$ of the equation $\Delta = 0$ corresponds to:

(i) a singularity of C' of type A_k at the point

$$q = (\alpha : \beta : -B(\alpha, \beta)/2\alpha^2),$$

when the multiplicity of the root $(\alpha : \beta)$ is $k + 1 \geq 2$; or