Let X be a smooth projective variety over $\mathbb{C}$ of dimension n; Let D be a simple normal crossing divisor on X; Let V be a unitary local syetem on $U:=X-D$; Let E be its canonical extension. Let $\mathrm{DR}(D, \nabla, E)$ be the de Rham complex

$$0 \longrightarrow E \xrightarrow{\nabla} E \otimes \Omega_X(\log D) \xrightarrow{\nabla} E \otimes \Omega_X^2(\log D) \xrightarrow{\nabla} \cdots$$

$$\cdots \xrightarrow{\nabla} E \otimes \Omega_X^n(\log D) \longrightarrow 0$$

For $i = 1, \cdots, n$, let $\mathrm{DR}^i(D, \nabla, E)$ denote the i-th node of the complex $\mathrm{DR}(D, \nabla, E)$. Our main theorem is

Theorem 1. *Let be an ample line bundle, then*

$$H^q(X, DR^p(D, DR^p(D, \nabla, E) \otimes L) = 0$$

for $p + q > n$

To prove it, we will

1. Put an increasing filtration (weight filtration) $W_\cdot$ on $\mathrm{DR}(D, \nabla, E)$ where W begins at 0 and ends at n.
2. Show Theorem 1, with $\mathrm{DR}^p(D, \nabla, E)$ replaced by $W_0(\mathrm{DR}^p(D, \nabla, E)$
3. Show Theorem 1, with $\mathrm{DR}^p(D, \nabla, E)$ replaced by $W_m/W_{m-1}\mathrm{DR}^p(D, \nabla, E)$, for $i = 1, \cdots, n$
4. Take cohomology sequence of

$$0 \to W_{m-1} \to W_m \to W_m/W_{m-1} \to 0$$

for $i = 1, \cdots, n$. Then, bootstrap.

The weight filtration $W_\cdot$ has been studied in [2], we will review it in the first two sections.

1 Residue map

Let D_m be the union of m-fold intersection of components of D, for $m = 1, cdots, n$; Let $\tilde{D}_m$ be the disjoint union of components of D_m; Let $v_m : \tilde{D}_m \to X$ be the composition of projection onto D_m and the inclusion map. $\tilde{C}_m := v_m^* D_{m+1})$ is either empty or a normal
For each $m \leq p \leq \dim D_m$, there exists a residue map crossing divisor in $\tilde{D}_m$.

$$\mathrm{Res}_m : \Omega_X^p(\log D) \to v_{m*}(\Omega_{\tilde{D}_m}^{p-m})$$

This map is defined as the following:

Let D_{m1} be one component of D_m, as the intersection of $D_{i1}, \cdots, D_{im}$. Then, the map Res_m sends dz_i/z_i to 1 if i appears in $i_1, \cdots, i_m$, and Res_m sends all other 1-form to 0. This map is well-defined independent of the chosen coordinate. Futhermore, this map commutes with exterior derivative d, making it a homomorphism of complexes

$$\mathrm{Res}_m : \Omega^{\cdot}_X(\log D) \to v_{m*}\Omega^{\cdot}_{\tilde{D}_m}(\log \tilde{C}_m)[-m]$$

Theorem 2. *[2]*

1. *$V_m := j_*V|_{D_m - D_{m+1}}$ is a unitary local system on $D_m - D_{m+1}$.*
2. *There exist a unique subvectorbundle E_m of E and a unique holomorphic integrable connection ∇_m on E_m with logarithmic poles along D_{m+1} such that*
$$\ker \nabla_m|_{\tilde{D}_m - \tilde{C}_m} = v_m^{-1} V_m$$
3. *There exists a unique subvectorbundle E_m^* of E with*
$$E_m \oplus E_m^* = E$$

Proof. 1. Cover X by polydisk open sets U_i the sheaf j_*V on X can be recovered by $j_*V|_{U_i}$ and the gluing morphisms

$$\phi_{ij} : j_*V_{U_i \cap U_j} \to j_*V_{U_j \cap U_i}$$

As V is locally constant on U, for each point $x \in D_m - D_{m-1}$, the stalk of j_*V at x is the same as the stalk to $j_*V|_{D_m - D_{m-1}}$. Therefore, to construct $j_*V|_{D_m - D_{m-1}}$, we can construct $j_*V|_{W_i}$, blue where $W_m = U_m \cap (D_m - D_{m-1})$, then glue $j_*V|_{W_i}$ using ϕ_{ij}.
Now, let $x \in D_m - D_{m-1}$ and let $\Delta = \Delta_1 \times \cdots \times \Delta_n$ be a polydisk open neighorhood in X such that D is union of coordinate hyperplanes. Choose Δ_i in the way so that D is defined as

$$z_1 \times z_2 \cdots \times z_s = 0$$

blue where z_i are coordinates on D_i.
The local system V on $\Delta \cap U$ is equivalent to an unitary representation

$$T : \pi_1(\Delta - D) \to \mathrm{GL}(r, \mathbb{C})$$

As $\pi_1(\Delta - D)$ is Abelian and T is unitary, we can simultaneously diagonalize all $T(\gamma_i)$, blue where γ_i form a generating set of $\pi_1(\Delta - D)$(see Appendix 1).
Therefore, we can assume that V is the direct sum of rank 1 local system on $\Delta - D$. Write

$$V = V^1 \oplus V^2 \oplus \cdots \oplus V^r$$

For each V^i, $\gamma_j \in \pi_1(\Delta - D)$ acts on it by $\lambda_{i,j}$. Therefore, V^i extends to D_j if and only if $\lambda_{i,j} = 1$.

Suppose $x \in D_{j1} \cap \cdots \cap D_{jm}$, then near x, V_m is

$$\bigoplus_{\lambda_{i,j1}=\lambda_{i,j2}=\cdots=\lambda_{i,jm}} V^i$$

This shows that V_m is a local system. The unitariness of V_m is clear.

2. The uniqueness of the subvectorbundle E_m follows from the uniqueness of canonical connection. Therefore, we can show its exitences locally. Use the notation from part 1, and assume V decomposes as direct sum of rank 1 unitary local system V^i. Let E^i be the canonical connection of V^i. Then, it is clear that

$$E_m := \bigoplus_{\lambda_{i,j1}=\lambda_{i,j2}=\cdots=\lambda_{i,jm}} E^i$$

is the canonical extension of V_m.

3. E inherits a flat Hermitian from V. Define E_m^* as the complement of E_m with respect to this metric. On Δ, E_m^* is the direct sum of E^i not appearing in the definition of E_m. Therefore, E_m^* is a bundle. $\square$

Remark 1. *$\tilde{E}_m$ could have different dimension on different component of $\tilde{D}_m$.*

Consider the following variation of the residue map Res_m

$$\begin{aligned}\mathrm{Res}_m(E) : \Omega_X^p(\log D) \otimes E &\xrightarrow{\mathrm{Res}_m \otimes \mathrm{id}} v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m)) \otimes E \\ &= v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes v_m^* E) \\ &\xrightarrow{\mathrm{id} \otimes p_m} v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes \tilde{E}_m)\end{aligned}$$

where $p_m : v_m^* E \to \tilde{E}_m$ is the projection onto the $\tilde{E}_m$ component.

Lemma 1. *[2] $\mathrm{Res}_m(E) \circ \nabla = \nabla_m \circ \mathrm{Res}_m(E)$,* i.e. *$\mathrm{Res}_m(E)$ is homomorphism of complexs*

$$DR(D, E, \nabla) \to v_{m*} DR(\tilde{C}_m, E_m, \nabla_m)[-m]$$

2 Weight filtration on DR(D, ∇, E)

The weight filtration W_m on DR(D, E, ∇) is defined as

$$W_m(\mathrm{DR}(D, E, \nabla)) = \ker \mathrm{Res}_{m+1}(E) \text{ if } m \geq 0$$
$$W_m(\mathrm{DR}(D, E, \nabla)) = 0 \text{ if } m < 0$$

We give some local description of $W_m(\mathrm{DR}(D, E, \nabla))$. Let $\Delta = \Delta_1 \otimes \cdots \otimes \Delta_n$ be a polydisk of X with coordinate $z_1, \cdots, z_n$. Suppose D is defined as

$$z_1 \times \cdots \times z_s = 0$$

As in part 1 of Theorem 2, we assume V is the direct sum of rank 1 unitary local systems on Δ, and write

$$V = V^1 \oplus \cdots \oplus V^r$$

Definition 1. *We say $\frac{dz_j}{z_j}$ acts on V^i by identity if $\lambda_{i,j} = 1$, i.e. the monodromy of V^i by a small circle around D_j is the identity.*

Let E^i be be canonical extension of V^1 on Δ;
Let μ_i be a section of E^i, then

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p} \otimes \mu_i$$

is in W_m if and only if there are at most m log forms acting on V^i by identity.

Proposition 1. *[2]*

1. *$W_.(DR(D, E, \nabla))$ is an increasing filtration.*
2. *$Res_m(E)$ induces an isomorphism*

$$Gr_m^W(DR(D, \nabla, E)) \to v_{m*}(W_0(DR(\tilde{C}_m, \tilde{E}_m, \tilde{\nabla}_m))[-m])$$

Proof. The statements are local. We can assume X is a polydisk and V is a unitary local system of rank 1.

1. From the local description of $W_m(\mathrm{DR}(D, E, \nabla))$, it is clear that $W_.$ is an increasing filtration.

2. Let s be a section $W_m(\mathrm{DR}(D, E, \nabla))$. Use the local description above, s is of the form

$$\omega \otimes \mu$$

where

$$\omega = \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p}$$

and ω has at most m log 1-forms acting on V by identity. μ is a generating section of E.
First, we show $\mathrm{Res}_m(E)(s) \in W_0(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes E_m$.

$$\mathrm{Res}_m(E)(s) = \mathrm{Res}_m(\omega) \otimes \mu_m$$

By the construction of ω, $\mathrm{Res}_m(\omega)$ does not log form

$$\frac{dz_j}{z_j}$$

where γ_j acts on V by identity. This shows that

$$\mathrm{Res}_m(E)(s) \in W_0(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes E_m)$$

If $\omega_0 \otimes \mu_m \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m)$, to get a preimage in $W_m(\Omega^p_X(\log D) \otimes E)$, simply take

$$\omega_m \wedge \omega_0 \otimes \mu$$

where ω_m is any m-form. And $\omega_m \wedge \omega_0 \otimes \mu \in W_m(\Omega^p_X(\log D) \otimes E)$ by the construction of ω_0. This shows that

$$\text{Res}_m(E) : W_m(\text{DR}(D, E, \nabla)) \to W_0(\text{DR}(\tilde{C}_m, \tilde{E}_m, \tilde{\nabla}_m))$$

is surjective.
If $\text{Res}_m(E)(s) = 0$, that means in ω, there are at most $m-1$ log forms acting on V by identity. This is precisely the description of local sections in $W_{m-1}(\Omega^p_X(\log D) \otimes E)$. ☐

$W_0(\text{DR}_D(D, E, \nabla))$ was studied by Timmerscheidt in the Appendix D of [1].

Theorem 3. *[1]*

1. $W_0(\mathit{DR}(D, E, \nabla))$ *is a resolution of* $j_* V$.
2. *The spectral sequence*

   $$E_1^{p,q} := H^q(X, W_0^p(\mathit{DR}(D, E, \nabla))) => \mathbb{H}^{p+q}(X, j_* V)$$

   degenerates at E_1.
3. *There exists a conjugate linear isomorphism*

   $$H^q(X, W_0^p(\mathit{DR}(D, E, \nabla))) \cong H^p(X, W_0^q(\mathit{DR}(D, N, \nabla)))$$

   where N *is the canonical extension of* $V^\vee$.

Proposition 2. *[1](1.5) Suppose* U *is an affine manifold. Let* $\mathcal{L}$ *be any local system on* U*. Then*

$$H^k(U, \mathcal{L}) = 0$$

for $k > \dim X$.

Therefore, back in our situation

Corollary 1. *Suppose* $U := X - D$ *is affine. Then*

$$H^q(X, W_0^p(\mathit{DR}(D, E, \nabla))) = 0$$

for $p + q > \dim X$.

Proof. By Theorem 3

$$H^k(U, V) = \mathbb{H}^k(X, \text{DR}(D, E, \nabla)) = \bigoplus_{p+q=k} H^q(X, \text{DR}^p(D, E, \nabla))$$

By Propoistion 2, the result follows. ☐

Theorem 4. *Suppose $D_m - D_{m+1}$ is affine for $m = 1, \cdots, \dim X$. Then*

$$H^q(X, DR^p(D, E, \nabla)) = 0$$

for $p + q > \dim X$.

Proof. Consider the weight filtration $W_\cdot$ on $\mathrm{DR}(D, E, \nabla)$, we show that for all $m \geq 0$,

$$H^q(X, W_m^p) = 0$$

for $p + q > \dim X$.
For $m = 1$, consider the exact sequence

$$0 \to W_0 \to W_1 \to W_1/W_0 \to 0$$

by Theorem 1,

$$W_1/W_0 \cong W_0(\mathrm{DR}(\tilde{C}_1, \tilde{E}_1, \tilde{\nabla}_1))[-1]$$

Take cohomology sequence of the above exact sequence, one gets

$$\cdots \to H^q(X, W_0^p) \to H^q(X, W_1^p) \to H^q(\tilde{D}_1, W_0^{p-1}(\mathrm{DR}(\tilde{C}_1, \tilde{E}_1, \tilde{\nabla}_1))) \to \cdots$$

Therefore, by Corollary 1, one has

$$H^q(X, W_1^p) = 0$$

for $p + q > \dim X$.
Repeat this process for all $m = 2, 3, \cdots$ □

3 Vanishing Theorem

Lemma 2. *Suppose B is a smooth divisor transversal to D. Then, there is short exact sequence*

$$0 \to \Omega_X^p(\log D + B) \otimes O_X(-B) \xrightarrow{i} \Omega_X^p(\log D) \xrightarrow{r} \Omega_B^p(\log D \cap B) \to 0$$

where i is the inclusion map, and r is the restriction map.

Proof. For simplicity, we prove the case for $p = 1$. We may also assume X is affine. Let $X = \mathrm{Spec}A$, and let $f_1, \cdots, f_s$ be the regular sequence corresponding to D, and let b be the defining equation of B.
The basis of $\Omega_X^1(\log D + B) \otimes O_X(-B)$ as an A-module is

$$\frac{df_1}{f_1} \otimes b, \cdots, \frac{df_s}{f_s} \otimes b, \frac{db}{b} \otimes b$$

The basis of $\Omega_X(\log D)$ as an A-module is

$$\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}$$

The basis of $\Omega_B(\log D \cap B)$ as an $\frac{A}{b}$-module is

$$\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}$$

where by abuse of notation f_i are regarded as their image in $\frac{A}{b}$.
Then, it is clear how to define i and r show that the above sequence is exact $\square$

Lemma 3. *Suppose B is a smooth divisor transversal to D. Then, $E_B := E \otimes O_B$ is the canonical extension of $V_B := V|_{B-B\cap D}$.*

Proof. The statement is local, and we may assume X is a polydisk

$$\Delta_1 \otimes \cdots \otimes \Delta_n$$

such that the analytic coordinate of Δ_i, for $i = 1, \cdots, s$, are defining equation of D_i, and the analytic coordinate of Δ_n is the defining equation of B.
First, we study V_B by computing its monodromy representation:
Let $T : \pi_1(X - D, x) \to \mathrm{GL}(r, \mathbb{C})$ be the monodromy representation of V. For each generator γ_i of $\pi_1(X - D, x)$, let $\Gamma_i = T(\gamma_i)$. As Γ_i are commuting and unitary, we can use one matrix to diagolize all of them. Therefore, we can assume all Γ_i are diagonal matrices. Moreover, as V is undefined only on D, so for each i, $\Gamma_i^{jj} = 1$, for $j = s+1, \cdots, n$.
Now, $B = \Delta_1 \times \cdots \times \Delta_{n-1}$, and the monodromy reprentation of $V|_{B-B\cap D}$ is given by

$$\pi_1(B - B \cap D) \xrightarrow{i} \pi_1(X - D) \xrightarrow{T} \mathrm{GL}(r, \mathbb{C})$$

where i is the natural inclusion map. It is clear that one can choose the basis of $\pi_1(B - B \cap D)$ and $\pi_1(X - D)$ such that i can be realized as the identity map. Therefore, the monodromy representations of $V_{B-B\cap D}$ are also Γ_i, for $i = 1, \cdots, s$.
To show $E|_B$ is the canonical extension of $V_{B-B\cap D}$, we compute the connection matrix of $E|_B$ and relate it to the monodromy representations of $V|_{B-B\cap D}$.
One can assume E is trivial over X. Choose a local frame of V on X, and use it as a trivialization of E. With respect to this trivialization, the connection ∇ can be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalues in the stripe

$$\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.
Now, restrict E to B, we see that the connection $\nabla|_B$ can still be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

As monodromy representations of $V_{B-B\cap D}$ are Γ_i, it follows that $E|_B$ is the canonical extension of $V_{B-B\cap D}$. $\square$

Theorem 5. *Suppose L is very ample on X. Then*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof. Let B be a smooth divisor transversal to D such that $L \cong O_X(B)$. By Lemma 2 we have the following exact sequence

$$0 \to \Omega_X^p(\log D + B) \xrightarrow{i} \Omega_X^p(\log D) \otimes O_X(B) \xrightarrow{r} \Omega_B^p(\log D \cap B) \otimes O_X(B) \to 0$$

Tensor it by E and take the cohomology sequence, we get:

$$\begin{aligned} &\cdots H^q(X, \Omega_X^p(\log D + B) \otimes E) \to H^q(X, \Omega_X^p(\log D) \otimes O_X(B) \otimes E) \\ &\to H^q(X, \Omega_B^p(\log B \cap D) \otimes O_X(B) \otimes E) \cdots \end{aligned}$$

Therefore, to prove the theorem, it is enough to show
Claim 1: $H^q(X, \Omega_X^p(\log D + B) \otimes E) = 0$
Claim 2: $H^q(X, \Omega_B^p(\log B \cap D) \otimes O_X(B) \otimes E) = 0$ for $p + q > \dim X$.
Proof of claim 1: The De Rham complex $\mathrm{DR}(D, \nabla, E)$ can be easily extended to the De Rham complex $\mathrm{DR}(D + B, \nabla, E)$. Let $W_.$ be the weight filtration defined $\mathrm{DR}(D+B, \nabla, E)$. Write T for $D+B$, and T_m for the union of m-fold intersection of components of T. As B is very ample, $T_m - T_{m+1}$ is affine for every m, so by Theorem 4, we know that

$$H^q(X, \mathrm{DR}^p(D + B, \nabla, E)) = 0$$

for $p + q > \dim n$. **End**
Proof of claim 2: Induction on the dimension of the variety. **End**

Now to finish the proof, it remains to show the base case of Claim 2, i.e. let X be a curve, then

$$H^1(X, \Omega_X(\log D) \otimes E \otimes L) = 0$$

But for the curve case, $\Omega_X(\log D) \otimes O_X(B) = \Omega_X(\log D + B)$. So the result follows again from Theorem 4 □

Now suppose L is any ample line bundle. Let m be an integer such that $L^{\otimes m}$ is very ample. Take a smooth divisor B transversal to D such that $L^{\otimes m} \cong O_X(B)$. Let φ be the local equation of B on some affine open set, and let $\pi : X' \to X$ be the normalization of X in $\mathbb{C}(X)(\varphi^{\frac{1}{m}})$.

Proposition 3. *Let $\pi : X' \to X$, B and L be as above*

1. *X' is smooth.*
2. *$\pi^* B = m\tilde{B}$, where $\tilde{B} = (\pi^* B)_{red}$.*
3. *$D' := \pi^* D$ is a normal crossing divisor on X'.*

4. *$\tilde{B}$ is transversal to π^*D.*

5. *$\pi^*\Omega^p_X(\log D) = \Omega^p_{X'}(\log D')$.*

6. *π^*E is the canonical extension of $\pi^{-1}V$.*

Proof. 1. We will construct X' by constructing its affine cover and specefiying the gluing morphisms. Let $U_i = \text{Spec}A_i$ be an affine cover of X, and let f_i be the defining equation of D in A_i.
For each A_i, $\frac{A_i[Y]}{(Y^m - f_i)}$ is integrally closed in $\mathbb{C}(X)(f_i^{1/m})$. Therefore,

$$U_i' := \text{Spec}\frac{A_i[Y]}{(Y^m - f_i)}$$

is the normalization of U_i in $\mathbb{C}(X)(f_i^{1/m})$
The same morphisms used to glue U_i into X can be used to glue U_i' into X'. Therefore, to show X' is smooth, it is enough to show $\frac{A_i[Y]}{(Y^m - f_i)}$ is a regular ring.

2. The local defining equation of $\tilde{B}$ is Y, and $\pi^*(f_i) = Y^m$

3. To see this, we describe π^*D in π^*U for any polydisk $U = \Delta_1 \times \cdots \times \Delta_n$. If $B \cap U \neq \varnothing$, then construct Δ_i such that defining equation of D_i, for $i = 1, \cdots, s$, are coordinates of D_i, for $i = 1, \cdots, s$; and the defining equation of B is the coordinate of D_n. Then,

$$\pi^*U = \Delta_1 \times \Delta_1 \cdots \Delta_{n-1} \times \Sigma^m$$

where Σ^m is the m-sheeted cover over a complex disk branched over the origin. In this case, π^*D is still defined by $z_1 \times z_2 \times \cdots z_s$.
If $B \cap U = \varnothing$, then π^*U is etale over U. Therefore, π^*D is etale over D. So π^*D is again a simple normal crossing divisor.

4.This is clear from the case 1 of part 3.

5. Straighforward computation. 6. We compute the monodromy representation

of $\pi^{-1}V$ first:
let $T : \pi_1(U - D, x) \to \text{GL}(r, \mathbb{C})$ be the representation corresponding to the local system V.
Case 1: Suppose $x \notin B$, then $\pi^{-1}(U)$ is etale over U. Let U' be an component of $\pi^{-1}(U)$, and let $x' \in U'$ be a preimage of x. Then,

$$T' : \pi_1(U' - D', x') \xrightarrow{\pi_*} \pi_1(U - D, x) \xrightarrow{T} \text{GL}(r, \mathbb{C})$$

is the representation corresponding to $\pi^{-1}V$.

Case 2: Suppose $x \in B$, then use the description from part 3, we know that

$$\pi^{-1}U = \Delta_1 \times \Delta_2 \times \cdots \times \Sigma^m$$

In both cases, $\pi^{-1}U - D'$ is homotopic to $S_1 \times S_2 \times \cdots \times S_s$ So we can define generators of $\pi_1(U' - D', x')$ and $\pi_1(U - D, x)$ such that π_* is the identity map. To show π^*E is the canonical extension of $\pi^{-1}V$, we only need to compute the connection matrix of π^*E and relate it to the monodromies of $\pi^{-1}V$:
Let γ_i be a small circle around D_i, and let Γ_i be the monodromy $T(\gamma_i)$. As $\pi_* : \pi_1(U' - D', x') \to \pi_1(U - D, x)$ is the identity map, Γ_i are also the monodromy representations of $\pi^{-1}V$. Next, we compute the connection matrix of E. Let U be small enough so that E is trivial over it. Choose a local frame of V, and use it as a trivialization of E. With respect to this trivialization, the connection ∇ can be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalue in the stripe

$$\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.
As $\pi^* z_i = z_i$, for $i = 1, \cdots, s$, we see that the $\pi^*\nabla$ over $\pi^{-1}U$ can be realized as:

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

This shows that π^*E is the canonical extension of $\pi^{-1}V$.

□

Corollary 2. *For any ample line bundle L on X,*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof. Let m, B and $\pi : X' \to X$ be as above. By 5

$$H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) = 0$$

for $p + q > \dim X' = \dim X$.
$\pi : X' \to X$ is a finite morphism, so for $i > 0$, $R^i {=}_* 0$ for any coherent sheaf. This implies

$$\begin{aligned} & H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) \\ & = H^q(X, \pi_*(\pi^*(E \otimes \Omega_X^p(\log D) \otimes L))) \\ & = H^q(X, \pi_*(O_Y) \otimes E \otimes \Omega_X^p(\log D) \otimes L) = 0 \end{aligned}$$

for $p + q > \dim X$. The second equality follows from the projection formula.
As $\pi_*(O_Y) \cong \bigoplus_{i=0}^{m-1} O_X(-L^{\otimes i})$, the result follows. □

4 Appendix

4.1 Linear algebra

Theorem 6. *Let U be an unitary matrix over $\mathbb{C}$, then U is diagonalizable.*

Theorem 7. *Let A and B be commuting diagonalizable $n \times n$ matrices over any field k, then A and B can be simultaneously diagolized.*

Proof. Let V be the vector space k^n. It is enough to show that A and B share the same eigenvectors.
Claim 1: A and B share at least one eigenvector.
Proof of Claim 1: Let v be an eigenvector of A with eigenvalue λ, then

$$ABv = BAv = B\lambda v = \lambda Bv$$

i.e. Bv is also an eigenvector of A with eigenvalue λ.
Let W be the subspace spanned by

$$v, Bv, \cdots, B^n v$$

Then, W is invariant under B. As V has a basis by eigenvectors of B, one can choose a vector $w \in W$ which is an eigenvector of B. Then, from the construction of W, w is also an eigenvector of A. **End**
Let w be as above, with $Bw = \mu w$; Let $e_1, \cdots, e_n$ be the standard basis of V; Let V' be the subspace spanned by $e_1, \cdots, e_{n-1}$; Let $\phi : V \to V$ be the linear map such that $\phi(e_n) = w$.

$$\phi^{-1} \circ A \circ \phi = A' \oplus \mathrm{Diag}(\lambda)$$
$$\phi^{-1} \circ B \circ \phi = B' \oplus \mathrm{Diag}(\mu)$$

where A' and B' are $n-1 \times n-1$ submatrices of A and B, representing the restriction of A and B on V'.
Now, A' and B' are diagonlizable, and they commute, therefore, by inducting on the size of the matrix, we are done. ☐

4.2 Canonical Extension of Unitary Local System

Let X be a smooth projective variety over $\mathbb{C}$; Let D be a simple normal crossing divisor on X; Let V be a unitary local system defined on $U{:=}X - D$; We will construct the canonical extension of V in this section:

References

[1] Eckart Viehweg Helene Esnault. Logarithmic de rham complexes and vanishing theorems. *Inventiones*, 1986.

[2] Klaus Timmerscheidt. Mixed hodge theory for unitary local system. *Journal für die reine und angewandte Mathematik*, 1987.