1 Problems

Let $\mathbb{P}^2$ be a smooth surface, let $D \subset \mathbb{P}^2$ be a non-SNC divisor, e.g. cusp in a surface. Let $p \in D$ be an isolated singularity. Let $\mathcal{L}$ be a local system on $\mathbb{P}^2 - D$. In case that D is defined by

$$y^2 = x^3$$

the local fundamental group of $\mathbb{P}^2 - D$ is

$$< a, b | a^2 = b^3 >$$

One can consider the resolution of D inside $\mathbb{P}^2$

$$\pi : \tilde{\mathbb{P}^2} \to \mathbb{P}^2$$

Let $\tilde{D}$ be the preimage of D. $\tilde{D}$ is a normal crossing divisor. Let $\tilde{V}$ be the canonical extension of $\pi^{-1}\mathcal{L}$. **Problem** What is the monodromy of $\tilde{V}$ along exceptional divisors?

Let $V := \pi_* \tilde{V}$. Then

1. What is V at p?
2. Is V independent of π?

   Should be. V can be constructed as the following: let p be the singular point of D. Remove p from $\mathbb{P}^2$, Then, $D - p$ is a normal crossing divisor on $\mathbb{P}^2 - p$; let Ł^o be the restriction Ł on $\mathbb{P}^2 - p$. Then, the theory of Deligne's canonical extension apply to Ł^o. Let V^o be the canonical extension of Ł^o; let

   $$j : \mathbb{P}^2 - p \hookrightarrow \mathbb{P}^2$$

   be the natural inclusion map. Then

   $$j_* V^o$$

   is the "correct" extension of Ł to $\mathbb{P}^2$
3. Is there a direct construction of V?
4. Is there a notion of residue?
5. If there is one, is it related to the local monodromy?

Simpler problem

Let $\mathbb{P}^2 = \mathbb{P}^2$, and let $C \subset \mathbb{P}^2$ be the cusp. Let $\pi : \tilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the two-step monoidal transformation that resolve the singularity of C. Let $\tilde{C}$ be the strict transform of C. Let $\mathcal{L}$ be the local system of rank 1 on $\tilde{\mathbb{P}^2}$ with monodromy Γ around $\tilde{C}$, and let $\tilde{V}$ be its canonical extension. Then, what is $\pi_* V$? Is it a bundle?

2 Flat bundles on blow-up

A local system L on $X - E$ cannot have any nontrivial monodromy around E, because $X - E = \mathbb{P}^2 - p$, which is simply connected.
Let $C_1 \subset X$ be the strict transform of $C \subset \mathbb{P}^2$. Let Ł be
Extension of a flat log connection by subdivisor Let $D \subset X$ be a simple nornal crossing divisor; let

$$\nabla : V \to V \otimes \Omega_X(\log D)$$

be a flat log connection. Let $D_1 \subset D$ be a subdivisor, and write $D = D_1 + D_2$.

Lemma 1. *There is an exact sequence*

$$0 \to \Omega_X(\log D_1 + D_2) \xrightarrow{i} \Omega_X(\log D_2) \otimes O_X(D_1) \to \Omega_{D_1} \otimes \Omega_{D_1}(\log D_1 \cap D_2) \to 0$$

Write $j : O_X \to O_X(D_1)$ for the inclusion map.
Define

$$\nabla^{D_1} : V \otimes O_X(D_1) \to V \otimes O_X(D_1) \otimes \Omega_X(\log D_2)$$

be the connection such that the following diagram commutes

$$\begin{tikzcd}
V \arrow[r, "\nabla"] \arrow[d, "j"] & V \otimes \Omega_X(\log D_1 + D_2) \arrow[d] \\
V \otimes O_X(D_1) \arrow[r, "\nabla^{D_1}"] & V \otimes O_X(D_1) \otimes \Omega_X(\log D_2)
\end{tikzcd}$$

If $\langle e_1, \cdots, e_n \rangle$ is a flat basis of V, then $\langle e_1 \otimes s, \cdots, e_n \otimes s \rangle$ is a flat basis of $V \otimes O_X(D_1)$, where s is a local generator of $O_X(D_1)$

Lemma 2. *Use the notation from above. The monodromy of ∇^{D_1} around D_2 is the same as the monodromy of ∇ around D_2*

Proposition 1. *Let X be be a smooth projective surface with a -1 curve E; let D be a simple normal crossing divisor transversal to E. Suppose*

$$\nabla : V \to V \otimes \Omega_X(\log E + D)$$

be a flat logarithmic connection. Then, the monodomy of ∇ around E is trivial.

Proof. Consider the extension of ∇ by D

$$\nabla^D : V \otimes O_X(D) \to V \otimes O_X(D) \otimes \Omega_X(\log E)$$

If ∇ has non-trivial monodromy around E, then so does ∇^D. This implies that the flat sections of ∇^D restricted to $X - E$ is a nontrivial local system. But $X - E$ is simply connected, as we can blow-down E to a point. This is a contradiction. ∎

Let $X = \mathbb{P}^2$, and let

$$\pi_1 X_1 \to X$$

be blow up at one point. Let E be the exceptional divisor. Let $D+E$ be a normal crossing divisor. Suppose (V, ∇) is a vector bundle with logarithmic singularity at $D+E$. Then, as we have seen above ∇ is in fact holomorphic near E, i.e. $\mathrm{Res}_E \circ \nabla = 0$. Write D as

$$\sum_{i=1}^{s} D_i$$

The following proposition can be used to compute the first Chern class of V

Proposition 2. *[?] Let $\Gamma_i = Res_i \circ \nabla$ and $[D_i]$ be the class of D_i in the Chow ring. Then,*

$$C_1(V) = -\sum_{i=1}^{s} Tr(\Gamma_i) \cdot [D_i]$$

Proposition 3. *If V is the canonincal of extension of a local system defined on $X_1 - (E+D)$, then*

$$Tr(\Gamma_i) \geq 0$$

3 Push-forward under blow up

Suppose L is a flat line bundle on X_1 with logarithmic singularity along $D+E$. We have seen that L is actually holomorphic along E. Use Proposition **??**, we see that

$$L \cong O_{X_1}(-\sum_{i=1}^{s} \mathrm{Tr}(\Gamma_i) \cdot D_i)$$

Now, write $L = O_{X_1}(\sum_i a_i D_i)$. Suppose $E \cdot D_i = b_i$. This means, for each D_i, there is an irreducible divisor C_i on X with multiplicity b_i at p such that

$$\pi_1{}^* C_i = D_i - b_i E$$

This means

$$L = O_{X_1}(\sum_i \pi_1{}^*(a_i C_i) + a_i b_i E)$$

Suppose $\sum_i a_i b_i = n$. Then

$$\pi_{1*} L = \begin{cases} O_X(\sum_i a_i C_i) & \text{if } n \geq 0 \\ O_X(\sum_i a_i C_i) \otimes m_p^n & \text{if } n < 0 \end{cases}$$

As we have seen in propsition 3, if L is the canonical extension of a local system on on X_1, then $a_i \leq 0$.

Proposition 4. *Suppose $\pi_{1*}L$ is a bundle on X. Then, if $E \cdot D_i > 0$, then monodromy of L around D_i must be trivial.*

The converse of the above statment is also true

Proposition 5. *If $D_i \cdot E > 0$, and L has nontrivial monodromy around D_i, then all section of $\pi_{1*}L$ vanishes at p.*

Consider the following commutative diagram

$$\begin{tikzcd} X_1 - E \arrow[r, "i"] \arrow[d, "\sigma^o"] & X_1 \arrow[d, "\sigma"] \\ X - p \arrow[r] & X \end{tikzcd}$$

Suppose (V, ∇) is a flat connection on X_1, where ∇ has at most logarithmic singularities along some normal crossing divisor. We have seen that ∇ is holomophic along E. We want to investigate what is

$$i_*(V|_{X_1 - E})$$

Proposition 6. *If (V, ∇) is a flat vector bundle on X_1, then*

$$\sigma_* V = j_* V|_{X_1 - E}$$

Proof. If (V, ∇) is a flat bundle on X_1, then V cannot have nontrivial monodromy around E. Then, we have

$$V = i_* V^o$$

where $V^o = V|_{X_1 - E}$. Then, by the commutativity of the diagram, we have

$$\sigma_* V = j_* V^o$$

□

Now, let X be any smooth surface, and let $\pi_1 : X_1 \to X$ be blow up of one point. Let (V, ∇) be a flat bundle on X_1, we want to compute $\pi_{1*}V$ and see how far it is different from being a vector bundle.
If (V, ∇) has logarithmic singularity along $D = \sum_i D_i$, and if Res $\circ \nabla$ is in the range of $[0, 1)$, then

$$C_1(V) = -\sum_i a_i D_i$$

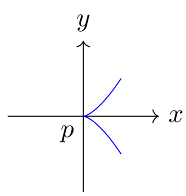
where $a_i = \text{Tr}(\Gamma_i)$ is nonnegativ.
Now, $V|_E$ is a vector bundle on $E \cong \mathbb{P}^1$. By a lemma of Grothendieck, every vector bundle on $\mathbb{P}^1$ splits as the sum of line bundles. Write $V = \sum_j O_E(c_j)$, we want to determine c_j.

4 Monodromy around the exceptional divisor

We address a specific question in this section: Let $X = \mathbb{P}^2$ and let C be the cusp with singularity p. Let Ł be a local system on $X - C$; let

$$X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$$

be the three-step blow up that resolves the singularity of C, and the pullback of C in X_3 is a normal crossing divisor.
The question is, what is the monodromy of $\pi^{-1}L$ around E_3, the exceptional divisor obtained from π_2?
Take a small circle δ around E_3, what is the image of δ under π?
Let $p_i \in X_i$ be the intersection of the exceptional divisor and the strict transform;
Local description of π_1 Let (x, y) be the local coordinate near p.

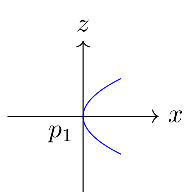


Blow-up in this small neigborhood is constructed as

$$\text{Proj}\frac{k[x,y][X,Y]}{(xY - yX)}$$

y-axis has the slope ∞, therefore y-axis will intersect the exceptional divisor of π_1 at point-of-infinity; Set $z = \frac{Y}{X}$ to be the affine coordinate of the finite part of E_1. In the xz-coordinate system C_1 is defined by the equation

$$x^3 = (xz)^2$$
$$x = z^2$$

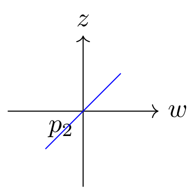


The map π_1 is given by

$$\pi_1(x, z) = (x, xz)$$

Now, blow up at p_1 and put x-axis to the point-at-infinity; Use $[X : Z]$ for the homogenous coordinate of E_2, use $\frac{X}{Z}$ for the affine coordinate of the exceptional divisor E_2; Write $w = \frac{X}{Z}$. In the wz-system, C_2 is defined by

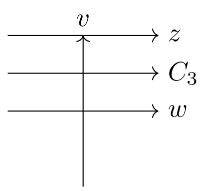
$$w = z$$



The map π_2 is given by

$$\pi_2(w, z) = (wz, z)$$

Now, blow up at p_2 and use $v = \frac{Z}{W}$ as the affine coordinate on E_3. Then, z-axis will be put into point at infinity. It is figuratively shown as below



Let $\pi : X_3 \to X$ be the composition of all the above blow-ups. $\pi^* C = Z + C_3 + W + V$ is a simple normal crossing divisor. (Z stands for z-axis and so on).

Image of a small circle around v-axis near w-axis Let $\delta(v)_r$ be a small circle around v-axis of height r, i.e. the v-coordinate of the δ_r is given by $v = r$. δ_r can be parametrically described as

$$\{(\epsilon e^{it}, r) | 0 \leq t \leq 2\pi\}$$

where ϵ is some small positive number. Using the local description of π_1, π_2, π_3, we see that the image of a small circle around E_3 under π is

$$\lambda = \{(r\epsilon^2 e^{2it}, r^2 \epsilon^2 e^{3it}) | 0 \leq t \leq 2\pi\}$$

Consider the projection onto the y-axis

$$p : \mathbb{C}^2 \ C \to \mathbb{C}$$

Let F be the fiber over $r\epsilon^2$. $\pi_1(F)$ generates $\pi_1(\mathbb{C}^2 \ C)$ the relation is given in Dimca's book.

We can deform λ so that it contracts to the fibre.

The map $y \mapsto |y|$ deforms D onto $[0, 1]$. Lift this map back to $\mathbb{C}^2$
C, the lift is given by

$$(x, y) \mapsto (x, |y|)$$

The homotopy $y \mapsto |y|$ is given by the arc homotopty (as the analogue of straight line homotopy). The map is given by

$$\lambda_s = (r\epsilon^2 e^{2it}, r^2 \epsilon^3 3its)$$

where s runs from 1 to 0. λ_s avoids the curve C as s goes from 1 to 0.
Now, the fiber of φ over $r\epsilon^2$ is given by

$$\mathbb{C} - \{q_1, q_2, q_3\}$$

where q_i are the cube roots of $r^4\epsilon^6$.
λ_0 includes q_i, and it wraps around them twice. Therefore, $\lambda = \beta^2$
Image of a small circle around v-axis above C_3 In this case $r > 1$, the circle λ_0 does not include the points q_i, so λ_0 is homotopically equivalent to 0.
Image of small circle around w-axis Let $\delta_r(w)$ be a small circle around w-axis. $\delta_r(w)$ can written as

$$\{(r, \epsilon e^{it}) | 0 \leq t \leq 2\pi\}$$

So the image of $\delta_r(w)$ under π is given by

$$\{(r^2\epsilon e^{it}, r^3\epsilon^2 e^{2it} | 0 \leq t \leq 2\pi\}$$

The fiber of p over $y = r^3\epsilon^2$ is given by

$$\mathbb{C} - \{q_1, q_2, q_3\}$$

where q_i are cube roots of $r^6\epsilon^4$. The λ is deformed to

$$\{(r^2\epsilon e^{it}, r^3\epsilon^2) | 0 \leq t \leq 2\pi\}$$

on $p^{-1}(r^3\epsilon^2)$
we see that the circle λ_0 includes q_i, and it wraps around them once. This means $\lambda = \beta$. (also, we can see the position of $\delta_r(w)$ does not matter, i.e. r does not matter)
Image of a small circle around C_3 Let $\delta_r(C_3)$ be a small circle around C_3. $\delta_r(C_3)$ is given by

$$\{(r, 1 + \epsilon e^{it}) | 0 \leq t \leq 2\pi\}$$

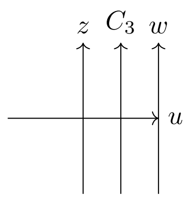
The image of it under π is given by

$$\{(r^2(1+\epsilon)e^{it}, r^3(1+\epsilon)^2 e^{2it}) | 0 \leq t \leq 2\pi\}$$

Deform λ to the fiber over $r^3(1+\epsilon)^2$, which is given by

$$\mathbb{C} - \{q_1, \cdots, q_3\}$$

where q_i are cube roots of $r^6(1+\epsilon)^4$. As $1+\epsilon > 1$, we see that λ_0 cannot includes q_i, i.e. λ_0 is homotopically equivalent to 0 in $\pi_1(\mathbb{C}^2$
$C)$.
Image of a small circle around z-axis Now in X_3, we look at the affine space given by the cooridate ring $k[z, \frac{W}{Z}]$. Write $u = \frac{W}{Z}$. In this affine space, C_3 is given by $u = 1$. The picture is



A small circle $\delta_r(z)$ around z-axis is given by

$$\{(\epsilon e^{it}, r) | 0 \leq t \leq 2\pi\}$$

Its image under π is

$$\{(\epsilon r^2 e^{it}, r^3 \epsilon e^{it}) | 0 \leq t \leq 2\pi\}$$

Deform it to the fiber over $r^3\epsilon$. The fiber is given by

$$\mathbb{C} - \{q_1, q_2, q_3\}$$

where q_i are the cube roots of $r^6\epsilon^2$. Since ϵ is small, $\epsilon^{2/3} > \epsilon$. Therefore, the cirle λ_0 cannot contain any of q_i. This means λ is homotopically equivalent to 0 in $\pi_1(\mathbb{C}^2$
$C)$.
λ_0 is given