1 Higgs Bundle Arising from Geometric Variation of Hodge Structure

Now suppose that (V^o, ∇^o) is part of a polarized variation of Hodge structure. Let F^o be the Hodge filtration on V^o. Suppose the monodromy of $\nabla^o)$ is unipotent, then we can extend the filtration F^o to a filtration F on V_* by a theorem of Schmid:

$$F^p V_* = j_*(F^{op} V^o) \cap V_*$$

Let $E = \mathrm{Gr}_F V_*$, and $\theta = \mathrm{Gr}_F \nabla_*$, then (E, θ) is a Higgs bundle with wht following properties

1. The Chern classes of E, in rational cohomology, all vanish.
2. The Higgs bundle is semistable in the sense that $\mu(E') \leq \mu(E) = 0$ for any proper coherent subsheaf stable under θ
3. The Higgs field θ is nilpotent. This follows from the Griffith transversality of ∇_*

If the monodromy of ∇^o is only quasi-unipotent, then we need an intermediate step to construct the extension of F^o. Let $\pi : Y \to X$ be the cyclic cover in section 2. The monodromy of $\pi^* \nabla^o$ is unipotent. Therefore, we can extend $\pi^* F^o$ to a filtration $\bar{F}$ on $\bar{V}$. Let

$$\phi : (\pi_* \bar{V})^G \to V_*$$

be the isomorphism section 2. Then, we define

$$F^p V_* := \phi((\pi_* \bar{F})^G)$$

We will prove that

Lemma 1. $(E, \theta) = (Gr_F V_*, Gr_F \nabla_*)$ *is a Higgs Bundle,* i.e. $\theta \wedge \theta = 0$.

Lemma 2. *The parabolic Chern classes of E is zero*

Lemma 3. *E is parabolic semistable.*

Lemma 4. *θ is nilpotent,* i.e. *∇_* has Griffith transversality with respect to F.*

Proof. Let α be a section of V_*, and let β be the section of $(\pi_* \bar{V})^G$, such that $\phi(\beta) = \alpha$. We will show that $\bar{\nabla}(\beta)$ is G-invariant, and $\phi(\bar{\nabla}(\beta)) = \nabla_*(\alpha)$.
It is enough to check locally. Use the notations from the proof of lemma 2.4. Locally on U, let α be given by

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

with respect to the frame $< s_1, \cdots, s_r >$. Then, β is given by

$$\begin{bmatrix} \alpha_1 \prod_{j=1}^{s} y_j^{d_1^j} \\ \vdots \\ \alpha_r \prod_{j=1}^{s} y_j^{d_r^j} \end{bmatrix}$$

with respect to the frame $< e_1, \cdots, e_r >$. To make the notation simpler, we write $\prod_{j=1}^{s} y_j^{d_i^j}$ as y^{δ_i} Then,

$$\bar{\nabla}(\beta) = \begin{bmatrix} y^{\delta_1} d\alpha_1 + \alpha_1 y^{\delta_1} \sum_{j=1}^{s} d_1^j \frac{dy_j}{y_j} \\ \vdots \\ y^{\delta_r} d\alpha_1 + \alpha_1 y^{\delta_r} \sum_{j=1}^{s} d_r^j \frac{dy_j}{y_j} \end{bmatrix} + N_1 \beta \frac{dy_1}{y_1} + \cdots N_s \beta \frac{dy_s}{y_s}$$

Reorganize the terms, we have

$$\bar{\nabla}(\beta) = \begin{bmatrix} y^{\delta_1} d\alpha_1 \\ \vdots \\ y^{\delta_r} d\alpha_r \end{bmatrix} + (N_1 + D_1)\beta \frac{dy_1}{y_1} + \cdots (N_s + D_s)\beta \frac{dy_s}{y_s}$$

We observe that in $\bar{\nabla}(\beta)$ every coefficient in $\bar{V}$ is G-invariant. Therefore, $\bar{\nabla}(\beta)$ is a G-invariant section.
Recall in the proof of lemma2.4,

$$N_i + D_i = B_i = k_i N A_i$$

So $\bar{\nabla}(\beta)$ can be written more compactly as

$$\bar{\nabla}(\beta) = \begin{bmatrix} y^{\delta_1} d\alpha_1 \\ \vdots \\ y^{\delta_r} d\alpha_r \end{bmatrix} + k_1 N A_1 \frac{dy_1}{y_1} + \cdots k_s N A_s \frac{dy_s}{y_s}$$

As $y_i^{k_i N} = x_i$, we have

$$\frac{dy_i}{y_i} = \frac{dx_i}{k_i N x_i}$$

Hence,

$$\begin{aligned} \phi(\bar{\nabla}(\beta)) &= d(\alpha) + A_1 \alpha \frac{dx_1}{x_1} + \cdots + A_s \alpha \frac{dx_s}{x_s} \\ &= \nabla_*(\alpha) \end{aligned}$$

As $\bar{\nabla}(\beta) \in \bar{F}^{p-1}\bar{V}$, we have proved that $\nabla_* \alpha \in F^{p-1} V_*$.

$\square$