1 Better description of the orbifold bundle corresponding a parabolic bundle in case of the VSH

Let X be a smooth projective \mathbb{C} -variety, let D be a normal crossing divisor on X. Let \mathcal{V} be a VHS on U := X - D. Let V be the Deligne canonical extension of \mathcal{V} . Then, by a theorem of Schmid, the Hodge filtration F on \mathcal{V} extends to a filtration on V. Take associated graded of the connection

$$\nabla: V \to \Omega_X(\log D) \otimes V$$

We get a O_X -linear map

$$\theta: E \to \Omega_X(\log D) \otimes E$$

Since ∇ is integral, (E, θ) is a Higgs bundles. In case that the monodromy of $\mathcal V$ is unipotent, the parabolic structure on E is trivial. However, if we monodromy of $\mathcal V$ is only quasi-unipotent, E will have non-trivial parabolic structure.

Question 1. Is there a better description of the orbifold bundle corresponding to *E*?

Let $p: Y \to X$ be the cyclic covered branched over D such that we can construct an orbifold bundle W (equivariant bundle) on Y, corresponding to the parabolic bundle E on X. Then, Y should have a better description in this case: pull-back the local system \mathcal{V} , and take the Deligne canonical extension.

1.1 Defining a parabolic structure on Deligne canonical extension

Assume D is irreducible and smooth. We have a map

$$\operatorname{Res} \circ \nabla : V \to V|_D$$

 $V \otimes O_X(-D)$ is contained in the kernel of Res $\circ \nabla$. So Res $\circ \nabla$ defines an endomorphism

$$\phi: V|_D \to V|_D$$

1.2 Main ideas

Think of V as a parabolic bundle on X. Let V' denote the Deligne canonical extension of $p^{-1}\mathcal{V}$. To show the orbifold bundle on Y corresponding to the parabolic bundle V on X, we show that

$$(p_*V)^{\Gamma} = V$$

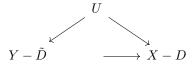
Several things to prove

Lemma 1. The action Γ on Y lifts to a compatible action on V' which makes V' an orbifold bundle.

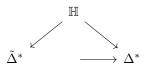
Proof. Assume X is a curve, and $D \subset X$ is a point. Let T be the monodromy of $\mathcal V$. Suppose T^N is unipotent. Let $p:Y\to X$ be the cyclic cover of degree N branched over D. Then the monodromy of $p^{-1}\mathcal V$ is unipotent. Let $\tilde D=p^{-1}(D)$. Then,

$$p: Y - \tilde{D} \to X - D$$

is etale, and we have the following commutative diagram



where U is the universal cover of X-D. Let $\tilde{\Delta}$ and Δ be small disks around \tilde{D} and D respectively. Then, locally, the above diagram looks like



Let \tilde{z} and z be the local analytic coordinates on $\tilde{\Delta}$ and Δ , and let t be the coordinate on \mathbb{H} . Then, we have

$$\tilde{z}=e^{\frac{2\pi it}{N}}, z=e^{2\pi it}$$

Suppose V' has rank r. Let e_1, \dots, e_r be the standard basis of \mathbb{C}^r . Let s_i be the multivalued functions on $\tilde{\Delta}$ such that pullback of s_i to the universal cover looks like

$$t \mapsto e^{Mt}e_i$$

where $M = -\log T^N$. Then, we use s_1, \dots, s_r to trivialize V' over \tilde{D}

$$V'|_{\tilde{\Delta}} \cong \bigoplus_{i=1}^r O_{\tilde{\Delta}} s_i$$

 Γ acts on s_i . Hence, Γ acts on V'.

On Y, we have V' and p^*V . Let $M = \text{Monodromy}(\mathcal{V})$, then near D

$$V \cong \bigoplus_{i=1}^r O_{\Delta} s_i$$

where $s_i(t) = e^{Mt}e_i$, t is the coordinate on the universal cover. So near \tilde{D}

$$p^*V \cong = \bigoplus_{i=1}^r O_{\tilde{\Delta}} s_i$$

 $NM = \text{Monodromy}(\mathcal{V}^{-1})$. So near \tilde{D}

$$V' = \bigoplus_{i=1}^r O_{\tilde{\Delta}} \tilde{s_i}$$

where $\tilde{s_i} = e^{NMt}e_i$ So there is an obvious morphism

$$\alpha: p^*V \to V^prime$$

sending s_i to $\tilde{s_i}$. This morphism is not reversible, because taking N-th root is not well-defined.

2 Perturb parabolic structure with weights in \mathbb{R} to parabolic structure with weights in \mathbb{Q}

Let X be a smooth projective $\mathbb C$ -variety, and let D be a normal crossing divisor on X. Let E be a parabolic bundle with weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. In case that $\alpha_i \in \mathbb Q$, we can construct an orbifold bundle on some branched cover over X. In case that $\alpha_i \in \mathbb R$, what do we do? The natural thing is to perturb α_i a little bit to bring them back to $\mathbb Q$. Let Pb() be the pertabation operation (there is no reason to believe this operation is canonical). Several things should happen

- There should be a sequence $Pb_{\epsilon}()$ such that $\epsilon \to 0$ and $Pb_0()=\mathrm{id}$
- $\bullet\,$ parabolic chern classes of E should be equal to parabolic chern classes of Pb(E)
- If E is a Higgs sheaf with Higgs field θ such that

$$\mathbb{H}^{i}(X, \mathrm{DR}(Pb_{\epsilon}(E, \theta))) = 0$$

then, it should happen that

$$\mathbb{H}^i(X, \mathrm{DR}(Pb_0(E,\theta))) = 0$$

Question 2. What would be a operator Pb so that everything above is true?