

Let X be a complex manifold, and let $D \subset X$ be a normal crossing divisor. Let \mathcal{V} be a quasi-unipotent local sytem on $X - D$. Then, the canonical extension V_X of \mathcal{V} has nontrivial parabolic structure. Let $p : Y \rightarrow X$ be a branched cyclic cover such that $p^{-1}\mathcal{V}$ is unipotent. Then, we will show that the orbifold bundle on Y corresponding to V_X is the canonical extension of $p^{-1}\mathcal{V}$.

I will use the following example to illutrate the ideas. Let $X = \Delta$, D be the origin, and $\mathcal{V} = (\mathbb{Z}^2, T)$, where T is the diagonal quasi-unipotent matrix

$$T = \begin{bmatrix} \epsilon_3 & 0 \\ 0 & \epsilon_5 \end{bmatrix}$$

where ϵ_i is a primitive i -th root of unity.

1 Construction of the canonical extension of \mathcal{V}

In this note, by canonical extension, we mean the extension (V, ∇) of \mathcal{V} such that the eigenvalues of $\text{Res}(\nabla)$ lie in $[0, 1)$

Let

$$M = -\log T = -\begin{bmatrix} \log \epsilon_1 & 0 \\ 0 & \log \epsilon_5 \end{bmatrix}$$

And we use the branch one step before the principal branch for both log functions, *i.e.*

$$M = -\begin{bmatrix} \frac{2\pi i}{3} - 2\pi i & 0 \\ 0 & \frac{2\pi i}{5} - 2\pi i \end{bmatrix}$$

Let $\mathbb{H} \rightarrow \Delta^*$ be the universal covering map. Let z and t be the coordinate on Δ and t , we have $z = \exp 2\pi i t$. Let $< e_1, e_2 >$ be the standard basis for \mathbb{Z}^2 . Consider the vector valued functions on \mathbb{H}

$$s_1(t) = \exp(Mt) \otimes e_1, s_2(t) = \exp(Mt) \otimes e_2$$

i.e.

$$s_1(t) = \begin{bmatrix} \exp \frac{4\pi i t}{3} \\ 0 \end{bmatrix}, s_2(t) = \begin{bmatrix} 0 \\ \exp \frac{8\pi i t}{5} \end{bmatrix}$$

Use $z = \exp 2\pi i t$, we can regard s_1 and s_2 as multi-valued functions on Δ^*

$$s_1(x) = \begin{bmatrix} x^{\frac{2}{3}} \\ 0 \end{bmatrix}, s_2(x) = \begin{bmatrix} 0 \\ x^{\frac{4}{5}} \end{bmatrix}$$

Let $V = O_X \cdot s_1(x) \oplus O_X \cdot s_2(x)$. Define $\nabla : V \rightarrow V \otimes \Omega_X(\log D)$, such that the connection matrix is given by

$$\frac{M}{2\pi i} \cdot \frac{dx}{x}$$

Then,

$$\text{Res}(\nabla) = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$$

This verifies that the (V, ∇) is the canonical extension of \mathcal{V} . Use $\langle e_1, e_2 \rangle$ as the standard basis of \mathbb{Z}^2 , the inclusion map $\mathcal{V} \rightarrow V|_{\Delta^*}$ is given by

$$e_1 \mapsto x^{-\frac{1}{3}} s_1, e_2 \mapsto x^{-\frac{4}{5}} s_2$$

And we can check that the images of e_1 and e_2 are indeed flat sections of ∇ .

1.1 The action of x on V_X

This section will explain how x acts on V_X and send V_X to an extension of \mathcal{V} whose residue has eigenvalues in $[1, 2)$.

x has a natural action on s_1 and s_2

$$x \cdot s_1 = \begin{bmatrix} x^{\frac{4}{3}} \\ 0 \end{bmatrix}, x \cdot s_2 = \begin{bmatrix} 0 \\ x^{\frac{9}{5}} \end{bmatrix}$$

We would have ended up those sections if we used $-4\pi i$ branch for our M . More precisely, if we set

$$M = - \begin{bmatrix} \frac{2\pi i}{3} - 4\pi i & 0 \\ 0 & \frac{2\pi i}{5} - 4\pi i \end{bmatrix}$$

at the very begining. Then, we will get an extention (V^1, ∇^1) with the generating sections

$$x \cdot s_1 \text{ and } x \cdot s_1$$

The connection ∇^1 is defined by

$$\frac{M}{2\pi i} \cdot \frac{dx}{x}$$

So

$$\text{Res}(\nabla) = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{9}{5} \end{bmatrix}$$

1.2 The parabolic structure on V_X

$\text{Res}(\nabla)$ defines an endomorphism of $V_X|_D$, the parabolic structure is given by the generalized eigenspaces of this endomorphism. In our example, the story is quite simple. The eigenvalues of $\text{Res}(\nabla)$ are $\frac{2}{3}$ and $\frac{4}{5}$, the corresponding eigenspaces are $\mathbb{C} \cdot s_1$ and $\mathbb{C} \cdot s_2$. The filtration on V_X is

$$V_X = F_1 \supset F_2 = O_X \cdot x \cdot s_1 \oplus O_X \cdot s_2 \supset V_X(-D)$$

and the weights are $\frac{2}{3}, \frac{4}{5}$.

2 On the branch Cover

Let $p : Y \rightarrow X$ be the branched coverd defined by $y^{15} = x$. Let Γ be the Galois group of this covering map. Then, $p^{-1}\mathcal{V} = (\mathbb{Z}^2, T^{15})$ is the trivial local system on Y . So the canonical extension V_Y of $p^{-1}\mathcal{V}$ is the trivial bundle with the trivial connection, *i.e.*

$$V_Y = O_Y \cdot e_1 \oplus O_Y \cdot e_2$$

where $\langle e_1, e_2 \rangle$ is the standard basis of \mathbb{Z}^2 . The connection ∇_Y on V_Y has the trivial connection matrix.

V_Y inherits an Γ -action from O_Y , and clearly this makes V_Y into an equivariant bundle.

The Γ -invariant part of p_*V_Y

e_1 and e_2 are constant sections on Y , so Γ has trivial action on them. So to compute $(p_*V_Y)^\Gamma$, we only need to compute $(p_*O_Y)^\Gamma$. As an O_X -module, p_*O_Y looks like

$$O_X \cdot 1 \oplus O_X \cdot y \oplus O_X \cdot y^2 \cdots \oplus O_X \cdot y^{14}$$

where y is a local coordinate on Y . Each y^i has a nontrivial Γ -action, *i.e.* if g is a generator of Γ , then $g \cdot y^i = (g \cdot y)^i$. So the only Γ -invariant part of p_*O_Y is $O_X \cdot 1$. Hence,

$$(p_*V_Y)^\Gamma = O_X \cdot e_1 \oplus O_X \cdot e_2$$

Here is what troubles me

Now, we have proved that V_Y is the trivial rank 2 vector bundle. The next step would be to construct a morphism

$$\alpha : (p_*V_Y)^\Gamma \rightarrow V_X$$

and show it is a morphism of parabolic bundles. But it seems that the parabolic structure on $(p_*V_Y)^\Gamma$ is finer, *i.e.* the filtration on it is finer than the one on V_X . According to Biswas's recipe, the parabolic structure on $(p_*V_Y)^\Gamma$ is defined as

$$(p_*V_Y)_t^\Gamma = (p_*V_Y \otimes O_Y(\lfloor -15t \rfloor \tilde{D}))^\Gamma$$

But for $i = 1, 2, \dots, 14$, we have

$$(p_*V_Y \otimes O_Y(-i\tilde{D}))^\Gamma \subsetneq (p_*V_Y \otimes O_Y(-(i+1)\tilde{D}))^\Gamma$$

For example, sections of $(p_*V_Y \otimes O_Y(-\tilde{D}))^\Gamma$ looks like

$$w = \begin{bmatrix} f_1(y) \\ f_2(y) \end{bmatrix}$$

where f_1 and f_2 vanishes at \tilde{D} to the order at least 1. So if w is invariant under Γ , monomials in f_1 and f_2 must at least be of degree 15. But there are more invariant sections in $(p_*V_Y)^\Gamma$, for example, nonzero constant sections. So between $(p_*V_Y)^\Gamma$ and $(p_*V_Y)^\Gamma \otimes O_X(-D)$ there is a filtration of length 14, whereas between V_X and $V_X \otimes O_X(-D)$ there is filtration of length only 1.