Variation of Hodge structure consists of the following data

1. A connected complex manifold M
2. A flat complex vector bundle $H_{\mathbb{C}} \to M$ with a flat real structure $H_{\mathbb{R}}$, and a flat bundle of lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, i.e. transition functions of $H_{\mathbb{C}}$ a locally constant and real-valued. The flat connection of $H_{\mathbb{C}}$ has real valued connection matrix
3. An integer k
4. A flat, nondegenerate bilinear form S on $H_{\mathbb{C}}$ which is rational with respect to $H_{\mathbb{Z}}$, *i.e* restricting to each fibre, S is a locally constant matrix of rational coefficient with repect to the chosen basis $H_{\mathbb{Z}}$
5. A decreasing filtration
$$H_{\mathbb{C}} \supset F^1 \supset \cdots \supset 0$$
by holomorphic subbundle.

The objects need to satisfy the following

1. Over each point t, $H_{\mathbb{C}}, F, S$ restrict to a polarized Hodge structure of weight k.
2. $\nabla(F^p) \subset F^{p-1}$

1 Classifying space for Hodge Structure

Let $H_{\mathbb{Z}}$ be some lattice over $\mathbb{Z}$, and let $H_{\mathbb{C}}$ be its complexification. Fix an integer k, and a collection of nonzero integers $h^{p,q}$ such that $p+q=k$. First objective is to give a manifold structure to the parameter space of all Hodge structure of weight k on $H_{\mathbb{C}}$. Let $D_p = dimF^p = \sum_{i \geq p} h^{p,q}$. Then, we consider the product of Grassmannian
$$Grass = G(H_{\mathbb{C}}, D_0) \times G(H_{\mathbb{C}}, D_1) \times \cdot \times$$
A possible filtration on $H_{\mathbb{C}}$ can be considered as a point in $Grass$. All possible filtrations on $H_{\mathbb{C}}$ can be identified with the incidence variety I on G(why incidence variety is smooth)

The general linear subgroup of $H_{\mathbb{C}}$ operates on I transitively. The subset $F \subset I$ satisfying
$$H_{\mathbb{C}} = F^p \oplus \bar{F}^{k-p+1}$$
is an Zariski open subset of I. So F parametrizes the Hodge structure on $H_{\mathbb{C}}$.

Let S be a nondegenerate bilinear form on $H_{\mathbb{C}}$, symmetric or skew, depending on the parity of k, such that
$$\begin{aligned} S(H^{p,q}, H^{r,s}) = & \quad 0, \text{unless} p=s, q=r \\ i^{p-q} S(v, \bar{v}) > & \quad 0, \text{if} v \in H^{p,q}, v \neq 0 \end{aligned}$$

Let $\check{D}$ be the points of I satisfy condition 1.

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) | s(gu, gv) = S(u, v)\}$$

operates on $\check{D}$ transitively.
Let $D \subset I$ be the subset of filtration satisfy the above two conditions. D is a open subvariety.

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) | S(gu, gv) = S(u, v)\}$$

acts transitively on D. Therefore, D is smooth. D parametrize Hodge structure of weight k on $H_{\mathbb{C}}$ with the polarization S
Choose a reference point $o \in D$, Let H be the corresponding filtration. Let $B \subset G_{\mathbb{C}}$ that fixes H, *i.e.* $gF^p(H) = F^p(H)$ for all $g \in B$ and all p. Then, we have a set-theoretical identification

$$G_{\mathbb{C}}/B \cong \check{D}$$

One obtains analogous identification

$$G_{\mathbb{R}}/V \cong D, V = G_{\mathbb{R}} \cap B$$

1.1 Group invariant subtangent bundle of $\check{D}$ and D

The Lie algebras of $G_{\mathbb{C}}$ can be described as

$$g = \{X \in GL(H_{\mathbb{C}}) | S(Xu, v) + S(u, Xv) = 0\}$$

How does an orthogonal matrix M to an real-valued symmetric matrix look like, regarded as a bilinear form.
g contains

$$g_0 = \{X \in g | XH_{\mathbb{R}} \subset H_R\}$$

$g = g_0 \oplus ig_0$. Via the containment $G_{\mathbb{R}} \subset G_{\mathbb{C}}$, g_0 becomes the Lie algebra of $G_{\mathbb{R}}$.

$$g = \bigoplus_p g^{p,-p}$$

such that if $X \in g^{p,-p}$, $XH^{r,s} \subset H^{r+p,s-p}$.
The Lie algebra b of B consists of all those $X \in g$ that preserves the reference Hodge filtration. i.e

$$b = \bigoplus_{p \geq 0} g^{p,-p}$$

Let v be the Lie algebra of $V = G_{\mathbb{R}} \cap B$, then

$$v = g_0 \cap b = g_0 \cap b \cap \bar{b} = b \cap g^{0,0}$$

The holomorphic tangent space of $\check{D} \cong G_{\mathbb{C}}/B$ at the base point is naturally isomorphic to g/b. Under this iso, the action of the isotropy group B on the tangent space correspond to the adjoint action of B on g/b. Consequently, the holomorphic tangent bundle $T \to \check{D}$ coincides with the vector bundle associated to the holomorphic principal bundle

$$B \to G_{\mathbb{C}} \to G_{\mathbb{C}}/B \cong \check{D}$$

$$b \oplus g^{-1,1}/b$$

Defines a $B-$invariant subspace of g/b. Therefore, it gives rise to a B invariant subbundle $T_h(\check{D})$. Call it horizontal tangent bundle.
A holomorphic map $\Psi : M \to \check{D}$ is said to be horizontal if at each point M, the induced map on tangent space takes value in $T_h(\check{D})$.
Construct $T_h(D)$ by restriction. $G_{\mathbb{C}}$-invariance of $T_h(\check{D})$ implies $G_{\mathbb{R}}$-invariance of $T_h(D)$.

1.2 Alternate description of the horizontal tagent bundle

Lemma 1. *At each point $c \in \check{D}$, correpsonding to the filtration $F(c)$, a vector field $X \in g$ takes value in the horizontal tangent space if and only if X regarded as an endomorphism of $H_{\mathbb{C}}$, maps F^p to F^{p-1}*

Proof. At the base point, the value of X lie in

$$b \oplus g^{-1,1}/b$$

Therefore, the value of X can be represented by somebody in $g^{-1,1}$ □

By construction, $\check{D}$ carries a tautological complex vector bundle $H_{\mathbb{C}}(\check{D})$, with fibre $H_{\mathbb{C}}$. The global filtration $\mathbb{F}$ of the bundle, restrict to the fitration of $H_{\mathbb{C}}$ corresponding to the base point. Let

$$\nabla : \mathbb{H}_{\mathbb{C}} \to \mathbb{H}_{\mathbb{C}} \otimes \Omega_{\check{D}}$$

be the flat connection. Let $p : H_{\mathbb{C}\check{D}} \to H_{\mathbb{C}\check{D}}/F^p$, the composition

$$p \circ \nabla F^p \to H_{\mathbb{C}\check{D}}/F^p \otimes \Omega_{\check{D}}$$

is $O_{\check{D}}$-linear.
Consider the holomorphic map $\Psi : M \to \check{D}$ of a complex manifold M into $\check{D}$. The vector bundle $H_{\mathbb{C}}(\check{D})$ pulls back to the trivial bundle $\mathbb{H}_{\mathbb{C}} \to M$
A holomorphic map $\Psi : M \to D$ is horizontal if and only if

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega_D$$

Now, consider a variation of Hodge structure $\{M, \mathbb{H}_{\mathbb{C}}, F\}$. Let $\pi : \tilde{M} \to M$ denote the universal covering of M. The vector bundle $\mathbb{H}_{\mathbb{C}}$ pullback to a trivial bundle $\tilde{M} \times H_{\mathbb{C}}$. The filtration also pulls back. Therefore, for each point $m \in \tilde{M}$, we get a Hodge structure on $H_{\mathbb{C}}$ of weight rank $\mathbb{H}_{\mathbb{C}}$. So we get a map

$$\tilde{\Phi} : \tilde{M} \to D$$

Variation of Hodge structure has the attribute of transversality. Therefore, the map $\tilde{\Phi}$ is automatically horizontal.
$\mathbb{H}_{\mathbb{C}}.H_{\mathbb{Z}}$ and $\mathbb{H}_{\mathbb{C}}.S$ are flat, therefore the action of $\pi_1(M)$ preserves both.
One can think of $\mathbb{H}_{\mathbb{C}}$ on M as a quotient from the universal cover in the first place.
The subgroup

$$\Gamma = \phi(\pi_1(M)) \subset G_{\mathbb{Z}}$$

is called the monodromy group of the variation of Hodge structure.
By construction of $\tilde{\Phi}$, if two points of $\tilde{M}$ are related by some $\sigma \in \pi_1(M)$, the corresponding Hodge structures are related by $\psi(\sigma)$.

$$\tilde{\sigma a} = \psi(\sigma) \circ \tilde{\Psi}(a), a \in \tilde{M}, \sigma \in \pi_1(M)$$

Therefore, $\tilde{\Phi}$ descends to a mapping

$$\Phi : M \to D/\Gamma$$

This map is called Griffith's period mapping for variaiton of Hodge structure.

Theorem 1. *(Griffith) The period mapping is holomorphic, locally liftable to D, and local liftings are horizontal*