Let X be a complex manifold, and let $D \subset X$ be a normal crossing divisor. Let $\mathcal{V}$ be a quasi-unipotent local sytem on $X - D$. Then, the canonical extension V_X of $\mathcal{V}$ has nontrivial parabolic structure. Let $p : Y \to X$ be a branched cyclic cover such that $p^{-1}\mathcal{V}$ is unipotent. Then, we will show that the orbifold bundle on Y corresponding to V_X is the canonical extension of $p^{-1}\mathcal{V}$.
I will use the following example to illutrate the ideas. Let $X = \Delta$, D be the origin, and $\mathcal{V} = (\mathbb{Z}^2, T)$, where T is the diagonal quasi-unipotent matrix

$$T = \begin{bmatrix} \epsilon_3 & 0 \\ 0 & \epsilon_5 \end{bmatrix}$$

where ϵ_i is a primitive i-th root of unity.

1 Construction of the canonical extension of $\mathcal{V}$

In this note, by canonical extension, we mean the extension (V, ∇) of $\mathcal{V}$ such that the eigenvalues of Res(∇) lie in $[0, 1)$
Let

$$M = -\log T = -\begin{bmatrix} \log \epsilon_1 & 0 \\ 0 & \log \epsilon_5 \end{bmatrix}$$

And we use the branch one step before the principal branch for both log functions, *i.e.*

$$M = -\begin{bmatrix} \frac{2\pi i}{3} - 2\pi i & 0 \\ 0 & \frac{2\pi i}{5} - 2\pi i \end{bmatrix}$$

Let $\mathbb{H} \to \Delta^*$ be the universal covering map. Let z and t be the coordinate on Δ and t, we have $z = \exp 2\pi i t$. Let $< e_1, e_2 >$ be the standard basis for $\mathbb{Z}^2$. Consider the vector valued functions on $\mathbb{H}$

$$s_1(t) = \exp(Mt) \otimes e_1, s_2(t) = \exp(Mt) \otimes e_2$$

i.e.

$$s_1(t) = \begin{bmatrix} \exp \frac{4\pi i t}{3} \\ 0 \end{bmatrix} s_2(t) = \begin{bmatrix} 0 \\ \exp \frac{8\pi i t}{5} \end{bmatrix}$$

Use $z = \exp 2\pi i t$, we can regard s_1 and s_2 as multi-valued functions on Δ^*

$$s_1(x) = \begin{bmatrix} x^{\frac{2}{3}} \\ 0 \end{bmatrix}, s_2(x) = \begin{bmatrix} 0 \\ x^{\frac{4}{5}} \end{bmatrix}$$

Let $V = O_X \cdot s_1(x) \oplus O_X \cdot s_2(x)$. Define $\nabla : V \to V \otimes \Omega_X(\log D)$, such that the connection matrix is given by

$$\frac{M}{2\pi i} \cdot \frac{dx}{x}$$

Then,

$$\text{Res}(\nabla) = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$$

This verifies that the (V, ∇) is the canonical extension of $\mathcal{V}$. Use $< e_1, e_2 >$ as the standard basis of $\mathbb{Z}^2$, the inclusion map $\mathcal{V} \to V|_{\Delta^*}$ is given by

$$e_1 \mapsto x^{-\frac{1}{3}} s_1, e_2 \mapsto x^{-\frac{4}{5}} s_2$$

And we can check that the images of e_1 and e_2 are indeed flat sections of ∇.

1.1 The action of x on V_X

This section will explain how x acts on V_X and send V_X to an extension of $\mathcal{V}$ whose residue has eigenvalues in $[1, 2)$. We will this action latter to explain why the morphism I talked about in our last meeting is a morphism of *parabolic* bundles.

x has a natural action on s_1 and s_2

$$x \cdot s_1 = \begin{bmatrix} x^{\frac{4}{3}} \\ 0 \end{bmatrix}, x \cdot s_2 = \begin{bmatrix} 0 \\ x^{\frac{9}{5}} \end{bmatrix}$$

We would have ended up those sections if we used $-4\pi i$ branch for our M. More precisely, if we set

$$M = -\begin{bmatrix} \frac{2\pi i}{3} - 4\pi i & 0 \\ 0 & \frac{2\pi i}{5} - 4\pi i \end{bmatrix}$$

at the very begining. Then, we will get an extention (V^1, ∇^1) with the generating sections

$$x \cdot s_1 \text{and} x \cdot s_1$$

The connection ∇^1 is defined by

$$\frac{M}{2\pi i} \cdot \frac{dx}{x}$$

So

$$\text{Res}(\nabla) = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{9}{5} \end{bmatrix}$$

1.2 The parabolic structure on V_X

Res(∇) defines an endomorphism of $V_X|_D$, the parabolic structure is given by the generalized eigenspaces of this endomorphism. In our example, the story is quite simple. The eigenvalues of Res(∇) are $\frac{2}{3}$ and $\frac{4}{5}$, the corresponding eigenspaces are $\mathbb{C} \cdot s_1$ and $\mathbb{C} \cdot s_2$. The filtration on V_X is

$$V_X = F_1 \supset F_2 = O_X \cdot s_2 \supset V_X(-D)$$

and the weights are $\frac{2}{3}, \frac{4}{5}$.

2 On the branch Cover

Let $p : Y \to X$ be the branched coverd defined by $y^{15} = x$. Let Γ be the Galois group of this covering map. Then, $p^{-1}\mathcal{V} = (\mathbb{Z}^2, T^{15})$ is the trivial local system on Y. So the canonical extension V_Y of $p^{-1}\mathcal{V}$ is the trivial bundle with the trivial connection, *i.e.*

$$V_Y = O_Y \cdot e_1 \oplus O_Y \cdot e_2$$

where $< e_1, e_2 >$ is the standard basis of $\mathbb{Z}^2$. The connection ∇_Y on V_Y has the trivial connection matrix.

V_Y inherits an Γ-action from O_Y, and clearly this makes V_Y into an equivariant bundle.

The Γ-invariant part of $p_* V_Y$

e_1 and e_2 are constant sections on Y, so Γ has trivial action on them. So to compute $(p_* V_Y)^\Gamma$, we only need to compute $(p_* O_Y)^\Gamma$. As an O_X-module, $p_* O_Y$ looks like

$$O_X \cdot 1 \oplus O_X \cdot y \oplus O_X \cdot y^2 \cdots \oplus O_X \cdot y^{14}$$

where y is a local coordinate on Y. Each y^i has a nontrivial Γ-action, *i.e.* if g is a generator of Γ, then $g \cdot y^i = (g \cdot y)^i$. So the only Γ-invariant part of $p_* O_Y$ is $O_X \cdot 1$. Hence,

$$(p_* V_Y)^\Gamma = O_X \cdot e_1 \oplus O_X \cdot e_2$$

The morphism $\alpha : (p_* V_Y)^\Gamma \to V_X$

It is natural to define α as

$$e_1 \mapsto s_1, e_2 \mapsto s_2$$

Let $\tilde{D} = p^{-1} D$. The parabolic structure on $(p_* V_Y)^\Gamma$ is given by

$$(p_* V_Y)^\Gamma_t = (p_* V_Y \otimes O_Y(\left\lfloor t\tilde{D} \right\rfloor))^\Gamma$$