As Prof.Arapura pointed out during our last meeting, the cyclic group action on the equivariant bundle on Y should remember the corresponding local system downstair. I will make it clear soon in this note, and I will define a group action on the equivariant bundle that does the job, and we will see how the Biswas's recipe recovers the parabolic structure downstair. We will see the line bundle case first, so that idea can be easily presented without messy notation, then we will go to general vector bundle case

Line bundle case Let $X = \Delta$ with coordinate x. Let D be the origin. Consider the parabolic line bundle $L = O_X \cdot x^{i/n}$. Let the connection ∇ on L to be the natural one, then the flat sections are $\mathcal{V} = \mathbb{C} < x^{-i/n} \cdot x^{i/n} >$. To simplify things, we write $\mathcal{V} = \mathbb{C} < x^{i/n} >$ so that we have enough symbols to preserve the monodromy infomation.

Let $p : Y \to X$ be defined by $y^n = x$. Let $\tilde{D} = (p^*D)_{\text{red}}$. The Galois group Γ of p is the cyclic group generated by μ_n. $p^{-1}\mathcal{V}$ is the trivial local system, so its canonical extension V_Y is the trivial bundle O_Y with the trivial connection. We define an action of Γ on O_Y such that it encodes the information that V_Y is the canonical extension of $p^{-1}\mathcal{V}$ but not p^{-1}(something like the trivial local system). Let y be the local coordinate on Y. The action we want is

$$\mu_n \cdot y^j = \mu_n^{j/i} \times y^j$$

Then, the Γ-invariant part of O_Y is $O_X \cdot y^i$. Hence

$$(p_* V_Y)^\Gamma = L$$

General vector bundle case Now let $V_X = \bigoplus_{i=1}^{r} O_X \cdot x^{a_i/n_i}$ be a vector bundle of r with the natural connection ∇. It is the canonical extension of $\mathcal{V} = \mathbb{C} < x^{a_1/n_1}, \cdots, x^{a_r/n_r} >$. Let m be the product of n_i and let $\alpha_i \in \mathbb{Z}$ such that

$$\frac{\alpha_i}{m} = \frac{a_i}{n_i}$$

Assume for simplicity that α_i are not redundant and

$$\alpha_1 < \alpha_2 < \cdots < \alpha_r$$

Let $p : Y \to X$ be defined by $y^m = x$. The canonical extension V_Y of $p^{-1}\mathcal{V}$ is the trivial bundle $\bigoplus_{i=1}^{r} O_Y \cdot e_i$ with the trivial connection. Let $(y^{j_1} \cdot e_1, y^{j_2} \cdot e_2, \cdots, y^{j_r} \cdot e_r)$ be a section of V_Y. The Γ-action on V_Y we want should look like

$$\mu_m \cdot y^{j_k} e_k = (\mu_m^{m j_k/\alpha_k} \times y^{j_k}) \cdot e_k$$

Just like the line bundle case, we can see that

$$(p_*(V_Y))^\Gamma = V_X$$

The parabolic structure on $(p_* V_Y)^\Gamma$ is given by $(p_* V_Y \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^\Gamma$.

Now, we see how $(p_* V_Y \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^{\Gamma}$ recovers the parabolic structure on V_X which is defined via the generalized eigenspace of Res∇. In our case, Res∇ is the diagonal matrix

$$\mathrm{diag}(\alpha_1/m, \alpha_2/m, \cdots, \alpha_r/m)$$

Let $A(i)$ the the eigenspace of Res∇, and let $F_i = \bigoplus_{j=i}^{r} A(j)$. Define the subsheaf $\bar{F}_i$ of V_X via the exact sequence

$$0 \to \bar{F}_i \to V_X \to V_X|_D / F_i \to 0$$

We can see that

$$\begin{aligned}
\bar{F}_1 &= V_X \\
\bar{F}_2 &= O_X(-D) \cdot x^{\alpha_1/m} \bigoplus_{i=2}^{r} O_X \cdot x^{\alpha_i/m} \\
\bar{F}_3 &= O_X(-D) \cdot x^{\alpha_1/m} \oplus O_X(-D) \cdot x^{\alpha_2/m} \bigoplus_{i=3}^{r} O_X \cdot x^{\alpha_i/m} \\
&\vdots \\
\bar{F}_k &= \bigoplus_{i=1}^{k-1} O_X(-D) \cdot x^{\alpha_i/m} \bigoplus_{i=k}^{r} O_X \cdot x^{\alpha_i/m}
\end{aligned}$$

The parabolic structure on V_X is given by

$$V_X = \bar{F}_1 \supset \bar{F}_2 \supset \cdots \supset \bar{F}_r \supset \bar{F}_{r+1} = V_X(-D)$$

with weights $\alpha_1/m, \alpha_2/m, \cdots, \alpha_r/m$.
To investigate the parabolic structure on $(p_* V_Y)^{\Gamma}$, we consider first the parabolic structure on each component $(p_* O_Y \cdot e_i)^{\Gamma}$, *i.e.* given the group action Γ on $p_* O_Y \cdot e_i$, we need to figure out when jumps happen for

$$(p_* O_Y \cdot e_i \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^{\Gamma}$$

Set $i = 1$. The Γ-action is given by

$$\mu_m \cdot y^j = \mu_m^{mj/\alpha_1} \times y^j$$

As we let j increase and $j \leq \alpha_1$, the first class of invariant sections showing up are

$$f(x) y^{\alpha_1}$$

where $f(x)$ comes from downstair. Those sections push-forward to $O_X \cdot x^{\alpha_i/m}$. But as soon as $j > \alpha_1$, we will have to wait till $j = m + \alpha_1$ to see next class of invariant sections

$$f(x) x y^{\alpha_1}$$

which push-forward to $O_X(-D)\cdot x^{\alpha_1/m}$
Putting everything together, we see that for $t \leq \frac{\alpha_1}{m}$,

$$(p_*V_Y \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^\Gamma = V_X$$

for $\frac{\alpha_1}{m} < t \leq \frac{\alpha_2}{m}$

$$(p_*V_Y \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^\Gamma = \bar{F}_1$$

Iterate this line of argument, we see that for $\frac{\alpha_k}{m} < t \leq \frac{\alpha_{k+1}}{m}$

$$(p_*V_Y \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^\Gamma = \bar{F}_k$$

This proves that $(p_*V_Y \otimes O_Y(\lfloor -m \times t \rfloor \tilde{D}))^\Gamma$ does recover the parabolic structure on V_X.