# 1 Better description of the orbifold bundle corresponding a parabolic bundle in case of the VSH

Let $X$ be a smooth projective $\mathbb{C}$-variety, let $D$ be a normal crossing divisor on $X$. Let $\mathcal{V}$ be a VHS on $U := X - D$. Let $V$ be the Deligne canonical extension of $\mathcal{V}$. Then, by a theorem of Schmid, the Hodge filtration $F$ on $\mathcal{V}$ extends to a filtration on $V$. Take associated graded of the connection

$$\nabla : V \to \Omega_X(\log D) \otimes V$$

We get a $O_X$-linear map

$$\theta : E \to \Omega_X(\log D) \otimes E$$

Since $\nabla$ is integral, $(E, \theta)$ is a Higgs bundles. In case that the monodromy of $\mathcal{V}$ is unipotent, the parabolic structure on $E$ is trivial. However, if we monodromy of $\mathcal{V}$ is only quasi-unipotent, $E$ will have non-trival parabolic structure.

**Question 1.** *Is there a better description of the orbifold bundle corresponding to $E$?*

Let $p : Y \to X$ be the cyclic covered branched over $D$ such that we can construct an orbifold bundle $W$ (equivariant bundle) on $Y$, corresponding to the parabolic bundle $E$ on $X$. Then, $Y$ should have a better description in this case: pull-back the local system $\mathcal{V}$, and take the Deligne canonical extension.

## 1.1 Defining a parabolic structure on Deligne canonical extension

Assume $D$ is irreducible and smooth. We have a map

$$\text{Res} \circ \nabla : V \to V|_D$$

$V \otimes O_X(-D)$ is contained in the kernel of Res $\circ\, \nabla$. So Res $\circ\, \nabla$ defines an endomorphism

$$\phi : V|_D \to V|_D$$

## 1.2 Main ideas

Think of $V$ as a parabolic bundle on $X$. Let $V'$ denote the Deligne canonical extension of $p^{-1}\mathcal{V}$. To show the orbifold bundle on $Y$ corresponding to the parabolic bundle $V$ on X, we show that

$$(p_*V)^\Gamma = V$$

Several things to prove

**Lemma 1.** *The action $\Gamma$ on $Y$ lifts to a compatible action on $V'$ which makes $V'$ an orbifold bundle.*

*Proof.* Assume $X$ is a curve, and $D \subset X$ is a point. Let $T$ be the monodromy of $\mathcal{V}$. Suppose $T^N$ is unipotent. Let $p : Y \to X$ be the cyclic cover of degree $N$ branched over $D$. Then the monodromy of $p^{-1}\mathcal{V}$ is unipotent. Let $\tilde{D} = p^{-1}(D)$. Then,

$$p : Y - \tilde{D} \to X - D$$

is etale, and we have the following commutative diagram

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ Y - \tilde{D} \arrow[rr] & & X - D \end{tikzcd}$$

where $U$ is the universal cover of $X - D$. Let $\tilde{\Delta}$ and $\Delta$ be small disks around $\tilde{D}$ and $D$ respectively. Then, locally, the above diagram looks like

$$\begin{tikzcd} & \mathbb{H} \arrow[dl] \arrow[dr] & \\ \tilde{\Delta}^* \arrow[rr] & & \Delta^* \end{tikzcd}$$

Let $\tilde{z}$ and $z$ be the local analytic coordinates on $\tilde{\Delta}$ and $\Delta$, and let $t$ be the coordinate on $\mathbb{H}$. Then, we have

$$\tilde{z} = e^{\frac{2\pi i t}{N}}, z = e^{2\pi i t}$$

Suppose $V'$ has rank $r$. Let $e_1, \cdots, e_r$ be the standard basis of $\mathbb{C}^r$. Let $s_i$ be the multivalued functions on $\tilde{\Delta}$ such that pullback of $s_i$ to the universal cover looks like

$$t \mapsto e^{Mt} e_i$$

where $M = -\log T^N$. Then, we use $s_1, \cdots, s_r$ to trivialize $V'$ over $\tilde{D}$

$$V'|_{\tilde{\Delta}} \cong \bigoplus_{i=1}^{r} O_{\tilde{\Delta}} s_i$$

$\Gamma$ acts on $s_i$. Hence, $\Gamma$ acts on $V'$.

∎

On $Y$, we have $V'$ and $p^*V$. Let $M = \text{Monodromy}(\mathcal{V})$, then near $D$

$$V \cong \bigoplus_{i=1}^{r} O_{\Delta} s_i$$

where $s_i(t) = e^{Mt} e_i$, $t$ is the coordinate on the universal cover. So near $\tilde{D}$

$$p^*V \cong = \bigoplus_{i=1}^{r} O_{\tilde{\Delta}} s_i$$

$NM = \text{Monodromy}(\mathcal{V}^{-1})$. So near $\tilde{D}$

$$V' = \bigoplus_{i=1}^{r} O_{\tilde{\Delta}} \tilde{s_i}$$

where $\tilde{s_i} = e^{NMt} e_i$
So there is an obvious morphism

$$\alpha : p^* V \to V^p rime$$

sending $s_i$ to $\tilde{s_i}$. This morphism is not reversible, because taking $N$-th root is not well-defined.

# 2 Perturb parabolic structure with weights in $\mathbb{R}$ to parabolic structure with weights in $\mathbb{Q}$

Let $X$ be a smooth projective $\mathbb{C}$-variety, and let $D$ be a normal crossing divisor on $X$. Let $E$ be a parabolic bundle with weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. In case that $\alpha_i \in \mathbb{Q}$, we can construct an orbifold bundle on some branched cover over $X$. In case that $\alpha_i \in \mathbb{R}$, what do we do? The natural thing is to perturb $\alpha_i$ a little bit to bring them back to $\mathbb{Q}$. Let $Pb()$ be the pertabation operation (there is no reason to believe this operation is canonical). Several things should happen

- There should be a sequence $Pb_\epsilon()$ such that $\epsilon \to 0$ and $Pb_0() = \text{id}$
- parabolic chern classes of $E$ should be equal to parabolic chern classes of $Pb(E)$
- If $E$ is a Higgs sheaf with Higgs field $\theta$ such that

  $$\mathbb{H}^i(X, \text{DR}(Pb_\epsilon(E, \theta))) = 0$$

  then, it should happen that

  $$\mathbb{H}^i(X, \text{DR}(Pb_0(E, \theta))) = 0$$

**Question 2.** *What would be a operator $Pb$ so that everything above is true?*