1 Higgs Bundle Arising from Geometric Variation of Hodge Structure

Now suppose that (V^o, ∇^o) is part of a polarized variation of Hodge structure. Let F^o be the Hodge filtration on V^o. Suppose the monodromy of $\nabla^o)$ is unipotent, then we can extend the filtration F^o to a filtration F on V_* by a theorem of Schmid:

$$F^p V_* = j_*(F^{op} V^o) \cap V_*$$

Let $E = \mathrm{Gr}_F V_*$, and $\theta = \mathrm{Gr}_F \nabla_*$, then (E, θ) is a Higgs bundle with wht following properties

1. The Chern classes of E, in rational cohomology, all vanish.
2. The Higgs bundle is semistable in the sense that $\mu(E') \leq \mu(E) = 0$ for any proper coherent subsheaf stable under θ
3. The Higgs field θ is nilpotent. This follows from the Griffith transversality of ∇_*

If the monodromy of ∇^o is only quasi-unipotent, then we need an intermediate step to construct the extension of F^o. Let $\pi : Y \to X$ be the cyclic cover in section 2. The monodromy of $\pi^* \nabla^o$ is unipotent. Therefore, we can extend $\pi^* F^o$ to a filtration $\bar{F}$ on $\bar{V}$. Let

$$\phi : (\pi_* \bar{V})^G \to V_*$$

be the isomorphism section 2. Then, we define

$$F^p V_* := \phi((\pi_* \bar{F})^G)$$

We will prove that

Lemm 1. *$(E, \theta) = (Gr_F V_*, Gr_F \nabla_*)$ is a Higgs Bundle,* i.e. $\theta \wedge \theta = 0$.

Lemm 2. *The parabolic Chern classes of E is zero*

Lemm 3. *E is parabolic semistable.*

Lemm 4. *θ is nilpotent,* i.e. *∇_* has Griffith transversality with respect to F.*