NOTES ON PARABOLIC HIGGS BUNDLES

1. PARABOLIC BUNDLES

Let X be a smooth projective complex variety with a reduced simple normal crossing divisor $D = \sum D_i$. Let $j : U = X - D \to X$ denote the inclusion of the complement. We fix this notation throughout the paper. For our purposes, a parabolic bundle on (X, D) consists of a vector bundle E on X with an increasing $\mathbb{R}$-indexed filtration $E_\alpha \subset E(*D)$, by locally free $\mathcal{O}_X$-modules such that

P1. $E_0 = E$
P2. $E_{\alpha+1} = E_\alpha(D)$
P3. $E_{\alpha+c} = E_\alpha$ for some $c > 0$ independent of α.
P4. $Gr_\alpha E := E_\alpha/E_{\alpha-\epsilon}$, $0 < \epsilon \ll 1$, is a locally free $\mathcal{O}_D$-module.

This definition is equivalent, with minor changes in notation, to the definition by Yokogawa [**?**, 3.1]. These conditions ensure that the filtration has a finite number of jumps in an interval, i.e. values α such that $Gr_{\alpha_i} E \neq 0$. We arrange the jumps in $[0, 1)$ in increasing order $0 \leq \alpha_1 < \alpha_2 < \ldots \alpha_\ell < 1$. These numbers are called weights. The subsheaves $E_{\alpha_i} \subset E(D)$ determine the whole filtration. Setting $E_i = E_{\alpha_i}$ gives a finite filtration on $E(D)$ called a quasiparabolic structure. From this point of view, a parabolic bundle is a bundle with a quasiparabolic structure plus a choice of weights. Parabolic bundles will be denoted by E_*.

We describe a few basic examples.

Example 1.1. *Any vector bundle E can be given a parabolic structure with integral weights and $E_i = E(iD)$. We refer to this as a trivial parabolic bundle.*

Example 1.2. *Choose a line bundle L and coefficients $\beta_i \in [0, 1)$ and let*

$$L_\alpha = L(\sum \lfloor \alpha + \beta_i \rfloor D_i) \tag{1}$$

Any parabolic line bundle is of this form. Zariski locally, any parabolic bundle is a sum of parabolic line bundles.

Example 1.3. *Suppose that (V^o, ∇^o) is a vector bundle with an integrable connection with regular singularities. By Deligne [**?**] there exists a unique extension*

$$\nabla_\alpha : V_\alpha \to \Omega^1_X(\log D) \otimes V_\alpha$$

with residues having real part in $[-\alpha, 1 - \alpha)$. This again forms a parabolic bundle, that we refer to as the Deligne parabolic bundle. If the monodromy of ∇^o around components of D is unipotent, then V_* has trivial parabolic structure. This is because the eigenvalues of the monodromy of ∇^o around components of D can is given by $\exp(2\pi i\alpha)$. So if the monodromy is unipotent, α must be integers.

A parabolic Higgs bundle on (X, D) is a parabolic bundle E_* together with holomorphic map

$$\theta : E \to \Omega^1_X(\log D) \otimes E$$

such that

$$\theta \wedge \theta = 0$$

and

$$\theta(E_\alpha) \subseteq E_\alpha$$

2. Biswas's correspondence

We will assume in this section that the weights are rational with denominator dividing a fixed positive integer N. Recall that Kawamata [**?**, thm17] has constructed a smooth projective variety Y, and a Galois covering $\pi : Y \to X$, such that $\pi^* D_i = k_i N(\tilde{D}_i)$ for some $k_i > 0$, where $\tilde{D}_i = (\pi^* D_i)_{red}$. Let G denote the Galois group. A G-equivariant vector bundle on Y, is a bundle $p : V \to Y$ (viewed geometrically rather than as a sheaf) on which G acts compatibly with p.

We list some basic classes of examples.

Example 2.1. *X, then $\pi^* V'$ can made into a G-equivariant bundle, so that the projections p*

$$\begin{tikzcd} \pi^* V' \arrow[r] \arrow[d, "p"] & V' \arrow[d, "p"] \\ Y \arrow[r, "\pi"] & X/ \end{tikzcd}$$

are compatible with the G-action.

Example 2.2. *The line bundle $\mathcal{O}_Y(\tilde{D}_i)$ has an equivariant structure compatible with the one on $\pi^* \mathcal{O}_X(D_i)$ under the isomorphism $\mathcal{O}_Y(\tilde{D}_i)^{\otimes k_i N} \cong \pi^* \mathcal{O}_X(D_i)$.*

Theorem 2.3 (Biswas [**?**])**.** *There is an equivalence $E_* \to \tilde{E}$ between the category of parabolic bundles on X with weights in $\frac{1}{N}\mathbb{Z}$ and G-equivariant bundles on Y.*

We recall the construction in one direction. Given an equivariant bundle $\tilde{E}$ on Y, we obtain a parabolic bundle

$$E_\alpha = \pi_*(\mathcal{E} \otimes \mathcal{O}_Y(\lfloor \alpha \pi^* D \rfloor))^G$$

where $\lfloor \alpha \pi^* D \rfloor = \sum_i \lfloor \alpha k_i N \rfloor \tilde{D}_i$.

Suppose that (V^o, ∇^o) is a vector bundle with connection satisfying the assumptions of example 1.3. In addition suppose that the eigenvalues of the monodromy around D are Nth roots of unity. Then the weights of the Deligne parabolic bundle lie in $\frac{1}{N}\mathbb{Z}$. Furthermore $(\tilde{V}^o, \Box^o) = (\pi^* V^o, \pi^* \nabla^o)$ has unipotent local monodromies. Let (V_*, ∇_*) and $(\tilde{V}, \Box)$ denote Deligne's extensions of V^o and $\tilde{V}^o$.

We prove the following

Lemma 2.4. *There is an isomorphism of vector bundle*

$$\phi : \pi^* V_* \to \tilde{V}$$

sending flat sections of $\pi^* \nabla_*$ to flat sections of $\Box$.

Proof. We will describe this morphism locally, and show it glues. Take a point $y \in Y$, and let $x = \pi(y)$. Let W and U be polydisc neighborhoods of y and x, with coordinates $(y_1, y_2, \cdots, y_d)$ and $(x_1, x_2, \cdots, x_d)$, respectively. Without loss of generality, we may assume that $\tilde{D}_i$ is locally defined by y_i and D_i is locally defined by x_i. As $\pi^* D_i = k_i N \tilde{D}_i$, we have

$$x_i = y_i^{k_i N}$$

On $U-D$ let $< s_1, \cdots, s_r >$ be a free basis of V^o,the connection matrix of ∇_* be

$$A_1\frac{dx_1}{x_1}+A_2\frac{dx_2}{x_2}+\cdots+A_s\frac{dx_s}{x_s}$$

The generalized eigenvalues of A_i are α_i, the weights of V_* on D_i. So on W, with respect to the frame $< e_1, \cdots, e_r >= \pi^* < s_1, \cdots, s_r >$ the connection matrix of $\pi^*\nabla_*$ is

$$k_1NA_1\frac{dw_1}{w_1}+k_2NA_2\frac{dw_2}{w_2}+\cdots+k_sNA_s\frac{dw_s}{w_s}$$

Write $B_i = k_iNA_i$. Let $J(B_i)$ be the Jordan canonical form of B_i, and let D_i be the invertible part of $J(B_i)$. Write $N_i = B_i - D_i$. Then, N_i is nilpotent, and the conection matrix of $\tilde{V}$, with respect to the frame $< e_1, \cdots, e_r >$ is

$$N_1\frac{dy_1}{y_1}+N_2\frac{dy_2}{y_2}+\cdots+N_s\frac{dy_s}{y_s}$$

Let

$$F=\exp-(N_1\log y_1+N_2\log y_2+\cdots+N_s\log y_s)$$

Then, the flat sections of $\tilde{V}$ has coordinates

$$F_1, F_2, \cdots, F_r$$

where F_i is the i-th column of F.

Similarly, let

$$\begin{aligned}G&=\exp-(B_1\log y_1+B_2\log y_2+\cdots+B_s\log y_s)\\&=\prod_{i=1}^{s}y_i^{-D_i}F\end{aligned}$$

Then, the flat sections of $\pi^*\nabla_*$ has coordinates

$$G_1, G_2, \cdots, G_r$$

where G_i is the i-th column of G.

Let d_i^j be the j-th eigenvalue of D_i. The map

$$\begin{aligned}\pi^*V_*&\to\tilde{V}\\e_j&\mapsto\prod_{j=1}^{s}y_j^{d_i^j}e_j\end{aligned}$$

sends flat sections to flat sections. Hence, it glues to a morphism of vector bundles

$$\phi:\pi^*V_*\to\tilde{V}$$

□

Lemma 2.5. *$\tilde{V}$ admits a G-equivariant action, and Biswas' construction applied to $\tilde{V}$ yields V_*.*

Proof. Let $\phi : \pi^*V_* \to \tilde{V}$ be the morphism from 2.4. Set d_i to be the largest eigenvalue of D_i. Then,

$$\phi\otimes O_Y(\sum_{i=1}^{s}d_i\tilde{D}_i):\pi^*V_*\otimes O_Y(\sum_{i=1}^{s}d_i\tilde{D}_i)\to\tilde{V}$$

is an isomorphism. Therefore, we can define an equivariant G-action on $\tilde{V}$ via ϕ. Explicitly, write the Galois group of $\pi : Y \to X$ as

$$G = \bigoplus_{i=1}^{s} \mathbb{Z}/k_i N$$

Let μ_i be a generator of $\mathbb{Z}/k_i N$, *i.e.* a primitive $k_i N$-th root of unity. Then

$$\mu_i \cdot e_j = y_i^{d_i^j} e_j$$

where d_i^j is the j-th eigenvalue of D_i.

The G-invariant sections of $\tilde{V}$ are precisely the G-invariant sections of $\pi^* V_*$, and G-invariant section of $\pi^* V_*$ are V_*. Therefore, we have the identification

$$(\pi_* \tilde{V})^G = V_*$$

To show $(\pi_* \tilde{V} \otimes O_Y(\lfloor \alpha \pi^* D \rfloor))^G$ recovers the parabolic structure of V_*, we show it locally by decomposing them into sum of line bundles. Hence, it is enough to assume V_* is a line bundle.

Use the notation from 2.4. $A_i = \alpha_i$ for some rational number α_i. The parabolic structure of V_* is thus

$$V_{*\alpha} = O_X(\lfloor \alpha - \alpha_1 \rfloor D_1 + \cdots + \lfloor \alpha - \alpha_s \rfloor D_s)$$

which is the parabolic structure of $(\pi_* \tilde{V} \otimes O_Y(\lfloor \alpha \pi^* D \rfloor))^G$. □

Biswas's correspondence extends to Higgs bundles as well [**?**, thm 5.5]. Now suppose that (V^o, ∇^o) is part of a polarized variation of Hodge structure etc.

3. Parabolic Chern classes

Given a parabolic line bundle with notation as in 1.2

$$\text{par-c}_1(L_*) = c_1(L) \pm \sum \beta_i [D_i]$$

At this point, we need to check signs. The correct sign is the one which makes $\pi^* \,\text{par-c}_1(L_*) = c_1(\mathcal{L})$ true, where $\mathcal{L}$ corresponds to L_* under Biswas.

Given a parabolic bundle E_*, the top exterior power $\det E$ carries an induced parabolic structure. Set $\text{par-c}_1(E_*) = \text{par-c}_1(\det E_*)$. Fix an ample line bundle H on X. Let $d = \dim X$. The parabolic degree of a parabolic bundle E_* is $c_1(\det E_*) \cdot H^{d-1}$. We can define (semi)stability of parabolic and parabolic Higgs bundles using this [**?**, **?**]. Under Biswas's correspondence semistable parabolic bundles (Higgs bundles) with rational weights correspond to semistable equivariant (Higgs) bundles.

Given a parabolic bundle E_*, let $p : Fl(E) \to X$ denote the full flag bundle of E. The pullback $p^* E$ carries a filtration $F^i \subset E$ by subbundles such that associated graded $G^i = F^i / F^{i+1}$ are line bundles. The parabolic structure on E can be pulled back to a parabolic structure on $p^* E$ along $\pi^* D$, and G^i carry induced parabolic structures.

Lemma 3.1. *The classes c_i defined below*

$$1 + c_1 + c_2 + \ldots = \prod (1 + \text{par-c}(G_*^i))$$

are pullbacks of classes $\text{par-c}_i(E_*) \in H^{2i}(X, \mathbb{R})$.

Since the map $H^*(X) \to H^*(Fl(E))$ is injective, the above property determines the above classes. The following is stated in [**?**, 4.6].

Lemma 3.2. *Suppose that E_* has weights in $\frac{1}{N}\mathbb{Z}$. Let $p : Y \to X$ and $\mathcal{E}$ be as in theorem 2.3, then $p^* \operatorname{par-c}_i(E_*) = c_i(\mathcal{E})$.*

Proof. Fill in ☐

4. Vanishing

Proposition 4.1. *Let (E_*, θ) be a semistable Higgs bundle with zero parabolic Chern classes. There exists a parabolic bundle (E'_*, θ') with the same properties and rational weights with $(E, \theta) = (E', \theta')$.*

Proof. Fill in ☐

Given a Higgs bundle (E, θ), we have complex

$$DR(E, \theta) = E \xrightarrow{\theta} \Omega_X(\log D) \otimes E \to \dots$$

Theorem 4.2. *Let (E_*, θ) be a semistable parabolic Higgs bundle on X with vanishing Chern classes and with θ nilpotent. Then*

$$H^i(DR(E, \theta) \otimes L) = 0$$

for $i > d$.

Sketch. Use above prop to reduce to case, where E has rational weights. By Biswas, it corresponds to G-equivariant Higgs bundle $(\tilde{E}, \tilde{\theta})$ on Y. We can apply the first main theorem of [?] to conclude

$$H^i(Y, DR(\tilde{E}, \theta) \otimes \pi^* L) = 0$$

Now take G-invariants. ☐

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