

1 Better description of the orbifold bundle corresponding a parabolic bundle in case of the VSH

Let X be a smooth projective \mathbb{C} -variety, let D be a normal crossing divisor on X . Let \mathcal{V} be a VHS on $U := X - D$. Let V be the Deligne canonical extension of \mathcal{V} . Then, by a theorem of Schmid, the Hodge filtration F on \mathcal{V} extends to a filtration on V . Take associated graded of the connection

$$\nabla : V \rightarrow \Omega_X(\log D) \otimes V$$

We get a \mathcal{O}_X -linear map

$$\theta : E \rightarrow \Omega_X(\log D) \otimes E$$

Since ∇ is integral, (E, θ) is a Higgs bundles. In case that the monodromy of \mathcal{V} is unipotent, the parabolic structure on E is trivial. However, if we monodromy of \mathcal{V} is only quasi-unipotent, E will have non-trivial parabolic structure.

Question 1. *Is there a better description of the orbifold bundle corresponding to E ?*

Let $p : Y \rightarrow X$ be the cyclic covered branched over D such that we can construct an orbifold bundle W (equivariant bundle) on Y , corresponding to the parabolic bundle E on X . Then, Y should have a better description in this case: pull-back the local system \mathcal{V} , and take the Deligne canonical extension.

1.1 Defining a parabolic structure on Deligne canonical extension

Assume D is irreducible and smooth. We have a map

$$\text{Res} \circ \nabla : V \rightarrow V|_D$$

$V \otimes \mathcal{O}_X(-D)$ is contained in the kernel of $\text{Res} \circ \nabla$. So $\text{Res} \circ \nabla$ defines an endomorphism

$$\phi : V|_D \rightarrow V|_D$$

1.2 Main ideas

Think of V as a parabolic bundle on X . Let V' denote the Deligne canonical extension of $p^{-1}\mathcal{V}$. To show the orbifold bundle on Y corresponding to the parabolic bundle V on X , we show that

$$(p_* V)^\Gamma = V$$

Several things to prove

Lemma 1. *The action Γ on Y lifts to a compatible action on V' which makes V' an orbifold bundle.*

Proof. Assume X is a curve, and $D \subset X$ is a point. Let T be the monodromy of \mathcal{V} . Suppose T^N is unipotent. Let $p : Y \rightarrow X$ be the cyclic cover of degree N branched over D . Then the monodromy of $p^{-1}\mathcal{V}$ is unipotent. Let $\tilde{D} = p^{-1}(D)$. Then,

$$p : Y - \tilde{D} \rightarrow X - D$$

is etale, and we have the following commutative diagram

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ Y - \tilde{D} & \longrightarrow & X - D \end{array}$$

where U is the universal cover of $X - D$. Let $\tilde{\Delta}$ and Δ be small disks around \tilde{D} and D respectively. Then, locally, the above diagram looks like

$$\begin{array}{ccc} & \mathbb{H} & \\ \swarrow & & \searrow \\ \tilde{\Delta}^* & \longrightarrow & \Delta^* \end{array}$$

Let \tilde{z} and z be the local analytic coordinates on $\tilde{\Delta}$ and Δ , and let t be the coordinate on \mathbb{H} . Then, we have

$$\tilde{z} = e^{\frac{2\pi i t}{N}}, z = e^{2\pi i t}$$

Suppose V' has rank r . Let e_1, \dots, e_r be the standard basis of \mathbb{C}^r . Let s_i be the multivalued functions on $\tilde{\Delta}$ such that pullback of s_i to the universal cover looks like

$$t \mapsto e^{Mt} e_i$$

where $M = -\log T^N$. Then, we use s_1, \dots, s_r to trivialize V' over \tilde{D}

$$V'|_{\tilde{\Delta}} \cong \bigoplus_{i=1}^r O_{\tilde{\Delta}} s_i$$

□

2 Perturb parabolic structure with weights in \mathbb{R} to parabolic structure with weights in \mathbb{Q}

Let X be a smooth projective \mathbb{C} -variety, and let D be a normal crossing divisor on X . Let E be a parabolic bundle with weights $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$. In case that $\alpha_i \in \mathbb{Q}$, we can construct an orbifold bundle on some branched cover over X . In case that $\alpha_i \in \mathbb{R}$, what do we do? The natural thing is to perturb α_i a little bit to bring them back to \mathbb{Q} . Let $Pb()$ be the pertabation operation (there is no reason to believe this operation is canonical). Several things should happen

- There should be a sequence $Pb_\epsilon()$ such that $\epsilon \rightarrow 0$ and $Pb_0() = \text{id}$
- parabolic chern classes of E should be equal to parabolic chern classes of $Pb(E)$
- If E is a Higgs sheaf with Higgs field θ such that

$$\mathbb{H}^i(X, \text{DR}(Pb_\epsilon(E, \theta))) = 0$$

then, it should happen that

$$\mathbb{H}^i(X, \text{DR}(Pb_0(E, \theta))) = 0$$

Question 2. *What would be a operator Pb so that everything above is true?*