Analysis linear elastic isotropic material with continuum mechanics

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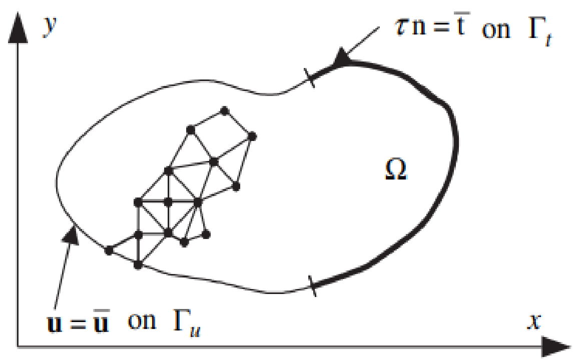
1 Introduction

The main goal of the project is solving elastostaics problems by Python based on the learning of the lecture. Its solution is expected to be same with that of commercial software such as ANSYS or Abaqus. The relation between stress and displacement is defined by constitutive relation. Materials are assumed as a linear elastic isotropic material. Small strain theory is used for kinematic relation. Input of the algorithm is known displacement (Dirichlet boundary condition) and force (Neumann boundary condition). Based on constitutive relation, kinematic relation and boundary conditions, unknown displacement vector and force vector as well as stress vectors are determined. For 2-D model, plane stress theory will be applied to the model. Various visualizations of displacement and stress will be introduced and discuss the trend of stresses. Designed algorithm's answer will be verified by analytical solution. Finally, future work will be discussed.

2 Explain linear elastic isotropic material model



<Figure 1. Boundary conditions and elements on domain>

Figure 1 is general domain of material with boundary conditions. $\Gamma_u (= \partial\Omega_u)$ is Dirichlet boundary related with displacements, and $\Gamma_t (= \partial\Omega_t\ )$ is Neumann boundary related with force. And Ω is whole domain of a model.

2.1 Partial differential equation

$$\int \frac{\partial \sigma_{ij}}{\partial x_j} dV + \int f_i \, dV = \int \bar{t}_i \, dS$$

$$where, \int \frac{\partial \sigma_{ij}}{\partial x_j} dV - \ \ stress\ term$$

$$\int f_i \, dV - \ \ Internal\ force\ term$$

$$\int \bar{t}_i \, dS - External\ force\ term$$

In the equation, it is possible to know there must be the balance among internal stress, internal force and external force. Since stress and internal force occur in whole domain, they are integrated by volume (dV). But external forces are applied at specific surfaces, so, it is integrated by surface (dS).

2.2 Constitutive relation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

In the equation, it is possible to know the relation between stress and strain. Where, σ_{ij} is stress, C_{ijkl} is material tensor which is composed of Young's modulus and Poisson's ratio, and ε_{kl} is strain.

2.3 Kinematic relation

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

It is small strain equation. That is, it is not rigid body and deformed by both displacement and rotation. Where ε_{kl} is strain, and u_k is displacement.

2.4 Dirichlet BC (Displacement)

$$u_i = u_i^g \, on \; \partial\Omega_{u_i}$$

$on\ \partial\Omega_{u_i}$, the amount of displacements (u) are known. It is required to solve the PDE equation.

2.5 Neumann BC (Traction)

$$\sigma_{ij}n_j = h_i \; on \; \partial\Omega_{h_i}$$

$On\ \partial\Omega_{h_i}$, the magnitude of forces (f) are known. It is required to solve the PDE equation.

2.6 Combine equations

As combining all previous equations, following equation can be achieved.

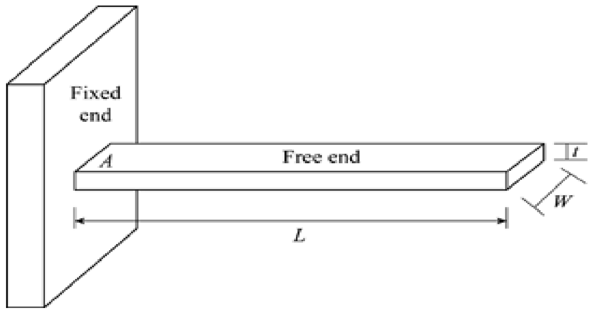
$$\int \frac{1}{2} \cdot \frac{\partial}{\partial x_j}\left(C_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right) dV + \int f_i \, dV = \int \sigma_{ij}n_j \, dS$$

The 2nd order PDE equation will be solved with two boundary conditions to get required answer such as unknown displacement. Force and stress vectors can be calculated by the displacement vector. In the project, internal force is ignored. So, equation becomes

$$\int \frac{1}{2} \cdot \frac{\partial}{\partial x_j}\left(C_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right) dV = \int \sigma_{ij}n_j \, dS$$

3.Define problem

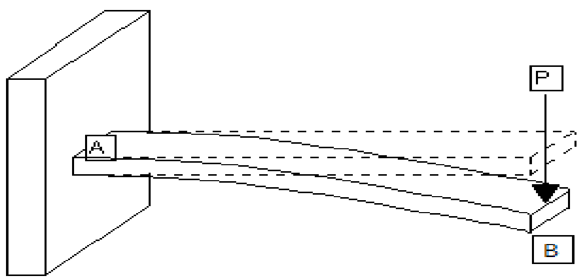
3.1 Geometry



<Figure 2. Geometry of the beam>

Dimension of the material would be 10 by 2 by 2 inches. In the figure, L is 10, W and t are 2 inches. Young's modulus is 29e6 psi and Poisson's ratio is 0.3. In Python, 3D array is designed. Python numpy library has function for generating mesh with the length of the material. The type of the mesh is rectangle/hexahedron mesh.

3.2 Boundary condition (BC)



<Figure 3. Boundary conditions of the beam>

Cantilever beam will be analyzed. There are two types of boundary conditions such as Dirichlet boundary conditions related with displacement and Neumann boundary conditions related with force. In figure 3, there is not any displacement at point A (x = 0 inches) which is Dirichlet BC. That is, $u_x = u_y = u_z = 0$. And There is down direction force at point B (x = 10 inches) which is Neumann BC. For example, $f_x = f_y = 0 \; and \; f_z = -p$. The magnitude of the p is 10e3 lbf.

3.3 Elasticity matrix (D)

In the case of 2-D, the model is assumed as plane stress case. In the case, stress component normal to the element plane is zero. It is suitable for the thin geometry such as beam element.

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

In the case of 3-D, it has only one elasticity matrix. So, it is not necessary to think whether it is plane stress or plane strain.

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

3.4 Principal stress

$$\begin{vmatrix} \sigma_x - \lambda & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \lambda & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \lambda \end{vmatrix} = 0$$

In order to calculate principal stress, it is required to solve above determinant equation. Then, three eigen values of the system is calculated. Then they are three principal stress maximum principal (σ_1), intermediate principal (σ_2) and minimum principal (σ_3) stress of the system. Maximum shear stress can be achieved by principal stress, such as σ_1 *and* σ_3 .

$$\tau_{max} = \frac{|\sigma_1 - \sigma_3|}{2}$$

3.5 Von Mises stress

$$\sigma_{von} = \sqrt{\frac{1}{2}\left[\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \left(\sigma_{yy} - \sigma_{zz}\right)^2 + (\sigma_{xx} - \sigma_{zz})^2\right] + 3\left(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2\right)}$$

$$\sigma_{von} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]}$$

Von Mises stress is a value used to determine whether a given material will yield or fracture. It is commonly used for ductile materials, such as metals. The von Mises yield criterion states that if the von Mises stress of a material under load is equal or greater than the yield limit of the same material under simple tension then the material will yield. First equation is derived by original stress tensor, and second equation is derived by principal stress tensor.

3.6 Deviatoric stress

$$[\sigma_{DEV}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} - \begin{bmatrix} \sigma_{HYD} & 0 & 0 \\ 0 & \sigma_{HYD} & 0 \\ 0 & 0 & \sigma_{HYD} \end{bmatrix}$$

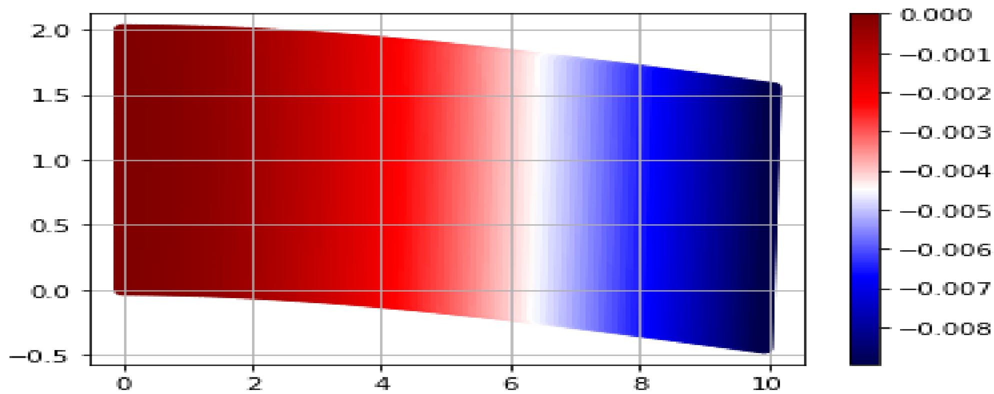
$$where, \sigma_{HYD} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

First, hydrostatic stress is calculated, which is average of principal stress. Hydrostatic stress tensor's diagonal values are average of principal stresses (σ_{HYD}), as rest elements are zero. Deviatoric stress is a difference between stress tensor and hydraulic stress tensor.

4 Result

In the section, results of 2-D and 3-D will be discussed. All results are achieved through previous sections theories and equations. First, displacement shape will be shown. Second, various stress distributions' result will be discussed.

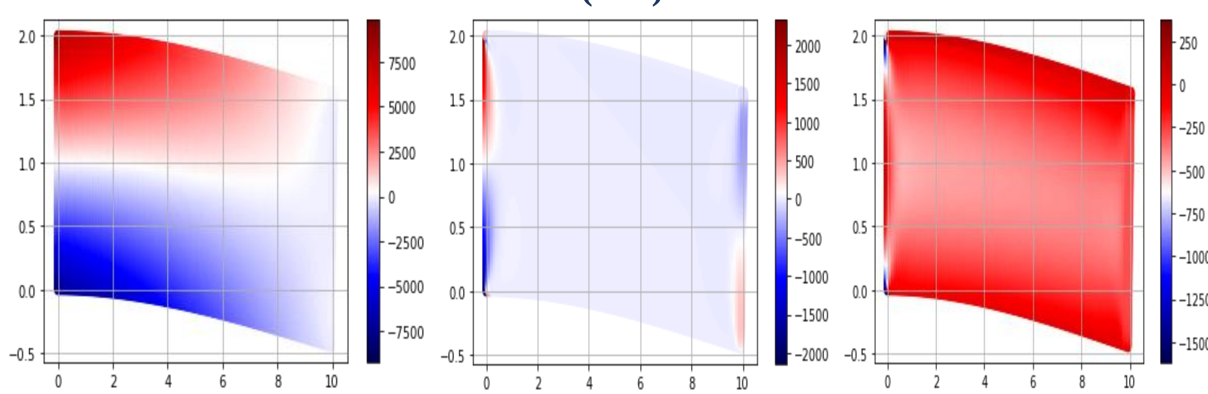
4.1 Results of 2-D case



<Figure 4. Displacement of 2-D beam>

Figure 4 is the deformed shape as it is under the point load at the end of the beam. Total number of nodes are 10000. The color distribution means y-direction displacement. As x value increases, displacement increases. Unit of the result is inches.

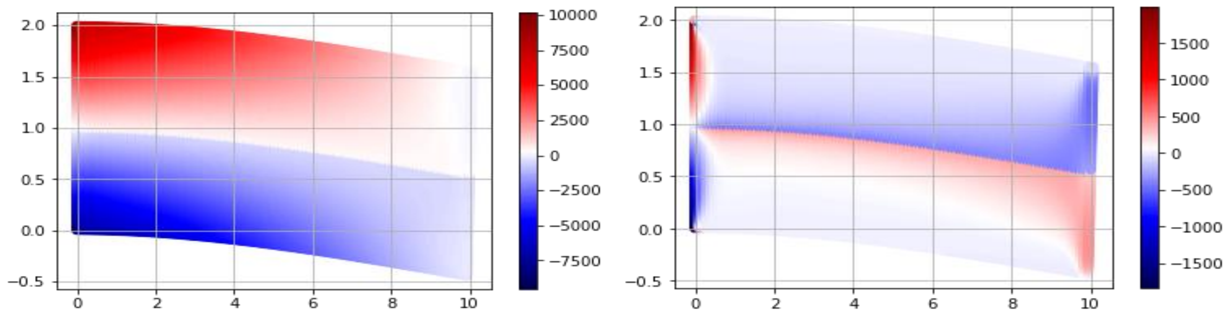
4.1.1 Normal and shear stress result (2-D)



<Figure 5. Normal and shear stress of the beam>

From left to right, x-normal stress (σ_x), y-normal stress (σ_y) and shear stress (τ_{xy}) at figure 5. Trend of σ_x is very clear. Upper half of the beam get traction and lower half of the beam get compression. For σ_y, it is possible to check stresses occur each end edges, and their trends are opposite. Except fixed end's top and bottom points, most of positions' shear stress looks very small. Unit of the results are psi.

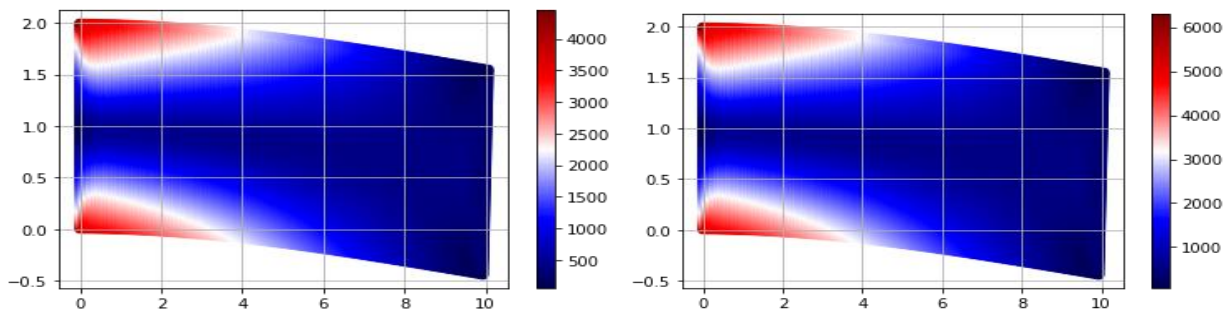
4.1.2 Principal stress result (2-D)



<Figure 6. Principal stresses>

Figure 6 is the result of principal stress. From left to right, maximum principal stress (σ_1) and minimum principal stress (σ_2). σ_1 seems same with σ_x, but its magnitude is larger than σ_x. And σ_2's trend looks similar with σ_y. But its magnitude is smaller than σ_y. Unit of the results are psi.

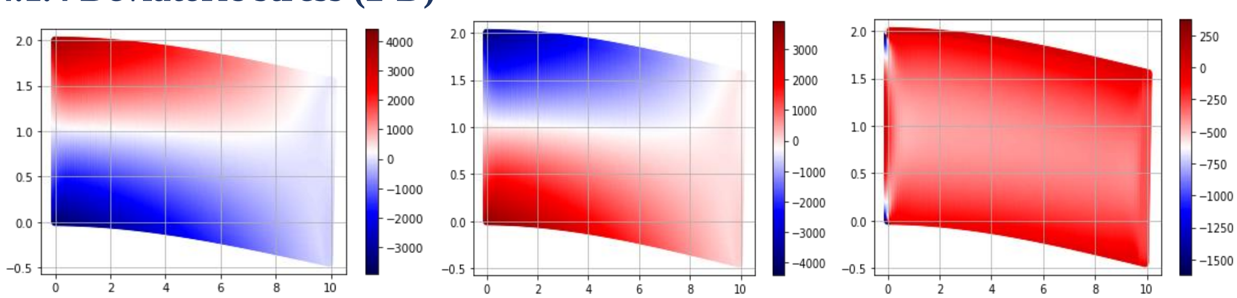
4.1.3 Maximum shear and Von Mises stress result (2-D)



<Figure 7. Maximum shear and Von Mises stress>

From left to right, maximum shear stress (τ_{max}) and Von Mises stress (σ_{Von}). Their stress distribution is almost same, but the value of Von Mises stress is larger than that of maximum shear stress. The magnitude of stress at top and bottom points of fixed end are very high. That is, fracture will start from top or bottom points of fixed end.

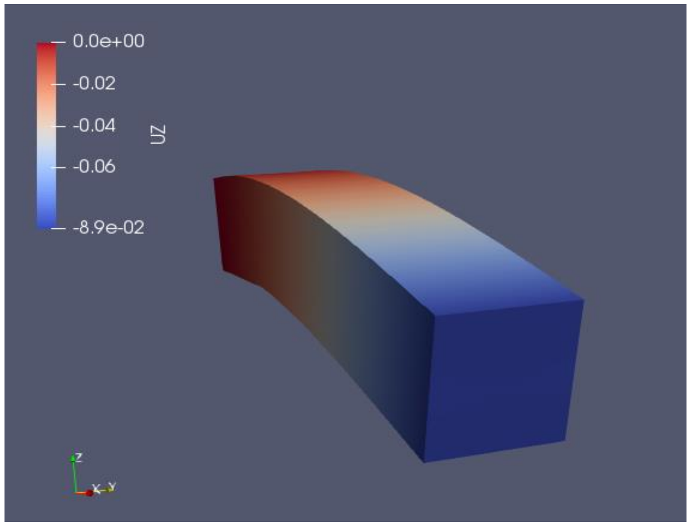
4.1.4 Deviatoric stress (2-D)



<Figure 8. Deviatoric stress>

Figure 8 is deviatoric stress distribution at the beam. From left to right, x-normal stress (σ_x), y-normal stress (σ_y) and shear stress (τ_{xy}) of deviatoric stress. It is easy to check the trend of σ_x and σ_y are opposite. τ_{xy} of deviatoric stress is same with that of normal stress. Their unit is psi.

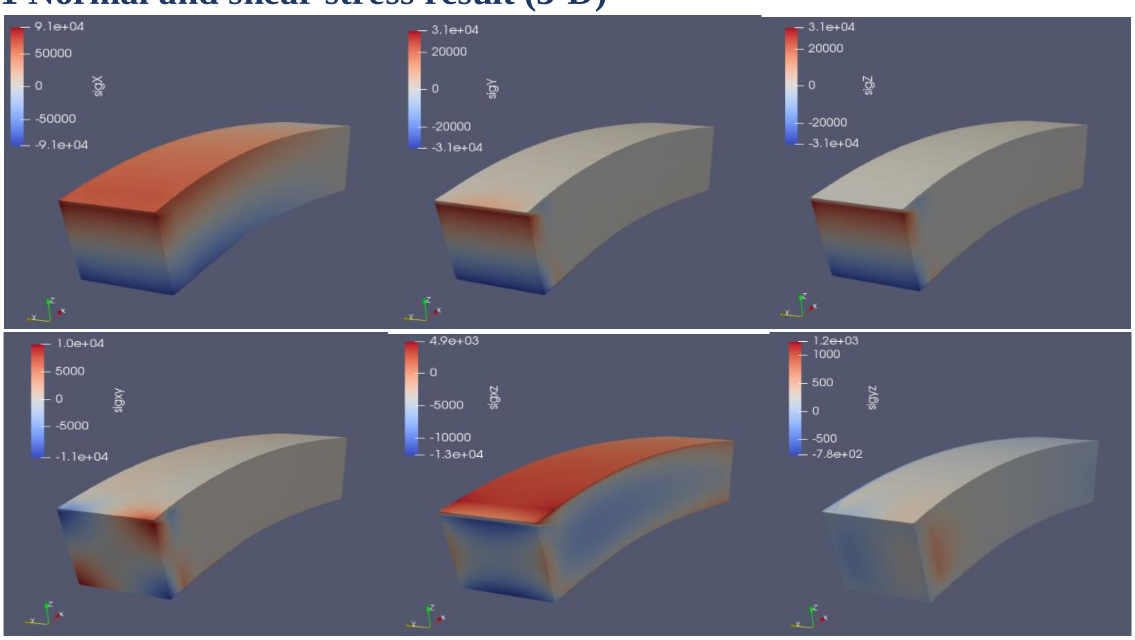
4.2 Results of 3-D case



<Figure 9. Displacement of 3-D beam >

Figure 9 is the deformed shape as it is under the point load at the end of the beam. The color distribution means z-direction displacement and unit is inches. As x value increases, displacement increases. The number of nodes is 20 by 20 by 20. It is visualized by Paraview.

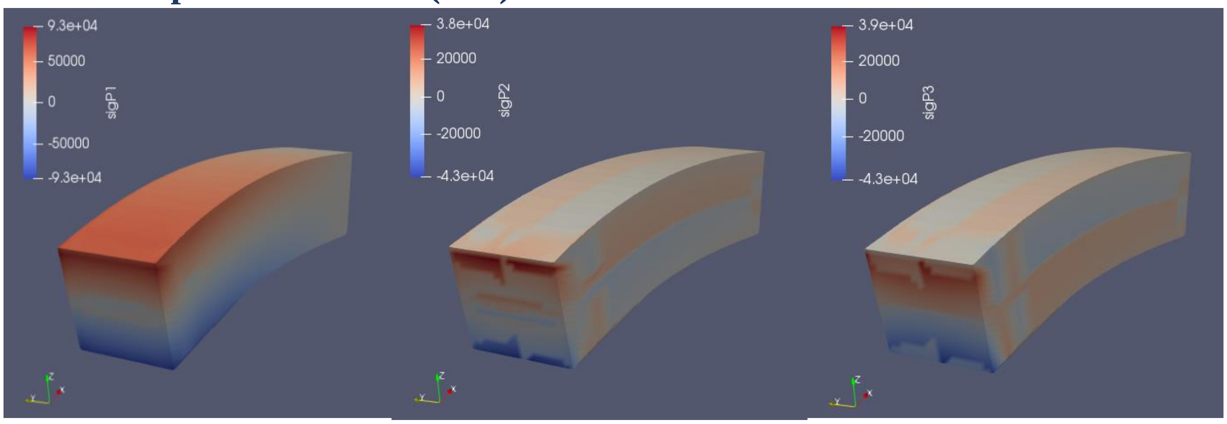
4.2.1 Normal and shear stress result (3-D)



<Figure 10. Normal stress of 3-D beam >

From left to right, x-normal stress (σ_x), y-normal stress (σ_y) and z-normal stress (σ_z) at first line and from left to right, xy-shear stress (τ_{xy}), xz-shear stress (τ_{xz}) and yz-shear stress (τ_{yz}) at second line. Trend of σ_x is very clear. Upper half of the beam get traction and lower half of the beam get compaction. The stresses of σ_y and σ_z occur on the fixed surface, and free ends stress values look like zero. Their unit is psi.

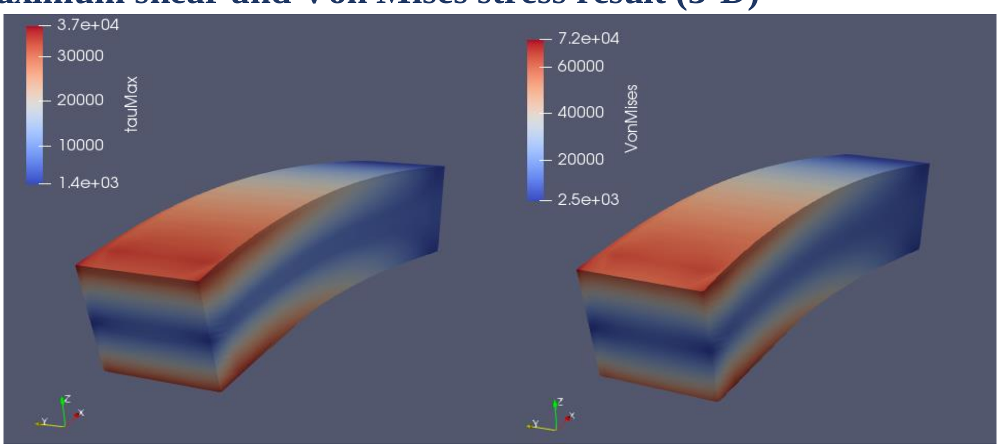
4.2.2 Principal stress result (3-D)



<Figure 11. Principal stress>

From left to right, maximum principal stress (σ_1), middle principal stress (σ_2) and minimum principal stress (σ_3). σ_1's trend is same with that of σ_x. Upper half of the beam get traction and lower half of the beam get compaction. The trend of stress distribution of σ_2 and σ_3 is opposite each other. Their unit is psi.

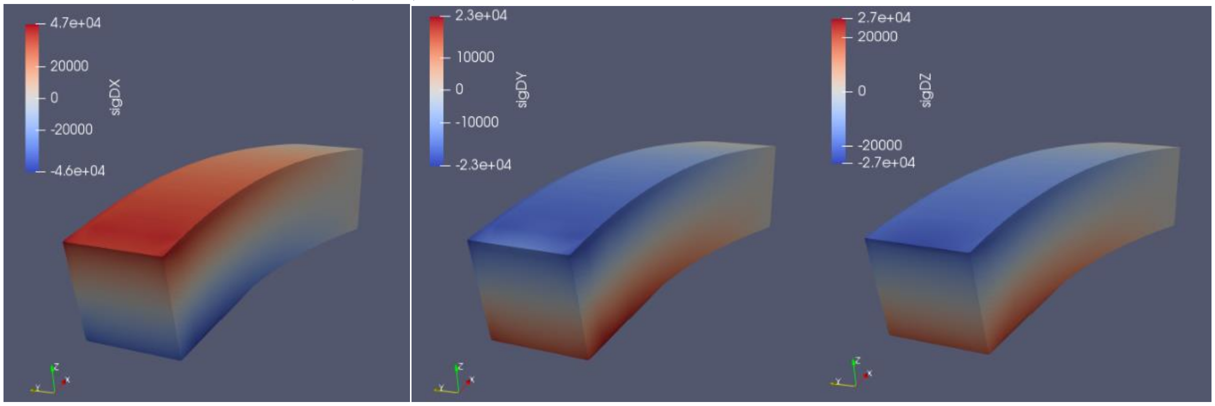
4.2.3 Maximum shear and Von Mises stress result (3-D)



<Figure 12. Maximum shear and Von Mises stress >

Left figure of figure 12 is maximum shear stress distribution. Right figure is Von Mises stress distribution. It is easy to see their distribution trends are same. Fixed end surface's lower and upper edges' stresses are strong. But Von Mises's values are twice of those of maximum shear. The magnitude of stress at top and bottom edges of fixed end surface are very high. That is, fracture will start from top or bottom edges of fixed end. Their unit is psi.

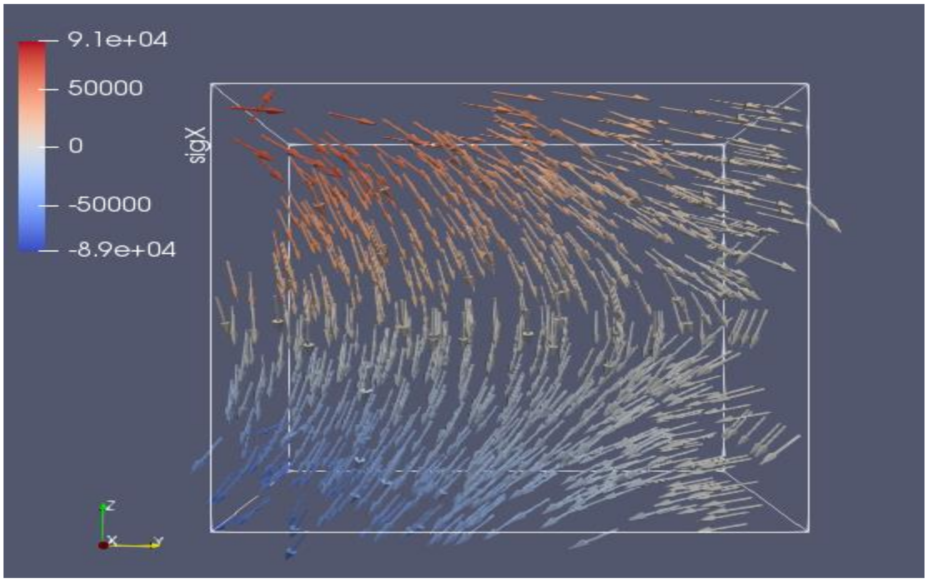
4.2.4 Deviatoric stress (3-D)



<Figure 13. Deviatoric stress >

Figure 13 is deviatoric stress distribution at the beam. From left to right, x-normal stress (σ_x), y-normal stress (σ_y) and z-normal stress (σ_Z) of deviatoric stress. The shear stress such as τ_{xy}, τ_{xz} and τ_{yz} are omitted because they are same with those of original stress. σ_y and σ_z show same trend and they are opposite with σ_x. Their unit is psi.

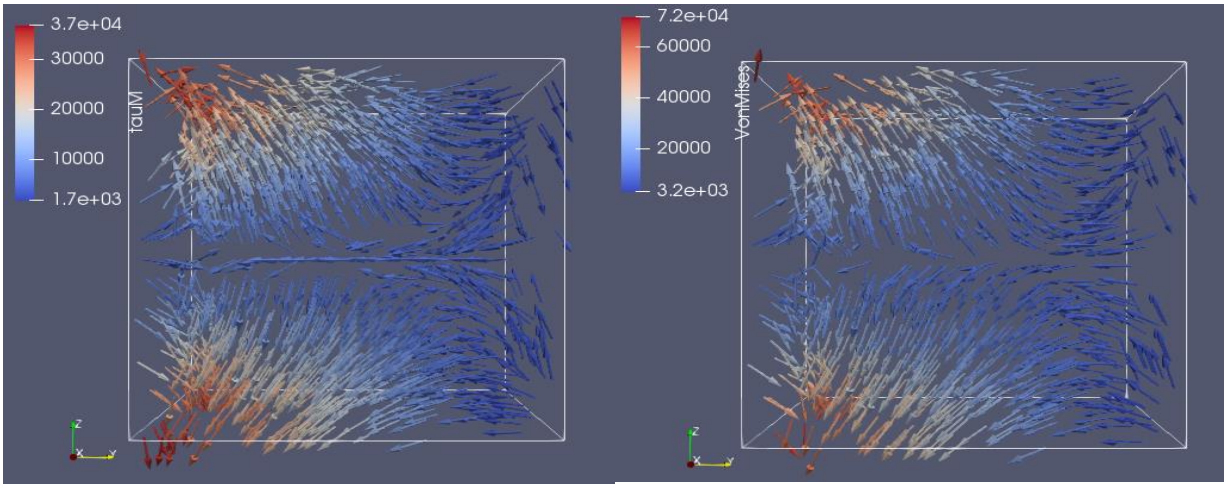
4.2.5 Vector field of x-normal stress



<Figure 14. Vector field of σ_x>

Figure 14 is vector field of x-normal stress (σ_x). Its direction is normal of contour surfaces of the x-normal stress. It is very obvious its direction is same with displacement direction of the model. The direction of gradient of contour surfaces is opposite direction of figure 14.

4.2.6 Vector field of maximum shear and Von Mises stress



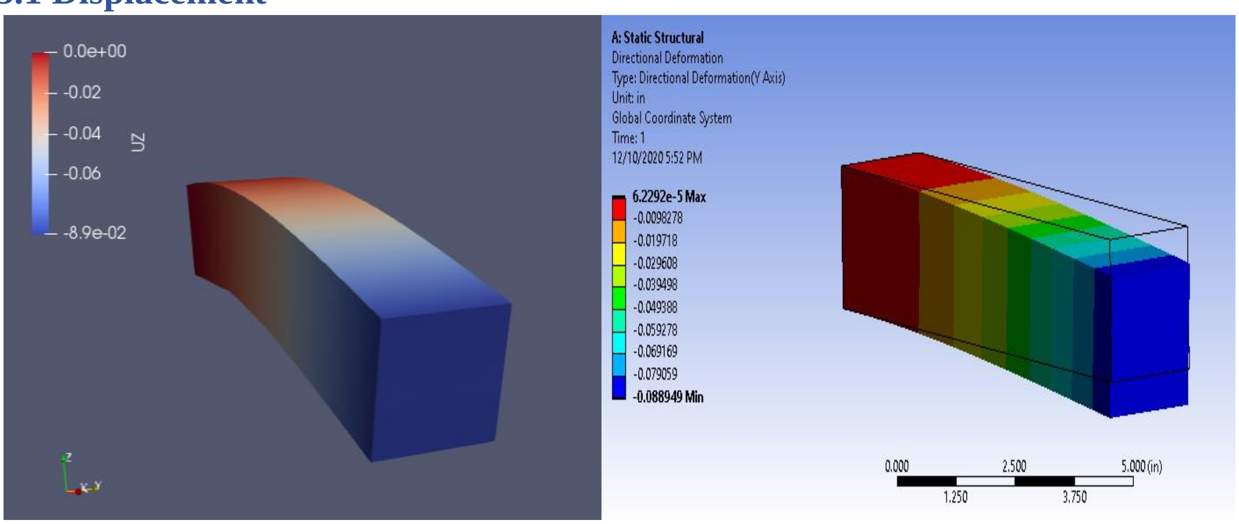
<Figure 15. Vector field of maximum shear (τ_{max}) and Von Mises (σ_{von}) stress>

Figure 15 is Vector field of maximum shear (τ_{max}) and Von Mises (σ_{von}) stress. Left figure is τ_{max} and right figure is σ_{von}. Their direction is gradient of contour surfaces of the stress. Their trend looks very similar. There are big flows toward fixed end's top and bottom edges of the model. In addition, there is a flow in the middle of the model.

5 Verification

In order to check designed model works well, its result will be compared with commercial software such as ANSYS and analytical solution.

5.1 Displacement



<Figure 16. Results of designed algorithm and ANSYS>

Designed 3-D models' displacement result is -0.089 inches and result of ANSYS is -0.0889 inches. They are almost same value.

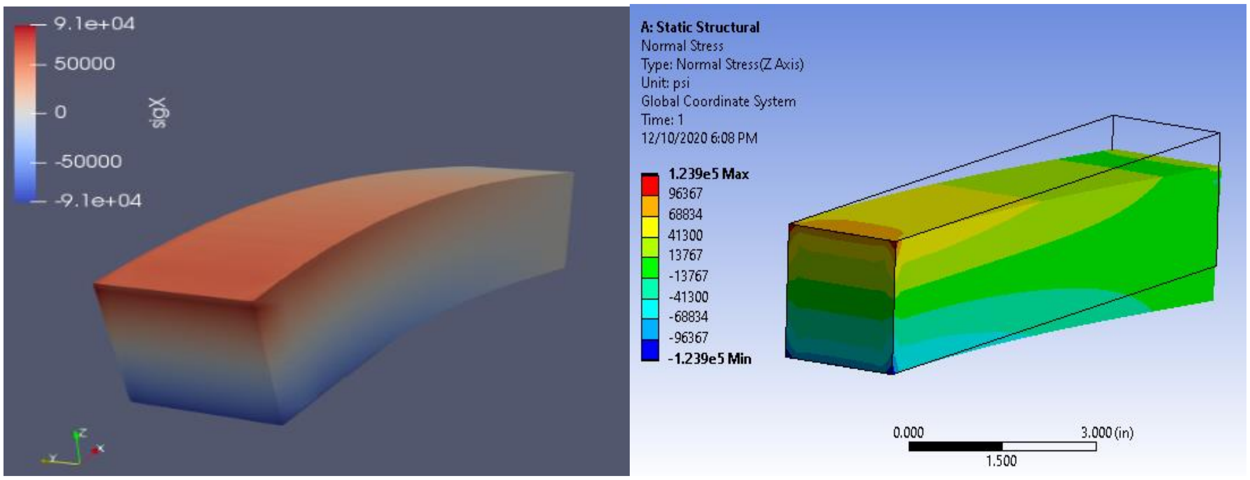
$$\delta = \frac{pL^3}{3EI} = \frac{-10e3 \cdot 10^3}{3 \cdot 29e6 \cdot 1.333} = -0.086\ in$$

$$\delta: Analytical\ displacement$$

$$where, I = \frac{bh^3}{12} = \frac{2^4}{12} = 1.333\ in^4$$

Designed model's result is -0.089 inches and analytical result is -0.086 inches. They are almost same value. Thus, it is possible to say it works well.

5.1 Stress



<Figure 17. Results of designed algorithm and ANSYS>

Designed 3-D models' normal stress result is ± 91000 psi and result of ANSYS is ± 123900 psi. Difference are about 35 %. It might be due to convergence of mesh.

$$\sigma = \frac{My}{I} = \pm \frac{-10000 \cdot 10 \cdot 1}{1.333} = \pm 75000\ psi$$

$$\sigma: Normal\ stress$$

Designed model's result is ± 91000 psi and analytical result is ± 75000 psi. Difference are about 17.5 %. Compared with the answer of ANSYS, designed algorithm's solution is much closer to analytical solution. Its difference is not small but it is possible to say it works well.

6. Future work

First, it is very difficult for users to define boundary conditions on the system. Users are required to have both mechanical and computer science knowledge to define them. So, it must be improved. Second, computational speed is not very good. Parallel algorithm and numerical linear algebra knowledge will be useful to improve it. Finally, the system causes memory problem due to the large size of stiffness matrix. In the case of 20 by 20 by 20 system, the size of stiffness matrix is 24000 by 24000. Sparse matrix solving method will be applied to reduce the size of the matrix.

7 Conclusion

In bending problem, x-normal stress (σ_x) dominates trend. Its value is almost three times larger than that of y-normal stress (σ_y) and z-normal stress (σ_z). In addition, it is possible to check its normal vector of contour surface is same with displacement shape of the model. So, σ_x dominates the trend. The stress distribution of x-normal stress (σ_x), maximum principal stress (σ_1) and x-normal stress of deviatoric stress ($\sigma_{x_{dev}}$) are same. Among them, σ_1 shows the largest value. The trend of maximum shear stress (τ_{max}) and Von Mises (σ_{von}) are same. but the value of σ_{von}'s value is almost twice of that of τ_{max}. In all cases, it is very obvious fixed end's stress is huge and free end's stress is almost zero. In the case of displacement, designed algorithm's solution is almost same with that of ANSYS and analytical solution. But there is some difference at answer for normal stress about 17.5%. Considering both displacement and stress, it is possible to say designed algorithm works well.

8 Reference

https://www.brown.edu/Departments/Engineering/Courses/En123/Exams/SGquizes/SG_QUIZ_11.htm

https://physics.stackexchange.com/questions/357082/euler-bernoulli-equation-for-a-cantilever-strained-by-a-force